On Computing Homological Hitting Sets

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Abstract

Cut problems form one of the most fundamental classes of problems in algorithmic graph theory. In this paper, we initiate the algorithmic study of a *high-dimensional* cut problem. The problem we study, namely, Homological Hitting Set (HHS), is defined as follows: Given a nontrivial r-cycle z in a simplicial complex, find a set $\mathcal{S}$ of r-dimensional simplices of minimum cardinality so that $\mathcal{S}$ meets every cycle homologous to z. Our first result is that HHS admits a polynomial-time solution on triangulations of closed surfaces. Interestingly, the minimal solution is given in terms of the cocycles of the surface. Next, we provide an example of a 2-complex for which the (unique) minimal hitting set is not a cocycle. Furthermore, for general complexes, we show that HHS is $\mathbf{W[1]}$-hard with respect to the solution size p. In contrast, on the positive side, we show that HHS admits an **FPT** algorithm with respect to $p + \Delta$, where Δ is the maximum degree of the Hasse graph of the complex K.

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1 Introduction

A *graph cut* is a partition of the vertices of a graph into two disjoint subsets. The set of edges that have one vertex lying in each of the two subsets determines a so-called *cut-set*. Typically, the objective function to optimize involves the size of the cut-set. Graph cuts have a ubiquitous presence in theoretical computer science, and they have also found many real-world applications in clustering, shape matching, image segmentation, and energy minimization problems in computer vision. We begin with the observation that graphs are 1-dimensional simplicial complexes. As we will observe in the ensuing discussion, cuts have a natural homological interpretation. Then, it is natural to mull over how would the notion of cuts have been defined had it first emerged in the context of simplicial complexes. Guided by the intuition that comes from topology, cuts as constructs need not be limited merely to achieve *separation*. For instance, viewing graph cuts as arising from coboundaries of a subset of vertices has led to the emergence of a rich and powerful theory of high-dimensional expansion [17, 18, 33]. In this work, we instead aim to extend the notion of cuts to achieve *homology modification* via removal of simplices.

Specifically, we study the problem of removing the minimum number of r-simplices from a complex so that an entire homology class is destroyed. Formally, the problem Homological Hitting Set (HHS) can be described as follows: given a nontrivial $\mathbb{Z}_2$ r-cycle ζ in a simplicial complex K, find a set $\mathcal{S}$ of r-dimensional simplices of minimum cardinality so that $\mathcal{S}$ meets every cycle homologous to ζ. HHS on graphs can be described as follows: Suppose we are given a graph G with k components. Let C be one of the components of G. Then, $\beta_0(G) = |k|$, and each component determines a 0-cycle. So the question of HHS is to determine the minimum number of vertices you need to remove so that C is not a component anymore. The answer is trivial! One needs to remove all the vertices in C. For example in Figure 1, C_2 ceases to be a component if and only if all four vertices in C_2 are removed. The global variant of HHS, denoted by GHHS, asks for the minimum number of r-simplices that need to be removed so that the class of *some* r-cycle is destroyed.

We note that it is the unidimensionality of graphs that makes the problem trivial. Moreover, even the "cut" aspect of the problem is not immediately visible for graphs. In contrast, for higher-dimensional complexes, the problem has a distinct cut flavor. For instance, consider the planar complex shown in Figure 2. The minimum number of edges that need to be removed so that every cycle homologous to ζ is destroyed is three (shown in red), whereas a solution to GHHS comprises of only two edges. In Figure 3, the cocycle ϑ is the minimal hitting set for the cycle ζ. Note that the edges in the minimal hitting sets happen to be in the "thin" portions of the respective complexes. So although HHS does not generalize any cut problems from graph theory, we view it as a *high-dimensional* cut problem that concerns homology modification.

1.1 Related work

Duval et al. [20] studied the vector spaces and integer lattices of cuts and flows associated to CW complexes and their relationships to group invariants. Ghrist and Krishnan [29] proved a topological version of the max-flow min-cut theorem using methods from sheaf theory. There is also a long line of work on cuts in surface embedded graphs [4, 5, 10, 11, 12, 13, 23, 24], which is closely related to our work. For a mathematical introduction to graphs on surfaces, we refer the reader to [32, 37], and for an up-to-date algorithmic introduction to the subject, we recommend the lecture notes and surveys by Colin de Verdière [14, 16] and Erickson [21, 22], respectively. There is a growing body of work on parameterized complexity in topology [1, 3, 6, 7, 8, 9, 10, 30, 34, 35, 39], and much of this paper can be characterized as such. Independently of our work, and prior to it, Maxwell and Nayyeri [36] defined the problem of finding combinatorial cuts in [36] which is, in some sense, dual to Homological Hitting Set.

1.2 Summary of results

Surfaces

Our first result, expounded in Section 3, is the following: Homological Hitting Set (HHS) admits a polynomial-time algorithm on triangulations of closed surfaces. At the heart of our proof lies an appealing characterization of the minimal solutions in terms of the cocycles of

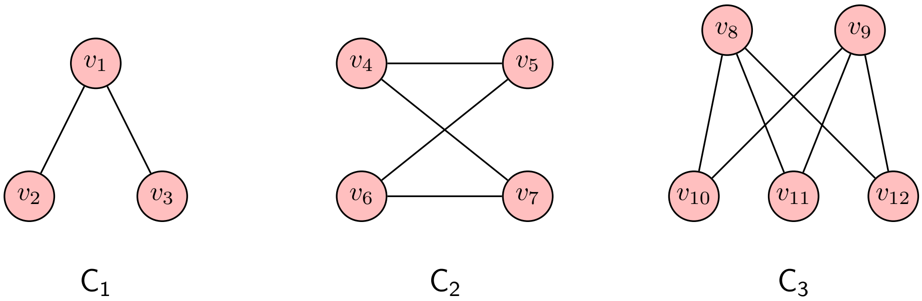

Figure 1 Graph G with three components.

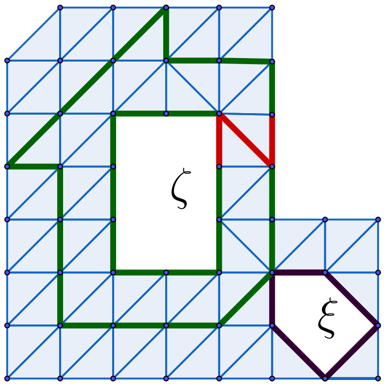

Figure 2 The figure shows two (simple) cycles that belong to $[\zeta]$ in green. Note that any cycle in $[\zeta]$ must pass through at least one of the three red edges. Thus, the set of red edges constitutes a minimal hitting set of ζ. On the other hand, ξ has a minimal hitting set of size 2.

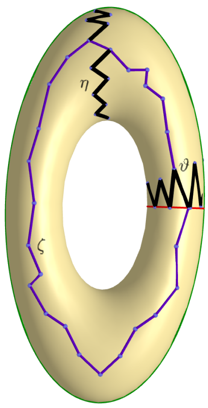

Figure 3 The figure shows a triangulated torus. The cycle ζ is shown in purple. The cocycles η and ϑ “transversal” to ζ are shown in black. Any cycle homologous to ζ passes through at least one of the edges of η (resp. ϑ). ϑ has a smaller support, and is, therefore, a more desirable “hitting set”.

the surface, which is of independent interest. Specifically, we show that a minimal solution set is necessarily a nontrivial cocycle. Eventually, we arrive at a very simple 3-step algorithm for HHS on surfaces.

Counterexample

In Section 4, we show that there exists a 1-cycle in a non-orientable 2-complex (which is not a surface) whose minimal homological hitting set is not a cocycle.

W[1]-hardness and NP-hardness

For general complexes, in Section 5, we show that HHS is **W[1]**-hard with respect to the solution size k as the parameter, (and hence, it is also **NP**-hard). The proof is based on a reduction from MULTICOLORED CLIQUE. Here, the reduction shows the essence of hardness: its description is short, but its proof exposes various "behaviors" that we find interesting. The forward direction requires a nontrivial parity based argument, while the reverse direction shows how to "trace" a solution through the complex.

Fixed-parameter tractability

On the positive side, in Section 6, we show that HHS admits an **FPT** algorithm with respect to $k + \Delta$, where Δ is the maximum degree of the Hasse graph of the complex K. Here, the main insight is that a minimal solution must be connected. Having this insight at hand, the algorithm follows: If we search across the geodesic ball of every r-simplex in the complex K, we will find a solution.

Problem definition

We now provide a formal definition for HOMOLOGICAL HITTING SET (HHS).

▶ **Problem 1** (HOMOLOGICAL HITTING SET).

INSTANCE:	A d-dimensional simplicial complex K, a natural number k, a natural number $r < d$ and a non-bounding cycle $\zeta \in \mathsf{Z}_r(\mathsf{K})$.
PARAMETER:	k.
QUESTION:	Does there exist a set $\mathcal{S}$ of r-dimensional simplices with $\lvert\mathcal{S}\rvert \leq k$ such that $\mathcal{S}$ meets every cycle homologous to ζ?

More generally, one can ask for a hitting set of minimum weight on a weighted complex K. In Section 3, we study the weighted version of the problem on surfaces.

Let $\mathsf{K}_{\mathcal{S}}$ denote the complex obtained from K upon removal of the set of r-simplices $\mathcal{S}$ along with all the cofaces of the simplices in $\mathcal{S}$. In particular, the homology class $[\zeta]$ does not survive in $\mathsf{K}_{\mathcal{S}}$. The inclusion map $\iota : \mathsf{K}_{\mathcal{S}} \hookrightarrow \mathsf{K}$ induces a map $\tilde{\iota} : \mathsf{H}_r(\mathsf{K}_{\mathcal{S}}) \to \mathsf{H}_r(\mathsf{K})$. Then, $\mathcal{S}$ is a homological hitting set if and only if $[\zeta]$ is not in the image of $\tilde{\iota}$. Let $\hat{\alpha}_i = \hat{\iota}(\alpha_i)$. Let $\mathbf{A}$ denote the matrix with nontrivial r-cycles $\hat{\alpha}_i$ as its columns. Let $\mathbf{M}$ denote the matrix $[\mathbf{A} \mid \partial_{r+1}(\mathsf{K})]$ and $C(\mathbf{M})$ the column space of $\mathbf{M}$. The following lemma ensures polynomial time verification for the decision variant of HOMOLOGICAL HITTING SET.

▶ **Lemma 1.** *$\zeta \notin$ column space of $\boldsymbol{M}$ if and only if $\mathcal{S}$ meets every cycle homologous to ζ.*

Proof. ($\Longrightarrow$) Let ρ be a cycle homologous to ζ such that $\mathcal{S}$ does not meet ρ. Then, $\rho \in C(\mathbf{M})$ since it survives in $\mathsf{K}_{\mathcal{S}}$. The claim follows from observing that ζ is homologous to ρ.

($\Longleftarrow$) Suppose that $\mathcal{S}$ meets every cycle that is homologous to ζ. Thus, at least one simplex is removed from every cycle homologous to ζ. Then, a cycle homologous to ζ (in K) is not present in $\mathsf{K}_{\mathcal{S}}$. The claim follows. ◀

Lemma 1 provides an easy way to check if a set constitutes a feasible solution. Let ω denote the exponent of matrix multiplication, and n denote the size of the input complex. Checking if a set $\mathcal{S}$ is a feasible solution to HHS amounts to first building the matrix $\mathbf{M}$, and then solving a linear system of equations. Both operations can also be done in $O(n^{\omega})$ time.

▶ **Lemma 2.** *HHS is in* **NP**, *and in* **XP** *with respect to the solution size k as the parameter.*

2 Notation and Preliminaries

Simplicial homology

Given a simplicial complex K, we will denote by $\mathsf{K}^{(p)}$ the set of p-dimensional simplices in K, and n_p the number of p-dimensional simplices in K. The complex induced by $\mathsf{K}^{(p)}$ is called the *p-dimensional skeleton* of K, and is denoted by K_p. We denote by H_{K}, the *Hasse graph* of K, which is simply the graph that has a node for every simplex σ of the complex, and an edge for every pair of simplices (σ, τ) if σ is incident on τ, and their dimensions differ by 1. We say that a complex K is *weighted* if it comes equipped with a non-negative weight function on its simplices. The *link* of a simplex σ in a complex K is the subcomplex of K, denoted by lk_{σ}, wherein a simplex τ belongs to lk_{σ} if an only if τ is contained in a simplex that contains σ and τ is disjoint from σ.

We consider formal sums of simplices with $\mathbb{Z}_2$ coefficients, that is, sums of the form $\sum_{\sigma\in\mathsf{K}^{(p)}} a_{\sigma}\sigma$, where each $a_{\sigma} \in \{0,1\}$. The expression $\sum_{\sigma\in\mathsf{K}^{(p)}} a_{\sigma}\sigma$ is called a *p-chain*. Since chains can be added to each other, they form an Abelian group, denoted by $\mathsf{C}_p(\mathsf{K})$. Since we consider formal sums with coefficients coming from $\mathbb{Z}_2$, which is a field, $\mathsf{C}_p(\mathsf{K})$, in this case, is a vector space of dimension n_p over $\mathbb{Z}_2$. The p-simplices in K form a (natural) basis for $\mathsf{C}_p(\mathsf{K})$. This establishes a natural one-to-one correspondence between elements of $\mathsf{C}_p(\mathsf{K})$ and subsets of $\mathsf{K}^{(p)}$. Thus, associated with each chain is an incidence vector v, indexed on $\mathsf{K}^{(p)}$, where $v_{\sigma} = 1$ if σ is a simplex of v, and $v_{\sigma} = 0$ otherwise. The *boundary* of a p-simplex is a $(p-1)$-chain that corresponds to the set of its $(p-1)$-faces. This map can be linearly extended from p-simplices to p-chains, where the boundary of a chain is the $\mathbb{Z}_2$-sum of the boundaries of its elements. Such an extension is known as the *boundary homomorphism*, and denoted by $\partial_p : \mathsf{C}_p(\mathsf{K}) \to \mathsf{C}_{p-1}(\mathsf{K})$. A chain $\zeta \in \mathsf{C}_p(\mathsf{K})$ is called a *p-cycle* if $\partial_p\zeta = 0$, that is, $\zeta \in \ker\partial_p$. The group of p-dimensional cycles is denoted by $\mathsf{Z}_p(\mathsf{K})$. As before, since we are working with $\mathbb{Z}_2$ coefficients, $\mathsf{Z}_p(\mathsf{K})$ is a vector space over $\mathbb{Z}_2$. A chain $\eta \in \mathsf{C}_p(\mathsf{K})$ is said to be a p-boundary if $\eta = \partial_{p+1}c$ for some chain $c \in \mathsf{C}_{p+1}(\mathsf{K})$, that is, $\eta \in \operatorname{im}\partial_{p+1}$. The group of p-dimensional boundaries is denoted by $\mathsf{B}_p(\mathsf{K})$. In our case, $\mathsf{B}_p(\mathsf{K})$ is also a vector space, and in fact a subspace of $\mathsf{C}_p(\mathsf{K})$. Thus, we can consider the quotient space $\mathsf{H}_p(\mathsf{K}) = \mathsf{Z}_p(\mathsf{K})/\mathsf{B}_p(\mathsf{K})$. The elements of the vector space $\mathsf{H}_p(\mathsf{K})$, known as the p-th *homology group* of K, are equivalence classes of p-cycles, where p-cycles are equivalent if their $\mathbb{Z}_2$-difference is a p-boundary. Equivalent cycles are said to be *homologous*. For a p-cycle ζ, its corresponding homology class is denoted by $[\zeta]$. Bases of $\mathsf{B}_p(\mathsf{K})$, $\mathsf{Z}_p(\mathsf{K})$ and $\mathsf{H}_p(\mathsf{K})$ are called *boundary bases*, *cycle bases*, and *homology bases* respectively.

The *coboundary of a p-simplex* is a $(p+1)$-cochain that corresponds to the set of its $(p+1)$-cofaces. The coboundary map is linearly extended from p-simplices to p-cochains, where the *coboundary of a cochain* is the $\mathbb{Z}_2$-sum of the coboundaries of its elements. This extension is known as the *coboundary homomorphism*, and is denoted by $\delta_p : \mathsf{C}^p(\mathsf{K}) \to \mathsf{C}^{p+1}(\mathsf{K})$. A cochain $\eta \in \mathsf{C}^p(\mathsf{K})$ is called a *p-cocycle* if $\delta_p\eta = 0$, that is, $\eta \in \ker \delta_p$. The collection of p-cocycles forms the p-th cocycle group of K, denoted by $\mathsf{Z}^p(\mathsf{K})$, which is also a vector space under $\mathbb{Z}_2$ addition. A cochain $\eta \in \mathsf{C}^p(\mathsf{K})$ is said to be a p-coboundary if $\eta = \delta_{p-1}\xi$ for some chain $\xi \in \mathsf{C}^{p-1}(\mathsf{K})$, that is, $\eta \in \operatorname{im} \delta_{p-1}$. The collection of p-coboundaries forms the p-th coboundary group of K, denoted by $\mathsf{B}^p(\mathsf{K})$ which is also a vector space under $\mathbb{Z}_2$ addition. The three vector spaces are related as follows: $\mathsf{B}^p(\mathsf{K}) \subset \mathsf{Z}^p(\mathsf{K}) \subset \mathsf{C}^p(\mathsf{K})$. Therefore, we can define the quotient space $\mathsf{H}^p(\mathsf{K}) = \mathsf{Z}^p(\mathsf{K})/\mathsf{B}^p(\mathsf{K})$, which is called the p-th cohomology group of K. The elements of the vector space $\mathsf{H}^p(\mathsf{K})$, known as the p-th *cohomology group* of K, are equivalence classes of p-cocycles, where p-cocycles are equivalent if their $\mathbb{Z}_2$-difference is a p-coboundary. Equivalent cocycles are said to be *cohomologous*. For a p-cocycle η, its corresponding cohomology class is denoted by $[\eta]$. The p-th *Betti number* of K, denoted by $\beta^p(\mathsf{K})$ is defined as $\beta^p(\mathsf{K}) = \dim \mathsf{H}^p(\mathsf{K}) = \dim \mathsf{H}_p(\mathsf{K})$. The identity $\langle \delta\eta, \zeta \rangle = \langle \eta, \partial\zeta \rangle$ holds for the maps δ and ∂.

Using the standard bases for $\mathsf{C}_p(K)$ and $\mathsf{C}_{p-1}(K)$, the matrix $[\partial_p\sigma_1\, \partial_p\sigma_2 \cdots \partial_p\sigma_{n_p}]$ whose column vectors are boundaries of p-simplices is called the p-th boundary matrix. Likewise, using standard bases for $\mathsf{C}^p(K)$ and $\mathsf{C}^{p+1}(K)$, the matrix $[\delta^p\sigma_1\, \delta^p\sigma_2 \ldots \delta^p\sigma_{n_p}]$ whose column vectors are coboundaries of p-simplices is called the p-th coboundary matrix. Abusing notation, we denote the p-th boundary and coboundary matrices by ∂_p and δ^p, respectively.

We say that a p-cycle ζ is incident on a p-simplex σ if ζ contains σ. A set of p-cycles $\{\zeta_1, \ldots, \zeta_g\}$ is called a *homology cycle basis* if the set of classes $\{[\zeta_1], \ldots, [\zeta_g]\}$ forms a homology basis. For brevity, we abuse notation by using the term *(p-th) homology basis* for $\{\zeta_1, \ldots, \zeta_g\}$. Similarly, a set of p-cocycles $\{\eta_1, \ldots, \eta_g\}$ is called a *(p-th) cohomology cocycle basis* if the set of classes $\{[\eta_1], \ldots, [\eta_g]\}$ forms a cohomology basis. Given a weighted complex K, the weight of a cycle is the sum of the weights of its simplices, and the weight of a homology basis is the sum of the weights of the basis elements.We call the problem of computing a minimum weight basis of $\mathsf{H}_p(\mathsf{K})$ the *minimum p-homology basis*, and the problem of computing a minimum weight basis of $\mathsf{H}^p(\mathsf{K})$, the *minimum p-cohomology basis*.

For a triangulated surface (more generally a 2-manifold) L, we denote by D_{L}, the *dual graph* of L, which is simply the graph that has a node for every 2-simplex (d-simplex) and an edge connecting two nodes if the corresponding 2-simplices (d-simplices) are incident on a common edge ($(d-1)$-simplex) in the complex L.

▶ **Notation 3.** *Since there is a 1-to-1 correspondence between the p-chains of a complex* K *and the subsets of* $\mathsf{K}^{(p)}$*, we abuse notation by writing* $\partial\mathcal{C}$ *in place of* $\partial(\sum_{\sigma\in\mathcal{C}} \sigma)$*, for* $\mathcal{C} \subset \mathsf{K}^{(p)}$*. Likewise, for p-cochains* $\delta(\sum_{\tau\in\mathcal{C}'} \tau)$*, we often write* $\delta\mathcal{C}'$*.*

We also abuse notation in the other direction. That is, we treat chains and cochains as sets. For instance, sometimes we say that a (co)chain γ *intersects a (co)chain* ζ*, when we actually mean that the corresponding sets of simplices of the respective (co)chains intersect. We say that a simplex* $\sigma \in \zeta$*, when indeed the simplex* σ *belongs to the set associated to* ζ*.*

▶ Remark 4. For the chain and cochain groups $\mathsf{C}_p(\mathsf{K})$ and $\mathsf{C}^p(\mathsf{K})$, there is a map

$$\mathsf{C}_p(\mathsf{K}) \times \mathsf{C}^p(\mathsf{K}) \to \mathbb{Z}_2, \quad (\zeta, \eta) \mapsto \eta(\zeta).$$

This bilinear map is referred to as the *evaluation map*. Furthermore, it induces another bilinear map, which is also non-degenerate.

$$\mathsf{H}_p(\mathsf{K}) \times \mathsf{H}^p(\mathsf{K}) \to \mathbb{Z}_2$$

In particular, if we evaluate the r-cocycle η on a r-cycle ζ, then by linearity,

$$\eta(\zeta) = \eta(\sum_{\sigma_i\in\zeta} \sigma_i) = \sum_{\sigma_i\in\zeta} \eta(\sigma_i).$$

Owing to $\mathbb{Z}_2$ addition, $\eta(\zeta) \in \{0, 1\}$. By non-degeneracy, we mean that:
1. η is trivial if and only if $\eta(\zeta) = 0$ for all p-cycles ζ.
2. ζ is trivial if and only if $\eta(\zeta) = 0$ for all p-cocycles η.

As a consequence of non-degeneracy, the evaluation map is well-defined. In particular, for any cocycle $\kappa \in [\eta]$ and any cycle $\xi \in [\zeta]$, $\eta(\zeta) = \kappa(\xi)$.

Parameterized complexity

Let Π be an **NP**-hard problem. In the framework of Parameterized Complexity, each instance of Π is associated with a *parameter* k. Here, the goal is to confine the combinatorial explosion in the running time of an algorithm for Π to depend only on k. Formally, we say that Π is *fixed-parameter tractable (*__FPT__*)* if any instance (I, k) of Π is solvable in time $f(k) \cdot |I|^{O(1)}$, where f is an arbitrary computable function of k. A weaker request is that for every fixed k, the problem Π would be solvable in polynomial time. Formally, we say that Π is *slice-wise polynomial (*__XP__*)* if any instance (I, k) of Π is solvable in time $f(k) \cdot |I|^{g(k)}$, where f and g are arbitrary computable functions of k. In other words, for a fixed k, Π has a polynomial time algorithm, and we refer to such an algorithm as an **XP** algorithm for Π. Nowadays, Parameterized Complexity supplies a rich toolkit to design **FPT** and **XP** algorithms [15, 19, 27]. Parameterized Complexity also provides methods to show that a problem is unlikely to be **FPT**. The main technique is the one of parameterized reductions analogous to those employed in classical complexity. Here, the concept of **W[1]**-hardness replaces the one of NP-hardness, and for reductions we need not only construct an equivalent instance in **FPT** time, but also ensure that the size of the parameter in the new instance depends only on the size of the parameter in the original one.

▶ **Definition 5** (Parameterized Reduction). *Let Π and Π' be two parameterized problems. A* parameterized reduction *from Π to Π' is an algorithm that, given an instance (I, k) of Π, outputs an instance (I', k') of Π' such that:*
- *(I, k) is a yes-instance of Π if and only if (I', k') is a yes-instance of Π'.*
- *$k' \leq g(k)$ for some computable function g.*
- *The running time is $f(k) \cdot |\Pi|^{O(1)}$ for some computable function f.*

If there exists such a reduction transforming a problem known to be **W[1]**-hard to another problem Π, then the problem Π is **W[1]**-hard as well. Central **W[1]**-hard problems include, for example, Clique parameterized by solution size, and Independent Set parameterized by solution size. To show that a problem Π is not **XP** unless **P** = **NP**, it is *sufficient* to show that there exists a fixed k such that Π is **NP**-hard. If the problem Π is in **NP** for a fixed k then it is said to be in para-**NP**, and if it is **NP**-hard for a fixed k then it is para-**NP**-hard.

3 Homological Hitting Set on Surfaces

Recall that the star of a vertex v of a complex K, written $\mathrm{star}_{\mathsf{K}}(v)$, is the subcomplex consisting of all faces of K containing v, together with their faces. A triangulated surface S in $\mathbb{R}^d$ is a simplicial complex consisting of a finite set of triangles such that every vertex of the surface belongs to at least one triangle and the star of each vertex is a simplicial disk. A closed surface is a surface that is compact and without boundary.

In this section, we describe a polynomial time algorithm for HHS on triangulated closed surfaces. The algorithm has a very simple high-level description. See Algorithm 1. Throughout, ζ denotes a nontrivial 1-cycle in a triangulated closed surface S equipped with a weight function on edges.

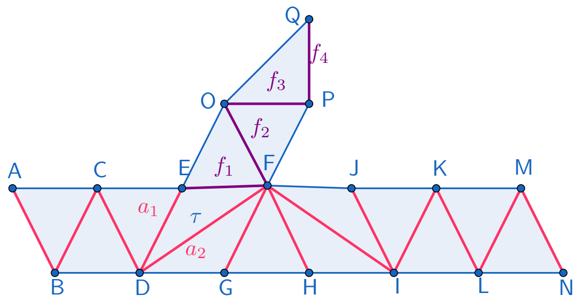

Figure 4 The figure illustrates a subcomplex of a closed surface. If $\mathcal{C}$ is a minimal solution set and $a_1 \in \mathcal{C}$, then either $f_1 \in \mathcal{C}$ or $a_2 \in \mathcal{C}$. Assuming $a_2 \in \mathcal{C}$ and iterating the argument, we obtain a cocycle $\eta \subset \mathcal{C}$ shown in red. Finally, we show $\eta = \mathcal{C}$ and hence the edges $f_1, f_2, f_3, f_4 \notin \mathcal{C}$.

Algorithm 1 An algorithm for HHS on surfaces with input cycle ζ.

1: Find a minimum 1-cohomology basis of S.
2: Arrange the cocycles in the basis in the ascending order of weight.
3: Pick the smallest weight cocycle η with $\eta(\zeta) = 1$.

▶ **Lemma 6.** *A minimal homological hitting set is a nontrivial cocycle that induces a* cycle graph *in D_{S}.*

Proof. Let $\mathcal{C}$ be a minimal solution set, and let $D_{\mathsf{S}}(\mathcal{C})$ be the subgraph of D_{S} induced by $\mathcal{C}$.

Targeting a contradiction, we assume that there exists a vertex in $D_{\mathsf{S}}(\mathcal{C})$ with degree 1. Let τ be the 2-simplex corresponding to a vertex in $D_{\mathsf{S}}(\mathcal{C})$ with degree 1 in $D_{\mathsf{S}}(\mathcal{C})$. Let e_1 be the unique edge in $D_{\mathsf{S}}(\mathcal{C})$ incident on τ. Let e_2 and f_1 be the edges incident on τ that are not in $\mathcal{C}$. From the minimality of $\mathcal{C}$, we conclude that there exists a cycle $\gamma \in [\zeta]$ with $\mathcal{C} \cap \gamma = \{e_1\}$, for if this were not true then $\mathcal{C} \setminus \{e_1\}$ would also be a hitting set, contradicting the minimality of $\mathcal{C}$. Then, for the cycle $\gamma' = \gamma + \partial\tau$, we have $e_2, f_1 \in \gamma'$ and $e_1 \notin \gamma'$. Since γ' differs from γ only on a simplex boundary, and since by assumption on the degree of τ in $D_{\mathsf{S}}(\mathcal{C})$, $e_2, f_1 \notin \mathcal{C}$, we conclude that $\gamma' \cap \mathcal{C} = \emptyset$, a contradiction. Therefore, all the vertices in $D_{\mathsf{S}}(\mathcal{C})$ have degree at least 2. See Figure 4 for an example.

Hence, there exists a connected 1-cocycle η for which $\eta \subset \mathcal{C}$, and C_η is a subgraph of $D_{\mathsf{S}}(\mathcal{C})$. In other words, for surfaces, there always exists a 1-cocycle that is supported by a minimal solution set. By construction, there always exists a γ such that $\eta(\gamma) = 1$. Let γ' be *any* cycle homologous to γ. Then, $\gamma' = \gamma + \kappa$, where κ is a boundary. From Remark 4, $\eta(\gamma') = \eta(\gamma) + \eta(\kappa)$, and $\eta(\kappa) = 0$. Hence, $\eta(\gamma') = 1$. That is, η itself is a hitting set, and $\eta = \mathcal{C}$. ◀

▶ **Corollary 7.** *Let $k > 1$ be an integer. Let η_i for $i \in [k]$ be nontrivial 1-cocycles. If η_i for $i \in [k]$ are not minimal hitting sets for the input cycle ζ, then any cocycle ϑ cohomologous to $\sum_{i=1}^{k} \eta_i$ is also not a minimal hitting set.*

Proof. A cocycle ϑ cohomologous to $\sum_{i=1}^{k} \eta_i$ can be written as $\sum_{i=1}^{k} \eta_i + \delta(S)$ where S is a collection of vertices. Then, by linearity,

$$\vartheta(\zeta) = \sum_{i=1}^{k} \eta_i(\zeta) + \delta(S)(\zeta) = \sum_{i=1}^{k} \eta_i(\zeta) + \sum_{v \in S} \delta(v)(\zeta).$$

Using Lemma 6 and Remark 4, $\delta(v)(\zeta) = 0$ for every $v \in S$, and $\eta_i(\zeta) = 0$ for $i \in [k-1]$. The claim follows. ◀

Although this is fairly well known, for the sake of completeness, we describe an algorithm for computing minimum cohomology basis of a triangulated surface that uses the minimum homology basis algorithm as a subroutine.

▶ **Lemma 8.** *The minimum cohomology basis problem on surfaces can be solved in the same time as the minimum homology basis problem on surfaces.*

Proof. Let S be a surface with a weight function w on its edges. Let $\hat{\mathsf{S}}$ be the dual cell complex of S. Then, to every edge e of S there is a unique corresponding edge $\hat{e}$ in $\hat{\mathsf{S}}$. We now define a weight function on the edges of $\hat{\mathsf{S}}$ in the obvious way: $w(\hat{e}) = w(e)$. Let $\hat{\mathsf{S}}'$ be the simplicial complex obtained from the stellar subdivision of each of the 2-cells of $\hat{\mathsf{S}}$. The weight function on edges of $\hat{\mathsf{S}}$ is extended to a weight function on edges of $\hat{\mathsf{S}}'$ by assigning weight ∞ to every newly added edge during the stellar subdivision. Such a complex $\hat{\mathsf{S}}'$ can be computed in linear time. It is easy to check that the cocycles of S are in one-to-one correspondence with the cycles of $\hat{\mathsf{S}}$, and the cminimalycles of $\hat{\mathsf{S}}$ are in one-to-one correspondence with finite weight cycles of $\hat{\mathsf{S}}'$. Moreover, if η is a cocycle of S, and if $\hat{\eta}$ and $\hat{\eta}'$ are the corresponding cycles in $\hat{\mathsf{S}}$ and $\hat{\mathsf{S}}'$, respectively, then $w(\eta) = w(\hat{\eta}) = w(\hat{\eta}')$. Hence, computing a minimum homology basis for $\hat{\mathsf{K}}'$ gives a minimum cohomology basis for S. ◀

▶ **Theorem 9.** *Let n denote the number of simplices in a surface, and β denote the rank of its first homology group. Then, Algorithm 1 computes a minimal solution for* HHS *of a surface in $O(\beta^3 n \log^2 n)$ time.*

Proof. Let $\{\nu_i \mid i \in [\beta]\}$ be a minimum 1-cohomology basis for S. Then, by Lemma 6, any minimal solution set is a cocycle. So, we can let k be the smallest integer for which $\nu_k \mid k \in [\beta]$ evaluates to 1 on the input cycle ζ. By Corollary 7, every cocycle cohomologous to a combination of cocycles in $\nu_i \mid [i] \in [k-1]$ evaluates to 0 on ζ.

On the other hand, because of the matroid structure on cohomology bases [25, Section 4], ν_k is the minimum weight cocycle independent of $\{\nu_i \mid i \in [k-1]\}$. Since the weights are positive, ν_k is a simple connected graph cycle in D_{S}. By Remark 4, ν_k evaluates to 1 on every cycle in $[\zeta]$. Therefore, ν_k is a minimal solution.

Step-1 of Algorithm 1 requires $O(\beta^3 n \log^2 n)$ time in the worst case if one uses the minimum (co)homology basis algorithm by Borradaile et al. [4]. Using a standard sorting algorithm, Step-2 can be computed in $O(\beta \log \beta)$ time. Finally, Step-3 can be implemented in $O(\beta n \log n)$. ◀

3.1 Improved algorithm for Homological Hitting Set

We note that to compute the minimum hitting set what we truly need is not a minimal cohomology basis, but only the smallest weight cocycle that evaluates to 1 on the input cycle ζ. This can be achieved by opening the black box from [4]. Given a surface $\mathcal{S}$ with n vertices, let β denote the rank of the first homology group of $\mathcal{S}$.

Algorithm 2 An improved algorithm for HHS on surfaces with input cycle ζ.

1: Construct the dual surface $\mathring{\mathcal{S}}$. The dual of ζ, denoted by $\mathring{\zeta}$, is a nontrivial cocycle in $\mathring{\mathcal{S}}$.
2: A topological space $\mathring{\mathcal{S}}_\zeta$ called the *cyclic double cover* [4], which is a covering space of $\mathring{\mathcal{S}}$, is constructed.
3: A multiple source shortest path procedure is run $O(\beta)$ times on $\mathring{\mathcal{S}}_\zeta$ to find the minimum weight cycle $\mathring{\eta}$ such that $\mathring{\zeta}(\mathring{\eta}) = 1$.

In the surface $\mathcal{S}$, the dual of $\mathring{\eta}$, denoted by η, is a cocycle in $\mathcal{S}$ satisfying $\eta(\zeta) = 1$. The costliest step in Algorithm 2 is Step-3. Using from [4, Lemma 6.8], Step-3 takes at most $O(\beta^2 n \log^2 n)$ time. This gives an improvement by a factor of β over Algorithm 1.

4 Minimal hitting sets are not necessarily cocycles

In this section, we introduce an example of a complex whose minimum hitting set is not a cocycle. We will first introduce a special complex that we call the *cactus complex* of an edge.

The cactus complex

Given an edge e, the *cactus complex* of e of height h, denoted by $C_h(e)$, is defined inductively as follows.

1. The complex $C_1(e)$ is merely the edge e itself. Edge e is assigned height 1.
2. Given $C_k(e)$, for some $k \geq 1$, we now describe the procedure to obtain $C_{k+1}(e)$: Identify a 2-simplex, denoted by σ_f, to every edge f of height k in $C_k(e)$. The complex obtained at the end of all identifications is $C_{k+1}(e)$. Every edge in $C_{k+1}(e)$ that has a unique 2-simplex incident on it is assigned height $k+1$. For every f, the associated simplex σ_f is assigned height k.

See Figure 5 for an illustration of a cactus complex of the edge bc of height 5.

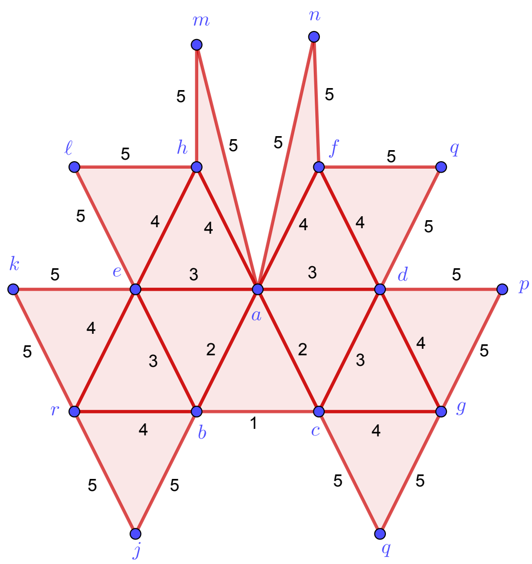

Figure 5 The figure depicts a cactus complex of the edge bc of height 5. The edges of the complex are labeled with their respective heights.

The twisted key complex

The twisted key complex can be described as follows. Start with an annulus formed by the red 2-simplices as shown in Figure 6. Attach a pencil-like complex made of blue 2-simplices as shown in the figure. Identify a cactus complex of height 22 to every red edge such that the red edge has height 1 in the cactus subcomplex. Finally, identify the blue edges with the same labels along orientations shown by directed arrows. The resulting complex is called the *twisted key complex*.

Next, we recall a lemma from Munkres [38, Lemma 3.2] that will be used to provide a guarantee that the twisted key complex is, in fact, a simplicial complex.

▶ **Lemma 10** (Munkres, [38, Lemma 3.2]). *Let $\mathcal{L}$ be a finite set of labels. Let K be a simplicial complex defined on a set of vertices V. Also, let $f : V \to \mathcal{L}$ be a surjective map associating to each vertex of K a label from $\mathcal{L}$. The labeling f extends to a simplicial map $g : \mathsf{K} \to \mathsf{K}_f$ where K_f has vertex set V and is obtained from K by identifying vertices with the same label.*

If for all pairs $v, w \in V$, $f(v) = f(w)$ implies that their stars $\mathrm{star}_{\mathsf{K}}(v)$ and $\mathrm{star}_{\mathsf{K}}(w)$ are vertex disjoint, then, for all faces $\eta, \omega \in \mathsf{K}$ we have that

- *η and $g(\eta)$ have the same dimension, and*
- *$g(\eta) = g(\omega)$ implies that either $\eta = \omega$ or η and ω are vertex disjoint in K.*

Lemma 10 provides a way of gluing faces of a simplicial complex by a simplicial quotient map obtained from vertex identifications. In particular, Lemma 10 provides conditions under which the gluing does not create unwanted identifications, and the resulting complex thus obtained is also a simplicial complex. Since the stars of the vertices that are identified are vertex disjoint (See Figure 6), the twisted key complex is a simplicial complex by Lemma 10.

Let $\zeta = \{\mathsf{DL}, \mathsf{LN}, \mathsf{NR}, \mathsf{RS}, \mathsf{SV}, \mathsf{VH}, \mathsf{HF}, \mathsf{FD}\}$ be the input cycle. Next, we will show that the minimum hitting set of ζ is not a cocycle.

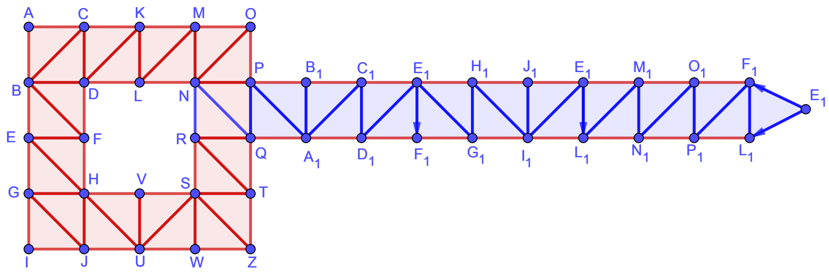

Figure 6 The figure illustrates some of the simplices of the *twisted key complex*. To begin with one starts with the unlabeled complex in the figure which looks like a key. To every red edge in this complex, one attaches a cactus complex of height 22. Then, the complex is labeled as shown. The edges labeled $\mathsf{E}_1\mathsf{F}_1$ are identified with each other with the arrow depicting the orientation of the identification. The edges labeled $\mathsf{E}_1\mathsf{L}_1$ are also identified with each other with the arrow depicting the orientation of the identification. The resulting complex is the twisted key complex. Let $\zeta = \{\mathsf{DL}, \mathsf{LN}, \mathsf{NR}, \mathsf{RS}, \mathsf{SV}, \mathsf{VH}, \mathsf{HF}, \mathsf{FD}\}$ be the input cycle. Then, a minimal hitting set of $[\zeta]$ has a minimum of 21 edges. In this case, the unique minimum hitting set comprises of the blue edges of the twisted key complex, which we denote by $\mathcal{S}$. The edges $\mathsf{E}_1\mathsf{L}_1, \mathsf{L}_1\mathsf{F}_1$ and $\mathsf{F}_1\mathsf{E}_1$ belong to $\mathcal{S}$ and are incident on the 2-simplex $\mathsf{E}_1\mathsf{L}_1\mathsf{F}_1$. Hence, $\mathcal{S}$ is not a cocycle.

▶ **Lemma 11.** *If a red edge of the twisted key complex belongs to a minimum hitting set of ζ, then the minimum hitting set has cardinality at least* 22.

Proof. Without loss of generality, let AC be the red edge which belongs to a minimum hitting set $\mathcal{S}$ of ζ. By minimality of $\mathcal{S}$, there is a cycle $\gamma \in [\zeta]$ such that $\gamma \cap \mathcal{S} = \{\mathsf{AC}\}$. In the cactus complex identified to edge AC, AC is of height 1. Then, the support of $\gamma + \partial\sigma_{\mathsf{AC}}$ differs from the support of γ only on the two edges of height 2 that are incident on σ_{AC}. Hence, one of the edges of height 2 incident on σ_{AC} belongs to $\mathcal{S}$. Repeating this argument inductively, we see that if AC is in $\mathcal{S}$, then there is at least one edge of height h for every $1 \leq h \leq 22$ in the cactus complex incident on AC that also belongs to $\mathcal{S}$, proving the claim. ◀

▶ **Theorem 12.** *The cycle ζ of the twisted key complex* T *has a unique minimum hitting set* $\mathcal{C}$*, where* $\mathcal{C}$ *is not a cocycle of* T*.*

Proof. If the set $\mathcal{C}$ does not contain any red or blue edges of T, then $\mathcal{C}$ is not a hitting set of ζ. Since we aim to find a hitting set of size smaller than 22, from Lemma 11, we hypothesize that $\mathcal{C}$ has at least one blue edge and no red edges. Without loss of generality let $\mathsf{C}_1\mathsf{D}_1$ be a blue edge in $\mathcal{C}$. By minimality of $\mathcal{C}$, there exists a cycle γ such that $\gamma \cap \mathcal{C} = \{\mathsf{C}_1\mathsf{D}_1\}$. Since the cycles $\gamma + \partial(\mathsf{C}_1\mathsf{D}_1\mathsf{E}_1)$ and $\gamma + \partial(\mathsf{C}_1\mathsf{D}_1\mathsf{A}_1)$ are also hit by $\mathcal{C}$, we conclude that $\mathsf{A}_1\mathsf{C}_1 \in \mathcal{C}$ and $\mathsf{E}_1\mathsf{D}_1 \in \mathcal{C}$. Inductively applying this argument, we conclude that all blue edges must lie in $\mathcal{C}$. Therefore, $|\mathcal{C}| = 21$. Since the choice of $\mathsf{C}_1\mathsf{D}_1$ at the beginning of the argument is arbitrary, it follows that ζ has a unique minimal hitting set, namely, one that comprises of all the blue edges. Finally, $\delta\mathcal{C} = \delta\sum_{e\in\mathcal{C}} e = \{\mathsf{F}_1\mathsf{E}_1\mathsf{L}_1\} \neq 0$. In other words, since the edges $\mathsf{E}_1\mathsf{F}_1$, $\mathsf{F}_1\mathsf{L}_1$ and $\mathsf{E}_1\mathsf{L}_1$ all belong to $\mathcal{C}$, and $\mathsf{F}_1\mathsf{E}_1\mathsf{L}_1$ is a 2-simplex in T, $\mathcal{C}$ is not a cocycle. ◀

5 W[1]-hardness for Homological Hitting Set

In this section, we obtain **W[1]**-hardness results for HHS with respect to the solution size k as the parameter via parameterized reductions from Multicolored Clique. We begin this section by recalling some common notions from graph theory.

A *k-clique* in a graph G is a complete subgraph of G with k vertices. A *k-coloring* of a graph G is an assignment of one of k possible colors to every vertex of G (that is, a vertex coloring) such that vertices of the same color do not share an edge. A graph G equipped with a k-coloring is called a *k-colored graph*. Then, a *multicolored k-clique* in a colored graph is a k-clique with a k-coloring. Multicolored Clique (MCC) asks for the existence of a multicolored k-clique in a k-colored graph G. We remark that reducing from MCC is a highly effective tool for showing **W[1]**-hardness [26]. Formally, MCC is defined as follows:

▶ **Problem 2** (Multicolored Clique (MCC)).

Instance:	A graph $G = (V, E)$, and a vertex coloring $c : V \to [k]$.
Parameter:	k.
Question:	Does there exist a multicolored k-clique H in G?

▶ **Theorem 13** (Fellows et al. [26]). *MCC is* **W[1]***-complete.*

For $i \in [k]$, the subset of vertices of color i is denoted by V_i. Clearly, the vertex coloring c induces a partition on V:

$$V = \cup_{i=1}^{k} V_i, \quad \text{and} \quad V_i \cap V_j = \emptyset \text{ for all } i, j \in [k].$$

We now provide a parameterized reduction from MCC to HHS. For $r = |V| - 1$, we define an $(r+1)$-dimensional complex $\mathsf{K}(G)$ associated to the given colored graph G as follows.

Vertices. The set of vertices of $\mathsf{K}(G)$ contains the disjoint union of the vertices V in the graph G, the set of colors $[k]$, and an additional dummy vertex d. Altogether, we have $r + k + 2$ vertices in $\mathsf{K}(G)$ so far. In what follows, further vertices are added to $\mathsf{K}(G)$.

Simplices. Below, we describe the simplices that constitute the complex $\mathsf{K}(G)$.

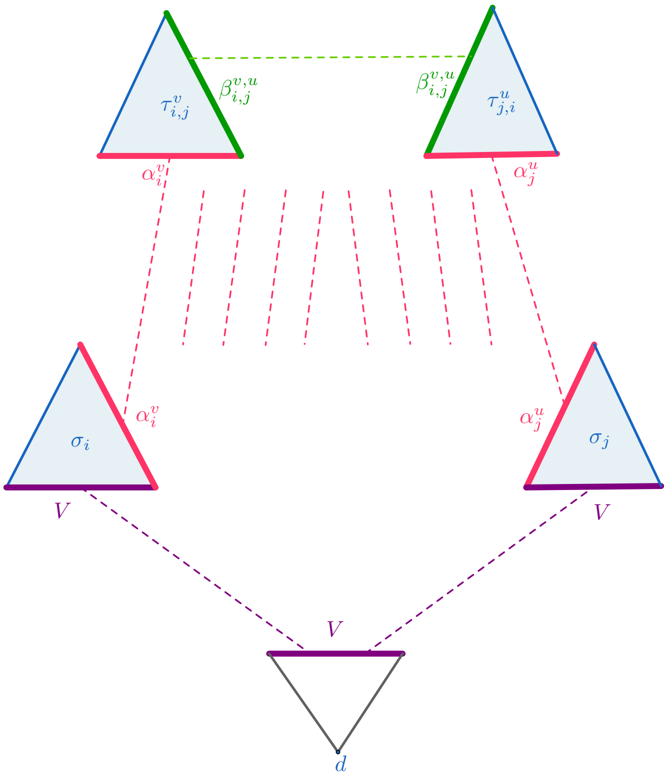

Figure 7 The figure shows simplices in $\mathcal{X} \subsetneq \mathsf{K}(G)$. In particular, α^v_i is the common face of $\tau^v_{i,j}$ and σ_i, α^u_j is the common face of $\tau^u_{j,i}$ and σ_j, and $\beta^{v,u}_{i,j}$ is the common face of $\tau^v_{i,j}$ and $\tau^u_{j,i}$. The dashed lines indicate identical facets. The set of r-simplices supported by the vertices $V \cup \{d\}$ form a nontrivial r-cycle in $\mathsf{K}(G)$.

The cycle ζ. First, add the r-simplex V corresponding to vertex set V of the graph G. Next, add the r-simplices $(V \setminus \{u\}) \cup \{d\}$ for every $u \in V$. The collection of these $r + 2$ simplices of dimension r forms a nontrivial r-cycle ζ. The simplices $(V \setminus \{u\}) \cup \{d\}$ for every $u \in V$ are deemed *undesirable*, whereas the simplex V is classified as *admissible*. As explained later, we wish to make the inclusion of undesirable simplices in solutions prohibitively expensive.

Next, we describe the sets $\mathcal{X}_1$ and $\mathcal{X}_2$, which are collections of $(r + 1)$-simplices.

The set $\mathcal{X}_1$. For every $i \in [k]$, add an $(r + 1)$-simplex $\sigma_i = V \cup \{i\}$ to $\mathcal{X}_1$.

- That is, $\mathcal{X}_1 = \{\sigma_i = V \cup \{i\} \mid i \in [k]\}$.

▶ **Definition 14** (Admissible and undesirable facets of σ_i). *The admissible facets of σ_i are:*

- *The facet V.*
- *The facets $(V \setminus \{v\}) \cup \{i\}$ of σ_i for all $v \in V_i$.*

The facet $(V \setminus \{v\}) \cup \{i\}$ for all $v \notin V_i$ are the undesirable *facets of σ_i.*

The idea here is that including an admissible simplex of the form $(V \setminus \{v\}) \cup \{i\}$ in $\mathcal{S}$ is akin to picking the vertex v of color i for constructing the colorful clique, whereas the coloring specified by undesirable facets is incompatible with the coloring c on vertices V.

The set $\mathcal{X}_2$. For every color $i \in [k]$, for every vertex v in V_i and every color $j \in [k] \setminus \{i\}$, add an $(r+1)$-simplex $\tau^v_{i,j} = (V \setminus \{v\}) \cup \{i, j\}$ to $\mathcal{X}_2$.

- That is, $\mathcal{X}_2 = \left\{\tau^v_{i,j} \mid i \in [k],\ v \in V_i,\ j \in [k] \setminus \{i\}\right\}$.

▶ **Definition 15** (Admissible and undesirable facets of $\tau^v_{i,j}$)**.** *The* admissible *facets of $\tau^v_{i,j}$ are:*

- *$(V \setminus \{v, u\}) \cup \{i, j\}$ with $u \in V_j$ and $\{u, v\} \in E$, and*
- *$(V \setminus \{v\}) \cup \{i\}$,*

A facet of $\tau^v_{i,j}$ that is not admissible is said to be undesirable*.*

By design, picking an admissible facet of the form $(V \setminus \{v, u\}) \cup \{i, j\}$ is akin to picking the edge $\{u, v\}$ of color $\{i, j\}$ for constructing the colorful clique, whereas the admissible facet $(V \setminus \{v\}) \cup \{i\}$ is common with σ_i. Undesirable simplices of $\tau^v_{i,j}$ correspond either to coloring that is incompatible with c or with edges that are not even present in E.

▶ **Notation 16.** *The admissible facets $(V \setminus \{v\}) \cup \{i\}$ and $(V \setminus \{v, u\}) \cup \{i, j\}$ are denoted by α^v_i and $\beta^{v,u}_{i,j}$, respectively. For every vertex $v \in V$ of color i, there is a facet α^v_i. For every edge $\{u, v\} \in E$, there is a facet $\beta^{v,u}_{i,j}$, where i is the color of v and j is the color of u.*

▶ Remark 17 (Meaning of superscripts and subscripts of simplices). A simple mnemonic for remembering the meaning of the notation for simplices is as follows: the indices in the subscript are the included colors, and the vertices in the superscript indicate the vertices excluded from V. For instance, $\beta^{v,u}_{i,j}$ is the full simplex on the vertex set $(V \setminus \{v, u\}) \cup \{i, j\}$. In this case, colors i and j are included and vertices u and v are excluded. The same notational rule applies for α^v_i , σ_i and $\tau^v_{i,j}$.

▶ Remark 18 (Correspondence between colors and vertices in α^v_i $\beta^{v,u}_{i,j}$ and $\tau^v_{i,j}$). In our notation, the first color corresponds to the first vertex, the second color to the second vertex.

- In α^v_i, vertex v is of color i, whereas in $\beta^{v,u}_{i,j}$, v is of color i and u is of color j.
- In $\tau^v_{i,j}$, v is of color i and the vertex associated to color j is not specified. It is, in fact, chosen through a facet $\beta^{v,u}_{i,j} \prec \tau^v_{i,j}$.

Now, let $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$. The $(r+1)$-simplices in $\mathcal{X}$ along with incident facets are shown in Figure 7. We intend to make the inclusion of undesirable simplices in the solution prohibitively expensive. For this purpose, we add an additional set $\mathcal{Y}$ of $(r+1)$-simplices (called *inadmissible simplices*).

Undesirable and inadmissible simplices. The undesirability of certain r-simplices is implemented as follows: Let $m = n^3$. Then, to every undesirable r-simplex $\omega = \{v_1, v_2, \ldots, v_{r+1}\}$, associate m new vertices $\mathcal{U}^\omega = \{u^\omega_1, u^\omega_2, \ldots, u^\omega_m\}$. Now introduce m new $(r+1)$-simplices

$$\Upsilon^\omega = \{\mu_i(\omega) = \{v_1, v_2, \ldots, v_{r+1}, u^\omega_i\} \mid i \in [m]\}$$

that are cofacets of ω.

▶ **Definition 19** (Set of inadmissible simplices associated to an undesirable simplex ω)**.** *The set of r-simplices in $\{\{$facets of $\mu_i(\omega)\} \mid i \in [m]\}$ is denoted by $[\omega]$. The simplices in the set $[\omega]$ are said to be* inadmissible*. In particular, ω itself is inadmissible.*

See Figure 8 for an illustrative example. Every edge in Figure 8 is a facet of some simplex $\mu_i(\omega)$, and hence inadmissible, whereas ω is undesirable (as well as inadmissible).

Further, note that the set of vertices in $\mathcal{U}^\omega$ and r-simplices in Υ^ω are unique to ω. As we observe later, introducing these new simplices makes inclusion of ω in the solution set prohibitively expensive. Denote by $\mathcal{Y}$ the set of all $(r+1)$-simplices added in this step.

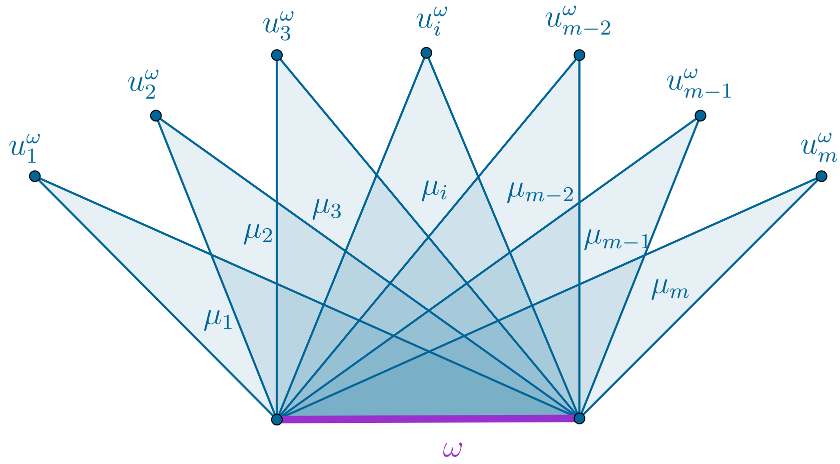

Figure 8 For every undesirable simplex $\omega = \{v_1, v_2, \dots, v_{r+1}\}$ m new vertices $\mathcal{U}^\omega = \{u_1^\omega, u_2^\omega, \dots, u_m^\omega\}$ are added to $\mathsf{K}(G)$. Also, m new $(r+1)$-simplices $\Upsilon^\omega = \{\mu_i(\omega) = \{v_1, v_2, \dots, v_{r+1}, u_i^\omega\} \mid i \in [m]\}$ are added to $\mathsf{K}(G)$ as cofacets of ω. The facets of $\mu_i(\omega)$ for every $i \in [m]$ are the inadmissible simplices associated to ω and denoted by $[\omega]$.

The final complex $\mathsf{K}(G)$ used in the reduction is the complex obtained by taking closure of simplices in $\mathcal{X} \cup \mathcal{Y}$ (under taking subsets). It is easy to check that the inadmissible and admissible simplices of $\mathsf{K}(G)$ partition the set of r-simplices of $\mathsf{K}(G)$.

▶ Remark 20 (Size of $\mathsf{K}(G)$). We note that every subset of vertices of G is a simplex in $\mathsf{K}(G)$. However, $\mathsf{K}(G)$ is represented implicitly, and the simplices of dimensions other than r and $(r+1)$ are not used in the reduction. Thus, although $\mathsf{K}(G)$ as a simplicial complex is exponential in the size of G, the reduction itself is polynomial in the size of G because the number of r and $(r+1)$ dimensional simplices of $\mathsf{K}(G)$ are polynomial in size of G.

Choice of parameter. Let $(k + \binom{k}{2} + 1 = \binom{k+1}{2} + 1)$ be the parameter for HHS on $\mathsf{K}(G)$.

▶ **Lemma 21.** *If there exists a multicolored k-clique $H = (V_H, E_H)$ of G, then there exists a homological hitting set $\mathcal{S}$ for ζ consisting of $\binom{k+1}{2} + 1$ r-simplices.*

Proof. We construct a set $\mathcal{S}$ of r-simplices that mimics the graphical structure of H. First, we define

$$\mathcal{S}_\alpha = \left\{\alpha_i^v \mid v \in V_i \bigcap V_H\right\}$$

$$\mathcal{S}_\beta = \left\{\beta_{i,j}^{v,u} \mid v \in V_i,\, u \in V_j,\, \{i,j\} \in E_H\right\}$$

Then, we set $\mathcal{S} = \{V\} \cup \mathcal{S}_\alpha \cup \mathcal{S}_\beta$. Next, note that every cycle $\zeta' \in [\zeta]$ can be expressed as

$$\zeta' = \zeta + \sum_{\nu_i \in \mathcal{X}'} \partial \nu_i + \sum_{\mu_j \in \mathcal{Y}'} \partial \mu_j$$

for some $\mathcal{X}' \subset \mathcal{X}_1 \cup \mathcal{X}_2$ and $\mathcal{Y}' \subset \mathcal{Y}$. Let $\mathcal{X}_1' = \mathcal{X}' \bigcap \mathcal{X}_1$, and $\mathcal{X}_2' = \mathcal{X}' \bigcap \mathcal{X}_2$. Now, we claim that removing $\mathcal{S}$ from $\mathsf{K}(G)$ destroys every cycle $\zeta' \in [\zeta]$. We show this by establishing that in every $\zeta' \in [\zeta]$ the coefficient of at least one of the simplicies of $\mathcal{S}$ is 1. In other words, $\mathcal{S} \cap \zeta' \neq \emptyset$ for every $\zeta' \in [\zeta]$.

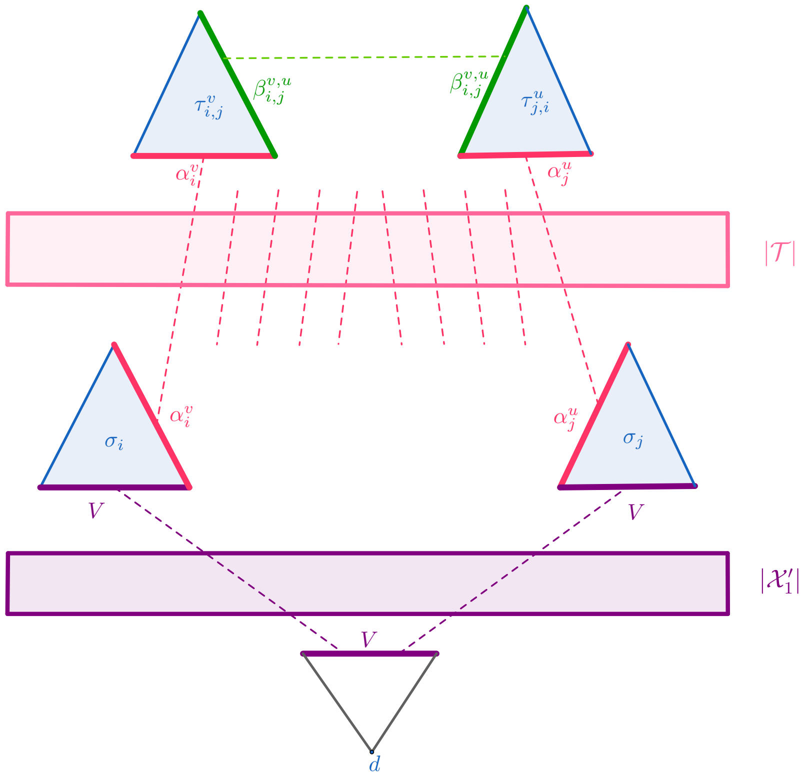

Figure 9 In this figure, $\tau^v_{i,j}$ and $\tau^u_{j,i}$ belong to $\mathcal{X}_2''$, σ_i and σ_j belong to $\mathcal{X}_1'$, and α^v_i and α^u_j belong to $\mathcal{S}_\alpha$. $\mathcal{T}$ accounts for all the incidences of the boundaries of simplices in $\mathcal{X}_2''$ on simplices in $\mathcal{S}_\alpha$. The final part of the argument in Case 4 of Lemma 21 is depicted here. Since $|\mathcal{T}|$ is even and the coefficient of all the simplices of $\mathcal{S} \setminus \{V\}$ have coefficient 0 in some cycle $\zeta' \in [\zeta]$, we have $\mathcal{X}_1' = \mathcal{O}$, and $\mathcal{O}$ has even cardinality. But if this is so, then the coefficient of V in ζ' is $(1 + |\mathcal{O}|) \bmod 2 = 1$.

Case 1: $\mathcal{X}' = \emptyset$. Then, $V \in \mathcal{S}$ has coefficient 1 in cycle ζ'. This is because simplices in $\mathcal{Y}$ are not incident on V, and $V \in \zeta$.

Case 2: $\mathcal{X}_1' \neq \emptyset, \mathcal{X}_2' = \emptyset$. Then, the cycle ζ' can be written as

$$\zeta' = \zeta + \sum_{\sigma_j \in \mathcal{X}_1'} \partial\sigma_j + \sum_{\mu_\ell \in \mathcal{Y}'} \partial\mu_\ell$$

Then, every $\alpha^v_j \in \mathcal{S}$ for $\sigma_j \in \mathcal{X}_1'$ and $v \in V_H \bigcap V_j$ has coefficient 1 in cycle ζ'. This is because $\alpha^v_j \in \partial\sigma_j$ for every $\sigma_j \in \mathcal{X}_1'$, but $\alpha^v_j \notin \zeta$ and $\alpha^v_j \notin \partial\mu_\ell$ for any $\mu_\ell \in \mathcal{Y}'$.

Case 3: $\mathcal{X}_1' = \emptyset, \mathcal{X}_2' \neq \emptyset$. This case is identical to Case 1 since $V \in \mathcal{S}$ has coefficient 1 in ζ'.

Case 4: $\mathcal{X}_1' \neq \emptyset$, $\mathcal{X}_2' \neq \emptyset$. If every simplex $\tau^v_{p,q} \in \mathcal{X}_2'$ is such that $v \in V_p \setminus V_H$, then this case becomes identical to Case 2. So we will assume without loss of generality that the set $\mathcal{X}_2'' = \left\{\tau^v_{p,q} \mid p, q \in [k], v \in V_p \cap V_H, \tau^v_{p,q} \in \mathcal{X}_2'\right\}$ is non-empty. For some $\{u, v\} \in E_H$ and $u \in V_q$, $v \in V_p$, if $\tau^v_{p,q} \in \mathcal{X}_2''$ and $\tau^u_{q,p} \notin \mathcal{X}_2''$, then the coefficient of $\beta^{v,u}_{p,q} \in \mathcal{S}$ in ζ' is 1 because the only two $(r+1)$-simplices incident on $\beta^{v,u}_{p,q}$ are $\tau^v_{p,q}$ and $\tau^u_{q,p}$. So, without loss of generality assume that the symmetric simplex $\tau^u_{q,p}$ is also in $\mathcal{X}''_2$. In other words, $|\mathcal{X}''_2|$ is even. Note that for every $\tau^v_{p,q} \in \mathcal{X}''_2$, exactly one facet of $\tau^v_{p,q}$ lies in $\mathcal{S}_\alpha$, namely α^v_p. Hence the cardinality of the multiset $\mathcal{T} = \{\partial\tau^v_{i,j} \cap \mathcal{S}_\alpha \mid \tau^v_{i,j} \in \mathcal{X}''_2\}$ is even. Let $\mathcal{C}(\sigma_i) = \left|\{\tau^v_{i,j} \in \mathcal{X}''_2 \mid \alpha^v_i \in \partial\tau^v_{i,j} \cap \mathcal{S}_\alpha\}\right|$. Also, let $\mathcal{E} = \{\sigma_i \mid \mathcal{C}(\sigma_i) \text{ is even}\}$ and $\mathcal{O} = \{\sigma_i \mid \mathcal{C}(\sigma_i) \text{ is odd}\}$.

Note that $|\mathcal{T}| = \sum_{\sigma_i \in \mathcal{E}} \mathcal{C}(\sigma_i) + \sum_{\sigma_i \in \mathcal{O}} \mathcal{C}(\sigma_i)$.

Since $|\mathcal{T}|$ is even, $|\mathcal{O}|$ must be even. By construction, $\mathcal{X}'_1 \subseteq \mathcal{O} \cup \mathcal{E}$. For some i, let $v \in V_i \cap V_H$. Now, if $\sigma_i \in \mathcal{E} \cap \mathcal{X}'_1$, then the coefficient of $\alpha^v_i \in \mathcal{S}$ in ζ' is 1 because the only $(r+1)$-simplices incident on α^v_i are $\{\tau^v_{i,j} \in \mathcal{X}''_2 \mid \alpha^v_i \in \partial\tau^v_{i,j}\} \cup \{\sigma_i\}$, and $|\{\tau^v_{i,j} \in \mathcal{X}''_2 \mid \alpha^v_i \in \partial\tau^v_{i,j}\}|$ is even when $\sigma_i \in \mathcal{E}$. So, without loss of generality assume that $\mathcal{E} \cap \mathcal{X}'_1$ is empty. That is, we assume that $\mathcal{X}'_1 \subseteq \mathcal{O}$. But if $\sigma_i \in \mathcal{O} \setminus \mathcal{X}'_1$, then the coefficient of $\alpha^v_i \in \mathcal{S}_\alpha$ in ζ' is 1 because in that case the only $(r+1)$-simplices incident on α^v_i are $\{\tau^v_{i,j} \in \mathcal{X}''_2 \mid \alpha^v_i \in \partial\tau^v_{i,j}\}$ and $\left|\{\tau^v_{i,j} \in \mathcal{X}''_2 \mid \alpha^v_i \in \partial\tau^v_{i,j}\}\right|$ is odd. So, we assume that $\mathcal{O} = \mathcal{X}'_1$. But if $\mathcal{O} = \mathcal{X}'_1$, then $V \in \mathcal{S}$ has coefficient 1 in ζ' because

1. $\mathcal{O}$ is even, and the simplices in $\mathcal{O}$ are incident on V
2. $V \in \zeta$.

This completes the proof. Figure 9 illustrates the final part of the argument. ◀

While the forward direction (Lemma 21) of the reduction is unaffected by the addition of inadmissible simplices, the inadmissible simplices are indispensable for proving the reverse direction (Lemma 22). We defer the proof of Lemma 22 to the full version of the paper.

▶ **Lemma 22.** *If $\mathcal{R}$ is a feasible solution for HHS on $\mathsf{K}(G)$ and $|\mathcal{R}| = \binom{k+1}{2} + 1$, then one can obtain a k-clique H of G from $\mathcal{R}$.*

Theorem 13 and Lemmas 21 and 22 combine to give the following theorem.

▶ **Theorem 23.** *HHS is $\mathbf{W}[\mathbf{1}]$-hard.*

6 FPT Algorithm for Homological Hitting Set

In Section 5 we showed that HHS is $\mathbf{W}[\mathbf{1}]$-hard with the solution size k as the parameter. This motivates the search of other meaningful parameters that make the problem tractable. With that in mind, in this section, we prove an important structural property about the connectivity of the minimal solution sets for HHS. First, we start with a definition.

▶ **Definition 24** (Induced subgraphs in Hasse graphs). *Given a d-dimensional complex K with Hasse graph H_K, and a set $\mathcal{S}$ of r-simplices for some $r < d$, the subgraph of H_K* induced by *$\mathcal{S}$ is the union of $\mathcal{S}$ with the set of $(r+1)$-dimensional simplices incident on $\mathcal{S}$.*

▶ **Lemma 25.** *Given a d-dimensional complex K, a minimal solution of HHS for a non-bounding cycle $\zeta \in \mathsf{Z}_r(\mathsf{K})$ for some $r < d$ induces a connected subgraph of H_K.*

Proof. Let $H_\mathcal{S}$ be the subgraph of the Hasse graph H_K induced by a minimum homological hitting set $\mathcal{S}$ of a non-bounding cycle $\zeta \in \mathsf{Z}_r(\mathsf{K})$. Targeting a contradiction, assume that $H_\mathcal{S}$ is a disjoint union of C_1 and C_2. That is, C_1 and C_2 are subgraphs of $H_\mathcal{S}$ that do not share any $(r+1)$-simplices. Since $\mathcal{S}$ is minimal, there exists an r-cycle $\phi \in [\zeta]$ that contains an r-simplex in C_1 but not any of the r-simplices in C_2, and an r-cycle $\psi \in [\zeta]$ that contains an r-simplex in C_2 but not any of the r-simplices in C_1. Then, $\phi = \psi + \partial b$, for some $(r+1)$ chain b. Let b' be an $(r+1)$-chain obtained from b by removing exactly those $(r+1)$-simplices that belong to C_1. Now, let $\phi' = \psi + \partial b'$. By construction, ϕ' is disjoint from C_1. Also, because C_1 and C_2 are disjoint, the simplices removed from b to obtain b' do not belong to C_2. Hence, ϕ' is disjoint from C_2. In other words, $\phi' \in [\zeta]$ does not meet $\mathcal{S}$, and $\mathcal{S}$ is not a hitting set, a contradiction. Therefore, the induced subgraph of $\mathcal{S}$ is connected. ◀

Note that the path from any r-simplex to another r-simplex passing through a shared $(r+1)$-simplex in the Hasse graph is of size 2. So it follows from Lemma 25 that any minimal solution of size at most k lies in some geodesic ball of radius $2k$ of some r-simplex in the Hasse graph. If we enumerate all the connected sets in the geodesic ball of every r-simplex in the complex K, then we will find a solution if one exists. Algorithm 3 provides a pseudocode for this algorithm.

Algorithm 3 FPT Algorithm for HOMOLOGICAL HITTING SET with $k + \Delta$ as the parameter.

1: SOL $= \mathsf{K}$;
2: **for each** r-simplex τ of K **do**
3: Consider the set S_τ of all simplices within the graph distance $2k$ (in H_K) of τ.
4: **for every** connected subset $S \subseteq S_\tau$ with $|S| \leq k$ **do**
5: **if** S is a hitting set of ζ and $|S| < |\,\text{SOL}\,|$ **then**
6: SOL $= S$;
7: **if** $|\,\text{SOL}\,| < k$ **then return** SOL;

Note that in Line 4 of Algorithm 3, we need to enumerate only the connected subsets S of cardinality less than or equal to k. We use Lemmas 27 and 28 by Fomin and Villanger [28] that provide very good bounds for enumerating connected subgraphs of graphs. First, we introduce some notation.

▶ **Notation 26.** *The neighborhood of a vertex v is denoted by* $\text{nbd}(v) = \{u \in V : u, v \in E\}$*, whereas the neighborhood of a vertex set $S \subseteq V$ is set to be* $\text{nbd}(S) = \cup_{v \in S} \text{nbd}(v) \setminus S$.

▶ **Lemma 27** ([28, Lemma 3.1]). *Let $G = (V, E)$ be a graph. For every $v \in V$, and $b, d \geq 0$, the number of connected vertex subsets $\mathcal{C} \subseteq V$ such that*
1. $v \in \mathcal{C}$*,*
2. $|\mathcal{C}| = b + 1$*, and*
3. $|\,\text{nbd}(\mathcal{C})| = d$

is at most $\binom{b+d}{b}$.

▶ **Lemma 28** ([28, Lemma 3.2]). *All connected vertex sets of size $b + 1$ with d neighbors of an n-vertex graph G can be enumerated in time $O(n^2 \cdot b \cdot (b + d) \cdot \binom{b+d}{b})$ by making use of polynomial space.*

In Algorithm 3, $b = O(k)$ and $d = O(k\Delta)$. Therefore,

$$\binom{b+d}{b} = \binom{O(k\Delta)}{O(k)} \leq (k\Delta)^{O(k)} = 2^{O(k \log(k\Delta))}. \tag{1}$$

Hence, by Lemma 28, for a single r-simplex, the number of connected sets enumerated in Line 4 is $O(n^2 \cdot O(k) \cdot O(k\Delta) \cdot 2^{O(k \log(k\Delta))}) = O(n^5 \cdot 2^{O(k \log(k\Delta))})$ time. Since we do this for every r-simplex τ in K, all candidate sets in Lines 4-6 can be enumerated in at most $O(n^6 \cdot 2^{O(k \log(k\Delta))})$ time. Using Lemma 2, one can check if the set is a feasible solution in time $O(n^\omega)$, where ω is the exponent of matrix multiplication. Checking feasibility for every enumerated connected set takes $O(n^{\omega+1} \cdot 2^{O(k \log(k\Delta))})$ time. Hence, the algorithm runs in $O(n^6 \cdot 2^{O(k \log(k\Delta))})$ time, which is fixed parameter tractable in $k + \Delta$.

▶ **Theorem 29.** *HHS admits an FPT algorithm with respect to the parameter $k + \Delta$, where Δ is the maximum degree of the Hasse graph and k is the solution size. The algorithm runs in $O(n^6 \cdot 2^{O(k \log(k\Delta))})$ time.*

Proof. The correctness of Algorithm 3 follows immediately from Lemma 25. ◀

▶ Remark 30. Note that the degree Δ of the Hasse graph H_{K} is bounded when the dimension of the complex is bounded and the number of incident cofacets on every simplex (or cell) is bounded. It is easy to check that Lemma 25 and Algorithm 3 are also applicable for cubical sets. We describe all results in this paper for simplicial complexes only to keep the background requirements to minimum. Low dimensional cubical sets are commonly used in dynamical systems computations [31, Definition 2.9], and provide an example of a family of complexes for which Δ is bounded.

7 Conclusion and Discussion

In Section 3, we describe Algorithm 1 only for triangulated surfaces. In the full version of the paper, we will generalize Algorithm 1 to cellularly embedded graphs and manifolds.

In Section 5, we prove that Homological Hitting Set is **W[1]**-hard. While this may be discouraging at first, we note in Remark 30 that for important families of complexes like cubical sets, Homological Hitting Set is tractable in the natural parameter. Overall, we believe that an algorithmic study of high-dimensional cuts is fruitful.

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