Homflypt Skein Theory, String Topology and 2-Categories

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Abstract

We show that relations in Homflypt type skein theory of an oriented 3-manifold M are induced from a 2-groupoid defined from the fundamental 2-groupoid of a space of singular links in M. The module relations are defined by homomorphisms related to string topology. They appear from a *representation* of the groupoid into free modules on a set of model objects. The construction on the fundamental 2-groupoid is defined by the singularity stratification and relates Vassiliev and skein theory. Several explicit properties are discussed, and some implications for skein modules are derived.

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1 Introduction and summary of results

The first version of this article was written in 2003, following results by the author on homotopy skein modules [25], [26] and two-term skein modules of framed links[27]. The techniques used in this paper are extending the ideas of those papers and modifications of the original ideas of Lin [36], Kalfagianni [28], Kalfagianni-Lin [30] and Kirk-Livingston [34], who construct finite type invariants of links in 3-manifolds. They observed essential singular spheres and tori as the obstructions in the construction of finite type invariants. The intention of this article is to apply mapping space techniques in a more directl way to the calculation of skein modules. In this way, without passing to *dual* modules, we will be able to describe the possible torsion in the skein modules.

The problem to find presentations of skein modules naturally splits into **(1)** to determine a *minimal* set of generators, and **(2)** to find relations in terms

of these relations. In 2001, Charlie Frohman pointed out to the author that the relations described in [25] are related to the string topology operations just introduced by Chas-Sullivan [9]. While the ideas from string topology give a pretty good way to approach (**2**) the problem (**1**) still remains evasive: It turns out to be difficult to determine whether the obvious minimal set of generators is sufficient (It is a challenging conjecture that this is even true for hyperbolic 3-manifolds). It has been a lack of explicit applications, which put a hold on the project described here at that time. But the author kept the hope that there would be substantial progress with problem (**1**), thereby providing stronger impact to the methods developed in this article. The author's decision to revise the original manuscript for publication now is partly based on recent progress in this direction: in fact problem (**1**) for $L(p,q)$ was solved by [13], and later on by [19], even though in both cases a careful analysis of their arguments is necessary (and will not be given in this article). The result that the Homflypt skein modules of lens spaces $L(p,q)$ (except for $S^2 \times S^1$) are free on the expected set of generators for $q > 1$ is still not published until now. The two main approaches to find presentations of skein modules of $L(p,q)$ with respect to problem (**2**) are using projections [19] respectively braids [14], [15], [16]. This work is still in progress but until now completely successful only for $q = 1$. A second reason to revisit the manuscript from 2003 came from increasing interest in higher category theory in low dimensional topology during recent years, motivation from categorification but also new relations to the topology of stratified spaces [45]. Also note that, in contrast to our skein modules, the groundbreaking recent result of Gunningham, Jordan and Safronov [**?**], Ddefinition 2.2. and Theorem 4.8, stating that the skein modules $\mathrm{Sk}_G(M)$ are finite-dimensional $\mathbb{C}[q^{1/d}]$-vector spaces for each reductive group G and closed, oriented 3-manifold M.

In this article we will circumvent (**1**) and replace skein modules by adic completions, which in the cases where (**2**) gives only trivial relations actually turns out to be equivalent to the construction of power series invariants previously constructed from finite type invariants in [36], [28], [29], [30].

On a theoretical level we develop ideas using category theory to study the stratified topology of mapping spaces associated with skein modules. We will see how the structures of skein modules and string topology naturally combine on a categorical level. This gives both a theoretical understanding and new tools for the computation of skein modules of links in oriented 3-manifolds.

Recall that a skein module of an oriented 3-manifold M is the quotient of a free module with basis a set of links (possibly including singular links) in M by a submodule generated by linear combinations of links defined from local tangle modifications in oriented 3-balls. In quantum field theory the skein modules of M, or rather their duals, appear as modules of quantum observables. In link theory the skein module is the target of a *universal link invariant* satisfying given skein relations. Skein *algebras* have been studied in detail for cylinders over oriented surfaces, But not much is known about the structure of skein modules for general 3-manifolds. For an overview see [40].

We will construct various types of link invariants satisfying skein relations. Our construction is algebraic and follows from the structure of a 2-groupoid defined in section 3 using the general ideas from section 2.

Throughout let M be a compact connected oriented 3-manifold. For $k \geq 0$ let $\mathcal{L}[k]$ be the set of k-links in M. The elements of $\mathcal{L}[k]$ are the isotopy class of k-embeddings, i. e. immersions of circles in M with singularity given by precisely k double-points without tangencies. Thus $\mathcal{L}[0]$ is the usual set of oriented links in M (including the empty link). For $K_* \in \mathcal{L}[1]$ let $K_\pm, K_0$ be the *Conway resolutions*. Similarly let $\mathcal{L}_{\mathsf{f}}[k]$ denote the set of isotopy classes of framed oriented links with precisely k double-points without tangencies. If a fixed 3-manifold is clear from the context we often just use notation $\mathcal{L}$.

$$K_+ \qquad\qquad K_- \qquad\qquad K_* \qquad\qquad K_0$$
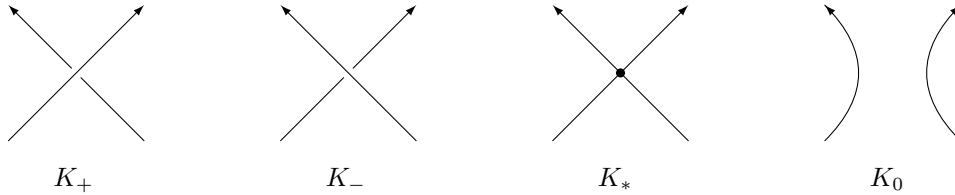
Let $\mathcal{R}$ be a commutative ring with 1 and let $\mathcal{R}^\times$ be the corresponding set of units. Let $q, v \in \mathcal{R}^\times$ (which may very well be 1). Let $I \subset J$ be proper ideals of $\mathcal{R}$. For each $\mathcal{R}$-module W let $W[[I]]$ denote the I-adic completion of W, i. e. the inverse limit of the system $W/I^i W$ for $i \geq 0$.

A map $\sigma : \mathcal{L}[1] \to \mathcal{R}\mathcal{L}[0]$ is called a *skein potential* (with respect to I) if the image of a j-component link is a linear combination of links with $< j$ components and coefficients in $J \setminus I$, and links with $\geq j$ components and coefficients in I.

We want to study skein relations of oriented links in M of the form $q^{-1}K_+ - qK_- = \sigma(K_*)$, and skein relations for framed oriented links: $q^{-1}K_+ - qK_- = \sigma(K_*)$, $K^{(+)} = qv^{-1}K$, where $K^{(+)}$ is the result of introducing a positive twist into a component of K.

In Chern Simons theory the *classical* observables are the homotopy classes of oriented links in M (holonomy along loops). Thus it is natural to consider skein modules as *deformations* of free objects generated by homotopy classes of links in M. This is our starting point.

Let $\mathfrak{b}[0]$ denote the set of homotopy classes of oriented links in M. We will choose *geometric models*, also called *standard links*, i. e. a map $\mathfrak{b}[0] \to \mathcal{L}[0]$ assigning to each homotopy class of link *some* oriented link with the given homotopy class. Let

$$\partial : \mathcal{L}[k] \to \mathcal{L}[k-1],$$

defined by $\partial(K_*) = q^{-2}K_+ - K_-$. ($*$ denotes the last double-point), be the

Jones boundary. For $q = 1$ it is called the *Conway boundary.* Let

$$\mathcal{S}(\sigma) := \mathcal{RL}[0]/im(\partial - q^{-1}\sigma)$$

be the *skein module* defined by σ.

In section 2 we will define the notion of *insensitive* skein potential. The usual oriented or framed oriented *local* skein relations are defined by insensitive skein potentials.

The idea of the skein module is a *finite formal expansion* of links using the skein relation repeatedly. We will study expansions of links in terms of standard links by possibly infinite applications of the skein relation. This is formalized in the following result.

Theorem 1.1. Let σ be a skein potential and $\mathfrak{s}$ be a choice of geometric models. Then there is defined a submodule

$$U = U(\sigma, \mathfrak{s}) \subset \mathcal{R}(\mathfrak{b}[0])[[I]],$$

snd the I-adic link invariant

$$\rho = \rho(\sigma, \mathfrak{s}) : \mathcal{L}[0] \to \mathcal{R}(\mathfrak{b}[0])[[I]]/U.$$

The submodule U is generated by (i) expansion of differentiability relations (see section 4), and (ii) expansion of elements in the image of a transversal string topology homomorphism (defined by σ). This homomorphism is defined on 1-dimensional homology groups of the space of immersions in M. If σ is insensitive then U is determined only by (ii). The map ρ induces the module isomorphism

$$S(\sigma)[[I]] \cong \mathcal{R}\mathfrak{b}[[I]]/U.$$

The composition

$$\rho(\sigma, \mathfrak{s}) \circ \mathfrak{s} : \mathfrak{b}[0] \to \mathcal{R}(\mathfrak{b}[0])[[I]]/U$$

is the composition of the natural map $\mathfrak{b}[0] \to \mathcal{R}(\mathfrak{b}[0])[[I]]$ with the natural projection.

We will show that for insensitive σ the submodule U is completely determined by loops in the space of immersions defined from kink crossings, and by essential singular tori in M. This corresponds to previous results of Kalfagianni [28], [29] and Kalfagianni and Lin [30].

The module $\mathcal{R}(\mathfrak{b}[0])[[I]]$ is an algebra with multiplication defined by formal multiplication of homotopy classes of maps. This multiplication induces on $S(\sigma)[[I]]$ the structure of a *skein algebra* if U is an ideal in $\mathcal{R}(\mathfrak{b}[0])$. The precise geometric relevance of such an algebra structure seems not yet have been studied.

Question. For which 3-manifolds M and skein relations σ can we choose $\mathfrak{s}$ such that $U(\sigma, \mathfrak{s})$ is an ideal? How can we modify the ring $\mathcal{R}$ to define a multiplication on $\mathcal{S}(\sigma)$, which *deforms* the multiplication of free homotopy classes in M. (This is known to be true for cylinders over surfaces, [41], [46].)

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From Theorem 1.1 some explicit results concerning classical skein modules can be deduced. $\mathcal{R} := \mathbb{Z}[q^{\pm 1}, z, h]$. Then $z \neq h \in \mathcal{R}$ are non-invertible elements. Let $\sigma(K_*) = hK_0$ for a self-crossing and $\sigma(K_*) = zK_0$ for a crossing of different components of K_*. The skein module $\mathcal{J}(M)$ is defined by the free $\mathcal{R}$-module with basis isotopy classes of oriented links, and relations $q^{-1}K_+ - qK_- = \sigma(K_*)$ is the called the *generalized Jones module* and has been considered in [46], [42]. The skein module $\mathcal{H}(M)$ is defined by framed oriented links with $\mathcal{R} := \mathbb{Z}[q^{\pm 1}, v^{\pm 1}, z, h]$ and relations $q^{-1}K_+ - qK_- = \sigma(K_*), K^{(+)} = qv^{-1}K$. This is the *(variant)* Homflypt *skein module* (with split variables). For $z = h = s^{-1} - s$ this module has been discussed e. g. in [38], see also [20], [21] for results about of $S^2 \times S^1$ and connected sums.

The skein modules defined by imposing additional vaccuum relations $(q^{-1} - q)\emptyset = hU$ respectively $(v - v^{-1})\emptyset = hU$ for the unknot U and the empty link $\emptyset$ will be denoted $\mathfrak{J}(M)$ and $\mathfrak{H}(M)$ respectively.

The skein modules of oriented links $\mathfrak{J}(M)$ respectively framed links $\mathfrak{H}(M)$ are modules over the corresponding skein modules of the 3-ball, with module actions defined by disjoint union with links in a separated 3-ball (here is where we use that M is connected). Let $\mathfrak{R}$ be the corresponding skein module of the 3-ball. It is known ([41] and [46]) that

$$\mathfrak{R} \cong \mathbb{Z}[q^{\pm 1}, z, h, \frac{q^{-1} - q}{h}]$$

and

$$\mathfrak{R} \cong \mathbb{Z}[q^{\pm}, v^{\pm}, z, h, \frac{v - v^{-1}}{h}],$$

respectively. Let $\mathfrak{b}_0$ be the set of homotopy classes of links in M without homotopically trivial components. Then there are defined $\mathfrak{R}$-homomorphisms $\mathfrak{s}$ from the free modules with basis $\mathfrak{b}_0$ into the modules $\mathfrak{J}(M)$ respectively $\mathfrak{H}(M)$. Then we have

$$\mathfrak{R}\mathfrak{b}_0 \cong S\mathfrak{R}\hat{\pi}^0,$$

where $\hat{\pi}^0$ is the set of non-trivial conjugacy classes of the fundamental group of M. Here the *geometric model* $\mathfrak{s}$ assigns to a monomial in $\hat{\pi}^0$ the oriented link with the homotopy classes of components given by the monomial. Note that there is also the trivial monomial $1 \in \mathfrak{b}_0$. The geometric model is called *nice* if the following holds: If some element of $\hat{\pi}^0$ appears repeatedly in the sequence we assume that there exist self-isotopies of the representing link, which arbitrarily change the order of components with the same free homotopy classes. Moreover we assume that multiplication by $\frac{q^{-1}-q}{h}$ respectively $\frac{v-v^{-1}}{h}$ corresponds to adding some unknotted unlinked component to the corresponding standard link.

A 3-manifold M is called *atoroidal* respectively *aspherical* if each essential (i. e. the induced homomorphism of fundamental groups is injective) map of a torus $S^2 \times S^1$ respectively map of a 2-sphere S^2 in M is homotopic into the boundary of M.

Theorem 1.2. Suppose that M is atoroidal and aspherical. Then the submodules of $\mathfrak{J}(M)$ respectively $\mathfrak{H}(M)$, which are generated by the image of a nice geometric model $\mathfrak{s}$, are isomorphic to $S\mathfrak{R}\hat{\pi}^0$.

In section 9 we will see that the following result can be used to reproduce some of the results of Kalfagianni [28].

Theorem 1.3. Suppose that M is atoroidal and aspherical. Then for each $\beta \in \mathfrak{b}_0$ there is defined a $\mathfrak{R}$-homomorphisms ι_β from $\mathfrak{J}(M)$ respectively $\mathfrak{H}(M)$ into $\mathfrak{R}[[h]]$ satisfying $\iota_\beta \circ \mathfrak{s}(\beta') = \delta_{\beta,\beta'}$ for all $\beta, \beta' \in \mathfrak{b}_0$.

If a geometric model map is onto and there are no relations then a basis is given by the geometric models. Using that the arguments contained in the proofs in [19] actually show that there are nice geometric models generating the skein module, we deduce the following result.

Theorem 1.4. Let $M = L(p,q)$ be a Lens space and $p \neq 0$. Then $\mathfrak{J}(M)$ respectively $\mathfrak{H}(M)$ is isomorphic to $S\mathfrak{R}\hat{\pi}^0$.

It follows now easily from Przytycki's universal coefficient theorem for skein modules that the usual variant Homflypt module with $z = h =: s - s^{-1}$ is a free module over $\mathbb{Z}[q^{\pm 1}, v^{\pm 1}, \frac{v-v^{-1}}{s-s^{-1}}]$, and similarly the result for the usual Homflypt skein module defined over $\mathbb{Z}[q^{\pm 1}, h^{\pm 1}]$.

The combined results of sections 2 and 3 prove the following result, similarly to Theorem 1.1 for the *Vassiliev relation* as described below.

Let $\mathcal{L} := \cup_{k \geq 0} \mathcal{L}[k]$ be the set of all singular links in M.

Let $\mathcal{R} := \mathbb{Z}[q^{\pm 1}, h]$. We define the *Vassiliev potential* (not a skein potential according to the definition above) by $\sigma_V : \mathcal{L} \to I\mathcal{L} \subset \mathcal{R}\mathcal{L}$ by $\sigma_V(K_*) = hK_*$, where I is the ideal generated by h. Then we can define the *infinite Vassiliev module*

$$\mathcal{S}(\sigma_V) := \mathcal{R}\mathcal{L}/im(\partial - q^{-1}\sigma).$$

Let $\mathfrak{b}[k]$ denote the set of homotopy classes of singular k-links, see [2] and [12]. Then $\mathfrak{b}$ is in one-to-one correspondence with a certain set of chord diagrams in M up to homotopy, with specified ordering of the chords. We choose geometric models $\mathfrak{s}$, i. e. singular links realizing the given homotopy classes. Let $\mathfrak{b} := \cup_{k \geq 0} \mathfrak{b}[k]$.

Theorem 1.5. Let $\mathfrak{s}$ be a given choice of geometric models. There is defined a submodule $U_V \subset \mathcal{R}\mathfrak{b}[[I]]$ and map

$$\rho(\sigma_V, \mathfrak{s}) : \mathcal{L} \to \mathcal{R}\mathfrak{b}[[h]]/U_V,$$

which induces the isomorphism

$$\mathcal{S}(\sigma_V)[[h]] \to \mathcal{R}\mathfrak{b}[[h]]/U_V.$$

The submodule U_V is generated by (i) expansions of differentiability relations, $4T$-relations and tangency relations (defined from the local stratification of the

space of immersions), and (ii) expansions of images of generalized string topology homomorphisms defined on the 1-dimensional homology of spaces of k-immersions. The composition

$$\rho(\sigma_V, \mathfrak{s}) \circ \mathfrak{s} : \mathcal{R}\mathfrak{b} \to \mathcal{R}\mathfrak{b}[[h]]/U_V$$

is the composition of the natural map $\mathfrak{b} \to \mathcal{R}\mathfrak{b}[[h]]$ with the natural projection.

The expansion map $\rho(\sigma_V, \mathfrak{s})$ in Theorem 1.5 is defined on *all* singular links. The *local* differentiability and tangency relations can be subsumed in terms of the ordering of chords. But the geometric $4T$-relations are very hard to control explicitly. This has been the obstruction for a direct topological construction of Kontsevitch's integral from mapping space topology in the early $90's$ [5], [24] which is still open.

Question. For which 3-manifolds M is the homomorphism

$$\mathcal{S}(\sigma_V) \to \mathcal{S}(\sigma_V)[[h]]$$

injective? The map $\rho(\sigma_V, \mathfrak{s})$ appears as a kind of limit of Vassiliev type invariants and thus could be related to questions about the strength of Vassiliev imvariants.

The relation between skein theory and Vassiliev theory has been observed by Przytyki, see [43], [31] and [6]. For $i \geq 0$ let $\mathcal{G}_i(M; \mathcal{R})$ be the quotient of $\mathcal{RL}$ by the submodule generated by relations $K_* = K_+ - K_-$ for *all* $K_* \in \mathcal{L}$ and $*$ *any* double-point, and $K = 0$ for each singular link K with $> i$ double-points. Note that $\mathcal{G}_i(M; \mathcal{R})$ is generated by either $\mathcal{L}[0]$ or by the set of homotopy classes of singular k-links with $k \leq i$. The dual of $\mathcal{G}_i(M; \mathcal{R})$ is the module of $\mathcal{R}$-valued Vassiliev invariants of order $\leq i$ for links in M. In particular $\mathcal{G}_0(M; \mathcal{R})$ is dual to the module of type 0-invariants. This module is isomorphic to $S\mathcal{R}\hat{\pi}$, where $\hat{\pi}$ is the set of conjugacy classes of the fundamental group of M. The module $\mathcal{G}_1(M; \mathcal{R})$ is the dual of the module of type 1-invariants, and has been discussed in [34]. It has been observed in [12] that a generalized string topology homomorphism provides the indeterminancy of the universal type 1-invariant. The observation for homotopy skein modules has been made by the author in [25].

Let

$$\mathcal{G}(M; \mathcal{R}) := \varprojlim \mathcal{G}_i(M; \mathcal{R})$$

be the *finite Vassiliev module* of M. It has been observed by Przytycki that each skein relation defined by a skein potential σ (for $\partial(K_*) = K_+ - K_-$) defines an epimorphism

$$\mathcal{G}(M; \mathcal{R}) \to \mathcal{S}(\sigma)[[I]],$$

by mapping $K_* \in \mathcal{L}[1]$ to $K_+ - K_- - \sigma(K_*)$.

The relation between the infinite and finite Vassiliev module is subtle. The fact that h is non-invertible is important here. Assume we are in the Conway case $q = 1$. For $i \geq 0$ let $\mathcal{W}_i(M; \mathcal{R})$ be the quotient of $\mathcal{RL}$ by relations $K_+ -$

$K_- - K_*$ for $*$ the *last* double-point, and $K = 0$ for singular links with $\geq i$ double-points. Applying the relation to K_{**} we get $K_{**} = K_{*+} - K_{*-}$. This is equal to $K_{+*} - K_{-*}$, which is the skein relation applied to another double-point, if and only if the differentiability relations hold. Note that $\mathcal{S}(\sigma_V)/h^i\mathcal{S}(\sigma_V)$ is *not* generated by links because we can only use $hK_* = K_+ - K_-$ to reduce the number of double-points. Roughly, this results in the fact that the order of chords in chord diagrams representing homotopy classes of singular links, can be changed only in corresponding powers of h.

Our Theorems 1.1 and 1.5 are linearized versions of a general construction on 2-groupoids equpped with special structure. These 2-groupoids are constructed from certain 3-stage stratifications in spaces of k-immersions (immersions with k fixed double points, see section 3 for the precise definitions). We will construct a type of *deformation* of the usual homotopy 2-groupoid of the space of k-immersions of circles in M. It will be shown that the resulting algebraic 2-groupoid induces the usual skein and Vassiliev structures. The isotopy classes of links appear as the objects of a category, transversal deformations between links are the 1-morphisms of the category. It should be possible to generalize the 2-groupoids constructed in this article to n-groupoids, deforming the usual homotopy n-groupoids of the spaces of k-immersions. It could be a future goal to construct deformation structures of string topology in this setting, which could be a future project. In the language of 2-groupoids the *string topology* homomorphisms are the structure, which relate 2-morphisms and equivalence classes of 1-morphisms.

It will be important that the categorical structure above can be *linearized* similarly to the quantum invariant setting. Our viewpoint in this paper is much more general than necessary to deduce the explicit results above. But it is interesting to realize that *the structures inherent in skein modules and Vassiliev invariants, are a necessary consequence of the groupoid structure*.

Most results of this article could be extended easily to suitable tangles in 3-manifold following the definitions in [23], section 6. We decided to keep with the case of links to keep the notation less difficult.

2 Formal skein theory

For details about category and higher category theory see [4], ([33], 0.2), [35] and [37].

A 2-*category* is a triple $(\mathfrak{ob}, \hom, \mathrm{mor}) = (\hom_0, \hom_1, \hom_2)$, for which both $(\mathfrak{ob}, \hom)$ and $(\hom, \mathrm{mor})$ are categories in the usual sense. More precisely, the following structures are given. For $x, y \in \mathfrak{ob}$ there is defined $\hom(x, y)$ with composition:

$$\circ : \hom(x, y) \times \hom(y, z) \to \hom(x, z).$$

The elements in $\hom(x, y)$ are called 1-morphisms. Composition is often abbreviated $u \circ v =: uv$. The morphisms between 1-morphisms are called 2-morphisms.

The set of 2-morphisms from u to v is denoted $\mathrm{mor}(u,v)$, and is non-empty only for *parallel* objects $u,v \in \mathrm{hom}(x,y)$. There are two compositions of 2-morphisms. For $u,v,w \in \mathrm{hom}(x,y)$ there is defined:

$$\circ_1 : \mathrm{mor}(u,v) \times \mathrm{mor}(v,w) \to \mathrm{mor}(u,w).$$

For $x,y,z \in \mathfrak{ob}$, $u_1,u_2 \in \mathrm{hom}(x,y)$ and $v_1,v_2 \in \mathrm{hom}(y,z)$ there is defined

$$\circ_2 : \mathrm{mor}(u_1,u_2) \times \mathrm{mor}(v_1,v_2) \to \mathrm{mor}(v_1 \circ u_1, v_2 \circ u_2).$$

Recall the there are the natural source and target maps denoted by sour and targ, defined $\mathrm{hom} \to \mathfrak{ob}$ and $\mathrm{mor} \to \mathrm{hom}$ respectively.

A 2-category in which *all* morphisms are equivalences is called a 2-*groupoid*. Equivalence for 1-morphisms is in the weak sense [35]. Thus for $u \in \mathrm{hom}(x,y)$ there is exists $v \in \mathrm{hom}(y,x)$ such that both $u \circ v$ and $v \circ u$ are equivalent to the corresponding identity morphisms id. Equivalence means here that $v \circ u$ and id are related by a 2-morphism, i. e. $\mathrm{mor}(v \circ u, \mathrm{id}) \neq \emptyset$. The 2-morphisms have to be invertible *on the nose*, i. e. for a 2 morphism $h \in \mathrm{mor}(u,v)$ there exists a 2-morphism $h^{-1} \in \mathrm{mor}(v,u)$ such that $h \circ_1 h^{-1} = \mathrm{id}_u$ and $h^{-1} \circ_1 h = \mathrm{id}_v$, where $\mathrm{id}_u, \mathrm{id}_v$ are the corresponding identity 2-morphisms in $\mathrm{mor}(u,u)$ respectively $\mathrm{mor}(v,v)$. A 2-groupoid *with inverses* is a small 2-groupoid with a choice of $v := u^{-1} \in \mathrm{hom}(y,x)$ for each $u \in \mathrm{hom}(x,y)$. Recall that an n-category is small if the class of objects and all classes of morphisms are sets. In the following without mentioning *we will assume that all our* 2-*groupoids are small and with inverses.*

The composition of 1-morphisms is associative up to the action of 2-morphisms. The set of 2-morphisms $\mathrm{mor}(u,u)$ is a group under $\circ_1$. For $x \in \mathfrak{ob}$ the set of equivalence classes of elements in $\mathrm{hom}(x,x)$ under the action of mor is a group with composition defined by $[u] \cdot [v] := [u \circ v]$. (This is well-defined because of the composition $\circ_2$.) Here $u,v \in \mathrm{hom}(x,x)$ are equivalent if $\mathrm{mor}(u,v) \neq \emptyset$.

Definition 2.1. A 2-*category with models* is a 2-categeory with a distinguished subset $\mathfrak{m} \subset \mathfrak{ob}$ such that for each $x \in \mathfrak{ob}$ there exists $b \in \mathfrak{m}$ such that $\mathrm{hom}(x,b) \neq \emptyset$. If $\mathrm{hom}(b_1,b_2) = \emptyset$ for all $b_1 \neq b_2$ then $\mathfrak{m}$ is called a *minimal set of models* of the 2-category.

For each set X let $F(X) =: F^{(0)}(X)$ be the free group generated by the elements of X. Let $F^{(j)}(X) := F(F^{j-1}(X))$ be defined inductively for $j \geq 1$. Let

$$\hat{F}(X) := \prod_{j \geq 0} F^{(j)}(X)$$

be the group defined by the infinite product. The group $\hat{F}(X)$ is not a free group but inverse limit of free groups.

Example 2.1. The group $\hat{F}(X)$ is always non-commutative for $X \neq \emptyset$. Let $X = \{*\}$. Then $F(X) = \mathbb{Z}$ but already $F^{(2)}(X) = F(\mathbb{Z})$ is the free group on an infinite number of generators. Consider the normal subgroup $2\mathbb{Z} \subset \mathbb{Z}$. Then $F(2\mathbb{Z}) \subset F(\mathbb{Z})$ is obviously not a normal subgroup.

There are natural inclusion *maps* $F^{(j-1)}(X) \subset F^{(j)}(X)$ defined by the inclusion of the basis. These maps are not homomorphisms because the product of the elements x_1, x_2 in $F^{(j-1)}$ maps to the basis element x_1x_2. The inclusions combine to define the natural *shift map* $\widehat{\mathrm{sh}} : \hat{F}(X) \to \hat{F}(X)$. Note that each map $Y \to F(X)$ comes with the usual extension, usually denoted by the same letter, $F(Y) \to F(X)$ from the universal property of the free group functor. Note that the extension of the inclusion $sh : F^{(j-1)}(X) \subset F^{(j)}(X)$ is the identity homomorphism. Now consider a homomorphism $h : F(Y) \to F(X)$. We can consider it as a map from the set $F(Y)$ into the group $F(X)$ and extend to the homomorphism $F(F(Y)) \to F(X)$. By composition with the inclusion $F(X) \subset F^{(2)}(X)$ we have defined the *map* $h_\bullet : F^{(2)}(Y) \to F^{(2)}(X)$. Note that there is also the homomorphism $F(h) : F^{(2)}(Y) \to F^{(2)}(X)$ defined by mapping a word $w_1 \ldots w_r$ with $w_i \in F(Y)$ to $h(w_1) \ldots h(w_r)$, which is different from $h_\bullet$.

Remark 2.1. The group $\hat{F}(X)$ has a second natural product reminiscent of the multiplication of polynomials. In fact if $(a_i), (b_j)$ are two elements in $\hat{F}(X)$ let (c_k) be the sequence defined by

$$c_k = \sum_{i+j=k} a_i b_j,$$

where the multiplication is performed in $F^{(k)}$ using the inclusion maps $F^{(i)} \subset F^{(k)}$ and $F^{(j)} \subset F^{(k)}$.

Moreover the shift map defines a natural action of the group $\hat{F}(X)$ on the set $\hat{F}(X)$ defined by $a \star b := a \cdot \widehat{\mathrm{sh}}(b)$ where the multiplication on the right hand side is the usual multiplication in $\hat{F}(X)$

There is a related natural construction. Note that the identity map from the set $F(X)$ into the group $F(X)$ extends to the homomorphism $F(F(X)) \to F(X)$. By iteration there are naturally defined homomorphisms $F^{(j)}(X) \to F(X)$ for each j. Thus we have defined the homomorphism:

$$c : \hat{F}(X) \to \widetilde{F}(X) := \prod_{j=0}^{\infty} F(X).$$

Note that there is defined a natural shift homomorphism $\widetilde{\mathrm{sh}}$ on $\widetilde{F}(X)$ such that $c \circ \widehat{\mathrm{sh}} = \widetilde{\mathrm{sh}} \circ c$. The shift homomorphism $\widetilde{\mathrm{sh}}$ is a kind of *universal operator action* representing the multiplication by a fixed ring element in the commutative module case.

There is the unique natural *functor*

$$\mathfrak{d} : \hom \to F(\mathfrak{ob})$$

defined by

$$\mathfrak{d}(u) = \mathrm{targ}(u)\mathrm{sour}(u)^{-1},$$

where $F(X)$ is considered as a monoid.

A *potential* of a 2-groupoid is a *contravariant* functor

$$\mathfrak{a} : \mathrm{hom} \to F(\mathfrak{ob}).$$

We will use $\mathfrak{a}$ to define inductively *power series expansions* of objects in terms of models. Note that $\mathfrak{a}$ uniquely extends to homomorphisms

$$F^{(j)}(\mathfrak{a}) : F^{(j)}(\mathrm{hom}) \to F^{(j+1)}(\mathfrak{ob})$$

for $j \geq 1$.

Theorem 2.1. Let $\mathcal{C}$ be a 2-groupoid with models $\mathfrak{m} \subset \mathfrak{ob}$. Let $\mathfrak{a}$ be a potential. Then there exists a natural normal subgroup $\mathcal{A} \subset \hat{F}(\mathfrak{m})$, and the map

$$\rho : \mathfrak{ob} \to \hat{F}(\mathfrak{m})/\mathcal{A},$$

such that the induced homomorphism:

$$\rho : \hat{F}(\mathfrak{ob}) \to \hat{F}(\mathfrak{m})/\mathcal{A}$$

satisfies the formal skein relation:

$$\rho(\mathfrak{d}(u)) = \widehat{\mathrm{sh}}(\rho(\mathfrak{a}(u^{-1})))$$

for all $u \in \mathrm{hom}_1$ and the induced map

$$\widehat{\mathrm{sh}} : \hat{F}(\mathfrak{m})/\mathcal{A} \to \hat{F}(\mathfrak{m})/\mathcal{A}.$$

Proof. The normal subgroup $\mathcal{A}$ and the homomorphism ρ is constructed from a sequence of normal subgroups $\mathcal{A}_j \subset F^{(j)}(\mathfrak{m})$ and homomorphisms $\rho_j : \mathfrak{ob} \to F^{(j)}(\mathfrak{m})/\mathcal{A}_j$. Let $\mathcal{A}_0$ be the trivial group. Define the map $\rho_0 : \mathfrak{ob} \to F^{(0)}(\mathfrak{m})$ by assigning to to each $x \in \mathfrak{ob}$ its unique model $m \in \mathfrak{m}$. We define $\rho_1 : \mathfrak{ob} \to F^{(1)}(\mathfrak{m})/\mathcal{A}_1$ in the following way. Consider $u \in \mathrm{hom}(x,m)$. There is defined $\mathfrak{a}(u) \in F(\mathfrak{ob})$. So we can apply the induced homomorphism

$$F(\rho_0) : F(\mathfrak{ob}) \to F^{(1)}(\mathfrak{m}).$$

Let $\lambda_1(u)$ be the resulting element. Define $\mathcal{A}_1 \subset F^{(1)}(\mathfrak{m})$ to be the normal subgroup generated by all elements $\lambda_1(u)\lambda_1(v)^{-1}$ for all $u, v \in \mathrm{hom}(x,m)$. Then the image of $\lambda_1(u)$ in the quotient $F^{(1)}(\mathfrak{m})/\mathcal{A}_1$ does not depend on $u \in \mathrm{hom}(x,m)$ but only on x. Next suppose that we have defined $\rho_j : \mathfrak{ob} \to F^{(j)}/\mathcal{A}_j$. Let ρ_j^1 be a lift of ρ_j to $F^{(j)}(\mathfrak{m})$. Note that the set of lifts is an orbit of of $\mathcal{A}_j$, and we can write ρ_j^a for the lift differing from ρ_j^1 by the action of $a \in \mathcal{A}_j$. For $u \in \mathrm{hom}(x,m)$ define $\lambda_{j+1}^a(u) \in F^{(j+1)}(\mathfrak{m})$ by application of

$$F(\rho_j^a) : F(\mathfrak{ob}) \to F(F^{(j)}(\mathfrak{m}) = F^{(j+1)}(\mathfrak{m})$$

to the element $\mathfrak{a}(u)$. Then define $\mathcal{B}_{j+1}$ by the normal subgroup of $F^{(j+1)}(\mathfrak{m})$ defined by all elements

$$\lambda_{j+1}^{a_1}(u)(\lambda_{j+1}^{a_2}(v))^{-1}$$

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for all $u, v \in \hom(x, m)$ and all $a_1, a_2 \in \mathcal{A}_j$. Then the image $\rho_{j+1}(u)$ of $\lambda_1^a(u)$ in the quotient $F^{(j+1)}(\mathfrak{m})/\mathcal{B}_{j+1}$ does not depend on the choice of $a \in \mathcal{A}_j$ or $u \in \hom(x, m)$ and ρ_{j+1} is well-defined. Finally define $\mathcal{A}_{j+1}$ by the normal subgroup generated by $\mathcal{B}_{j+1}$ and all elements of the form $\widehat{\mathrm{sh}}(x^{-1})\widehat{\mathrm{sh}}(xa)$ for all $x \in F^{(j)}(\mathfrak{m})$ and all $a \in \mathcal{A}_j$. Then obviously $\widehat{\mathrm{sh}}$ induces the map $\widehat{\mathrm{sh}} : \hat{F}(\mathfrak{m})/\mathcal{A} \to \hat{F}(\mathfrak{m})/\mathcal{A}$. The second claim follows comparing $\rho(\mathrm{targ}(u))$ and $\rho(\mathrm{sour}(u))$. In the definition of ρ any morphism can be chosen so for a choice of $v \in \hom(\mathrm{targ}(u), m)$ in the first case we can choose $vu \in \hom(\mathrm{sou}(u), m)$ in the second case where m is the model for $\mathrm{targ}(u)$ and $\mathrm{sour}(u)$. Then the equality follows immediately from $\mathfrak{a}(vu) = \mathfrak{a}(u)\mathfrak{a}(v)$. □

The subgroup $\mathcal{A}$ depends inductively on all possible choices of morphisms in the sets $\hom(x, m)$. This is difficult to describe in the most abstract setting, in particular because of the lack of structure of the shift $\widehat{\mathrm{sh}}$. Thus we consider the composition of ρ with the projection onto $\widetilde{F}(\mathfrak{m})$.

For each set X and subset $U \subset \widetilde{F}(X)$ let $(U)_{\widetilde{\mathrm{sh}}}$ denote the smallest normal subgroup of $\widetilde{F}(X)$ containing all subsets $\widetilde{\mathrm{sh}}^j(U)$ for all $j \geq 0$.

Theorem 2.2. The map ρ induces the isomorphism

$$\widetilde{\rho} : \widetilde{F}(\mathfrak{ob})/(\mathfrak{d}(u)\widetilde{\mathrm{sh}}(\mathfrak{a}(u^{-1})), u \in \hom)_{\mathrm{sh}} \cong \widetilde{F}(\mathfrak{m})/\widetilde{\mathcal{A}}.$$

Proof. The homomorphism c induces the epimorphism

$$\hat{F}(\mathfrak{m})/\mathcal{A} \to \widetilde{F}(\mathfrak{m})/\widetilde{\mathcal{A}}$$

with $\widetilde{\mathcal{A}} := c(\mathcal{A})$. Thus by composition we have defined

$$\mathfrak{ob} \to \widetilde{F}(\mathfrak{m})/\widetilde{\mathcal{A}}.$$

We extend this map to the homomorphism $\overline{\rho}$ defined on $\widetilde{F}(\mathfrak{ob}) = \prod_{j=0}^{\infty} F(\mathfrak{ob})$ by mapping the sequence (u_j) to $\prod_{j\geq 0} \widetilde{\mathrm{sh}}^j(u_j)$. This is an infinite product but finite in each factor of $\widetilde{F}(\mathfrak{m})$, thus well-defined. Obviously

$$((\mathfrak{d}(u)\widetilde{sh}(\mathfrak{a}(u^{-1}), u \in \hom)) \subset ker(\overline{\rho}).$$

Let $\widetilde{\rho}$ denote the homomorphism defined on the quotient. By construction $\overline{\rho}$ maps $\widetilde{F}(\mathfrak{m}) \subset \widetilde{F}(\mathfrak{ob})$ by projection onto $\widetilde{F}(\mathfrak{m})/\widetilde{\mathcal{A}}$. This implies that $\widetilde{\rho}$ is onto. Moreover the composition $\widetilde{\mathcal{A}} \subset \widetilde{F}(\mathfrak{m}) \subset \widetilde{F}(\mathfrak{ob}) \to \widetilde{F}(\mathfrak{ob})/(\mathfrak{d}(u)\widetilde{\mathrm{sh}}(\mathfrak{a}(u^{-1}), u \in \hom)_{\widetilde{\mathrm{sh}}}$ is the trivial homomorphism by the very construction of $\mathcal{A}$. This easily implies the injectivity of $\widetilde{\rho}$. □

Remark 2.2. The group $\widetilde{F}(\mathfrak{ob})/((\mathfrak{d}(u)\widetilde{\mathrm{sh}}(\mathfrak{a}(u^{-1})), u \in \hom))$ is the completion of the $\mathfrak{a}$-*skein group* defined by

$$\mathcal{S}(\mathfrak{a}) := \bigoplus_{j\geq 0} F(\mathfrak{ob}))/(\mathfrak{d}(u)\mathrm{sh}(\mathfrak{a}(u^{-1})))_{\mathrm{sh}},$$

where sh is the shift homomorphism on the direct sum and $(\ )_{\mathrm{sh}}$ is defined as above.

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The advantage of working with $\widetilde{F}(\mathfrak{m})$ comes from the easier description of the indeterminancy subgroup $\mathcal{A}$ resulting from the homomorphism property of $\widetilde{\mathrm{sh}}$. Note that the contributions to $\widetilde{\mathcal{A}}_{j+1}$ resulting from lower orders now can be describe by $\widetilde{\mathrm{sh}}(\mathcal{A}_j)$.

We will *split* the contributions to $\mathcal{A}$ now in a natural way associated to the two origins of relations. Actually such a splitting is only possible in each degree but we can argue inductively. The idea is that, because of the 2-groupoid structure, we can consider all $u \in \mathrm{hom}(m, m)$ with $\mathrm{mor}(u, 1) \neq \emptyset$ and all those with $\mathrm{mor}(u, 1) = \emptyset$, considered up to the action of mor. This suffices to dscribe $\mathcal{A}_1$ and thus by induction all $\mathcal{A}_j$. Note that the change from $u : x \to m$ to $v : x \to m$ is provided only by uv^{-1} and there is $h \in \mathrm{mor}(u, v)$ if and only if there is $h' \in \mathrm{mor}(uv^{-1}, 1)$.

Recall that the functor $\mathfrak{d} : \mathrm{hom} \to F(\mathfrak{ob})$ induces the homomorphism $\mathfrak{d} : F(\mathrm{hom}) \to F(\mathfrak{ob})$ by the free group property. Also recall that $\rho_0 : \mathfrak{ob} \to F(\mathfrak{m})$ assigns to each object its model, and let ρ_0 also denote the extension to $F(\mathfrak{ob})$.

Definition 2.2. Let $\mathfrak{r} \subset \mathrm{mor}$ be a subset. An potential $\mathfrak{a}$ is called *insensitive with respect to* $\mathfrak{r}$ if for each $m \in \mathfrak{m}$ and $r \in \mathrm{hom}(m, m)$ such that there exists $r \in \mathrm{mor}(u, 1) \cap \mathfrak{r}$, the following holds: (i) $\rho_0(\mathfrak{a}(u)) = 0$, and (ii) there exists $w \in F(\mathrm{hom})$ such that $\mathfrak{d}(w) = \mathfrak{a}(u)$ and $\mathfrak{a}(w) = 1$. The potential is called *insensitive* if it is insensitive with respect to $\mathfrak{r} = \mathrm{mor}$.

Theorem 2.3. Suppose that $\mathfrak{a}$ is insensitive. Then $\widetilde{\mathcal{A}}$ is the normal group generated by the images of maps

$$\mathrm{hom}(m, m)/\mathrm{mor} \to \widetilde{\mathrm{F}}(\mathfrak{m})$$

for all $m \in \mathfrak{m}$ and all their shifts. These maps are defined inductively by arbitrary expansion of elements in $\mathrm{hom}(m, m)$ using the skein relation and following the idea of the proof of Theorem 2.4. It suffices to consider the image of a generating set of $\mathrm{hom}(m, m)/\mathrm{mor}$.

Proof. Let $u \in \mathrm{hom}(m, m)$ with $\mathrm{mor}(u, 1) \neq \emptyset$. The contribution from u to $\mathcal{A}_1$ is provided by applying ρ_0 from the proof of Theorem 2.4 (respectively its c-projection) to $\mathfrak{a}(u)$. This involves a choice of homomorphism w from $\mathfrak{a}(u)$ to some element of $F(\mathfrak{m})$, such that $\mathfrak{d}(w) = \mathfrak{a}(u)$. Note that because of (i) of insensitivity we know that the contribution in $\mathcal{A}_1$ vanishes. Moreover because of (ii) we know that the contributions in higher order vanish too up to indeterminancies in order ≥ 2. (We also need that $\mathcal{A}_0$ is the trivial group). By induction it can be proved that the contributions can be assumed to vanish in all orders, up to the indeterminancies resulting from the choices described by elements of $u \in \mathrm{hom}(m, m)$ with $\mathrm{mor}(u, 1) = \emptyset$. $\square$

Consider $h \in \mathrm{mor}(u, v)$ with $u, v : m \to m$. Let

$$d(h) := \mathfrak{a}(uv^{-1}) \in F(\mathfrak{ob}).$$

Note that $d(h)$ actually only depends on its source and target. Let $\mathcal{D} \subset F(\mathfrak{ob})$ be the normal closure of the image of d. Then there is well-defined homomorphisms

$$\gamma : \mathrm{hom}(m, m)/mor \to F(\mathfrak{ob})/\mathcal{D},$$

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which assigns to $u : m \to m$ the element $\mathfrak{a}(u)$. Here we use the group structure on $\hom(m, m)/\mathrm{mor}$ given by $u \cdot v := vu$, where on the right hand side the operation is composition. The homomorphism γ seems to be a kind of *categorical* version of a string topology homomorphisms of Chas and Sullivan.

Next we will linearize the above structures.

Definition 2.3. A *model complexity* is a map

$$\mathfrak{c} : \mathfrak{m} \to \mathbb{N},$$

where $\mathbb{N}$ is the set of non-negative integers.

If $\mathfrak{m}$ is minimal then the model complexity first extends to $\mathfrak{ob}$ by $\mathfrak{c}(x) = \mathfrak{c}(m)$, where m is the unique model of x. Then extend to hom by defining $\mathfrak{c}(u) = \mathfrak{c}(\mathrm{targ}(u)) = \mathfrak{c}(\mathrm{sour}(u))$. Finally define $\mathfrak{c}(h) = \mathfrak{c}(u)$ for each 2-homomorphism $h : u \to v$. Note that $\mathfrak{c}(u) = \mathfrak{c}(v)$. There is always defined the trivial complexity, which is a constant. If there is no explicit mentioning of a complexity then we will assume that we work with the trivial complexity. Let $\mathfrak{ob}(j) := \mathfrak{c}^{-1}(j)$ and similarly for hom and mor.

Definition 2.4. Let $R \supset \mathbb{Z}$ be a commutative ring with 1.
(i) A *linear potential* is a contravariant functor:

$$\mathfrak{a} : \hom \to R\mathfrak{ob},$$

where $R\mathfrak{ob}$ is considered as a groupoid (abelian group), and hom is the category with objects $\mathfrak{ob}$ and morphisms hom.
(ii) Let $I \subset R$ be an ideal and $\mathfrak{c}$ be a model complexity. Then $\mathfrak{a}$ is a linear potential with respect to I if it maps $\hom(j)$ into the direct sum of

$$R(\bigcup_{i<j} \mathfrak{ob}(i)) \bigoplus I(\bigcup_{j \leq i \leq \ell} \mathfrak{ob}(i))$$

for each $j \geq 0$ and a fixed number ℓ independent of j.

Let $\partial : R(\hom) \to R\mathfrak{ob}$ be defined by

$$\partial(u) = \mathrm{targ}(u) - \mathrm{sour}(u).$$

For a trivial model complexity we have a linear potential with respect to I if it maps *all* homomorphisms into $I(\mathfrak{ob})$.

Now replace in the constructions above for each set X the group product $\widetilde{F}(X)$ by the infinite product of free modules

$$\prod_{j \geq 0} I^j X.$$

Replace the potential by the linear potential. Then the results above admit analogs in the commutative framework of R-modules.

$$\prod_{i \geq 0} I^i X \cong RX[[I]].$$

The isomorphism is defined from the sequence of homomorphisms mapping $(u_i) \in \prod_{i \geq 0} I^i X$ to $\sum_{i \geq 0} u_i \in RX/I^j X$ for $j \geq 0$. This is compatible with projections and thus defines a homomorphism into the inverse limit, which is easily be seen to be an isomorphism. Similarly we call $\mathfrak{a}$ insensitive if for each $u \in \hom(m, m)$ with $\mathrm{mor}(u, 1) \neq \emptyset$ we have that (i) $\rho_0(\mathfrak{l}(u)) = 0$ and (ii) there exists $w \in \hom$ such that $\mathfrak{a}(u) = \mathfrak{d}(w)$ and $\mathfrak{a}(w) = 0$.

Example 2.2. Let R be a commutative ring with 1 and let $I = (h)$. Let X be a set. Let $\mathfrak{c}$ be the trivial complexity. Then there exists the natural *linearization functor*

$$\widetilde{F}(X) \to RX[[I]]$$

by defining it on the i-th factor of $\widetilde{F}(X)$ such that $x \in X$ maps to $h^i x \in I^i X$. Note that this map factors through $\mathbb{Z}(X)$ on each factor. By composition with the linearization functor a potential becomes linear.

Theorem 2.4. Let $\mathfrak{a}$ be a linear potential with respect to I and model complexity $\mathfrak{c}$. Suppose that $\mathfrak{a}$ is insensitive. Then there exist a map

$$\mathfrak{ob} \to R\mathfrak{m}/\mathcal{I},$$

where the submodule $\mathcal{I}$ is generated by the images and I-translates of generators under maps

$$\hom(m, m)/\hom_2 \to R\mathfrak{m}[[I]] \cong \prod_{i=o}^{\infty} I^i \mathfrak{m}.$$

Moreover there is an induced homormphism:

$$R\mathfrak{ob}/(\partial(u) - \mathfrak{a}(u), u \in \hom) \cong R\mathfrak{m}[[I]]/\mathfrak{I},$$

and an induced isomorphism

$$\left(R\mathfrak{ob}/(\partial(u) - \mathfrak{a}(u), u \in \hom)\right)[[I]] \cong R\mathfrak{m}[[I]]/\mathfrak{I}.$$

Proof. If $\mathfrak{c}$ is a trivial model complexity then the proof is analogous to the proofs above of similar results. Otherwise consider the subset $\mathfrak{ob}(i) \subset \mathfrak{ob}$. Then we will change the construction of Theorem 2.4 into the k-th factor in $\prod_i R\mathfrak{m}$. Let ℓ be the nonnegative integer defined in Definition 2.4. Then for a given complexity i, all elements of $\mathfrak{ob}(i)$ map into $I^{k-j}\mathfrak{m}$ for some $j \leq i + (k-j)\ell$. This can easily be proved by induction. Thus $j(\ell + 1) \leq i + k\ell$ and

$$k - j \geq k - \frac{i + k\ell}{\ell + 1} =: \phi(i, \ell, k).$$

The sequence of numbers $\phi(i, \ell, k)$ converges to infinity for k to infinity. (Essentially we can replace the inverse limit over k by an inverse limit over $\phi(i, \ell, k)$, see the remark below). Now one can construct submodules $\mathfrak{I}_j$ and homomorphisms

$$\mathfrak{ob} \to \bigoplus_{j \leq k} R\mathfrak{m}/\mathfrak{I}_k =: \mathfrak{M}_k$$

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for each fixed i, k. We may have to apply the inductive procedure of Theorem 4 more than k times to define the image in $\mathfrak{M}_k$. In fact just apply it k' times with $\phi(i, \ell, k') \geq k$, but do the computation mod I^{k+1}. $\qquad\square$

Remark 2.3. It is possible to construct submodules $\mathcal{C}_k \subset R\mathfrak{m}/I^k\mathfrak{m}$ and compatible maps

$$\mathfrak{ob} \to \left(R\mathfrak{m}/I^k\mathfrak{m}\right)/\mathcal{C}_k$$

directly in the following way. The maps will be constructed by application of the main procedure from Theorem 2.4, i. e. by replacing objects by model objects, and elements in the image of $\mathfrak{a}$ based on the *skein relation* $\partial(u) = \mathfrak{a}(u)$ for $u \in$ hom. More precisely construct maps on $\mathfrak{ob}(i)$ for each k by induction on the complexity i. Then we can assume that we have the expansion in $R\mathfrak{ob}$:

$$x = \sum_j a_j m_j + b + r,$$

with $a_j \in R, m_j \in \mathfrak{m}$ and r a linear combintaion of skein relations. Let $b = \sum_n b_n x_n$ with $b_n \in I^k$ and $x_n \in \mathfrak{ob}$. Note that we can assume inductively that each object in the linear combination b has complexity at most $i + \ell k$. Now apply the skein relations to x_n. This will give rise to a linear combination containing models, elements of the form ay with $a \in I$ and $y \in \mathfrak{ob}$ and elements of the form $a'y'$ with $a' \in R$ but $c(y') \leq i + \ell - 1$. Thus after at most $\ell k + 1$ applications of the skein relation we will have written x as a linear combination

$$x = \sum_j a'_j m'_j + b + b' + r',$$

with $b \in I^{k+1}\mathfrak{ob}$, $c(b') < i$ and r' a linear combination of skein relations. Now we can apply the inductive adssumption to expand b' in the form we want, and finally get an expansion we intend. The indeterminancies result from all possible choices in the procedure of applications of the skein relation.

We need one more definition for further discussions.

Definition 2.5. We say that the 2-groupoid is *free on a set of elementary homomorphisms S* if the set hom is in one-to-one correspondence with the *subset* of $F(S)$ defined by the products of composable words in $S^{\pm}$ such that the product of words corresponds multiplication in the free group. In this case we write hom $= F(S, \circ)$.

3   Mapping spaces and Vassiliev groupoids

General references for this section are [7], [47] and [48] for the topology and smooth structure of the mapping spaces. For some necessary results concerning transversality also see [50].

Let M be an oriented compact 3-manifold. Let $\mathrm{Diff}(M)$ be the group of diffeomorphisms of M, which are isotopic to the identity. Let $\mathrm{iso}(M)$ be the

space of paths $\gamma : I \to \mathrm{Diff}(M)$ with $\gamma(0) = id$. (There is the fibration

$$\Omega\mathrm{Diff}(M) \to \mathrm{iso}(M) \xrightarrow{ev} \mathrm{Diff}(M),$$

with $\mathrm{ev}(\gamma) = \gamma(1)$, which is closely related to the 2-groupoids discussed in this section.)

In this section we describe the 2-groupoids used in this paper. Our approach could generalize to spaces X with a suitable manifold stratification in codimensions $\leq n$ (generalizing spaces of immersions of circles into a 3-manifold). In fact the easiest application here could be to spaces of immersions of circles into higher dimensional manifolds. For different results relating stratified spaces and 2-catgeory theory see [45].

The idea is to replace the fundamental n-groupoid by a subgroupoid consisting of maps transversal with respect to the stratification. The resulting n-groupoid is *quantized* with respect to certain morphisms, i. e. we form a quotient groupoid in which these morphisms are identities. For $n = 2$ and X a space of immersions of circles into the 3-manifold M, the quanitzation is with respect to 1-morphisms, which are isotopies (traces of the action of $\mathrm{Diff}(M)$). Here we only describe the resulting 2-groupoid. It turns out that forming the quotient groupoid is a quite delicate problem on its own.

Let $\widetilde{\mathrm{imm}}(j)$ be the set of *singular links* $f : \cup_j S^1 \to M$ as defined for knots in [7], chapter 3. These are immersions, which induce embeddings on $\cup_j S^1 \setminus A$ for a finite subset A of $\cup_j S^1$ and for any two distinct points $x_1, x_2 \in A$ the branches of f at $f(x_1)$ and $f(x_2)$ can have tangency of at most finite order. We will also assume that all small diagonals are deleted (i. e. immersions containing identical components). It is shown in [7] that $\widetilde{\mathrm{imm}}(j)$ is an open subset of the space of all immersions $\cup_j S^1 \to M$, where the last space is equipped with a smooth structure based on an infinite limit of Hilbert spaces. Moreover the proof in ([7], 3.2.1 Lemma) generalizes easily from knots to links showing that the inclusion of singular links into the space of all immersions is a weak homotopy equivalence.

Note that the diagonals form a subset of infinite codimension which can be ignored in most discussions. The symmetric group Σ_j acts smoothly and freely on $\widetilde{imm}(j)$ by permuting the circles in the domain. Let $\mathrm{imm}(j)$ be the resulting quotient manifold.

For each space X let $C_i(X)$ denote the space of ordered confiquations of i points in X. An ordered i-configuration is just an embedding of the space $*_1 \cup \ldots \cup *_i \hookrightarrow X$. Let $k \geq 1$. Consider the evaluation map

$$ev : \widetilde{\mathrm{imm}}(j) \times C_{2k}(\cup_j S^1) \to M^{2k}$$

defined by $\mathrm{ev}(f, x) := f(x)$. Let $\Delta \subset M^{2k}$ be the submanifold of those points $(x_1, y_1, x_2, y_2, \ldots, x_k, y_k)$ with $x_i = y_i$ for $i = 1, \ldots k$ but $x_i \neq x_j$ for $i \neq j$. Note that Δ is a locally closed submanifold of M^{2k}, and is diffeomorphic to the configuration space $C_k(M)$ of ordered k-configurations in M.

Let $\widetilde{\text{imm}}[k](j) := ev^{-1}(\Delta)$. It follows e.g. from [22] and standard *local* arguments that this is an infinite dimensional submanifold of $\widetilde{\text{imm}}(j) \times C_{2k}(\cup_j S^1)$ of codimension k. The symmetric group acts on $\widetilde{\text{imm}}(j) \times C_{2k}(\cup_j S^1)$ by simultaneously permuting the circles in the domain of immersions and the circles in the codomain of the embeddings in $C_{2k}(\cup_j S^1)$. The group $\mathbb{Z}_2^k$ acts by permuting the points in the domain of embeddings $*_1 \cup \ldots *_{2k} \to \cup_j S^1$. Let $\text{imm}[k](j)$ be the quotient by the actions. (Again we may delete diagonals as above to consider a free action.)

Now for $k \geq 0$ let

$$\text{imm}[k] := \bigcup_{j \geq 0} \text{imm}[k](j)$$

be the space of *k-immersions* in M. Note that a k-immersion always has k distinguished *ordered* double-points.

Remark 3.1. The spaces $\text{imm}[k](j)$ are related to spaces of mappings of graphs into M. In fact, one can use the ordering of the configuration space coordinates, orientation and minimal distance between the $2k$ points in $\cup_j S^1$ to define a map from $\cup_j S^1$ onto oriented standard graphs, such that immersions with the given configuration factor through mappings of graphs. This can be done *continously* over $\text{imm}[k](j)$. The graphs are equipped with an orientation around the vertices and basepoints on components away from the vertices. This will define a continuous mapping from $\text{imm}[k](j)$ onto a space of mappings of graphs into M. Using length parameters on edges of the graphs it is possible to construct an inverse map.

Let $\text{emb}[k] \subset \text{imm}[k]$ be the subset representing immersions with precisely k-double-points without tangencies. These immersions are called k-embeddings.

For $k \geq 0$ let $\mathcal{L}[k]$ denote the set of *isotopy* classes of k-embeddings. Here two k-embeddings are isotopic if there is a continous path in $\text{emb}[k]$ joining the two k-embeddings.

Then $\mathcal{L}[0]$ is the usual set of isotopy classes of oriented embeddings of circles also called oriented links in M. The elements of $\mathcal{L}[k]$ in general are called oriented k-links in M.

Let $\text{imm}[k,1]$ denote the subspace of elements in $\text{imm}[k]$ representing immersions with precisely $(k+1)$ double-points without tangencies.

Lemma 3.1. The space $\text{imm}[k,1] \subset \text{imm}[k]$ is naturally homeomorphic to the space $\text{emb}[k+1]$.

Proof. There is a natural ordering of the double-points by putting the distinguished double-point in the last place. Thus the sets are in one-to-one correspondence, which is easily seen to be a homeomorphism. $\square$

Let $\text{imm}[k,2]$ denote the union of the set of elements in $\text{imm}[k]$ consisting of (i) immersions with $(k+2)$ double-points without tangency, (ii) immersions with $(k-1)$ double points without tangency and a triple point, and (iii) immersions with k double-points, $(k-1)$ of which without tangency and a tangency at one of the double-points. Note that the immersions with (i) in $\text{imm}[k,2]$ have

an ordering of double-points up to permutations of the last two double-points. Thus there is a natural 2-fold covering map $\mathrm{emb}[k+2] \to \mathrm{imm}[k,2]_\bullet$, where $\mathrm{imm}[k,2]_\bullet \subset \mathrm{imm}[k,2]$ is the subspace satisfying (i). In general let $\mathrm{imm}[k,j]_\bullet \subset \mathrm{imm}[k]$ be the subspace represented by immersions, whose singularities are the k ordered double-points without tangency, and j additional double-points without tangency.

We will now describe the sequence of 2-groupoids $\mathcal{C}[k]$ for $k \geq 0$. Skein theory and Vassiliev theory should be considered as a construction on these groupoids.

The set of objects of $\mathcal{C}[k]$ will be the set $\mathcal{L}[k]$ of singular links with k double points, i. e. isotopy classes ($\mathrm{Diff}(M)$-orbits) of immersions of circles in M with precisely k double points without tangencies.

The 1-morphisms in the category $\mathcal{C}[k]$ will be the *isotopy classes of transversal paths* in $\mathrm{imm}[k]$. A path $\gamma : I \to \mathrm{imm}[k]$ is *transversal* if $\gamma(t) \in \mathrm{emb}[k]$ except for finitely many t where $\gamma(t)$ maps to an immersion with precisely $(k+1)$ double points without tangencies. Moreover, in the neighbourhood of such t the image of $\gamma(s)$ runs through a standard crossing change in some oriented 3-ball in M. Note that a homotopy between two elements of $\mathrm{emb}[k]$ can be approximated by a transversal homotopy. The set of points in $imm[k]$ with a tangency at one of the k double points has codimension 2 in $\mathrm{imm}[k]$, also compare [36], 3.2. In [36] Lin works in the PL-setting. The general reference for this transversality arguments and those in the following arguments we refer to [47]. (Here the mapping spaces are replaced by finite dimensional approximations using an embedding $M \subset \mathbb{R}^N$ for some N and considering *projections* of polynomial approximations in a tubular neighborhood of the embedding.) Vassiliev's theory will allow us to focus on the codimension of singularities and apply transversality like in a finite dimensional setting.

Next we define *isotopy* of transversal paths. Let $\chi : I \to \mathrm{iso}(M)$ be a *piecewise* continuous path. Assume that $t \mapsto \chi(t)(1) \in \mathrm{Diff}(M)$ is continuous. Note that $\chi(t)(0) = id$ for all t. The *adjoint* of χ is the map $\chi' : I \times I \to \mathrm{Diff}(M)$ defined by $\chi'(t,t') \to \chi(t)(t')$. Then χ acts as a *deformation* $I \times I \to \mathrm{imm}[k]$ of γ by

$$(t,t') \mapsto \chi'(t,t') \circ \gamma(t).$$

For a given transversal path γ we assume that χ only jumps at parameters t away from the singular parameters of γ. We assume that χ is a jump function in the following way. At a jump parameter t the limit of the isotopies from the left or right is different from the isotopy at t. But we always assume continuity from left or right. Then we say that χ defines the *isotopy* $\gamma \to \gamma'$ with $\gamma'(t) = \chi(t)(1) \circ \gamma(t)$. The singular parameters of γ' and γ coincide. In this way isotopy can change γ between singular parameters by arbitrary loops in $\mathrm{emb}[k]$. We will allow a second class of isotopies of a more trivial nature. Two transversal paths γ, γ' are also isotopic if there is a transversal homotopy $I \times I \to \mathrm{emb}[k] \cup imm[k,1]$ restricting to γ respectively γ' on $I \times \{0\}$ respec-

tively $I \times \{1\}$. Transversality of the homotopy means that the preimage of the set $\mathrm{imm}[k] \setminus \mathrm{emb}[k]$ is a 1-dimensional manifold properly embedded in M.

Note that composition of 1-morphisms is well-defined. In fact, let $\gamma : K_1 \to K_2$ and $\gamma' : K_2 \to K_3$ are homomorphisms for singular links K_i, $i = 1, 2, 3$. Here we choose representative paths γ, γ' between corresponding representing immersions. We can choose any isotopy between representatives $\gamma(1), \gamma'(0)$ of K_2, and take the usual composition of paths. The result of the composition does not depend on the choice of isotopy up to isotopy of transversal paths. The composition $\circ$ of transversal paths is associative because isotopy includes reparamatrizations of the interval. But in general $\gamma \circ \gamma^{-1}$ is not isotopic to the constant path because the embedding of the singular parameters in I is well-defined up to isotopy.

Finally the 2-morphisms between any two isotopy classes of transversal paths will be defined by *homotopy classes* of *transversal homotopies* between representing paths in $\mathrm{imm}[k]$. The appropiate notion of homotopy class is defined below. There are obvious ways of horizontal and vertical composition of maps $I \times I \to \mathrm{imm}[k]$ with certain boundary restrictions coinciding (corresponding to $\circ_2$ and $\circ_1$. A *transversal homotopy* is the result of *horizontal and vertical composition* of
(i) continous maps
$$I \times I \to \mathrm{imm}[k],$$
which are constant on $I \times \{0, 1\}$, and
(ii) maps $I \times I \to \mathrm{imm}[k]$, which are adjoint to piecewise continuous paths $I \to \mathrm{iso}(M)$ as defined above. (in general this is not constant on $I \times \{0, 1\}$, but the restriction to $I \times \{0, 1\}$ is two continuous paths in emb), and
(iii) transversal maps $I \times I \to \mathrm{emb}[k] \cup \mathrm{imm}[k, 1]$.

The transversality of 2-morphisms will be in the spirit of Lin [36]. We can assume that the preimage of the set of those elements in $imm[k]$, which are not in $emb[k]$, is a graph with vertices of valence 1, 2 or 4. All vertices of valence 1 will be contained in the boundary $\{0, 1\} \times I$ and be mapped to immersions with precisely $k + 1$ double-points without tangencies. All the open edges of the graph are mapped to immersions with precisely $k + 1$ double-points without tangencies. The vertices of valence 4 or 1 are contained in the interior of $I \times I$. Those of valence 4 are mapped to either (i) immersions with $k + 2$ double-points without tangency, or (ii) immersions with one transversal triple point (with a distinguished branch) and $k - 1$ double points without tangency. The vertices of valence 2 are mapped to immersions with k double points but with a tangency at one of those double-points. Note that for $k = 0$ both the tangency contributions and triple points vanish.

Now we explain homotopy of transversal homotopies. Note that a 2-morphism has source and target *isotopy* classes of transverse paths. We will generate homotopy of 2-morphisms by homotopy of (i), (ii) and compositions $\circ_1, \circ_2$. A

homotopy of (i) is a continous map

$$(I \times I) \times I \to \operatorname{imm}[k],$$

which is constant on $(I \times \{0,1\}) \times I$. Finally we *define* any two transversal homotopies of type (ii) mapping the transversal path γ to the transversal path γ' to be homotopic to the identity on the class of γ. It is now easy to see that the set of 2-morphisms *mor* is well-defined. Note that the defintions are straightforward in order to have well-defined compositions operations on the quotient groupoid.

Lemma 3.2. Each homotopy between transversal paths joining two k-embeddings can be approximated relative to the boundary by a transversal homotopy.

Proof. We consider the finite dimensional approximation model for the space of immersions respectively singular links, see [47],[48]. It can be assumed that evaluation maps restricted to $\operatorname{ev}^{-1}(\Delta) \subset \widetilde{\operatorname{imm}}(j) \times C_{2k}(\cup_j S^1)$ corresponding to higher order singularities and jet singularities are transversal. Then the result follows from known local models of the singularity strata and their codimensions in these spaces, for details see [48]. Obviously the open stratum in $\operatorname{imm}[k]$ is $\operatorname{emb}[k]$, the codimension-1 stratum consists of immersions with precisely $(k+1)$ ordered double-points without tangency. The codimension 2-stratum is constructed from the codimension-1 generic degeneracies in the limit set of the immersions in the codimension-1 stratum. $\square$

Remark 3.2. (i) Lin's transversality results [36] are the piecewise linear versions of the results of the lemma, see also the results by Stanford [44], Bar-Natan and Stoimenov [5] and Hutchings [24].
(ii) The result of the lemma holds for any map of a surface $F \to \operatorname{imm}[k]$ with given transversal map on the boundary ∂F.
(iii) It is easy but tedious to prove the groupoid property of the 2-category $\mathcal{C}[k]$. The inverse of a transverse path is the inverse path γ^{-1} with $\gamma^{-1}(t) = \gamma(1-t)$. The homotopy $I \times I \to \operatorname{emb}[k] \cup \operatorname{imm}[k,1], (s,t) \mapsto \gamma(t)$ then is transversal according to the definition above and shows $\hom([\gamma^{-1}] \circ [\gamma], *_1) \neq \emptyset$ for the constant path $*_1$ at $\gamma(0)$, after suitable reparametrization. Similarly we deduce from this $\hom([\gamma] \circ [\gamma^{-1}], *_2) \neq \emptyset$ for $*_2$ the constant path in $\gamma(1)$. The inverse of a 2-morphism represented by a transversal homotopy $h : I \times I \to \operatorname{emb}[k] \cup \operatorname{imm}[k,1]$ is similary defined by the transversal homotopy $h^{-1}(s,t) = h(s, 1-t)$. Note that in this case the composition represents the corresponding identity because 2-morphisms are by definition homotopy classes. Some care has to be taken of the fact that 1-morphisms are not necessarily represented by homotopic paths because corresponding transversal path representatives can differ by nontrivial loops in $\operatorname{emb}[k]$.

Definition 3.1. The *graded* 2-groupoid

$$\mathcal{C} = \mathcal{C}(M) := \bigcup_{k \geq 0} \mathcal{C}(k)$$

is called the *homotopy Vassiliev groupoid* of M.

The following two basic results describe the structure of the 2-groupoids $\mathcal{C}[k]$. The first theorem is immediate from the definitions. The proof of the second one follows from Lemma 3.1.

Theorem 3.1. For each $k \geq 0$ the morphism set hom$[k]$ is $\circ$-generated by the elementary morphisms $K_+ \to K_-$ defined by crossing changes, and their inverses $K_- \to K_+$, for all $K_* \in \mathcal{L}[k+1]$ with $*$ indicating the last double-point. By identification of $\mathcal{L}[k+1]$ with the elementary morphisms in $\mathcal{C}[k]$ we have

$$\text{hom}[k] = F(\mathcal{L}[k+1], \circ).$$

$\square$

Note that for any $K \in \mathcal{L}[k]$ there is the unique 1-morphism 1, which is represented by *any* isotopy (path in emb$[k]$) between representing embeddings of K.

We describe resolutions of triple points. Consider immersions in $imm[k]$ with precisely $(k-1)$ double points without tangencies and a transverse triple point. There is a distinguished branch and an ordering of the remaining two branches at the triple point. Let $\mathcal{T}[k]$ denote the set of isotopy classes of these immersions. Let $K \in \mathcal{T}[k]$. We will define resolutions $K_\pm^{1/2} \in \mathcal{L}[k+1]$ in the following way: $K_\pm^{1/2}$ ist the positive respectively negative resolution of the double point, which appears when the first respectively second branch is ignored and only the remaining and the distinguished branch are considered to intersect at the triple point. The geometric $4T$-relation will be built with these singular links. See [5], [24] and [47] for the description of geometric $4T$-relations.

Theorem 3.2. The morphism set mor$[k]$ is generated via compositions $\circ_1$ and $\circ_2$ by elementary 2-morphisms of the following form:

(i) morphisms $K_{-*}^{-1} K_{*+}^{-1} K_{+*} K_{*-} \to 1$ (1 is the 1-morphism for K_{--}) defined for each $K_{**} \in \mathcal{L}[k+2]$. (differentiablity relations)

(ii) for $k > 0$, morphisms $(K_+^1)^{-1}(K_+^2)^{-1}K_-^1 K_-^2 \to 1$ defined for each $K \in \mathcal{T}[k]$ (geometric 4T-relation).

(iii) for $k > 0$, morphisms $K \to K'$ for $K, K' \in \mathcal{L}[k+1]$ defined by changing the order of the two double-points appearing in the natural two-point resolution of a tangency.

Proof. The result is immediate from the definitions and Lemma 3.2. $\square$

Remark 3.3. It is possible to define a slight variation of the 2-groupoid $\mathcal{C}$ by changing the definition of isotopy of 1-morphisms given above in the following way. Instead of allowing arbitrary transversal homotopies in emb$[k] \cup \text{imm}[k, 1]$ only allow homotopies defined by reparametrizations of the interval (use homotopies from id into monotone functions $I \to I$). Then of course we have to introduce the isotopies of the above form into the 2-morphisms. In this case the 1-morphisms hom$[k]$ will be identified with the *monoid* generated by $\mathcal{L}[k+1]^{\pm 1}$. Then in the statement of Theorem 3.2 we have introduce additionally $uu^{-1} \leftrightarrow 1$.

We can construct a set of models $\mathfrak{m}$ for the category $\mathcal{C}$ by choosing an element in $\mathrm{emb}[k]$ for each path component of the spaces $\mathrm{imm}[k]$ and $k \geq 0$. Note that the set of path-components of $\mathrm{imm}[k]$ is the set of homotopy classes of singular k-links. $\mathfrak{b} = \cup_{k \geq 0}\mathfrak{b}[k]$. Note that $\mathfrak{b}[0]$ is in one-to-one correspondence with the set of monomials in $\hat{\pi}$, where $\hat{\pi}$ is the set of conjugacy classes of $\pi_1(M)$.

For $k \geq 1$, the homotopy classes of singular k-links are in one-to-one correspondence with the set of chord diagrams in M with k chords. These are usual chord diagrams equipped with free homotopy classes of maps into M assigned to each component of the complement of the set of endpoints of chords in $\cup_j S^1$. We will require an ordering of the chords for each representative chord diagram. Note that it is possible that chord diagrams with different orderings are homotopic, depending on the topology of the space of immersions into M.

The differentiablility relations impose *commutativity on the level of models*. This means that indeterminacies of expansions defined by abstract skein potentials in $\tilde{F}(\mathfrak{ob})$ in section 2 contain the commutators of $\mathfrak{ob}$. Thus it suffices to consider skein potentials like in the usual skein or Vassiliev theory.

Next we will define a certain 2-groupoid $\mathcal{B}$ from suitable bordism classes of mappings of i-dimensional manifolds into imm for $i = 0, 1, 2$. In this category the 1-morphisms are *linearized* in a certain way, as is suggested by the notion of skein potential in a commutative ring with 1. The precise definition of this 2-groupoid, called the *homology Vassiliev groupoid* turns out to be quite interesting and subtle on its own. In the following note that bordism and homology coincide in dimensions ≤ 2, see ([32], page 319)

The objects of $\mathcal{B}[k]$ are the same as those of $\mathcal{C}[k]$, i. e. the isotopy classes of embeddings into $\mathrm{emb}[k]$.

The 1-morphisms $x \to y$, for $x, y \in \mathcal{L}[k]$, are the oriented bordism classes u of maps of oriented compact 1-manifolds $W \to \mathrm{emb}[k] \cup \mathrm{imm}[k, 1]$, i. e. elements of $H_1(\mathrm{emb}[k] \cup \mathrm{imm}[k, 1], \mathrm{emb}[k])$ such that $\partial(u) = y - x$, where

$$\partial : H_1(\mathrm{emb}[k] \cup \mathrm{imm}[k], \mathrm{emb}[k]) \to H_0(\mathrm{emb}[k])$$

is the usual boundary operator. Note that this is well-defined since $H_0(\mathrm{emb})$ is the free abelian group on $\mathcal{L}[k]$. Thus the set of 1-morphisms hom of $\mathcal{B}$ is a certain *subset* of the homology group. Note that W can be represented by a map

$$(I \cup \cup_i S^1, \partial I) \to (\mathrm{emb}[k] \cup \mathrm{imm}[k], \mathrm{emb}[k]).$$

There can be *arbitrary* maps of closed components into $\mathrm{emb}[k] \cup \mathrm{imm}[k, 1]$. Note that a representative map may very well contain other component maps $I \to \mathrm{emb}[k] \cup \mathrm{imm}[k, 1]$. But the homological boundary of all these components vanishes in $\mathrm{emb}[k]$.

The 2-morphisms $u_1 \to u_2$ for 1-morphisms $u_i : x \to y$ and $i = 1, 2$, are more difficult to describe. See ([49], page 6) for a description of the general bordism

set-up used here. We consider *bordism classes* of quadruples of maps ($j = 1, 2$)

$$g : (F, \partial_j F, \partial_1 F \cap \partial_2 F) \to (\mathrm{imm}[k], \mathrm{emb}[k] \cup \mathrm{imm}[k, 1], \mathrm{emb}[k]),$$

where F is an oriented compact surface with boundary $\partial F = \partial_1 F \cup \partial_2 F$ such that $\partial_1 F \cap \partial_2 F$ is a disjoint union of two points. We require that $g|(\partial_j F)$ represents $u_j \in H_1(\mathrm{imm}[k] \cup \mathrm{emb}[k], \mathrm{emb}[k])$ for $j = 1, 2$. Suppose that we have given two quadruples g_i, $i = 1, 2$, as above:

$$g_i : (F_i, \partial_j F_i, \partial_1 F_i \cap \partial_2 F_i) \to (\mathrm{imm}[k], \mathrm{emb}[k] \cup \mathrm{imm}[k, 1], \mathrm{emb}[k]).$$

A bordism from g_1 to g_2 as above is a quadruple of maps

$$G : (W, \partial_j W, \partial_1 W \cap \partial_2 W) \to (\mathrm{imm}[k], \mathrm{emb}[k] \cup \mathrm{imm}[k, 1], \mathrm{emb}[k]),$$

where W is an oriented compact 3-manifold with corners. We have that $\partial_j(W) \subset \partial W$ is a 2-manifold with corners for $j = 1, 2$. More precisely $\partial W = \partial_1 W \cup \partial_2 W \cup F_1 \cup F_2$, $\partial_j W \cap F_i = \partial_j F_i$ for $i, j = 1, 2$, $\partial_j W$ is a bordism between $\partial_j F_1$ and $\partial_j F_2$, and $\partial_1 W \cap \partial_2 W$ is a bordism from $\partial_1 F_1 \cap \partial_2 F_1$ to $\partial_1 F_2 \cap \partial_2 F_2$. The map of W restricts to the maps given by g_i on the corresponding strata.

The definition of horizontal and vertical compositions of 2-morphisms requires cut, paste and smoothing arguments but is straightforward.

There is an obvious surjection (mor in the category $\mathcal{B}$)

$$\mathrm{mor} \to H_2(\mathrm{imm}[k], \mathrm{emb}[k] \cup \mathrm{imm}[k, 1]),$$

where we identify elements in $H_2(\mathrm{imm}[k], \mathrm{emb}[k] \cup \mathrm{imm}[k, 1])$ with bordism classes of maps of oriented compact surfaces

$$(F, \partial F) \to (\mathrm{imm}[k], \mathrm{emb}[k] \cup \mathrm{imm}[k, 2]).$$

The proof of the next result is immediate. We omit the tedious details because we will not use this result in the rest of the article.

Proposition 3.1. There exists the natural forgetful functor $\mathcal{C}(M) \to \mathcal{B}(M)$. $\quad\square$

Obviously we can use the images of the elementary category generators of $\mathcal{C}$ for the category $\mathcal{B}$. There has to included the attaching of certain handles. We will not describe this in detail at this point, see the next remark.

Remark 3.4. The 2-groupoid $\mathcal{B}$ is a *linearization* of $\mathcal{C}$ in the following way: Let hom$[k]$ denote the set of 1-morphisms in the skein groupoid. Then hom$[k] \subset \mathbb{Z}\mathcal{L}[k + 1]$ using the obvious identification. This is e. g. contained in the proof of Proposition 4.1 in the next section. It also follows from Remark 3.2 (ii). The additional commutation relations are of course induced by suitable maps of tori into imm. We have hom$[k] \neq \mathbb{Z}\mathcal{L}[k + 1]$. The composition of elements in hom$[k]$ corresponds to the usual addition of homology classes. But it is only defined for classes $u, v \in H_1(\mathrm{emb}[k] \cup \mathrm{imm}[k, 1])$, for which $\partial(u) = z - y$,

$\partial(v) = y - x$ such that $\partial(u + v) = z - x$. Then $u \circ v$ is a morphims from x to z and is represented by the homology class $u + v$. We will write $\hom[k] = A(\mathcal{L}[k+1], \circ) \subset \mathbb{Z}\mathcal{L}[k+1]$ to indicate that the 1-morphisms are the subset of the free abelian group corresponding to *composable* morphisms of the 2-groupoid. Note that by the excision property of homology the explicit insertion of loops in *emb* is not necessary in the category $\mathcal{B}$.

In the next sections we will study the structure of the skein groupoid and its variations relevant for skein modules. We will see that all the necessary facts needed are consequences of the exact homology sequences of the pairs $(\mathrm{imm}[k], \mathrm{emb}[k])$. More details about $\mathcal{B}$ will be given in a future article.

In order to keep notation short we will consider the graded spaces $\mathrm{imm} = \cup_{k \geq 0}\mathrm{imm}[k]$. We let $\mathcal{L} := \cup_{k \geq 0}\mathcal{L}[k]$. We use the shift notation $\mathcal{L}[+1]$ and define $\mathcal{L}[+1][k] := \mathcal{L}[k + 1]$.

Definition 3.2. Let $\mathcal{R}$ be a commutative ring with 1. A *skein potential in $\mathcal{R}$* is a map
$$\sigma : \mathcal{L}[+1] \to \mathcal{R}\mathcal{L}.$$
Sometimes we will only consider
$$\sigma : \mathcal{L}[1] \to \mathcal{L}[0]$$
and call this also a skein potential.

Lemma 3.3. Each skein potential defines a linear potential for the 2-categories $\mathcal{C}$ and $\mathcal{B}$.

Proof. This follows from the description of the 1-morphisms in $\mathcal{C}$ and $\mathcal{B}$ respectively. In fact, the skein potential extends to a homomorphism of abelian groups $\mathbb{Z}\mathcal{L}[+1] \to \mathcal{R}\mathcal{L}$. But $\hom \subset \mathbb{Z}\mathcal{L}[+1]$ and the inclusion maps composition into sum. $\square$

Note that a skein potential defined $\mathcal{L}[1] \to \mathcal{R}\mathcal{L}[0]$ can easily be extended trivially to a full skein potential. So usually we will not have to distinguish between the two cases.

In sections 7 and 8 we will define two further modifications of the 2-groupoid $\mathcal{B}$. This will explain the passage to Jones type skein relations and skein relations for framed oriented links.

The skein groupoid above is the result of applying a homology functor to a 2-groupoid defined by chain groups in imm, see [9] for a set-up of homology in this framework. It should be pointed out that this preprint of Chas and Sullivan never was published. The reason has been that transversality on the chain level could not be established. Many later approaches to define string topology on the chain level but in the bordism setting. In the low dimensional range we need for our 2-categorical setting bordism and homology coincide. But a more general n-category setting could not work on the level of homology and instead

use bordism or other modifications, see [8] and [39]. The chain groups *linearize* the deformation 2-groupoid constructed at the beginning of this section. It is also interesting to study skein theory from this viewpoint.. Then skein modules are the 0-dimensional homology modules of a chain complex with $\mathcal{R}$-coefficients and suitably σ-deformed boundary operator. The author intens to study this in the future.

4   Link theory interpretation of homology exact sequences

We consider the exact homology sequence of the pair $(\mathrm{imm}, \mathrm{emb})$ (remember the grading convention from section 3):

$$H_1(\mathrm{emb}) \to H_1(\mathrm{imm}) \to H_1(\mathrm{imm}, \mathrm{emb}) \to H_0(\mathrm{emb}) \to H_0(\mathrm{imm})$$

with $i_* : H_0(\mathrm{emb} \to \mathrm{H}_0(\mathrm{imm})$ surjective by transversality.

We want to describe the geometric meaning of the groups and homomorphisms of the above exact sequence. Obviously

$$H_0(\mathrm{emb}) \cong \mathbb{Z}\mathcal{L},$$

Note that $H_0(\mathrm{imm}[0]) \cong H_0(\mathrm{map})$, where *map* is the space of all smooth maps in M. This group is isomorphic to the free abelian group on the set of monomials in the set $\hat{\pi}$ of free homotopy classes of loops in M. So

$$H_0(\mathrm{imm}[0]) \cong S\mathbb{Z}\hat{\pi},$$

where S denotes the symmetric algebra. Using the above isomorphisms, the homomorphism i_* corresponds to the map $\mathfrak{h}$ defined by assigning to each oriented singular link its homotopy class.

The description of the relative homology group is more interesting.

Proposition 4.1. There is a natural isomorphism

$$H_1(\mathrm{imm}, \mathrm{emb}) \cong \mathbb{Z}\mathcal{L}[+1]/\mathcal{D}$$

for subgroups $\mathcal{D}[k] \subset \mathcal{L}[k+1]$, $k \geq 0$. For $k = 0$ the subgroup $\mathcal{D}[k]$ is generated by all elements

$$K_{*+} - K_{*-} - K_{+*} + K_{-*},$$

for all $K_{**} \in \mathcal{L}[k+2]$ (differentiability relations). For $k \geq 1$ the subgroup $\mathcal{D}[k+1]$ additionally has generators corresponding to all geometric $4T$-relations and tangency relations (Theorem 3.2, (ii) and (iii)).

Proof. It follows from Lemma 3.2 that for $k \geq 0$ the homomorphism

$$H_1(\mathrm{imm}[k, 1] \cup \mathrm{emb}[k], \mathrm{emb}[k]) \to H_1(\mathrm{imm}[k], \mathrm{emb}[k])$$

induced by the inclusion

$$\text{imm}[k, 1] \cup \text{emb}[k] \subset \text{imm}[k]$$

is surjective. Thus we can represent each homology class by a chain, which maps into $\text{imm}[k, 1] \cup \text{emb}[k]$ with the boundary mapping into $\text{emb}[k]$. It can be assumed that the mappings of 1-simplices are transverse in the sense of section 3. Thus all parameters values of the intervals map into $\text{emb}[k]$, except for a finite number mapping into $\text{imm}[k, 1]$. The orientation of the 1-simplex and the usual coorientation of $\text{imm}[k, 1]$ in $\text{imm}[k]$ define a sign for each *singular* parameter. This defines an integral linear combination of elements of $\text{imm}[k, 1]$ and thus an element in $H_0(\text{imm}[k, 1])$. Next consider a relative boundary. This is given by a 2-chain with boundary mapping into $\text{emb}[k]$. Now we apply Lemma 3.1, see also Remark 3.2(ii). Thus we can perturb each mapping of an oriented surface F into $\text{imm}[k]$ relative to the boundary such the set of parameters in F mapping into $\text{imm}[k] \setminus \text{emb}[k]$ is a 1-complex, which is properly embedded in F with vertices of valence 4 or 2 in the interior mapping into $\text{imm}[k, 2]$. The contribution of a boundary element thus is a sum of *monodromies* around elements of $\text{imm}[k, 2]$. These elements generate the subgroup $\mathcal{D}[k + 2]$ and are computed by abelianizing the relations described in Theorem 3.2. This proves the claim. $\square$

We summarize the discussion in the following theorem.

Theorem 4.1. The homology exact sequence of the pair (imm, emb) can be identified with the following exact sequence:

$$H_1(\text{emb}) \to H_1(\text{imm}) \xrightarrow{\mu} \mathbb{Z}\mathcal{L}[+1]/\mathcal{D} \xrightarrow{\partial} \mathbb{Z}\mathcal{L} \xrightarrow{\mathfrak{h}} \mathbb{Z}\mathfrak{b} \to 0$$

The homomorphism $\mu[k]$ is defined by the signed sum of all terms in $\mathcal{L}[k + 1]$ along transversal paths in $imm[k, 1] \cup emb[k]$. The homomorphism $\partial[k]$ is the Vassiliev resolution of the last double-point

$$\mathcal{L}[k + 1] \to \mathbb{Z}\mathcal{L}[k].$$

Proof. The result follows easily from proposition 4.1 and its proof. $\square$

The above sequence provides a description of the kernel of the $\mathfrak{h}$. This is the forgetful homomorphism from isotopy to homotopy, mapping *quantum obserbales* to their semi-classical limits.

Corollary 4.1. There is the isomorphism

$$\mathbb{Z}\mathcal{L} \cong \mathbb{Z}\mathfrak{b} \oplus coker(\mu).$$

The result of the corollary is just a different description of the 2-groupoid structure discussed in section 3. It shows in a neater way the distinction in terms of *local* relations and string topology homomorphism.

Recall that we have special *geometric* splitting homomorphisms

$$\mathfrak{s} : \mathbb{Z}\mathfrak{b} \to \mathbb{Z}\mathcal{L}.$$

These are defined by realizing chord diagrams in M by corresponding immersions. Then for $k = 0$, a sequence of free homotopy classes is mapped to a link with components realizing those free homotopy classes. Recall that $\mathfrak{b}[0]$ is the set of monomials in the set $\hat{\pi}$, and

$$\mathfrak{s}[0] : \mathfrak{b} \to \mathcal{L}[0].$$

Let $\mathfrak{s}[0](\alpha) =: K_\alpha \in \mathcal{L}[0]$ be the standard link corresponding to the monomial $\alpha \in \mathfrak{b}$.

In the following let σ denote either a skein potential or the Vassiliev potential σ_V.

Proof. [**of Theorem 1.1 and Theorem 1.5**] (in the Conway boundary case) The claims follow from Theorems 2.1, 2.2, 2.3 and 3.1 applied to the categories $\mathcal{B}[0]$ respectively $\mathcal{B}$ (or $\mathcal{C}[0]$ respectively $\mathcal{C}$), see the discussion at the end of section 3. The models are $\mathfrak{b}[0]$ respectively $\mathfrak{b}$. The exact sequence of Theorem 4.1 (and the splitting in Corollary 4.1) describe in a systematic way *all* homology classes of 1-cycles with boundary $K - \mathfrak{s} \circ \mathfrak{h}(K)$ for $K \in \mathcal{L}$. The additional information we have here is that the relations coming from expansion of *closed* 1-morphism up to 2-morphisms, *factor* through the homomorphism μ. In the inductive argument we actually have to consider the lift of $\widetilde{\sigma} \circ \mu$ to $R\mathcal{L}$. Here we have defined:

$$\widetilde{\sigma} : \mathbb{Z}\mathcal{L}[+1]/\mathcal{D} \to R\mathcal{L}/\langle\sigma(\mathcal{D})\rangle,$$

where $\langle\sigma(\mathcal{D})\rangle$ is the submodule generated by the subgroup $\sigma(\mathcal{D})$. $\qquad \square$

Remark 4.1. Let $\iota : R\mathcal{L}[0] \to \mathcal{S}(\sigma)$ be the projection. Then $\iota \circ \widetilde{\sigma} \circ \mu = 0$. Here we use that ι factors through $\mathcal{R}/\langle\mathcal{D}\rangle$ and in $\mathcal{S}(\sigma)$ the following relation holds:

$$\sigma(K_{*+} - K_{*-} - K_{+*} - K_{-*}) = \partial(K_{*+} - K_{*-} - K_{+*} - K_{-*}) = K_{++} - K_{-+}....$$

But for $K_+ \neq K_-$ a skein relation of the form $K_+ - K_- - \sigma(K_*)$ is not contained in the image of $\widetilde{\sigma} \circ \mu$, even if we consider the extended homomorphism

$$\widetilde{\sigma} \circ \mu : H_1(\text{imm}) \otimes \mathcal{R} \to R\mathcal{L}/\langle\sigma(\mathcal{D})\rangle.$$

In fact, there is a more natural way of associating a string topology homomorphism to a given skein relation or Vassiliev relation. This will be discussed in section 10.

5   Some general results about skein potentials and skein modules

We work in the graded 2-groupoid $\mathcal{B}$ or in the 2-groupoid $\mathcal{B}[0]$. Then hom[k] is naturally identified with $\mathbb{Z}\mathcal{L}[k+1]$. Because of the functoriality property in the definition, a skein potential is determined by its values on the generators $\mathcal{L}[k+1]$. Thus we consider a skein potential as a map $\mathcal{L}[+1] \to \mathcal{L}$ using the usual graded notation.

For some of the following results we have to restrict the choice of geometric models. Let

$$\sigma : \mathcal{L}[1] \to \mathcal{RL}[0]$$

be a skein potential with respect to the ideal $I \subset \mathcal{R}$. Let $K_\alpha := \mathfrak{s}(\alpha)$ be the *standard link* with homotopy class α.

Definition 5.1. A choice of geometric models

$$\mathfrak{s} : \mathfrak{b}[0] \to \mathcal{L}[0]$$

is called *nice* if the following two conditions hold: (i) for free homotopy classes which are multiple times contained in a monomial $\alpha \in \mathfrak{b}[0]$ there exists a self-isotopy of the standard link K_α, which arbitrarily changes the order of the corresponding components. (ii) for each trivial free homotopy class in α, the standard link K_α contains some unlinked and unknotted circles in a 3-ball separated from the rest of the link.

Lemma 5.1. It is always possible to choose nice geometric models.

Proof. If α contains a free homotopy class $\beta \in \hat{\pi}$ with multiplicity n_β then we can choose n_β parallel copies in the link K_α. Of course trivial homotopy classes can be represented by unlinked and unknotted components separated from the link. □

Remark 5.1. Let M be the solid torus $S^1 \times D^2$, or a Lens space $L(p,q)$. Then $L(p,q)$ contains a solid torus and each link is isotopic into this torus. Then we can choose nice standard links contained in $S^1 \times D^2$, which are descending. In fact, there is an isotopy of the solid torus which changes the order of components with multiple free homotopy classes of components.

Definition 5.2. A skein potential $\sigma : \mathcal{L}[+1] \to \mathcal{RL}$ is called *local* if it is of the form

$$\sigma(K_*) = \sum_{i=1}^{n} h_i K_i,$$

where for $1 \leq i \leq n$, $a_i \in \mathcal{R}$ and K_i are singular links defined by replacing the two intersecting arcs in the oriented 3-ball centered about the last double-point $*$ by the 2-tangles t_i. We assume that $a_i \in I$ if the number of components of K_i is not smaller than the number of components of K_*.

The following is obvious.

Lemma 5.2. Each local skein potential

$$\sigma : \mathcal{L}[1] \to \mathcal{L}[0]$$

extends to a local skein potential

$$\mathcal{L}[+1] \to \mathcal{L}.$$

In fact for each $1 \le i \le k+1$ there exist maps

$$\sigma_i : \mathcal{L}[k+1] \to \mathcal{L}[k]$$

defined by applying σ to the i-th double-point of a singular link.   □

Similarly we can define Jones and Conway boundaries on $\mathcal{L}[k+1]$ for $1 \le i \le k+1$ by applying ∂ to the i-th double point. Note that for $i < j$

$$\partial_i \circ \sigma_j = \sigma_{j-1} \circ \partial_i.$$

Recall that a skein relation is insensitive (with respect to differentiability relations for $k \ge 1$ if for each differentiability element $d \in \mathcal{L}[k+1]$ we can *choose* a preimage $\delta^{-1}(\sigma(d))$ such that $\sigma(\delta^{-1}(\sigma(d)) = 0$.

Remark 5.2. Global insensitivity for the skein potentials of the graded category is rare because of the $4T$-relations. In fact, the usual Vassiliev potential σ_V is *not* robust. Also the extensions of skein relations according to Lemma 5.1 are not robust. In particular the following result only applies to the usual skein potentials.

Proposition 5.1. Each local skein potential $\mathcal{L}[1] \to \mathcal{RL}[0]$ is robust. Also, each local skein potential $\mathcal{L}[+1] \to \mathcal{RL}$ is robust with respect to the differentiablity morphisms.

Proof. Each differentiability element d is in the image of

$$\partial_{k+2} - \partial_{k+1} : \mathcal{L}[k+2] \to \mathbb{Z}\mathcal{L}[k],$$

which maps $K_{**} \in \mathcal{L}[k+2]$ to

$$K_{*+} - K_{*-} - K_{+*} - K_{-*}.$$

We have to show that for each $y \in \mathcal{L}[k+2]$ we can choose a preimage, denoted $\partial^{-1}(y)$, such that

$$\sigma \circ \partial^{-1} \circ \sigma \circ \partial_{k+2}(y) = \sigma \circ \partial^{-1} \circ \sigma \circ \partial_{k+1}(y).$$

Recall that σ without index operates on the last double-point. Of course we can choose the preimage on the left hand side of the equation such that $\partial^{-1} \circ \sigma \circ \partial_{k+2}(y) = \sigma_{k+1}(y)$ and similarly on the right hand side $\partial^{-1} \circ \sigma \circ \partial_{k+1}(y) = \sigma(y)$ This follows from locality, e. g. in the first case: $\sigma \circ \partial_{k+2} = \partial \circ \sigma_{k+1}$. Then the claim follows using locality:

$$\sigma \circ \sigma_{k+1} = \sigma \circ \sigma_{k+2}.$$

□

In the following it will sometimes turn out to be useful to describe skein modules $\mathcal{S}(\sigma)$ as modules over the corresponding skein module $\mathcal{S}(D^3)$.

Proposition 5.2. For each local skein potential σ the module $\mathcal{S}(M; \sigma)$ is a module over the commutative skein algebra $\mathcal{S}(D^3; \sigma)$, and a module over the skein algebra $\mathcal{S}(\partial M; \sigma)$.

Proof. The disjoint union with links in a 3-ball separated from a given link is a well-defined operation. Locality of the skein potential implies that this defines a pairing

$$\mathcal{S}(D^3; \sigma) \otimes \mathcal{S}(M; \sigma) \to \mathcal{S}(M; \sigma).$$

The same holds for $\mathcal{S}(\partial M; \sigma)$ using a collar of ∂M. In fact, the case of D^3 is a special case where we replace M by the complement of some open 3-ball. Because of locality this does not change the skein module. Also this allows to identify the skein modules of D^3 and $S^2 \times [0, 1]$. The algebra structures on $\mathcal{S}(M; \sigma)$ is defined in the usual way using the $[0, 1]$-structure. $\square$

If ∂M is not a union of tori then the skein algebra usually is not commutative.

Let σ be the *Conway skein potential* $\sigma(K_*) = hK_0$ respectively zK_0 for a double-point of the same respectively different components. In this case we add the vaccuum relation $(q^{-1} - q)\emptyset = hU$. Note that $hU \neq \sigma(K_*)$ for any singular link $K_* \in \mathcal{L}[1]$. Recall that by definition our category contains the empty link and thus the empty 1-morphism in $\mathcal{L}[1]$, which is the identity morphism of the empty link. When working with vaccum relations we actually only replace the trivial skein relation $\sigma(\emptyset) = (q^{-1} - q)\emptyset$ e. g. in the Jones case (such that $(\partial - \sigma)(\emptyset) = 0$ automatically holds) by the relation above. This will have the advantage that the kink relations (see section 6) can be already included into the structure of the ring $\mathfrak{R} := \mathcal{S}(D^3; \sigma)$.

6   The homomorphism μ and the topology of M

For $k \geq 1$ let $\mathcal{K}[k] \subset \mathcal{L}[k]$ denote the set of k-immersions with a distinguished self-crossing denoted $*$ such that one of the lobes of the singular component of K_* bounds a disk intersecting K_* only in its boundary along the lobe. We call this a *kink k-immersion*. There is a unique homology class $\gamma(K_*) \in H_1(\mathrm{imm}[k-1])$ represented by the transversal path in $\mathrm{imm}[k-1]$, which is defined by running through the natural isotopy from K_+ to K_- and the crossing change at K_*. The disk bounding the trivial component in the smoothing K_0 defines a null-homology of this loop in $\mathrm{map}[k-1]$. Here $\mathrm{map}[k]$ is the space of all smooth maps of circles in M with exactly k fixed double points, with definition analogous to the definition of $\mathrm{imm}[k]$ is section 3. This null-homology contains a single point contained in $\mathrm{map}[k-1] \setminus \mathrm{imm}[k-1]$, where the corresponding map contains the mapping of a circle which embeds except at a single point with vanishing tangent vector. The corresponding element is non-trivial in $H_1(\mathrm{imm}[k-1])$. Note that

$$(\mu[k-1])(\gamma(K_*)) = K_*,$$

and

$$(\partial[k-1])(K_*) = K_+ - K_- = 0$$

by construction for each $K_* \in \mathcal{K}[k]$.

Remark 6.1. Let $\widetilde{\mathcal{L}}[k]$ denote the set of isotopy classes of ordered singular k-links (the set of path components of the space $\widetilde{\mathrm{emb}}[k]$ of the space of ordered k-embeddings). Then there is a well-defined *onto* map

$$\chi[k] : \widetilde{\mathcal{L}}[k-1] \to \mathcal{K}[k],$$

which assigns to each isotopy class of ordered $(k-1)$-singular link the isotopy class of the immersion with an additional kink in the *first* component away from possible preimages of double-points.

Lemma 6.1. The kernel of the epimorphism

$$j_* : H_1(\mathrm{imm}) \to H_1(\mathrm{map})$$

is generated by all elements of the form $\gamma(K_*)$ for all $K_* \in \mathcal{K}[+1]$. In particular the linear extension of χ maps $\mathbb{Z}\widetilde{\mathcal{L}}$ onto $ker(j_*)$.

Proof. This is another application of Lin transversality, respectively a modification of Lemma 3.1, see also Remark 3.2 (i). Consider a mapping of a surface F into map$[k]$, which is transversal along ∂F. It can be approximated relative ∂F to the following way: The set of points in F, which map into map$[k] \setminus \mathrm{emb}[k]$ consists of a 1-complex embedded in F, with vertices of possible orders $2, 4$ or 1 in the interior. Those of order 4 or 2 are mapped to immk$[k, 2]$. Those of order 1 are mapped to a smooth map, which is an immersion with k double points and a single point in the complement of the double points with vanishing tangent. The point with vanishing tangent appears in the boundary of imm$[k, 1] \subset \mathrm{imm}[k]$. Now cut out small disks from F around the vertices of order 1. The restriction to the the boundary of a disk represents an element in $H_1(\mathrm{imm}[k])$ of the form $\gamma(K_*)$ for $K_* \in \mathcal{K}[k]$. $\qquad\square$

Let in the presentation $K_{**} \in \mathcal{K}[2]$ the first place indicate a self-crossing with a bounding lobe. Then K_{*+}, K_{*-} are contained in $\mathcal{K}[1]$. This is not necessarily the case for K_{+*}, K_{-*} but $K_{+*} = K_{-*}$. Thus we have the following commuting diagram with exact rows, where $\mathcal{D}_\bullet$ is the subgroup of $\mathbb{Z}(\mathcal{L}[1] \setminus \mathcal{K}[1])$, which is generated by the differentiability relations with *all* four terms in $\mathcal{L}[1] \setminus \mathcal{K}[1]$:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{D} \cap \mathbb{Z}\mathcal{K}[1] & \longrightarrow & \mathcal{D}[0] & \longrightarrow & \mathcal{D}_\bullet & \longrightarrow & 0 \\
& & \subset\downarrow & & \subset\downarrow & & \subset\downarrow & & \\
0 & \longrightarrow & \mathbb{Z}\mathcal{K}[1] & \longrightarrow & \mathbb{Z}\mathcal{L}[1] & \longrightarrow & \mathbb{Z}(\mathcal{L}[1] \setminus \mathcal{K}[1]) & \longrightarrow & 0
\end{array}
$$

and the diagram of homomorphisms

$$
\begin{array}{ccc}
H_1(\mathrm{imm}[0]) & \xrightarrow{\;\mu[0]\;} & \mathbb{Z}\mathcal{L}[1]/\mathcal{D}[0] \\
i_*[0]\downarrow & & \downarrow \\
H_1(\mathrm{map}[0]) & \xrightarrow{\;\mu_\bullet\;} & \mathbb{Z}(\mathcal{L}[1] \setminus \mathcal{K}[1])/\mathcal{D}_\bullet
\end{array}
$$

with the vertical right map defined by projection.

Corollary 6.1. There is the induced boundary homomorphism $\partial_\bullet$ and the exact sequence

$$H_1(\mathrm{emb}[0]) \to H_1(\mathrm{map}[0]) \xrightarrow{\mu_\bullet} \mathbb{Z}(\mathcal{L}[1] \setminus \mathcal{K}[1])/\mathcal{D}_\bullet \xrightarrow{\partial_\bullet} \mathbb{Z}\mathcal{L}[0] \xrightarrow{\flat} S\mathbb{Z}\hat{\pi}.$$

There is the induced isomorphism

$$\mathrm{coker}(\mu[0]) \cong \mathrm{coker}(\mu_\bullet).$$

Proof. Most of the claims follow by the construction of the homomorphisms. The exactness at $H_1(\mathrm{imm}[0])$ follows because $\mu[0]$ restricts to an epimorphism

$$\mathcal{K}[0] \to \mathbb{Z}\mathcal{K}[1]/(\mathcal{D}[1] \cap \mathbb{Z}\mathcal{K}[1]).$$

$\square$

The main link theoretic consequences will follow from Theorem 1.1.

Definition 6.1. Let M be a compact 3-manifold. M is called *atoroidal* (respectively *aspherical*) if each π_1-injective map of a torus (respectively map of a 2-sphere) in M is homotopic into ∂M

Each irreducible 3-manifold is aspherical. Note that each hyperbolic 3-manifold is aspherical and atoroidal.

Theorem 6.1. Suppose M is aspherical and atoroidal. Then $\mu_\bullet = 0$.

Proof. Most of the arguments are already contained in [25] and [26]. Fix a component map_α of the space $\mathrm{map}[0]$ corresponding to $\alpha \in \mathfrak{b}[0]$. Note that

$$H_*(\mathrm{map}[0]) \cong \bigoplus_{\alpha \in \mathfrak{b}[0]} H_*(\mathrm{map}_\alpha).$$

The Hurewicz homomorphism

$$\pi_1(\mathrm{map}[0], f_\alpha) \to H_1(\mathrm{map}_\alpha)$$

is onto. Let $\widetilde{\mathrm{map}}$ be the space of ordered smooth maps with the fat diagonal excluded. Let $\widetilde{\mathrm{map}}_\alpha$ denote the preimage of map_α under the covering projection

$$\widetilde{\mathrm{map}} \to \mathrm{map}.$$

Note that the path components of $\widetilde{\mathrm{map}}_\alpha$ are labelled by the orderings of α. Let a be an ordered sequence of elements in $\hat{\pi}$ corresponding to α. We choose a representative embedding $\widetilde{f_a} \in \widetilde{\mathrm{map}}_a$. Note that f_a is ordered. If a does not contain multiple elements of $\hat{\pi}$ then the component $\widetilde{\mathrm{map}}_a$ is homeomorphic to map_α. But multiple occurences of elements of $\hat{\pi}$ in a imply that there are

homotopies joining ordered links with the same underyling unordered link. Then the covering of the component is non-trivial and the injection

$$\pi_1(\widetilde{\mathrm{map}}_a, f_a) \to \pi_1(\mathrm{map}[0], f_\alpha)$$

is not necessarily onto. This means that a loop in imm_α does not necessarily lift to a loop in imm_a. In this case we choose the embeddings f_a in the following symmetric way: If some element in a appears multiple times then we choose corresponding parallel components for f_a. Thus there exists an isotopy of M which changes the order of components. We can compose a given loop γ in map_α with loops in emb changing the order in such a way that the composition of loops lifts to a loop in map_a. Note that the composition still has the same image under $\mu_\bullet$ as γ. So the image of $\mu_\bullet$ on the component $H_1(\mathrm{map}_\alpha)$ corresponds to the image of a homomorphism defined on

$$\pi_1(\mathrm{map}[0]; f_a) \cong \prod_i \pi_1(\mathrm{map}_{a_i}, f_{a_i}).$$

Note that we can fill in the infinite codimensional fat diagonal without changing the fundamental group and then identify our mapping space with the product of single component mapping spaces. This shows that the image of $\mu_\bullet$ is generated by the images of elements represented by loops in $\mathrm{imm}[0]$ which fix all but one component. Let L denote the union of those components which are fixed during the homotopy. In the following note that free homotopy implies homology in our case.

Next consider the singular torus map given by the adjoint of the non-constant component of such a loop in $\mathrm{imm}[0]$. First assume that this map is *not* essential. Then the image of $\pi_1(S^1 \times S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$ in $\pi_1(M)$ is cyclic (for details see [25]). Thus on a neighbourhood of wedge $S^1 \vee S^1$ in $S^1 \times S^1$ the map can be homotoped into the tubular neighbourhood T of a knot embedded in M. The full torus map is homotopic in T up to a map of a 2-sphere. We can easily arrange that the singular torus can be represented as a connected sum of singular torus in N and parallel copies of 2-spheres contained in the collar neighbourhood of the union of 2-sphere components in ∂M. Now consider a singular torus contained in T. We want to apply the homotopy exact sequence of the fibration (see [48], Appendix):

$$\Omega M \to \mathrm{map}[0] \to M,$$

where Ω is the usual based loop functor for $M = T$ a solid torus. It is easy to see that all elements of $\pi_1(\mathrm{map}[0]; f)$ can be represented by loops in $\mathrm{imm}[0, 1]$. (Each generator of $\pi_1(\mathrm{map}[0]; f)$ can be represented by longitudinal rotation of the knot f in T.) The possible connected sum arcs for the 2-sphere contribution can be homotoped along. But since the two-spheres and connecting tubes can be assumed embedded the resulting loop is still contained in $\mathrm{imm}[0, 1]$. Finally since the core of T and connecting arcs are 1-dimensional and the 2-sphere maps into a collar of the boundary, the intersections with L can be avoided and the loop of maps is contained in *emb*. Thus the image under $\mu_\bullet$ vanishes.

It remains to discuss the case of an essential torus map. If this map is homotopic into the boundary then the boundary contains a torus and we can homotope into a collar neighbourhood $N \cong S^1 \times S^1 \times [0, 1]$ of a torus boundary component, in particular avoiding L. As before we can argue in the mapping space of N and find a free homotopy of the torus map thus inducing a homology of the given loop into emb[0]. $\hfill\square$

Example 6.1. (a) Let $M = S^3$ and consider a *connected sum* element in $\mathcal{L}[1]$. Thus $*$ is a self-crossing and K_0 is the union of two nonempty links contained in disjoint balls. Then $K_+ = K_-$ by rotation. The full 2π-rotation defines a loop ℓ in *imm*. Note that $\mu(\ell)$ is represented by K_* so obviously not trivial in $\mathcal{L}[1]$. Now consider a path in imm, which joins one of the two link pieces in K_0 with trivial link by only crossing changes. Since the rotation can be performed along this deformation there exists a free homotopy of the loop ℓ into a loop for which the rotation is a kink rotation. Thus this phenomenon is not measured by $\mu_\bullet$.

(b) Suppose M is not aspherical. Then either M contains $S^2 \times S^1$ or M is a connected sum. In each case it is easy to show that $\mu_0 \neq 0$, at least modulo the Poincare conjecture. For example in the second case take a connected sum immersion with each lobe homotopically non-trivial. A natural isotopy across the S^2 easily changes K_+ into K_-. But now K_* cannot be trivial in $(\mathbb{Z}\mathcal{L}[1] \setminus \mathcal{K}[1])/\mathcal{D}_\bullet$. In order to show this map into $S\mathbb{Z}\hat{\pi}$ by taking homotoppy classes.

7 The Jones deformation

In this section we describe the passage from the Conway boundary to the Jones boundary case. There is a natural way to introduce the q-structure abstractly on the groupoid level. In the topological case this amounts to a lift of twisted homology to the deformed fundamental groupoid.

Let $\mathcal{C}$ be a 2-groupoid and let

$$\varepsilon : \mathrm{hom}/\mathrm{mor} \to \mathbb{Z}$$

be a *twist* homomorphism (or abstract local system), i. e.

$$\varepsilon(u \circ v) = \varepsilon(u) + \varepsilon(v)$$

for all $u, v \in hom$. (We identify elements in *hom* with its equivalence classes under the action of mor.) It follows that

$$\varepsilon(u^{-1}) = -\varepsilon(u)$$

since $\mathrm{mor}(u \circ u^{-1}, 1) \neq \emptyset$ and $\varepsilon(1) = 0$ for each identity 1-morphism $1 \in \mathrm{hom}(x, x)$.

We describe a 2-groupoid $\mathcal{C}_q$, the *Jones deformation* of $\mathcal{C}$. To avoid confusion we will write the compositions in the category $\mathcal{C}_q$ as $\diamond$.

Let for $i = 0, 1, 2$

$$(\hom_i)_q := \{q^j w | w \in \hom_i, j \in \mathbb{Z}\},$$

where we use the notation $q^j w := (j, w)$, i. e. $(\hom_i)_q$ is the subset of scalar multiples of basis elements in free abelian group on $\hom_i$. To avoid confusion we write $q^0 u$ for the image of $u \in \hom_i$ in $(\hom_i)_q$ for all i.

There is the natural action of $\mathbb{Z}$ on $(\hom_i)_q$ defined by $q^k w := q^{j+k} u \in (\hom_i)_q$ for $w = q^j u$, $u \in \hom_i$ and $j, k \in \mathbb{Z}$ and $i = 0, 1, 2$.

We define the target and source maps and the compositions. If $u \in \hom(x, y)$ then let $q^0 u \in \hom_q(x, q^{-2\varepsilon(u)} y)$ and $q^i u \in hom_q(q^i x, q^{i-2\varepsilon(u)y})$ for $i \in \mathbb{Z}$. Thus

$$\mathrm{sour}_q(q^i u) = q^i \mathrm{sour}(u) \in \mathfrak{ob}_q$$

and

$$\mathrm{targ}_q(q^i u) = q^{i-2\varepsilon(u)} \mathrm{targ}(u) \in \mathfrak{ob}_q$$

for each $u \in \hom$. The source and target maps are *equivariant* with respect to the $\mathbb{Z}$-action.

Let $v \in \hom(x, y)$ and $u \in \hom(y, z)$. It follows from the *homomorphism* property of ε that, if $u \circ v$ is defined then

$$q^{-2\varepsilon(v)} u \diamond q^0 v := q^0 (u \circ v)$$

is defined, and is a 1-morphism from x to $q^{-2\varepsilon(u)-2\varepsilon(v)} y = q^{-2\varepsilon(u \circ v)} y$ in $\mathcal{C}_q$. A composition with $q^i v$ is defined such that equivariance holds:

$$q^i(a \diamond b) = (q^i a) \diamond (q^i b)$$

for $a, b \in \hom_q$. Note that $\hom_q(q^i x, q^j y) = \emptyset$ if $j - i$ is odd. Thus $\mathfrak{ob}_q$ naturally splits into a disjoint union of two sets with homomorphisms only between objects in each of the two sets.

Finally we define horizontal and vertical composition of 2-morphisms. Note that if $\mathrm{mor}(u, v) \neq \emptyset$ then $\varepsilon(u) = \varepsilon(v)$. Thus after identification of $u, v \in \hom$ with the corresponding elements in $\hom_q$ (note that the targets have changed) we can identify each $h \in \mathrm{mor}(u, v)$ with the corresponding 2-morphism in mor_q. We will have $q^i h \in \mathrm{mor}_q(q^i u, q^i v)$ for $h \in \mathrm{mor}(u, v)$. The composition in mor induces obvious compositions $\diamond_1, \diamond_2$ in mor_q.

Theorem 7.1. (i) $\mathcal{C}_q = (\mathfrak{ob}_q, \hom_\mathrm{q}, \mathrm{mor}_\mathrm{q})$ defines a 2-groupoid.
(ii) There is an obvious functor of 2-groupoids $p_q : \mathcal{C}_q \to \mathcal{C}$ defined by $q \mapsto 1$, which is onto on all sets $\hom_\mathrm{j}$ for $j = 0, 1, 2$.
(iii) There is a natural ε-twisted functor $i_q : \mathcal{C} \to \mathcal{C}_q$, where $i_q(a) := q^0 a$ for $a \in \hom_\mathrm{i}$. This means that for $u, v \in \hom$:

$$i_q(u \circ v) = q^{-2\varepsilon(v)} u \diamond q^0 (v).$$

For $v_1, v_2 \in \hom(x, y)$, $u_1, u_2 \in \hom(y, z)$ and $g \in \mathrm{mor}(u_1, u_2)$, $h \in \mathrm{mor}(v_1, v_2)$ we have

$$i_q(g \circ_2 h) = (q^{-2\varepsilon(v_1)}g) \diamond_2 q^0 h.$$

For $h_1 \in \mathrm{mor}(v_2, v_3)$ and $h_1 \in \mathrm{mor}(v_1, v_2)$ with $v_i \in \hom(x, y)$ for $i = 1, 2, 3$ we have the untwisted identity

$$i_q(h_1 \circ_1 h_2) = (q^0 h_1) \diamond_1 (q^0 h_2).$$

(iv) The equation of functors holds: $p_q \circ i_q = id$.

Proof. Let $x, y \in \mathfrak{ob}$. and $u \in \hom(x, y)$ with uniquely determined $u^{-1} \in \hom(y, x)$. Then $(q^0 u)^{-1} := q^{-2\varepsilon(u)}u^{-1} \in \hom_q(q^{-2\varepsilon(u)}y, x)$. We are using that $\varepsilon(u^{-1}) = -\varepsilon(u)$. The inverse in general is defined by equivariance

$$(q^i u)^{-1} := q^i (q^0 u)^{-1}.$$

for $i \in \mathbb{Z}$. The remaining arguments are straightforward and left to the reader. $\square$

Remark 7.1. In terms of graphs (see [4] for the graphical explanation of categories) this means that we take the original graph corresponding to the category $\mathcal{C}$ and take $\mathbb{Z} \times \mathfrak{ob}$. Then define the edges according to the above procedure determined by the twist. This is like lifting paths into a covering space corresponding to ε. Note that the twist is a local system on the graph associated to the 2-groupoid.

Theorem 7.2. Let $\mathcal{R} \supset \mathbb{Z}$ be a commutative ring with 1. Let $\mathcal{C}$ be a 2-groupoid with linear potential $\mathfrak{a} : \hom \to \mathcal{R}\mathfrak{ob}$. Suppose that $\hom = F(S, \circ)$ or $\hom = A(S, \circ)$ for a subset $S \subset \hom$. There exists the skein potential in $\mathcal{R}[q^{\pm 1}]$:

$$\mathfrak{a}_q : \hom_q \to \mathcal{R}\mathfrak{ob}_q \cong \mathcal{R}[q^{\pm 1}]\mathfrak{ob},$$

defined uniquely by functoriality and $\mathbb{Z}$-equivariance:

$$\mathfrak{a}_q(q^0 s) := \mathfrak{a}(s)$$

for all $s \in S$

By assumption the following equation holds:

$$\mathfrak{a}_q((q^0 s)^{-1}) = \mathfrak{a}(s^{-1}) = -q^n \mathfrak{a}(s).$$

The next useful formula is easily proved from the definitions.

Proposition 7.1. Let $\mathfrak{a}, S$ be as in the last theorem and and let the 2-groupoid $\mathcal{C}$ be the the homotopy Vassiliev groupoid $\mathcal{C}(M)$ or the homology Vassiliev groupoid $\mathcal{B}(M)$. Let $s_j \in S$ and $\epsilon_j \in \{\pm 1\}$ for $j = 1, \ldots, r$. Then the following formula holds:

$$\mathfrak{a}_q(i_q(s_r^{\epsilon_r} \circ s_{r-1}^{\epsilon_{r-1}} \circ \ldots \circ s_1^{\epsilon_1})) = \sum_{j=1}^{r} \epsilon_j q^{-\epsilon_j \varepsilon(s_j)} \mathfrak{a}(s_j)$$

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Remark 7.2. We have for $\mathcal{C}(M)$ respectively $\mathcal{B}(M)$, $\hom[k] = F(\mathcal{L}[k+1], \circ)$ respectively $\hom[k] = A(\mathcal{L}[k+1], \circ)$. We define $\varepsilon(K^{\pm}) = \pm 1$ for $K \in \mathcal{L}[k+1]$. Note that K_* corresponds to the morphism s_* defined by the crossing change from $-$ to $+$ at $*$. We have $\hom_q[k] = \hom[k]_k \subset \mathbb{Z}[q^{\pm}]\mathcal{L}[k+1]$ in a similar way (see also the description of $\mathbb{Z}[q^{\pm}]\mathcal{L}[k+1]$ as twisted homology below). The following is important. In the q-deformed case K_* corresponds to the morphism defined by $K_- \mapsto q^{-2}K_+$ and K_*^{-1} corresponds to the morphism $q^{-2}K_+ \mapsto K_-$, which the q^{-2}-multiple of the usual morphism $K_+ \mapsto q^2 K_-$. It is this correspondence which makes $\hom_q \subset \mathbb{Z}[q^{\pm}]\mathcal{L}[+1]$ a morphism.

Let σ be a skein potential in $\mathcal{R}$. By Theorem 7.2 σ induces a skein potential $\mathcal{L}[+1] \to \mathcal{R}\mathcal{L}$ in the following way. Note that σ induces a linear potential in the usual way. This linear potential induces a linear potential in the deformed category $\mathcal{B}_q$. This defines a skein potential

$$\sigma_q : \mathcal{L}[+1] \to \mathcal{R}[q^{\pm}]\mathcal{L}.$$

Then

$$\sigma_q\big(i_q\big(s_r^{\epsilon_r} \circ s_{r-1}^{\epsilon_{r-1}} \circ \ldots \circ s_1^{\epsilon_1}\big)\big) = \sum_{j=1}^{r} \epsilon_j q^{-\epsilon_j} \sigma(s_j)$$

where $s_j = K_{*,j}$ is a composable sequence of elements in $\mathcal{L}[k+1]$.

Definition 7.1. We define that $(\mathcal{C}, \varepsilon)$ has *kinks* if for each $x \in \mathfrak{ob}$ there exists a kink morphism $k_x \in \hom(x, x)$ such that $\varepsilon(k_x) = 1$.

Now suppose $(\mathcal{C}, \varepsilon)$ has kinks and let $\mathfrak{m}$ be a set of models of $\mathcal{C}$. Then obviously $\mathfrak{m}_q := \mathfrak{m} \cup q\mathfrak{m}$ is a set of models of the category $\mathcal{C}_q$. Let $i \in \mathbb{Z}$. For $x \in \mathfrak{ob}$ we can find $u \in \hom(x, m)$. By possibly iterated composition with kink morphisms we can arrange that $\varepsilon(u) \in \{-i, -i-1\}$. Thus $q^i u \in \hom(q^i x, m)$ or $q^i u \in \hom(q^i x, qm)$. Now assume that $I \subset R$ is an ideal and $\mathfrak{c}$ is a model complexity on $\mathfrak{m}$. Then we define $\mathfrak{c}_q$ by $\mathfrak{c}_q(m) = \mathfrak{c}_q(qm) = \mathfrak{c}(m)$. Let $I_q \subset \mathcal{R}[q^{\pm 1}]$ be the ideal generated by I in the extension. Then it is easy to see that if $\mathfrak{a}$ is linear with respect to I and $\mathfrak{c}$ then $\mathfrak{a}_q$ is linear with respect to I_q and $\mathfrak{c}_q$.

Thus we can apply Theorem 2.4 to $(\mathcal{C}_q, \mathfrak{a}_q, \mathfrak{m}_q)$. and deduce the map

$$\mathfrak{ob}_q \to (\mathcal{R}[q^{\pm 1}]\mathfrak{m}_q)[[I_q]]/\mathcal{I}_q,$$

with the submodule $\mathcal{I}_q$ defined in the obvious way.

In the topological case, which we consider here, the passage from $q = 1$ to Jones theory is more easily described by introducing a *local system* on the space imm. This viewpoint has already been used in [26]. We refer to there and to [51] for some of the technical details. This local system appear naturally in trying to work the kink contributions into the theory as will be seen in the following.

Throughout this section we assume from now on $k = 0$ and omit the grading index.

Let $R := \mathbb{Z}[q^{\pm 1}]$. The idea is that a *universal* Jones type relation should be of the form:

$$q^{-1}K_+ - qK_- - K_*$$

respectively

$$q^{2\epsilon} K_\epsilon - K_{-\epsilon} - \epsilon K_* = 0$$

for $\epsilon = \pm 1$.

Consider the trivial bundle over imm with fiber R. Let γ be a loop in imm which is tansversal (see section 3). Then the *oriented intersection number with the discriminant* $\delta'(\gamma)$ is well-defined. Formally it can be defined as the homomorphism:

$$\delta' : H_1(\mathrm{imm}) \xrightarrow{\ \mu\ } \mathbb{Z}\mathcal{L}[1]/\mathcal{D} \longrightarrow \mathbb{Z},$$

where the second homomorphism is defined my mapping each element of $\mathbb{Z}\mathcal{L}[1]$ to the generator $1 \in \mathbb{Z}$. For fixed basepoints f_α in the components of imm (possibly represented by $\mathfrak{s}(\mathfrak{b})$) let

$$\pi_1(\mathrm{imm}) := \bigcup_{\alpha \in \mathfrak{b}} \pi_1(\mathrm{imm}; f_\alpha)$$

Then define thefollowing map, which restricts to homomorphisms on the fundamental groups of the path components of imm:

$$\delta : \pi_1(\mathrm{imm}) \to \mathrm{Aut}(\mathbb{Z}[q^{\pm 1}])$$

is defined by mapping a loop γ to the multiplication by $q^{-2(\delta' \circ H)(\gamma))}$. Here we use the Hurewicz homomorphism

$$H : \pi_1(\mathrm{imm}) \to H_1(\mathrm{imm}).$$

The map δ induces the local system $\widetilde{R}$ on imm. It is called the *Jones local system* since it is related to the Jones respectively Homfly polynomial skein relation.

Remark 7.3. It is possible to describe the local system *directly* as a functor on the *transversal fundamental groupoid* of imm. The objects of this category are points in emb and the morphisms are homotopy classes of paths in imm joining two embeddings. Now represent a path in imm with endpoints in emb relative to the boundary by a transversal path. Then compute the oriented intersection number with the discriminant. So in this case we apply the homomorphism

$$H_1(\mathrm{imm}, \mathrm{emb}) \to \mathbb{Z}\mathcal{L}[1]/\mathcal{D} \to \mathbb{Z}$$

with the first homomorphism defined in the proof of proposition 1. Now the boundary operator on chains with coefficients in R is easily defined as usual using the *parallel transport* on suitable paths. Thus our construction actually given by a transversal chain construction as usual in string topology.

Note that the local system is trivial over emb. Also by using the kink loops described above we see that

$$H_0(\mathrm{imm}; \widetilde{R}) \cong SR\hat{\pi}/(q^2 - 1) \cong S\mathbb{Z}\hat{\pi} \oplus qS\mathbb{Z}\hat{\pi}.$$

Let $\widetilde{\mathcal{D}}$ be the submodule of $R\mathcal{L}[1]$ generated by all elements of the form

$$q^{-1}K_{*+} - qK_{*-} - q^{-1}K_{+*} + qK_{-*}$$

and let

$$\widetilde{\partial} : R\mathcal{L}[1]/\widetilde{\mathcal{D}} \to R\mathcal{L}$$

be defined by

$$\widetilde{\partial}(K_*) = q^{-2}K_+ - K_-.$$

Note that

$$\widetilde{\partial}(q^{-1}K_{*+} - qK_{*-} - q^{-1}K_{+*} + qK_{-*}) = 0.$$

The following theorem is proved similarly to Theorem 4.1.

Theorem 7.3. The homology exact sequence for the local system $\widetilde{R}$ on the pair of spaces $(\mathrm{imm}, \mathrm{emb})$ is isomorphic to the exact sequence:

$$H_1(\mathrm{emb}; R) \to H_1(\mathrm{imm}; \widetilde{R}) \xrightarrow{\widetilde{\mu}} R\mathcal{L}[1]/\widetilde{\mathcal{D}} \xrightarrow{\widetilde{\partial}} R\mathcal{L} \xrightarrow{\widetilde{\mathfrak{h}}} SR\widehat{\pi}/(q^2 - 1)$$

with $\widetilde{\mathfrak{h}}$ onto and the homomorphism $\widetilde{\mu}$ described below.

The homomorphism $\tilde{\mu}$ has the following description: Consider a 1-chain representing a homology class in $H_1(\mathrm{imm}; \widetilde{R})$. We can approximate corresponding maps on 1-simplices by transversal maps, i. e. with the boundaries mapping into emb. Now let a simplex s be given by a transversal map

$$\beta : ([0,1], \{0,1\}) \to (\mathrm{imm}, \mathrm{emb})$$

and let $0 < t_1 < t_2 < \ldots t_n < 1$ be the parameters which map into $\mathrm{imm}[1]$. Let the sign at t_i be $\epsilon_i \in \{0,1\}$, with $\epsilon_i = -1$ if the path crosses the discriminant from the *positive to the negative side.*

$$\tilde{\mu}(s) = \sum_{i=1}^{n} \epsilon_i q^{2(\epsilon_1 + \ldots \epsilon_{i-1}) + \epsilon_i} K_{*,i},$$

where $K_{*,i}$ is the isotopy class of $\beta(t_i)$.

Remark 7.4. Combining 1-simplices with cancelling boundary terms in a 1-chain it follows easily that there is an epimorphism

$$R\pi_1^0(\mathrm{imm}) \to H_1(\mathrm{imm}; \widetilde{R}),$$

where $\pi_1^0(\mathrm{imm})$ is the union (over $\mathfrak{h}$) of the subgroups of homotopy classes of those loops with trivial index (thus the 1-simplex representing the loop has trivial boundary with local coefficients). Note that the effect of changing the basepoint of a loop in $f \in \mathrm{emb}$ representing an element in $\pi_1^0(\mathrm{imm}; f)$ to a basepoint g by pre- and postcomposition with a path from g to f and the reverse path multiplies by a power of q. Thus we can restrict to a single basepoint in each component.

From the exact sequence in theorem we conclude:

Corollary 7.1. There is the short exact sequence of R-modules:

$$0 \to \left(R\mathcal{L}[1]/\widetilde{\mathcal{D}} \right)/\mathrm{im}(\widetilde{\mu}) \to R\mathcal{L} \to SR\hat{\pi}/(q^2 - 1) \to 0,$$

This sequence is *never* split in the category of R-modules (but in the category of abelian groups). Note that, for a given $K \in \mathcal{L}$, there is a unique $\alpha \in \mathfrak{b}$ such that for each integer number n

$$K - q^{2n} K_\alpha \in \ker(\widetilde{\mathfrak{h}}).$$

In physics terms this could probably be interpreted as follows: *For each classical state we can choose a quantum state up to a certain phase.*

By application of the homotopy Conway map (defined in section 4) the description of the homomorphism $\tilde{\mu}$ given above precisely corresponds to the map on paths in [25].

Proposition 7.2. Suppose that M is aspherical and atoroidal. Then

$$\mathrm{im}(\widetilde{\mu}) \subset (R\mathcal{K}/(\mathcal{D} \cap R\mathcal{K})) \cap \ker(\widetilde{\mathfrak{h}})$$

More precisely, each element in the image of $\tilde{\mu}$ is represented by a sum of elements of the form

$$q^{n_i}(K_{*,i} - K'_{*,i}),$$

where for fixed i all $K_{*,i}, K'_{*,i}$ are immersions in $\mathcal{K}$ resulting from the same link $K \in \mathcal{L}$ by introducing a kink in one of its components.

Proof. First define homomorphisms for each $\alpha \in \mathfrak{b}$

$$\widetilde{\mu}_\alpha : \pi_1(\mathrm{imm}; f_\alpha) \to R\mathcal{L}[1]/\widetilde{\mathcal{D}}$$

using the explicit formula given above. Note that the resulting homomorphism does not factor through a map defined on the set of free homotopy classes of loops in imm. Now assume we have given a loop ρ in imm with index $\varepsilon(\rho) = 0$. It is easy to see that there is a homotopy to a product

$$\prod_{1 \leq i \leq n} \ell_i \rho_i \ell_i^{-1}.$$

Here ℓ_i is a transverse arc joining $\rho(*) \in \mathrm{emb}$ (representing a link $K \in \mathcal{L}$) to the basepoint $*_i$ on ρ_i, and each ρ_i is a kink loop (representing $\gamma(K_*)$ for some K_* in $\mathcal{L}[1]$). Now each path $\ell_i \rho_i \ell_i^{-1}$ is homotopic inside imm to a kink path for the basepoint embedding. Note that the order of crossing changes can be altered up to $\widetilde{\mathcal{D}}$. This allows to inductively change each of the paths. Thus finally we have a composition of kink loops for the embedding $*$ with vanishing index because the index is homotopy invariant. This proves the claim. Note that the kinks can still be contained in different components of K. $\qquad\square$

8 The framing local systems and its homology sequences

In this section we use a local system of coefficients R' on imm$[k]$ naturally related to the study of framed links in M. The geometric interpretation of the corresponding homology exact sequence is more delicate than the discussion in the previous section. Note that emb$[0]$ is just the usual space of embeddings of oriented links in M with $\mathcal{L}[0]$ the set of isotopy classes.

The following extends the set-up discussed in [27]. A *total framing* of $f \in \widetilde{\text{imm}}[0]$ is a choice of equivalence class $[v]$ of a normal vectorfield of f (section of the normal bundle), where we will have $v_1 \equiv v_2$ if v_1 is homotopic to v_2, or if v_1 differs from v_2 by twisting the framings of components of f such that the total number of those twists add up to zero. We use the natural projections

$$\widetilde{\text{imm}}(j) \times C_{2k}(\cup_j S^1) \to \widetilde{\text{imm}}(j)$$

to define the total framing of $f \in \widetilde{\text{imm}}[k]$ for $k \geq 1$. Obviously the set of total framings is in $1-1$ correspondence with the integer numbers. The total framings are defined for unordered immersions by dividing by the actions of symmetric groups in the obvious way. Now for each $f \in \text{imm}[k]$ let R'_f denote the free abelian group gnerated by all total framings on f. Then $R'_f \cong \mathbb{Z}[q^{\pm 1}] = R$ with q^i and q^j indicating two framings of f, which differ by $j - i$ twists.

Then a local system $R_{\mathfrak{f}}$ on (imm, emb) with bundle of groups given by

$$\bigcup_{f \in \text{imm}} R_f$$

is defined from the collection of homomorphisms, for $f \in \text{emb}$:

$$\delta_{\mathfrak{f}} : \pi_1(\text{imm}; f) \to \text{Aut}(R_f)$$

by assigning to a loop γ in f the multiplication by $q^{\delta'_{\mathfrak{f}}(\gamma)}$. Here $\delta'_{\mathfrak{f}}(\gamma) \in \mathbb{Z}$ is the sum of $2(\varepsilon \circ H)(\gamma)$ and the framing change induced by the loop. It is shown in [27], using Lin transversality, that the homomorphism is well-defined and thus represents a local system $R_{\mathfrak{f}}$ on imm$[0]$. Moreover, the homology module $H_0(\text{imm}[0]; R_{\mathfrak{f}})$ is isomorphic to the skein module $\mathcal{S}_{\mathfrak{f}}$ defined by dividing $R\mathcal{L}_{\mathfrak{f}}[0]$ by the submodule generated by all elements $q^{-1}K_+ - qK_-$ and $q^{-1}K^{(+)} - K$ for $K, K_\pm \in \mathcal{L}_{\mathfrak{t}}[0]$. Here $\mathcal{L}_{\mathfrak{f}}[0]$ is the usual set of isotopy classes of framed oriented links and for each framed oriented link K the framed link $K^{(+)}$ is the framed link defined from K by introducing a positive twist into the framing of any of its components. The correspondingly defined set of isotopy classes of totally framed singular links, equipped with k self-intersections, in M will be denoted $\mathcal{L}_{\mathfrak{t}}[k]$ and similarly we have the singular framed links $\mathcal{L}_{\mathfrak{f}}[k]$.

Remark 8.1. The local system above admits an interpretation in terms of the following general construction: Let $\tilde{X} \to X$ be a (not necessarily regular) covering space over a connected space X. Let F be the fiber over the basepoint

$* \in X$ and let $\pi_1(X; *) \to \mathrm{homeo}(F)$ be the associated monodromy map. Then there is the natural map

$$\mathrm{homeo}(F) \to \mathrm{Aut}(\mathbb{Z}F),$$

where $\mathrm{homeo}(F)$ is the group of homeomorphisms of F. So we can form the associated covering space with fiber $\mathbb{Z}F$ and define the *induced* monodromy by composition in the obvious way. Now assume that F is a group acting on itself by translation. Thus we have a natural map $F \to \mathrm{homeo}(F)$. Assume that, using a stratification of X, there is defined a second monodromy map:

$$\pi_1(X; *) \to F \to \mathrm{homeo}(F).$$

The two monodromies into $\mathrm{homeo}(F)$ can be multiplied, and then mapped to a homomorphism into $\mathrm{Aut}(\mathbb{Z}F)$ as above. The fiber of the induced covering is the abelian group $\mathbb{Z}F$ and the monodromy defines a local system, see [51].

Lemma 8.1. There are the following isomorphisms

$$H_0(\mathrm{imm}; R_{\mathfrak{f}}) \cong R\mathcal{L}_{\mathfrak{t}}/(q^{-1}K^{(+)} - K) \cong R\mathcal{L}_{\mathfrak{f}}/(q^{-1}K^{(+)} - K) \cong \mathbb{Z}\mathcal{L}_{\mathfrak{t}}.$$

The right hand isomorphism is an isomorphism of abelian groups.

Proof. The fiber R_f over a given k-embedding f is the free abelian group generated by total framings of f. R_f is identified with R by a choice of total framing of f. Multiplication by q corresponds to a positive twist. Note that

$$H_0(\mathrm{emb}; R_{\mathfrak{f}}) \cong \bigoplus_{K \in \mathcal{L}} H_0(\mathrm{emb}_K, R_{\mathfrak{f}}),$$

where emb_K is the set of embeddings in the isotopy class $K \in \mathcal{L}_{\mathfrak{t}}$. Recall that by definition emb_K is the $\mathrm{Diff}(M, \mathrm{id})$-orbit of a representative k-embedding f. Here $\mathrm{Diff}(M, \mathrm{id})$ is the group of diffeomorphisms of M, which are isotopic to the identity. Then each element of $H_0(\mathrm{emb}_K; R_{\mathfrak{f}})$ is represented by an element in the fiber over some representative embedding of the isotopy class. Now the result follows from the definitions of $\delta_{\mathfrak{f}}$ and $\mathcal{L}_{\mathfrak{t}}$ and the monodromy interpretation of 0-dimensional homology for local systems [51]. $\square$

We use the notation $\mathcal{F}[k] := R\mathcal{L}_{\mathfrak{t}}[k]/(q^{-1}K^{(+)} - K)$ for $k \geq 0$, and as usual $\mathcal{F} := \cup_k \mathcal{F}[k]$.

Theorem 8.1. The homology exact sequence for the local system $R_{\mathfrak{f}}$ on the pair $(\mathrm{imm}, \mathrm{emb})$ is isomorphic with the exact sequence:

$$H_1(\mathrm{emb}; R_{\mathfrak{f}}) \to H_1(\mathrm{imm}; R_{\mathfrak{f}}) \xrightarrow{\mu_{\mathfrak{f}}} \mathcal{F}[+1]/\mathcal{D}_{\mathfrak{t}}$$

$$\xrightarrow{\partial_{\mathfrak{f}}} \mathcal{F} \xrightarrow{\mathfrak{h}_{\mathfrak{f}}} \mathcal{S}_{\mathfrak{f}} \to 0$$

The submodule $\mathcal{D}_{\mathfrak{t}}$, the homomorphisms $\mu_{\mathfrak{f}}$, $\partial_{\mathfrak{f}}$ and $\mathfrak{h}_{\mathfrak{f}}$ are defined as before by replacing $\mathcal{L}$ with $\mathcal{L}_{\mathfrak{t}}$. In the case of $\mu_{\mathfrak{f}}$ one has to consider the transportation of framings.

Proof. Using Lin transversality and transportation of framings as described above we get the exact sequence

$$H_1(\text{emb}; R_{\mathfrak{f}}) \to H_1(\text{imm}; R_{\mathfrak{f}}) \to H_0(\text{imm}[+1]; R_{\mathfrak{f}})/\partial'(H_0(\text{imm}[+2]; R_{\mathfrak{f}}))$$

$$\to H_0(\text{emb}; R_{\mathfrak{f}}) \to \mathcal{S}_{\mathfrak{f}} \to 0$$

and then apply Lemma 8.1. Note that $\mathcal{S}_{\mathfrak{f}}$ is isomorphic to the quotient of $R\mathcal{L}_{\mathfrak{t}}$ by the submodules generated by elements $q^{-1}K^{(+)} - K$ and $q^{-1}K_+ - qK_-$ (see [26]). $\qquad\square$

Remark 8.2. It is interesting to note that the exact sequence above maps into the sequence of Theorem 4.1 under the the coefficient homomorphism $R \to \mathbb{Z}$, which maps q to 1. Then the coefficient system becomes trivial and $\mathcal{F}[k]$ maps onto $\mathbb{Z}\mathcal{L}[k]$. In this way Theorem 8.1 can be interpreted as a *framing quantization* of Theorem 4.1.

Corollary 8.1. There is the short exact sequence of R-modules

$$0 \to (\mathcal{F}[+1]/\mathcal{D}_{\mathfrak{t}})/\text{im}(\mu_{\mathfrak{f}}) \to \mathcal{F} \to \mathcal{S}_{\mathfrak{f}} \to 0$$

It is known that the homomorphism $\mathfrak{h}_{\mathfrak{f}}$ splits in the category of R-modules if and only if each mapping $S^1 \times S^1 \to M$ is homotopic into ∂M.

Theorem 8.2. (i) There is the isomorphism of R-modules

$$\mathcal{F}[0] \cong \bigoplus_{\alpha \in \mathfrak{b}[0]} R(\mathfrak{h}[0]^{-1}(\alpha))/(q^{2\varepsilon_0(\alpha)} - 1),$$

where for $k \geq 1$ $\mathfrak{h}[k] : \mathcal{L}[k] \to \mathfrak{b}[k]$ is the map taking homotopy classes of components. The index $\varepsilon_0(\alpha) \in \mathbb{N}$ is defined by the the *gcd* over all absolute values of total oriented intersection numbers with 2-spheres in M, see [27] for details.
(ii) There is the isomorphism of R-modules

$$\mathcal{S}_{\mathfrak{f}} \cong \bigoplus_{\alpha \in \mathfrak{b}} R/(q^{2\varepsilon(\alpha)} - 1),$$

where $\varepsilon(\alpha)$ is the *gcd* of absolute values of intersection numbers of singular tori (defined by sweeping a component α_i through M) with a link realizing α.

Proof. This is just Chernov's result [11], see also the author's approach in [26]. It easily extends to the case $k \geq 1$. $\qquad\square$

Remark 8.3. By defining suitable torus maps from 2-spheres as in (i) above it can be proved that $\epsilon_0(\alpha)$ is a multiple of $\epsilon(\alpha)$ (see [26]). The homomorphism $\mathfrak{h}_{\mathfrak{f}}$ thus is defined by composition of $\mathfrak{h}$ with some obvious projection. (Use that $x^{nm} - 1 = (x^n - 1)(1 + x^n + x^{2n} + \ldots + x^{nm})$.)

Corollary 8.2. (i) Suppose that M does not contain any non-separating 2-spheres. Then $\mathcal{F}[0] \cong R\mathcal{L}[0]$.
(ii) Suppose that each mapping from $S^1 \times S^1$ into M is homologous into ∂M. $\mathcal{S}_\mathfrak{f} \cong SR\hat{\pi}$. $\qquad\square$

Proof. By choosing total framings we can define a section $\psi : \mathcal{L} \to \mathcal{L}_\mathfrak{t}$ of the forget map $\phi : \mathcal{L}_\mathfrak{t} \to \mathcal{L}$. The induced homomorphism $\chi : R\mathcal{L} \to R\mathcal{L}_\mathfrak{t}$ is obviously onto. It follows from Chernov's [11] results that the fibers of ϕ are in 1-correspondence with $\mathbb{Z}$. This implies that χ is injective. (If $\chi(q^n K - K) = 0$ then $\psi(K)$ and its n-fold twist would be isotopic.) $\qquad\square$

Corollary 8.3. Suppose that each map of a torus into M is homologous into ∂M. Then the homology exact sequence for the local system $R_\mathfrak{f}$ on $(\mathrm{imm}[0], \mathrm{emb}[0])$ is isomorphic to the exact sequence:

$$H_1(\mathrm{emb}[0]; R) \to H_1(\mathrm{imm}[0]; R) \xrightarrow{\mu_\mathfrak{f}} R\mathcal{L}[1]/\widetilde{\mathcal{D}}$$

$$\xrightarrow{\partial_\mathfrak{f}} R\mathcal{L}[0] \xrightarrow{\mathfrak{h}_\mathfrak{f}} SR\hat{\pi} \to 0$$

$\qquad\square$

Proof. The assumption implies that the monodromy homomorphism $\delta_\mathfrak{f}$ is trivial and therefore the homology for the local system is the usual homology with coefficients in R. $\qquad\square$

Remark 8.4. The homomorphism $\mathfrak{h}_\mathfrak{f}$ composed with the projection homomorphism $SR\hat{\pi} \to SR\hat{\pi}/(q^2 - 1)$ is the homomorphism $\widetilde{\mathfrak{h}}$ in theorem 7.3. Note that $\mu_\mathfrak{f}$ is defined on $H_1(\mathrm{imm}[0]; R) \cong H_1(\mathrm{imm}[0]) \otimes R$ while $\widetilde{\mu}$ is defined on the homology with local coefficients defined by $\widetilde{R}$.

Proposition 8.1. Suppose that M is apherical and atoroidal. Then $\mu_\mathfrak{f} = 0$ for $k = 0$.

Proof. It follows from the assumption that in particular each torus is homologous into the boundary. Thus Corollary 8.2 applies. Since $H_1(\mathrm{imm}[0]; R) \cong H_1(\mathrm{imm}[0]) \otimes R$ it suffices to show that $\mu_\mathfrak{f}$ vanishes on the image of the natural homomorphism of abelian groups $H_1(\mathrm{imm}[0]) \to H_1(\mathrm{imm}[0]) \otimes R$. That $\mu_\mathfrak{f}$ vanishes on a kink isotopy holds by definition of the local system. (In fact the kink isotopy induces a change of framing cancelling the contribution of the crossing.) Thus we actually consider a homomorphism

$$H_1(\mathrm{map0}) \to \mathcal{F}[1]/\mathcal{D}_\mathfrak{f} \cong R\mathcal{L}[1]/\widetilde{\mathcal{D}}.$$

The rest of the argument follows word by word the geometric reasoning in the proof of Theorem 6.1. $\qquad\square$

9 Proofs of the results on skein modules

In this section we prove Theorems 1.2, 1.3. and 1.4. concerning the skein module $\mathfrak{J}(M)$ of oriented links in M, considered as module over the skein algebra of the ball $\mathfrak{J}(D^3)$, which is isomorphic to $\mathfrak{R} = \mathbb{Z}[q^{\pm 1}, z, h, \frac{q^{-1}-q}{h}]$ ([42] and [46]). The framed case $\mathfrak{H}(M)$ is similar and is proved by using the results of section 8, and will not be discussed here in detail.

Recall that $\mathfrak{b}$ is the set of monomials in $\hat{\pi}$ and $\mathfrak{b}_0$ is the set of monomials in $\hat{\pi}_0$, where $\hat{\pi}$ and $\hat{\pi}_0$ respectively is the set of free homotopy classes and non-trivial free homotopy classes of oriented loops in M respectively. Each of these sets of monomials will contain the trivial monomial 1 corresponding to a vaccuum free homotopy class. Then for the corresponding symmetric algebras we have module equalities $S\mathcal{R}\hat{\pi} = \mathcal{R}\mathfrak{b}$ (and $S\mathfrak{R}\hat{\pi}_0 = S\mathfrak{R}\hat{\pi}_0$). Throughout let $u \in \hat{\pi}$ denote the trivial free homotopy class. We can identify $\mathcal{R}\mathfrak{b} = (\mathcal{R}[u])\mathfrak{b}_0$. Note that there is the projection:

$$\mathcal{R}[u] \to \mathcal{R}[u]/(hu - (q^{-1} - q)) \cong \mathfrak{R} := \mathbb{Z}[q^{\pm 1}, z, h, \frac{q^{-1} - q}{h}] \subset \mathbb{Z}[q^{\pm 1}, z, h^{\pm 1}].$$

The inclusion is induced by the epimorphism of $\mathbb{Z}[q^{\pm 1}, z, h]$-modules:

$$\mathcal{R}[u] \to \mathbb{Z}[q^{\pm 1}, z, h, \frac{q^{-1} - q}{h}]$$

defined by $u \mapsto \frac{q^{-1}-q}{h}$. Note that each element in $\mathcal{R}[u]$ is *uniquely* represented, modulo the ideal generated by $hu - (q^{-1} - q)$, in the form $b_0 + \sum b_i u^i$ with $b_0 \in \mathbb{Z}[q^{\pm 1}, z, h]$ and $b_i \in \mathbb{Z}[q^{\pm 1}, z]$. Note that $\mathfrak{R}$ is the Rees algebra over the ring $\mathbb{Z}[q^{\pm 1}, z]$ for the ideal generated by $q^{-1} - q$, see definition in [17], 6.5.

The proofs will be carried out in two steps. We will first prove the corresponding results for the module $\mathcal{J}(M)$, which is defined without vacuum relations and over the ring $\mathcal{R} := \mathbb{Z}[q^{\pm 1}, z, h]$. We have skein relations $q^{-1}K_+ - qK_- = \sigma(K_*)$ with $\sigma(K_*) = hK_0$ for a self-crossing and $\sigma(K_*) = ZK_0$ for a crossing of distinct components.

Note also that there is the ring homomomorphism defined by the composition of the inclusion $\mathfrak{R} \subset \mathbb{Z}[q^{\pm 1}, z, h^{\pm 1}]$ with the homorphism into the ring $\mathbb{Z}[q^{\pm 1}, h^{\pm 1}]$ mapping z to h. It is over this ring for which standard Homflypt skein theory is usually discussed. Note that it is a priori not clear that the *model* homomorphism into the corresponding skein module:

$$\mathbb{Z}[q^{\pm 1}, h^{\pm 1}]\mathfrak{b}_0 \to \mathcal{S}(M; \mathbb{Z}[q^{\pm 1}, h^{\pm 1}])$$

with skein relations $q^{-1}K_+ - qK_- = hK_0$ for all crossings, including the vaccuum relation.

We will apply Theorem 1.1 with geometric models corresponding to a homomorphism

$$\mathfrak{s} : \mathcal{R}\mathfrak{b} \to \mathcal{J}(M),$$

which is injective and thereby the gives a minimal set of models in a corresponding category. This follows by mapping $q \mapsto 1$ and $z, h \mapsto 0$. We will

then relate $\mathcal{J}(M)$ to $\mathfrak{J}(M)$. Note that there is there natural $\mathcal{R}$-homomorphism $\mathcal{J}(M) \to \mathfrak{J}(M)$ defined by the identity on $\mathcal{L}(M)$. The main point is that this homomorphism is compatible with the $\mathcal{R}$-homomorphism $\mathcal{R}[u] \to \mathfrak{R}$.

Furthermore we would like to point out that our theorems together with Przytycki's universal coefficient theorem [41] will imply the corresponding results in usual Homflypt skein theory, in particular module isomorphism with $S\mathbb{Z}[q^{\pm 1}, h^{\pm 1}]\hat{\pi}_0$.

Note that we have the free $\mathcal{R}$-module $\mathcal{R}\mathfrak{b} \cong S\mathcal{R}\hat{\pi}$, where $\mathfrak{b}$ is the set of monomials in free homotopy classes of loops $\hat{\pi}$ in the fundamental group $\pi_1(M)$ and S denotes the symmetric algebra.

Proof. [**of Theorem 1.2**]. We apply Theorem 1.1 in the situation above for the boundary operator and linear skein potential as defined above. The potential is insensitive, see e. g. [25] or [28]. Thus 2.12 applies. It follows from 6.5, and 7.11, which shows that *kink relations* are of the form $(q^{-1} - q)K = h(\mathcal{U} \amalg K)$, where $\amalg$ is disjoint union in a 3-ball and $\mathcal{U}$ is the unknot, that U is generated by only kink relations. Now we consider the commutative diagram with I the ideal generated by h:

$$
\begin{array}{ccccc}
\mathcal{R}\mathfrak{b} & \longrightarrow & \mathcal{J}(M) & \longrightarrow & \mathcal{R}\mathfrak{b}[[I]]/U \\
\downarrow & & \downarrow & & \downarrow \\
\mathfrak{R}\mathfrak{b}_0 & \xrightarrow{\mathfrak{s}} & \mathfrak{J}(M) & \longrightarrow & \mathfrak{R}\mathfrak{b}_0[[I]]
\end{array}
$$

It follows from the diagram and the definitions that the composition of the bottom two maps is the inclusion $\mathfrak{R}\mathfrak{b}_0 \to \mathfrak{R}\mathfrak{b}_0[[I]]$.. Now $\mathfrak{R}$ is noetherian as a subring of a noetherian ring. It follows that the inclusion of a free module into the completion is injective, see the corollary of Krull's theorem ([3], p. 110). Therefore the image of $\mathfrak{s}$ in the skein module $\mathfrak{J}(M)$ is isomorphic to $\mathfrak{R}\mathfrak{b}_0$. $\square$

Remark 9.1. A proof of Theorem 1.2 can also be given by extending the 2-category $\mathcal{C}_q$ from Theorem 7.1 to a category $\mathcal{C}_q^u$ so that objects are of the form $u^i q^j K$ for i nonnegative integers and $j \in \mathbb{Z}$, with K having no trivial components separated in a 3-ball. Similarly 1 and 2-morphisms can be extended (see the argument below concerning the action of i on the set of singular links). Let $\mathcal{R}_0 := \mathbb{Z}[z, h]$ so that $\mathcal{R} = \mathcal{R}_0[q^{\pm 1}]$. Let $\mathcal{L}_0(M)$ denote the set of isotopy classes of oriented links in M without any trivial components in separated 3-balls. Note that isotopy of links in M is in one-to-one correspondence with isotopy in $M \setminus D$ for an open 3-ball D. Then, after fixing a 3-ball near the boundary there is a well-defined action of a non-negative integer i on spaces of singular links in $M \setminus D$ defined by taking a disjoint union with an i-component unlink in that ball. This actions descends to isotopy classes. In particular we can identify, denoting $M \setminus D$ by just M, $\mathcal{L}(M) = \cup_i u^i \mathcal{L}_0(M)$. Now note that there isomorphisms of $\mathcal{R}_0$-modules $\mathcal{R}_0 \mathrm{ob}(\mathcal{C}_q^u) \cong \mathcal{R}[u]\mathcal{L}_0(M) \cong \mathcal{R}\mathcal{L}(M)$. Now for each $K \in \mathcal{L}(M)$ let $\chi(K) \geq 0$ be the number of components of a maximal unlink separated from the remaining component of K. Then the $\mathcal{R}$-module $\mathfrak{J}(M)$ is isomorphic to the quotient of the free $\mathcal{R}[u]$-module with basis $\mathcal{L}_0(M)$

by the submodule generated by $q^{-1}K_+ - u^{\chi(K_-)}qK_- = \sigma(K_*)u^{\chi(K_0)}K_0$ with $K_+ \in \mathcal{L}_0(M)$, a similar relation for $K_- \in \mathcal{L}_0(M)$, and $q^{-1} - q = hu$. In this way $\mathfrak{J}(M)$ is in fact an $\mathcal{R}[u]$-module. Now the mapping $\mathcal{R}[u] \to \mathfrak{R}$, $u \mapsto \frac{q^{-1}-q}{h}$ is an $\mathcal{R}[u]$-isomomorphism to the $\mathfrak{R}$-module $\mathfrak{J}(M)$. Note that each $\mathfrak{R}$-module is also a $\mathcal{R}[u]$-module using the mapping above. We can now apply Theorem 2.4 to the category $\mathcal{C}_q^u$ and Theorem 6.1 and Proposition 7.1 as before.

Proof. [**of Theorem 1.3**]. Using the homomorphism defined in Theorem 1.1 and the Proof of Theorem 1.2 above we get for each $\beta \in \mathfrak{b}_0$ the homomorphism

$$\iota_\beta : \mathfrak{J}(M) \to \mathfrak{R}\mathfrak{b}_0[[h]] \to \mathfrak{R}[[h]]$$

with the right hand map induced by the projection $\mathfrak{R}\mathfrak{b}_0 \to \mathfrak{R}$ defined by β. $\square$

Remark 9.2. Recall that there is a ring homomorphism $\mathfrak{R} \to \mathbb{Z}[q^{\pm 1}, h^{\pm 1}]$, which then can be mapped in various ways to the power series ring $\mathbb{Q}[[x]]$. These ring homomorphisms extend to
$$\mathfrak{R}[[h]] \to \mathbb{Q}[[x]]$$
and we recover the power series invariants of links in 3-manifolds as defined in [**?**], [29].

Proof. [**of Theorem 1.4**]. Lens spaces $M = L(p, q)$ are aspherical and atoroidal for $p \neq 0$. Thus Theorems 1.2 and 1.3 apply. A careful checking of the reduction arguments in ([13], Proof of Theorem 1.1), or in [19]), shows that the homomorphism $\mathfrak{R}\mathfrak{b}_0 \to \mathfrak{J}(M)$ is onto. $\square$

Remark 9.3. (i) The argument in [46], 5.3 shows that the skein module over $\mathbb{Z}[q^{\pm 1}, h^{\pm 1}]$, which is not distinguishing between self-crossings and mixed crossings, does not contain more information about links than the module $\mathcal{L}(M)$.
(ii) The study of Homflypt modules for Lens spaces $L(p, 1)$ has a long history, in particular the question of the minimal generating set has been settled a long time ago, see [14], [15] and [16]. The result for $L(p, 1)$ has been first proved in [19].

10   Relations with string topology

In order to relate the homomorphisms μ respectively μ_0 with string topology operations we have to deal with both a passage from isotopy to homotopy, and a multiplication (respectively transversely a smoothing) operation. In turns out to be interesting to describe this in the two possible ways of applying these operations in different order. Recall that string topology operations in our case are defined on the chain level using bordism instead of homology, i. e. representing homology classes by mappings oriented manifolds for which the usual transversality arguments hold.

The ad hoc arguments used in the proofs of our main results hint at difficulties in a passage from ordered to unordered maps. The Chas Sullivan construction is in fact a construction in the homology of *ordered* maps of circles into M.

We first recall basic features of their set-up and restrict our viewpoint at this moment to the two fundamental string topology operations in the case of 3-manifolds. Let $\widetilde{\text{top}}(j)$ denote the space of continuous (or piecewise smooth) mappings $\cup_j S^1 \to M$. Moreover let $\widetilde{\text{top}}(j)_o$ denote the subspace of those maps with at least one constant component. The group $(S^1)^j$ acts on the space $\widetilde{\text{top}}(j)$ preserving the subspace $\widetilde{\text{top}}(j)_o$. This can be used to define equivariant homology groups $H_*^{\text{eq}}(\widetilde{\text{top}}(j))$ and relative equivariant homology groups $H_*^{\text{eq}}(\widetilde{\text{top}}(j), \widetilde{\text{top}}(j)_o)$.

In [9] and [10] the following two basic string operations c and s are defined, which we call the *collison* and *self-collision* operators. These are homomorphisms

$$c : H_1^{\text{eq}}(\widetilde{\text{top}}(2)) \to H_0(\text{top}(1))$$

and

$$s : H_1^{\text{eq}}(\text{top}(1)) \to H_0(\widetilde{\text{top}}(2), \widetilde{\text{top}}(2)_o)$$

Remark 10.1. In the case of rational coefficients we have the Künneth isomorphisms

$$H_*^{\text{eq}}(\widetilde{\text{top}}(j), \widetilde{\text{top}}(j)_o) \cong \bigotimes_j H_*^{\text{eq}}(\widetilde{\text{top}}(1), \widetilde{\text{top}}(1)_o).$$

In [10] the operations c and s are described geometrically in terms of $H_*^{\text{eq}}(\widetilde{\text{top}}(j), \widetilde{\text{top}}(j)_o)$. But Chas and Sullivan prefer to use the Künneth identification to express the operations in terms of $H_*(\text{top}(1), \text{top}(1)_0)$ denoted $\mathbb{L}$ in [10]. We have already identified the equivariant and non-equivariant 0-dimensional homology groups.

To avoid cluttering of notation we will not indicate the grading in spaces of immersions in the rest of the paper so that imm is used denote **only** imm[0], and similarly for the ordered versions. It follows easily from the definitions that operations, also denoted s, c here, can be defined as follows

$$s : H_1^{\text{eq}}(\text{imm}(1)) \to H_0(\widetilde{\text{imm}}(2)).$$

and similarly

$$c : H_1^{\text{eq}}(\widetilde{\text{imm}}(2)) \to H_0(\text{imm}(1))$$

Here we used that the inclusion from continuous maps into immersions is an isomorphism in 0 dimensional homology.

The Chas Sullivan construction allows to introduce *dummy* components, which are insensitive to collisions or self-collisions. An example would be

$$s_i : H_1^{\text{eq}}(\widetilde{\text{imm}}(j)) \to H_0(\widetilde{\text{imm}}(j+1)),$$

which measures self-collisions in the i-th component, $1 \leq i \leq j$. In the same way there are defined for $j \geq 2$:

$$c_{k\ell} : H_1^{\text{eq}}(\widetilde{\text{imm}}(j)) \to H_0(\widetilde{\text{imm}}(j-1)),$$

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measuring the collisions between the k-th and ℓ-th component for $1 \leq k < \ell \leq j$. By taking the sum of the s_i respectively $c_{k\ell}$ we get well-defined homomorphisms with domain and target as above. Finally by summation over all j we have defined the homomorphisms:

$$s, c : H_1^{\mathrm{eq}}(\widetilde{\mathrm{imm}}) \to H_0(\widetilde{\mathrm{imm}})$$

and thus the homomorphism (s, t). This can be precomposed with the surjective homomorphism

$$H_1(\widetilde{\mathrm{imm}}) \to H_1^{\mathrm{eq}}(\widetilde{\mathrm{imm}})$$

to define

$$\widetilde{\mathfrak{j}} : H_1(\widetilde{\mathrm{imm}}) \to H_0(\mathrm{imm}) \oplus H_0(\mathrm{imm}),$$

where we have in the domain we have have composed with the mapping induced by the projection $\widetilde{\mathrm{imm}} \to \mathrm{imm}$. (The surjectivity is proved as follows: First decompose

$$H_1^{\mathrm{eq}}(\widetilde{\mathrm{imm}}) \cong \bigoplus_{a \in \widetilde{\mathfrak{b}}} H_1^{eq}(\widetilde{\mathrm{imm}}_a),$$

where the $\widetilde{\mathfrak{b}}$ is the set of ordered sequences in $\hat{\pi}$. Then apply the Künneth theorem to decompose for a of length n:

$$H_1^{eq}(\widetilde{\mathrm{imm}_a}) \cong \bigotimes_{1 \leq i \leq n} H_1(\mathrm{imm}_{a_i}/S^1).$$

Consider the Gysin epimorphisms, see [26] and [9]

$$H_1(\mathrm{imm}_{a_i}) \to H_1(\mathrm{imm}_{a_i}/S^1).$$

Finally apply the Künneth isomorphism in non-equivariant homology to deduce the result.)

Consider the decomposition

$$H_1(\widetilde{\mathrm{imm}}) \cong \bigoplus_{a \in \widetilde{\mathfrak{b}}} H_1(\mathrm{imm}_a).$$

and the Hurewicz epimorphisms for $a \in \mathfrak{b}$:

$$\pi_1(\widetilde{\mathrm{imm}}; f_a) \to H_1(\widetilde{\mathrm{imm}}_a).$$

The definitions of c and s imply that $\widetilde{\mathfrak{j}}$ factors through the image of $\pi_1(\widetilde{\mathrm{imm}}_a)$ in $\pi_1(\mathrm{imm}_\alpha)$, where $\alpha \in \mathfrak{b}$ is the unordered sequence underlying a. We know that this homomorphism *naturally* extends to a *homomorphism* on the full groups $\pi_1(\mathrm{imm}_\alpha)$. Because commutators map trivially it factors through the homology groups $H_1(\mathrm{imm}_\alpha)$. Using linearity and summarizing we thus have defined the homomorphism:

$$\mathfrak{j} : H_1(\mathrm{imm}) \to H_0(\mathrm{imm}) \oplus H_0(\mathrm{imm}).$$

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We want to consider $\mathfrak{j}$ with the homomorphism μ. We first show that $\mathfrak{j}$ results from $\mathfrak{mu}$ by applying a smoothing construction followed by the passage from isotopy to homotopy.

Let

$$\mathfrak{c} : \mathbb{Z}\mathcal{L}[1] \to \mathbb{Z}\mathcal{L} \oplus \mathbb{Z}\mathcal{L}$$

be defined by mapping basis elements K_* to the *Conway smoothing* K_0, placed into the first summand for a self-crossing and into the second summand for a crossing of dictinct components of K_*.

Recall that for $j \geq 0$ let $\mathcal{L}[k](j)$ denotes the subset of $\mathcal{L}[k]$ given by immersions with j components. Then more precisely the homomorphism defined above is graded by homomorphisms:

$$\mathfrak{c}(j) : \mathbb{Z}\mathcal{L}[1](j) \to \mathbb{Z}\mathcal{L}(j+1) \oplus \mathbb{Z}\mathcal{L}(j-1)$$

for $j \geq 1$. Of course $\mathfrak{c}$ induces the homomorphism

$$\mathbb{Z}\mathcal{L}[1]/\mathcal{D} \to \mathbb{Z}\mathcal{L} \oplus \mathbb{Z}\mathcal{L}/\mathfrak{c}(\mathcal{D}).$$

It follows that there is the well-defined homomorphism

$$\mathfrak{c} \circ \mu : H_1(\mathrm{imm}) \to (\mathbb{Z}\mathcal{L} \oplus \mathbb{Z}\mathcal{L})/\mathfrak{c}(\mathcal{D}).$$

assigning to each transversal loop in imm the oriented sum of Conway smoothings at singular parameters. We call this map the *Lin homomorphism*.

Now note that

$$\mathfrak{h} \oplus \mathfrak{h} : \mathbb{Z}\mathcal{L} \oplus \mathbb{Z}\mathcal{L} \to \mathbb{Z}\mathfrak{b} \oplus \mathbb{Z}\mathfrak{b}$$

maps $\sigma(\mathcal{D})$ to 0. Thus we have defined the homomorphism

$$(\mathfrak{h} \oplus \mathfrak{h}) \circ \mathfrak{c} \circ \mu : H_1(\mathrm{imm}) \to \mathbb{Z}\mathfrak{b} \oplus \mathbb{Z}\mathfrak{b} \cong H_0(\mathrm{imm}) \oplus H_0(\mathrm{imm}).$$

The following result is now obvious from the definitions and compares our constructions with those of Chas and Sullivan .

Theorem 10.1.

$$(\mathfrak{h} \oplus \mathfrak{h}) \circ \mathfrak{c} \circ \mu = \mathfrak{j} : H_1(\mathrm{imm}) \to H_0(\mathrm{imm}) \oplus H_0(\mathrm{imm})$$

References

[1] J. E. Andersen, J. Mattes, N. Reshitikin, Quantization of the algebra of chord diagrams *Math. Proc. Camb. Phil. Soc.*,**124** (1998), 451–467

[2] V. I. Arnold, The Vassiliev theory of discriminants and knots, *in Proceedings of the European Congress of Mathematics*

[3] M. F. Atiyah, I. G. MacDonald, *Introduction to Commutative Algebra*, Addison-Wesley Series in Mathematics, 1969

[4] J. C. Baez, J. Dolan, Categorification *in in Contemp. Math. ,***230***, eds. Ezra Getzler and Mikhail Kapranov,American Mathematical Society, Providence, Rhode Island*, (1998), 1–36

[5] D. Bar-Natan, A. Stoimenow, The fundamental theorem of Vassiliev invariants, *in Geometry and Physics*, eds. J. E. Andersen, J. Dupont, H. Pedersen and A. Swann, lecture notes in pure and applied mathematics 184, Marcel-Dekker, New York 1997

[6] J. Birman, X. S. Lin, Knot polynomials and Vassiliev's invariants *Inventiones math.* **111** (1993), 225–270

[7] J. Brylinski, *Loop Spaces, Characteristic Classes, and Geometric Quantization*, Progress in Mathematics **107**, Birkhäuser Boston 1993

[8] D. Chateur, A Bordism Approach to String Topology, *International Mathematics Research Notices* **46**, (2005), 2829–2875

[9] M. Chas, D. Sullivan, String topology, preprint math.GT/9911159, http://xxx.lanl.gov, 1999

[10] M. Chas, D. Sullivan, Closed string operators in topology leading to bialgebras and higher string algebra, in *The legacy of Niels Henrik Abel*, ed. Olav Arnfinn Laudal, Ragni Piene, (2004), Springer, 771–784

[11] V. Chernov, Framed knots in 3-manifolds and affine self-linking numbers, *J. Knot Theory Ramifications*, **14** (23005), no. 6, 791–818

[12] V. Chernov, Y. Rudyak, Toward a general theory of linking invariants, *Geometry and Topology* **9**, (2005), 1881–1913

[13] C. Cornwell, A polynomial invariant for links in Lens spaces, *J. Knot Theory Ramifications* **21**, No. 6, 21 pp.

[14] I. Diamantis, S. Lambropoulou, J. H. Przytycki, Topological steps toward the Homflypt skein module of the lens space $L(p, 1)$ via braids *J. Knot Theory and its Ramifications* **25** (2016), No. 14, 26 pp.

[15] I. Diamantis, S. Lambropoulou, The braid approach to the HOMFLYPT skein module of the Lens spaces $L(p, 1)$, *in Algebraic modeling of topological and computational structures and applications, Springer Proc. Math. Stat.* **219**, Springer, Cham (2017), 143–176

[16] I. Diamantis, S. Lambropoulou, A new basis for the Homflypt skein module of the solid torus, *J. Pure and Applied Algebra* **220**, (2016), 577–605

[17] D. Eisenbud, *Commutative Algebra, with a view toward Algebraic Geometry*, Graduate Texts in Mathematics **150**, Springer, New York, 1994

[18] B. Gabrovsek, E. Horvat, Knot invariants in Lens Spaces, in *Knots, Low-Dimensional Topology and Applications, Knots in Hellas 2016*, Springer Proc. in Math. and Stat. , Springer (2019), 347–362

[19] B. Gabrovsek, M. Mroczkowski, The Homflypt skein module of the lens spaces $L(p,1)$, *Topology and its Applications*,**175** (2014)

[20] P. Gilmer, J. Zhong *On the Homflypt Skein Module of $S^2 \times S^1$*, Math. Zeit. 237, 2001, 769–814

[21] P. Gilmer, J. Zhong The Homflypt skein module of a connected sum of 3-manifolds, *Algebraic and Geometric Topology* **1**, (2001), 605–625

[22] H. Glöckner, Implicit Functions from Topoloical Vector Spaces to Banach Spaces, *Israel Journal of Mathematics* **155**, (2006), 205–252

[23] S. Gunningham, D. Jordan, P. Safronov, The finiteness conjecture for skein modules, arXiv: 1908.05233v2 [math.QA]

[24] K. Habiro, Claspers and finite type invariants of links, *Geometry and Topology* Volume **4**, (2000), 1–83

[25] M. Hutchings, Integration of singular braid invariants and graph cohomology, *Transactions of the American Mathematical Society*, **350**, no. 5, (1998), 1791–1809

[26] U. Kaiser, Presentations of q-homotopy skein modules of oriented 3-manifolds, *J. Knot Theory Ramifications*, **10** (2001), no. 3, 461–491

[27] U. Kaiser, Deformation of string topology into homotopy skein modules, *Algebr. Geom. Topol.* , **3** (2003), 1005–1035

[28] U. Kaiser, Quantum deformations of the fundamental group of oriented 3-manifolds, *Trans. Amer. Math. Soc.* , **356** (2004), no. 10, 3869–3880

[29] E. Kalfagianni, Finite type invariants for knots in 3-manifolds, *Topology* **37**, no. 3 (1998), 673–707

[30] E. Kalfagianni, Power series link invariants and the Thurston norm, *Topology and its Apllications*, vol. **101** (2) (2000), 107–119

[31] E. Kalfagianni, X. S. Lin, The HOMFLY polynomial for links in rational homology spheres, *Topology* **38**, no. 1 (1999), 95–115

[32] C. Kassel, *Quantum Groups*, Graduate Texts in Mathematics **155**, Springer Verlag, New York, 1995

[33] L. Kauffman, *On Knots*, Annals of Mathematical Studies **115**, Princeton University Press, 1987

[34] T. Kerler, V. V. Lybashenko, *Non-semisimple Topological Quantum Field Theories for 3-Manifolds with Corners*, Lecture Notes in Mathematics **1765**, Springer Verlag, New York, 2001

[35] P. Kirk, C. Livingston, Knot invariants in 3-manifolds and essential tori, *Pacific Journal of Mathematics* **197**, No. 1, (2001), 73–96

[36] S. Lack, A 2-categories companion, in *Baez J., May J. (eds) Towards Higher Categories The IMA Volumes in Mathematics and its Applications*, vol. **152**, Springer, New York, 2010

[37] X. S. Lin, Finite type link invariants of 3-manifolds, *Topology* **33**, no.1 (1994), 45–71

[38] S. MacLane, *Categories for the Working Mathematician*, Graduate Texts in Mathematics Vol. **5**, Springer Verlag, New York, 1971

[39] H. Morton, Skein theory and the Murphy operators, *Journal of Knot Theory and its Ramifications* **11** (2002), 475–492

[40] L. Meier, Eine geometrische Betrachtung der String-Topologie (German), Thesis Mathematisch-Naturwissenschaftliche Universität der Rheinischen Friedrich-Wilhelms Universität Bonn, 2009

[41] J. Przytycki, Algebraic topology based on knots: an introduction, in *Proceedings of Knots 96*, edited by Shin'ichi Suzuki, 1997, 279–297

[42] J. Przytycki, Skein modules of 3-manifolds, *Bull. Ac. Pol. Math.* **39**(1-2) (1991), 91–100

[43] J. Przytycki, Skein module of links in a handlebody in *Topology 90, Proceedings of the Research Semester in Low Dimensional Topology at OSU*, eds. B. Apanasov, W. Neumann, A. Reid, L. Siebenmann, de Gruyter Verlag 1992, 315–342

[44] J. Przytycki, Vassiliev-Gusarov skein modules of 3-manifolds and criteria for periodicity of knots, in *Low Dimensional Topology*, ed. K. Johannson, International Press Co. Cambridge, MA 02238, 1994, 143–162

[45] T. Stanford, Finite type invariants of knots, links and graphs, *Topology* **35**, no. 4 (1996), 1027–1050

[46] D. Treumann, Exit paths and constructible stacks, *Compositio Math.* **145**, (2009), 1504–1532

[47] V. Turaev, Skein quantization of Poisson algebras of loops on surfaces, *Annales scientifiques de l'E. N. S.* 4^e serie, tome **24**, no. 6, 1991, 635–704

[48] V. Vassiliev, On invariants and homology of spaces of knots in arbitrary manifolds, in *American Math. Soc. Translations* (2), Vol. 185, 1998

[49] V. Vassiliev, *Complements of discriminants of smooth maps: topology and applications*, *Transl. Math. Monographs*, vol. **98**, AMS, Providence, RI 1994

[50] C. T. C. Wall, *Surgery on Compact Manifolds*, 2nd Edition, *Mathematical Surveys and Monographs* Vol. **69**, American Mathematical Society, 1999

[51] C. T. C. Wall, Transversality in families of mappings, *Proc. Lond. Math. Soc. (3)*, **99** (2009), no. 1, 67-99

[52] G. W. Whitehead, *Elements of Homotopy Theory*, Graduate Texts in Mathematics Vol. **61**, Springer Verlag, New York 1978