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Research article

Euler's totient function applied to complete hypergroups

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Abstract: We study the Euler's totient function (called also the Euler's phi function) in the framework of finite complete hypergroups. These are algebraic hypercompositional structures constructed with the help of groups, and endowed with a multivalued operation, called hyperoperation. On them the Euler's phi function is multiplicative and not injective. In the second part of the article we find a relationship between the subhypergroups of a complete hypergroup and the subgroups of the group involved in the construction of the considered complete hypergroup. As sample application of this connection, we state a formula that relates the Euler's totient function defined on a complete hypergroup to the same function applied to its subhypergroups.

Keywords: Euler's totient function; complete hypergroup; period of an element; heart of a hypergroup
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1. Introduction

The Euler's totient function, introduced by Leonhard Euler in 1763, as an arithmetic function that counts all positive integers up to a given positive integer n and relatively prime to n, has many applications not only in number theory or in RSA encryption system used for security purposes, but also in group theory, and recently in hypergroup theory. Knowing the decomposition of the positive integer n in prime factors, i.e., $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where all p_i, $i \in \{1, \ldots, k\}$, are prime numbers, one calculates the value of the phi function as $\varphi(n) = n\left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_k}\right)$. Moreover, the Euler's totient function shows the order of the multiplicative group of integers modulo n. It can be calculated also using only group theory elements, through the formula $\varphi(n) = |\{\bar{a} \in \mathbb{Z}_n \mid o(\bar{a}) = \exp(\mathbb{Z}_n)\}|$, where $o(\bar{a})$ denotes the order of the element $\bar{a}$ in $\mathbb{Z}_n$ and $\exp(\mathbb{Z}_n)$ is the exponent of the group $\mathbb{Z}_n$. Naturally, this formula can be extended to an arbitrary group G, defining $\varphi(G) = |\{a \in G \mid o(a) = \exp(G)\}|$ [1].

Clearly, $\varphi(n) = \varphi(\mathbb{Z}_n)$, which leads to the investigation of the relation between $\varphi(G)$ and $\varphi(|G|)$, getting the equality when G is a cyclic group. More details on this topic can be found in Tărnăuceanu [1].

Since hypergroups are meaningful generalizations of groups, it is natural to ask about the behaviour of the Euler's totient function on different types of hypergroups. Extending the operation on groups to hyperoperations, i.e., to multivalued operations that associate to any pair of elements x and y in the underlying set H a subset $x \circ y$ of H, one obtains a hypercomposition structure $(H, \circ)$ having particular properties. If the hyperoperation is associative, meaning that $x \circ (y \circ z) = (x \circ y) \circ z$, for any elements x, y, z in H, and reproductive, i.e., $x \circ H = H \circ x = H$ for all x in H, then the hypercompositional structure $(H, \circ)$ is a hypergroup. Notice that in a generic hypergroup the existence of a unit or of an inverse element is not required, but there exist some hypergroups satisfying this property. These are, for example, the canonical hypergroups, studied by Mittas [2] in 1972 for their properties related to homomorphisms and subhypergroups and nowadays for their connections with Krasner hyperfields [3, 4], or the quasi-canonical hypergroups introduced in 1981 by Bonansinga [5], called also polygroups by Comer [6] and then by other researchers [7–9]. Recently, Sonea and Davvaz [10] have studied the Euler's totient function in the framework of canonical hypergroups. It is defined as an arithmetic function on a hypergroup, counting the number of the elements in H having the period equal to the exponent of the hypergroup, i.e., $\varphi(H) = |\{x \in H \mid p(x) = \exp(H)\}|$, where the period $p(x)$ of an arbitrary element x in H is the minimum natural number k such that the k-power x^k is a subset of the heart of H. In other words, the definition of the Euler's totient function for groups is slightly changed, substituting the order of an element with its period.

In this paper we study several properties of the phi function defined on finite complete hypergroups. They were defined in 1978 using the notion of complete part by Corsini [11], who also characterised them in [12], by connecting their hyperoperation with the operation of a group, called the underlying group of the complete hypergroup (see Theorem 2.1). Fundamental properties of complete hypergroups are recalled in Preliminaries, after a short overview on hypergroup theory. Moreover, we prove that the cartesian product of two complete hypergroups is a complete hypergroup, too (see Proposition 2.3), having the heart equal to the cartesian product of the two hearts of the composing hypergroups (see Proposition 2.4). The second part of Section 2 presents some fundamental properties of the Euler's totient function on groups, while Section 2.2 is dedicated to the study of the phi function applied to finite complete hypergroups. First we calculate its value using the partition of the complete hypergroup. Then we prove that it is a multiplicative function (see Proposition 3.5) and not injective. In Section 4 we concentrate on some properties of the subhypergroups of the complete hypergroups, relating them to the subgroups of the underlying groups. This connection is then used to calculate the Euler's totient function of the subhypergroups of the complete hypergroups having $\mathbb{Z}_n$ as their underlying group (see Proposition 4.4). The manuscript ends with several conclusive ideas and future works.

2. Preliminaries

We begin by recalling the basic notions and results about the complete hypergroups and Euler's function in group theory, necessary to understand the rest of this paper. For a more complete overview of the hypergroup theory we refer to the books [13, 14] and articles [15–17].

2.1. Complete hypergroups

After reviewing some fundamental definitions in hypergroup theory, we focus on complete hypergroups, in particular on the direct product of finite complete hypergroups.

Given a nonempty set H and denoting by $\mathcal{P}^*(H)$ the set of all nonempty subsets of H, we define a binary hyperoperation on H as a function $\circ : H \times H \longrightarrow \mathcal{P}^*(H)$. Then the couple $(H, \circ)$ is called a hypergroupoid, and in particular it is

(i) a semihypergroup if the associativity holds, i.e., for all $(a, b, c) \in H^3$, $(a \circ b) \circ c = a \circ (b \circ c)$,
(ii) a quasihypergroup if the reproductive law holds, i.e., for all $a \in H$, $H \circ a = a \circ H = H$,
(iii) a hypergroup if it is a semihypergroup and a quasihypergroup.

An element $e \in H$ is called a left identity or left unit of the hypergroupoid H if, for any $a \in H$, it satisfies the relation $a \in e \circ a$. Similarly, a right identity is defined. We say that $e \in H$ is a bilateral identity (sometimes called just identity or unit) if there is $a \in a \circ e \cap e \circ a$. An element a of a hypergroupoid H endowed with at least one bilateral identity e is called invertible, if there exists at least one element a' in H such that $e \in a \circ a' \cap a' \circ a$. Such an element a' is called an inverse of a. It is worth mentioning that an arbitrary hypergroup may have elements with zero inverses, or with only one inverse, ore with more inverses. A hypergroup H having at least one bilateral identity and with all elements having at least one inverse is called regular. Moreover, a regular hypergroup is called reversible if for all $a, b, c \in H$ such that $a \in b \circ c$, it follows that $b \in a \circ c'$ and $c \in b' \circ a$, for some inverses b' of b and c' of c.

The natural connection between groups and hypergroups is established with the help of the equivalence $\beta = \bigcup_{n\geq 1} \beta_n$, where β_1 is the diagonal relation on H and for any integer $n > 1$, β_n is defined as follows:

$$a\beta_n b \Leftrightarrow \exists n \in \mathbb{N}, \exists\, (x_1, x_2, \ldots, x_n) \in H^n : \{a, b\} \subseteq \prod_{i=1}^{n} x_i.$$

If H is a hypergroup, then the quotient H/β is a group and β is the smallest equivalence relation on H with this property, called a fundamental relation. Denoting by $\pi_H : H \to H/\beta$ the canonical projection, we define the heart of a hypergroup H as the set $\omega_H = \{x \in H | \ \pi_H(x) = 1\}$, where 1 is the identity of the group H/β.

A nonempty subset A of a semihypergroup H is called a complete part of H if the following implication holds: for any natural number n and for any arbitrary elements $x_1, \ldots, x_n \in H$, such that $\prod_{i=1}^{n} x_i \cap A \neq \emptyset$, it follows that $\prod_{i=1}^{n} x_i \subseteq A$. The complete closure $C(A)$ of A in H is then defined as the intersection of all complete parts of H containing A. It is known that $C(A) = A \circ \omega_H = \omega_H \circ A$, for any nonempty subset A of H.

Based on the notion of complete part [18], one may define the complete hypergroups. A semihypergroup $(H, \circ)$ is complete if, for any $(x, y) \in H^2$, $C(x \circ y) = x \circ y$. Since in practice this definition is not used very much, we recall here the characterization theorem.

Theorem 2.1. *[12,13] A hypergroup $(H, \circ)$ is complete if and only if there exist some nonempty subsets A_g of H, for all $g \in G$, such that $H = \bigcup_{g\in G} A_g$, where G and A_g satisfy the following conditions:*

(1) $(G, \cdot)$ is a group.

(2) For all $g_1 \neq g_2 \in G$*, there is* $A_{g_1} \cap A_{g_2} = \emptyset$*.*
(3) If $(a,b) \in A_{g_1} \times A_{g_2}$*, then* $a \circ b = A_{g_1 \cdot g_2}$*.*

We refer to G as the underlying group of the complete hypergroup H. Notice that several non-isomorphic complete hypergroups of the same cardinality can be constructed with the same underlying group G, depending on the cardinalities of the subsets A_g, with $g \in G$, that partition the hypergroup H. It is clear that any group can be seen as a complete hypergroup, while if G and H have the same cardinality, then the complete hypergroup H is a group.

Theorem 2.2. *[13, 19] Let* $(H, \circ)$ *be a complete hypergroup with the underlying group G, where* $H = \bigcup_{g \in G} A_g$*. Then:*

(1) *The heart* ω_H *of the hypergroup H is the set of all bilateral identities of H.*
(2) *If e is the identity of the group G, then* $\omega_H = A_e$*.*
(3) *The* β*-classes of H are the sets* A_g*, with* $g \in G$*.*
(4) *H is a reversible and regular hypergroup.*

The next result shows that the cartesian product of finite complete hypergroups is again a finite complete hypergroup.

Proposition 2.3. *If* $(H_1, \circ_1)$ *and* $(H_2, \circ_2)$ *are two finite complete hypergroups, then the cartesian product* $(H_1 \times H_2, \otimes)$ *is also a finite complete hypergroup with respect to the hyperproduct* $\otimes : (H_1 \times H_2) \times (H_1 \times H_2) \to \mathcal{P}^*(H_1 \times H_2)$*, defined as follows:*

$$(a_1, b_1) \otimes (a_2, b_2) = \{(a, b) \in H_1 \times H_2 |\ a \in a_1 \circ_1 a_2,\ b \in b_1 \circ_2 b_2\}.$$

Proof. Let H_i, $i = 1, 2$, be a complete hypergroup with the underlying group G_i, $i = 1, 2$. Using the characterization theorem, we will prove that $(H_1 \times H_2, \otimes)$ is a complete hypergroup having the underlying group $G_1 \times G_2$.

Writing $H_1 = \bigcup_{g \in G_1} A_g$ and $H_2 = \bigcup_{h \in G_2} A_h$, it follows that $H_1 \times H_2 = \bigcup_{(g,h) \in G_1 \times G_2} \left(A_g \times A_h\right)$. It is enough to show that such a partition of $H_1 \times H_2$ satisfies the conditions in Theorem 2.1.

(1) $G_1 \times G_2$ is indeed a group.

(2) For any $(g_1, h_1) \neq (g_2, h_2) \in G_1 \times G_2$, we prove that $(A_{g_1} \times A_{h_1}) \cap (A_{g_2} \times A_{h_2}) = \emptyset$. By absurd, let us consider $(a, b) \in (A_{g_1} \times A_{h_1}) \cap (A_{g_2} \times A_{h_2})$. This means that $a \in A_{g_1} \cap A_{g_2}$ and $b \in A_{h_1} \cap A_{h_2}$, leading to the relations $g_1 = g_2$ and $h_1 = h_2$, contradicting the hypothesis.

(3) For any arbitrary elements $(a_1, b_1) \in A_{g_1} \times A_{h_1}$ and $(a_2, b_2) \in A_{g_2} \times A_{h_2}$, it holds

$$\begin{aligned} (a_1, b_1) \otimes (a_2, b_2) &= \{(a, b) \in H_1 \times H_2 \mid a \in a_1 \circ_1 a_2, b \in b_1 \circ_2 b_2\} \\ &= \left\{(a, b) \in H_1 \times H_2 \mid a \in A_{g_1 g_2}, b \in A_{h_1 h_2}\right\} \\ &= A_{g_1 g_2} \times A_{h_1 h_2}. \end{aligned}$$

Concluding, $H_1 \times H_2$ is a complete hypergroup. □

The next result determines a useful connection between the hearts of the complete hypergroups H_1, H_2 and the one of their cartesian product $H_1 \times H_2$.

Proposition 2.4. *Let H_1 and H_2 be two complete hypergroups. Then*

$$\omega_{H_1\times H_2} = \omega_{H_1} \times \omega_{H_2}.$$

Proof. According with Theorem 2.2 applied to the complete hypergroup $H_1 \times H_2$, we know that its heart $\omega_{H_1\times H_2}$ is the set of the bilateral identities of $H_1 \times H_2$, i.e.,

$$\omega_{H_1\times H_2}=\{(a,b) \mid (a,b)\in(a,b)\otimes(x,y) \cap (x,y)\otimes(a,b) \text{ for any } (x,y)\in H_1\times H_2\}.$$

Using the definition of the hyperproduct $\otimes$, the relation $(a,b) \in (a,b) \otimes (x,y) \cap (x,y) \otimes (a,b)$ for any $(x,y) \in H_1 \times H_2$ is equivalent with $a \in a \circ_1 x \cap x \circ_1 a$, for any $x \in H_1$, and $b \in b \circ_2 y \cap y \circ_2 b$, for any $y \in H_2$, meaning that $a \in \omega_{H_1}$ and $b \in \omega_{H_2}$. Clearly, $\omega_{H_1\times H_2} = \{a \in H_1 \mid a \in \omega_{H_1}\} \times \{b \in H_2 \mid b \in \omega_{H_2}\} = \omega_{H_1} \times \omega_{H_2}$. □

2.2. *The Euler's totient function in groups*

This subsection gathers some properties of Euler's totient function in group theory. For more details about this topic we refer the readers to [1].

For an arbitrary finite group $(G,\cdot)$, the Euler's totient function is defined as follows:

$$\varphi(G) = |\{a \in G \mid o(a) = \exp(G)\}|, \tag{2.1}$$

where by $o(a)$ we denote the order of an arbitrary element a from G, while the exponent $\exp(G)$ of the group G is defined as the least common multiple of the orders of all elements of the group. If there is no least common multiple, the exponent is considered to be zero. For example, in the group $G = (\mathbb{Z}_n, +)$, the order of an arbitrary element $\bar{a}$ is $o(\bar{a}) = \frac{n}{\gcd(a,n)}$.

Recall now some well known properties of Euler's totient function from group theory, needed in the next sections of the article.

(1) The function φ is not injective, i.e., if G_1 and G_2 are two groups such that $\varphi(G_1) = \varphi(G_2)$, then it doesn't imply that $G_1 = G_2$. For example, $\varphi(\mathbb{Z}_3) = \varphi(\mathbb{Z}_4) = 2$.
(2) If G is a cyclic group, then $\varphi(G) = \varphi(|G|)$.
(3) Let G be a finite group with $\exp(G) = m$, $m \in \mathbb{N}$. Then $\varphi(G) = \varphi(m)k$, where k represents the number of the cyclic subgroups of order m in G.
(4) The function φ is multiplicative, i.e., if $\{G_i\}_{i=1,\dots,k}$ is a family of finite groups of relatively prime orders, then
$$\varphi\left(\prod_{i=1}^{k} G_i\right) = \prod_{i=1}^{k}\varphi(G_i).$$

3. The Euler's totient function for complete hypergroups

In this section we study the form of the Euler's totient function on finite complete hypergroups. First, we show that its form is strictly related to the partition of the complete hypergroup obtained through the characterization theorem. Then we prove that this function is a multiplicative one, as it is in group theory, and that it is not a one-to-one function. In the second part of the section we present a connection between the subhypergroups of a complete hypergroup and the subgroups of the underlying group.

We start with some results related to the period of an element in a hypergroup. Its role is similar with the one of the order of an element in group theory. This notion was introduced by Vougiouklis [20] for cyclic hypergroups, clearly reviewed in [21], but it can be naturally extended to an arbitrary hypergroup.

Definition 3.1. Let H be a hypergroup with the heart ω_H. An element x from H is called periodic, if there exists $k \in \mathbb{N}$ such that $x^k \subseteq \omega_H$. The period of x, denoted $p(x)$, is then defined by the formula

$$p(x) = \min\{k \in \mathbb{N} \mid x^k \subseteq \omega_H\}. \tag{3.1}$$

Similarly to group theory, the Euler's totient function on a hypergroup H [10] is defined as follows:

$$\varphi(H) = |\{x \in H \mid p(x) = \exp(H)\}|,$$

where $\exp(H)$ represents the least common multiple of the periods of all elements of the hypergroup H. In other words,

$$exp(H) = l.c.m.\{p_1, p_2, \ldots, p_k\},\ p(a_i) = p_i, a_i \in H,\ i \in \{1, \ldots, k\}.$$

The next result presents the relation between the period of the elements of a complete hypergroup H and the order of the elements of the underlying group G.

Proposition 3.2. *Let $(H, \circ)$ be a complete hypergroup with the underlying group G, i.e., $H = \bigcup_{g \in G} A_g$. The period of any element $x \in A_g$ is equal to the order of the element g in the group G, i.e., we have*

$$p(x) = o(g).$$

Proof. According to Theorem 2.1, the complete hypergroup H can be represented as the disjoint union $H = \bigcup_{g \in G} A_g$. Thus, for any $x \in H$ it exists and it is unique an element $g \in G$ such that $x \in A_g$. If $p(x) = m$, then, on one hand, we get $x^m \subseteq \omega_H = A_e$, where e denotes the neutral element of the group G. On the other hand, based on the definition of the hyperoperation of the complete hypergroup, we get $x^m \subseteq A_{g^m}$. Therefore $A_e \cap A_{g^m} \neq \emptyset$, meaning that $g^m = e$, where m is the smallest natural number with this property, equivalently with $o(g) = m$.

Thereby we conclude that $p(x) = o(g) = m$. □

As an immediate corollary, we obtain that for any complete hypergroup H with the underlying group G, there is $exp(H) = exp(G)$ and thus the Euler's totient function has the particular form

$$\varphi(H) = \left| \left\{ x \in \bigcup_{g \in G} A_g \mid o(g) = \exp(G), x \in A_g \right\} \right| = \sum_{o(g) = \exp(G)} |A_g|. \tag{3.2}$$

We better illustrate this formula in the following example.

Example 3.3. Let $G = \left(\mathbb{Z}_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}, +\right)$ and H a complete hypergroup with 9 elements, given by the following partition:

$$A_{\bar{0}} = \{a_0, a_1\},\ A_{\bar{1}} = \{a_2, a_3, a_4\},\ A_{\bar{2}} = \{a_5\},\ A_{\bar{3}} = \{a_6, a_7, a_8\}.$$

Based on the characterization theorem, the Cayley table of the complete hypergroup H has the form

$\circ$	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
a_0	$A_{\hat{0}}$	$A_{\bar{0}}$	$A_{\bar{1}}$	$A_{\bar{1}}$	$A_{\bar{1}}$	$A_{\bar{2}}$	$A_{\bar{3}}$	$A_{\bar{3}}$	$A_{\bar{3}}$
a_1	$A_{\bar{0}}$	$A_{\bar{0}}$	$A_{\bar{1}}$	$A_{\bar{1}}$	$A_{\bar{1}}$	$A_{\bar{2}}$	$A_{\bar{3}}$	$A_{\bar{3}}$	$A_{\bar{3}}$
a_2	$A_{\bar{1}}$	$A_{\bar{1}}$	$A_{\bar{2}}$	$A_{\bar{2}}$	$A_{\bar{2}}$	$A_{\bar{3}}$	$A_{\bar{0}}$	$A_{\bar{0}}$	$A_{\bar{0}}$
a_3	$A_{\bar{1}}$	$A_{\bar{1}}$	$A_{\bar{2}}$	$A_{\bar{2}}$	$A_{\bar{2}}$	$A_{\bar{3}}$	$A_{\bar{0}}$	$A_{\bar{0}}$	$A_{\bar{0}}$
a_4	$A_{\bar{1}}$	$A_{\bar{1}}$	$A_{\bar{2}}$	$A_{\bar{2}}$	$A_{\bar{2}}$	$A_{\bar{3}}$	$A_{\bar{0}}$	$A_{\bar{0}}$	$A_{\bar{0}}$
a_5	$A_{\bar{2}}$	$A_{\bar{2}}$	$A_{\bar{0}}$	$A_{\bar{0}}$	$A_{\bar{0}}$	$A_{\bar{0}}$	$A_{\bar{1}}$	$A_{\bar{1}}$	$A_{\bar{1}}$
a_6	$A_{\bar{3}}$	$A_{\bar{3}}$	$A_{\bar{0}}$	$A_{\bar{0}}$	$A_{\bar{0}}$	$A_{\bar{1}}$	$A_{\bar{2}}$	$A_{\bar{2}}$	$A_{\bar{2}}$
a_7	$A_{\bar{3}}$	$A_{\bar{3}}$	$A_{\bar{0}}$	$A_{\bar{0}}$	$A_{\bar{0}}$	$A_{\bar{1}}$	$A_{\bar{2}}$	$A_{\bar{2}}$	$A_{\bar{2}}$
a_8	$A_{\bar{3}}$	$A_{\bar{3}}$	$A_{\bar{0}}$	$A_{\bar{0}}$	$A_{\bar{0}}$	$A_{\bar{1}}$	$A_{\bar{2}}$	$A_{\bar{2}}$	$A_{\bar{2}}$

We know that $\omega_H = A_{\bar{0}} = \{a_0, a_1\}$. We calculate now the periods of all elements of H. It is clear that those elements situated in the same set A_g, with $g \in \mathbb{Z}_4$, have the same period. Since $a_0, a_1 \in \omega_H$, it follows that $p(a_0) = p(a_1) = 1$. By simple computations we get

$$\begin{aligned} a_2 \circ a_2 &= A_{\bar{2}}; (a_2 \circ a_2) \circ a_2 = A_{\bar{2}} \circ a_2 = a_5 \circ a_2 = A_{\bar{3}}; \\ (a_2 \circ a_2 \circ a_2) \circ a_2 &= A_{\bar{3}} \circ a_2 = \{a_6, a_7, a_8\} \circ a_2 \\ &= \bigcup_{i=6}^{8} (a_i \circ a_2) = A_{\bar{0}} = \omega_H, \end{aligned}$$

meaning that $p(a_2) = 4 = p(a_3) = p(a_4)$. Similarly, $a_5 \circ a_5 = A_{\bar{0}} = \omega_H$ implies that $p(a_5) = 2$.

Finally,

$$\begin{aligned} a_6 \circ a_6 &= A_{\bar{2}}; (a_6 \circ a_6) \circ a_6 = A_{\bar{2}} \circ a_6 = a_5 \circ a_6 = A_{\bar{1}}; \\ (a_6 \circ a_6 \circ a_6) \circ a_6 &= A_{\bar{1}} \circ a_6 = \{a_2, a_3, a_4\} \circ a_6 \\ &= \bigcup_{i=2}^{4} (a_i \circ a_6) = A_{\bar{0}} = \omega_H, \end{aligned}$$

concluding that $p(a_6) = 4 = p(a_7) = p(a_8)$.

Therefore $exp(H) = l.c.m.\{1, 4\} = 4 = exp(\mathbb{Z}_4)$ and the value of the Euler's totient function is

$$\begin{aligned} \varphi(H) &= |\{x \in H |\ p(x) = \exp(H)\}| \\ &= |\{x \in H |\ p(x) = 4\}| \\ &= |\{a_2, a_3, a_4, a_6, a_7, a_8\}| \\ &= 6. \end{aligned}$$

Moreover, calculating the orders of the elements of the group $\mathbb{Z}_4$, we get $o(\bar{0}) = 1$, $o(\bar{1}) = o(\bar{3}) = 4$, $o(\bar{2}) = 2$ and we notice that

$$\varphi(H) = |A_{\bar{1}}| + |A_{\bar{3}}| = 6.$$

Remark 3.4. Based on the definitions of the Euler's totient function for groups and complete hypergroups, there is always $\varphi(H) \geq \varphi(G)$, where G is the underlying group of the complete hypergroup H.

The next result shows that the Euler's totient function on complete hypergroups is multiplicative, under a certain condition. Recall that the same property is satisfied for this function defined on groups.

Theorem 3.5. *Let* $(H_1, \circ_1)$ *and* $(H_2, \circ_2)$ *be two finite complete hypergroups having the underlying groups* $(G_1, \cdot_1)$ *and* $(G_2, \cdot_2)$*, respectively, with relatively prime orders n and m. Then the Euler's totient function* φ *is multiplicative, i.e.,* $\varphi(H_1 \times H_2) = \varphi(H_1)\varphi(H_2)$.

Proof. Based on the characterization theorem for complete hypergroups, there exist the nonempty sets $\{A_g\}_{g\in G_1}$ and $\{A_h\}_{h\in G_2}$ such that

- $H_1 = \bigcup\limits_{g\in G_1} A_g$ and $H_2 = \bigcup\limits_{h\in G_2} A_h$.
- For any $g_1 \neq g_2$ and $h_1 \neq h_2$, we have $A_{g_1} \cap A_{g_2} = \emptyset$ and $A_{h_1} \cap A_{h_2} = \emptyset$.
- For $(a_1, b_1) \in A_{g_1} \times A_{g_2}$, there is $a_1 \circ_1 b_1 = A_{g_1 \cdot_1 g_2}$ and
- for $(a_2, b_2) \in A_{h_1} \times A_{h_2}$, there is $a_2 \circ_2 b_2 = A_{h_1 \cdot_2 h_2}$.

We know that the cartesian product $H_1 \times H_2$, endowed with the hyperproduct $\otimes : (H_1 \times H_2) \times (H_1 \times H_2) \to \mathcal{P}^*(H_1 \times H_2)$ defined by

$$\begin{aligned}(a_1, a_2) \otimes (b_1, b_2) &= \{(a, b) \mid a \in a_1 \circ_1 b_1,\ b \in a_2 \circ_2 b_2\} \\ &= \left\{(a, b) \in A_{g_1 \cdot_1 g_2} \times A_{h_1 \cdot_2 h_2}\right\},\end{aligned}$$

is a complete hypergroup partitioned as $H_1 \times H_2 = \bigcup\limits_{(g,h)\in G_1\times G_2} A_g \times A_h$.

Calculating the Euler's totient functions related to H_1, H_2 and $H_1 \times H_2$ based on formula (3.2), we obtain

$$\varphi(H_1) = \sum_{o(g)=\exp(G_1)} |A_g|,\ \ \varphi(H_2) = \sum_{o(h)=\exp(G_2)} |A_h|$$

and similarly

$$\varphi(H_1 \times H_2) = \sum_{o(g,h)=\exp(G_1)\cdot\exp(G_2)} |A_g \times A_h|.$$

By hypothesis, the groups G_1 and G_2 have relatively prime orders, therefore any two arbitrary elements $g \in G_1$ and $h \in G_2$ have relatively prime orders, too, meaning that $o(g, h) = l.c.m.\{o(g), o(h)\} = o(g) \cdot o(h)$. Thus the equality $o(g, h) = \exp(G_1) \cdot \exp(G_2)$ holds if and only if $o(g) = \exp(G_1)$ and $o(h) = \exp(G_2)$. Then the Euler's totient function related to $H_1 \times H_2$ has the form

$$\varphi(H_1 \times H_2) = \sum_{\substack{o(g)=\exp(G_1)\\ o(h)=\exp(G_2)}} |A_g \times A_h| = \sum_{\substack{o(g)=\exp(G_1)\\ o(h)=\exp(G_2)}} |A_g| \cdot |A_h| = \varphi(H_1)\varphi(H_2),$$

proving that the Euler's totient function is multiplicative. □

Example 3.6. With the help of the groups G_1=($\mathbb{Z}_3$=$\{\hat{0}, \hat{1}, \hat{2}\}, +)$ and $G_2 = (\mathbb{Z}_5=\{\bar{0}, \ldots, \bar{4}\}, +)$, we construct two complete hypergroups $H_1=\{a_0, a_1, \ldots, a_4\}$ and $H_2 = \{b_0, b_1, b_2, b_3, b_4, b_5, b_6\}$, considering the following representations $H_1 = \bigcup_{g\in\mathbb{Z}_3} A_g$ and, respectively, $H_2 = \bigcup_{h\in\mathbb{Z}_5} B_h$, where

$$\begin{aligned} A_{\hat{0}} &= \{a_0, a_1\};\ A_{\hat{1}} = \{a_2\},\ A_{\hat{2}} = \{a_3, a_4\}; \\ B_{\bar{0}} &= \{b_0\},\ B_{\bar{1}} = \{b_1\},\ B_{\bar{2}} = \{b_2, b_3\},\ B_{\bar{3}} = \{b_4, b_5\},\ B_{\bar{4}} = \{b_6\}. \end{aligned}$$

Since $(\mathbb{Z}_3, +)$ and $(\mathbb{Z}_5, +)$ are cyclic groups, accordingly with [1], we know that $\varphi(\mathbb{Z}_3) = \varphi(3) = 2$ and $\varphi(\mathbb{Z}_5) = \varphi(5) = 4$. Based on Theorem 3.5, we compute

$$\begin{aligned} \varphi(H_1) &= \sum_{ord(g)=\exp(\mathbb{Z}_3)=3} |A_g| = |A_{\hat{1}}| + |A_{\hat{2}}| = 1 + 2 = 3; \\ \varphi(H_2) &= \sum_{ord(H)=\exp(\mathbb{Z}_5)=5} |B_h| = |B_{\bar{1}}| + |B_{\bar{2}}| + |B_{\bar{3}}| + |B_{\bar{4}}| \\ &= 1 + 2 + 2 + 1 = 6. \end{aligned}$$

Therefore, $\varphi(H_1)\cdot\varphi(H_2) = 18$. Calculating the Euler's totient function for the complete hypergroup $H_1 \times H_2$, we obtain

$$\begin{aligned} &\varphi(H_1 \times H_2) = \sum_{o(g,h)=\exp(\mathbb{Z}_3)\cdot\exp(\mathbb{Z}_5)} |A_g \times B_h| = \sum_{o(g)=3, o(h)=5} |A_g \times B_h| \\ &= \sum_{g\in\{\hat{1},\hat{2}\}, h\in\{\bar{1},\ldots,\bar{4}\}} |A_g \times B_h| \\ &= |\{(a_2, b_1), \ldots, (a_2, b_6)\}| + |\{(a_3, b_1), \ldots, (a_3, b_6)\}| + |\{(a_4, b_1), \ldots, (a_4, b_6)\}| \\ &= 6 + 6 + 6 = 18. \end{aligned}$$

The condition saying that the orders of the groups G_1 and G_2 must be relatively prime is a necessary one for the multiplicity property of the Euler's totient function, as shown by the following example.

Example 3.7. Let us consider the groups $G_1 = \left(\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}, +\right)$ and the Klein four-group $G_2 = (K = \{e, a, b, c\}, \cdot)$ as the underlying groups of the complete hypergroups $H_1 = \{a_1, a_2, a_3\}$ and $H_2 = \{b_1, b_2, \ldots, b_6\}$ partitioned as follows $H_1 = \bigcup_{g\in\mathbb{Z}_2} A_g$, and $H_2 = \bigcup_{h\in K} B_h$, with

$$\begin{aligned} A_{\bar{0}} &= \{a_1\},\ A_{\bar{1}} = \{a_2, a_3\}; \\ B_e &= \{b_1, b_2\},\ B_a = \{b_3\},\ B_b = \{b_4, b_5\},\ B_c = \{b_6\}. \end{aligned}$$

We immediately notice that $\exp(\mathbb{Z}_2) = 2$, and $\exp(K) = 2$. Thereby

$$\begin{aligned} \varphi(H_1) &= \sum_{o(g)=\exp(\mathbb{Z}_2)=2} \left|A_g\right| = |A_{\bar{1}}| = 2, \\ \varphi(H_2) &= \sum_{o(h)=\exp(K)=2} |B_h| = |B_a| + |B_b| + |B_c| = 4, \end{aligned}$$

thus $\varphi(H_1) \cdot \varphi(H_2) = 8$.

In order to determine the Euler's totient function associated to the cartesian product $H_1 \times H_2$, we first calculate $\exp(G_1 \times G_2) = l.c.m.\{o(g, h) \mid g \in \mathbb{Z}_2, h \in K\}$. The identity element of the group $G_1 \times G_2$

is $\left(\bar{0}, e\right)$, having the order 1, while all the other elements have the order 2, so $\exp(G_1 \times G_2) = 2 = \exp(H_1 \times H_2)$ and $p(a, b) = 2$ for any $(a, b) \in (H_1 \times H_2) \setminus (A_{\bar{0}} \times B_e)$. It follows that $\varphi(H_1 \times H_2) = |\{(a_1, b_3), (a_1, b_4), (a_1, b_5), (a_1, b_6)\}| + |\{(a_2, b_1), (a_2, b_2), \ldots, (a_2, b_6)\}| + |\{(a_3, b_1), (a_3, b_2), \ldots, (a_3, b_6)\}| = 4 + 6 + 6 = 16 \neq 8 = \varphi(H_1) \cdot \varphi(H_2)$, confirming that the function φ is not multiplicative in this case.

Moreover, the Euler's totient function is not one-to-one, as illustrated in the following example.

Example 3.8. Let $G_1 = (\mathbb{Z}_3 = \{\hat{0}, \hat{1}, \hat{2}\}, +)$ and $G_2 = (\mathbb{Z}_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}, +)$ be the underlying groups of the complete hypergroups $H_1 = \{a_0, a_1, a_2, a_3, a_4\}$ and $H_2 = \{b_0, b_1, b_2, b_3, b_4, b_5\}$, partitioned as $H_1 = \bigcup\limits_{g \in \mathbb{Z}_3} A_g$, $H_2 = \bigcup\limits_{h \in \mathbb{Z}_4} A_h$ where

$$\begin{aligned} A_{\hat{0}} &= \{a_0, a_1\},\ A_{\hat{1}} = \{a_2\},\ A_{\hat{2}} = \{a_3, a_4\}, \\ A_{\bar{0}} &= \{b_0\},\ A_{\bar{1}} = \{b_1\},\ A_{\bar{2}} = \{b_2, b_3\},\ A_{\bar{3}} = \{b_4, b_5\}. \end{aligned}$$

Calculating the Euler's totient functions we get

$$\begin{aligned} \varphi(H_1) &= \sum_{ord(g)=\exp(\mathbb{Z}_3)=3} |A_g| = |A_{\hat{1}}| + |A_{\hat{2}}| = 1 + 2 = 3, \\ \varphi(H_2) &= \sum_{ord(H)=\exp(\mathbb{Z}_4)=4} |A_h| = |B_{\bar{1}}| + |B_{\bar{3}}| = 1 + 2 = 3. \end{aligned}$$

Therefore $\varphi(H_1) = \varphi(H_2) = 3$, but $H_1 \neq H_2$.

4. Properties of the subhypergroups of a complete hypergroup

In this section we focus on some properties of the subhypergroups of complete hypergroups, determining a connection between them and the subgroups of their underlying group. This connection will help us to compute, at the end of this section, the Euler's totient function of the complete hypergroups having the additive group $\mathbb{Z}_n$ as their underlying group.

Proposition 4.1. *Let $(H, \circ)$ be a complete hypergroup with the underlying group G, i.e., $H = \bigcup\limits_{g \in G} A_g$. If K is a subgroup of G, then $\mathcal{K} = \bigcup\limits_{k \in K} A_k$ is a subhypergroup in H.*

Proof. $\mathcal{K}$ is a subhypergroup in H if and only if it satisfies the two conditions:

(1) For any $a, b \in \mathcal{K}$, $a \circ b \subseteq \mathcal{K}$.
(2) For any $a \in \mathcal{K}$, $a \circ \mathcal{K} = \mathcal{K} \circ a = \mathcal{K}$.

To check the first condition, let a, b be two arbitrary elements in $\mathcal{K}$. Then there exist, and they are unique, $k_1, k_2 \in K$ such that $a \in A_{k_1}$, $b \in A_{k_2}$, implying that $a \circ b = A_{k_1 k_2} \subseteq \mathcal{K}$, because K is a subgroup in G.

The second condition can be written in the following way: there exists $k \in K$ such that $a \in A_k$ and then

$$a \circ \mathcal{K} = \bigcup_{b \in \mathcal{K}} (a \circ b) = \bigcup_{i=1}^{|K|} A_{kk_i} = \bigcup_{j=1}^{|K|} A_{k_j} = \mathcal{K},$$

$$\mathcal{K} \circ a \;=\; \bigcup_{b \in \mathcal{K}} (b \circ a) = \bigcup_{i=1}^{|K|} A_{k_i k} = \bigcup_{j=1}^{|K|} A_{k_j} = \mathcal{K}.$$

Indeed, since the function $f : K \to K$ defined by $f(k_i) = kk_i$, with $i = 1, \ldots, |K|$, is a bijection, it follows that, for any element $k_j \in K$, there exists $k_i \in K$ such that $kk_i = k_j$ and thereby $\bigcup_{i=1}^{|K|} A_{kk_i} = \bigcup_{j=1}^{|K|} A_{k_j}$, and similarly for the second equality. □

The converse implication holds too, as shown below.

Proposition 4.2. *Let* $(H, \circ)$ *be a complete hypergroup with the underlying group* G. *If* K *is a non empty subset of* G *such that* $\mathcal{K} = \bigcup_{k \in K} A_k$ *is a subhypergroup in* H, *then* K *is a subgroup in* G.

Proof. Let K be a non empty subset of G such that $\mathcal{K} = \bigcup_{k \in K} A_k$ is a subhypergroup of the complete hypergroup H. This means that, for any $a, b \in \mathcal{K}$, it holds $a \circ b \subseteq \mathcal{K}$ and, for any $a \in \mathcal{K}$, we have $a \circ \mathcal{K} = \mathcal{K} \circ a = \mathcal{K}$.

By the first condition, there exist $k_i, k_j \in K$, with $i, j \in \{1, \ldots, |K|\}$, such that $a \circ b = A_{k_i k_j} \subseteq \mathcal{K}$, implying that $k_i k_j \in K$. Since the condition holds for any a and b in H, it follows that $k_i k_j \in K$, for any $k_i, k_j \in K$. In order to prove that K is a subgroup of G, it remains to show that for any $k \in K$, the inverse k^{-1} is an element of K, too. For doing that, first we prove that the heart ω_H of the hypergroup H is a subset of $\mathcal{K}$. Let k be an arbitrary element in K and $a \in A_k$. Since $a \circ \mathcal{K} = \mathcal{K}$, it follows that there exists $b \in \mathcal{K}$ such that $a \in a \circ b$, so there exists $k_1 \in K$ such that $b \in A_{k_1}$, where now $A_k \cap A_{kk_1} \neq \emptyset$. This leads to the equality $k = kk_1$, thus $k_1 = e \in K$. Thereby $A_e = \omega_H \subseteq \mathcal{K}$. Besides, for an arbitrary a_0 in $A_e = \omega_H \subseteq \mathcal{K}$, there exists $a' \in \mathcal{K}$ such that $a_0 \in a \circ a'$, equivalently, there exists $k' \in K$ such that $a' \in A_{k'}$ and $A_e = A_{kk'}$. This implies that $e = kk'$, i.e., $k' = k^{-1} \in K$.

Concluding, K is a subgroup of the group G. □

As a consequence of Propositions 4.1 and 4.2, we obtain the following result.

Theorem 4.3. *Let* $(H, \circ)$ *be a complete hypergroup having the underlying group* G. *Then* K *is a subgroup in* G *if and only if* $\mathcal{K} = \bigcup_{k \in K} A_k$ *is a subhypergroup in* H.

Based on Theorem 4.3, we may state a relation between the Euler's totient function associated to a complete hypergroup H with the underlying group $G = (\mathbb{Z}_n, +)$, $n \geq 2$, for which we know its subgroups, and the Euler's totient function associated to the subhypergroups of H. The elements of $\mathbb{Z}_n$ are denoted by $\overline{0}, \overline{1}, \ldots, \overline{n-1}$, while the subgroups of the cyclic group $(\mathbb{Z}_n, +)$ are $C_d =< \overline{\frac{n}{d}} >$, with $|C_d| = d$, and $d \mid n$, $d \in \mathbb{N}$. Applying now Theorem 4.3, we obtain that the subhypergroups of H have the form

$$\mathcal{K}_d = \bigcup_{\alpha=0}^{d-1} A_{\left(\alpha \cdot \overline{\frac{n}{d}}\right)}. \tag{4.1}$$

We start with the value of the Euler's totient function associated to a subhypergroup $\mathcal{K}_d$.

Proposition 4.4. *Let $(H, \circ)$ be a complete hypergroup having $(\mathbb{Z}_n, +)$ as the underlying group and let $\{\mathcal{K}_d\}_{d|n}$, $d \neq \{1, n\}$, be its subhypergroups. Then*

$$\varphi(\mathcal{K}_d) = \sum_{\substack{d|n \\ (\alpha,d)=1}} \left| A_{\left(\alpha\cdot\overline{\frac{n}{d}}\right)} \right|.$$

Proof. For any element $a \in A_{\left(\alpha\cdot\overline{\frac{n}{d}}\right)}$, we know by Proposition 3.2 that $p(a) = o(\alpha \cdot \overline{\frac{n}{d}})$, which is a divisor of d, for any $\alpha \in \{0, 1, \ldots, d-1\}$.

The Euler's totient function associated to the subhypergroup $\mathcal{K}_d$ is

$$\begin{aligned} \varphi(\mathcal{K}_d) &= \left|\{a \in \mathcal{K}_d \mid p(a) = \exp(\mathcal{K}_d)\}\right| \\ &= \left|\left\{a \in A_{\left(\alpha\cdot\overline{\frac{n}{d}}\right)} \mid o\left(\alpha \cdot \overline{\frac{n}{d}}\right) = \exp(C_d) = d\right\}\right|. \end{aligned}$$

In order to prove the statement of the theorem, we need to show that $o(\alpha \cdot \overline{(\frac{n}{d})}) = d$ if and only if $(\alpha, d) = 1$. On one side, let first suppose by absurd that $(\alpha, d) = \beta > 1$. It means that there exist $a_1, a_2 \in \mathbb{Z}$ such that $\alpha = \beta a_1$, $d = \beta a_2$, and $(a_1, a_2) = 1$. Then we calculate

$$ord\left(\alpha \cdot \overline{(\frac{n}{d})}\right) = ord\left(\beta a_1 \cdot \overline{(\frac{n}{\beta a_2})}\right) = ord\left(a_1 \cdot \overline{(\frac{n}{a_2})}\right) = a_2 \le d,$$

which is a contradiction.

On the other side, suppose that $(\alpha, d) = 1$. It is clear that $\alpha \cdot \overline{(\frac{n}{d})} \cdot d = \overline{\alpha \cdot n} = \bar{0}$. It remains to prove that d is the smallest positive integer with this property. If there exists another integer $b > 0$ such that $\alpha \cdot \overline{(\frac{n}{d})} \cdot b = \bar{0}$, then there exists $t \in \mathbb{Z}$ such that $\alpha \cdot \frac{n}{d} \cdot b = n \cdot t$, equivalently with $\frac{\alpha \cdot b}{d} = t \in \mathbb{Z}$. Since $(\alpha, d) = 1$, it follows that $d \mid b$, concluding that $o(\alpha \cdot \overline{(\frac{n}{d})}) = d$. □

We have excluded the particular cases of the improper divisors of n, i.e., $d = 1$ and $d = n$, because they lead to immediate results. Indeed, we get the subhypergroups $\mathcal{K}_1 = A_{\bar{0}} = \omega_H$, with $\varphi(\mathcal{K}_1) = |\omega_H|$, and $\mathcal{K}_n = H$, when $\varphi(\mathcal{K}_n) = \varphi(H)$, that are connected through one formula stated in the next result.

Theorem 4.5. *If $(H, \circ)$ is a complete hypergroup having the cyclic group $(\mathbb{Z}_n, +)$ as the underlying group, then*

$$\sum_{\substack{d \mid n \\ d \neq \{1, n\}}} \varphi(\mathcal{K}_d) = |H| - |\omega_H| - \varphi(H). \tag{4.2}$$

Proof. Based on Proposition 4.4, we know that

$$\sum_{\substack{d \mid n \\ d \neq \{1, n\}}} \varphi(\mathcal{K}_d) = \sum_{\substack{d \mid n \\ d \neq \{1, n\}}} \left(\sum_{(\alpha,d)=1} \left| A_{\left(\alpha\cdot\overline{\frac{n}{d}}\right)} \right| \right).$$

Using the definition of the Euler's function, we get

$$\varphi(H) = \sum_{\substack{g\in\mathbb{Z}_n \\ o(g)=\exp(\mathbb{Z}_n)=n}} |A_g| = \sum_{\substack{g\in\mathbb{Z}_n \\ (g,n)=1}} |A_g|.$$

We then calculate

$$|H| - |\omega_H| - \varphi(H) = \sum_{g\in\mathbb{Z}_n} \left|A_g\right| - |A_{\bar{0}}| - \sum_{\substack{g\in\mathbb{Z}_n \\ (g,n)=1}} |A_g| = \sum_{\substack{(g,n)\neq 1 \\ g\neq 1}} \left|A_g\right| = \sum_{g\in D^*(\mathbb{Z}_n)} \left|A_g\right|,$$

where by $D^*(\mathbb{Z}_n)$ we denote the set of all non-zero divisors of zero of $\mathbb{Z}_n$.

It remains to prove the equality

$$\sum_{\substack{d \mid n \\ d\neq\{1,n\}}} \left(\sum_{(\alpha,d)=1} \left|A_{\left(\alpha\cdot\overline{\frac{n}{d}}\right)}\right|\right) = \sum_{g\in D^*(\mathbb{Z}_n)} \left|A_g\right|. \tag{4.3}$$

To simplify the writing, we introduce the notation $g_{\alpha,d} = \alpha\cdot\overline{(\frac{n}{d})}$, with $(\alpha, d) = 1$, $d \neq 1$, $d \neq n$. First we show, under these hypotheses, that $g_{\alpha,d} \in D^*(\mathbb{Z}_n)$. Indeed, since $g_{\alpha,d}\cdot\overline{d} = \alpha\cdot\bar{n} = \bar{0}$, it follows that $g_{\alpha,d}$ is a zero-divisor. Let us suppose now that $g_{\alpha,d} = \bar{0}$, equivalently $\alpha\cdot\overline{(\frac{n}{d})} = \bar{0}$. This means that there exists $t \in \mathbb{Z}$ such that $\alpha\cdot\frac{n}{d} = n\cdot t$, leading to the fact that $\frac{\alpha}{d} \in \mathbb{Z}$, with $(\alpha, d) = 1$ and $d \neq 1$. This is a contradiction, therefore $g_{\alpha,d} \neq \bar{0}$.

Consequently, relation (4.3) can be written in the following way

$$\sum_{\substack{d|n \\ d\neq\{1,n\}}} \left(\sum_{g_{\alpha,d}\in D^*(\mathbb{Z}_n)} \left|A_{\left(\alpha\cdot\overline{\frac{n}{d}}\right)}\right|\right) = \sum_{g\in D^*(\mathbb{Z}_n)} \left|A_g\right|,$$

where the equality holds if and only if $\left\{\alpha\cdot\overline{\frac{n}{d}} \mid (\alpha, d) = 1, d \notin \{1, n\}\right\} = D^*(\mathbb{Z}_n)$.

To start with, denote $E_{\alpha,d} = \left\{\alpha\cdot\overline{\frac{n}{d}} \mid (\alpha, d) = 1, d \neq \{1, n\}\right\}$. We have already proved that $E_{\alpha,d} \subseteq D^*(\mathbb{Z}_n)$, so it remains to show the inverse inclusion. Let $\overline{x} \in D^*(\mathbb{Z}_n)$. It results that $(x, n) \neq 1$. Take $(x, n) = a$, $a \neq 1$. This means that there exist $b, c \in \mathbb{Z}$ such that $x = a\cdot b$, $n = a\cdot c$, with $(b, c) = 1$. Therefore

$$\overline{x} = b\cdot\overline{a} = b\cdot\overline{(\frac{a\cdot c}{c})} = b\cdot\overline{(\frac{n}{c})},$$

so $\overline{c}$ has the form of one element in $E_{\alpha,d}$, for $\alpha = b$ and $d = c$, implying that $D^*(\mathbb{Z}_n) \subseteq E_{\alpha,d}$.

Concluding, we proved the formula

$$\sum_{\substack{d \mid n \\ d\neq\{1,n\}}} \varphi(\mathcal{K}_d) = |H| - |\omega_H| - \varphi(H).$$

□

In the particular case when n is a prime number, we get that

$$\varphi(H) = \sum_{(g,n)=1} |A_g| = \sum_{g\neq\bar{0}} |A_g| = |H| - |\omega_H|.$$

Example 4.6. Let us consider the complete hypergroup H with the underlying group $G = (\mathbb{Z}_8, +)$ and having the following representation

$$\begin{aligned} A_{\bar{0}} &= \{a_0, a_1\}; A_{\bar{1}} = \{a_2, a_3, a_4\}, A_{\bar{2}} = \{a_5, a_6\}, \\ A_{\bar{3}} &= \{a_7, a_8, a_9, a_{10}\},\ A_{\bar{4}} = \{a_{11}\},\ A_{\bar{5}} = \{a_{12}, a_{13}\}, \\ A_{\bar{6}} &= \{a_{14}\},\ A_{\bar{7}} = \{a_{15}, a_{16}\}. \end{aligned}$$

We will check that formula stated in Theorem 4.5 holds. Indeed, we immediately calculate

$$\varphi(H) = \sum_{(g,n)=1} \left|A_g\right| = |A_{\bar{1}}| + |A_{\bar{3}}| + |A_{\bar{5}}| + |A_{\bar{7}}| = 3 + 4 + 2 + 2 = 11,$$

while $|H| - |\omega_H| - \varphi(H) = 17 - 2 - 11 = 4$. The proper subhypergroups of H are $\mathcal{K}_d = \bigcup_{\alpha=0}^{d-1} A_{\left(\alpha\cdot\overline{\left(\frac{n}{d}\right)}\right)}$, with $d \mid n$, and $\varphi(\mathcal{K}_d) = \sum_{(\alpha,d)=1} \left|A_{\left(\alpha\cdot\overline{\left(\frac{n}{d}\right)}\right)}\right|$. Therefore,

$$\begin{aligned} \mathcal{K}_2 &= \bigcup_{\alpha=0}^{2-1} A_{\left(\alpha\cdot\overline{\frac{8}{2}}\right)} = A_{\bar{0}} \cup A_{\bar{4}}; \\ \varphi(\mathcal{K}_2) &= \sum_{(\alpha,2)=1} \left|A_{\left(\alpha\cdot\overline{\frac{8}{2}}\right)}\right| = |A_{\bar{4}}| = 1; \\ \mathcal{K}_4 &= \bigcup_{\alpha=0}^{4-1} A_{\left(\alpha\cdot\overline{\frac{8}{4}}\right)} = A_{\bar{0}} \cup A_{\bar{2}} \cup A_{\bar{4}} \cup A_{\bar{6}}; \\ \varphi(\mathcal{K}_4) &= \sum_{(\alpha,4)=1} \left|A_{\left(\alpha\cdot\overline{\frac{8}{4}}\right)}\right| = |A_{\bar{2}}| + |A_{\bar{6}}| = 3. \end{aligned}$$

Concluding, $\varphi(\mathcal{K}_2) + \varphi(\mathcal{K}_4) = |H| - |\omega_H| - \varphi(H)$.

5. Conclusions

Complete hypergroups have proved to be a fertile environment to investigate several types of combinatorial problems related to some arithmetic functions. We recall here the study concerning the fuzzy grade of a complete hypergroup [15, 19], or the recent ones on the commutativity degree [22] and completeness degree [23]. This paper has considered the Euler's totient function defined by Sonea and Davvaz on hypergroups in [10] and applied here to finite complete hypergroups. We have proved that it is a multiplicative and not injective function. Moreover, we have established a formula that relates the phi function defined on a complete hypergroup to the same function defined on its subhypergroups.

This study has arisen some open questions. The first one is connected with the cyclic hypergroups. It is known that for a cyclic group G, the equality $\varphi(G) = \varphi(|G|)$ holds, so it is natural to investigate the validity of this formula also for cyclic hypergroups, considering all types of cyclicity, as discussed in [21]. The second open question is related to the particular case of complete hypergroups having the underlying group $\mathbb{Z}_p$, with p a prime number. We have noticed that the quantity $|H| - |\omega_H| - \varphi(H)$ is zero. Are there other types of hypergroups for which this expression is zero? Could we find some invariants related to the Euler's totient function on hypergroups?

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Conflict of interest

The authors declare no conflict of interest.

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