2013

A study of finite symmetrical groups

Patrick Kevin Martinez

A Study of Finite Symmetrical Groups

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

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Abstract

In this thesis, we have discovered several important groups that involve the classical and sporadic groups. These groups have appeared as finite homomorphic images of the progenitors $3^{*8} : PGL_2(7)$, $2^{*14} : L_3(2)$, $5^{*3} : S_3$ and $7^{*2} :_m S_3$. We used the technique of manual double coset enumeration to give a by hand construction of several groups, including $(M_{21} \times 4) : S_3$, $U_3(3) : 3$, and A_7 . For some of the groups we have given computer-based proofs of their isomorphism types. The symmetric presentations given in this thesis for the groups $L_2(7)$, $U_3(3) : 3$, $(M_{21} \times 4) : S_3$ and $S_4(5)$ are original to the best of our knowledge.

Acknowledgements

No journey of this academic magnitude can be walked alone. I wish to thank my wonderful wife for supporting me in this major endeavor. Without her love and support, my success would not have been possible. I wish to thank Dr. Stanton for his patience and understanding. I thank my Graduate Commitee members Dr. Laura Wallace and Dr. Shawnee McMurran, who have been instrumental in my success here at CSUSB. I wish to thank my fellow students for helping me succeed, in particular Rebeca Blanquet, Ramon Ontiveros, Curtis Koch, Grace Gerken and May Majid. Lastly, I wish to thank my graduate mentor Dr. Zahid Hasan. His guidence and support in my understanding of algerba has been unparalleled. He helped me realize my potential for graduate level mathematics and his belief in my ability has helped me become a better mathematician, and a better person.

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Chapter 1

Introduction

The successful exploration of finite groups requires multiple methods, due to their varied properties. We will investigate and construct several types of groups throughout this thesis. The first type is *symmetric*, which we will discuss in Chapter 2. In Chapter 3, we will construct an *alternating* group, which we will then find and factor by the center. In Chapter 4, we will construct a *projective general linear* group, denoted *PGL*. In some instances, we utilize multiple types of groups within a single construction. Such is the case in Chapter 5, where we will construct a *unitary* group as the homomorphic image of a *general linear* group. In Chapter 6, we examine similar presentations of two different types of groups via their homomorphic images by constructing a monomial presentation of an *alternating* group as a homomorphic image of another monomial presentation of a *general linear* group. We will also define and discuss the *lifting process* by *induction*, which we utilize to determine multiple homomorphic images of a monomial progenitor in Chapter 7.

1.1 Definitions

Group

A *group* is a set, G, combined with an operation $*$, such that:

(1) An identity element exists:

There exists an $e \in G$ such that for all $g \in G$, $e * g = g * e = g$

(2) The inverse element exists in G:

There exists an $h \in G$ such that $g * h = h * g = e$

We say H is a *subset* of G if every element of H is also an element of G.

Semi-Direct Product

Lemma 1.1. *Let K be a group and $A \leq AutK$ be a subgroup of the automorphism group of K. Then the cartesian product $A \times K$ becomes a group under the binary operation "$\circ$" defined by $(a, x) \circ (b, y) = (ab, x^b y)$ where $a, b \in A$ and $x, y \in K$.*

The group constructed from a cartesian product of two groups A and K, as described in the lemma above, is called a *semi-direct product* and is denoted by $K : A$. A *progenitor* is a semi-direct product of the form:

$P \cong m^{*n} : N = \{\pi\omega | \pi \in N$ and ω is a reduced word in $t_i\}$

where m^{*n} denotes the free product of n copies of the cyclic group of order m generated by t_i for $i = 1, 2, ..., n$, of order n, and N is a transitive permutation group of degree n which acts on the free product by permuting the generators (i.e. *joins*), t_i's.

Group Action

Let G be a group and X be a nonempty set. We say that G acts on X if there exists a mapping $\alpha : G \times X \to X$ defined as $(g, x) \to xg$ such that:

(1) $x \cdot 1 = x, \forall x \in X$

(2) For each $x \in X$, $x(gh) = (gh)x, \forall g, h \in X$.

The mapping α is called an *action of G on X*.

If G is a group and $a \in G$, then a *conjugate* of a is any element in G of the form $g^{-1}ag$, where $g \in G$. We also write $g^{-1}ag = a^g$. If G act on X, then $f : G \to S_X$ is a homomorphism. We have $xf(g) = xg, \forall x \in X$

Right Coset

Let G be a group and H be a subgroup of G then a *right coset* of $h \in G$ is a set $Ha = \{ha | a \in G\}$, where $a \in G$. The cosets partition the set G into disjoint subsets. We note that:

(1) Either $Ha = Hb$ or $Ha \cap Hb = \oslash$

(2) $Ha = H$ if and only if $a \in H$.

Orbits

Let G be a permutation group on the finite set X and let $x \in X$. The *orbit* of x is the set

$$X^G = \{x^\alpha | \alpha \in G\}$$

Double Coset

Let H be a subgroup of G. Let $x \in G$. Then $HxH = \{Hxh | h \in H\}$ is a *double coset* of H in G. Notice that double cosets are composed of right cosets, i.e. single cosets. The *index* of a subgroup $H \in G$, denoted by $[G : H] = \frac{|G|}{|H|}$, is the number of single cosets of H in G. In particular, the number of single cosets in the double coset NwN is $\frac{|N|}{|N^{(w)}|}$. To determine the distinct *single cosets* in a double coset NwN, you take Nw and conjugate it by its coset stabiliser $N^{(w)}$. If $N^{(w)}$ has several elements, you do this for each element. The *orbits* of $N^{(w)}$ on the symmetric generators are obtained through conjugation of each generator by $N^{(w)}$. The orbits are disjoint.

Permutation Group

In some of the following chapters, we will be dealing with groups in which the control subgroup, N, is a permutation group. The *permutation group* S_n is the group of permutations of (01234....n). The order of S_n is $|S_n| = n!$

Let $X = \{1, 2, 3, ...\}$. Then S_X, the set of all one-to-one and onto mappings from X to X, called permutations of X, forms a group under function composition. S_X is called the *permutation group* of X. If $X = \{1, 2, 3, ..., n\}$, then $S_X = S_n$ is called the *symmetric group of degree n.*

1.2 Types of Representations

In group theory, we have different ways to characterize and define a specific group. We define these different methods of expressing groups as a **representation.** We will discuss four different types of representations known as *symmetric*, *permutation*, *matrix* and *monomial.* We will first discuss *symmetric* representation.

Symmetric Representation

We define a symmetric representation of a group G of the form

$$G \cong \frac{p^{*n} : N}{\pi w_1, \pi w_2, ...}$$

where p^{*n} denotes a free product of n copies of the cyclic group of order p, N is a transitive permutation group of degree n which permutes the n generators of the cyclic groups by conjugation, which defines a semi-direct product factored by the relators, denoted $\pi w_1, \pi w_2, ...$

The progenitor $p^{*n} : N$ represents an infinite group, so to produce finite images of G, we must factor by some relation represented by $\pi w_1, \pi w_2, ...$

Permutation Representation

We will now dicuss a permutation representation of a group.

Let G be a group, denoted by

$$G = \{a, b | a^2 = b^2 = (ab)^2 = 1\}$$

where $ab = ba$. The elements within this group are $\{e, a, b, ab\}$. We will demonstrate the permutation representaion of G by denoting the elements a and b as two cycle permutaions

$$a = (1,2)(3,4) \text{ and } b = (1,3)(2,4).$$

Then the permutaion representation of this particular G is

$$P =< (1,2)(3,4), (1,3)(2,4) >$$

where all the elements within G are generated by these two cycle permutations via right hand multiplication. This gives us

$$\{e, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$$

Matrix Representation

We define a Matrix representation of a group G as

$G \to GL(n, \mathbb{C})$, where $GL(n, \mathbb{C})$ is a general linear group of degree n

defined by

$$x \mapsto A(x)$$

where

$$A : G \to GL(n, \mathbb{C})$$

is a representation of G if A is a homomorphism. Thus, A is an $n \times n$ matrix. Now, since matrix multiplication is associative, we have

$$A(x) \cdot A(y) = A(xy), \forall x, y \in G$$

which implies that

$$A(x^{-1}) = A(x)^{-1}.$$

Since

$$A(x) \cdot A(x^{-1}) = A(e) = I_n$$
$$\Rightarrow A(x) = A(x)^{-1}$$

Now,

$$A(x) = A(x \cdot e) = A(x) \cdot A(e) = A(x)$$

which implies

$$A(e) = I$$

Now, the matrix representation of this same group G would be denoted

$$\mathbf{a} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and similarly,

$$\mathbf{b} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

If we square the matrix denoted a, we have

$$\mathbf{a}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and repeat the process with b, which gives us

$$\mathbf{b}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

So, by right hand multiplication, we have

$$a^2 = b^2 = I.$$

Thus, a and b satisfy $< a, b | a^2 = b^2 = (ab)^2 = 1 >$. This process implies that

$$\left\langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\rangle$$

is a **matrix representation** of G.

Monomial Representation

A *monomial representation* of a group G is a homomorphism from G into $GL_n(F)$, the group of non singular $n \times n$ matrices over the field F, in which the image of every element of G is a monomial matrix over F. The *action* of the image of a monomial representation on the underlying vector space is to permute the vectors of a basis while multiplying them by scalars.

Every monomial representation of G in which G acts transitively on the 1-dimensional subspaces generated by the basis vectors is obtained by inducing a linear representation of a subgroup H up to G. If this linear representation is trivial, we obtain the permutation representation of G acting on the cosets of H. Otherwise we obtain a proper monomial representation.

An ordinary linear representation of H is a homomorphism of H onto C_m, where C_m is a cyclic multiplicative subgroup of the complex numbers $\mathbb{C}$, and the resulting monomial matrices will involve comples mth roots of unity. Similarly, we can define a linear representation into any field F which possesses mth roots of unity.

1.3 Methods and Applications

The Manual Double Coset Enumeration

Building on the above definitions of right coset and double coset, we define the process of *manual double coset enumeration*, which is the process of determining the decomposition of single and double cosets within a finite group G factored by some relation.

The Lifting Process

Let G be a group, let N be a normal subgroup of G and χ_0 be a character of G/N. Then we define

$$\chi(g) = \chi_0(Ng), \forall g \in G.$$

χ is a character of G lifted from the character χ_0 of G/N.
Note:

$$\chi(n) = \chi_0(Nn) = \chi_0(N)$$

An example of the lifting proces will be performed in Chapter 7.

Factoring by the Center

As we complete the double coset enumeration of a finite group factored by a relation, we determine the permutation representation of that group. The three resulting permutations $xx = f(x)$, $yy = f(y)$ and $tt_i = f(t)$ generate our group G. Factoring by the center is the process by which we find the centralizing elements (denoted Z) of our group to determine which double coset (or double cosets) represent blocks of impermiability that are at the center of the group. We then find the double coset(s) that contain a single coset farthest from our identity coset (denoted[$*$]). We then determine what our Z is by setting the coset representative of that double coset equal to the identity to find our new relation based on the equation

$$z = n \cdot w, \text{ where } w \text{ is a word in the } t_i's \text{ and } m = n^{-1}.$$

Once we have determined our new relation, we perform double coset enumeration of our group with the new relation, which will collapse the group into a smaller Cayley diagram configuration. We utilize this property to find the centraliser of our group.

Chapter 2

Construction of $2^5 : S_5$

We have a computer-based proof that G is isomorphic to $2^5 : S_5$. This proof is obtained as follows: We first us MAGMA to obtain the composition factors of a permutation representation of G. This is done as follows:

```
f,G1,k:=CosetAction(G,sub<G|x,y>);
CompositionFactors(G1);
```

gives

```
G
| Cyclic(2)
*
| Alternating(5)
*
| Cyclic(2)
*
| Cyclic(2)
*
| Cyclic(2)
*
| Cyclic(2)
*
| Cyclic(2)
1
```

We now write a presentation of the group $2^5 : S_5$ (obtained based on the composition factors above) and verify that $G \cong 2^5 : S_5$.

$$\frac{2^{*5} : S_5}{t_0 t_1 = t_1 t_0}$$

We will perform a double coset enumeration on the group $2^{*5} : S_5$ factored by the relation $t_0 t_1 = t_1 t_0$, denoted by the following group representation:

$$G = < x, y, t | x^5, y^2, t^2, (xy)^4, (x, y)^3, (t, y), (t, x^2 y x^{-1} y), (t t^x)^2 >$$

where $N = < x, y > \cong S_5$, $x \sim (01234)$ and $y \sim (01)$. We know $N \cong S_5$ has 120 elements, or $|N| = 120$.

2.1 Relations

We are given the relation $t_0 t_1 = t_1 t_0$. This relation can be used to determine equal cosets with words of length two. We take $N t_0 t_1 = N t_1 t_0$ and conjugate it by every element in our control group S_5 to get the following relations:

$$\begin{array}{lll} t_0 t_2 = t_2 t_0, & t_0 t_3 = t_3 t_0, & t_0 t_4 = t_4 t_0, \\ t_1 t_2 = t_2 t_1, & t_1 t_3 = t_3 t_1, & t_1 t_4 = t_4 t_1, \\ t_2 t_3 = t_3 t_2, & t_2 t_4 = t_4 t_2, & t_3 t_4 = t_4 t_3. \end{array}$$

To utilize our relations for words of length three, we must use right coset multiplication by the $t_i' s$ to increase the length of the relations. Then we use the above relations to manipulate the relations of length three:

$$\begin{aligned} t_0 t_1 t_2 &= t_1 \underline{t_0 t_2} \\ &= \underline{t_1 t_2} t_0 \\ &= t_2 \underline{t_1 t_0} \\ &= t_2 t_0 t_1 \end{aligned}$$

Using this method, we can determine all the relations with words of length three:

$$012 \sim 102 \sim 120 \sim 210 \sim 201 \sim 021$$
$$031 \sim 301 \sim 310 \sim 130 \sim 103 \sim 013$$

$$041 \sim 401 \sim 410 \sim 140 \sim 104 \sim 014$$
$$241 \sim 421 \sim 412 \sim 142 \sim 124 \sim 214$$
$$231 \sim 321 \sim 312 \sim 132 \sim 123 \sim 213$$
$$341 \sim 431 \sim 413 \sim 143 \sim 134 \sim 314$$

To utilize our relations for words of length greater than three we repeat the aforementioned process, adding the appropriate amount of letters as needed.

2.2 Double Coset Enumeration

NeN

NeN is a double coset made up of words of length zero. We know $NeN = \{N\}$, which is the first double coset [*]. There is one single coset within the first double coset. The coset representative for [*] is Ne. We find the orbits of N on $0, 1, 2, 3, 4$ by permuting each element by $g \in N$ as follows:

$$0^g = \{01234\}$$
$$1^g = \{01234\}$$
$$2^g = \{01234\}$$
$$3^g = \{01234\}$$
$$4^g = \{01234\}$$

Thus, we see that the orbit on N on $\{0, 1, 2, 3, 4\}$ is $\{0, 1, 2, 3, 4\}$ When we apply a representative t_i from each orbit to the coset representative Ne we see that all five of the elements in orbit $\{0, 1, 2, 3, 4\}$ extend to a new double coset Nt_0N, called [0]. This double cosets will be made up of words of length one.

$\mathbf{Nt_0N}$

We must first determine the coset stabilizer, denoted $N^{(0)}$. We look at permutations in N and find those that "fix" the the element 0. So, $N^{(0)} =< (1234), (12) >$, is the point stabiliser in N of 0. At this point in the process, our relation $t_0t_1 = t_1t_0$ is not needed, since it does not affect words of less than length two. Thus, our point stabilizer, denoted N^0 is also our coset stabilizer, denoted $N^{(0)}$. Since N is transitive on

{01234}, the 24 permutations that "fix" the element (0) represent the coset satbilizers. Thus, $|N^{(0)}| = 24$. We now will determine the number of single coset in the double coset [0] by this formula $\frac{|N|}{|N^{(0)}|}$ This gives us $\frac{120}{24} = 5$. The coset representative for [0] is Nt_0. We now identify the orbits of $N^{(0)}$ and determine their action. Since the element (0) is fixed, our two orbits are {0} and {1,2,3,4}. When we apply a representative t_i from each orbit to the coset representative Nt_0 we see that the following results:

$$Nt_0 \cdot t_0 = N(t_0)^2 \in NeN$$

So the the orbit {0} takes one generator back to the double coset [*].

$$Nt_0 \cdot t_1 = Nt_0t_1 \in Nt_0t_1N$$

So the the orbit $\{1, 2, 3, 4\}$ extends four generators to a new double coset Nt_0t_1N, denoted [01].

$\mathbf{Nt_0t_1N}$

We now have a double coset with word of length two, so our relation $t_0t_1 = t_1t_0$ must be utilized to help us accurately determine the coset stabilizer. The following equations will tell us what permutation(s) increase the coset stabilizer by taking the representative coset back to itself:

$$Nt_0t_1 = Nt_1t_0 \Rightarrow Nt_0t_1^{(01)} = Nt_1t_0 = Nt_1t_0$$

So $(01) \in N^{(01)}$. Thus, the generators of $N^{(01)}$ are: $< N^{01}, (01) >$. The six elements of the coset stabilizer $N^{(01)}$ are : $\{e, (23), (24), (34), (234), (243)\}$. The permutation (01) will double this number, so $|N^{(01)}| = 12$. We may now determine the number of single cosets in the double coset [01] by our formula:

$$\frac{|N|}{|N^{(01)}|}$$

This gives us:

$$\frac{120}{12} = 10.$$

So there are ten single cosets in the double coset [01] .

Next, we will determine the orbits of Nt_0t_1. Since $N^{01} =< (234), (23) >$, our orbits of N^{01} on $\{0, 1, 2, 3, 4\}$ are $\{0, 1\}$ and $\{2, 3, 4\}$. We take a representative coset from [01] and a representative t_i from each orbit to determine the action:

$$Nt_0t_1 \cdot t_1 = Nt_0(t_1)^2 = Nt_0 \in Nt_0N$$

So the the orbit$\{0,1\}$ takes two generators back to the double coset [0].

$$Nt_0t_1 \cdot t_2 = Nt_0t_1t_2 \in Nt_0t_1t_2N$$

So the the orbit$\{2,3,4\}$ extends three generators to a new double coset $Nt_0t_1t_2N$, denoted [012].

$\mathbf{Nt_0t_1t_2N}$

We wish to determine the single cosets in the double coset [012] and to do this, we will "fix" our next point (2). Thus, the point stabilizer will be :

$$N^{012} =< (34) >.$$

Using our relation, we can expand the point stabilizer to our desired coset stabilizer:

$$Nt_0t_1 = Nt_1t_0 \Rightarrow Nt_0t_1t_2 = Nt_1t_0t_2 = Nt_1t_0 \text{ (right hand coset multiplication of } t_2)$$
$$\Rightarrow Nt_0t_1t_2^{(01)} = Nt_1t_0t_2 = Nt_0t_1t_2 \Rightarrow \text{the permutation } (12)^{(01)} \in N^{(012)}$$

So $(01) \in N^{(012)}$. Thus, the generators of $N^{(0,1,2)}$ are: $< (34), (01)^{(12)} >$ The six elements of the coset stabilizer $N^{(0,1,2)}$ are : $\{e, (01), (02), (12), (012), (021)\}$. The permutation (34) will double this number, so $|N^{(012)}| = 12$

We may now determine the number of single cosets in the double coset [012] by our formula:

$$\frac{|N|}{|N^{(012)}|}$$

This gives us:

$$\frac{120}{12} = 10.$$

So there are ten single cosets in the double coset [012] .

Next, we will determine the orbits of $Nt_0t_1t_2$. Since the elements 0,1 and 2 are fixed, our orbits of $N^{(012)}$ on $\{0,1,2,3,4\}$ are $\{0,1,2\}$ and $\{3,4\}$. We take a representative coset from [012] and a representative t_i from each orbit to determine the action:

$$Nt_0t_1t_2 \cdot t_2 = Nt_0t_1(t_2)^2 = Nt_0t_1 \in Nt_0t_1N$$

So the the orbit$\{0, 1, 2\}$ takes three generators back to the double coset [01].

$$Nt_0t_1t_2 \cdot t_3 = Nt_0t_1t_2t_3 \in Nt_0t_1t_2t_3N$$

So the the orbit$\{3, 4\}$ extends two generators to a new double coset $Nt_0t_1t_2t_3N$, denoted [0123].

$\mathbf{Nt_0t_1t_2t_3N}$

We determine the single cosets of [0123]. We begin by "fixing" our next point (3). Thus, the point stabilizer will be :

$$N^{0123} =< e >.$$

Expanding our relations (as shown in section 2.1), we can increase the point stabilizer. We note:

$$\begin{array}{lll} 02 \sim 20, & 03 \sim 30, & 04 \sim 40, \\ 12 \sim 21, & 13 \sim 31, & 14 \sim 41, \\ 23 \sim 32, & 24 \sim 42, & 34 \sim 43. \end{array}$$

The above relations are used to determine the elements in $N^{(0123)}$. We do this by conjugating the representative coset $Nt_0t_1t_2t_3$ by generators that will take the point stabilizer back to itself:

$$\begin{aligned} Nt_0t_1t_2t_3^{(01)} &= Nt_1t_0t_2t_3 = Nt_0t_1t_2t_3 \to (01) \in N^{(0123)} . \\ Nt_0t_1t_2t_3^{(012)} &= Nt_1t_2t_0t_3 = Nt_0t_1t_2t_3 \to (0,1,2) \in N^{(0123)} \\ Nt_0t_1t_2t_3^{(0123)} &= Nt_1t_2t_3t_0 = Nt_0t_1t_2t_3 \to (0123) \in N^{(0123)} \end{aligned}$$

So $N^{(0123)} =< (01), (012), (0123) >$. Now $|N^{(0123)}| = |S_4| = 4! = 24$

We may now calculate the number of single cosets in [0123] by our formula:

$$\frac{|N|}{|N^{(0123)}|}$$

This gives us:

$$\frac{120}{24} = 5.$$

So there are five single cosets in the double coset [0123] .

Next, we will determine the orbits of $Nt_0t_1t_2t_3$. Since the elements 0,1,2 and 3 are fixed, our orbits of $N^{(0123)}$ on $\{0,1,2,3,4\}$ are $\{0,1,2,3\}$ and $\{4\}$. We take a representative coset from [0123] and a representative t_i from each orbit to determine the action:

$$Nt_0t_1t_2t_3 \cdot t_3 = Nt_0t_1t_2(t_3)^2 = Nt_0t_1t_2 \in Nt_0t_1t_2N$$

So the the orbit$\{0,1,2,3\}$ takes four generators back to the double coset [012].

$$Nt_0t_1t_2t_3 \cdot t_4 = Nt_0t_1t_2t_3t_4 \in Nt_0t_1t_2t_3t_4N$$

So the the orbit$\{4\}$ extends one generator to a new double coset $Nt_0t_1t_2t_3t_4N$, denoted [01234].

$\mathbf{Nt_0t_1t_2t_3t_4N}$

We determine the single cosets in the double coset [01234]. First note that all of the five elements are"fixed". Thus, the point stabilizer is :

$$N^{01234} =< e >.$$

As before, we must apply the appropirate permutation to the point stabilizer to send it back to itself:

$$Nt_0t_1t_2t_3t_4^{(01)} = Nt_0t_1t_2t_3t_4 \rightarrow (0,1) \in N^{(01234)}$$
$$Nt_0t_1t_2t_3t_4^{(01234)} = Nt_1t_2t_3t_4t_0 = Nt_0t_1t_2t_3t_4 \rightarrow (0,1,2,3,4) \in N^{(01234)}$$

Now $|N^{(01234)}| =< (01), (01234) >= |S_5| = 5! = 120$

We may now calculate the number of single cosets in [01234] by our formula:

$$\frac{|N|}{|N^{(01234)}|}$$

This gives us:

$$\frac{120}{120} = 1.$$

So there is one single coset in the double coset [01234] .

Next, we will determine the orbits of $Nt_0t_1t_2t_3t_4$. Since all of the elements are fixed, our single orbit of $N^{(0,1,2,3,4)}$ on $\{0,1,2,3,4\}$ is $\{0,1,2,3,4\}$. We take a representative coset from [01234] and a representative t_i from this orbit to determine the action:

$$Nt_0t_1t_2t_3t_4 \cdot t_4 = Nt_0t_1t_2t_3(t_4)^2 = Nt_0t_1t_2t_3 \in Nt_0t_1t_2t_3N$$

Thus, the the orbit$\{0, 1, 2, 3, 4\}$ takes all five generators back to the double coset [0123] Since we have no generators extending to new double cosets, our double coset enumeration is complete. All this information is summarized in the following cayley diagram Figure 2.1:

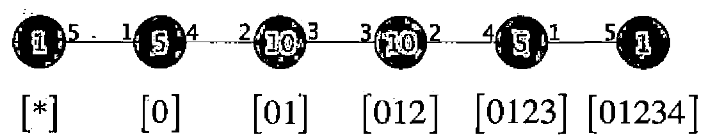

Figure 2.1: Cayley diagram for $2^5 : S_5$

In Table 2.1, we first label each single coset. We then compute the action of xx, yy, and tt_0. We will use the information in the table to determine $f(x)$, $f(y)$, and $f(t)$.

Table 2.1: Labeling of and Actions on the Single Cosets

Labeling	Single Cosets	xx		yy		tt_0	
1	N	1	N	1	N	2	Nt_0
2	Nt_0	3	Nt_1	3	Nt_1	1	N
3	Nt_1	4	Nt_2	2	Nt_0	7	Nt_0t_1
4	Nt_2	5	Nt_3	4	Nt_2	8	Nt_0t_2
5	Nt_3	6	Nt_4	5	Nt_3	9	Nt_0t_3
6	Nt_4	2	Nt_0	6	Nt_4	10	Nt_0t_4
7	Nt_0t_1	11	Nt_1t_2	7	Nt_0t_1	3	Nt_1
8	Nt_0t_2	12	Nt_1t_3	11	Nt_1t_2	4	Nt_2
9	Nt_0t_3	13	Nt_1t_4	12	Nt_1t_3	5	Nt_3
10	Nt_0t_4	7	Nt_0t_1	13	Nt_1t_4	6	Nt_4
11	Nt_1t_2	14	Nt_2t_3	8	Nt_0t_2	17	$Nt_0t_1t_2$
12	Nt_1t_3	15	Nt_2t_4	9	Nt_0t_3	18	$Nt_0t_1t_3$
13	Nt_1t_4	8	Nt_0t_2	10	Nt_0t_4	19	$Nt_0t_1t_4$
14	Nt_2t_3	16	Nt_3t_4	14	Nt_2t_3	20	$Nt_0t_2t_3$
15	Nt_2t_4	9	Nt_0t_3	15	Nt_2t_4	21	$Nt_0t_2t_4$
16	Nt_3t_4	10	Nt_0t_4	16	Nt_3t_4	22	$Nt_0t_3t_4$
17	$Nt_0t_1t_2$	23	$Nt_1t_2t_3$	17	$Nt_0t_1t_2$	11	Nt_1t_2
18	$Nt_0t_1t_3$	24	$Nt_1t_2t_4$	18	$Nt_0t_1t_3$	12	Nt_1t_3
19	$Nt_0t_1t_4$	17	$Nt_0t_1t_2$	19	$Nt_0t_1t_4$	13	Nt_1t_4
20	$Nt_0t_2t_3$	25	$Nt_1t_3t_4$	23	$Nt_1t_2t_3$	14	Nt_2t_3
21	$Nt_0t_2t_4$	18	$Nt_0t_1t_3$	24	$Nt_1t_2t_4$	15	Nt_2t_4
22	$Nt_0t_3t_4$	19	$Nt_0t_1t_4$	25	$Nt_1t_3t_4$	16	Nt_3t_4
23	$Nt_1t_2t_3$	26	$Nt_2t_3t_4$	20	$Nt_0t_2t_3$	27	$Nt_0t_1t_2t_3$
24	$Nt_1t_2t_4$	20	$Nt_0t_2t_3$	21	$Nt_0t_2t_4$	28	$Nt_0t_1t_2t_4$
25	$Nt_1t_3t_4$	21	$Nt_0t_2t_4$	22	$Nt_0t_3t_4$	29	$Nt_0t_1t_3t_4$
26	$Nt_2t_3t_4$	22	$Nt_0t_3t_4$	26	$Nt_2t_3t_4$	30	$Nt_0t_2t_3t_4$
27	$Nt_0t_1t_2t_3$	31	$Nt_1t_2t_3t_4$	27	$Nt_0t_1t_2t_3$	23	$Nt_1t_2t_3$
28	$Nt_0t_1t_2t_4$	27	$Nt_0t_1t_2t_3$	28	$Nt_0t_1t_2t_4$	24	$Nt_1t_2t_4$
29	$Nt_0t_1t_3t_4$	28	$Nt_0t_1t_2t_4$	29	$Nt_0t_1t_3t_4$	25	$Nt_1t_3t_4$
30	$Nt_0t_2t_3t_4$	29	$Nt_0t_1t_3t_4$	31	$Nt_1t_2t_3t_4$	26	$Nt_2t_3t_4$
31	$Nt_1t_2t_3t_4$	30	$Nt_0t_2t_3t_4$	30	$Nt_0t_2t_3t_4$	32	$Nt_0t_1t_2t_3t_4$
32	$Nt_0t_1t_2t_3t_4$	32	$Nt_0t_1t_2t_3t_4$	32	$Nt_0t_1t_2t_3t_4$	31	$Nt_1t_2t_3t_4$

Thus:

$$f(x) = (2,3,4,5,6)(7,11,14,16,10)(8,12,15,9,13)(17,23,26,22,19)$$
$$(18,24,20,25,21)(27,31,30,29,28)$$

$$f(y) = (2,3)(8,11)(9,12)(10,13)(20,23)(21,24)(22,25)(30,31)$$

$$f(t) = (1,2)(3,7)(4,8)(5,9)(6,10)(11,17)(12,18)$$
$$(13,19)(14,20)(15,21)(16,22)(23,27)$$

2.3 Factoring by the Center

We will find the centralizer of $2^5 : S_5$ and factor by its center. The order of our blocks of impermiability is two,since we have 2 double cosets that contain only one coset. We see that these permutatons occur on 32 letters because there exist 32 single cosets in this group. So we have our central element

$$f(t) = (1,2)(3,7)(4,8)(5,9)(6,10)(11,17)(12,18)(13,19)$$
$$(14,20)(15,21)(16,22)(23,27)(24,28)(25,29)(26,30)(31,32).$$

We examine our Cayley diagram and determine the double coset $[0,1,2,3,4]$ contains only one coset (excluding the identity coset). We then determine our centralizer Z by setting the coset representative of that double coset equal to the identity

$$Nt_0t_1t_2t_3t_4 = e$$

We let $z = n \cdot w$, where w is

$$t_0t_1t_2t_3t_4 = e$$
$$t_0t_1t_2t_3 = t_4^{-1}$$
$$t_0t_1t_2 = t_4^{-1}t_3^{-1}.$$

Since $t^2 = e$ then $t = t^{-1}$. Hence, our relation is $t_0t_1t_2 = t_4t_3$
Using our prior relation $t_0t_1 = t_1t_0$, we also have $t_0t_1t_2 = t_3t_4$
We now repeat the double coset enumeration with our new relation.

$$\frac{2^5 : S_5}{t_0t_1t_2 = t_3t_4}$$

We will perform a double coset enumeration on the group $2^5 : S_5$
factored by the relation $t_0t_1t_2 = t_3t_4$, denoted by the
following group representation:

$$G \cong < x, y, t | x^5, y^2, t^2, (xy)^4, (x, y)^3, (t, y), (t, x^2yx^{-1}y), (tt^x)^2, tt^xt^(x^2)t^(x^3)t^(x^4) >$$

where $N =< x, y >\cong S_5$, $x \sim (01234)$ and $y \sim (01)$. We know $N \cong S_5$ has 120 elements, or $|N| = 120$.

2.4 Double Coset Enumeration

NeN and Nt$_0$N

Our procees for this double coset enumeration will be repeated exactly as in the above steps for double coset NeN and Nt_0N due to the fact that our new relation will not increase the coset stabilzer $N^{(0)}$. Recall:

$$NeN = \{N\}$$
$$Nt_0N = \{Nt_0, Nt_1, Nt_2, Nt_3, Nt_4, \}$$

Nt$_0$t$_1$N

We now have a double coset with word of length two, so our relation $t_0t_1t_2 = t_3t_4$ must be utilized to help us accurately determine the coset stabilizer.

$$Nt_0t_1 = Nt_1t_0 \Rightarrow Nt_0t_1^{(01)} = Nt_1t_0 = Nt_1t_0$$

So $(01) \in N^{(01)}$. Thus, the generators of $N^{(01)}$ are $< N^{01}, (01) >$. The six elements of the coset stabilizer $N^{(01)}$ are : $\{e, (23), (24), (34), (234), (243)\}$. The permutation (01) will double this number, so $|N^{(01)}| = 12$. We may now determine the number of single cosets in the double coset [01] by our formula $\frac{|N|}{|N^{(01)}|}$. This gives us: $\frac{120}{12} = 10$.

So there are ten single cosets in the double coset [01] .

Now we determine the orbits of [01] to be $\{0, 1\}$ and $\{2, 3, 4\}$. We will take a representative t_i from each of these orbits and apply right hand multiplication to the coset Nt_0t_1:

1. $Nt_0t_1 \cdot t_0 = Nt0t_1^2 = Nt_0 \in Nt_0N$ denoted [0], so this orbit takes 2 generators back to the double coset [0].

2. $Nt_0t_1 \cdot t_2 = Nt_0t_1t_2 = Nt_4t_3 \in Nt_0t_1N$ denoted [01], so this orbit takes 3 generators back to itself (the double coset [01]).

Since we have no generators extending to new double cosets, our double coset enumeration is complete. All this information is summarized in the following cayley diagram Figure 2.2

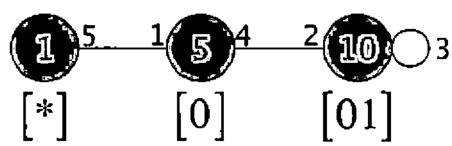

Figure 2.2: Cayley diagram for $2^5 : S_5$ Factored by $t_0t_1t_2 = t_3t_4$

Chapter 3

Construction of $3^{*3} : A_3$

$$G = \frac{3^{*3} : A_3}{t_0 t_1 = t_1 t_0}$$

We will perform a double coset enumeration of the group $3^{*3} : A_3$ factored by the relation $t_0 t_1 = t_1 t_0$, given by:

$$G \cong < x, t | x^3, t^3, tt^x = t^x t >$$

We have a computer-based proof that $G \cong C_3 \times C_3 \times C_3 \times C_3$.

where $N =< x, t >\cong A_3$, $x \backsim (0, 1, 2)(\bar{0}\bar{1},\bar{2})$. We know $N \cong A_3$ has 3 elements , or $|N| = 3$.

3.1 Relations

Since this group has three generators, we let $t = t_3$. Our given relation is $tt^x = t^x t$. We can substitute the values for $t \sim t_3$ and $x \sim (123)(\bar{1}\bar{2}\bar{3})$ and obtain $t_3 t_3^{(123)(\bar{1}\bar{2}\bar{3})} = t_3^{(123)(\bar{1}\bar{2}\bar{3})} t_3$, thus:

$$t_3 t_1 = t_3 t_1.$$

We prefer to to write $t_3 = t_0$. Thus, our relation is:

$$t_0 t_1 = t_0 t_1.$$

We have the three generators t_0, t_1, t_2 and their inverses, denoted by $\bar{t}_0, \bar{t}_1, \bar{t}_2$, respectively.

3.2 Double Coset Enumeration

NeN

NeN is a double coset made up of words of length zero. We know $NeN = \{N\}$, which is the first double coset [*]. The coset representative for [*] is N. We find that the orbits of N on $\{0,1,2,\bar{0},\bar{1},\bar{2}\}$ are $\{0,1,2\}$ and $\{\bar{0},\bar{1},\bar{2}\}$. When we apply a representative t_i from each orbit to the double coset representative N we see that the elements in orbit $\{0,1,2\}$ extend to a new double coset Nt_0N, denoted [0], and the elements in the orbit $\{\bar{0},\bar{1},\bar{2}\}$ extend to another new double coset $N\bar{t}_0N$, denoted $[\bar{0}]$. These double cosets will be made up of words of length one. Unlike $2^5 : S_5$ in the previous chapter, this Cayley diagram splits from [*] and extends to two new double cosets denoted [0] and $[\bar{0}]$ as shown in Figure 3.1 below:

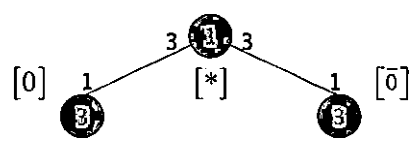

Figure 3.1: Partial Cayley diagram of G over A_3

Nt$_0$N

We now will determine the number of single coset in the double coset [0] by this formula $\frac{|N|}{|N^{(0)}|}$ which gives us $\frac{3}{1} = 3$. The coset representative for [0] is Nt_0. We now identify the orbits of $N^{(0)}$ and determine where they go. We see that the orbits of N on $\{0,1,2,\bar{0},\bar{1},\bar{2}\}$ are $\{0\}$, $\{1\}$, $\{2\}$,$\{\bar{0}\}$,$\{\bar{1}\}$, $\{\bar{2}\}$. When we apply a representative t_i from each orbit to the coset representative Nt_0 we see the following results:

1. $Nt_0t_0 = N\bar{t}_0$ which means this orbit extends Nt_0 to a new double coset $N\bar{t}_0N$, denoted $[\bar{0}]$.

2. $Nt_0t_1 = Nt_0t_1$ which means this orbit extends Nt_0 to a new double coset Nt_0t_1N, denoted [01].

3. Nt_0t_2, which means this orbit extends Nt_0 to a new double coset Nt_0t_2N, denoted [02]. Our relation tells us the double coset [02] is equivalent to [01]. Hence, this orbit takes Nt_0 to the double coset [01]

4. $Nt_0\bar{t}_0 = Ne = N$, which takes this coset back to the double coset [*].

5. $Nt_0\bar{t}_1$, which means this orbit extends Nt_0 to a new double coset $Nt_0\bar{t}_1N$, denoted $[0\bar{1}]$.

6. $Nt_0\bar{t}_2$, which means this orbit extends Nt_0 to a new double coset $Nt_0\bar{t}_2N$, denoted $[0\bar{2}]$.

$\mathbf{N\bar{t}_0N}$

We now will determine the number of single coset in the double coset $[\bar{0}]$ by this formula $\frac{|N|}{|N^{(\bar{0})}|}$ which gives us $\frac{3}{1} = 3$. $N^{(\bar{0})}$ is the stabiliser of the coset $N\bar{t}_0$. We now identify the orbits of $N^{(\bar{0})}$ and determine where they go. We see that the orbits of $N^{(\bar{0})}$ on $\{0, 1, 2, \bar{0}, \bar{1}, \bar{2}\}$ are $\{0\}$, $\{1\}$, $\{2\}$,$\{\bar{0}\}$,$\{\bar{1}\}$, $\{\bar{2}\}$. When we apply a representative t_i from each orbit to the coset representative $N\bar{t}_0$ we see that the following results:

1. $N\bar{t}_0t_0 = Ne$=N which means this orbit takes $N\bar{t}_0$ back to the double coset [*].

2. $N\bar{t}_0\boldsymbol{t_1}$. Our relation tells us the double coset $[\bar{0}1]$ is equivalent to $[0\bar{2}]$. Hence, this orbit takes $N\bar{t}_0$ to the double coset $[0\bar{2}]$.

3. $N\bar{t}_0t_2$. Our relation tells us the double coset $[\bar{0}2]$ is equivalent to $[0\bar{1}]$. Hence, this orbit takes $N\bar{t}_0$ to the double coset $[0\bar{1}]$

4. $N\bar{t}_0\bar{t}_0 = Nt_0$, which takes this coset back to the double coset [0].

5. $N\bar{t}_0\bar{t}_1$, which means this orbit extends $N\bar{t}_0$ to a new double coset $N\bar{t}_0\bar{t}_1N$, denoted $[\bar{0}\bar{1}]$.

6. $N\bar{t}_0\bar{t}_2$, which means this orbit extends $N\bar{t}_0$ to a new double coset $N\bar{t}_0\bar{t}_2N$, denoted $[\bar{0}\bar{2}]$.

From this point on in our process, we will be dealing with words of length two or more, so we must utilize our relation $t_0t_1 = t_1t_0$ to find our remaining cosets. In addition, we will use our relation to determine which double and single cosets (if any) exist in other single or double cosets. First, we must calculate all our relations. Conjugation by elements of A_3 gives rise to the following relations:

$$01 \sim 10,\ 20 \sim 02,\ 12 \sim 21, 0\bar{1} \sim \bar{1}0, 1\bar{2} \sim \bar{2}1, 2\bar{0} \sim \bar{0}2, \bar{1}\bar{2} \sim \bar{2}\bar{1}, \bar{2}\bar{0} \sim \bar{0}\bar{2}, \bar{0}\bar{1} \sim \bar{1}\bar{0}$$

Furthermore, we use the above relations to seek out which double cosets are actually elements of other double cosets. We call these "equal cosets". We will show that many double cosets in this group are equal. MAGMA confirms what we will show by hand:

Remembering our notation;

3=0, 1 = 1, 2 = 2, 6 =$\bar{0}$, 4 =$\bar{1}$, 5 = $\bar{2}$.

We prove that [02] = [01];

Note: $A_3 =< (012), (\bar{0}\bar{1}\bar{2}), (021)(\bar{0}\bar{2}\bar{1}), e >$.

pf:

[01] = [02]

$Nt_0t_1N= \{N(t_0t_1)^n \in N\} = \{Nt_0t_1, Nt_1t_2, Nt_2t_0\}$

But our relation tells us that $Nt_2t_0 = Nt_0t_2 \in Nt_0t_2N$

So $Nt_0t_1N = Nt_0t_2N$.

[$\bar{0}$1] = [0$\bar{2}$]

$N\bar{t}_0t_1N = Nt_0\bar{t}_2N$

$N\bar{t}_0t_1N= \{N(\bar{t}_0t_1)^n \in N\} = \{N\bar{t}_0t_1, N\bar{t}_2t_0, Nt_{\bar{1}}t_2\}$

But our relation tells us that $N\bar{t}_2t_0 = Nt_0\bar{t}_2 \in Nt_0\bar{t}_1N$

So $N\bar{t}_0t_1N = Nt_0\bar{t}_2N$.

Similarly, we prove that the remaining ten double coset equalities listed below:

[02] = [01], [$\bar{0}$1] = [$\bar{0}$2], [$\bar{0}$2] = [0$\bar{1}$], [0$\bar{2}$] = [$\bar{0}\bar{1}$],

[0$\bar{1}\bar{2}$] = [$\bar{0}\bar{1}$2], [$\bar{0}$12] = [0$\bar{1}$2], [01$\bar{2}$] = [0$\bar{1}$2], [$\bar{0}$1$\bar{2}$] = [$\bar{0}\bar{1}$2]

[01$\bar{2}$1] = [02], [0$\bar{1}$2$\bar{1}$] = [012], [0$\bar{1}$2$\bar{0}$] = [0$\bar{2}$], [0$\bar{1}$20] = [$\bar{0}\bar{1}$2].

We will now use these relations to help us find our remaining cosets.

$\mathbf{Nt_0t_1N}$

We now will determine the number of single coset in the double coset [01] by this formula $\frac{|N|}{|N^{(01)}|}$ which gives us $\frac{3}{1} = 3$. The coset representative for [01] is Nt_0t_1. We now identify the orbits of $N^{(01)}$ and determine where they go. We see that the orbits of $N^{(01)}$ on $\{0, 1, 2, \bar{0}, \bar{1}, \bar{2}\}$ are {0}, {1}, {2},{$\bar{0}$},{$\bar{1}$}, {$\bar{2}$}. When we apply a representative t_i from each orbit to the coset representative Nt_0 we see the following results:

1. $Nt_0t_1t_0 = Nt_0t_0t_1 = N\bar{t}_0t_1$ which means this orbit extends Nt_0t_1 to a new double coset $N\bar{t}_0t_1N$, denoted $[\bar{0}1]$.
   We have proven that $[\bar{0}1] = [0\bar{2}]$, so this orbit extends Nt_0t_1 to $[0\bar{2}]$.

2. $Nt_0t_1t_1 = Nt_0\bar{t}_1$ which means this orbit extends Nt_0t_1 to a new double coset $Nt_0\bar{t}_1N$, denoted $[0\bar{1}]$.

3. $Nt_0t_1t_2$ which means this orbit extends Nt_0t_1 to a new double coset $Nt_0t_1t_2N$, denoted $[012]$.

4. $Nt_0t_1\bar{t}_0 = Nt_1, \in [0]$.

5. $Nt_0t_1\bar{t_1} = Nt_0$, which means this orbit extends Nt_0t_1 back to the double coset Nt_0N, denoted $[0]$.

6. $Nt_0t_1\bar{t_2}$, which means this orbit extends Nt_0t_1 to a new double coset $Nt_0t_1t_{\bar{2}}N$, denoted $[01\bar{2}]$.
   We have proven that $[01\bar{2}] = [0\bar{1}2]$, so this orbit extends Nt_0t_1 to $[0\bar{1}2]$.

$\mathbf{Nt_0t_2N}$

We now will determine the number of single coset in the double coset $[02]$ by this formula $\frac{|N|}{|N^{(02)}|}$ which gives $\frac{3}{1} = 3$. The coset representative for $[02]$ is Nt_0t_2. We now identify the orbits of $N^{(02)}$ and determine where they go. We see that the orbits of $N^{(02)}$ on $\{0, 1, 2, \bar{0}, \bar{1}, \bar{2}\}$ are $\{0\}$, $\{1\}$, $\{2\}$,$\{\bar{0}\}$,$\{\bar{1}\}$, $\{\bar{2}\}$. When we apply a representative t_i from each orbit to the coset representative Nt_0t_2 we see the following results:

1. $Nt_0t_2t_0 = N\bar{t}_0t_2$ which means this orbit extends Nt_0t_2 to a new double coset $N\bar{t}_0t_2N$, denoted $[\bar{0}2]$.

2. $Nt_0t_2t_1 = Nt_0t_1t_2$ which means this orbit sends Nt_0t_2 to the double coset $Nt_0t_1t_2N$, denoted $[012]$.

3. $Nt_0t_2t_2 = Nt_0\bar{t}_2$, which means this orbit extends Nt_0t_2 to a new double coset $Nt_0\bar{t}_2N$, denoted $[0\bar{2}]$.

4. $Nt_0t_2\bar{t}_0 = Nt_0\bar{t}_0t_2 = Nt_2, \in [0]$.

5. $Nt_0t_2\bar{t}_1 = Nt_0\bar{t}_1t_2$, which means this orbit extends Nt_0t_2 to a new double coset $Nt_0\bar{t}_1t_2N$, denoted $[0\bar{1}2]$.

6. $Nt_0t_2\bar{t}_2 = Nt_0$, which means this orbit sends Nt_0t_2 back to the double coset $[0]$.

$\mathbf{Nt_0\bar{t}_1N}$

We now will determine the number of single coset in the double coset $[0\bar{1}]$ by this formula $\frac{|N|}{|N^{(0\bar{1})}|}$ which gives us $\frac{3}{1} = 3$. The coset representative for $[0\bar{1}]$ is $Nt_0\bar{t}_1$. We now identify the orbits of $N^{(0\bar{1})}$ and determine where they go. We see that the orbits of $N^{(0\bar{1})}$ on $\{0, 1, 2, \bar{0}, \bar{1}, \bar{2}\}$ are $\{0\}$, $\{1\}$, $\{2\}$,$\{\bar{0}\}$,$\{\bar{1}\}$, $\{\bar{2}\}$. When we apply a representative t_i from each orbit to the coset representative $Nt_0\bar{t}_1$ we see the following results:

1. $Nt_0\bar{t}_1t_0 = N\bar{t}_0\bar{t}_1$ which means this orbit extends Nt_0 to a new double coset $N\bar{t}_0N$, denoted $[\bar{0}]$.

2. $Nt_0\bar{t}_1t_1 = Nt_0$ which means this orbit takes $Nt_0\bar{t}_1$ back to the double coset $[0]$.

3. $Nt_0\bar{t}_1t_2$, which means this orbit extends $Nt_0\bar{t}_1$ to a new double coset $Nt_0\bar{t}_1t_2N$, denoted $[0\bar{1}2]$.

4. $Nt_0\bar{t}_1\bar{t}_0 = N\bar{t}_1$, which takes this coset back to the double coset $[\bar{0}]$.

5. $Nt_0\bar{t}_1\bar{t}_1 = Nt_0t_1$, which means this orbit takes $Nt_0\bar{t}_1$ back to a the double coset $[01]$.

6. $Nt_0\bar{t}_1\bar{t}_2$, which extends $Nt_0\bar{t}_1$ to a new double coset $[0\bar{1}\bar{2}] = [\bar{0}\bar{1}2]$.

$\mathbf{Nt_0\bar{t}_2N}$

We now will determine the number of single coset in the double coset $[0\bar{2}]$ by this formula $\frac{|N|}{|N^{(0\bar{2})}|}$ which gives us $\frac{3}{1} = 3$. The coset representative for $[0\bar{2}]$ is $Nt_0\bar{t}_2$. We now identify the orbits of $N^{(0\bar{2})}$ and determine where they go. We see that the orbits of $N^{(0\bar{2})}$ on $\{0, 1, 2, \bar{0}, \bar{1}, \bar{2}\}$ are $\{0\}$, $\{1\}$, $\{2\}$,$\{\bar{0}\}$,$\{\bar{1}\}$, $\{\bar{2}\}$. When we apply a representative t_i from each orbit to the coset representative $Nt_0\bar{t}_2$ we see that the following results:

1. $Nt_0\bar{t}_2t_0 = N\bar{t}_0\bar{t}_2$ which means this orbit extends $Nt_0\bar{t}_2$ to a new double coset $N\bar{t}_0\bar{t}_2N$, denoted $[\bar{0}\bar{2}]$.

2. $Nt_0\bar{t}_2t_1$ $=Nt_0t_1\bar{t}_2$ which means this orbit extends $Nt_0\bar{t}_2$ to a new double coset $Nt_0t_1\bar{t}_2N$, denoted $[01\bar{2}] \in [0\bar{1}2]$.

3. $Nt_0\bar{t}_2t_2 = Nt_0$, which means this orbit sends $Nt_0\bar{t}_2$ back to [0].

4. $Nt_0\bar{t}_2\bar{t}_0 = N\bar{t}_2$, which takes this coset back to the double coset $[\bar{0}]$.

5. $Nt_0\bar{t}_2\bar{t}_1 = Nt_0\bar{t}_1\bar{t}_2$, which means this orbit takes $Nt_0\bar{t}_2$ to a new double coset $[0\bar{1}\bar{2}]$.

6. $Nt_0\bar{t}_2\bar{t}_2 = Nt_0t_2$, which means this orbit extends $Nt_0\bar{t}_2$ to the double coset [02].

$\mathbf{N\bar{t}_0t_1N}$

We now will determine the number of single coset in the double coset $[\bar{0}1]$ by this formula $\frac{|N|}{|N^{(\bar{0}1)}|}$ which gives us $\frac{3}{1} = 3$. The coset representative for $[\bar{0}1]$ is $N\bar{t}_0t_1$. We now identify the orbits of $N^{(\bar{0}1)}$ and determine where they go. We see that the orbits of $N^{(\bar{0}1)}$ on $\{0,1,2,\bar{0},\bar{1},\bar{2}\}$ are $\{0\}$, $\{1\}$, $\{2\}$,$\{\bar{0}\}$,$\{\bar{1}\}$, $\{\bar{2}\}$. When we apply a representative t_i from each orbit to the coset representative $N\bar{t}_0t_1$ we see the following results:

1. $N\bar{t}_0t_1t_0 = Nt_1$ which means this orbit takes $N\bar{t}_0t_1$ back to the double coset [0].

2. $N\bar{t}_0t_1t_1 = N\bar{t}_0\bar{t}_1$, which means this orbit takes $N\bar{t}_0t_1$ to the double coset $[\bar{0}\bar{1}]$.

3. $N\bar{t}_0t_1t_2$, which means this orbit extends $N\bar{t}_0t_1$ to a new double coset $N\bar{t}_0t_1t_2N$, denoted $[\bar{0}12] = [0\bar{1}2]$.

4. $N\bar{t}_0t_1\bar{t}_0 = Nt_0t_1$, which takes this coset back to the double coset [01].

5. $N\bar{t}_0t_1\bar{t}_1 = N\bar{t}_0$, which means this orbit takes $N\bar{t}_0t_1$ back to the double coset $[\bar{0}]$.

6. $N\bar{t}_0t_1\bar{t}_2$, which means this orbit extends $N\bar{t}_0t_1$ to a new double coset $N\bar{t}_0t_1\bar{t}_2N$, denoted $[\bar{0}1\bar{2}] = [\bar{0}\bar{1}2]$.

$\mathbf{N\bar{t}_0t_2N}$

We now will determine the number of single coset in the double coset $[\bar{0}2]$ by this formula $\frac{|N|}{|N^{(\bar{0}2)}|}$ which gives us $\frac{3}{1} = 3$. The coset representative for $[\bar{0}2]$ is $Nt_{\bar{0}}t_2$. We now identify the orbits of $N^{(\bar{0}2)}$ and determine where they go. We see that the orbits of $N^{(\bar{0}2)}$ on $\{0,1,2,\bar{0},\bar{1},\bar{2}\}$ are $\{0\},\{1\},\{2\},\{\bar{0}\},\{\bar{1}\},\{\bar{2}\}$. When we apply a representative t_i from each orbit to the coset representative $N\bar{t}_0t_2$ we see the following results:

1. $N\bar{t}_0t_2t_0 = Nt_2$ which means this orbit takes $N\bar{t}_0t_2$ back to the double coset [0].

2. $N\bar{t}_0t_2t_1 = N\bar{t}_0t_1t_2$, which means this orbit extends $N\bar{t}_0t_2$ to the double coset $[\bar{0}12] = [0\bar{1}2]$.

3. $N\bar{t}_0t_2t_2 = N\bar{t}_0\bar{t}_2$, which means this orbit takes $N\bar{t}_0t_2$ to the double coset $[\bar{0}\bar{2}]$.

4. $N\bar{t}_0t_2\bar{t}_0 = Nt_0t_2$, which takes this coset back to the double coset [02].

5. $N\bar{t}_0t_2\bar{t}_1 = N\bar{t}_0\bar{t}_1t_2$, which means this orbit extends $N\bar{t}_0t_2$ to a new double coset $[\bar{0}\bar{1}2]$.

6. $N\bar{t}_0t_2\bar{t}_2 = N\bar{t}_0$, which means this orbit takes $N\bar{t}_0t_2$ back to the double coset $[\bar{0}]$.

$\mathbf{N\bar{t}_0\bar{t}_1N}$

We now will determine the number of single coset in the double coset $[\bar{0}\bar{1}]$ by this formula $\frac{|N|}{|N^{(\bar{0}\bar{1})}|}$ which gives us $\frac{3}{1} = 3$. The coset representative for $[\bar{0}\bar{1}]$ is $N\bar{t}_0\bar{t}_1$. We now identify the orbits of $N^{(\bar{0}\bar{1})}$ and determine where they go. We see that the orbits of $N^{(\bar{0}\bar{1})}$ on $\{0,1,2,\bar{0},\bar{1},\bar{2}\}$ are $\{0\},\{1\},\{2\},\{\bar{0}\},\{\bar{1}\},\{\bar{2}\}$. When we apply a representative t_i from each orbit to the coset representative $N\bar{t}_0\bar{t}_1$ we see the following results:

1. $N\bar{t}_0\bar{t}_1t_0 = N\bar{t}_1$ which means this orbit takes $N\bar{t}_0\bar{t}_1$ back to the double coset $[\bar{0}]$.

2. $N\bar{t}_0\bar{t}_1t_1 = N\bar{t}_0$, which means this orbit takes $N\bar{t}_0\bar{t}_1$ to the double coset $[\bar{0}]$.

3. $N\bar{t}_0\bar{t}_1t_2$, which means this orbit extends $N\bar{t}_0\bar{t}_1$ to a new double coset $N\bar{t}_0\bar{t}_1t_2N$, denoted $[\bar{0}\bar{1}2]$.

4. $N\bar{t}_0\bar{t}_1\bar{t}_0 = Nt_0\bar{t}_1$, which takes this coset back to the double coset $[0\bar{1}]$.

5. $N\bar{t}_0\bar{t}_1\bar{t}_1 = N\bar{t}_0t_1$, which means this orbit extends $N\bar{t}_0\bar{t}_1$ to a new double coset $N\bar{t}_0t_1N$, denoted $[\bar{0}1] = [0\bar{2}]$.

6. $N\bar{t}_0\bar{t}_1\bar{t}_2$, which means this orbit extends $N\bar{t}_0\bar{t}_1$ to a new double coset $N\bar{t}_0\bar{t}_1\bar{t}_2N$, denoted $[\bar{0}\bar{1}\bar{2}]$.

$\mathbf{N\bar{t}_0\bar{t}_2N}$

We now will determine the number of single coset in the double coset $[\bar{0}\bar{2}]$ by this formula $\frac{|N|}{|N^{(\bar{0}\bar{2})}|}$ which gives us $\frac{3}{1} = 3$. The coset representative for $[\bar{0}\bar{2}]$ is $N\bar{t}_0\bar{t}_2$. We now identify the orbits of $N^{(\bar{0}\bar{2})}$ and determine where they go. We find that the orbits of $N^{(\bar{0}\bar{2})}$ on $\{0, 1, 2, \bar{0}, \bar{1}, \bar{2}\}$ are $\{0\}, \{1\}, \{2\}, \{\bar{0}\}, \{\bar{1}\}, \{\bar{2}\}$. When we apply a representative t_i from each orbit to the coset representative $N\bar{t}_0\bar{t}_2$ we see the following results:

1. $N\bar{t}_0\bar{t}_2t_0 = N\bar{t}_2$ which means this orbit takes $N\bar{t}_0\bar{t}_2$ back to the double coset $[\bar{0}]$.

2. $N\bar{t}_0\bar{t}_2t_1 = N\bar{t}_0t_1\bar{t}_2$, which means this orbit extends $Nt\bar{t}_0\bar{t}_2$ to a new double coset $N\bar{t}_0t_1\bar{t}_2N$, denoted $[\bar{0}1\bar{2}]$.

3. $N\bar{t}_0\bar{t}_2t_2 = N\bar{t}_0$, which means this orbit takes $N\bar{t}_0\bar{t}_2$ to the double coset $[\bar{0}]$.

4. $N\bar{t}_0\bar{t}_2\bar{t}_0 = Nt_0\bar{t}_2$, which means this orbit extends $N\bar{t}_0\bar{t}_2$ to a new double coset $Nt_0\bar{t}_2N$, denoted $[0\bar{2}]$.

5. $N\bar{t}_0\bar{t}_2\bar{t}_1 = N\bar{t}_0\bar{t}_1\bar{t}_2$, which means this orbit extends $N\bar{t}_0\bar{t}_2$ to a new double coset $N\bar{t}_0\bar{t}_1\bar{t}_2N$, denoted $[\bar{0}\bar{1}\bar{2}]$.

6. $N\bar{t}_0\bar{t}_2\bar{t}_2 = N\bar{t}_0t_2$, which means this orbit extends $N\bar{t}_0\bar{t}_2$ to a new double coset $N\bar{t}_0t_2N$, denoted $[\bar{0}2] = [0\bar{1}]$.

$\mathbf{Nt_0t_1t_2N}$

We now will determine the number of single coset in the double coset $[012]$ by this formula $\frac{|N|}{|N^{(012)}|}$ which gives us $\frac{3}{3} = 1$. The coset representative for $[012]$ is $Nt_0t_1t_2$. We now identify the orbits of $N^{(012)}$ and determine where they go. We find that the orbits of $N^{(012)}$ on $\{0, 1, 2, \bar{0}, \bar{1}, \bar{2}\}$ are $\{0\}, \{1\}, \{2\}, \{\bar{0}\}, \{\bar{1}\}, \{\bar{2}\}$.

When we apply a representative t_i from each orbit to the coset representative $Nt_0t_1t_2$ we see the following results:

1. $Nt_0t_1t_2t_0 = N\bar{t}_0t_1t_2$ which means this orbit takes $Nt_0t_1t_2$ to the double coset $[\bar{0}12] = [0\bar{1}2]$.

2. $Nt_0t_1t_2t_1 = Nt_0\bar{t}_1t_2$ which means this orbit extends $Nt_0t_1t_2$ to the double coset $[0\bar{1}2]$.

3. $Nt_0t_1t_2t_2 = Nt_0t_1\bar{t}_2$ which means this orbit extends $Nt_0t_1t_2$ to a new double coset $Nt_0t_1\bar{t}_2N$, denoted $[012] = [0\bar{1}2]$.

4. $Nt_0t_1t_2\bar{t}_0 = Nt_1t_2, \in [01]$.

5. $Nt_0t_1t_2\bar{t}_1 = Nt_0t_2$, which means this orbit takes $Nt_0t_1t_2$ back to the double coset $[02] = [01]$.

6. $Nt_0t_1t_2\bar{t}_2 = Nt_0t_1$, which means this orbit takes $Nt_0t_1t_2$ back to the double coset $[01]$.

$\mathbf{Nt_0t_1\bar{t}_2N}$

We now will determine the number of single coset in the double coset $[01\bar{2}]$ by this formula $\frac{|N|}{|N^{(01\bar{2})}|}$ which gives us $\frac{3}{3} = 3$. The coset representative for $[01\bar{2}]$ is $Nt_0t_1\bar{t}_2$. We now identify the orbits of $N^{(01\bar{2})}$ and determine where they go. We see that the orbits of $N^{(01\bar{2})}$ on $\{0, 1, 2, \bar{0}, \bar{1}, \bar{2}\}$ are $\{0\}, \{1\}, \{2\}, \{\bar{0}\}, \{\bar{1}\}, \{\bar{2}\}$. When we apply a representative t_i from each orbit to the coset representative $Nt_0t_1\bar{t}_2$ we see the following results:

1. $Nt_0t_1\bar{t}_2t_0 = N\bar{t}_0t_1\bar{t}_2$ which means this orbit takes $Nt_0t_1\bar{t}_2$ to the double coset $[\bar{0}1\bar{2}] = [\bar{0}\bar{1}2]$.

2. $Nt_0t_1\bar{t}_2t_1 = Nt_0\bar{t}_1\bar{t}_2$ which means this orbit takes $Nt_0t_1\bar{t}_2$ to the double coset $[0\bar{1}\bar{2}] = [\bar{0}\bar{1}2]$.

3. $Nt_0t_1\bar{t}_2t_2 = Nt_0t_1$ which means this orbit takes $Nt_0t_1\bar{t}_2$ back to the double coset $[01]$.

4. $Nt_0t_1\bar{t}_2\bar{t}_0 = Nt_1\bar{t}_2, \in [0\bar{1}]$

5. $Nt_0t_1\bar{t}_2\bar{t}_1 = Nt_0\bar{t}_2$, which means this orbit takes $Nt_0t_1\bar{t}_2$ back to the double coset $[0\bar{2}]$.

6. $Nt_0t_1\bar{t}_2\bar{t}_2 = Nt_0t_1t_2$, which means this orbit takes $Nt_0t_1\bar{t}_2$ back to the double coset $[012]$.

$\mathbf{Nt_0\bar{t}_1t_2N}$

We now will determine the number of single coset in the double coset $[0\bar{1}2]$ by this formula $\frac{|N|}{|N^{(0\bar{1}2)}|}$ which gives us $\frac{3}{1} = 3$. The coset representative for $[0\bar{1}2]$ is $Nt_0\bar{t}_1t_2$. We now identify the orbits of $N^{(0\bar{1}2)}$ and determine where they go. We see that the orbits of $N^{(0\bar{1}2)}$ on $\{0, 1, 2, \bar{0}, \bar{1}, \bar{2}\}$ are $\{0\}, \{1\}, \{2\}, \{\bar{0}\}, \{\bar{1}\}, \{\bar{2}\}$. When we apply a representative t_i from each orbit to the coset representative $Nt_0\bar{t}_1t_2$ we see that the following results:

1. $Nt_0\bar{t}_1t_2t_0 = N\bar{t}_0\bar{t}_1t_2$ which means this orbit takes $Nt_0\bar{t}_1t_2$ to the double coset $[\bar{0}\bar{1}2]$.

2. $Nt_0\bar{t}_1t_2t_1 = Nt_0t_2$ which means this orbit takes $Nt_0\bar{t}_1t_2$ to the double coset $[02] = [01]$.

3. $Nt_0\bar{t}_1t_2t_2 = Nt_0\bar{t}_1\bar{t}_2$ which means this orbit takes $Nt_0\bar{t}_1t_2$ to the double coset $[0\bar{1}\bar{2}] = [\bar{0}\bar{1}2]$.

4. $Nt_0\bar{t}_1t_2\bar{t}_0 = N\bar{t}_0\bar{t}_1t_2, \in [\bar{0}\bar{1}2]$.

5. $Nt_0\bar{t}_1t_2\bar{t}_1 = Nt_0t_1t_2$, which means this orbit takes $Nt_0\bar{t}_1t_2$ back to the double coset $[012]$.

6. $Nt_0\bar{t}_1t_2\bar{t}_2 = Nt_0\bar{t}_1$, which means this orbit takes $Nt_0\bar{t}_1t_2$ back to the double coset $[0\bar{1}]$.

$\mathbf{N\bar{t}_0\bar{t}_1t_2N}$

We now will determine the number of single coset in the double coset $[\bar{0}\bar{1}2]$ by this formula $\frac{|N|}{|N^{(\bar{0}\bar{1}2)}|}$ which gives us $\frac{3}{1} = 3$. The coset representative for $[\bar{0}\bar{1}2]$ is $N\bar{t}_0\bar{t}_1t_2$. We now identify the orbits of $N^{(\bar{0}\bar{1}2)}$ and determine where they go. We see that the orbits of $[\bar{0}\bar{1}2]$ on $\{0, 1, 2, \bar{0}, \bar{1}, \bar{2}\}$ are $\{0\}, \{1\}, \{2\}, \{\bar{0}\}, \{\bar{1}\}, \{\bar{2}\}$. When we apply a representative t_i from each orbit to the coset representative $N\bar{t}_0\bar{t}_1t_2$ we see the following results:

1. $N\bar{t}_0\bar{t}_1t_2t_0 = N\bar{t}_1t_2$ which means this orbit takes $N\bar{t}_0\bar{t}_1t_2$ back to the double coset $[0\bar{1}]$.

2. $N\bar{t}_0\bar{t}_1t_2t_1 = N\bar{t}_0$, which means this orbit takes $N\bar{t}_0\bar{t}_1t_2$ to the double coset $[\bar{0}]$.

3. $N\bar{t}_0\bar{t}_1t_2t_2$, which means this orbit extends $N\bar{t}_0\bar{t}_1t_2$ to a new double coset $N\bar{t}_0\bar{t}_1\bar{t}_2N$, denoted $[\bar{0}\bar{1}\bar{2}]$.

4. $N\bar{t}_0\bar{t}_1t_2\bar{t}_0 = Nt_0\bar{t}_1$, which takes $N\bar{t}_0\bar{t}_1t_2$ back to the double coset $[0\bar{1}]$.

5. $N\bar{t}_0\bar{t}_1t_2\bar{t}_1 = N\bar{t}_0t_1t_2$, which means this orbit takes $N\bar{t}_0\bar{t}_1t_2$ to the double coset $[\bar{0}12] = [0\bar{1}2]$.

6. $N\bar{t}_0\bar{t}_1t_2\bar{t}_2 = N\bar{t}_0\bar{t}_1$, which means this orbit takes $N\bar{t}_0\bar{t}_1t_2$ back to double coset $[\bar{0}\bar{1}]$.

$\mathbf{N\bar{t}_0\bar{t}_1\bar{t}_2N}$

We now will determine the number of single coset in the double coset $[\bar{0}\bar{1}\bar{2}]$ by this formula $\frac{|N|}{|N^{(\bar{0}\bar{1}\bar{2})}|}$ which gives us $\frac{3}{3} = 1$. The coset representative for $[\bar{0}\bar{1}\bar{2}]$ is $N\bar{t}_0\bar{t}_1\bar{t}_2$. We now identify the orbits of $N^{(\bar{0}\bar{1}\bar{2})}$ and determine where they go. We find that the orbits of $N^{(\bar{0}\bar{1}\bar{2})}$ on $\{0, 1, 2, \bar{0}, \bar{1}, \bar{2}\}$ are $\{0\}, \{1\}, \{2\}, \{\bar{0}\}, \{\bar{1}\}, \{\bar{2}\}$. When we apply a representative t_i from each orbit to the coset representative $N\bar{t}_0\bar{t}_1\bar{t}_2$ we see the following results:

1. $N\bar{t}_0\bar{t}_1\bar{t}_2t_0 = N\bar{t}_1\bar{t}_2$ which means this orbit takes $N\bar{t}_0\bar{t}_1\bar{t}_2$ back to the double coset $[\bar{0}\bar{1}]$.

2. $N\bar{t}_0\bar{t}_1\bar{t}_2t_1 = N\bar{t}_0\bar{t}_2$, which means this orbit takes $N\bar{t}_0\bar{t}_1\bar{t}_2$ to the double coset $[\bar{0}\bar{2}] = [\bar{0}\bar{1}]$.

3. $N\bar{t}_0\bar{t}_1\bar{t}_2t_2 = N\bar{t}_0\bar{t}_1$, which means this orbit takes $N\bar{t}_0\bar{t}_1\bar{t}_2$ back to the double coset $[\bar{0}\bar{1}]$.

4. $N\bar{t}_0\bar{t}_1\bar{t}_2\bar{t}_0 = Nt_0\bar{t}_1\bar{t}_2$, which takes $N\bar{t}_0\bar{t}_1\bar{t}_2$ back to the double coset $[0\bar{1}\bar{2}] = [\bar{0}\bar{1}2]$.

5. $N\bar{t}_0\bar{t}_1\bar{t}_2\bar{t}_1 = N\bar{t}_0t_1\bar{t}_2$, which means this orbit takes $N\bar{t}_0\bar{t}_1\bar{t}_2$ to the double coset $[\bar{0}1\bar{2}] = [\bar{0}\bar{1}2]$.

6. $N\bar{t}_0\bar{t}_1\bar{t}_2\bar{t}_2 = N\bar{t}_0\bar{t}_1t_2$, which means this orbit takes $N\bar{t}_0\bar{t}_1\bar{t}_2$ back to double coset $[\bar{0}\bar{1}2]$.

As we continue this process, we determine that we have a total of eleven double cosets that survive the enumeration via the aforementioned relation. Those double cosets are [*], [0], $[\bar{0}]$,[01],$[0\bar{1}]$, $[0\bar{2}]$, $[\bar{0}\bar{1}]$, [012], $[0\bar{1}2]$, $[\bar{0}\bar{1}2]$,and $[\bar{0}\bar{1}\bar{2}]$. The results are summarized in the following Cayley graph Figure 3.2:

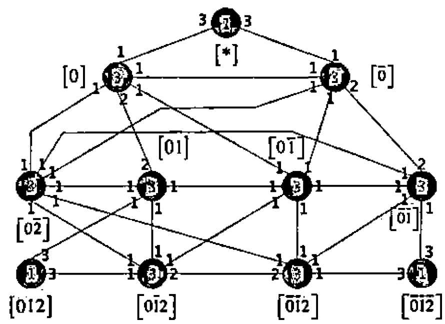

Figure 3.2: Cayley diagram of G over A_3 Factored by $t_0t_1 = t_1t_0$

Factoring this group by the center yields us the following Cayley diagram Figure 3.3:

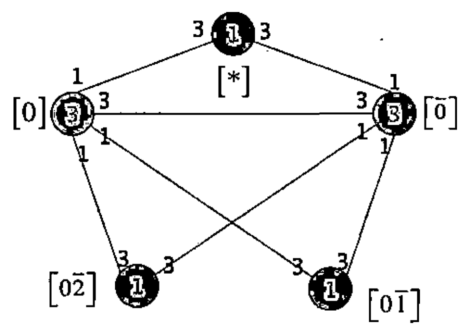

Figure 3.3: Cayley diagram of G over A_3 Factored by the Center

Chapter 4

Construction of $(M_{21} \times 4){:}S_3$

We have a computer-based proof that G is isomorphic to $(M_{21} \times 4){:}S_3$. This proof is obtained as follows: We first use MAGMA to obtain the composition factors of a permutation representation of G. This is done as follows:

```
f,G1,k:=CosetAction(G,sub<G|x,y>);
CompositionFactors(G1);
```

gives

```
 The below results indicate a semi direct product:
    G
    |  Cyclic(3)
    *
    |  Cyclic(2)
    *
    |  A(2, 4)                    = L(3, 4)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    1
```

Note: The above progenitor has produced another group in addition to the presentation we will construct below. From MAGMA, we have

```
for a in [0..10] do for  b in [0..10] do  for c in [0..10] do
for d in [0..10] do
 for e in [0..10 ] do
G<x,y,t>:=Group<x,y,t|x^8 , y^2 , (x*y)^3 , (x,y)^4,t^3,(t,y), (t,x^3 *
y * x^3 * y * x^-1),(t,y * x^-2 * y * x^3 * y * x^-2),(x*t)^a,
(x^3*t)^b,(x^-2 * y * x^2 * y*t^(x^6))^c,(x^4 * y*t)^d,(x^2*t)^e>;
 if  Index(G,sub<G|x,y>) ge 3 then a,b,c,d,e,
Index(G,sub<G|x,y>); end if;  end for; end for; end for; end for; end
for;
G<x,y,t>:=Group<x,y,t|x^8 , y^2 , (x*y)^3 , (x,y)^4,t^3,(t,y), (t,x^3 *
y * x^3 * y * x^-1),(t,y * x^-2 * y * x^3 * y * x^-2),(x*t)^0,
(x^3*t)^0,(x^-2 * y * x^2 * y*t^(x^6))^0,(x^4 * y*t)^6,(x^2*t)^0>;

 CompositionFactors(G1);
    G
    |  Cyclic(3)
    *
    |  Cyclic(2)
    *
    |  A(2, 4)                 = L(3, 4)
    *
    |  A(1, 7)                 = L(2, 7)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    1
```

which gives rise to the group $L_3(4)$.

We now write a presentation of the group $(M_{21} \times 4){:}S_3$ (obtained based on the composition factors above) and verify that $G \cong (M_{21} \times 4) : S_3$.
We will perform a double coset enumeration of the group $(M_{21} \times 4){:}S_3$ factored by the relation $t_3t_6t_2 \sim \bar{t}_8\bar{t}_4\bar{t}_7$, given by:

$$G \cong < x, y, t | x^8, y^2, (xy)^3, (xy)^4, t^3, (t, y), (t, x^3yx^3yx^{-1}), (t, yx^{-2}yx^3yx^{-2}), (x^3t)^6 >$$

where

$$N =< x, y > \cong PGL_2(7)$$
$$x \backsim (8, 2, 5, 4, 6, 1, 7, 3)(\bar{8}, \bar{2}, \bar{5}, \bar{4}, \bar{6}, \bar{1}, \bar{7})$$
$$y \sim (1, 6)(2, 5)(3, 4)(\bar{1}, \bar{6})(\bar{2}, \bar{5})(\bar{3}, \bar{4}).$$

We know $N \cong PGL_2(7)$ has 168 elements , or $|N| = 168$.
We have a computer-based proof that $G \cong C_3 \times C_2 \times A_2(40) = L_3(4) \times C_2 \times C_2$

4.1 Relations

Since this group has eight generators, we let $t = t_8$.
The first relation we must expand is

$$(x^3t)^6 = 1$$

Let $\pi = x^3$, then our relation becomes

$$(\pi t)^6 = 1$$

expanding our relation, we have

$$\pi^6 t^{\pi^5} t^{\pi^4} t^{\pi^3} t^{\pi^2} t^{\pi} t = 1$$

$$x^{18} t^{x^{15}} t^{x^{12}} t^{x^9} t^{x^6} t^{x^3} t = 1$$

since we are using t_∞, this relation becomes

$$x^2 t^{x^7} t^{x^4} t^{x^1} t^{x^6} t^{x^3} t = 1$$

The permutation representation of our group is

$$f(x) = xx = (1, 7, 3, 8, 2, 5, 4, 6)(\bar{1}, \bar{7}, \bar{3}, \bar{8}, \bar{2}, \bar{5}, \bar{4}, \bar{6})$$

So, using xx, we have

$$x^2 t_3 t_6 t_2 t_7 t_4 t_8 = 1$$

Using right-hand multiplication and the property $t_i^{-1} = \bar{t}_i$, we obtain the following

$$t_3 t_6 t_2 \sim \bar{t}_8 \bar{t}_4 \bar{t}_7 \qquad (4.1)$$

4.2 Double Coset Enumeration

NeN

We start our double coset enumeration by evaluating our first double coset, denoted [*], containing words of length zero. This double coset has one single coset, which is the identity $NeN = N$. Since $t = t_8$ and t has eight conjugates, there are two orbits that extend from [*]. The first orbit includes the generators $\{1, 2, 3, 4, 5, 6, 7, 8\}$, and the second orbit includes the generators $\{\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}, \bar{8}\}$.

Now we examine the double cosets containing words of length one. We do this by taking a representative t_i from each orbit and apply right hand multiplication to the coset N.

$$N \cdot t_8 = Nt_8 \in Nt_8N$$

Denote this double coset as [8]

$$N \cdot \bar{t}_8 = N\bar{t}_8 \in N\bar{t}_8N$$

Denote this double coset as [$\bar{8}$]

$\mathbf{Nt_8N}$

Consider the double coset [8]. We now compute the coset stabilizer $N^{(8)}$. Note that in this case the coset stabiliser equals the point stabilizer N^8. Using MAGMA we found the order of the coset stabiliser, $|N^{(8)}| = 42$. Next, we find the number of cosets in the double coset [8] by using the formula

$$|Nt_8N| = \frac{|N|}{|N^{(8)}|} \qquad (4.2)$$

Hence, $|Nt_8N| = \frac{336}{42} = 8$.

Now we compute orbits for the double coset [8] using MAGMA. It tells us that there are four orbits on [8]. The first orbit contains the generator {8}, the second orbit contains the generator $\{\bar{8}\}$, the third orbit contains the generators {1,2,3,4,5,6,7} and the fourth orbit contains the generators $\{\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}, \bar{8}\}$. We will now examine the two "singleton" orbits containing the generators 8 and 16. We will take a representative t_i from each of these two orbits and apply right hand multiplication to the double coset [8].

$$Nt_8 \cdot t_8 = N\bar{t}_8 \in N\bar{t}_8N$$

So this orbit takes one generator over to the double coset $[\bar{8}]$.

$$Nt_8 \cdot \bar{t}_8 = Ne \in NeN$$

So this orbit takes one generator back to the double coset [*]. Now we examine the remaining two orbits in this double coset:

$$Nt_8 \cdot t_1 = Nt_8t_1 \in Nt_8t_1N$$

Denote this double coset as $[8, 1]$.

$$Nt_8 \cdot \bar{t}_1 = Nt_8\bar{t}_1 \in Nt_8\bar{t}_1N$$

Denote this double coset as $[8, \bar{1}]$.

N$\bar{\mathbf{t}}_8$N

Consider the double coset $[\bar{8}]$. We now compute the coset stabilizer $N^{(\bar{8})}$. Note that in this case the coset stabiliser equals the point stabilizer $N^{\bar{8}}$. Using MAGMA we found the order of the coset stabiliser, $|N^{(\bar{8})}| = 42$. Next, we find the number of cosets in the double coset $[\bar{8}]$ by using the formula

$$|N\bar{t}_8N| = \frac{|N|}{|N^{(\bar{8})}|} \tag{4.3}$$

Hence, $|N\bar{t}_8N| = \frac{336}{42} = 8$

Now we compute orbits for the double coset $[\bar{8}]$ using MAGMA. It tells us that there are four orbits on $[\bar{8}]$. The first orbit contains the generator {8}, the second orbit contains the generator $\{\bar{8}\}$, the third orbit contains the generators {1,2,3,4,5,6,7} and the fourth orbit contains the generators $\{\bar{1},\bar{2},\bar{3},\bar{4},\bar{5},\bar{6},\bar{7}\}$. We will now examine the two "singleton" orbits containing the generators {8} and $\{\bar{8}\}$. We will take a representative t_i from each of these two orbits and apply right hand multiplication to the double coset $[\bar{8}]$.

$$N\bar{t}_8 \cdot t_8 = Ne \in NeN$$

So this orbit takes one genrator back to the double coset [*].

$$N\bar{t}_8 \cdot \bar{t}_8 = Nt_8 \in Nt_8N$$

So this orbit takes one generator to the double coset $[\bar{8}]$. Now we examine the remaining two orbits in this double coset.

$$N\bar{t}_8 \cdot t_1 = N\bar{t}_8 t_1 \in N\bar{t}_8 t_1 N$$

Denote this double coset as $[\bar{8},1]$.

$$N\bar{t}_8 \cdot \bar{t}_1 = N\bar{t}_8\bar{t}_1 \in N\bar{t}_8\bar{t}_1 N$$

Denote this double coset as $[\bar{8},\bar{1}]$.

We have four new double cosets with words of length two. Note that the orbits not only extend the generators to double cosets with words of increased length, they also take the generators back to double coests with words of reduced length. They can also take generators to other double cosets with words of equal length. We will now consider these four double cosets.

$\mathbf{Nt_8t_1N}$

Consider the double coset [8, 1]. We now compute the coset stabilizer $N^{(81)}$. Note that in this case the coset stabiliser equals the point stabilizer N^{81}. Using MAGMA we found the order of the coset stabiliser, $|N^{(81)}| = 42$. Next, we find the number of cosets in the double coset [8, 1] by using the formula

$$|Nt_8t_1N| = \frac{|N|}{|N^{(81)}|} \tag{4.4}$$

Hence, $|Nt_8t_1N| = \frac{336}{6} = 56$. Therefore, we have 56 cosets in Nt_8t_1N. This implies that we have 56 equal names for these cosets. Please note

$$Nt_1t_2 = Nt_2t_1 \Rightarrow 12 \sim 21$$

Thus we have the following 56 relations for $Nt_8t_1N = \{N(t_8t_1)^n | n \in N\}$:

$$12 \sim 13 \sim 14 \sim 15 \sim 16 \sim 17 \sim 18$$

$$\sim 21 \sim 23 \sim 24 \sim 25 \sim 26 \sim 27 \sim 28$$

$$\sim 31 \sim 32 \sim 34 \sim 35 \sim 36 \sim 37 \sim 38$$

$$\sim 41 \sim 42 \sim 43 \sim 45 \sim 46 \sim 47 \sim 48$$

$$\sim 51 \sim 52 \sim 53 \sim 54 \sim 56 \sim 57 \sim 58$$

$$\sim 61 \sim 62 \sim 63 \sim 64 \sim 65 \sim 67 \sim 68$$

$$\sim 71 \sim 72 \sim 73 \sim 74 \sim 75 \sim 76 \sim 78$$

$$\sim 81 \sim 82 \sim 83 \sim 84 \sim 85 \sim 86 \sim 87$$

MAGMA confirms these relations, which we will use to find equal double cosets with words of length three and greater within this group.

Now we compute orbits for the double coset Nt_8t_1N by conjugating elements in Nt_8t_1N by the coset stabilizer $N^{(81)}$. Note that the permutation for Nt_8t_1N is

$$N^{(81)} =< (265734)(\bar{2}\bar{6}\bar{5}\bar{7}\bar{3}\bar{4}) >$$

1. $1^{N^{(81)}} = \{1\}$
2. $2^{N^{(81)}} = \{6, 5, 7, 3, 4, 2\}$
3. $8^{N^{(81)}} = \{8\}$
4. $\bar{1}^{N^{(81)}} = \{\bar{1}\}$
5. $\bar{2}^{N^{(81)}} = \{\bar{6}, \bar{5}, \bar{7}, \bar{3}, \bar{4}, \bar{2}\}$
6. $\bar{8}^{N^{(81)}} = \{\bar{8}\}$

To examine these orbits we take a representative t_i from each of these orbits and apply right hand multiplication to the double coset [8, 1]. We will now examine orbits 1 and 4:

$$Nt_8t_1 \cdot t_1 = Nt_8\bar{t}_1 \in Nt_8\bar{t}_1N$$

So this orbit takes one generator over to the double coset $[8, \bar{1}]$.

$$Nt_8t_1 \cdot \bar{t}_1 = Nt_8 \in Nt_8N$$

So this orbit takes one genrator back to the double coset [8].

We will now consider the remaining four orbits, which extend the generators to new double cosets with words of length three.

$$Nt_8t_1 \cdot t_2 = Nt_8t_1t_2 \in Nt_8t_1t_2N$$

So this orbit extends 6 generators to the double coset [8, 1, 2].

$$Nt_8t_1 \cdot t_8 = Nt_8t_1t_8 \in Nt_8t_1t_8N$$

So this orbit extends one generator to the double coset [8, 1, 8].

$$Nt_8t_1 \cdot \bar{t}_2 = Nt_8t_1\bar{t}_2 \in Nt_8t_1\bar{t}_2N$$

So this orbit extends six generators to the double coset $[8, 1, \bar{2}]$.

$$Nt_8t_1 \cdot \bar{t}_8 = Nt_8t_1\bar{t}_8 \in Nt_8t_1\bar{t}_8N$$

So this orbit extends one generator to the double coset $[8, 1, \bar{8}]$.

$\mathbf{Nt_8\bar{t}_1N}$

Now we consider the double coset $[8, \bar{1}]$. We now compute the coset stabilizer $N^{(8\bar{1})}$. Note that in this case the coset stabiliser equals the point stabilizer $N^{8\bar{1}}$. Using MAGMA we found the order of the coset stabiliser, $|N^{(8\bar{1})}| = 42$. Next, we find the number of cosets in the double coset $[8, \bar{1}]$ by using the formula

$$|Nt_8\bar{t}_1N| = \frac{|N|}{|N^{(8\bar{1})}|} \tag{4.5}$$

Hence, $|Nt_8\bar{t}_1N| = \frac{336}{6} = 56$.

Now we compute orbits for the double coset $Nt_8\bar{t_1}N$ by conjugating elements in $Nt_8\bar{t_1}N$ by the coset stabilizer $N^{(8\bar{1})}$.

1. $1^{N^{(8\bar{1})}} = \{1\}$
2. $2^{N^{(8\bar{1})}} = \{2, 4, 3, 7, 5, 6\}$
3. $8^{N^{(8\bar{1})}} = \{8\}$
4. $\bar{1}^{N^{(8\bar{1})}} = \{\bar{1}\}$
5. $\bar{2}^{N^{(8\bar{1})}} = \{\bar{2}, \bar{4}, \bar{3}, \bar{7}, \bar{5}, \bar{6}\}$
6. $\bar{8}^{N^{(8\bar{1})}} = \{\bar{8}\}$

We repeat the process to examine the above orbits:

$$Nt_8\bar{t}_1 \cdot t_1 = Nt_8 \in Nt_8N$$

So this orbit takes one generator back to the double coset [8].

$$Nt_8\bar{t}_1 \cdot \bar{t}_1 = Nt_8t_1 \in Nt_8t_1N$$

So this orbit takes one generator over to the double coset $[8, 1]$.

$$Nt_8\bar{t}_1 \cdot t_8 = Nt_8\bar{t}_1t_8 \in Nt_8\bar{t}_1t_8N$$

So this orbit extends one generator to the double coset $[8, \bar{1}, 8]$.

$$Nt_8\bar{t}_1 \cdot \bar{t}_8 = Nt_8\bar{t}_1\bar{t}_8 \in Nt_8\bar{t}_1\bar{t}_8N$$

So this orbit extends one generator to the double coset $[8, \bar{1}, \bar{8}]$.

$$Nt_8\bar{t}_1 \cdot t_2 = Nt_8\bar{t}_1t_2 \in Nt_8\bar{t}_1t_2N$$

So this orbit extends six generators to the double coset $[8, \bar{1}, 2]$.

$$Nt_8\bar{t}_1 \cdot \bar{t}_2 = Nt_8\bar{t}_1\bar{t}_2 \in Nt_8\bar{t}_1\bar{t}_2N$$

So this orbit extends six generators to the double coset $[8, \bar{1}, \bar{2}]$.

$\mathbf{N\bar{t}_8 t_1 N}$

Consider the double coset $[\bar{8}, 1]$. We now compute the coset stabilizer $N^{(\bar{8}1)}$. Note that in this case the coset stabiliser equals the point stabilizer $N^{\bar{8}1}$. Using MAGMA we found the order of the coset stabiliser, $|N^{(\bar{8}1)}| = 42$ Next, we find the number of cosets in the double coset $[\bar{8}, 1]$ by using the formula

$$|N\bar{t}_8 t_1 N| = \frac{|N|}{|N^{(\bar{8}1)}|} \qquad (4.6)$$

Hence, $|N\bar{t}_8 t_1 N| = \frac{336}{6} = 56$.

Now we compute orbits for the double coset $[\bar{8}, 1]$ by repeating the same process detailed in the prior two double cosets. Our results tells us that there are six orbits on $[\bar{8}, 1]$:

1. the generator $\{1\}$
2. the generator $\{8\}$
3. the generator $\{\bar{1}\}$
4. the generator $\{\bar{8}\}$
5. the generators $\{2, 3, 4, 5, 6, 7\}$
6. the generators $\{\bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}\}$

We repeat the process of right hand multiplication to examine the above orbits:

$$N\bar{t}_8 t_1 \cdot t_1 = N\bar{t}_8 \bar{t}_1 \in N\bar{t}_8 \bar{t}_1 N$$

So this orbit takes one generator over to the double coset $[\bar{8}, \bar{1}]$.

$$N\bar{t}_8 t_1 \cdot \bar{t}_1 = N\bar{t}_8 \in N\bar{t}_8 N$$

So this orbit takes one generator back to the double coset $[\bar{8}]$.

$$N\bar{t}_8 t_1 \cdot t_8 = N\bar{t}_8 t_1 t_8 \in N\bar{t}_8 t_1 t_8 N$$

So this orbit extends one generator to a new double coset $[\bar{8}, 1, 8]$.

$$N\bar{t}_8 t_1 \cdot t_{16} = N\bar{t}_8 t_1 \bar{t}_8 \in N\bar{t}_8 t_1 \bar{t}_8 N$$

So this orbit extends one generator to a new double coset $[\bar{8}, 1, \bar{8}]$.

$$N\bar{t}_8 t_1 \cdot t_2 = N\bar{t}_8 t_1 t_2 \in N\bar{t}_8 t_1 t_2 N$$

So this orbit extends six generators to a new double coset $[\bar{8}, 1, 2]$.

$$N\bar{t}_8 t_1 \cdot \bar{t}_2 = N\bar{t}_8 t_1 \bar{t}_2 \in N\bar{t}_8 t_1 \bar{t}_2 N$$

So this orbit extends six generators to a new double coset $[\bar{8}, 1, \bar{2}]$.

$\mathbf{N\bar{t}_8\bar{t}_1N}$

Consider the double coset $[\bar{8}, \bar{1}]$. We now compute the coset stabilizer $N^{(\bar{8}\bar{1})}$. Note that in this case the coset stabiliser equals the point stabilizer $N^{\bar{8}\bar{1}}$. Using MAGMA we found the order of the coset stabiliser, $|N^{(\bar{8}\bar{1})}| = 42$. Next, we find the number of cosets in the double coset $[\bar{8}, \bar{1}]$ by using the formula

$$|N\bar{t}_8\bar{t}_1N| = \frac{|N|}{|N^{(\bar{8}\bar{1})}|} \tag{4.7}$$

Hence, $|N\bar{t}_8\bar{t}_1N| = \frac{336}{6} = 56$.

Now we compute orbits for the double coset $[\bar{8}, \bar{1}]$ using the aforementioned process. It tells us that there are six orbits on $[\bar{8}, \bar{1}]$:

1. the generator $\{1\}$
2. the generator $\{8\}$
3. the generator $\{\bar{1}\}$
4. the generator $\{\bar{8}\}$
5. the generators $\{2, 3, 4, 5, 6, 7\}$
6. the generators $\{\bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}\}$

repeating our process to examine the above orbits, we note the following:

$$N\bar{t}_8\bar{t}_1 \cdot t_1 = N\bar{t}_8 \in N\bar{t}_8 N$$

So this orbit takes one generator back to the double coset $[\bar{8}]$.

$$N\bar{t}_8\bar{t}_1 \cdot \bar{t}_1 = N\bar{t}_8 t_1 \in N\bar{t}_8 t_1 N$$

So this orbit takes one generator over to the double coset $[\bar{8}, 1]$.

$$N\bar{t}_8\bar{t}_1 \cdot t_8 = N\bar{t}_8\bar{t}_1 t_8 \in N\bar{t}_8\bar{t}_1 t_8 N$$

So this orbit extends one generator to a new double coset $[\bar{8}, \bar{1}, 8]$.

$$N\bar{t}_8\bar{t}_1 \cdot \bar{t}_8 = N\bar{t}_8\bar{t}_1\bar{t}_8 \in N\bar{t}_8\bar{t}_1\bar{t}_8 N$$

So this orbit extends one generator to a new double coset $[\bar{8}, \bar{1}, \bar{8}]$.

$$N\bar{t}_8\bar{t}_1 \cdot t_2 = N\bar{t}_8\bar{t}_1 t_2 \in N\bar{t}_8\bar{t}_1 t_2 N$$

So this orbit extends six generators to a new double coset $[\bar{8}, \bar{1}, 2]$.

$$N\bar{t}_8\bar{t}_1 \cdot \bar{t}_2 = N\bar{t}_8\bar{t}_1\bar{t}_2 \in N\bar{t}_8\bar{t}_1\bar{t}_2 N$$

So this orbit extends six generators to a new double coset $[\bar{8}, \bar{1}, \bar{2}]$.

We have sixteen new double cosets with words of length three. Again, note that the orbits not only extend the generators to double cosets with words of increased length, they also take the generators back to double cosets with words of reduced length. They can also take generators to other double cosets with words of equal length. We will apply the process described in the relations section of this chapter to find any equal double cosets with words of length three. By using our relations, we find that six of of the sixteen "new" double cosets are equivalent to existing double cosets. Thus, we have ten double cosets with words of length three:

$$[8,1,2], [8,1,\bar{2}], [8,1,8], [8,1,\bar{2}], [8,\bar{1},\bar{2}], [8,\bar{1},8], [8,\bar{1},\bar{8}], [\bar{8},1,8], [\bar{8},1,\bar{8}], [\bar{8},\bar{1},8]$$

We will now examine these ten double cosets.

$\mathbf{Nt_8t_1t_2N}$

Consider the double coset $[8, 1, 2]$. We now compute the coset stabilizer $N^{(812)}$. Using MAGMA we found the order of the coset stabiliser, $|N^{(812)}| = 2$. Next, we find the number of cosets in the double coset $[8, 1, 2]$ by using the formula

$$|Nt_8t_1t_2N| = \frac{|N|}{|N^{(812)}|} \tag{4.8}$$

We then calculate the number of cosets in the double coset [8, 1, 2] to be

$$|Nt_8t_1t_2N| = \frac{336}{2} = 168 \tag{4.9}$$

Now we compute orbits for the double coset $Nt_8t_1t_2N$ by conjugating elements in $Nt_8t_1t_2N$ by the coset stabilizer $N^{(812)}$. Our resulting sixteen orbits contain a single generator in each:

$$\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{\bar{1}\}, \{\bar{2}\}, \{\bar{3}\}, \{\bar{4}\}, \{\bar{5}\}, \{\bar{6}\}, \{\bar{7}\}, \{\bar{8}\}.$$

We will take a representative t_i from each of these orbits and apply right hand multiplication to the coset $Nt_8t_1t_2$. Note that we will use our relations to determine which double coset the orbits take their generators to:

$$Nt_8t_1t_2 \cdot t_1 = Nt_8t_1t_2t_1 = Nt_8\bar{t_2}\bar{t_1}\bar{t_2} = Nt_8\bar{t_1}t_2 = Nt_8t_1\bar{t_2} \in Nt_8t_1\bar{t_2}N$$

So this orbit takes one generator over to the double coset $Nt_8t_1\bar{t_2}N$ denoted $[8, 1, \bar{2}]$.

$$Nt_8t_1t_2 \cdot t_2 = Nt_8t_1\bar{t_2} \in Nt_8t_1\bar{t_2}N$$

So this orbit also takes one generator over to the double coset$Nt_8t_1\bar{t_2}N$ denoted $[8, 1, \bar{2}]$.

$$Nt_8t_1t_2 \cdot t_3 = Nt_8t_1t_2t_3 = Nt_8t_1t_2t_1 = Nt_8t_1t_8t_1 = Nt_8t_1t_1t_8 = Nt_8\bar{t_1}t_8 \in Nt_8\bar{t_1}t_8N$$

So this orbit takes one generator to the double coset $Nt_8\bar{t_1}t_8N$ denoted $[8, \bar{1}, 8]$.

$$Nt_8t_1t_2 \cdot t_4 = Nt_8t_1t_2t_4 = Nt_8t_1t_2t_1 = Nt_8t_1t_8t_1 = Nt_8t_1t_1t_8 = Nt_8\bar{t_1}t_8 \in Nt_8\bar{t_1}t_8N$$

So this orbit takes one generator to the double coset $Nt_8\bar{t_1}t_8N$ denoted $[8, \bar{1}, 8]$.

$$Nt_8t_1t_2 \cdot t_5 = Nt_8t_1t_2t_5 = Nt_8t_1t_2t_1 = Nt_8t_1t_8t_1 = Nt_8t_8t_1t_1 = N\bar{t_8}\bar{t_1} \in N\bar{t_8}\bar{t_1}N$$

So this orbit takes one generator back to the double coset $N\bar{t_8}\bar{t_1}N$ denoted $[\bar{8}, \bar{1}]$.

$$Nt_8t_1t_2 \cdot t_6 = Nt_8t_1t_2t_6 = Nt_8t_1t_2t_1 = Nt_8t_1t_8t_1 = Nt_8t_8t_1t_1 = N\bar{t_8}\bar{t_1} \in N\bar{t_8}\bar{t_1}N$$

So this orbit takes one generator back to the double coset $N\bar{t_8}\bar{t_1}N$ denoted $[\bar{8}, \bar{1}]$.

$$Nt_8t_1t_2 \cdot t_7 = Nt_8t_1t_2t_7 = Nt_8t_1t_2t_1 = Nt_8\bar{t_2}\bar{t_1}\bar{t_2} = Nt_8\bar{t_1}t_2 = Nt_8t_1\bar{t_2} \in Nt_8t_1\bar{t_2}N$$

So this orbit takes one generator over to the double coset $Nt_8t_1\bar{t_2}N$ denoted $[8,1,\bar{2}]$.

$$Nt_8t_1t_2 \cdot t_7 = Nt_8t_1t_2t_8 = Nt_8t_1t_2t_1 = Nt_8\bar{t_2}\bar{t_1}\bar{t_2} = Nt_8\bar{t_1}t_2 = Nt_8t_1\bar{t_2} \in Nt_8t_1\bar{t_2}N$$

So this orbit takes one generator over to the double coset $Nt_8t_1\bar{t_2}N$ denoted $[8,1,\bar{2}]$. Using a similar process with their inverse counter parts, the remaining orbits behave in a similar fashion:

$\{\bar{1}\}$ takes one generator to the double coset $N\bar{t_8}t_1\bar{t_8}N$ denoted $[\bar{8},1,\bar{8}]$
$\{\bar{2}\}$ takes one generator to the double coset Nt_8t_1N denoted $[8,1]$
$\{\bar{3}\}$ takes one generator to the double coset $Nt_8\bar{t_1}\bar{t_2}N$ denoted $[8,\bar{1},\bar{2}]$
$\{\bar{4}\}$ takes one generator to the double coset $Nt_8\bar{t_1}\bar{t_2}N$ denoted $[8,\bar{1},\bar{2}]$
$\{\bar{5}\}$ takes one generator to the double coset $Nt_8\bar{t_1}\bar{t_2}N$ denoted $[8,\bar{1},\bar{2}]$
$\{\bar{6}\}$ takes one generator to the double coset $Nt_8\bar{t_1}\bar{t_2}N$ denoted $[8,\bar{1},\bar{2}]$
$\{\bar{7}\}$ takes one generator to the double coset Nt_8t_1N denoted $[8,1]$
$\{\bar{8}\}$ takes one generator to the double coset $N\bar{t_8}t_1\bar{t_8}N$ denoted $[\bar{8},1,\bar{8}]$

Showing Equal Double Cosets

We now have double cosets consisting of words of at least length three. Since our relation is based on three letters, we must now apply the relation to our double cosets to verify the existence of equal cosets.

We derived our original relation

$$x^2t_3t_6t_2 = \bar{t}_8\bar{t}_4\bar{t}_7$$

from our symmetric presentation. Through conjugation by elements of our control subgroup $PGL_2(7)$ we obtain the relation

$$t_8t_1t_8 = \bar{t}_1\bar{t}_8\bar{t}_1$$

By using this relation, we will now verify that the double coset $Nt_8t_1t_8t_1N$, denoted $[8,1,8,1]$, is equal to the double coset $N\bar{t}_8\bar{t}_1N$, denoted $[\bar{8},\bar{1}]$.

$$\underline{t_8t_1t_8}t_1 = \bar{t}_1\bar{t}_8\underline{\bar{t}_1t_1} = \bar{t}_1\bar{t}_8$$

Now we must show that the coset $N\bar{t}_1\bar{t}_8$ belongs to $N\bar{t}_8\bar{t}_1N$

Lets examine the double coset $[\bar{8}\bar{1}]$:

$$N\bar{t}_8\bar{t}_1N = \{N(\bar{t}_8\bar{t}_1)^n | n \in N\}$$

Recall: $N =< x, y >=< (17382546)(\bar{1}\bar{7}\bar{3}\bar{8}\bar{2}\bar{5}\bar{4}\bar{6}), (16)(25)(34)(\bar{1}\bar{6})(\bar{2}\bar{5})(\bar{3}\bar{4}) >$. We must find a permutation, n, in N such that

$$N(\bar{t}_8\bar{t}_1)^n = N\bar{t}_1\bar{t}_8$$

or n takes the coset $N\bar{t}_8\bar{t}_1$ to $N\bar{t}_1\bar{t}_8$. We found the desired permutation to be:

$$n = (x^y)^4 = (65)(72)(43)(18)(\bar{6}\bar{5})(\bar{7}\bar{2})(\bar{4}\bar{3})(\bar{1}\bar{8})$$

So, $N(\bar{t}_8\bar{t}_1)^n = N\bar{t}_1\bar{t}_8$. Hence, the single coset $N\bar{t}_1\bar{t}_8$ belongs to the double coset $N\bar{t}_8\bar{t}_1N$. Since two different double cosets are disjoint, we can conclude that $N\bar{t}_1\bar{t}_8N = N\bar{t}_8\bar{t}_1N$. Hence,

$$N\underline{t_8t_1t_8}t_1N = N\bar{t}_1\bar{t}_8\underline{\bar{t}_1t_1}N = N\bar{t}_1\bar{t}_8N = N\bar{t}_8\bar{t}_1N$$

Thus, we have verified the following double coset equality

$$Nt_8t_1t_8t_1N = N\bar{t}_8\bar{t}_1N.$$

Repeating this process, we can verify the existence of other equal double cosets within our group.

$$[8,1,8,1] = [\bar{8},\bar{1}]$$
$$[8,1,8,8] = [8,1,\bar{8}]$$
$$[8,1,8,\bar{1}] = [\bar{8},\bar{1},8]$$
$$[8,1,8,\bar{8}] = [8,1]$$
$$[8,1,8,2] = [8,1,2]$$
$$[8,1,8,\bar{2}] = [8,1,\bar{8}]$$

Due to times constraints, we were not able to finish the manual construction of this group. However, we have utilized algorithms in MAGMA that provided us with an accurate Cayley diagram, which is provided in Figure 4.1 below:

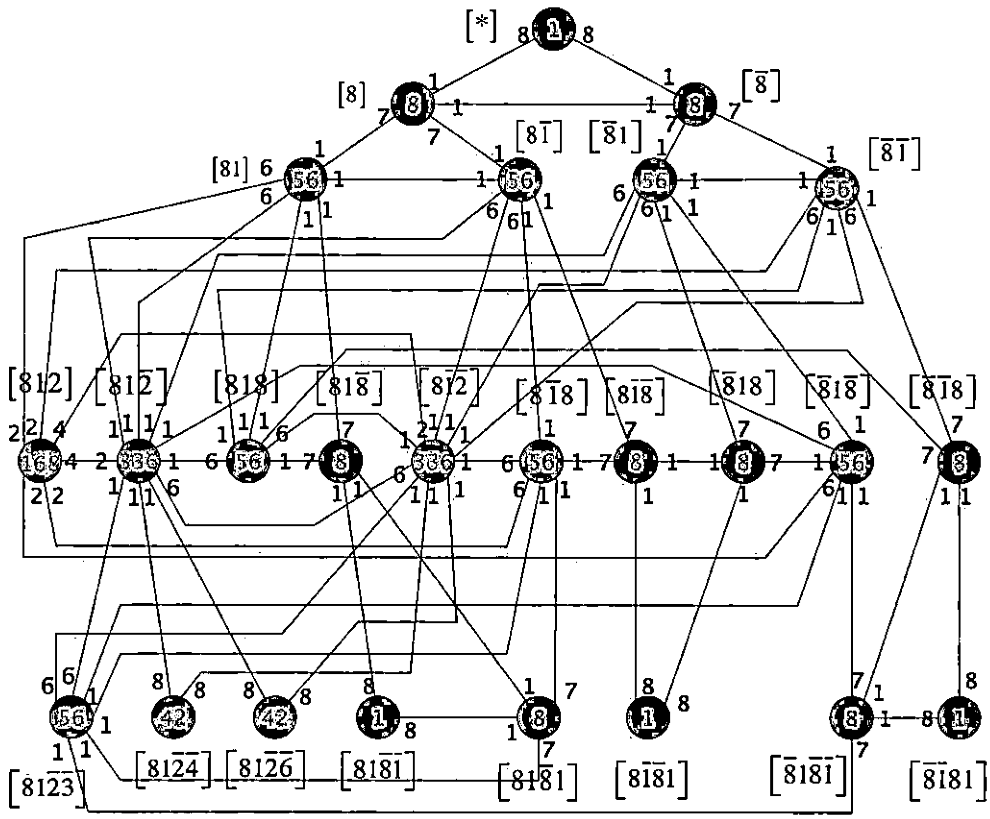

Figure 4.1: Completed Cayley diagram of $(M_{21} \times 4){:}S_3$

4.3 Factoring by the Center

Due to times constraints, we were not able to manually construct this group factored by its centralizer. However, we have utilized algorithms in MAGMA that provide us with an accurate Cayley diagram as seen in Figure 4.2. This illustrates the efficiency in finding and factoring larger groups by their center:

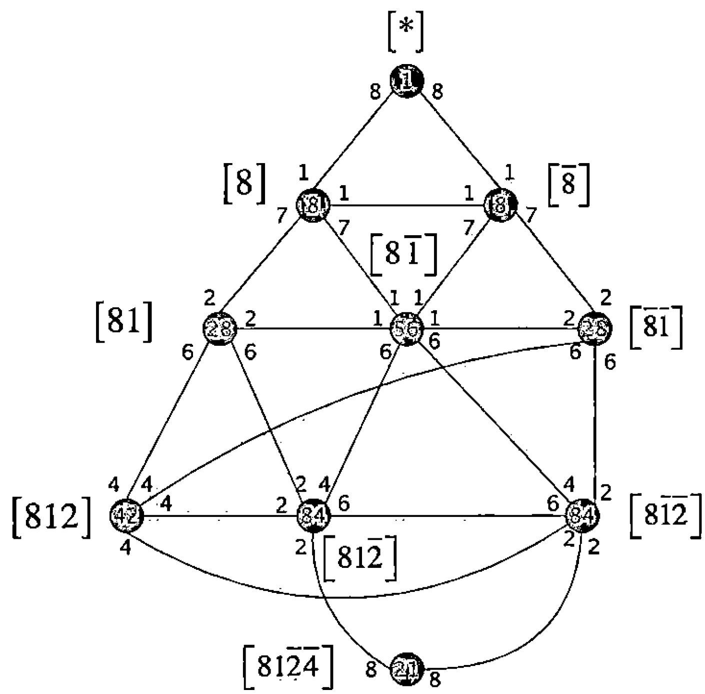

Figure 4.2: Completed Cayley diagram of $(M_{21} \times 4){:}S_3$ Factored by the Center

The MAGMA algorithms used to generate the figure above are listed in Appendix C.

Chapter 5

Construction of $U_3(3):3$ as a Homomorphic Image of $2^{*14}:L_3(2)$

We have a computer-based proof that

$$G \simeq \frac{2^{*14}:L_3(2)}{32=71} \xrightarrow{homo} U_3(3):3$$

This proof is obtained as follows: We first use MAGMA to obtain the composition factors of a permutation representation of G. This is done as follows:

```
f,G1,k:=CosetAction(G,sub<G|x,y>);
CompositionFactors(G1);
```

which gives the following results.

```
G
| Cyclic(2)
*
| 2A(2, 3) = U(3, 3)
1
```

We now write a presentation of the group $U_3(3):3 \xrightarrow{homo} 2^{*14}:L_3(2)$ (obtained based on the composition factors above) and verify that $G \cong U_3(3):3 \xrightarrow{homo} 2^{*14}:L_3(2)$.
We will perform a double coset enumeration on the group $U_3(3):3 \xrightarrow{homo} 2^{*14}:L_3(2)$ factored by the relation $t_3t_2 = \bar{t}_7\bar{t}_1$, denoted by the following group representation:

$$G \cong < x, y, t | x^7, y^2, (xy)^3, (x,y)^4, t^2, (t, x^{-3}yx^2), (t,y), t^x t^{(xy)}, (xyt^{-1})^8, (xyt^{x^2})^6 >$$

We have a computer-based proof that $G \cong C_2 \times 2A_2(3) = U_3(3)$.

where $|G| = 12096$ and $N =< x, y > \cong L_3(2)$.
The generators are represented by:

$$x \backsim (1,2,3,4,5,6,7)(\bar{1},\bar{2},\bar{3},\bar{4},\bar{5},\bar{6},\bar{7})$$

and

$$y \sim (1,\bar{1})(3,\bar{7})(2,\bar{6})(4,5)(\bar{4},\bar{5})(\bar{2},6).$$

We know $N \cong L_3(2)$ has 168 elements , or $|N| = 168$.

5.1 Relations

The relation we must expand is

$$(xyt^{x^{-1}})^8 = 1$$

Let $\pi = xy$. We then expand our relation

$$(xyt^{x^{-1}})(xyt^{x^{-1}})(xyt^{x^{-1}})(xyt^{x^{-1}})(xyt^{x^{-1}})(xyt^{x^{-1}})(xyt^{x^{-1}})(xyt^{x^{-1}}) = 1$$

We also know that $t \sim t_7$ and $t_7^{-1} = t_{14}$. We use the insertion of identity inverses $\pi^{-1}t\pi = t^\pi$ to convert our relation to a relation involving the $t_i's$:

$$\pi^8(t_7^{-1})^{\pi^7}(t_7^{-1})^{\pi^6}(t_7^{-1})^{\pi^5}(t_7^{-1})^{\pi^4}(t_7^{-1})^{\pi^3}(t_7^{-1})^{\pi^2}(t_7^{-1})^{\pi}t^{-1} = 1$$

Now, we consider our permutation xy which we have transformed into π which becomes

$$\pi = xy \sim (1234567)(\bar{1}\bar{2}\bar{3}\bar{4}\bar{5}\bar{6}\bar{7})(1\bar{1})(3\bar{7})(2\bar{6})(45)(\bar{4}\bar{5})(\bar{2}6).$$

We then apply our permutation π to our relation which gives us our permutations

$$\pi^2 = (1357246)(\bar{1}\bar{3}\bar{5}\bar{7}\bar{2}\bar{4}\bar{6}),$$

$$\pi^3 = (1473625)(\bar{1}\bar{4}\bar{7}\bar{3}\bar{6}\bar{2}\bar{5}),$$

$$\pi^4 = (1526374)(\bar{1}\bar{5}\bar{2}\bar{6}\bar{3}\bar{7}\bar{4}).$$

$$\pi^5 = (1357246)(\bar{1}\bar{3}\bar{5}\bar{7}\bar{2}\bar{4}\bar{6}),$$

$$\pi^6 = (1473625)(\bar{1}\bar{4}\bar{7}\bar{3}\bar{6}\bar{2}\bar{5}),$$

$$\pi^7 = (1526374)(\bar{1}\bar{5}\bar{2}\bar{6}\bar{3}\bar{7}\bar{4}).$$

$$\pi^8 = (1473625)(\bar{1}\bar{4}\bar{7}\bar{3}\bar{6}\bar{2}\bar{5}),$$

We then convert our permutations back into $t_i's$ to get our relation

$$\pi^4 t_3 t_2 t_1 t_7 = 1$$

Utilizing right multiplication of our $t_i's$, we have a relation based on two letters

$$\pi^4 t_3 t_2 = \bar{t_7}\bar{t_1}.$$

We can use this relation to evaluate cosets and double cosets within our group.

5.2 Double Coset Enumeration

NeN

We start our double coset enumeration by evaluating our first double coset, denoted [*], containing words of length zero. This double coset has one single coset, which is the identity $NeN = N$. Since our presentation group is $U3(3)$, we have $t = t_7$ and $t_7{}^{-1} \sim \bar{t}_7$. This means our first orbit contains all fourteen generators

$$\{1, 2, 3, 4, 5, 6, 7, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}\}.$$

When we apply a representative t_i from this orbit, say t_7 to the coset representative Ne to get a new coset Nt_7. We see that all fourteen generators extend to a new double coset Nt_7N, denoted [7]. This double cosets will be made up of words of length one.

Nt$_7$N

We now consider the double coset $Nt_7N = \{Nt_7^n | n \in N\}$. We must first determine the coset stabilizer, denoted $N^{(7)}$. We look at permutations in $N = L_3(2)$ and find those that "fix" the the element (7) and permute all others. We determine

$$N^7 =< (1\bar{1})(2\bar{6})(3\bar{3})(45)(6\bar{2})(\bar{4}\bar{5}), (16\bar{5})(2\bar{4}3)(4\bar{3}\bar{2})(5\bar{1}\bar{6}) > .$$

Since there are no additional relations, our point stabilizer *is* our coset stabilizer. Thus we have

$$N^7 = N^{(7)}.$$

Please note that

$$|N^7| = |N^{(7)}| = 2^2 \cdot 3 = 12$$

We now determine the number of cosets in [7] by using our equation

$$|Nt_7N| = \frac{|N|}{|N^{(7)}|} \tag{5.1}$$

which gives us

$$|Nt_7N| = \frac{168}{12 \cdot 2} = 7.$$

This is true, since each coset in [7] has two equal names. We now determine the orbits on [7], which are

$$\{7\}, \{\bar{7}\}, \text{and } \{1, 2, 3, 4, 5, 6, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}\}.$$

We will take a representative t_i from each of these orbits and apply right hand multiplication to the coset Nt_7:

1. $Nt_7t_7 = Nt_7^{-1} = Nt_7\bar{t}_7 \in NeN$ denoted [*], so this orbit takes 1 generator to the double coset [*].

2. $Nt_7\bar{t}_7 = Ne \in NeN$ denoted [*], so this orbit takes 1 generator to the double coset [*].

3. $Nt_7t_1 = Nt_7t_1 \in Nt_7t_1N$ denoted $[7, 1]$, so this orbit takes 12 generators to the double coset $[7, 1]$.

$\mathbf{Nt_7t_1N}$

We now consider the double coset Nt_7t_1N. Through Magma, we determine there are 8 equal names in this double coset:

$$[\bar{1}, 7], [7, \bar{1}], [7, 1], [1, \bar{7}], [\bar{7}, 1], [1, 7], [\bar{7}, \bar{1}], [\bar{1}, \bar{7}]$$

We must determine the coset stabilizer, denoted $N^{(7,1)}$. We look at permutations in $N = L_3(2)$ and find those that "fix" the the elements 7 and 1 while permuting all others. We determine the coset stabilizer by using our relations:

$$N^{(7,1)} =< (17\bar{1}\bar{7})(2\bar{4}56)(3\bar{3})(4\bar{5}\bar{6}\bar{2}), (16\bar{7})(2\bar{2})(4\bar{3}\bar{2})(46)(5\bar{5})(7\bar{1})(\bar{4}\bar{6}) > .$$

We determine $|N^{(7,1)}| = 8$. Thus, we may now determine the number of cosets within the double coset $[7,1]$:

$$|Nt_7t_1N| = \frac{|N|}{|N^{(7,1)}|} \tag{5.2}$$

which gives us

$$|Nt_7t_1N| = \frac{168}{8} = 21.$$

We now determine the orbits on $[7,1]$, which are

$$\{3,\bar{3}\},\ \{1,7,\bar{1},\bar{7}\},\ \text{and}\ \{2,4,5,6,\bar{2},\bar{4},\bar{5},\bar{6}\}.$$

We will take a representative t_i from each of these orbits and apply right hand multiplication to the coset Nt_7t_1:

1. $Nt_7t_1\bar{t}_1 = Nt_7 \in Nt_7N$ denoted [7], so this orbit takes 4 generators back to the double coset [7].
2. $Nt_7t_1t_3 = Nt_7t_1t_3 \in Nt_7t_1t_3N$ denoted $[7,1,3]$, so this orbit extends 2 generators to a new double coset $[7,1,3]$.
3. $Nt_7t_1t_2 = Nt_7t_1t_2 \in Nt_7t_1t_2N$ denoted $[7,1,2]$, so this orbit extends 8 generators to a new double coset $[7,1,2]$.

$\mathbf{Nt_7t_1t_2N}$

We now consider the double coset $Nt_7t_1t_2N$. As in the prior double cosets, there are eight equal names:

$$[5,9,8],[8,14,9],[14,8,5],[5,9,8],[9,5,14],[12,2,7],[1,7,12],[2,12,1]$$

We must determine the coset stabilizer, denoted $N^{(7,1,2)}$. We look at permutations in $N = L_3(2)$ and find those that "fix" the the elements 7, 1, and 2 while permuting all others. We determine the coset stabilizer by utilizing our relations to increase the stabilizer:

$N^{(7,1,2)} =< (17)(2\bar{5})(4\bar{4})(5\bar{2})(6\bar{6})(\bar{1}\bar{7}), (1\bar{5}72)(3\bar{3})(4\bar{6}\bar{4}6)(5\bar{7}\bar{2}\bar{1}), (15)(2\bar{7})(3\bar{3})(4\bar{4})(7\bar{2})(\bar{1}\bar{5}) >$.

We determine $|N^{(7,1,2)}| = 8$. We now calculate the number of cosets within $[7, 1, 2]$ by using our equation:

$$|Nt_7t_1t_2N| = \frac{|N|}{|N^{(7,1,2)}|} \tag{5.3}$$

which gives us

$$|Nt_7t_1t_2N| = \frac{168}{8} = 21.$$

Again, we have 21 cosets in $[7, 1, 2]$. We now determine the orbits on $[7, 1, 2]$, which are

$$\{3, \bar{3}\},\ \{4, 6, \bar{4}, \bar{6}\},\text{ and } \{1, 2, 5, 7, \bar{1}, \bar{2}, \bar{5}, \bar{7}\}.$$

We will take a representative t_i from each of these orbits and apply right hand multiplication to the coset $Nt_7t_1t_2$:

1. $Nt_7t_1t_2t_3 = Nt_7t_1t_2t_3 \in Nt_7t_1t_2t_3N$ denoted $[7, 1, 2, 3]$, so this orbit extends 2 generators to a new double coset $[7, 1, 2, 3]$.
2. $Nt_7t_1t_3 = Nt_7t_1t_3 \in Nt_7t_1t_3N$ denoted $[7, 1, 3]$, so this orbit extends 2 generators to a new double coset $[7, 1, 3]$.
3. $Nt_7t_1t_2 = Nt_7t_1t_2 \in Nt_7t_1t_2N$ denoted $[7, 1, 2]$, so this orbit extends 8 generators to a new double coset $[7, 1, 2]$.

$\mathbf{Nt_7t_1t_3N}$

We now consider the double coset $Nt_7t_1t_3N$. There are 24 cosets within $[7, 1, 3]$ having equal names:

$$[1, 10, 14], [14, 3, 8], [1, 14, 10], [8, 10, 7], [14, 8, 3], [8, 3, 14], [7, 8, 10], [1, 3, 7],$$
$$[10, 8, 7], [7, 10, 8], [3, 7, 1], [7, 3, 1], [3, 1, 7], [10, 7, 8], [14, 1, 10], [10, 14, 1],$$
$$[3, 8, 14], [8, 14, 3], [7, 1, 3], [8, 7, 10], [14, 10, 1], [1, 7, 3], [3, 14, 8], \text{ and } [10, 1, 14].$$

We then increase our coset stabilizer to account for the equally named cosets. The permutation that achives this is:

$$(1,3,8,10)(2,12,13,11)(4,9,5,6)(7,14)(1,10,7)(2,13,5)(3,14,8)(6,12,9)$$

We must determine the coset stabilizer, denoted $N^{(7,1,3)}$. We look at permutations in $N = L_3(2)$ and find those that "fix" the the elements 7, 1, and 3 while permuting all others. We determine the coset stabilizer by utilizing our relations to increase the stabilizer:

$$N^{(7,1,3)} =< (13\bar{1}\bar{3})(2\bar{5}\bar{6}\bar{4})(4\bar{2}56)(7\bar{7}), (1\bar{3}7)(2\bar{6}5)(3\bar{7}\bar{1})(6\bar{5}\bar{2}) > .$$

We determine $|N^{(7,1,3)}| = 24$. We now calculate the number of cosets within $[7,1,3]$ by using our equation:

$$|Nt_7t_1t_3N| = \frac{|N|}{|N^{(7,1,3)}|} \tag{5.4}$$

which gives us

$$|Nt_7t_1t_3N| = \frac{168}{24} = 7.$$

We now determine the orbits on $[7,1,3]$, which are

$$\{1,3,7,\bar{1},\bar{3},\bar{7}\} \text{ and } \{2,4,5,6,\bar{2},\bar{4},\bar{5},\bar{6}\}.$$

We will take a representative t_i from each of these orbits and apply right hand multiplication to the coset $Nt_7t_1t_3$:

1. $Nt_7t_1t_3t_3 = Nt_7t_1(t_3)^2 = Nt_7t_1 \in Nt_7t_1N$ denoted $[7,1]$, so this orbit takes 6 generators backto the double coset $[7,1]$.

2. $Nt_7t_1t_3t_2 = Nt_7t_1t_3t_2 \in Nt_7t_1t_3t_2N$ denoted $[7,1,3,2]$, so this orbit extends 8 generators to the new double coset $[7,1,3,2]$.

We now consider the double coset $Nt_7t_1t_3t_2N$. There are 24 cosets within $[7,1,3,2]$ having equal names:

$[1,10,14,13], [14,3,8,12], [1,14,10,6], [8,10,7,4], [14,8,3,5], [8,3,14,2], [7,8,10,13], [1,3,7,5], [10,8,7,12], [7,10,8,6], [3,7,1,4], [7,3,1,9], [3,1,7,11], [10,7,8,5], [14,1,10,4], [10,14,1,2], [3,8,14,6], [8,14,3,9], [7,1,3,2], [8,7,10,11], [14,10,1,11], [1,7,3,12], [3,14,8,13]$, and $[10,1,14,9]$.

We then increase our coset stabilizer to account for the equally named cosets. The relation that we achive this with is:

(1, 3, 8, 10)(2, 12, 13, 11)(4, 9, 5, 6)(7, 14)(1, 10, 14)(2, 4, 13)(3, 7, 8)(6, 9, 11)

We determine $|N^{(7,1,3,2)}| = 24$. We now calculate the number of cosets within $[7, 1, 3, 2]$ by using our equation:

$$|Nt_7t_1t_3N| = \frac{|N|}{|N^{(7,1,3)}|} \tag{5.5}$$

which gives us

$$|Nt_7t_1t_3N| = \frac{168}{24} = 7.$$

We have 7 cosets in $[7, 1, 3, 2]$. We now determine the orbits on $[7, 1, 3, 2]$, which are

$$\{1, 3, 7, 8, \bar{3}, \bar{7}\} \text{ and } \{2, 4, 5, 6, \bar{2}, \bar{4}, \bar{5}, \bar{6}\}.$$

We will take a representative t_i from each of these orbits and apply right hand multiplication to the coset $Nt_7t_1t_3$:

1. $Nt_7t_1t_3t_2t_1 = Nt_7t_1t_3 = Nt_7t_1t_2t_3 \in Nt_7t_1t_2t_3N$ denoted $[7, 1, 2, 3]$, so this orbit extends 6 generators to the double coset $[7, 1, 2, 3]$.

2. $Nt_7t_1t_3t_2t_2 = Nt_7t_1t_3 \in Nt_7t_1t_3N$ denoted $[7, 1, 3]$, so this orbit takes 8 generators back to the double coset $[7, 1, 3]$.

We now use MAGMA to confirm that we have an increase in the total count of single cosets thus far in our group. We determine that the total count of cosets *do not* increase with $[7, 1, 3, 2]$, which indicates that we have equal double cosets. We now confirm and identify our equal double cosets by conjugating the double coset $[7, 1, 3, 2]$ by the permutation that stabilizes the coset, and compare the result to existing double cosets. We find that

$$(Nt_7t_1t_3t_2N)^{(1,3,8,10)(2,12,13,11)(4,9,5,6)(7,14)(1,10,7)(2,13,5)(3,14,8)(6,12,9)}$$
$$= Nt_7t_1t_2t_3N.$$

Thus, the double coset $[7, 1, 3, 2] = [7, 1, 2, 3]$. Therefore, the orbit $\{2, 4, 5, 6, \bar{2}, \bar{4}, \bar{5}, \bar{6}\}$. takes 6 generators back to the double coset $[7, 1, 2]$.

$\mathbf{Nt_7t_1t_2t_4N}$

We now consider the double coset $Nt_7t_1t_2Nt_4$. There are 24 cosets within $[7, 1, 2, 4]$ having equal names:

$$[2,8,7,10],[3,7,4,5],[8,2,5,3],[7,3,9,12],[9,10,7,8],[14,4,2,1],[5,9,8,4],[2,11,14,5],$$
$$[4,12,10,2],[11,2,10,12],[12,4,8,9],[12,14,1,10],[10,9,11,1],[8,11,12,7],[7,1,2,4],$$
$$[11,8,3,14],[14,12,9,3],[3,1,11,9],[4,14,3,8],[10,5,4,7],[1,7,12,11],[5,10,1,14],$$
$$[1,3,5,2],[9,5,14,11].$$

We then increase our coset stabilizer to account for the equally named cosets. The permutation that we achieve this with are:

$$(1,8)(2,7)(3,11)(4,10)(5,12)(9,14),$$
$$(1,7,3)(2,4,5)(8,14,10)(9,11,12),$$
$$(1,2,5,14)(3,11,10,4)(6,13)(7,8,9,12).$$

We must determine the coset stabilizer, denoted $N^{(7,1,2,4)}$. We look at permutations in $N = L_3(2)$ and find those that "fix" the the elements 7, 1, 2and 4 while permuting all others. We determine the coset stabilizer by utilizing our relations to increase the stabilizer:

$$N^{(7,1,2,4)} =< (1\bar{1})(27)(3\bar{4})(4\bar{3}),(5\bar{5})(\bar{2}\bar{7}),(173)(245),(\bar{1}\bar{7}\bar{3})(\bar{2}\bar{4}\bar{5}),(125\bar{7})(3\bar{4}\bar{3}4)(6\bar{6})(7\bar{1}\bar{2}\bar{5}) >$$

.

We determine $|N^{(7,1,2,4)}| = 24$. We now calculate the number of cosets within $[7,1,2,4]$ by using our equation:

$$|Nt_7t_1t_2t_4N| = \frac{|N|}{|N^{(7,1,2,4)}|} \tag{5.6}$$

which gives us

$$|Nt_7t_1t_2t_4N| = \frac{168}{24} = 7.$$

We now determine the orbits on $[7,1,2,4]$, which are

$$\{1,2,3,4,5,7,\bar{1},\bar{2},\bar{3},\bar{4},\bar{5},\bar{7}\} \text{ and } \{6,13\}$$

We will take a representative t_i from each of these orbits and apply right hand multiplication to the coset $Nt_7t_1t_2t_4$:

1. $Nt_7t_1t_2t_4t_4 = Nt_7t_1t_2(t_4)^2 = Nt_7t_1t_2 \in Nt_7t_1t_2N$ denoted $[7,1,2]$, so this orbit sends 12 generators back to the double coset $[7,1,2]$.

2. $Nt_7t_1t_2t_4t_6 = Nt_7t_1t_2t_4t_6 \in NNt_7t_1t_2t_4t_6N$ denoted $[7,1,2,4,6]$, so this orbit extends 2 generators to a new double coset $[7,1,2,4,6]$.

$\mathbf{Nt_7t_1t_2t_4t_6N}$

We now consider the double coset $Nt_7t_1t_2t_4t_6$. There are 168 cosets within $[7,1,2,4,6]$ having equal names. Utilizing MAGMA, we obtain the permutation that increase the order of our coset stabilizer to 168:

$$(1,6,10)(3,8,13)(4,7,12)(5,11,14),\ (1,2,6)(4,7,5)(8,9,13)(11,14,12),$$
$$(1,3,11,14,6,12,2)(4,7,13,5,9,8,10).$$

We must determine the coset stabilizer, denoted $N^{(7,1,2,4,6)}$. We look at permutations in $N = L_3(2)$ and find those that "fix" the the elements 7, 1, 2, 4 and 6 while permuting all others. We determine the coset stabilizer by utilizing our relations to increase the stabilizer:

$$N^{(7,1,2,4,6)} =< (16\bar{3})(3\bar{1}\bar{6})(47\bar{5})(5\bar{4}\bar{7}), (126)(475), (\bar{1}\bar{2}\bar{6})(\bar{4}\bar{7}\bar{5}), (13\bar{4}\bar{7}6\bar{5}2)(47\bar{6}5\bar{2}\bar{1}\bar{3}) > .$$

We determine $|N^{(7,1,2,4,6)}| = 168$. We now calculate the number of cosets within $[7,1,2,4,6]$ by using our equation:

$$|Nt_7t_1t_2t_4t_6N| = \frac{|N|}{|N^{(7,1,2,4,6)}|} \tag{5.7}$$

which gives us

$$|Nt_7t_1t_2t_4t_6N| = \frac{168}{168} = 1.$$

We have 1 coset in $[7,1,2,4,6]$. We now determine the orbits on $[7,1,2,4,6]$, which are

$$\{1,2,3,4,5,6,7,\bar{1},\bar{2},\bar{3},\bar{4},\bar{5},\bar{6},\bar{7}\}$$

Thus, all the generators are within a single orbit. We will take a representative t_i from this orbit and apply right hand multiplication to the coset $Nt_7t_1t_2t_4t_6$:

1. $Nt_7t_1t_2t_4t_6t_6 = Nt_7t_1t_2t_4(t_6)^2 = Nt_7t_1t_2t_4 \in Nt_7t_1t_2t_4N$ denoted $[7,1,2,4]$, so this orbit sends all 14 generators back to the double coset $[7,1,2,4]$.

Since we have no orbits extending generators to new double cosets, this group is closed under right hand multiplication. Thus, we have completed the double coset enumeration process for $U_3(3):3$ as a homomorphic image of $2^{*14}:L_3(2)$. The results are submarized in the following cayley diagram Figure 5.1:

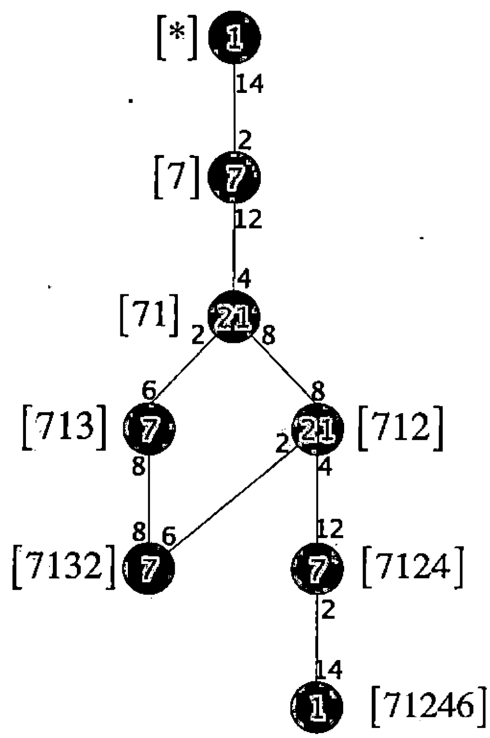

Figure 5.1: Cayley diagram of $U_3(3):3$ as a Homomorphic Image of $2^{*14}:L_3(2)$

Chapter 6

Construction of A_7 as a Homomorphic Image of the Monomial Progenitor $3^{*7} :_m L_3(2)$

We have a computer-based proof that

$$\frac{3^{*7} :_m L_3(2)}{t_3 t_2 = t_7 t_1} \xrightarrow{homo} A_7$$

This proof is obtained as follows: We first us MAGMA to obtain the composition factors of a permutation representation of G. This is done as follows:

```
f,G1,k:=CosetAction(G,sub<G|x,y>);
CompositionFactors(G1);
```

give

```
G
| Alternating(7)
1
```

We now write a presentation of the group A_7 as a Homomorphic Image of the Progenitor $3^{*7} :_m L_3(2)$ (obtained based on the composition factors above) and verify that G is isomorphic to A_7 as a Homomorphic Image of the Progenitor $3^{*7} :_m L_3(2)$.

$$A_7 \xrightarrow{homo} \frac{3^{*7}:L_3(2)}{t_3t_2=\bar{t}_7\bar{t}_1}$$

I will perform a double coset enumeration on the group $G \xrightarrow{homo} \frac{3^{*7}:L_3(2)}{t_3t_2=\bar{t}_7\bar{t}_1}$, denoted by the following group representation:

$$G \cong < x, y, t | x^7, y^2, (xy)^3, (x,y)^4, t^3, (t, x^{-3}yx^2), (t, y), t^x t^{xy}, (xt)^4 >$$

We have a computer-based proof that $G \cong A_7$.

where $|G| = 2520$ and $N = < x, y > \cong L_3(2)$.
The generators are represented by:

$$x \backsim (1,2,3,4,5,6,7)(\bar{1},\bar{2},\bar{3},\bar{4},\bar{5},\bar{6},\bar{7})$$

and

$$y \sim (1,\bar{1})(3,\bar{7})(2,\bar{6})(4,5)(\bar{4},\bar{5})(\bar{2},6).$$

We know $N \simeq L_3(2)$ has 168 elements , or $|N| = 168$.

6.1 Relations

The first relation we must expand is

$$(xt)^4 = 1$$

Let $\pi = x$, then our relation becomes

$$(\pi t)^4 = 1$$

We then expand our relation, giving us

$$\pi^4 t^{\pi^3} t^{\pi^2} t^{\pi} t = 1.$$

Note in this particular group, $t \sim t_7$ and $t_7{}^{-1} \sim \bar{t}_7$. This transforms our relation to

$$\pi^4 t_7^{\pi^3} t_7^{\pi^2} t_7^{\pi} t_7 = 1.$$

Now, we consider our permutation x which we have transformed into π which becomes

$$\pi = x \sim (1234567)(\bar{1}\bar{2}\bar{3}\bar{4}\bar{5}\bar{6}\bar{7}).$$

We then apply our permutation π to our relation which gives us our permutations

$$\pi^2 = (1357246)(\bar{1}\bar{3}\bar{5}\bar{7}\bar{2}\bar{4}\bar{6}),$$

$$\pi^3 = (1473625)(\bar{1}\bar{4}\bar{7}\bar{3}\bar{6}\bar{2}\bar{5}),$$

$$\pi^4 = (1526374)(\bar{1}\bar{5}\bar{2}\bar{6}\bar{3}\bar{7}\bar{4}).$$

We then convert our permutations back into $t_i's$ to get our relation

$$\pi^4 t_3 t_2 t_1 t_7 = 1$$

Utilizing right multiplication of our $t_i's$, we have a relation based on two letters

$$\pi^4 t_3 t_2 = \bar{t_7}\bar{t_1}.$$

We will use this relation to evaluate cosets and double cosets within our group.

6.2 Double Coset Enumeration

NeN

We start our double coset enumeration by evaluating our first double coset, denoted [*], containing words of length zero. This double coset has one single coset, which is the identity $NeN = N$. Since our presentation group is A_7, we have $t = t_7$ and ${t_7}^{-1} \sim \bar{t}_7$. This means our first orbit contains all fourteen generators

$$\{1, 2, 3, 4, 5, 6, 7, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}\}.$$

When we apply a representative t_i from this orbit, say t_7 to the coset representative Ne to get a new coset Nt_7. We see that all fourteen generators extend to a new double cosetNt_7N, denoted [7]. This double cosets will be made up of words of length one.

Nt$_7$N

We now consider the double coset $Nt_7N = \{Nt_7^n | n \in N\}$.We must first determine the coset stabilizer, denoted $N^{(7)}$. We look at permutations in $N = L_3(2)$ and find those that "fix" the the element (7) and permute all others. Using Magma, we found

$$N^7 =< (1\bar{1})(2\bar{6})(3\bar{3})(45)(6\bar{2})(\bar{4}\bar{5}), (16\bar{5})(2\bar{4}3)(4\bar{3}\bar{2})(5\bar{1}\bar{6}) > .$$

Since there are no additional relations, our point stabilizer *is* our coset stabilizer. Thus we have

$$N^7 = N^{(7)}.$$

We note that

$$|N^7| = |N^{(7)}| = 2^2 \cdot 3 = 12$$

We now determine the number of cosets in [7] by using our equation

$$|Nt_7N| = \frac{|N|}{|N^{(7)}|} \tag{6.1}$$

whic gives us

$$|Nt_7N| = \frac{168}{12} = 14.$$

Now we compute orbits of $N^{(7)}$ on $\{1,2,3,4,5,6,7,\bar{1},\bar{2},\bar{3},\bar{4},\bar{5},\bar{6},\bar{7}\}$ by conjugating elements in Nt_7N by the coset stabilizer $N^{(7)}$.

1. $7^{N^{(7)}} = \{7\}$

2. $\bar{7}^{N^{(7)}} = \{\bar{7}\}$

To determine the next orbit, we assign variables to the generators of $N = L_3(2)$:
Let $A = (1\bar{1})(2\bar{6})(3\bar{3})(45)(6\bar{2})(\bar{4}\bar{5})$ be the first permutation and let $B = (16\bar{5})(2\bar{4}3)(4\bar{3}\bar{2})(5\bar{1}\bar{6})$ be the second permutation.
Next, we multiply and conjugate the remaining elements by A and B to construct our orbit:

$$1^A = \{\bar{1}\}$$

$$1^{A^2} = \{1\}$$

$$1^B = \{6\}$$

$$1^{B^2} = \{\bar{5}\}$$

$$1^{B^A} = \{2\}$$

Now all the above generators are in the same orbit as 1. Since 1 and $\bar{1}$ share the same cycle within A, any generator within a cycle containing 1 or $\bar{1}$ will be in the same orbit. Similarly, any generator sharing the same cycle within B will also be in the same orbit. Having said that, we can finish the construction of this orbit.

$$1^{(B^A)^2} = \{4\}$$

$$1^{A^B} = \{3, \bar{3}\}$$
$$\bar{1}^B = \{\bar{6}\}$$
$$\bar{1}^{B^2} = \{\bar{5}\}$$
$$\bar{1}^{B^A} = \{\bar{2}\}$$
$$\bar{1}^{(B^A)^2} = \{\bar{4}\}$$

Thus, we have all the above elements in the final orbit:

$$\{1, 2, 3, 4, 5, 6, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}\}$$

Now we have three orbits for Nt_7N:

1. $\{7\}$
2. $\{\bar{7}\}$
3. $\{1, 2, 3, 4, 5, 6, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}\}$

We will examine orbits 1 and 2: We will take a representative t_i from each of these two orbits and apply right hand multiplication to the coset Nt_7:

$$Nt_7 \cdot t_7 = Nt_7^2 = N\bar{t}_7 \in Nt_7N$$

So this orbit takes one generator back to the same double coset Nt_7N denoted [7].

$$Nt_7 \cdot \bar{t}_7 = Ne \in NeN$$

So this orbit takes one generator back to the double coset NeN denoted [*].

$$Nt_7 \cdot t_1 = Nt_7t_1 \in Nt_7t_1N$$

At first glance, one would assume that this orbit extends the twelve generators to a new double coset Nt_7t_1N, but we must remember that the cosets $Nt_1 \in Nt_7N$ and $Nt_7 \in Nt_7N$. This implies that $Nt_1 = Nt_7$. By substitution, we have

$$Nt_7 \cdot t_1 = Nt_7 \cdot t_7 = Nt_7^2 = N\bar{t}_7 \in Nt_7N$$

so the third orbit also takes the twelve generators back to the same double coset Nt_7N denoted [7]. Since we have no orbits extending generators to new double cosets, this group is closed under right hand multiplication. Thus, we have completed the double coset enumeration process for A_7 as a homomorphic image of the progenitor $3^{*7} :_m L_3(2)$. The results are submarized in the following cayley diagram Figure 6.1:

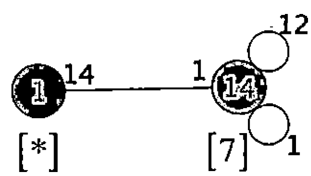

Figure 6.1: Cayley diagram of A_7 as a Homomorphic Image of $3^{*7} : L_3(2)$

Chapter 7

Finite Homomorphic Images of the Monomial Progenitor $7^{*2} :_m S_3$

7.1 An Irreducible Monomial Representation of S_3

We define the **monomial representation** of a group G as a homomorphism from G into $GL_n(F)$, the group of non-singular $n \times n$ matrices over the field F, in which the image of every element of G is a monomial matrix over F.

We define a **monomial matrix** as follows: An $n \times n$ matrix $M = [m_{ij}]$ over a field K is **monomial** if there is $\alpha \in S_n$ and (not necessarily distinct) nonzero elements $x_1, ..., x_n \in K$ such that

$$y = \begin{cases} x_i & \text{if } j = \alpha(i), \\ 0 & \text{otherwise.} \end{cases}$$

Monomial matrices thus have only one nonzero entry in any row or column. Of course, a monomial matrix in which each $x_i = 1$ is a permutation matrix over K.

We say G_χ is a **monomial character** of G if $\chi = \lambda^G$, where λ is a linear character of a subgroup (not necessarily proper) of G. Note: For a linear character $\lambda, \lambda(1) = 1$.

Induced linear characters of H become monomial characters of G. All linear characters of G are monomial, therefore a single entry in the monomial matrix.

To induce a progenitor from another group, we must utilize their respective character tables. Tables 7.1 and 7.2 are the character tables for both groups:

Table 7.1: Character Table for S_3

χ	C_1	C_2	C_3
$\chi^{(1)}$	1	1	1
$\chi^{(2)}$	1	-1	1
$\chi^{(3)}$	2	0	-1

Table 7.2: Character Table for Z_3

χ	C_1	C_2	C_3
$\chi^{(1)}$	1	(123)	$(123)^2 = (132)$
$\chi^{(2)}$	1	w	w^2
$\chi^{(3)}$	1	w^2	w

In this example, we will induce from the third character of the Z_3 table and write the permutations that generate S_3 in matrix form. We have:

$$S_3 =< (123)(12) >,$$

where the permutations (123) and (12) are represented respectively in this manner:

$$xx \ (123)$$
$$yy \ (12)$$

We find the right transversals through magma, which are:

$$e, (12)$$

Since the right transversal contains two elements, we a have a 2×2 matrix. We want the four possible entries based on xx using our right transversals $e, (12)$.

$$A(xx) = \begin{pmatrix} exxe^{-1} & exx(12) \\ (12)xxe^{-1} & (12)xx(12)^{-1} \end{pmatrix}$$

Now we substitute $xx = (123)$ and evaluate the four matrix entries by multiplying the permutations and comparing to the elements in $A_3 = \{e, (123), (132)\}$:

$$A(xx) = \begin{pmatrix} e(123)e^{-1} & e(123)(12) \\ (12)(123)e^{-1} & (12)(123)(12)^{-1} \end{pmatrix}$$

which gives rise to the following matrix:

$$A(xx) = \begin{pmatrix} (123) \in A_3 & (13) \notin A_3 \\ (13) \notin A_3 & (213) = (132) = (123)^2 \in A_3 \end{pmatrix}$$

These entries must be in A_3, else the entries $= 0$. Then we have the resulting matrix:

$$A(xx) = \begin{pmatrix} (123) & 0 \\ 0 & (123)^2 \end{pmatrix}$$

Now we will substitute the entries from the third row of the character table Z_3 (see Table 7.2) to find the final matrix configuration denoted A:

$$A = \begin{pmatrix} w^2 & 0 \\ 0 & w \end{pmatrix}$$

. We must now determine the second matrix, denoted B which pertains to the permutation $yy = (12)$. Using the same right transversals and matrix template, we have:

$$B(yy) = \begin{pmatrix} eyye^{-1} & eyy(12) \\ (12)yye^{-1} & (12)yy(12)^{-1} \end{pmatrix}.$$

Now we substitute $yy = (12)$ and evaluate the four matrix entries by multiplying the permutations and comparing to the elements in $A_3 = \{e, (123), (132)\}$:

$$B(yy) = \begin{pmatrix} e(12)e^{-1} & e(12)(12) \\ (12)(12)e^{-1} & (12)(12)(12)^{-1} \end{pmatrix}$$

which gives rise to the following matrix:

$$B(yy) = \begin{pmatrix} (12) \notin A_3 & (e) \in A_3 \\ (e) \in A_3 & (12) \notin A_3 \end{pmatrix}$$

These entries must be in A_3, else the entries $= 0$. By using the same steps as in determmining $A(xx)$, we have the resulting matrix, denoted B:

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

So we now have the following matrices A and B:

$$A = \begin{pmatrix} w^2 & 0 \\ 0 & w \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

We must now verify that we have a monomial representation of S_3 by checking the order of A, B, and their product, $A \cdot B$. From MAGMA, we have: $|A| = 3, |B| = 2, |A \cdot B| = 2$. Thus, A, B, given a monomial representation of S_3.

Now, w is a cube root of unity. We must find the smallest finite field with cube roots of unity. In other words, a finite field which has elements of order 3 in its multiplicative group. Since the matrices have cube roots of unity, we replace these by the cube root of 1 in the smallest field with cube roots of unity. Consider the field

$$Z_7 = \{0, 1, 2, 3, 4, 5, 6\}.$$

We take a group $H = \{1, 2, 3, 4, 5, 6\}$, which is a group under multiplication modulo seven. Then:

$$|2| = 3, \text{ (since } 2^3 = 8 = 1(mod)7)$$
$$|4| = 3, \text{ (since } 4^3 = 64 = 1(mod)7)$$

So we let $w = 2$, and $w^2 = 4$. Now, we substitute those values into the matrix A to generate out $t_i's$, which will generate our permutations for xx and yy.

Since we have 2×2 matrices over a field of seven elements, we will have $2t_i's$ of order 7. Thus, our progenitor will will be expressed as:

$$7^{*2} :_m S_3.$$

The m typifies this as a monomial presentation. We use our matrix A to generate permutations where each t_i goes:

$$A(x) = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} t_1 & t_2 \end{array} \\ \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \end{array}$$

For the A matrix, we see that entry $t_1 = 4$ and $t_2 = 2$, which implies:

$$t_1 \to t_1^4$$
$$t_2^1 \to t_2^2.$$

We now determine the action on the remaining $t_i's$ by multiplying the exponents of all the $t_1's$ by 4, and the exponents of all the $t_2's$ by 2. Note that we evaluate each new exponent by modulo 7. We now set up our $t_i's$ based on two sets of six elements (six elements for t_1 and six elements for t_2) = 12 letters:

Table 7.3: Labeling for Matrix $A(xx)$ to Determine $t_i's$

1	2	3	4	5	6	7	8	9	10	11	12
t_1	${t_1}^2$	${t_1}^3$	${t_1}^4$	${t_1}^5$	${t_1}^6$	t_2	${t_2}^2$	${t_2}^3$	${t_2}^4$	${t_2}^5$	${t_2}^6$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
${t_1}^4$	${t_1}^1$	${t_1}^5$	${t_1}^2$	${t_1}^6$	${t_1}^3$	${t_2}^2$	${t_2}^4$	${t_2}^6$	${t_2}^1$	${t_2}^3$	${t_2}^5$

The results of Table 7.3 give rise to our permutation

$$xx = (1,4,2)(3,5,6)(7,8,10)(9,12,11)$$

We use a similar method for determining the permuation for yy: We examine the matrix for $B(yy)$:

$$B(y) = \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} t_1 & t_2 \end{array} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array}$$

Since the B matrix has $1's$ for all entries for t_1 and t_2, we see that

$$t_1 \to t_2$$
$$t_2 \to t_1$$

for each respective exponent. As with the A matrix, we now set up our labeling table based on the action of the B matrix:

Table 7.4: Labeling for Matrix $B(yy)$ to Determine $t_i's$

1	2	3	4	5	6	7	8	9	10	11	12
t_1	${t_1}^2$	${t_1}^3$	${t_1}^4$	${t_1}^5$	${t_1}^6$	t_2	${t_2}^2$	${t_2}^3$	${t_2}^4$	${t_2}^5$	${t_2}^6$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
${t_2}^1$	${t_2}^2$	${t_2}^3$	${t_2}^4$	${t_2}^5$	${t_2}^6$	${t_1}^1$	${t_1}^2$	${t_1}^3$	${t_1}^4$	${t_1}^5$	${t_1}^6$

The results of Table 7.4 give rise to our permutation

$$yy = (1,7)(2,8)(3,9)(4,10)(5,11)(6,12)$$

We are now ready to write our progenito S_3, which is represented by the progenitor

$$S_3 =< x^3, y^2, (xy)^2 >.$$

Since our matrices are over Z_7, we add t^7 to our progenitor which gives us:

$$7^{*2} :_m S_3 =< x^3, y^2, (xy)^2, t^7 >$$

We must now find the normaliser for t in N (or, the permutation that takes the set $< t >$ back to itself). Then we must find the relations which commute with t. To find the normaliser for t we must first assign t to one of the $t_i's$. We let $t = t_1$. So by inspection, we want to determine the permutation cycles that keeps t_7 and its powers together, or fixes the $t_i's$. We find

$$xx = (1, 4, 2)(3, 5, 6)(7, 8, 10)(9, 12, 11), e, xx^{-1}$$

satisfies that condition. First we must determine the action within the 3-cycle permutation that contains t_1^1. We see that if we permute t_1 by xx, we get t_4. Note also that if we permute t_1 by $(xx)^{-1}$, we get t_2. Thus, these two actions represent the normaliser of t. Thus far, we have a progenitor:

$$7^{*2} :_m S_3 =< x^3, y^2, (xy)^2, t^7, t^x = t^4, t^{x^{-1}} = t^2 >.$$

We need to find relations that represent what t commutes with. From MAGMA, we have the relation

$$(xty)^3 = 1$$

So we add this relation to our progenitor to get the following:

$$7^{*2} :_m S_3 =< x^3, y^2, (xy)^2, t^7, t^x = t^4, t^{x^{-1}} = t^2, (xty)^3 = 1 >.$$

Having completed this monomial progenitor, we can now look at its composition factors. MAGMA tells us that the composition factors of this progenitor is:

$$A(1, 7) = L(2, 7)$$

which is a computer-based proof that we verify by constructing the following group.

7.2 The Construction of $7^{*2} :_m S_3$: The Relations

We will now construct our new progenitor

$$7^{*2} :_m S_3 =< x^3, y^2, (xy)^2, t^7, t^x = t^4, t^{x^{-1}} = t^2, (xty)^3 = 1 >.$$

by first expanding our relation

$$(xty)^3 = 1$$
$$\Longrightarrow (xty)(xty)(xty) = 1$$

We now use the identity principle

$$\pi\pi^{-1} = 1$$

and the property of conjugation

$$\pi^{-1}t\pi = 1$$

to expand our relation, which gives us

$$(xy)^3 t^{(xy)^2} t^{(xy)} t = 1.$$

Since we have the relation

$$(xy)^2 = 1$$

we now have

$$(xy)tt^{(xy)}t = 1.$$

We now examine the permutations x and y to further define our $t_i's$. Recall that we let $t = t_1$. We need to determine what the permutation (xy) does to 1, then assign the results to the corresponding t_i. The permutation (xy) takes 1 to 10. Therefore, our relation becomes

$$f(xy)t_1 t_{10} t_1 = 1.$$

We have two generators, each of order seven. Therefore

$$t_i{}^7 = 1$$
$$\Rightarrow t_i{}^1 = t_i{}^{-6}.$$

Using right hand multiplication, we can determine other relations:

$$f(xy)t_1t_{10}t_1 = 1$$
$$\Rightarrow f(xy)t_1t_{10}t_1 \cdot {t_1}^{-1} = 1 \cdot {t_1}^{-1}$$
$$\Rightarrow f(xy)t_1t_{10} = {t_1}^{-1}$$

We will expand on these relations to determine other relations as needed when we construct and perform double coset enumeration on this group.

7.3 Double Coset Enumeration

We have a computer-based proof that G is isomorphic to

$$7^{*2} :_m S_3$$

. This proof is obtained as follows: We first us MAGMA to obtain the composition factors of a permutation representation of G. This is done as follows:

```
f,G1,k:=CosetAction(G,sub<G|x,y>);
CompositionFactors(G1);
   G
    |  A(1, 7)                    = L(2, 7)
```

We now write a presentation of the group $7^{*2} :_m S_3$ (obtained based on the composition factors above) and verify that G is isomorphic to $7^{*2} :_m S_3$.

We will perform a double coset enumeration on the group $7^{*2} :_m S_3$ factored by the relation $t_1t_{10}t_1 = 1$, denoted by the following group representation:

$$7^{*2} :_m S_3 =< x^3, y^2, (xy)^2, t^7, t^x = t^4, t^{x^{-1}} = t^2, (xty)^3 = 1 >.$$

where $N =< x, y >\simeq S_3$, $x \backsim (1,2,3)$ and $y \sim (1,2)$. We know $N \simeq S_3$ has 6 elements , or$|N| = 6$.

Consider the following notation of our $t_i's$:

$$t_1 = f(t), t_2 = {t_1}^2, t_3 = {t_1}^3, t_4 = {t_1}^4, t_5 = {t_1}^5, t_6 = {t_1}^6$$
$$t_2 = f(t^y), t_8 = {t_2}^2, t_9 = {t_2}^3, t_{10} = {t_2}^4, t_11 = {t_2}^5, t_12 = {t_2}^6$$

which gives rise to:

$$t_1, \quad t_2 = {t_1}^2, \quad t_3 = {t_1}^3, \quad t_4 = {t_3}^{-1} = {t_1}^4, \quad t_5 = {t_2}^{-1} = {t_1}^5, \quad t_6 = {t_1}^{-1} = {t_1}^6$$
$$t_8 = {t_2}^2, \quad t_9 = {t_2}^3, \quad t_{10} = {t_9}^{-1} = {t_2}^4, \quad t_{11} = {t_8}^{-1} = {t_2}^5, \quad t_{12} = {t_7}^{-1} = {t_2}^6$$

Based on the above notation,

$$S_3 =< (1, 1^4, 2)(1^6, 1^3, 1^5)(1^7, 2^2, 2^4)(2^6, 2^5, 2^3) >.$$

We conjugate our relation $t_1 t_2^4 t_1 = e \Rightarrow 12^4 1 = e$ by the elements in S_3 to obtain our remaining relations:

$$11^5 1 \sim 11^4 1 \sim 12^6 1 \sim 12^5 1 \sim 12^4 1$$
$$\sim 1^2 1^6 1^2 \sim 1^2 1^4 1^2 \sim 1^2 2^6 1^2 \sim 1^2 2^5 1^2 \sim 1^2 2^4 1^2$$
$$\sim 1^3 1^6 1^3 \sim 1^3 1^5 1^3 \sim 1^3 2^6 1^3 \sim 1^3 2^5 1^3 \sim 1^3 2^4 1^3$$
$$\sim 1^6 2 1^6 \sim 1^6 1^3 1^6 \sim 1^6 2 1^6 \sim 1^6 2^2 1^6 \sim 1^6 2^3 1^6$$
$$\sim 1^5 1 1^5 \sim 1^5 1^3 1^5 \sim 1^5 2 1^5 \sim 1^5 2^2 1^5 \sim 1^5 2^3 1^5$$
$$\sim 1^4 1 1^4 \sim 1^4 1^2 1^4 \sim 1^4 2 1^4 \sim 1^4 2^2 1^4 \sim 1^4 2^3 1^4$$
$$\sim 2^6 1 2^6 \sim 2^6 2 2^6 \sim 2^6 1^3 2^6 \sim 2^6 2^2 2^6 \sim 2^6 2^3 2^6$$
$$\sim 21^6 2 \sim 21^5 2 \sim 21^4 2 \sim 22^5 2 \sim 22^4 2$$
$$\sim 2^2 1^6 2^2 \sim 2^2 1^5 2^2 \sim 2^2 1^4 2^2 \sim 2^2 2^6 2^2 \sim 2^2 2^4 2^2$$
$$\sim 2^3 1^6 2^3 \sim 2^3 1^5 2^3 \sim 2^3 1^4 2^3 \sim 2^3 2^6 2^3 \sim 2^3 2^5 2^3$$
$$\sim 2^5 1 2^5 \sim 2^5 1^2 2^5 \sim 2^5 1^3 2^5 \sim 2^5 2 2^5 \sim 2^5 2^3 2^5$$
$$\sim 2^4 1 2^4 \sim 2^4 1^2 2^4 \sim 2^4 1^3 2^4 \sim 2^4 2 2^4 \sim 2^4 2^2 2^4$$

MAGMA confirms these relations, which we will use to find equal double cosets with words of length two and greater within this group.

NeN

NeN is a double coset made up of words of length zero. We know $NeN = \{N\}$, which is the first double coset [*]. The coset representative for [*] is N. The number of cosets in [*] is 1. We find that the orbits of N on $\{1, 1^2, 1^3, 1^4, 1^5, 1^6, 2, 2^2, 2^3, 2^4, 2^5, 2^6\}$ are $\{1, 1^2, 1^4, 2, 2^2, 2^4\}$ and $\{1^3, 1^5, 1^6, 2^3, 2^5, 2^6\}$. When we apply a representative t_i from each orbit to the double coset representative N we see that the elements in orbit $\{1, 1^2, 1^4, 2, 2^3, 2^4\}$ extend to a new double coset Nt_1N, denoted [1], and the elements in the orbit $\{1^3, 1^5, 1^6, 2^3, 2^5, 2^6\}$ extend to another new double coset Nt_1^6N, denoted [1^6]. These double cosets will be made up of words of length one.

$\mathbf{Nt_1N}$

We now will determine the number of single coset in the double coset [1] by this formula $\frac{|N|}{|N^{(1)}|}$ which gives us $\frac{6}{1} = 6$. The coset representative for [1] is Nt_1. We now identify the orbits of $N^{(1)}$ and determine where they go. We see that the orbits of N on $\{1, 2, 1^4, 2, 2^2, 2^4\}$ are$\{1,1^3\}$,$\{1^6\}$,$\{1^2,1^4,1^5,2^4\}$,$\{2,2^3\}$, $\{2^2\}$,$\{2^5\}$ and$\{2^6\}$. When we apply a representative t_i from each orbit to the coset representative Nt_1 we see the following results:

1. $Nt_1t_1 = N(t_1)^2 = Nt_2 \in Nt_1N$, so this orbit sends 2 joins back to the same double coset Nt_1N, denoted [1].

2. $Nt_1t_1^5 = Nt_1{}^{-1} = Nt_1{}^6 \in Nt_1{}^6N$ which means this orbit sends 4 joins to a new double coset $Nt_1{}^6N$, denoted [1^6].

3. $Nt_1t_1^6 = NeN$, so this orbit sends 1 join back to the doublecoset [*].

4. $Nt_1t_2 \in Nt_1t_2N$, which means this orbit extends 2 joins to a new double coset Nt_1t_2N, denoted [1, 2].

5. $Nt_1t_2{}^2 \in Nt_1t_2{}^2N$, which means this orbit extends 1 join to a new double coset $Nt_1t_2{}^2N$, denoted [$1, 2^2$].

6. $Nt_1t_2{}^5 \in Nt_1t_2{}^5N$, which means this orbit extends 1 join to a new double coset $Nt_1t_2{}^5N$, denoted [$1, 2^5$].

7. $Nt_1t_2{}^6 \in Nt_1t_2{}^6N$, which means this orbit extends 1 join to a new double coset $Nt_1t_2{}^6N$, denoted $[1, 2^6]$.

$\mathbf{Nt_1{}^6N}$

We now will determine the number of single coset in the double coset $[1^6]$ by this formula $\frac{|N|}{|N^{(6)}|}$ which gives us $\frac{6}{1} = 6$. The coset representative for $[1^6]$ is Nt_6. We now identify the orbits of $N^{(6)}$ and determine where they go. We see that the orbits of N on $\{1^3, 1^5, 1^6, 2^3, 2^5, 2^6\}$ are $\{1\}$, $\{1^2, 1^3, 1^5, 2^3\}$, $\{1^4, 1^6\}$,$\{2\}$,$\{2^2\}$,$\{2^4, 2^6\}$, and $\{2^5\}$. When we apply a representative t_i from each orbit to the coset representative $Nt_1{}^6$ we see the following results:

1. $Nt_1{}^6t_1 = NeN$ so this orbit send 1 join back to $[*]$.
2. $Nt_1{}^6t_9$= $Nt_1{}^{-1}t_{10}^{-1} = Nt_1 \in Nt_1N$, so this orbit sends four joins back to the double coset Nt_1N, denoted [1].
3. $Nt_1{}^6t_1{}^6 = N(t_1{}^6)^2 = Nt_1{}^5 \in Nt_1{}^6N$, so this orbit sends two joins back to itself.
4. $Nt_1{}^6t_2{}^6 = Nt_1t_2 \in Nt_1t_2N$, so this orbit sends two joins to the double coset Nt_1t_2N, denoted $[1, 2]$.
5. $Nt_1{}^6t_2{}^5 = Nt_1{}^6t_2{}^5 = Nt_1t_8 \in Nt_1t_8N$, so this orbit sends 1 join to the double coset $Nt_1t_2{}^2N$, denoted $[1, 2^2]$.
6. $Nt_1{}^6t_2 = Nt_1t_2{}^6 \in Nt_1t_2{}^6N$, so this orbit sends 1 join to the double coset $Nt_1t_2{}^5N$, denoted $[1, 2^5]$.
7. $Nt_1{}^6t_8 = Nt_1t_2{}^6 \in Nt_1t_2{}^6N$, so this orbit sends 1 join to the double coset $Nt_1t_2{}^6N$, denoted $[1, 2^6]$.

We have now completed all double cosets with words of length one.

We will now determine the double cosets with words of length two. We will now utilize our relation

$$(xy)t_1t_{10}t_1 = 1$$

to determine the orbit paths for our joins within our double coset enumeration. We will also utilize MAGMA to determine the existence of equal double cosets, which may collapse part of our Cayley diagram by reducing the number of distinct double cosets.

$\mathbf{Nt_1t_2N}$

We now will determine the number of single coset in the double coset $[1, 2]$ by this formula $\frac{|N|}{|N^{(1,2)}|}$ which gives us $\frac{6}{1} = 6$. The coset representative for $[1, 2]$ is Nt_1t_2. We now identify the orbits of $N^(1, 2)$ and determine where they go. We see that the orbits of $N^{(1,2)}$ on $\{3, 1^5, 1^6, 2^3, 2^5, 2^6\}$

are $\{1^4, 2^6\}$, $\{1, 2^3\}$, $\{1^5, 2^2\}$,$\{1^6, 7\}$,$\{2, 2^4\}$ and$\{3, 2^5\}$. When we apply a representative t_i from each orbit to the coset representative Nt_1t_2 we see the following results:

1. $Nt_1t_2t_2{}^6 = Nt_1t_2t_2{}^{-1} = Nt_1 \in Nt_1N$ so this orbit send two joins back to the double coset Nt_1N, denoted [1].

2. $Nt_1t_2t_1 = Nt_1t_2t_6{}^{-1} = Nt_6 \in Nt_6N$ which means this orbit sends four joins back to the double coset Nt_6N, denoted [6].

3. $Nt_1t_2t_5 = Nt_1t_2 \in Nt_1t_2N$. So this orbit sends these two joins back to itself.

4. $Nt_1t_2t_2 = Nt_1t_2{}^2Nt_1t_8 \in Nt_1t_8N$. So this orbit sends these two joins to the double coset Nt_1t_8N, denoted $[1, 2^2]$.

5. $Nt_1t_2t_2{}^4 = Nt_1t_2{}^5 \in Nt_1t_2{}^5N$, which means this orbit sends these two joins to the double coset $Nt_1t_2{}^5N$, denoted $[1, 2^5]$.

6. $Nt_1t_2t_3 = Nt_1t_2{}^6 \in Nt_1t_2{}^6N$, which means this orbit sends these two joins to the double coset $Nt_1t_2{}^6N$, denoted $[1, 2^6]$.

$\mathbf{Nt_1t_2{}^2N}$

We now will determine the number of single coset in the double coset $[1, 2^2]$ by this formula $\frac{|N|}{|N^{(1,2^2)}|}$ which gives us $\frac{6}{2} = 3$. The coset representative for $[1, 2^2]$ is

$Nt_1t_2{}^2$. We now identify the orbits of $N^{(1,2^2)}$ and determine where they go. We see that the orbits of N on $\{3, 1^5, 1^6, 2^3, 2^5, 2^6\}$ are $\{1, 2\}$, $\{1^5, 2^5\}$, $\{2, 2^2\}$, $\{1^6, 2^6\}$,$\{3, 2^3\}$, and $\{1^4, 2^4\}$. When we apply a representative t_i from each orbit to the coset representative $Nt_1t_2{}^2$ we see that the following results:

1. $Nt_1t_2{}^2t_1 = Nt_1t_2{}^2t{=}Nt_1Nt_2{}^2t_2{}^{2^{-1}}t_1t_1^{-1}t_2{}^2 = Nt_1^{-1}t_2{}^2 \in Nt_1t_2N$ so this orbit send two joins back to the double coset Nt_1t_2N, denoted $[1, 2]$.

2. $Nt_1t_2{}^2t_2{}^5 = Nt_1t_2{}^2t_2{}^{2^{-1}} = Nt_1 \in Nt_1N$ so this orbit send two joins back to the double coset Nt_1N, denoted [1].

3. $Nt_1t_2{}^2t_2{}^2{=}\ Nt_1t_2{}^{2^2} = Nt_1{}^6 \in Nt_1{}^6N$ which means this orbit sends two joins back to the double coset $Nt_1{}^6N$, denoted $[1^6]$.

4. $Nt_1t_2{}^2t_2{}^6 = Nt_1t_2^2 = Nt_1t_2 \in Nt_1t_2N$. So this orbit sends these two joins to the double coset Nt_1t_2N, denoted $[1, 2]$.

5. $Nt_1t_2{}^2t_2{}^3 = Nt_1t_2{}^5 \in Nt_1t_2^5N$. Our relations indicate this orbit sends these two joins to the double coset $Nt_1t_2{}^5N$, denoted $[1, 2^5]$.

6. $Nt_1t_2{}^2t_2{}^4 = Nt_1t_2{}^6 \in Nt_1t_2{}^6N$, so this orbit sends 2 joins to the double coset $Nt_1t_2^6N$, denoted $[1, 2^6]$.

$\mathbf{Nt_1t_2{}^5N}$

We now will determine the number of single coset in the double coset $[1, 2^5]$ by this formula $\frac{|N|}{|N^{(1,2^5)}|}$ which gives us $\frac{6}{2} = 3$. The coset representative for $[1, 2^5]$ is $Nt_1t_2^5$. We now identify the orbits of $N^{(1,2^5)}$ and determine where they go. We see that the orbits of $N^{(1,2^5)}$ on $\{3, 1^5, 1^6, 2^3, 2^5, 2^6\}$ are $\{1^4, 2^2\}$, $\{1^5, 2^6\}$, $\{1^6, 2^3\}$,$\{1, 2^4\}$, $\{2, 7\}$ and $\{3, 2^5\}$. When we apply a representative t_i from each orbit to the coset representative $Nt_1t_2^5$ we see that the following results:

1. $Nt_1t_2^5t_2{}^2 = Nt_1t_2{}^{2^{-1}}t_2{}^2 = Nt_1 \in Nt_1N$ so this orbit send 2 joins back to the double coset Nt_1N, denoted [1].

2. $Nt_1t_2^5t_1^5 = Nt_1^6 \in Nt_1^6N$ which means this orbit sends 2 joins back to the double coset Nt_1^6N, denoted $[1^6]$.

3. $Nt_1t_2^5{t_2}^3 = Nt_1t_2 \in Nt_1t_2N$. So this orbit sends 2 joins to the double coset Nt_1t_2N, denoted $[1,2]$.

4. $Nt_1t_2^5{t_2}^4 = Nt_1{t_2}^2 \in Nt_1{t_2}^2N$. Our relations indicate this orbit sends 2 joins to the double coset $Nt_1{t_2}^2N$, denoted $[1,2^2]$.

5. $Nt_1t_2^5t_2 = Nt_1t_2^6 \in Nt_1t_2^6N$, so this orbit sends 2 joins to the double coset $Nt_1t_2^6N$, denoted $[1,2^6]$.

6. $Nt_1t_2^5t_2^5 = Nt_1t_2^5t_2^5 = Nt_1t_2^7t_2^3 = Nt_1t_2^3 = Nt_1{t_2}^2 \in Nt_1t_2N$, so this orbit sends 2 joins to the double coset Nt_1t_2N, denoted $[1,2]$.

$\mathbf{Nt_1{t_2}^6N}$

We now will determine the number of single coset in the double coset $[1,2^6]$ by this formula $\frac{|N|}{|N^{(1,2^6)}|}$ which gives us $\frac{6}{2} = 3$. The coset representative for $[1,2^6]$ is $Nt_1t_2^6$. We now identify the orbits of $N^{(1,2^6)}$ and determine where they go. We see that the orbits of $N^{(1,2^6)}$ on $\{3,1^5,1^6,2^3,2^5,2^6\}$ $\{1,2^4\}$, $\{2,7\}$, $\{3,2^5\}$, $\{1^4,2^2,\}$,$\{1^6,2^3\}$, and $\{1^5,2^6\}$. When we apply a representative t_i from each orbit to the coset representative $Nt_1t_2^6$ we see that the following results:

1. $Nt_1{t_2}^6t_1 = Nt_1t_2^6t_1 = Nt_1t_2 \in Nt_1t_2N$ so this orbit send 2 joins back to the double coset Nt_1t_2N, denoted $[1,2]$.

2. $Nt_1{t_2}^6t_2 = Nt_1t_2t_2^{-1} = Nt_1 \in Nt_1N$ so this orbit send 2 joins back to the double coset Nt_1N, denoted $[1]$.

3. $Nt_1{t_2}^6t_2^5 = Nt_6 \in Nt_6N$ which means this orbit sends 2 joins back to the double coset Nt_6N, denoted $[1^6]$.

4. $Nt_1{t_2}^6{t_2}^2 = Nt_1t_2 \in Nt_1t_2N$. So this orbit sends 2 joins to the double coset Nt_1t_2N, denoted $[1,2]$.

5. $Nt_1{t_2}^6{t_2}^3 = Nt_1{t_2}^2 \in Nt_1{t_2}^2N$. Our relations indicate this orbit sends 2 joins to the double coset $Nt_1{t_2}^2N$, denoted $[1,2^2]$.

6. $Nt_1{t_2}^6t_2^6 = Nt_1(t_2^6)^2 = Nt_1t_2^5 \in Nt_1t_2^5N$, so this orbit sends 2 joins to the double coset $Nt_1t_2^5N$, denoted $[1,2^6]$.

This concludes all words of length two. Since we did not extend any of the orbits to new double cosets, this group is closed under right hand multiplication. Thus, our double coset enumeration of $7^{*2} :_m S_3$ is complete.

The results are submarized in the following cayley diagram Figure 7.1:

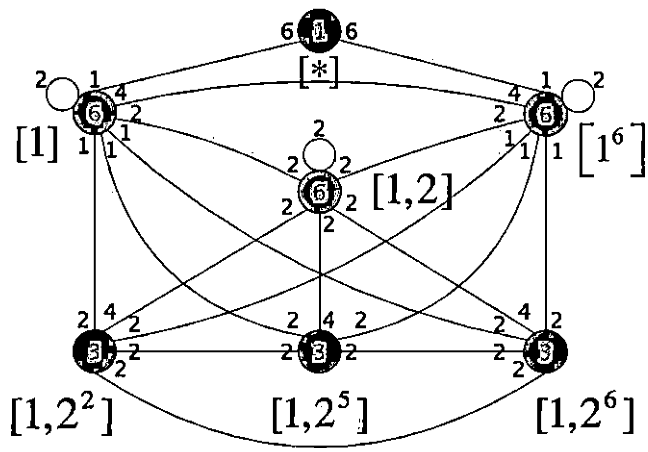

Figure 7.1: Cayley diagram of $7^{*2} :_m S_3$

In Table 7.5, we first label each single coset. We then compute the action of xx, yy and tt_1 to determine $f(x)$, $f(y)$, and $f(t)$:

$$f(x) = (2,4,6)(3,7,9)(5,11,10)(8,15,14)(12,21,23)(13,24,25)$$
$$(16,20,26)(17,27,18)(19,28,22)$$

$$f(y) = (2,5)(3,8)(4,10)(6,11)(7,14)(9,15)(12,17)(13,25)$$
$$(16,26)(18,21)(19,28)(23,27)$$

$$f(t) = (1,2,6,7,4,9,3)(5,12,22,17,8,16,13)$$
$$(10,14,23,26,28,25,18)(11,19,15,24,21,27,20)$$

Table 7.5: Labeling of and Actions on the Single Cosets

Labeling	Single Cosets	xx	yy	tt_1
1	N	1 N	1 N	2 Nt_1
2	Nt_1	4 Nt_1^4	5 Nt_2	6 Nt_1^2
3	Nt_1^6	7 Nt_1^3	8 Nt_2^6	1 N
4	Nt_1^4	6 Nt_1^2	10 Nt_2^4	9 Nt_1^5
5	Nt_2	11 Nt_2^2	2 Nt_1	12 Nt_1t_2
6	Nt_1^2	2 Nt_1	11 Nt_2^2	7 Nt_1^3
7	Nt_1^3	9 Nt_1^5	14 Nt_2^3	4 Nt_1^4
8	Nt_2^6	15 Nt_2^5	3 Nt_1^6	16 $Nt_1^4t_2^3$
9	Nt_1^5	3 Nt_1^6	15 Nt_2^5	3 Nt_1^6
10	Nt_2^4	5 Nt_2	4 Nt_1^4	14 Nt_2^3
11	Nt_2^2	10 Nt_2^4	6 Nt_1^2	19 $Nt_1^4t_2^4$
12	Nt_1t_2	21 $Nt_1^4t_2^2$	17 Nt_2t_1	22 $Nt_1t_2^2$
13	$Nt_1^4t_2^5$	24 $Nt_1^2t_2^3$	25 $Nt_1t_2^6$	5 Nt_2
14	Nt_2^3	8 Nt_2^6	7 Nt_1^3	23 $Nt_1^2t_2^4$
15	Nt_2^5	14 Nt_2^3	9 Nt_1^5	24 $Nt_1\bar{t}_8$
16	$Nt_1^4t_2^3$	20 $Nt_1^2t_2^6$	26 $Nt_1t_2^5$	13 $Nt_1^4t_2^5$
17	Nt_2t_1	27 $Nt_2^2t_1^4$	12 Nt_1t_2	8 Nt_2^6
18	$Nt_2^4t_1^2$	17 Nt_2t_1	21 $Nt_1^4t_2^2$	10 Nt_2^4
19	$Nt_1^4t_2^4$	28 $Nt_2t_1^3$	28 $Nt_2t_1^3$	15 Nt_2^5
20	$Nt_1^2t_2^6$	26 $Nt_1t_2^5$	20 $Nt_1^2t_2^6$	11 Nt_2^2
21	$Nt_1^4t_2^2$	23 $Nt_1^2t_2^4$	18 $Nt_2^4t_1^2$	27 $Nt_2^2t_1^4$
22	$Nt_1t_2^2$	19 $Nt_1^4t_2^4$	22 $Nt_1t_2^2$	17 Nt_2t_1
23	$Nt_1^2t_2^4$	12 Nt_1t_2	27 $Nt_2^2t_1^4$	26 $Nt_1t_2^5$
24	$Nt_1^2t_2^3$	25 $Nt_1t_2^6$	24 $Nt_1^2t_2^3$	21 $Nt_1^4t_2^2$
25	$Nt_1t_2^6$	13 $Nt_1^4t_2^5$	13 $Nt_1^4t_2^5$	18 $Nt_2^4t_1^2$
26	$Nt_1t_2^5$	16 $Nt_1^4t_2^3$	16 $Nt_1^4t_2^3$	28 $Nt_2t_1^3$
27	$Nt_2^2t_1^4$	18 $Nt_2^4t_1^2$	23 $Nt_1^2t_2^4$	20 $Nt_1^2t_2^6$
28	$Nt_2t_1^3$	22 $Nt_1t_2^2$	19 $Nt_1^4t_2^4$	25 $Nt_1t_2^6$

7.4 Additional Finite Homomorphic Images of the Monomial Progenitor $7^{*2} :_m S_3$

We found five other homomorphic images of the monomial progenitor $7^{*2} :_m S_3$, which can be further examined as we have done in the above chapter. From MAGMA, we have the following:

```
a:=0;b:=0;c:=0;d:=4;e:=0;f:=0;g:=0;          //Index = 168 //
G<x,y,t>:=Group<x,y,t|x^3,y^2,(x*y)^2,t^7,t^x=t^4,t^(x^-1)=t^2,
(x*t*t^x*t^(x^2))^a,(x*y*t)^b,(x^2*y*t^x)^c,
(x*y*t^y)^d,(x*t*t^(x^2))^e,
(x*t*t^(x^2))^e, (y*t*t^y*t^x*t^y*t^2*(t^x)^3)^f,(t*t^x)^g>;

G;

Finitely presented group G on 3 generators
Relations
    x^3 = Id(G)
    y^2 = Id(G)
    (x * y)^2 = Id(G)
    t^7 = Id(G)
    t^x = t^4
    x * t * x^-1 = t^2
    (x * t * y)^4 = Id(G)

 #G;

1,008

f,G1,k:=CosetAction(G,sub<G|x,y>);
```

```
CompositionFactors(G1);

G
   |  A(1, 7)                    = L(2, 7)
   *
   |  Cyclic(2)
   *
   |  Cyclic(3)
   1
```

```
a:=0;b:=0;c:=6;d:=7;e:=0;f:=3;g:=0;          //Index = 224 //
G<x,y,t>:=Group<x,y,t|x^3,y^2,(x*y)^2,t^7,t^x=t^4,t^(x^-1)=t^2,
(x*t*t^x*t^(x^2))^a,(x*y*t)^b,(x^2*y*t^x)^c,
(x*y*t^y)^d,(x*t*t^(x^2))^e,
(x*t*t^(x^2))^e, (y*t*t^y*t^x*t^y*t^2*(t^x)^3)^f,(t*t^x)^g>;

G;

Finitely presented group G on 3 generators
Relations
    x^3 = Id(G)
    y^2 = Id(G)
    (x * y)^2 = Id(G)
    t^7 = Id(G)
    t^x = t^4
    x * t * x^-1 = t^2
    (x^2 * y * x^-1 * t * x)^6 = Id(G)
```

```
    (x * t * y)^7 = Id(G)
    (y * t * y^-1 * t * y * x^-1 * t * x * y^-1 * t * y * t^2 * x^-1 * t^3 *
    x)^3 = Id(G)

#G;

1,344

f,G1,k:=CosetAction(G,sub<G|x,y>);

 CompositionFactors(G1);


 G
    |  A(1, 7)                = L(2, 7)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    1

------------------------------------------------------------------------------

a:=0;b:=0;c:=0;d:=0;e:=0;f:=2;g:=0;          //Index = 343 //
G<x,y,t>:=Group<x,y,t|x^3,y^2,(x*y)^2,t^7,t^x=t^4,t^(x^-1)=t^2,
(x*t*t^x*t^(x^2))^a,(x*y*t)^b,(x^2*y*t^x)^c,
(x*y*t^y)^d,(x*t*t^(x^2))^e,
(x*t*t^(x^2))^e, (y*t*t^y*t^x*t^y*t^2*(t^x)^3)^f,(t*t^x)^g>;

G;
```

```
Finitely presented group G on 3 generators
Relations
    x^3 = Id(G)
    y^2 = Id(G)
    (x * y)^2 = Id(G)
    t^7 = Id(G)
    t^x = t^4
    x * t * x^-1 = t^2
    (y * t * y^-1 * t * y * x^-1 * t * x * y^-1 * t * y * t^2 * x^-1 * t^3 *
    x)^2 = Id(G)



#G;

 2,058

f,G1,k:=CosetAction(G,sub<G|x,y>);

 CompositionFactors(G1);


 CompositionFactors(G1);
    G
    |  Cyclic(2)
    *
    |  Cyclic(3)
    *
    |  Cyclic(7)
```

```
*
|  Cyclic(7)
*
|  Cyclic(7)
1
```

```
a:=0;b:=0;c:=0;d:=5;e:=0;f:=4;g:=0;          //Index = 420 //
G<x,y,t>:=Group<x,y,t|x^3,y^2,(x*y)^2,t^7,t^x=t^4,t^(x^-1)=t^2,
(x*t*t^x*t^(x^2))^a,(x*y*t)^b,(x^2*y*t^x)^c,
(x*y*t^y)^d,(x*t*t^(x^2))^e,
(x*t*t^(x^2))^e, (y*t*t^y*t^x*t^y*t^2*(t^x)^3)^f,(t*t^x)^g>;

G;


Finitely presented group G on 3 generators
Relations
    x^3 = Id(G)
    y^2 = Id(G)
    (x * y)^2 = Id(G)
    t^7 = Id(G)
    t^x = t^4
    x * t * x^-1 = t^2
    (x * t * y)^5 = Id(G)
    (y * t * y^-1 * t * y * x^-1 * t * x * y^-1 * t * y * t^2 * x^-1 * t^3 *
    x)^4 = Id(G)
```

```
#G;
 2,520

f,G1,k:=CosetAction(G,sub<G|x,y>);

 CompositionFactors(G1);


 CompositionFactors(G1);
    G
    |  Alternating(7)
    1
```

```
a:=0;b:=0;c:=6;d:=7;e:=0;f:=0;g:=0;          //Index = 1792 //
G<x,y,t>:=Group<x,y,t|x^3,y^2,(x*y)^2,t^7,t^x=t^4,t^(x^-1)=t^2,
(x*t*t^x*t^(x^2))^a,(x*y*t)^b,(x^2*y*t^x)^c,
(x*y*t^y)^d,(x*t*t^(x^2))^e,
(x*t*t^(x^2))^e, (y*t*t^y*t^x*t^y*t^2*(t^x)^3)^f,(t*t^x)^g>;

G;



Finitely presented group G on 3 generators
Relations
    x^3 = Id(G)
    y^2 = Id(G)
    (x * y)^2 = Id(G)
    t^7 = Id(G)
```

```
    t^x = t^4
    x * t * x^-1 = t^2
    (x^2 * y * x^-1 * t * x)^6 = Id(G)
    (x * t * y)^7 = Id(G)



#G;

// 10,752  //


f,G1,k:=CosetAction(G,sub<G|x,y>);

 CompositionFactors(G1);


 CompositionFactors(G1);
    G
    |  A(1, 7)                  = L(2, 7)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
    *
    |  Cyclic(2)
```

```
    1
```

Lastly, we discovered another progenitor $2^5 : S_3$ which gives rise to a Symplectic group $S_4(5)$, as seen in our MAGMA code below:

```
 G<x,y,t>:=Group<x,y,t|x^3 , y^3 , (x*y)^2, t^5, (t,x),

>  (y*t)^5, (x*y*t)^0, (y*t*(t^y)^(x^2))^4>;

> #G;
4680000
> f,G1,k:=CosetAction(G,sub<G|x,y>);
> CompositionFactors(G1);
    G
    |  C(2, 5)                    = S(4, 5)
    1
```

The construction of this presentation gives us $5^3 : S_3$.

Appendix A: MAGMA Code for $\frac{3^{*}:A_3}{t_0t_1=t_1t_0}$.

```
Group 3 ^*3: A3/t0t1=t1t0
N:=Sym(6);
xx:=N!(1,2,3)(4,5,6);
N:=sub<N|xx>;
G<x,t>:=Group<x,t|x^3,t^3,t*t^x=t^x*t>;
Index(G,sub<G|x>);
 f,G1,k:=CosetAction(G,sub<G|x>);
 IN:=sub<G1|f(x)>;
ts:=[Id(G1) : i in [1..6]];
ts[3]:=f(t); ts[1]:=f(t^x); ts[2]:=f(t^(x^2));
ts[4]:=ts[1]^-1; ts[5]:=ts[2]^-1; ts[6]:=ts[3]^-1;
cst := [null : i in [1 .. 27]] where null is [Integers() | ];
prodim := function(pt, Q, I)
/*
Return the image of pt under permutations Q[I] applied sequentially.
 */
 v := pt;
for i in I do
 v := v^(Q[i]);
end for;
return v;
end function;
```

```
CompositionFactors(G1);
    G
    |  Cyclic(3)
    *
    |  Cyclic(3)
    *
    |  Cyclic(3)
    *
    |  Cyclic(3)
    1
```

```
N3:=Stabiliser(N,3);
S:={[3]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for n in IN do
if ts[3] eq n*(ts[(Rep(SSS[i]))[1]])
then  print Rep(SSS[i]);
end if;
end for;
end for;
T3:=Transversal(N,N3);
for i in [1..#T3] do
ss:=[3]^T3[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..27] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N3);
```

```
N6:=Stabiliser(N,6);
S:={[6]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SSS] do
for n in IN do
if ts[6] eq n*(ts[(Rep(SSS[i]))[1]])
then  print Rep(SSS[i]);
end if;
end for;
end for;
T6:=Transversal(N,N6);
for i in [1..#T6] do
ss:=[6]^T6[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0;
for i in [1..27] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N6);
```

```
N31:=Stabiliser(N3,1);
S:={[3,1]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[3]*ts[1]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if; end for; end for;
 T31:=Transversal(N,N31);
```

```
for i in [1..#T31] do
ss:=[3,1]^T31[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..27] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N31);
-----------------------------------------------------------------
N34:=Stabiliser(N,[3,4]);
S:={[3,4]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[3]*ts[4]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if; end for; end for;
T34:=Transversal(N,N34);
for i in [1..#T34] do
ss:=[3,4]^T34[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..27] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N34);
-----------------------------------------------------------------
N32:=Stabiliser(N,[3,2]);
S:={[3,2]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[3]*ts[2]
```

```
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if; end for; end for;
Orbits(N32);
T32:=Transversal(N,N32);
for i in [1..#T32] do
ss:=[3,2]^T32[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..27] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N32);
----------------------------------------------------------------------
N35:=Stabiliser(N,[3,5]);
S:={[3,5]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[3]*ts[5]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if; end for; end for;
Orbits(N35);
T35:=Transversal(N,N35);
for i in [1..#T35] do
ss:=[3,5]^T35[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..27] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N35);
----------------------------------------------------------------------
```

```
N61:=Stabiliser(N,[6,1]);
S:={[6,1]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[6]*ts[1]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if; end for; end for;
T61:=Transversal(N,N61);
for i in [1..#T61] do
ss:=[6,1]^T61[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..27] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N61);
----------------------------------------------------------------------
N64:=Stabiliser(N,[6,4]);
S:={[6,4]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[6]*ts[4]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if; end for; end for;
Orbits(N64);
T64:=Transversal(N,N64);
for i in [1..#T64] do
ss:=[6,4]^T64[i];
cst[prodim(1, ts, ss)] := ss;
```

```
end for;
m:=0; for i in [1..27] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N64);
```

```
N62:=Stabiliser(N,[6,2]);
S:={[6,2]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[6]*ts[2]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if; end for; end for;
T62:=Transversal(N,N62);
for i in [1..#T62] do
ss:=[6,2]^T62[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..27] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N62);
```

```
N65:=Stabiliser(N,[6,5]);
 S:={[6,5]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[6]*ts[5]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if; end for; end for;
```

```
T65:=Transversal(N,N65);
for i in [1..#T65] do
ss:=[6,5]^T65[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..27] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N65);
----------------------------------------------------------------------
N642:=Stabiliser(N,[6,4,2]);
 S:={[6,4,2]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[6]*ts[4]*ts[2]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];
end if; end for; end for;
T642:=Transversal(N,N642);
for i in [1..#T62] do
ss:=[6,4,2]^T642[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..27] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N642);
----------------------------------------------------------------------
N312:=Stabiliser(N,[3,1,2]);
S:={[3,1,2]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
```

```
 for g in IN do if ts[3]*ts[1]*ts[2]
 eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];
end if; end for; end for;
 for n in N do if [3,1,2]^n eq [1,2,3]
then N312:=sub<N|N312,n>; end if; end for;
// Determines equal double cosets //
for n in N do if [3,1,2]^n eq [2,3,1]
then N312:=sub<N|N312,n>; end if; end for;
T312:=Transversal(N,N312);
for i in [1..#T312] do
ss:=[3,1,2]^T312[i];
cst[prodim(1, ts, ss)] := ss;
end for;
 m:=0; for i in [1..27] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N312);
------------------------------------------------------------------------
N342:=Stabiliser(N,[3,4,2]);
 S:={[3,4,2]};
 SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
 for g in IN do if ts[3]*ts[4]*ts[2]
 eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
 then print SSS[i];
end if; end for; end for;
T342:=Transversal(N,N342);
for i in [1..#T342] do
ss:=[3,4,2]^T342[i];
 cst[prodim(1, ts, ss)] := ss;
end for;
```

```
m:=0; for i in [1..27] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N342);
------------------------------------------------------------------------
N3421:=Stabiliser(N,[3,4,2,1]);
S:={[3,4,2,1]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[3]*ts[4]*ts[2]*ts[1]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]*ts[R\
ep(SSS[i])[4]]
 then print SSS[i];
 end if; end for; end for;
T3421:=Transversal(N,N3421);
for i in [1..#T342] do
ss:=[3,4,2,1]^T3421[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..27] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N3421);
------------------------------------------------------------------------
N3424:=Stabiliser(N,[3,4,2,4]);
S:={[3,4,2,4]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
 for g in IN do if ts[3]*ts[4]*ts[2]*ts[4]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]*ts[R\
ep(SSS[i])[4]]
then print SSS[i];
```

```
 end if; end for; end for;
// determines equal cosets  //
for n in N do if [3,4,2,4]^n eq [1,5,3,5]
then N3424:=sub<N|N3424,n>; end if; end for;
for n in N do if [3,4,2,4]^n eq [2,6,1,6]
then N3424:=sub<N|N3424,n>; end if; end for;
T3424:=Transversal(N,N3424);
for i in [1..#T3424] do
ss:=[3,4,2,4]^T3424[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..27] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N3424);
--------------------------------------------------------------------
N3426:=Stabiliser(N,[3,4,2,6]);
 S:={[3,4,2,6]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[3]*ts[4]*ts[2]*ts[6]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]*ts[Rep(SSS[i])[4]]
then print SSS[i];
end if; end for; end for;
T3426:=Transversal(N,N3426);
for i in [1..#T3426] do
ss:=[3,4,2,6]^T3426[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..27] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N3426);
```

```
------------------------------------------------------------------------
N3423:=Stabiliser(N,[3,4,2,3]);
 S:={[3,4,2,3]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do if ts[3]*ts[4]*ts[2]*ts[3]
eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]*ts[Rep(SSS[i])[4]]
then print SSS[i];
end if; end for; end for;
T3423:=Transversal(N,N3423);
for i in [1..#T3423] do
ss:=[3,4,2,3]^T3423[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..27] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N3423);
------------------------------------------------------------------------
N645:=Stabiliser(N,[6,4,5]);
S:={[6,4,5]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
 for g in IN do if ts[6]*ts[4]*ts[5]
 eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]*ts[Rep(SSS[i])[3]]
then print SSS[i];
end if; end for; end for;
// determines equal cosets  //
for n in N do if [6,4,5]^n eq [5,6,4]
then N645:=sub<N|N645,n>; end if; end for;
for n in N do if [6,4,5]^n eq [4,5,6]
```

```
then N645:=sub<N|N645,n>; end if; end for;
T645:=Transversal(N,N645);
for i in [1..#T645] do
ss:=[6,4,5]^T645[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..27] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
Orbits(N645);
```

Appendix B: MAGMA Code for $\frac{7^{*2}:_m S_3}{t_0 t_1 0 t_1 = e}$

```
G<x,y,t>:=Group<x,y,t|x^3,y^2,(x*y)^2,t^7,t^x=t^4,t^(x^-1)=t^2,(x*y*t)^3>;

#G;

//168//

Index(G,sub<G|x,y>);

//  28 //

f,G1,k:=CosetAction(G,sub<G|x,y>);


f((y*x)^3*t^((y*x)^2)*t^(y*x)*t);

f((x*y)^3*t^((x*y)^2)*t^(x*y)*t);


 CompositionFactors(G1);
/*
    G
    |  A(1, 7)                    = L(2, 7)
```

```
*/


IN:=sub<G1|f(x),f(y)>;
ts:=[Id(G1) : i in [1..12]];
 ts[1]:=f(t); ts[2]:=(ts[1])^2; ts[3]:=(ts[1])^3;
 ts[4]:=(ts[1])^4; ts[5]:=(ts[1])^5; ts[6]:=(ts[1])^6;ts[7]:=f(t^y);
 ts[8]:=(ts[7])^2;ts[9]:=(ts[7])^3;ts[10]:=(ts[7])^4;ts[11]:=(ts[7])^5;ts

 [12]:=(ts[7])^6;
 S:=Sym(12);
 xx:=S!(1,4,2)(3,5,6)(7,8,10)(9,12,11);
 yy:=S!(1,7)(2,8)(3,9)(4,10)(5,11)(6,12);
 N:=sub<S|xx,yy>;
 xx*yy;
// (1, 10)(2, 7)(3, 11)(4, 8)(5, 12)(6, 9) //
f(x*y)*ts[1]*ts[10]*ts[1];
// Id(G1) //

f(x*y)*ts[7]*ts[2]*ts[7];

if f(x*y)*ts[1]*ts[10] eq ts[1]^-1 then print true;end if;
if f(x*y)*ts[1] eq ts[1]^-1*ts[10]^-1 then print true;end if;

cst := [null : i in [1 .. 28]] where null is [Integers() | ];
prodim := function(pt, Q, I)

v := pt;
for i in I do
 v := v^(Q[i]);
end for;
return v;
```

```
end function;
#DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);

// 7 //
DoubleCosets(G,sub<G|x,y>,sub<G|x,y>);
/*
{ <GrpFP, Id(G), GrpFP>, <GrpFP, t * y * x * t, GrpFP>, <GrpFP, t * y * t,
GrpFP>, <GrpFP, t, GrpFP>, <GrpFP, t^-1, GrpFP>, <GrpFP, t * y * t^-1, GrpFP>,
<GrpFP, y^t, GrpFP> }
*/
Setseq(DoubleCosets(G,sub<G|x,y>,sub<G|x,y>));
/*
[ <GrpFP, Id(G), GrpFP>, <GrpFP, t * y * x * t, GrpFP>, <GrpFP, t * y * t,
GrpFP>, <GrpFP, t, GrpFP>, <GrpFP, t^-1, GrpFP>, <GrpFP, t * y * t^-1, GrpFP>,
<GrpFP, y^t, GrpFP> ]
> { <GrpFP, Id(G), GrpFP>, <GrpFP, t * y * x * t, GrpFP>, <GrpFP, t * y * t,
> GrpFP>, <GrpFP, t, GrpFP>, <GrpFP, t^-1, GrpFP>, <GrpFP, t * y * t^-1, GrpFP\
>,
> <GrpFP, y^t, GrpFP> }
*/
----------------------  ti's and inverses  ------------------------------
ts[1]*ts[6];
// Id(G1) implies ts[1] = ts[6]^-1 //

ts[2]*ts[5];
// Id(G1) implies ts[2] = ts[5]^-1 //

ts[3]*ts[4];
// Id(G1) implies ts[3] = ts[4]^-1 //

ts[4]*ts[3];
//Id(G1)  implies ts[4] = ts[3]^-1//
```

```
ts[5]*ts[2];
//Id(G1)  implies ts[5] = ts[2]^-1//

ts[6]*ts[1];
//Id(G1)  implies ts[6] = ts[1]^-1 //

ts[7]*ts[12];
// Id(G1) implies ts[7] = ts[12]^-1//

ts[8]*ts[11];
//Id(G1)  implies ts[8] = ts[11]^-1//

ts[9]*ts[10];
//Id(G1) implies ts[9] = ts[10]^1//

ts[10]*ts[9];
//Id(G1) implies ts[10] = ts[9]^1//


ts[11]*ts[8];
//Id(G1) implies ts[11] = ts[8]^-1//

ts[12]*ts[7];
//Id(G1) implies ts[12] = ts[7]^-1//


       Checking orbits paths from DC to DC
----------------------------------------------------------------------

checking orbit paths from DC [1]
if ts[1]*ts[6] eq Id(G1) then print true; end if;
// true so ts[6] takes 1 to [*]//
```

```
if ts[1]*ts[3] eq ts[1] then print true; end if;
// ?? so either ts[1] or ts[3] takes 2 back to itself//
if ts[1]*ts[5] eq ts[6] then print true; end if;
//   so ts[5] takes 4 to [6]//
if ts[1]*ts[7] eq ts[1]*ts[7] then print true; end if;
//  so ts[7]  takes 2 to [1,7]//
if ts[1]*ts[] eq ts[1]*ts[8] then print true; end if;
// true so ts[8] takes 1 to [1,8]//
if ts[1]*ts[] eq ts[1]*ts[11] then print true; end if;
// true  so ts[11] takes 1 to [1,11]//
if ts[1]*ts[] eq ts[1]*ts[12] then print true; end if;
// true so ts[12] takes 1 to [1,12]//

checking orbit paths from DC [6]
if ts[6]*ts[1] eq Id(G1) then print true; end if;
// true so ts[1] takes 1 to [*]//
if ts[6]*ts[6] eq ts[6] then print true; end if;
// ?? so either ts[4] or ts[6] takes 2 back to itself//
if ts[6]*ts[2] eq ts[1] then print true; end if;
// true  so ts[2] takes 4 to [1]//
if ts[6]*ts[10] eq ts[1]*ts[7] then print true; end if;
//  ?? so either ts[10] or ts[12] takes 2 to [1,7]//
if ts[6]*ts[11] eq ts[1]*ts[8] then print true; end if;
// true so ts[8] takes 1 to [1,8]//
if ts[6]*ts[] eq ts[1]*ts[11] then print true; end if;
// true  so ts[11] takes 1 to [1,11]//
if ts[6]*ts[] eq ts[1]*ts[12] then print true; end if;
// true so ts[12] takes 1 to [1,12]//

checking orbit paths from DC [1,7]
if ts[1]*ts[7]*ts[12] eq ts[1] then print true; end if;
// true so ts[12] takes 2 to [1]//
```

```
if ts[1]*ts[7]*ts[9] eq ts[1]^(-1) then print true; end if;
// ??  so either ts[1] or ts[9] takes 2 to [6]//
if ts[1]*ts[7]*ts[8] eq ts[1]*ts[7] then print true; end if;
// ?? so either ts[5] or ts[8] takes 2 back to itself//
if ts[1]*ts[7]*ts[7] eq ts[1]*ts[8] then print true; end if;
// true so ts[7] takes 2 to [1,8]//
if ts[1]*ts[7]*ts[10] eq ts[1]*ts[11] then print true; end if;
// true  so ts[10] takes 2 to [1,11]//
if ts[1]*ts[7]*ts[11] eq ts[1]*ts[12] then print true; end if;
// true so ts[11] takes 2 to [1,12]//

checking orbit paths from DC [1,8]
if ts[1]*ts[8]*ts[12] eq ts[1]*ts[7] then print true; end if;
// true //
if ts[1]*ts[8]*ts[11] eq ts[1] then print true; end if;
// true //
if ts[1]*ts[8]*ts[8] eq ts[1]^-1 then print true; end if;
// ?? //
if ts[1]*ts[8]*ts[9] eq ts[1]*ts[11] then print true; end if;
// true //
if ts[1]*ts[8]*ts[10] eq ts[1]*ts[12] then print true; end if;
// true //

checking orbit paths from DC [1,11]
if ts[1]*ts[11]*ts[8] eq ts[1] then print true; end if;
// true //
if ts[1]*ts[11]*ts[12] eq ts[6] then print true; end if;
// ?? //
if ts[1]*ts[11]*ts[9] eq ts[1]*ts[7] then print true; end if;
// true //
if ts[1]*ts[11]*ts[10] eq ts[1]*ts[8] then print true; end if;
// true //
```

```
if ts[1]*ts[11]*ts[7] eq ts[1]*ts[12] then print true; end if;
// true //

checking orbit paths from DC [1,12]
if ts[1]*ts[12]*ts[7] eq ts[1] then print true; end if;
// true //
if ts[1]*ts[12]*ts[11] eq ts[6] then print true; end if;
// ?? //
if ts[1]*ts[12]*ts[8] eq ts[1]*ts[7] then print true; end if;
// true //
if ts[1]*ts[12]*ts[9] eq ts[1]*ts[8] then print true; end if;
// true //
if ts[1]*ts[12]*ts[12] eq ts[1]*ts[11] then print true; end if;
// true //

//------------------------ DC [1] ----------------------------------//
S:={[1]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do
if ts[1] eq g*ts[Rep(SSS[i])[1]]
then print SSS[i];
end if;
end for;
end for;

/*
{
    [ 1 ]
}
*/
```

```
N1:=Stabiliser(N,1);
#N1;
N1;
/*
Permutation group N1 acting on a set of cardinality 12
Order = 1
*/
T1:=Transversal(N,N1);#T1;

// 6 transversals //
T1;
/*
{@
    Id(N),
    (1, 4, 2)(3, 5, 6)(7, 8, 10)(9, 12, 11),
    (1, 7)(2, 8)(3, 9)(4, 10)(5, 11)(6, 12),
    (1, 2, 4)(3, 6, 5)(7, 10, 8)(9, 11, 12),
    (1, 10)(2, 7)(3, 11)(4, 8)(5, 12)(6, 9),
    (1, 8)(2, 10)(3, 12)(4, 7)(5, 9)(6, 11)
@}
*/
for i in [1..#T1] do
SS:=[1]^T1[i];
cst [prodim(1,ts,SS)]:=SS;
end for;
m:=0; for i in [1..#cst] do if cst[i] ne [] then m:=m+1; end if;
end for; m;
// 6, so 6 cosets in the DC [1]  //

Orbits(N1);
/*
```

```
[
    GSet{ 1 },
    GSet{ 2 },
    GSet{ 3 },
    GSet{ 4 },
    GSet{ 5 },
    GSet{ 6 },
    GSet{ 7 },
    GSet{ 8 },
    GSet{ 9 },
    GSet{ 10 },
    GSet{ 11 },
    GSet{ 12 }
]
*/
//------------------------ DC [6]= [1^6]------------------------------//
N6:=Stabiliser(N,6);
S:={[6]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do
if ts[6] eq g*ts[Rep(SSS[i])[1]]
then print SSS[i];
end if;
end for;
end for;
/*
{
    [ 6 ]
}
*/
```

```
 #N6;
  // 1 //

N6;
/*
Permutation group N6 acting on a set of cardinality 12
Order = 1
*/
T6:=Transversal(N,N6);#T6;
// 6, so 6 transversals //
T6;
/*
{@
    Id(N),
    (1, 4, 2)(3, 5, 6)(7, 8, 10)(9, 12, 11),
    (1, 7)(2, 8)(3, 9)(4, 10)(5, 11)(6, 12),
    (1, 2, 4)(3, 6, 5)(7, 10, 8)(9, 11, 12),
    (1, 10)(2, 7)(3, 11)(4, 8)(5, 12)(6, 9),
    (1, 8)(2, 10)(3, 12)(4, 7)(5, 9)(6, 11)
@}
*/
for i in [1..#T6] do
SS:=[6]^T6[i];
cst [prodim(1,ts,SS)]:=SS;
end for;
m:=0; for i in [1..#cst] do if cst[i] ne [] then m:=m+1; end if;
end for; m;
// 6, so 6 cosets in DC [6]   //

Orbits(N6);
/*
```

```
[
    GSet{ 1 },
    GSet{ 2 },
    GSet{ 3 },
    GSet{ 4 },
    GSet{ 5 },
    GSet{ 6 },
    GSet{ 7 },
    GSet{ 8 },
    GSet{ 9 },
    GSet{ 10 },
    GSet{ 11 },
    GSet{ 12 }
]
*/
------------------------ DC [1,7]----------------------------------
N17:=Stabiliser(N,[1,7]);
S:={[1,7]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do
if ts[1]*ts[7] eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if;
end for;
end for;
/*
{
    [ 1, 7 ]
}
*/
```

```
#N17;
//  1   //

N17;

/*
Permutation group N17 acting on a set of cardinality 12
Order = 1
*/
T17:=Transversal(N,N17);
for i in [1..#T17] do
ss:=[1,7]^T17[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..28] do if cst[i] ne []
then m:=m+1; end if; end for;  m;
// 18-12=6, so 6 cosets in DC [1,7]  //
#T17;
// 6 //

T17;
/*
{@
    Id(N),
    (1, 4, 2)(3, 5, 6)(7, 8, 10)(9, 12, 11),
    (1, 7)(2, 8)(3, 9)(4, 10)(5, 11)(6, 12),
    (1, 2, 4)(3, 6, 5)(7, 10, 8)(9, 11, 12),
    (1, 10)(2, 7)(3, 11)(4, 8)(5, 12)(6, 9),
    (1, 8)(2, 10)(3, 12)(4, 7)(5, 9)(6, 11)
@}
*/
Orbits(N17);
```

```
/*
[
    GSet{ 1 },
    GSet{ 2 },
    GSet{ 3 },
    GSet{ 4 },
    GSet{ 5 },
    GSet{ 6 },
    GSet{ 7 },
    GSet{ 8 },
    GSet{ 9 },
    GSet{ 10 },
    GSet{ 11 },
    GSet{ 12 }
]

*/
```

```
------------------------ DC [1,8]---------------------------------

S:={[1,8]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do
if ts[1]*ts[8] eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if;
end for;
end for;
```

```
{
    [ 1, 8 ]
}
{
    [ 7, 2 ]
}

N18:=Stabiliser(N,[1,8]);


/* Enter  [1,8] ~ [7,2]*/
for n in N do if 1^n eq 7 and 8^n eq 2
then N18c:=sub<N|N18,n>; end if;end for;
[1,8]^N18c;

#N18c;
// 2//
N18c;
/*

Permutation group N18c acting on a set of cardinality 12
Order = 2
    (1, 7)(2, 8)(3, 9)(4, 10)(5, 11)(6, 12)

*/

T18:=Transversal(N,N18c);
for i in [1..#T18] do
ss:=[1,8]^T18[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..28] do if cst[i] ne []
```

```
then m:=m+1; end if; end for;  m;
// 21-18=3, so 3 cosets in DC [1,8] //

#T18;
// 6 //
T18;
/*
{@
    Id(N),
    (1, 4, 2)(3, 5, 6)(7, 8, 10)(9, 12, 11),
    (1, 2, 4)(3, 6, 5)(7, 10, 8)(9, 11, 12)
@}

*/

Orbits(N18);
/*

[
    GSet{ 1, 7 },
    GSet{ 2, 8 },
    GSet{ 3, 9 },
    GSet{ 4, 10 },
    GSet{ 5, 11 },
    GSet{ 6, 12 }
]

So our orbit paths are:
1) Nt_1t_8t_7=
2) Nt_1t_8t_8=
3) Nt_1t_8t_
4) Nt_1t_8t_
```

```
5) Nt_1t_8t_
6) Nt_1t_8t_

*/

------------------------ DC [1,11]---------------------------------

S:={[1,11]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do
if ts[1]*ts[11] eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if;
end for;
end for;

/*
{
    [ 1, 11 ]
}
{
    [ 10, 3 ]
}

*/
N111:=Stabiliser(N,[1,11]);


/* Enter  [1,11] ~ [10,3]*/
for n in N do if 1^n eq 10 and 11^n eq 3
```

```
then N111c:=sub<N|N111,n>; end if;end for;
[1,11]^N111c;

#N111c;
// 2//
N111c;
/*
Permutation group N111c acting on a set of cardinality 12
Order = 2
    (1, 10)(2, 7)(3, 11)(4, 8)(5, 12)(6, 9)

*/
T111:=Transversal(N,N111c);
for i in [1..#T111] do
ss:=[1,11]^T111[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..28] do if cst[i] ne []
then m:=m+1; end if; end for;  m;

// 24 - 21 = 3, so 3 cosets in DC [1,11] //



#T111;
// 3 //
T111;

/*
{@
    Id(N),
    (1, 4, 2)(3, 5, 6)(7, 8, 10)(9, 12, 11),
```

```
    (1, 2, 4)(3, 6, 5)(7, 10, 8)(9, 11, 12)
@}


*/

Orbits(N111c);
/*
[

    GSet{ 1, 10 },
    GSet{ 2, 7 },
    GSet{ 3, 11 },
    GSet{ 4, 8 },
    GSet{ 5, 12 },
    GSet{ 6, 9 }

]

*/

------------------------ DC [1,12]---------------------------------

S:={[1,12]};
SS:=S^N;
SSS:=Setseq(SS);
for i in [1..#SS] do
for g in IN do
if ts[1]*ts[12] eq g*ts[Rep(SSS[i])[1]]*ts[Rep(SSS[i])[2]]
then print SSS[i];
end if;
end for;
```

```
end for;
/*
{
    [ 10, 5 ]
}
{
    [ 1, 12 ]
}

*/
N112:=Stabiliser(N,[1,12]);

/* Enter  [10,5] ~ [1,12]*/
for n in N do if 10^n eq 1 and 5^n eq 12
then N112c:=sub<N|N112,n>; end if;end for;
[1,12]^N112c;

#N112c;
// 2//
N112c;
/*
Permutation group N112c acting on a set of cardinality 12
Order = 2
    (1, 10)(2, 7)(3, 11)(4, 8)(5, 12)(6, 9)
*/

T112:=Transversal(N,N112c);
for i in [1..#T112] do
ss:=[1,12]^T112[i];
cst[prodim(1, ts, ss)] := ss;
end for;
m:=0; for i in [1..28] do if cst[i] ne []
```

```
then m:=m+1; end if; end for;  m;
// 27-24 = 3, so 3 cosets in DC [1,12] //

#N112;
// 1 //
N112;
/*
Permutation group N112 acting on a set of cardinality 12
Order = 1
*/
#T112;
// 3 //
T112;

/*

{@
    Id(N),
    (1, 4, 2)(3, 5, 6)(7, 8, 10)(9, 12, 11),
    (1, 2, 4)(3, 6, 5)(7, 10, 8)(9, 11, 12)
@}

*/

Orbits(N112C);
/*
[
    GSet{ 1, 10 },
    GSet{ 2, 7 },
    GSet{ 3, 11 },
    GSet{ 4, 8 },
    GSet{ 5, 12 },
```

```
    GSet{ 6, 9 }
]
*/
--process for finding relations used in determining equal double cosets--
NN<a,b>:=Group<a,b|a^3,b^2,(b*a)^2>;
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..6]];
for i in [2..6] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1
then P[j]:=xx^-1; end if;
if Eltseq(Sch[i])[j] eq 2
then P[j]:=yy; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
for n in IN do if ts[1]*ts[8] eq n*ts[7]*ts[2] then n; end if; end for;
/*
(2, 4, 6)(3, 7, 9)(5, 11, 10)(8, 15, 14)(12, 21, 23)(13, 24, 25)(16, 20, 26)(17,
    27, 18)(19, 28, 22)
*/
for i in [1..20] do i, cst[i]; end for;
/*
1 []
2 [ 1 ]
3 [ 6 ]
4 [ 4 ]
```

```
5 [ 7 ]
6 [ 2 ]
7 [ 3 ]
8 [ 12 ]
9 [ 5 ]
10 [ 10 ]
11 [ 8 ]
12 [ 7, 1 ]
13 [ 7, 6 ]
14 [ 9 ]
15 [ 11 ]
16 [ 7, 5 ]
17 [ 1, 7 ]
18 [ 2, 10 ]
19 [ 8, 1 ]
20 [ 8, 6 ]

So (2,6,4)-->(1,2,4): (3,9,7)-->(6,5,3): (5,10,11)-->(7,10,8)
: (8,14,15)-->(12,9,11): enter that into next loop.
We use [1..6} since S_3 has 6 elements...
*/
for i in [1..6] do if ArrayP[i] eq N!(1,2,4)(6,5,3)(7,10,8)(12,9,11)
then Sch[i]; end if; end for; // a^-1 , so we use (x^-1) as relation
that proves [1,8]=[7,2]//
ts[1]*ts[8] eq f(x^-1)*ts[7]*ts[2];
// true , so $t_1t_8=(x^-1)t_7t_2$  //

NN<a,b>:=Group<a,b|a^3,b^2,(b*a)^2>;
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..6]];
for i in [2..6] do
P:=[Id(N): l in [1..#Sch[i]]];
```

```
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1
then P[j]:=xx^-1; end if;
if Eltseq(Sch[i])[j] eq 2
then P[j]:=yy; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
for n in IN do if ts[1]*ts[11] eq n*ts[10]*ts[3] then n;
 end if; end for;
/*
(2, 6, 4)(3, 9, 7)(5, 10, 11)(8, 14, 15)(12, 23, 21)
(13, 25, 24)(16, 26, 20)(17,18, 27)(19, 22, 28)
*/
for i in [1..20] do i, cst[i]; end for;
/*
1 []
2 [ 1 ]
3 [ 6 ]
4 [ 4 ]
5 [ 7 ]
6 [ 2 ]
7 [ 3 ]
8 [ 12 ]
9 [ 5 ]
10 [ 10 ]
11 [ 8 ]
12 [ 7, 1 ]
```

```
13 [ 7, 6 ]
14 [ 9 ]
15 [ 11 ]
16 [ 7, 5 ]
17 [ 1, 7 ]
18 [ 2, 10 ]
19 [ 8, 1 ]
20 [ 8, 6 ]

So (2,6,4)-->(1,2,4): (3,9,7)-->(6,5,3): (5,10,11)-->(7,10,8):
 (8,14,15)-->(12,9,11): enter that into next loop.
We use [1..6} since S_3 has 6 elements...*/
for i in [1..6] do if ArrayP[i] eq N!(1,2,4)(6,5,3)(7,10,8)(12,9,11)
 then Sch[i]; end if; end for;
// a^-1 , so we use (x^-1) as relation that proves [1,11]=[10,3]//
ts[1]*ts[11] eq f(x^-1)*ts[10]*ts[3];
 // true , so $t_1t_11=(x^-1)t_10t_3$  //

NN<a,b>:=Group<a,b|a^3,b^2,(b*a)^2>;
Sch:=SchreierSystem(NN,sub<NN|Id(NN)>);
ArrayP:=[Id(N): i in [1..6]];
for i in [2..6] do
P:=[Id(N): l in [1..#Sch[i]]];
for j in [1..#Sch[i]] do
if Eltseq(Sch[i])[j] eq 1 then P[j]:=xx; end if;
if Eltseq(Sch[i])[j] eq -1
then P[j]:=xx^-1; end if;
if Eltseq(Sch[i])[j] eq 2
then P[j]:=yy; end if;
end for;
PP:=Id(N);
for k in [1..#P] do
```

```
PP:=PP*P[k]; end for;
ArrayP[i]:=PP;
end for;
for n in IN do if ts[10]*ts[5] eq n*ts[1]*ts[12] then n;
end if; end for;
/*
(2, 6, 4)(3, 9, 7)(5, 10, 11)(8, 14, 15)(12, 23, 21)(13, 25, 24)
(16, 26, 20)(17, 18, 27)(19, 22, 28)
*/
 for i in [1..20] do i, cst[i]; end for;
/*
1 []
2 [ 1 ]
3 [ 6 ]
4 [ 4 ]
5 [ 7 ]
6 [ 2 ]
7 [ 3 ]
8 [ 12 ]
9 [ 5 ]
10 [ 10 ]
11 [ 8 ]
12 [ 7, 1 ]
13 [ 7, 6 ]
14 [ 9 ]
15 [ 11 ]
16 [ 7, 5 ]
17 [ 1, 7 ]
18 [ 2, 10 ]
19 [ 8, 1 ]
20 [ 8, 6 ]
so we use this table's labeling to convert the above permutation
```

```
to another permutation and use the resulting permutation to determine
the relation: (2,6,4)->(1,2,4): (3,9,7)->(6,5,3): (5,10,11)->(7,10,8):
 (8,14,15)->(12,9,11) . We use [1..6} since S_3 has 6 elements...*/
 for i in [1..6] do if ArrayP[i] eq N!(1,2,4)(6,5,3)(7,10,8)(12,9,11)
 then Sch[i}; end if; end for;  // a^-1 , so we use (x^-1) as relation
 that proves [10,5]=[1,12]// ts[10]*ts[5] eq f(x^-1)*ts[1]*ts[12];
  // true , so $t_10t_5=(x^-1)t_1t_12$  //
// SO THE RELATION (x^-1) IS USED TO PROVE ALL EQUAL DOUBLE COSETS //
```

Appendix C: MAGMA Code for $(M_{21} \times 4){:}S_3$ Factored by Center

```
                    MAGMA CODE 3*8 : PGL _2 (7)
                    FACTORED BY NEW RELATIONS :
  ((t^(x^6))^-1)^3, x^-1 * y * x^2*(t^(x^3))^-1*t^(x^6)*t^(x^3)*(t
^(x^6))^-1, (x^-1 * y * x^2)^-1*t^(x^3)*t^(x^6)*(t^(x^3))^-1*(t^(x^6))^-1>;
*****************************************************************
 S:=Sym(16);
 xx:=S!(8,2,5,4,6,1,7,3)(16,10,13,12,14,9,15,11);
 yy:=S!(1,6)(2,5)(3,4)(9,14)(10,13)(11,12);
 N:=sub<S|xx,yy>;
 G<x,y,t>:=Group<x,y,t|x^8 , y^2 , (x*y)^3,
 (x,y)^4,t^3,(t,y),
 (t,x^3 * y * x^3 * y * x^-1),
 (t,y * x^-2 * y * x^3 * y * x^-2),
 (x^3*t)^6,
 ((t^(x^6))^-1)^3,
 x^-1 * y * x^2*(t^(x^3))^-1*t^(x^6)*t^(x^3)*(t^(x^6))^-1,
 (x^-1 * y * x^2)^-1*t^(x^3)*t^(x^6)*(t^(x^3))^-1*(t^(x^6))^-1
>;
IndexG:=Index(G,sub<G|x,y>);
f,G1,K:=CosetAction(G,sub<G|x,y>);
G1;
/*
```

```
Permutation group G1 acting on a set of cardinality 360
Order = 120960 = 2^7 * 3^3 * 5 * 7
where 360 is the number of double cosets.
*/
IN:=sub<G1|f(x)>;
ts:=[Id(G1) : i in [1..16]];
 ts[8]:=f(t);
ts[2]:=f(t^x);
ts[5]:=f(t^(x^2));
ts[4]:=f(t^(x^3));
ts[6]:=f(t^(x^4));
ts[1]:=f(t^(x^5));
ts[7]:=f(t^(x^6));
 ts[3]:=f(t^(x^7));
 ts[9]:=ts[1]^-1;
ts[10]:=ts[2]^-1;
ts[11]:=ts[3]^-1;
ts[12]:=ts[4]^-1;
ts[13]:=ts[5]^-1;
ts[14]:=ts[6]^-1;
 ts[15]:=ts[7]^-1;
ts[16]:=ts[8]^-1;
prodim := function(pt, Q, I)
v := pt;
for i in I do
v := v^(Q[i]);
end for;
return v;
end function;

cst := [null : i in [1 .. 360]] where null is [Integers() | ];
for i := 1 to 16 do
```

```
cst[prodim(1, ts, [i])] := [i];
end for;
m:=0; for i in [1..#cst] do if cst[i] ne [] then m:=m+1; end if;
end for; m;
//  16   so the origional 16 SC's in [8] and [16]    ///////
for a in [8,16] do
Stabil := Stabilizer(N,[a]);
trans := Transversal(N, Stabil);
   for i := 1 to #trans do
      ss := [a]^trans[i];
      cst[prodim(1, ts, ss)] := ss;
   end for;
n:=m;m:=0;for i in [1..#cst] do if cst[i] ne [] then m:=m+1; end if; end for;
if m-n ne 0 then "Number of Cosets in DC [",a,"]",m-n,m; end if;
if m-n ne 0 then Orbits(Stabil); end if;
end for;

// now we fix [8] & [16] and check all words of length two //

for a in [8,16],b in [1..16] do
Stabil := Stabilizer(N,[a,b]);
trans := Transversal(N, Stabil);
   for i := 1 to #trans do
      ss := [a,b]^trans[i];
      cst[prodim(1, ts, ss)] := ss;
   end for;
n:=m;m:=0;for i in [1..#cst] do if cst[i] ne [] then m:=m+1; end if; end for;
if m-n ne 0 then "Number of Cosets in DC [",a,b,"]: ",m-n;
"Total Cosets filled out of 360:",m; end if;
if m-n ne 0 then Orbits(Stabil); end if;
end for;
```

```
// below are all DC's, SC's and orbits of length two//

/*
Number of Cosets in DC [ 8 1 ]:  28
Total Cosets filled out of 360: 44
[
    GSet{ 1 },
    GSet{ 8 },
    GSet{ 9 },
    GSet{ 16 },
    GSet{ 2, 3, 4, 5, 6, 7 },
    GSet{ 10, 11, 12, 13, 14, 15 }
]
Number of Cosets in DC [ 8 9 ]:  56
Total Cosets filled out of 360: 100
[
    GSet{ 1 },
    GSet{ 8 },
    GSet{ 9 },
    GSet{ 16 },
    GSet{ 2, 3, 4, 5, 6, 7 },
    GSet{ 10, 11, 12, 13, 14, 15 }
]
Number of Cosets in DC [ 8 16 ]:  1
Total Cosets filled out of 360: 101
[
    GSet{ 8 },
    GSet{ 16 },
    GSet{ 1, 2, 3, 4, 5, 6, 7 },
    GSet{ 9, 10, 11, 12, 13, 14, 15 }
]
Number of Cosets in DC [ 16 9 ]:  28
```

```
Total Cosets filled out of 360: 129
[
    GSet{ 1 },
    GSet{ 8 },
    GSet{ 9 },
    GSet{ 16 },
    GSet{ 2, 3, 4, 5, 6, 7 },
    GSet{ 10, 11, 12, 13, 14, 15 }
]
*/

/* Now, we fix the third elements [1],[9],[16] & check for DC's of length three
*/

for a in [8,16],b in [1,9,16], c in [1..16] do
Stabil := Stabilizer(N,[a,b,c]);
trans := Transversal(N, Stabil);
   for i := 1 to #trans do
      ss := [a,b,c]^trans[i];
      cst[prodim(1, ts, ss)] := ss;
   end for;
n:=m;m:=0;for i in [1..#cst] do if cst[i] ne [] then m:=m+1; end if; end for;
if m-n ne 0 then "Number of Cosets in DC [",a,b,c,"]: ",m-n;
"Total Cosets filled out of 360:",m; end if;
if m-n ne 0 then Orbits(Stabil); end if;
end for;

// below are the results of that search  /////////

/*
Number of Cosets in DC [ 8 1 2 ]:  42
Total Cosets filled out of 360: 171
```

```
[
    GSet{ 1 },
    GSet{ 2 },
    GSet{ 3 },
    GSet{ 4 },
    GSet{ 5 },
    GSet{ 6 },
    GSet{ 7 },
    GSet{ 8 },
    GSet{ 9 },
    GSet{ 10 },
    GSet{ 11 },
    GSet{ 12 },
    GSet{ 13 },
    GSet{ 14 },
    GSet{ 15 },
    GSet{ 16 }
]
Number of Cosets in DC [ 8 1 10 ]:  84
Total Cosets filled out of 360: 255
[
    GSet{ 1 },
    GSet{ 2 },
    GSet{ 3 },
    GSet{ 4 },
    GSet{ 5 },
    GSet{ 6 },
    GSet{ 7 },
    GSet{ 8 },
    GSet{ 9 },
    GSet{ 10 },
    GSet{ 11 },
```

```
    GSet{ 12 },
    GSet{ 13 },
    GSet{ 14 },
    GSet{ 15 },
    GSet{ 16 }
]
Number of Cosets in DC [ 8 9 10 ]:  84
Total Cosets filled out of 360: 339
[
    GSet{ 1 },
    GSet{ 2 },
    GSet{ 3 },
    GSet{ 4 },
    GSet{ 5 },
    GSet{ 6 },
    GSet{ 7 },
    GSet{ 8 },
    GSet{ 9 },
    GSet{ 10 },
    GSet{ 11 },
    GSet{ 12 },
    GSet{ 13 },
    GSet{ 14 },
    GSet{ 15 },
    GSet{ 16 }
]
/*
 Now, we fix the fourth elements [2],[10] & check for DC's of length four
*/
for a in [8,16],b in [1,9,16], c in [2,10], d in [1..16] do
Stabil := Stabilizer(N,[a,b,c,d]);
trans := Transversal(N, Stabil);
```

```
   for i := 1 to #trans do
      ss := [a,b,c,d]^trans[i];
      cst[prodim(1, ts, ss)] := ss;
   end for;
n:=m;m:=0;for i in [1..#cst] do if cst[i] ne [] then m:=m+1; end if; end for;
if m-n ne 0 then "Number of Cosets in DC [",a,b,c,d,"]: ",m-n;
"Total Cosets filled out of 360:",m; end if;
if m-n ne 0 then Orbits(Stabil); end if; end for;

//Our resulting final DC, it's SC's and orbits are below  //////

Number of Cosets in DC [ 8 1 10 12 ]:  21
Total Cosets filled out of 360: 360
[
    GSet{ 1 },
    GSet{ 2 },
    GSet{ 3 },
    GSet{ 4 },
    GSet{ 5 },
    GSet{ 6 },
    GSet{ 7 },
    GSet{ 8 },
    GSet{ 9 },
    GSet{ 10 },
    GSet{ 11 },
    GSet{ 12 },
    GSet{ 13 },
    GSet{ 14 },
    GSet{ 15 },
    GSet{ 16 }
]
-------------------------------------------------------------------
```

The below code is to check the extensions from Double coset to connected double cosets. By inputting a specific DC, the below program compares the two lists and based on my input DC, outputs all the orbit paths to their respective DC's. From this information, I'm able to complete my Cayley diagram.

```
/*
    List of the Names of my Double Cosets
*/

mlist:=[
[8],
[16],
[8,1],
[8,9],
[8,16],
[16,9],
[8,1,2],
[8,1,10],
[8,9,10],
[8,1,10,12]
];
restore pkm;
a :=16;b:=9;
for c in [1..16]do
Stabil := Stabilizer(N,[a,b,c]);
trans := Transversal(N, Stabil);
   for i := 1 to #trans do
      ss := [a,b,c]^trans[i];
      cst[prodim(1, ts, ss)] := ss;
   end for;
n:=m;m:=0;for i in [1..#cst] do if cst[i] ne [] then m:=m+1; end if; end for;
if m-n ne 0 then "Number of Cosets in DC [",a,b,c,"]: ",m-n;
```

```
end if;
end for;
load pk;

"[",a,b,c,"]";
nlist:=[null : i in [1..10]] where null is [Integers() | ];
i:=1;j:=1;k:=1;
repeat
if mlist[i] eq dlist[j] then i:=i+1; j:=j+1; end if;
if mlist[i] ne dlist[j] then nlist[k]:=mlist[i]; k:=k+1; i:=i+1; end if;
until i gt 10 or j gt 10;
for i in [1..10] do if nlist[i] ne [] then nlist[i]; end if; end for;
```

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