

Rigidity in Mechanism Design and Its Applications

Shahar Dobzinski 

Weizmann Institute of Science, Rehovot, Israel

Ariel Shaulker 

Weizmann Institute of Science, Rehovot, Israel

Abstract

We introduce the notion of rigidity in auction design and use it to analyze some fundamental aspects of mechanism design. We focus on the setting of a single-item auction where the values of the bidders are drawn from some (possibly correlated) distribution \mathcal{F} . Let f be the allocation function of an optimal mechanism for \mathcal{F} . Informally, S is (linearly) rigid in \mathcal{F} if for every mechanism M' with an allocation function f' where f and f' agree on the allocation of at most x -fraction of the instances of S , it holds that the expected revenue of M' is at most an x fraction of the optimal revenue.

We start with using rigidity to explain the singular success of Cremer and McLean's auction assuming interim individual rationality. Recall that the revenue of Cremer and McLean's auction is the optimal welfare if the distribution obeys a certain "full rank" conditions, but no analogous constructions are known if this condition does not hold. We show that the allocation function of the Cremer and McLean auction has logarithmic (in the size of the support) Kolmogorov complexity, whereas we use rigidity to show that there exist distributions that do not obey the full rank condition for which the allocation function of every mechanism that provides a constant approximation is almost linear.

We further investigate rigidity assuming different notions of individual rationality. Assuming ex-post individual rationality, if there exists a rigid set then the structure of the optimal mechanism is relatively simple: the player with the highest value "usually" wins the item and contributes most of the revenue. In contrast, assuming interim individual rationality, there are distributions with a rigid set S where the optimal mechanism has no obvious allocation pattern (in the sense that its Kolmogorov complexity is high). Since the existence of rigid sets essentially implies that the hands of the designer are tied, our results help explain why we have little hope of developing good, simple and generic approximation mechanisms in the interim individual rationality world.

2012 ACM Subject Classification Theory of computation \rightarrow Algorithmic game theory and mechanism design

Keywords and phrases Revenue Maximization, Auctions

Digital Object Identifier 10.4230/LIPIcs.ITCS.2023.44

Funding Work supported by ISF grant 2185/19 and BSF-NSF grant (BSF number: 2021655, NSF number: 2127781).

1 Introduction

The Setting

We consider the following standard auction model: there is one item for sale and n bidders. The value of bidder i for getting the item is v_i , and 0 otherwise. v_i is the private information of each bidder i . The values are drawn from some joint distribution $\mathcal{F} \in \mathbb{R}^n$ that is publicly known. This paper aims to design (deterministic) dominant strategy mechanisms that maximize the revenue. The literature considers two main notions of individual rationality:

- *Ex-post individual rationality*: the payment of bidder i is at most v_i if he wins the item, and 0 otherwise.



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14th Innovations in Theoretical Computer Science Conference (ITCS 2023).

Editor: Yael Tauman Kalai; Article No. 44; pp. 44:1–44:21

Leibniz International Proceedings in Informatics



LIPICs Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

- *Interim individual rationality*: the expected payment of bidder i with value v_i is at most $x_i \cdot v_i$, where x_i is the probability that bidder i wins the item given that his value is v_i . Recall that the auction mechanism is deterministic, so the expectation is only over the distribution \mathcal{F} of instances.

When the values of the players are drawn from independent distributions, Myerson [11] provides a characterization of the optimal auction. In a sharp contrast, when the distribution of values is correlated, no such crisp characterization is known. Broadly speaking, the literature takes two approaches: developing approximation mechanisms, and identifying special cases where the optimal solution takes a simple form. The first approach is dominant when assuming ex-post individual rationality, whereas the second one is more prominent when assuming interim individual rationality.

We now give a brief survey of the most related literature. We start with surveying the state-of-the-art in the design of ex-post IR mechanisms. The jewel in the crown here is Ronen's 2001 paper [13] that can be seen as an early precursor to many later trends in Algorithmic Mechanism Design. In that paper, Ronen presents the lookahead auction: the revenue-maximizing auction in which the item can be sold only to the bidder with the highest value.

Remarkably, Ronen proves that this simple, easy-to-describe auction always guarantees half of the optimal revenue. In contrast, computing the optimal auction is NP-hard [12]. Subsequent work considered a natural extension of the lookahead auction – the k -lookahead auction. This is the revenue-maximizing auction in which in every instance, the item is sold to at most one of the k bidders with the highest values, where k is ideally some small constant [6, 3, 7, 2, 9]. The most relevant result is that the approximation ratio of the k -lookahead auction approaches $\frac{e}{e+1}$ [3] and that this is tight [7].

The literature on interim IR mechanisms is also quite rich. The stunning result here is that of Cremer and McLean [4, 5], which shows that when the joint distribution satisfies a particular full rank condition, the revenue that the auctioneer can extract equals to *all* the expected social welfare. This revenue can be extracted by running a simple second-price auction and charging the participating bidders appropriate fees. However, Albert, Conitzer, and Lopomo [1] show that this result cannot be extended to all distributions by showing that there are distributions for which the ratio between the optimal revenue and the optimal welfare goes to 0. Feldman and Lavi [8] further show that this gap exists even for distributions that have “almost full” rank.

We stress that there are no known interim IR mechanisms that play an analogous role to that of the lookahead auction in the ex-post IR universe: there are no natural, simple to describe auctions that for every distribution provide a significant fraction of the optimal revenue. To put it differently, if our goal is revenue maximization, ex-post mechanism designers can offer us a rich toolbox to work with, whereas interim IR mechanism designers either offer us a dream solution (but only if our distribution happen to obey the Cremer-McLean condition), or leave us empty handed (if we want our solution to generically work for all possible distributions). A primary goal of this paper is to understand whether the set of tools of interim IR mechanism design can be significantly extended.

Understanding the Success of Cremer-McLean via Kolmogorov Complexity

Our first set of results (Section 4) attempts to explain the singular success of Cremer-McLean in the interim IR universe. We would like to show that there are no generic, simple-to-describe allocation functions that extract a significant fraction of the optimal (interim IR) revenue for

all distributions. To put some technical sense to this statement, we analyze the Kolmogorov complexity of the allocation function of good mechanisms. Recall that, informally speaking, the Kolmogorov complexity (see, e.g., [10]) of a string is the size of the smallest Turing machine that generates it. We are interested in the Kolmogorov complexity of functions, so we view an allocation function as a string over the alphabet $[n]$ that its i 'th position specifies which of the n players receives the item in the i 'th instance. Our results hold whenever the indices of the string correspond to instances that are ordered in some “natural” order. Roughly speaking, an order is natural if it has the almost obvious property that there exists a small Turing machine that given i , prints the i 'th instance.

When the distribution obeys the Cremer-McLean condition, the optimal allocation function is simple: give the item to a highest-value player. It is not hard to see that this implies that the Kolmogorov complexity of the optimal mechanism is low (logarithmic in the size of the support). In contrast, for some distributions for which the Cremer-McLean condition does not hold, we prove that even if we settle on approximations, the allocation function has high Kolmogorov complexity.

► **Theorem.** *Fix some constant $0 < c < 1$. There are distributions for which the Kolmogorov complexity of the allocation function of every mechanism that guarantees a c -fraction of the revenue-maximizing interim IR mechanism is polynomial in the size of the support.*

Understanding the Limitations of Interim IR Mechanism Design via Rigidity

We prove the last theorem by introducing and studying the the novel concept of *rigidity* (Section 3). Informally, S is rigid in \mathcal{F} if for every mechanism M' with an allocation function f' where f and f' agree on the allocation of at most x -fraction of the instances of S , it holds that the expected revenue of M' is at most an x fraction of the optimal revenue.¹

We study rigidity under both notions of individual rationality. First, observe that rigid sets can be easily constructed: suppose that player 1 always has some very high value and the other bidders always have very low values. For simplicity assume a uniform distribution on the instances. The optimal mechanism always sells player 1 the item at high price. Clearly, any mechanism that allocates that item to player 1 in at most fraction x of the instances extracts at most x fraction of the optimal solution, hence the set of all instances is rigid with respect to this distribution.

Assuming ex-post individual rationality, Ronen's lookahead auction shows that in every rigid set the highest value player should be allocated the item at least half of the time, simply because only the player with the highest value can be allocated in the lookahead auction and because the lookahead auction extracts at least half of the optimal revenue. In contrast, assuming interim individual rationality, we show that there are distributions with rigid sets in which the optimal allocation function do not have a simple pattern. For example, the revenue that the optimal mechanism extracts from players with the i 'th highest value almost equals to the revenue that is extracted from players with the highest value, for all values of i . This is in fact a corollary of the previous theorem that more precisely shows that the Kolmogorov complexity of the allocation function is high, even when restricted to the rigid sets.

¹ As stated here, there is a linear dependency between x and the approximation ratio, but the definition and our treatment are more general.

Since the existence of rigid sets essentially implies that the hands of the designer are tied, our results help explain why we have little hope of developing generic constructions (a-la Ronen’s ex-post lookahead auction) in the interim individual rationality world.

Examples for Rigid Sets

We now describe two distributions with complex rigid sets. Both are obtained and analyzed using our main technical result, which is a generic way of obtaining complex rigid sets (Section B.1). We give a high-level description of our approach. We start with some set of instances S and construct “unnatural” allocations on it. We then use this set to define a probability distribution \mathcal{F} over a larger set of instances. The support of the distribution \mathcal{F} includes instances that are not all in S . Consider now an incentive-compatible mechanism for the distribution \mathcal{F} . Define the *agreement ratio* of the mechanism M with the allocation on the set of instances S to be the fraction of the instances of S such that the allocation function of the mechanism M coincides with the unnatural allocation function. Our key finding is that S is rigid: the revenue of M is at most $\alpha + \varepsilon$ of that of the optimal mechanism for the distribution \mathcal{F} , where α is the agreement ratio and $\varepsilon > 0$ is very small.

This paper provides two constructions of complex rigid sets (Section A). We now give an imprecise description of each construction. The allocation function is deterministic in both constructions, but the construction process is random. In both constructions, we focus on making the player that is allocated the item indistinguishable from a large fraction of the other players.

1. **Random High Values** (Section A.1). We construct some base set of values, where about half of the players have values that are distributed uniformly and independently between $\frac{1}{2}$ and 1 and the other players have values very close to 0. From this base set we generate $\frac{n}{2}$ instances, where in each such instance, one of the remaining $\frac{n}{2}$ players (the “active” player) has some value that is uniformly distributed between $\frac{1}{2}$ and 1. All other values are identical to their value in the base set. The partial allocation function allocates the item in each of these $\frac{n}{2}$ instances to the active player. We repeat this process m times (for some very large m), each time starting with a different base set, so in total, our partial allocation function is defined on $m \cdot \frac{n}{2}$ instances. Without taking a “global” view of the allocation function, it is hard to make sense of it: in every one of the instances, we expect $\frac{n}{2} + 1$ to have values that are distributed between $\frac{1}{2}$ and 1, and we allocate the item to an arbitrary one of them.
2. **Geometrically increasing base sets** (Section A.2). In this construction method, we start with some arbitrary base set of values v_1, \dots, v_n , e.g., for each i , $v_i = 1$. From this base set we generate n instances, where in the i ’th instance the i ’th player has value $r_i \cdot v_i$, for some r_i that is chosen uniformly at random from $[2, 4]$, say. Player i is allocated the item in that instance. We repeat this process m times, for some large m , each time the base set is obtained from the previous base set by multiplying each v_i by some r_i that is chosen independently and uniformly at random from $[2, 4]$. Note that after not too many repetitions, the instances will look “random”, as the value of the player that is allocated the item is essentially indistinguishable from the values of the rest of the players. All players are considered active for each base set in this construction method.

We show that each of the partial allocation functions constructed by the above two methods can be “embedded” into some distribution \mathcal{F} and that the approximation ratio of any mechanism deteriorates in essentially a linear fashion with the agreement ratio of the allocation function of the mechanism and the partial allocation function that we started with.

We note that the two constructions methods are just two concrete applications of a more general lemma (see the full version). This lemma takes some allocation function on a set of instances S , embeds the set S in a distribution \mathcal{F} , and gives an upper bound on the fraction of the optimal revenue that can be extracted in terms of the agreement ratio with S and the structure of S . Most of the technical difficulty in this paper is in proving this general lemma.

2 Preliminaries

We have one seller, one indivisible item and n bidders. Each bidder has a private value $v_i \in D_i$ for the item and these values are drawn from a joint distribution F over the set of possible instances $D = D_1 \times \dots \times D_n$.

A (deterministic) *mechanism* M is a tuple $M = (x, p)$ where $x : D \rightarrow \{0, 1\}^n$ is the allocation function and $p : D \rightarrow \mathbb{R}^n$ is the payment function. The allocation function satisfies the feasibility constraint: $\sum_{i=1}^n x_i(v) \leq 1$ for every $v \in D$.

A mechanism $M = (x, p)$ is *dominant strategy incentive compatible* if for every player i , every $v_{-i} \in D_{-i}$, and every $v_i, v'_i \in D_i$ it holds that $x_i(v_i, v_{-i}) \cdot v_i - p_i((v_i, v_{-i})) \geq x_i(v'_i, v_{-i}) \cdot v_i - p_i((v'_i, v_{-i}))$.

The LHS of the last inequality is the *profit* of bidder i (w.r.t. M) and is denoted $\pi^M(v_i, v_{-i})$. The *expected profit* of bidder i with value v_i (w.r.t. M) is $\mathbb{E}_{v_{-i} \sim \mathcal{F}_{-i} | v_i=v_i} [\pi_i^M(v_i, v_{-i})]$.

The *expected revenue* $REV(M, \mathcal{F})$ of a mechanism M in the distribution \mathcal{F} is the expected sum of the payments of the bidders.

► **Definition 1.** A mechanism $M = (x, p)$ is *interim individually rational* if for every $i \in [n]$ and every $v_i \in D_i$, the expected profit of bidder i who knows only his own value (and the underlying distribution F) and bids truthfully is non-negative, i.e., $\mathbb{E}_{v_{-i} \sim D_{-i} | v_i=v_i} [\pi_i^M(v_i, v_{-i})] \geq 0$.

A mechanism $M = (x, p)$ is *ex-post individually rational* if for every $i \in [n]$ and every $v \in D$, the ex-post profit of player i who bids truthfully is non-negative, i.e., $\pi_i^M(v_i, v_{-i}) \geq 0$.

► **Definition 2.** The *approximation ratio* of an *interim IR*, *deterministic* and *dominant strategy incentive compatible* mechanism M with respect to a distribution \mathcal{F} is at most $\beta \leq 1$ if:

$$\frac{REV(M, \mathcal{F})}{REV(OPT, \mathcal{F})} \leq \beta$$

where OPT is an *interim IR*, *deterministic* and *dominant strategy incentive compatible* mechanism that maximizes the revenue of \mathcal{F} .

► **Definition 3.** A *partial monotone allocation function* is a monotone allocation function that is defined for a subset of all instances and can be extended to all instances by a monotone function.

A *partial monotone allocation function* f can be specified by a partial monotone allocation set: this is a set whose elements are tuples $\{(\vec{v}, i)\}$ such that $f(\vec{v}) = i$ and if no player is allocated the in \vec{v} , we write $f(\vec{v}) = 0$.

Throughout this document we refer to Cremer and McLean's condition for a distribution, we give a formal definition for it, but before that we need to define the conditional probability matrix of a bidder.

► **Definition 4.** The conditional probability matrix of bidder i w.r.t a discrete distribution \mathcal{F} over a domain D is a matrix $CP_i(\mathcal{F})$ of dimensions $|D_i| \times |D_{-i}|$ where for every $1 \leq k \leq |D_i|$ and every $1 \leq j \leq |D_{-i}|$ we have:

$$[CP_i(\mathcal{F})]_{(k,j)} = \Pr_{v \sim \mathcal{F}}(v_{-i} = v_{-i}^j | v_i = v_i^k)$$

► **Definition 5.** A distribution \mathcal{F} over the set of instances satisfies the full rank condition, or Cremer and McLean's condition if for every player i , his conditional probability matrix $CP_i(\mathcal{F})$ has full rank.

3 Rigidity

In this section we introduce the notion of Rigidity. Roughly speaking, a distribution \mathcal{F} is rigid with respect to a partial monotone allocation set S if the approximation ratio of every mechanism M for the distribution \mathcal{F} depends on the fraction of instances in which M allocates the item as in S .

We start this section with definitions and notations for rigidity and some basic properties of it (Section 3.1), then in Section 3.2, we describe and prove several structural results of rigid sets.

3.1 Definitions and Basic Properties

We now define the notion of rigidity and study some basic properties of it. First, we require the following definitions:

► **Definition 6.** The disagreement ratio of a mechanism M with some partial allocation multi-set S is the fraction of instances in S for which the allocation of M is different than the allocation of S .

► **Definition 7.** A revenue disagreement function is a function $f : [0, 1] \rightarrow [0, 1]$ that is monotone non-increasing.

We are now ready to define rigidity:

► **Definition 8 (Rigidity).** Let \mathcal{F} be a distribution over a set of instances D , S a partial monotone allocation multi-set over D , and f some revenue disagreement function. Then, \mathcal{F} is ex-post (interim) IR f -rigid with respect to S if the approximation ratio of every dominant strategy incentive compatible, deterministic and ex-post (interim) IR mechanism M with disagreement ratio of at least x with S is at most $f(x)$.

We sometimes abuse notation and say that a partial monotone allocation set S is ex-post (interim) IR rigid. By that, we mean that there is a distribution and a revenue disagreement function f such that \mathcal{F} is ex-post (interim) IR f -rigid with respect to S , where f and \mathcal{F} are clear from the context.

In the definition of rigidity, S was allowed to be a multi-set so that different allocations could affect the disagreement ratio of a mechanism differently. Also, S may have instances where no player is assigned the item.

Now, we present the main family of revenue disagreement functions that we analyze in the paper.

► **Definition 9.** Fix $\varepsilon \in (0, 1)$. A revenue disagreement function is ε -almost linear if for every $x \in (\varepsilon, 1]$ it holds that $f(x) \leq 1 - x + \varepsilon$ and for every $x \in (0, \varepsilon]$ it holds that $f(x) < 1$.

Observe an ε -almost linear revenue disagreement function is strictly smaller than 1 for every value in $(0, 1]$. Therefore, even a disagreement of one instance implies that the revenue is smaller than the optimal revenue. Also observe that an ε -almost linear revenue disagreement function is also ε' -almost linear revenue disagreement function for every $\varepsilon < \varepsilon' < 1$.

3.1.1 Union of Rigid Sets

We now study a basic property of rigid sets. This set of results is not used directly in the technical parts of the paper, but will be helpful in understanding the concept of rigidity.

▷ **Claim 10.** Let \mathcal{F} be a distribution over set V of instances, f_1, f_2 revenue disagreement functions and S_1, S_2 partial monotone allocation sets over V . Assume \mathcal{F} is ex-post (interim) IR f_1 -rigid w.r.t S_1 and ex-post (interim) IR f_2 -rigid w.r.t S_2 . Let $x_1 := \max\{x \in [0, 1] \mid f_1(x) = 1\}$, $x_2 := \max\{x \in [0, 1] \mid f_2(x) = 1\}$. Define for every $x \in [0, 1]$, $f(x) = \max\{f_1(x \cdot (1 - x_1)), f_2(x \cdot (1 - x_2))\}$.

Then, there exists a partial monotone allocation set S composed only of elements from S_1 and S_2 such that \mathcal{F} is f -rigid w.r.t S and that at least $1 - x_1$ and $1 - x_2$ fractions of the elements from S_1 and S_2 respectively are in S .

Observe that the claim is tight in the sense that for some values of S_1 and S_2 , there is no partial *monotone* allocation set that contains more than $1 - x_1$ fraction of the elements from S_1 and at least $1 - x_2$ fraction of the elements from S_2 , as the allocations in S_1 might conflict with the allocations in S_2 . For example, consider the distribution \mathcal{F} supported on two instances of two players: $(1, 1)$ and $(\frac{1}{\varepsilon}, 1)$, where the first instance has probability $1 - \varepsilon$ and the second has probability ε . Let $S_1 = \{((1, 1), 2), ((\frac{1}{\varepsilon}, 1), 2)\}$, $S_2 = \{((1, 1), 1), ((\frac{1}{\varepsilon}, 1), 1)\}$, i.e., player 2 gets the item in both instances in S_1 and player 1 gets the item in both instances in S_2 . Then, $f_1(0) = f_1(\frac{1}{2}) = f_2(0) = f_2(\frac{1}{2}) = 1$, $f_1(1) = f_2(1) = \frac{1}{2}$ and $x_1 = x_2 = \frac{1}{2}$. Now, consider a set S that is composed of elements from S_1 and S_2 , for every element from S_1 in S another element from S_2 cannot be added to S and vice versa. However, we focus on the family of almost linear revenue disagreement functions (Definition 9) for which $x_1 = x_2 = 0$, i.e., all the elements from S_1 and S_2 are in S .

Proof of Claim 10. Consider an optimal dominant strategy incentive compatible, deterministic and ex-post (interim) IR mechanism O for the distribution \mathcal{F} . Let S be the set of elements from $S_1 \cup S_2$ that O agrees with their allocation. Note that S is a partial monotone allocation set as O is a dominant strategy incentive compatible mechanism. Now, by the definition of rigidity, O has disagreement ratio at most x_1 with S_1 where $x_1 = \max\{x \in [0, 1] \mid f_1(x) = 1\}$ and at most $x_2 = \max\{x \in [0, 1] \mid f_2(x) = 1\}$ with S_2 . Then, at most x_1 fraction of S_1 elements are not in S and at most x_2 fraction of S_2 elements are not in S . In the next lemma we prove that the set S we defined is indeed f -rigid which concludes the proof.

► **Lemma 11.** *Let M be some dominant strategy incentive compatible, deterministic and IR mechanism with disagreement ratio $x \in [0, 1]$ with S , then M has disagreement ratio at least $x \cdot (1 - x_1)$ with S_1 or at least $x \cdot (1 - x_2)$ with S_2 (or both).*

Proof. Let d be the number of allocations in S that M disagrees with, d_1 , the number of allocations in S_1 that M disagrees with and d_2 the number of allocations in S_2 that M disagrees with. Then, $d \leq d_1 + d_2$, $x = \frac{d}{|S|}$ and we want to show that either $\frac{d}{|S|} \leq \frac{d_1}{|S_1| \cdot (1 - x_1)}$ or $\frac{d}{|S|} \leq \frac{d_2}{|S_2| \cdot (1 - x_2)}$ (or both). Assume to the contrary that:

$$\frac{d_1}{|S_1| \cdot (1 - x_1)} < \frac{d}{|S|} \iff d_1 \cdot |S| < d \cdot |S_1| \cdot (1 - x_1) \leq (d_1 + d_2) \cdot |S_1| \cdot (1 - x_1) \quad (1)$$

$$\frac{d_2}{|S_2| \cdot (1 - x_2)} < \frac{d}{|S|} \iff d_2 \cdot |S| < d \cdot |S_2| \cdot (1 - x_2) \leq (d_1 + d_2) \cdot |S_2| \cdot (1 - x_2) \quad (2)$$

Then,

$$|S| < |S_1| \cdot (1 - x_1) + |S_2| \cdot (1 - x_2)$$

in contradiction to the construction of S such that it contains at least $1 - x_1$ fraction of the elements of S_1 and at least $1 - x_2$ fraction of the elements of S_2 . \blacktriangleleft

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We can further strengthen this claim when the functions are ε -almost linear:

► **Corollary 12.** *Let \mathcal{F} be a distribution that is f -rigid with respect to some set S and an ε -almost linear revenue disagreement function f . Then, there exists a rigid set S_c , such that every other rigid set S' of \mathcal{F} with f , belongs to S (i.e., $S' \subseteq S_c$).*

3.2 On the Structure of Ex-post IR Rigid Sets

In this section, we show that the family of Lookahead auctions restricts the set of ex-post IR rigid sets. We then study how Cremer and McLean's result restricts interim IR rigid sets. We show that for distributions that satisfy Cremer and McLean's condition, the only rigid sets are essentially those in which the highest player always gets the item. This section mostly considers ε -almost linear revenue disagreement functions, and so some of our results are stated for this class.

▷ **Claim 13.** Consider a distribution \mathcal{F} that is ex-post IR f -rigid w.r.t S . Let x be the fraction of the allocations in S to players that are not the highest player. Then, $f(x) \geq \frac{1}{2}$.

Proof. Consider the Lookahead auction [13], this auction can only sell the item to the highest player and so its disagreement ratio with S is at least x . The Lookahead auction has an approximation ratio of at least $\frac{1}{2}$ when considering dominant strategy incentive compatible and ex-post IR auctions. Therefore, $f(x) \geq \frac{1}{2}$. \triangleleft

► **Corollary 14.** *Consider a distribution \mathcal{F} that is ex-post IR f -rigid w.r.t S , for an ε -almost linear revenue disagreement function (for some $\varepsilon \in (0, \frac{1}{2})$). Then, the fraction of the allocations in S to players that are not the highest player is at most $\frac{1}{2} + \varepsilon$.*

We can use analogous claims by considering the 2-lookahead auction:

▷ **Claim 15.** Consider a distribution \mathcal{F} that is ex-post IR f -rigid w.r.t S . Let x be the fraction of the allocations in S to players that are not the highest or second highest players, then $f(x) \geq \frac{\sqrt{\varepsilon}}{\sqrt{\varepsilon+1}} \approx 0.622$.

▷ **Corollary 16 (Claim 15).** *Consider a distribution \mathcal{F} that is ex-post IR f -rigid w.r.t S , for an ε -almost linear revenue disagreement function (for some $\varepsilon \in (0, 0.622)$). Then, the fraction of the allocations in S to players that are not the highest or second highest players is at most $1 - \frac{\sqrt{\varepsilon}}{\sqrt{\varepsilon+1}} + \varepsilon \approx 0.377 + \varepsilon$.*

Proof of Claim 15. The deterministic 2-Lookahead auction is a generalization of the Lookahead auction and it can only sell the item to one of the 2 highest players. Then, its disagreement ratio with S is at least x . The approximation ratio of the 2-Lookahead was bounded by $\frac{3}{5}$ by [6] and later improved to $\frac{\sqrt{e}}{\sqrt{e+1}} \approx 0.622$ by [3]. Therefore, $f(x) \geq \frac{\sqrt{e}}{\sqrt{e+1}} \approx 0.622$. \triangleleft

We now consider interim IR mechanisms and distributions that satisfy Cremer and McLean's full rank condition.

\triangleright **Claim 17.** Consider a distribution \mathcal{F} that satisfies Cremer and McLean's full rank condition and assume there exist some revenue disagreement function f and a partial monotone allocation set S such that \mathcal{F} is interim IR f -rigid w.r.t S . Let x be the fraction of the allocations in S that are not to the highest player, then $f(x) = 1$.

Proof of Claim 17. Cremer and McLean [5] show that when the distribution satisfy some condition (full rank of the conditional probability matrix), the second price auction with some lotteries has an optimal revenue equal to the social welfare. Then, their auction has a disagreement ratio of at most x with S and hence $f(x) = 1$. \triangleleft

\blacktriangleright **Corollary 18.** Consider a distribution \mathcal{F} that satisfies Cremer and McLean's full rank condition. Suppose that there is some ε -almost linear revenue disagreement function f and a partial monotone allocation set S such that \mathcal{F} is interim IR f -rigid w.r.t S . Then, all the allocations in S are to the highest player.

In Claim 13 and Claim 15 we saw that ex-post IR rigid sets have a certain structure.

The next claim show that the same is not true for interim IR rigid sets.

\triangleright **Claim 19.** For every $k < n$, there exists an interim IR rigid set S_k with an $\frac{1}{n-k}$ -almost linear revenue disagreement function, such that no allocation in S is to one of the k -highest players.

Observe that for ex-post IR rigid set it is not possible. Every ex-post IR rigid set S with an ε -almost linear (for some $\varepsilon < \frac{1}{2}$) revenue disagreement function has **at most** $\frac{1}{2} + \varepsilon$ allocations to players that are not the highest (Corollary 14). While we show for every $\varepsilon > 0$ an interim IR rigid set S_ε with an ε -almost linear revenue disagreement function in which all allocations are not to the highest player.

Similarly, this also shows that Corollary 16 does not hold for interim IR rigidity. I.e., every ex-post IR rigid set S with an ε -almost linear (for some $\varepsilon < 0.622$) revenue disagreement function has at most **0.377** + ε allocations to players that are not the two highest players (Corollary 14). While we show for every $\varepsilon > 0$ an interim IR rigid set S_ε with an ε -almost linear revenue disagreement function in which all allocations are not to the two highest player.

The proof of Claim 19 is relegated to the full version

4 Kolmogorov Complexity

In this section we analyze the Kolmogorov complexity of optimal and approximately optimal allocation functions. We start with defining some needed basic objects (Section 4.1). Then, we show that for distributions that satisfy the Cremer and McLean's condition (Definition 5) the Kolmogorov complexity of the allocation function of an optimal auction is low (Section 4.2). In Section 4.3 we show the existence of distributions (that do not satisfy Cremer and McLean's condition) for which the Kolmogorov complexity of the allocation function of every mechanism with a constant approximation ratio is high.

4.1 Natural Orders

We would like to analyze the Kolmogorov complexity of allocation functions. However, Kolmogorov complexity is defined on strings, not functions. Thus, we represent allocation functions as strings with alphabet $[n]$. Index i of the string corresponds to the i 'th instance in the support according to some full order \prec . We will consider *natural* orders. We say that an order is natural if the Kolmogorov complexity of printing all the instances in the support when they're sorted by this order is logarithmic in the size of the support. Note that this is equivalent of having a small Turing machine that gets i as input and prints the corresponding instance. This seems to be a minimal requirement for "interpreting" the allocation function (if it is hard to understand what instances some of the positions corresponds to, the representation is almost useless).

► **Definition 20.** *A full order \prec over a domain D of instances of n players is natural if the Kolmogorov complexity of the string that equals to the concatenation of all instances in D sorted by \prec is $O(\log |D|)$.*

For example, the lexicographic order is natural in the domain $\{1, \dots, H\}^n$. To see this, consider the program that starts with a string $0 \dots 0$ and each time increases its value by one (as if it were a number in base H) and prints it.

We now define a specific order \prec' . We will analyze the Kolmogorov complexity of the strings that correspond to allocations ordered by \prec' and show that it is high. Our result will then be extended to all natural orders since we will show that transforming every string ordered by a natural order to a string ordered by \prec' can be done by a small Turing machine.

► **Definition 21.** *Let \prec' be the following order over domain D of instances of n players: instance $v \in D$ is before instance $v' \in D$ if $\arg \max_{i \in [n]} \{v_i\} \leq \arg \max_{i \in [n]} \{v'_i\}$. In case of equality, we break ties by lexicographic order (i.e., let j be the first index in which v and v' differ, then $v_j \prec' v'_j$).*

The next two claims show that a lower bound on the Kolmogorov complexity of strings ordered by \prec' implies a lower bound on the Kolmogorov complexity of strings ordered by some natural order.

▷ **Claim 22.** Fix some domain D of instances of n players. The encoding of a Turing machine that gets two instances from D and determines their order according to \prec' is at most $c_2 \cdot n$, for some constant c_2 . Specifically, given two instances v_1, v_2 , the Turing machine returns 0 if $v_1 \prec' v_2$ and 1 otherwise.

Proof of Claim 22. The operations of copying a string and comparing two numbers can be implemented using Turing machine with a linear number of states in the size of the input alphabet (which in our case is n). Consider the following Turing machine: first, the machine copies the two input strings (v_1, v_2) . For their first copy, we find the maximal value in each string and save its index. If the index of the first input is smaller (larger) than the index of the second input then $v_1 \prec' v_2$ ($v_1 \succ' v_2$) and we print 0 (1). If the two indices are equal we continue to the next stage. This stage takes at most $O(n)$ states. In the second stage, we work on the second copy of our input strings and compare the two strings until we find the first index that they are different on, if the value of v_1 in this index is smaller than v_2 's value in this index, we print 0. Otherwise we print 1. This stage takes at most $O(n)$ states as well. ◁

▷ **Claim 23.** Fix a distribution \mathcal{F} over a domain of instances D . Let \prec be some natural order over the instances in D and \prec' be the order specified in 21. Let f be an allocation function for the distribution \mathcal{F} , and s and s' be the allocation strings of f with respect to the orders \prec and \prec' respectively over the instances in D . Suppose that the Kolmogorov complexity of s' is at least m . Then, the Kolmogorov complexity of s is at least $m - c_1 \log |D| - c_2 n - c_3$ for some constant c_3 , a constant c_1 that depends only on the order \prec and a constant c_2 that only depends on the order \prec' .

Proof of Claim 23. Assume Claim 23 does not hold. Then, there exists a distribution \mathcal{F} over a domain of instances D , f an allocation function for the distribution \mathcal{F} , s and s' that are the allocation strings of f with respect to the orders \prec and \prec' respectively over the instances in D , such that the Kolmogorov complexity of s' is at least m but the Kolmogorov complexity of s is less than $m - c_1 \log |D| - c_2 n - c_3$.

Let P_1 be a Turing machine that prints s with encoding size less than $m - c_1 \log |D| - c_2 n - c_3$. Consider the following program for printing s' . First, generate s . Second, generate all the instances in D by order \prec . Then sort all the instances in D according to the order \prec' and at the same time sort the string s by the same swaps that are made for the instances in D and print the sorted s .

We now explain how to implement this program while maintaining encoding size of less than m , which will contradict our assumption. For the first step we use the program P_1 whose encoding size is less than $m - c_1 \log |D| - c_2 n - c_3$. For the second step, since \prec is a natural order, there exists a Turing machine P_2 whose encoding is at most some $c_1 \cdot \log |D|$ and its simulation gives us all instances in D ordered by \prec . For the third step, we need two components, the first is a “comparator” function P_3 for the order \prec' that gets two instances and returns their order according to \prec' and has an encoding of size at most some $c_2 \cdot n$. The second component is a sorting algorithm with small encoding, we use, say, Bubble Sort. The sorting algorithm will simultaneously sort the string s by using same swaps that it made for the instances in the list. Observe that bubble sort needs a constant number of states and together with P_3 we get an encoding which is linear in n . Now, this program has KC of at most $k + c_1 \log |V| + c_2 \cdot n + c_3$, where c_3 is some constant. This contradicts our assumption, as needed. ◁

Thus, from now on we will analyze the Kolmogorov complexity of the order \prec' and get, as a corollary, the Kolmogorov complexity of any natural order.

4.2 Low Kolmogorov Complexity of Cremer and McLean’s Distributions

In this section we state and prove the theorem about the low Kolmogorov complexity of the allocation function of an optimal auction of distributions that satisfy the Cremer and McLean’s condition (see Definition 5). We consider the Kolmogorov complexity of the allocation function with respect to the order \prec' (definition 21).

► **Theorem 24.** *Let \mathcal{F} be a distribution that satisfies the Cremer and McLean’s condition with support size k . There exists an optimal auction for \mathcal{F} for which the Kolmogorov complexity of the allocation function is $O(n \log k)$ with respect to the order \prec' .*

Proof of Theorem 24. Cremer and McLean showed that the optimal auction for every distribution that satisfies the full rank condition (Definition 5) is a second price auction with appropriate fees. Recall that the allocation function of the second price auction simply gives the item to a maximum-value player. Consider the following program for a distribution \mathcal{F} ; Print 1 some k_1 times, then 2 some k_2 times and so on, where for each i , k_i is the number of

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instances in the support of \mathcal{F} in which the i 'th player is maximal. Each k_i can be represented by a number of bits that is logarithmic in the size of the support, and so the length of an encoding of this program is of order $O(n \cdot \log k)$. This program indeed returns the correct string as we use the order \prec in which $v \preceq v'$ if $\arg \max_{i \in [n]} \{v_i\} \leq \arg \max_{i \in [n]} \{v'_i\}$. ◀

Observe that the Kolmogorov complexity of Myerson's optimal mechanism for independent distributions [11] is also logarithmic in the size of the support under natural orders: using the natural order we can go over all the instances in the support one by one and compute in each of them the virtual values of each player. The player with the maximal virtual value gets the item.

4.3 The Kolmogorov Complexity of Approximations

In this section we use rigid sets to analyze the Kolmogorov complexity of the allocation function of every mechanism that provides a good approximation ratio. The notion of a rigid set allows us to focus on the allocation of a mechanism in the instances of S , and so, if the mechanism has a good approximation, its allocation string restricted to the instances of S has to be relatively close to the allocation string of S .

► **Theorem 25.** *Consider distributions $\mathcal{F}_1, \dots, \mathcal{F}_k$, all with support size of at most r . Suppose each \mathcal{F}_i is f -rigid with respect to S_i , where f is some ε -almost linear revenue disagreement function (for some $\varepsilon \in [0, 1)$). Suppose that all the following conditions hold for some $c \in (\varepsilon, 1)$, \prec a full order, and $h \in \mathbb{N}$:*

1. $|S_1| = |S_2| = \dots = |S_k|$, and let $g = |S_1|$.
2. For every S_j , \prec imposes an order over the instances in S_j . Let s_j be the allocation string for the instances in S_j when they are sorted by this order and their allocations are as specified in S_j . Then, for every $i \neq j$ we have $s_i \neq s_j$.
3. Let $x = 1 - c + \varepsilon$, then $\frac{k}{\binom{g}{x}} > n^h \cdot \binom{g}{x} \cdot n^{x \cdot g}$.

Then, there exists some distribution \mathcal{F}_j , such that the Kolmogorov complexity of the allocation function of every auction that provides a c approximation for \mathcal{F}_j is at least h with respect to the order \prec .

Proof of Theorem 25. For every $j \in [k]$, consider an allocation string a for the distribution \mathcal{F}_j . a has exactly g indices that correspond to the instances from S_j and a is of size at most r so there are at most $\binom{r}{g}$ different subsets of g indices that can correspond to the instances of the set S_j . This is true for every $j \in [k]$ and so at least $\frac{k}{\binom{r}{g}}$ of the distributions $\mathcal{F}_1, \dots, \mathcal{F}_k$ share the exact same set of indices that correspond to the instances in their respective rigid sets S_1, \dots, S_k , we denote the set of these distributions by O and the set of the indices by I . Observe that by property 2, for every $\mathcal{F}_i, \mathcal{F}_j \in O$ with $i \neq j$ it holds that $s_i \neq s_j$.

► **Lemma 26.** *Fix an allocation string b . b corresponds to an allocation function of an optimal mechanism of at most one of the distributions in O .*

Proof of Lemma 26. Since f is an ε -almost linear revenue disagreement function, for every $\mathcal{F}_j \in \mathcal{F}_1, \dots, \mathcal{F}_k$ any optimal allocation must agree with all the allocations in S_j . Thus, for an allocation string b to be an optimal allocation to more than one distribution in O , it has to agree with at least two different allocations on the same set of indices I , which is a contradiction. ◀

Denote by L_h the set of all strings with Kolmogorov Complexity less than h . Let $B_h \subseteq O$ be the set of distributions in O such that a distribution $\mathcal{F}_j \in O$ is in B_h if it has a mechanism with approximation ratio at least c whose allocation string has Kolmogorov complexity less than h . Observe that the claim follows immediately by showing that B_h is a strict subset of O :

► **Lemma 27.** *B_h is a strict subset of O .*

Proof of Lemma 27. By Lemma 26, each string in L_h is an optimal allocation to at most one of the distributions in O . The size of L_h is at most the number of strings with Kolmogorov complexity less than h , which is $\sum_{j=0}^{h-1} n^j = \frac{n^h-1}{n-1} < n^h$, since the left hand side is the number of different Turing machines over the alphabet $[n]$ with encoding size less than h . Now, for every string $a \in L_h$ we bound the number of distributions \mathcal{F}_j in O for which a restricted to the indices in I (denoted a_I) and the allocation string of S_j are different in at most $x \cdot g$ places. We bound it by at most $\binom{g}{g \cdot x} \cdot n^{g \cdot x}$; There are $\binom{g}{g \cdot x}$ different sets of size $g \cdot x$ of locations in the string a_I of size g . The number of possible values in each location is n . Overall there are at most $\binom{g}{g \cdot x} \cdot n^{g \cdot x}$ such distributions (as each distribution in O has a different allocation in the indices in I). Therefore, there are at most $n^h \cdot \binom{g}{g \cdot x} \cdot n^{g \cdot x}$ distributions in B_h and at least $\frac{k}{\binom{r}{g}}$ distributions in O . Now, by Property 3 there are more distributions in O than in B_h . ◀

4.3.1 Distributions with High Kolmogorov Complexity for Every Approximation

Now that we have the right tools prepared, we can apply a specific construction to prove that there exist distributions for which the allocation function of every approximation mechanism has high Kolmogorov complexity for every natural order \prec (Definition 20). We will rely on the construction of Section A.1.1 that is given later.

► **Corollary 28.** *For every constant $0 < c < 1$ and every large enough $n, m \in \mathbb{N}$, there exists a distribution with support linear in $m \log^2 n$, for which the Kolmogorov complexity of the allocation function of every mechanism with approximation ratio c is at least m for every natural order.*

Proof of Corollary 28. We use the construction of Section A.1.1. We show that for every fixed $0 < \varepsilon < c$ and large enough n, m , Theorem 25 holds with S_1, \dots, S_k as the sets in $R_{n,m}$ (defined in Section A.1.1), $\mathcal{F}_1, \dots, \mathcal{F}_k$ as the distributions guaranteed by Theorem 45 when given m -divisible set from $R_{n,m}$, f as their ε -almost linear revenue disagreement function (from Corollary 47), \prec' as the order \prec , and $h = l \cdot m$ for some constant l to be specified later.

First, by Claim 33, each m -divisible set S in $R_{n,m}$ is of size $m \cdot \log n$, satisfies the uniqueness of thresholds property (44) and each of its m subsets has a set of $\log n$ active players. Then $g = m \log n$ and by Theorem 45 $r \leq 6m \log n + m \log^2 n$. Hence, Property 1 holds. By Claim 33, the parameter of an m -divisible set $S \in R_{n,m}$, c_S is at most $c_S \leq \frac{4n}{m \cdot \log n} + \frac{4}{\log n}$ and thus by Corollary 47 f is an ε -almost linear revenue disagreement function when $\frac{4n}{m \cdot \log n} + \frac{4}{\log n} \leq \varepsilon < c$ (which is true for large enough n, m).

In Lemma 29 and Lemma 30 we show that Property 2 and Property 3 hold respectively.

► **Lemma 29.** *For every S_j , \prec' imposes an order over the instances in S_j . Let s_j be the allocation string for the instances in S_j when they are sorted by this order and their allocations are as specified in S_j . Then, for every $i \neq j$ we have $s_i \neq s_j$.*

Proof of Lemma 29. Recall that S_j is an m -divisible set with respect to its subsets S_j^1, \dots, S_j^m (Definition 39). By Claim 33, in every instance in S_j , the first player has the maximal value and his value is the same for all the instances in the same subset S_j^l . Moreover, the values of the first player in the different m subsets are strictly increasing. Hence, in the support of the distribution, when considering the order \prec' the instances that belongs to the first subset S_j^1 appears first and then the instances from the second subset S_j^2 and so on until the instance from S_j^m . Furthermore, every two instances v, v' in the same subset S_j^l differ from each other in exactly two places, these places correspond to the players that get the item in these instances. When player i gets the item in v and player i' gets the item in v' we have $v_i > v'_i$ and $v_{i'} < v'_{i'}$, and so the order inside each subset is a decreasing order over the indices of the players that get the item (i.e., if i gets the item in $v \in S_j^l$ and $i' > i$ gets the item in $v' \in S_j^l$ then $v' \prec' v$), the players in the active set of S_j^l . Lastly, $R_{n,m}$ (as described in Section A.1.1) has $k = \binom{n}{\log n}^m$ different allocations, in every two different m -divisible sets $S_j, S_{j'} \in R_{n,m}$ the set of active players in some repetition $j \in [m]$ is different and so $s_j \neq s_{j'}$ and Property 2 holds. \blacktriangleleft

► **Lemma 30.** *Property 3 holds for large enough n, m .*

Proof of Lemma 30. We need to show that for large enough values of n, m it holds that $\frac{k}{\binom{r}{g}} > n^h \cdot \binom{g}{xg} \cdot n^{xg}$ where $k = \binom{n}{\log n}^m, g = m \log n, h = l \cdot m$ and $r < 6m \log n + m \log^2 n$. We use known bounds on the binomial coefficient² and get:

$$\frac{k}{\binom{r}{g}} = \frac{\binom{n}{\log n}^m}{\binom{(m \log n)(6 + \log n)}{m \log n}} \geq \left(\frac{n}{e \log n (6 + \log n)} \right)^{m \log n}$$

$$n^{l \cdot m} \cdot \binom{g}{xg} \cdot n^{xg} \leq n^{l \cdot m} \cdot \left(\frac{e}{x} \right)^{xm \log n} \cdot n^{xm \log n}$$

Therefore it is enough to show that for large enough values of n, m it holds that:

$$\frac{n}{e \log n (6 + \log n)} > n^{\frac{l}{\log n}} \cdot \left(\frac{e}{x} \right)^x \cdot n^x$$

Recall that $x < 1$ and so $e^{l+1} \cdot n^x \geq n^{\frac{l}{\log n}} \cdot \left(\frac{e}{x} \right)^x \cdot n^x$.

Finally, for large enough values of n we have (as l is a constant independent of n, m) $n^{1-x} > e^{l+2} \log n (6 + \log n)$. \blacktriangleleft

Now, we have that there is a distribution $\mathcal{F}_j \in \{\mathcal{F}_1, \dots, \mathcal{F}_k\}$ for which the Kolmogorov complexity of every allocation function of every auction with approximation ratio c is at least $l \cdot m$, with respect to the order \prec' . By Claim 23 we can get that the Kolmogorov complexity of every allocation function of every auction with approximation ratio c for \mathcal{F}_j is at least m , with respect to every natural order. We choose l to be large enough so $l \cdot m - c_1 \cdot \log r - c_2 \cdot n - c_3 \geq m$, where c_1, c_2, c_3 are constants from Claim 23 that depend on the specific natural order and a property of \prec' , so we choose l large enough to satisfy this for every natural order and the claim follows. \blacktriangleleft

² $\binom{n}{k} \leq \binom{n}{k} \leq \left(\frac{n \cdot e}{k} \right)^k$

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A Explicit Constructions of Interim IR Rigid Sets

In this section we present two explicit constructions of interim IR rigid sets. We construct interim IR rigid sets by using Theorem 45. First, we describe a way to generate partial monotone allocation sets S that have a certain structure, they are m -divisible sets (Definition 39). Then, Theorem 45 provides us with a distribution \mathcal{F}_S that is f_S interim IR rigid with respect to S . The revenue disagreement function f_S , is a c_S -almost linear revenue disagreement function where c_S is a parameter that depends on properties of the set S .

In both constructions we can choose $c_S > 0$ to be as small as we want by taking large enough values of n (the number of players) and the size of the rigid set S .

As mentioned, the partial monotone allocation sets that we construct have a certain structure which we call an m -divisible set. This term is quite technical and so we try to write the constructions in a way that is clear without the need to fully understand what an m -divisible set is. For that purpose we explain some notation that is used in the construction methods.

The partial monotone allocation sets S are constructed in some $m \in \mathbb{N}$ steps, each step $j \in [m]$ constructs a subset S_j of S such that $S = \bigcup_{j \in [m]} S_j$. For each subset S_j , we have a set of 'active players', denoted A_j . This is the set of players that are allocated the item in some instance in S_j . We show that these are monotone allocation sets by proving that each constructed set S is m -divisible, as m -divisible sets are monotone (see Section B).

The two methods for constructing partial allocation sets that are m -divisible are described next (Section A.1 and Section A.2). We also describe a construction for a set of partial allocation sets (Section A.1.1), this construction is based on the first construction method that constructs a single partial allocation set. This construction of sets of partial allocation sets is used to prove Corollary 28.

A.1 Construction I: Random High Values

All instances in this construction consist of $\frac{n}{2} + 1$ players with values that are uniformly and independently distributed between $\frac{1}{2}$ and 1. The remaining players have low values that are close to 0. The partial allocation function allocates the item to one of the (approximately) $\frac{n}{2}$ players with values in $[\frac{1}{2}, 1]$. Thus, by considering each instance by itself it is hard to guess which player the item should go to – a more “global” view of the allocation function is necessary. Formally:

- We start with some arbitrary vector \vec{v}_0 in which the values of all n players are in $(0, 1]$.
1. Each player i is added to the set of active players in the j 'th subset A_j with probability $\frac{1}{2}$, independently at random. If the set A_j is empty (which happens with probability $\frac{1}{2^n}$), we resample it.
 2. For every active player $i \in A_j$, sample $\vec{r}_i \sim U[2, 4]$, and let k_i be the index of the latest subset $k_i < j$ in which player i was active, if no such index exists, let $k_i = 0$. Then, player's i value in \vec{v}_j is set to:

$$(\vec{v}_j)_i = \frac{(\vec{v}_{k_i})_i}{r_i}.$$

3. For every non active player $i \in [n] \setminus A_j$, sample a value $v_i \sim U[\frac{1}{2}, 1]$ and set player's i value in \vec{v}_j to be equal to the samples value, i.e., $(\vec{v}_j)_i = v_i$.

We construct the allocations subset S_j by sampling a threshold (for receiving the item when $\vec{v}_{-i} = \vec{v}_{j,-i}$) for every active player $i \in A_j$ in the j 'th subset:

4. For every player $i \in A_j$ we sample a threshold $u_i \sim [\frac{1}{2}, 1]$.
5. For every player $i \in A_j$, we set the allocation in $(u_i + \varepsilon', (\vec{v}_j)_{-i})$ to be to player i by adding the tuple $((u_i + \varepsilon', (\vec{v}_j)_{-i}), i)$ to S_j , for some arbitrary small $\varepsilon' > 0$.³

▷ **Claim 31.** For every $n, m \in \mathbb{N}$, the construction outputs a set S of size at least $\frac{n \cdot m}{4}$ that is m -divisible w.r.t. S_1, \dots, S_m with with probability at least $1 - e^{(-\frac{n \cdot m}{16})}$.

³ Note that the role of ε' is to break ties – otherwise it is not clear who gets the item at the threshold value.

► **Corollary 32.** *For every small enough $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that for every $n \geq n_\varepsilon$ there exists $m \in \mathbb{N}$ for which the construction method above constructs an m -divisible set with $c_S \leq \varepsilon$ with high probability.*

The proof of Claim 31 and the proof of Corollary 32 can be found in the full version.

A.1.1 Construction of a Set of m -divisible Sets

In this section we describe a construction for a set of m -divisible Sets. This construction is used in Section 4 to prove Corollary 28. However, this construction is very similar to the Random High Values construction and uses some of its claims, for that reason we describe this construction here.

For every large enough n, m , we define the set $R_{n,m}$ of m -divisible sets. Each m -divisible set $S = S_1, \dots, S_m$ in $R_{n,m}$ will have a set of active players of size $\log n$ for every $j \in [m]$. Now, we define the set of allocations $O_{n,m}$ for m -divisible set that we want to consider and we'll make sure that for every allocation in O we have an m -divisible set in $R_{n,m}$ with this allocation.

$O_{n,m}$ is a set of size $\binom{n}{\log n}^m$ of all possible variations of m active players sets, each set of size $\log n$ and players can be from $\{2, \dots, n\}$.

For every allocation $o \in O_{n,m}$, we construct an m -divisible set S_o that will be added to $R_{n,m}$ in the following way:

We start with a vector $\vec{v}_0 = 1^n$. For every $j \in [m]$, we construct the j 'th subset S_j of S_o based on S_{j-1} and o_j .

1. Let $A_j = \{i \mid i \text{ appears in } o_j\}$.
2. For every active player $i \in A_j$, sample $\vec{r}_i \sim U[2, 4]$, and let k_i be the index of the latest subset $k_i < j$ in which player i was active, if no such index exists, let $k_i = 0$. Then, player's i value in \vec{v}_j is set to:

$$(\vec{v}_j)_i = \frac{(\vec{v}_{k_i})_i}{r_i}.$$

3. For every non active player $i \in [n] \setminus \{A_j \cup \{1\}\}$, sample a value $v_i \sim U[\frac{1}{2}, 1)$ and set player's i value in \vec{v}_j to be equal to the samples value, i.e., $(\vec{v}_j)_i = v_i$.

We construct the allocations subset S_j by sampling a threshold (for receiving the item when $\vec{v}_{-i} = \vec{v}_{j-i}$) for every active player $i \in A_j$ in the j 'th subset:

4. For every player $i \in A_j$ we sample a threshold $u_i \sim [\frac{1}{2}, 1)$.
5. Sample player's 1 value $v_1 \sim U[\max_{i \in A_j} u_i, 1]$ in \vec{v}_j to be the highest value among all threshold in the j -th iteration.
6. For every player $i \in A_j$, we set the allocation in $(u_i + \varepsilon', (\vec{v}_j)_{-i})$ to be to player i by adding the tuple $((u_i, (\vec{v}_j)_{-i}), i)$ to S_j , for some arbitrary small $\varepsilon' > 0$.⁴

If the object we constructed is not an m -divisible set or if it doesn't satisfy the uniqueness of thresholds assumption 44 we repeat the process of construction S_0 (the process will stop, see Claim 33). Otherwise, we add S_o to $R_{n,m}$.

► **Claim 33.** The process of constructing the set $R_{n,m}$ will stop and the following properties will hold:

⁴ Note that the role of ε' is to break ties – otherwise it is not clear who gets the item at the threshold value.

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1. Each $S \in R_{n,m}$ satisfy the uniqueness of thresholds assumption 44.
2. For every $S \in R_{n,m}$, in all its instances the maximal player is player 1.
3. For every $S \in R_{n,m}$, for every $j \in [m]$ the size of the set of active players in iteration j is exactly $\log n$.
4. For every allocation $o \in O_{n,m}$, there exists an m -divisible set $S_o \in R_{n,m}$ with the allocation o on its instances.
5. Each m -divisible set S in $R_{n,m}$ has parameter $c_S < \frac{4n}{m \log n} + \frac{4}{\log n}$

We relegate the proof of Claim 33 to the full version.

A.2 Construction II: Geometrically Increasing Base Sets

We start with the base vector in which all values equal 1 and generate n instances, where in the i 'th instance the i 'th player has value $r_i \cdot v_i$, for some r_i that is chosen uniformly at random from $[2, 4]$. Player i is allocated the item in the i 'th instance. We repeat this process m times, for some large m , each time the base set is obtained from the previous base set by multiplying each v_i by some r_i that is chosen independently and uniformly at random from $[2, 4]$. Note that after not too many iterations, the instances will look “random”, as the value of the player that is allocated the item is essentially indistinguishable from the values of the rest of the players. All players are active in each base set in this construction method. We construct the j 'th base vector after the previous $j - 1$ base vectors:

1. For every player $i \in [n]$, sample $\vec{r}_i \sim U[2, 4]$.
2. Define \vec{v}_j by $(\vec{v}_j)_i = (v_{j-1})_i \cdot r_i$ for every player $i \in [n]$.

We construct the allocations subset S_j by sampling for every player i , a threshold for $\vec{v}_{-i} = \vec{v}_j$:

3. For every player i we sample a threshold $u_i \in [(v_j)_i \cdot 2, (v_j)_i \cdot 4]$.
4. For every player i , we set the allocation in $(u_i + \varepsilon', (v_j)_{-i})$ to be to player i by adding the tuple $((u_i + \varepsilon', (v_j)_{-i}), i)$ to S_j , for some arbitrary small $\varepsilon' > 0$. (The addition of ε' is for tie breaking, similarly to the previous construction.)

▷ **Claim 34.** For every $n, m \in \mathbb{N}$, the construction method outputs a set $S = \bigcup_{j \in [m]} S_j$ of size $n \cdot m$ that is m -divisible w.r.t. S_1, \dots, S_m with with parameter $c_S \leq \frac{2n+2^m}{n \cdot m}$.

► **Corollary 35.** For every small enough $\varepsilon > 0$ there exists $m_\varepsilon, n \in \mathbb{N}$ such that for every $m \geq m_\varepsilon$ there exist $n \in \mathbb{N}$ for which the construction method above constructs an m -divisible set with parameter $c_S \leq \varepsilon$.

The proof of Claim 34 and the proof of Corollary 35 can be found in the full version.

B A General Construction of Interim IR Rigid Sets

In this section, we show how to “embed” a partial allocation function S into a distribution \mathcal{F}_S so that the revenue that can be extracted by a mechanism mostly depends on its agreement ratio and the structure of the allocation function, i.e., we show how to embed a set S into a distribution F_S so that the distribution \mathcal{F}_S will be rigid with respect to S . In Section A we present some concrete partial allocation functions for which the revenue that can be extracted by incentive compatible mechanisms depends on the agreement ratio in a way that is almost linear (i.e., the revenue disagreement function is ε -almost linear for every small enough ε). We begin with some definitions.

► **Definition 36.** A partial monotone allocation function is a monotone allocation function that is defined for a subset of all instances and can be extended to all instances by a monotone function.

A partial monotone allocation function f can be specified by a partial monotone allocation set: this is a set whose elements are tuples $\{(\vec{v}, i)\}$ such that $f(\vec{v}) = i$.

The construction of a partial monotone allocation function f can be described by repeating the following process several times: choose a vector \vec{v}_j that specifies the values of the n players. Then, for every player i , choose a threshold $u_{i,j}$ (it is possible that $u_{i,j} = \infty$) such that f gives player i the item in the instance $(t, (\vec{v}_j)_{-i})$ for every $t \geq u_{i,j}$. Each vector \vec{v}_j that is chosen in some iteration j of the process is called a base vector.

Observe that as described so far, this construction might not yield a feasible allocation function. Thus, we make two changes. The first change is that in the j 'th iteration, we require each threshold $u_{i,j}$ to be larger than $\vec{v}_{j,i}$. Otherwise, if there were two players i, i' with $u_{i,j} < \vec{v}_{j,i}$ and $u_{i',j} < \vec{v}_{j,i'}$, then the allocation set is not feasible since both i and i' have to be allocated the item in the instance \vec{v}_j .

The second change is that for every two different base vectors \vec{v}_j, \vec{v}_k and for every player i , we require that $(\vec{v}_j)_i \neq (\vec{v}_k)_i$. This ensures that the resulting monotone allocation function is feasible, i.e., that at most one player gets the item in every instance. To see this, consider an instance \vec{v} and suppose that the value of more than one player is higher than its threshold in this instance (without loss of generality, players 1 and 2). Then, for some iteration j of the construction, we have that $\vec{v}_{j-1} = \vec{v}_{-1}$ together with $u_{1,j} < \vec{v}_1$. Similarly, for some iteration k of the construction, we have $\vec{v}_{k-2} = \vec{v}_{-2}$ together with $u_{k,2} < \vec{v}_2$. Now, because whenever $j \neq k$ we required that $(\vec{v}_j)_i \neq (\vec{v}_k)_i$ for every player i , and in particular for player 3 (this is why we assume $n > 2$), and we have $\vec{v}_{j,3} = \vec{v}_{k,3}$, it must be that $j = k$. But, this means that in the same iteration, in the same base vector, we allocated the item to more than one player, which is not possible.

Based on this construction, we state some definitions.

► **Definition 37.** A base set is a partial monotone allocation set S such that there exists an instance \vec{v}_S for which every $(\vec{u}, i) \in S$ satisfies $v_{S-i} = u_{-i}$. The instance \vec{v}_S is called the base vector of the set. In a base set each player i has at most one instance \vec{u} such that $(\vec{u}, i) \in S$. In addition, we require monotonicity: for every $(\vec{u}, i) \in S$ it holds that $\vec{u}_i > \vec{v}_{S-i}$.

Consider some iteration j of the process and observe that it defines a base set with base vector \vec{v}_j .

► **Definition 38.** A set of base vectors V is sparse if for every two base vectors $\vec{w}, \vec{u} \in V, \vec{w} \neq \vec{u}$, and $i \in [n]$ it holds that $w_i \neq u_i$.

By the second change, the set of all the base vectors constructed by this process is sparse.

► **Definition 39.** A partial allocation set S is m -divisible with respect to S_1, \dots, S_m if S_1, \dots, S_m is a partition of S into m non-empty base sets such that the set of their corresponding base vectors $\vec{v}_1, \dots, \vec{v}_m$ is sparse and the instances in S are for $n > 2$ players.

Observe that the process above also constructs an m -divisible set. Each iteration j of the process yields a base set S_j (Definition 37) with base vector \vec{v}_j and the union of m iterations is an m -divisible set S with respect to S_1, \dots, S_m since the set of $\vec{v}_1, \dots, \vec{v}_m$ is sparse (Definition 38).

As we saw, partial monotone allocation functions and m -divisible sets are closely related objects. We later state Theorem 45 and prove it using the terminology of m -divisible sets.

B.1 The Main Technical Theorem

In this section we state our main technical theorem (Theorem 45). This theorem is a construction; it gets as input an m -divisible set S (with some parameters) and constructs a distribution D_S that is interim IR f -rigid with respect to S (see Definition 8). Where f is an ε -almost linear revenue disagreement function for ε that depends on the parameters of S . We first require some notations and definitions.

► **Definition 40.** *Let S be an m -divisible set with respect to S_1, \dots, S_m . Then, player i is active in a set S_j if there is an instance \vec{u} such that $(\vec{u}, i) \in S_j$. For every set S_j , we denote by A_j^S its set of active players. Let A^S the set of all active players, i.e., $A^S = \bigcup_{j \in [m]} A_j^S$.*

Sometimes it will be easier to consider the subsets in which a certain player is active. For that purpose, let $A_i^S = \{j \in [m] \mid i \in A_j^S\}$.

When considering an m -divisible set S with respect to S_1, \dots, S_m we need some notation for the values of the players in the instances of S . We denote the value of player i in the base vector \vec{v}_j (of S_j) by $v_{i,j}$. Similarly, for an active player i in the subset S_j , we denote by $u_{i,j}$ its value in the instance \vec{u} such that $(\vec{u}, i) \in S_j$.

The next two definitions of the parameters of an m -divisible set S are used to quantify the bound guaranteed by Theorem 45 on the approximation ratio of any interim IR mechanism m with agreement ratio of x with S .

► **Definition 41.** *Let S be an m -divisible set with respect to S_1, \dots, S_m . For every set S_j , let*

$$\alpha_j^S = \frac{\max_{i \in [n]} \{v_{i,j}\}}{\min_{k \in A_j^S} \{u_{k,j} - v_{k,j}\}}$$

Denote by α_{avg}^S their average, i.e., $\alpha_{avg}^S = \frac{1}{m} \sum_{j \in [m]} \alpha_j^S$.

The parameter α_j^S , defined for the subset S_j , measures the ratio between the highest valuation in the base vector v_j and the minimal gap between the threshold $u_{i,j}$ of an active player i in S_j and this player value in the base set $v_{i,j}$.

► **Definition 42.** *Let S be an m -divisible set with respect to S_1, \dots, S_m . For every $i \in A$, arrange by ascending order the values $v_{i,j}$ for sets S_j that i is active in (i.e., $i \in A_j$) and denote them by $y_{i,1}, \dots, y_{i,k_i}$. Let*

$$g_i^S = \frac{1}{1 - \max_{1 \leq j \leq k_i - 1} \left\{ \frac{y_{i,j}}{y_{i,j+1}} \right\}} = 1 + \frac{1}{\min_{1 \leq j \leq k_i - 1} \left\{ \frac{y_{i,j+1}}{y_{i,j}} \right\} - 1}$$

Denote by g_{avg}^S the average of (g_i^S) over all active players, i.e., $g_{avg}^S = \frac{1}{|A^S|} \sum_{i \in A^S} (g_i^S)$.

The parameter g_i^S , for an active player i , is related to the growth rate of player i 's values in the base vectors of S_1, \dots, S_m . Formally, this parameter equal to 1 plus 1 over the minimal growth in the values of player i in the base vectors that corresponds to subsets he is active in (the growth is measured between the sorted values), minus one.

► **Definition 43.** *The agreement ratio of a mechanism M with some partial allocation function $f(\cdot)$ is the fraction of the allocations that f is defined for which the allocation of M is identical to the allocation of f .*

To simplify the proof we require that the m -divisible set S satisfy the uniqueness of thresholds property (Definition 44).

► **Definition 44** (uniqueness of thresholds). *An m -divisible with respects to S_1, \dots, S_m has the uniqueness of thresholds property if for every player i , his thresholds values $u_{i,j}$ ($j \in A_i$) are distinct and different from his value in the base vectors (i.e., from the values $v_{i,1}, \dots, v_{i,m}$).*

We are now ready to state our main technical theorem:

► **Theorem 45.** *Let S be an m -divisible set with respect to S_1, \dots, S_m with parameters $|A^S|, g_{avg}^S$ and α_{avg}^S that satisfy the uniqueness of thresholds assumption. There exists a distribution \mathcal{F}_S on which the approximation ratio of every dominant strategy incentive compatible, interim IR, and deterministic mechanism with agreement ratio of at most x with S is at most:*

$$\min\{c_S + x, 1\}$$

where $c_S = \frac{|A^S| \cdot g_{avg}^S + m \cdot \alpha_{avg}^S}{|S|}$, and the size of \mathcal{F}_S 's support is at most $5|S| + m + \sum_{j \in [m]} |A_j|^2$.

In Section A we apply this theorem on sets for which the expression c_S is very small. Thus, the approximation ratio depends mostly on the agreement ratio.

▷ **Claim 46.** In Theorem 45, every interim IR, dominant strategy incentive compatible and deterministic mechanism with agreement ratio at most $x \in [1 - c_S, 1)$ has approximation ratio less than 1.

The proof of Claim 46 is relegated to the full version.

► **Corollary 47.** *Let S be an m -divisible set with parameter c_S , then there exists a distribution \mathcal{F}_S that is interim IR f_S -rigid with respects to S (Definition 8). Where f_S is c_S -almost linear revenue disagreement function (Definition 9).*