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Modeling and Stress Analysis of Rounded Rectangular Inclusion Enclosed by FGM Layer

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Abstract
The aim of the present work is to model and analyze stresses around rounded rectangular inclusion enclosed with functionally graded material (FGM) layer. The inclusion has been considered in an infinite plate which is subjected to far-field tensile stress. The extended finite element method (XFEM) has been used to model the inclusion with non-conformal mesh. The level set functions of circular and rectangular shapes have been used to trace the inclusion boundary with mesh. The FGM has been considered as continuous varying mixture of inclusion and plate materials with power law function along normal direction to the inclusion interface. Young's modulus has been assumed to vary within FGM layer, whereas Poisson's ratio is kept constant. The stress distribution and stress concentration factor (SCF) have been analyzed for different geometrical and FGM parameters. It has been observed that XFEM with level set method efficiently model the difficult shape inclusions such as rounded rectangle. Applying the FGM layer smoothens the stress distribution around rounded rectangular inclusion and significantly reduces SCF. The position of maximum stress shifted from the inclusion interface toward the FGM layer interface. The least SCF has been noted with power law index $n = 0.5$ and FGM layer thickness $t = r$.

Keywords- Inclusion, Functionally graded material, Extended finite element method, Level-set method.

1. Introduction

Recent development in engineering and space applications cultivates the need of materials that can satisfy these requirements without fail. Composites are the kind of materials that can fulfill such needs. Composite materials generally have two constituents, one as matrix and other as reinforcement. At the interface of these two, there is a property mismatch. This mismatching plays an important role in the characteristics of composites and may cause stress concentration. FGM layer around the discontinuity is one of the approaches which can reduce property mismatch and avoid stress concentration. FGM is a composite material concept consisting of varying volume fractions of its constituents that result in smooth variation in material properties along the direction of gradation (Goyat et al., 2021). Luo and Gao (2009) solved the stress analysis problem of arbitrary inclusion using complex variable method and Faber-Laurent series. Shen et al. (2010) modeled the inclusion in piezoelectric material for mechanical and electrical loadings. Wang and Chen (2013) analyzed the thermal stresses for arbitrary shape inclusion for hydrostatic type thermal loading. Sburlati et al. (2013) tested the FGM layer around circular hole and noticed a significant reduction in SCF. The concept of FGM layer then attracted many researchers for SCF reduction around different kinds of discontinuities. Yang and Gao (2013) investigated the circular inclusion reinforced with FGM layer for thermal stresses. They found that FGM can be suitable material to enclosed the inclusion for SCF reduction. Lee et al. (2016) solved the problem of Eshelby's arbitrary shape inclusion. Yang et al. (2018) extended their work for tensile load and tested the effect of FGM on SCF around circular, elliptical and rectangular inclusions. Chen (2019) investigated Eshelby's elliptical inclusion using complex variable method. Most of the work reported above used complex variable method to model and solve the stress analysis problem. The complex variable method required assistance of conformal mapping to map the interface on a circle. For simple and analytical shapes, it is convenient to map however, for complicated shapes, it is pretty cumbersome. Further, it may not guarantee the high accuracy of results as it consists some mapping errors. Goyat et al. (2018) presented a simple and accurate XFEM modeling of FGM reinforced hole. The hole interface was modeled by level set function in a non-conformal XFEM mesh. Goyat et al. (2022) proposed level set function-based graded FGM for SCF reduction near hole in a finite plate.

Yang et al. (2021) experimentally investigated the effect of FGM parameters on SCF around hole on the universal testing machine. They observed a good agreement between the result obtained from mathematical modeling and experimental investigation. Salasiya and Sundaram (2022) analyzed inclusions of different shapes for stress distribution and observed that rounded corner inclusions have a smooth stress distribution compared to sharp corner inclusions.

Literature evident that the use of FGM has proven itself as the best strategy to reduce the SCF around discontinuities such as holes, notches, inclusions, etc. By controlling FGM parameters, one can control the SCF significantly. Most of the work found in the literature is about SCF reduction around holes. A few articles are available on the application of FGM for SCF reduction at inclusion interface of circular or elliptical shapes. Further, the work on SCF reduction around rounded rectangular inclusion is not available in the literature and it is significant to analyze the rounded rectangular inclusion for SCF reduction because rounded rectangle is a usual inclusion shape (Jaiswal et al., 2022). Which is the prime motivation for the present work i.e., modeling and stress analysis of rounded rectangular inclusion enclosed by FGM layer.

2. Mathematical Modeling

The current problem is to analyze the stress distribution and stress concentration around rounded rectangular inclusion using XFEM. The FGM layer has been applied around the inclusion with the aim of getting smooth stress distribution as well as reduced stress concentration. Figure 1(a) shows the problem configuration, an infinite plate under uniaxial unit magnitude tensile stress and having central rounded rectangular inclusion enclosed by FGM layer. XFEM is selected to model the current problem. XFEM is an enriched finite element method (FEM) that can model the inclusion without conformal meshing (Moes et al., 1999). Goyat et al. (2018) explained the detailed mathematical modeling of XFEM for FGM enclosed holes in an infinite panel subjected to different load conditions.

In the current problem, a 2D infinite domain (Ω) i.e., plate is considered with central rounded rectangular inclusion with traction free interface Γ and enclosed with FGM layer. The FGM layer has been made up by mixing inclusion and plate material in a continuous varying volume fraction. The parameters of rounded rectangular inclusions are: r – radius, a – inclusion length and b – inclusion breadth ($b = 2r$). The plate width (W) and height (H) have been considered as 50 times the length of the rounded rectangular inclusion. To model the plate a non-conformal regular mesh of eight node quadrilateral elements was prepared. The inclusion boundary has been modeled by using enrichment function. Only nodes of elements having the inclusion boundary have been enriched. The displacement behavior within a simple element i.e., non-enriched element has been estimated by the standard FEM equation as:

$$u(x,y) = \sum_{i\in m} N_i(x,y)u_i \quad (1)$$

N_i is nodal shape function, u_i is nodal displacement and m is total nodes in element.

The displacement behavior of elements having inclusion boundary i.e., enriched elements and surrounding element having sharing enriched nodes can be expressed as (Sukumar et al., 2001):

$$u(x,y) = \sum_{i\in m} N_i(x,y)u_i + \sum_{j\in e} N_j(x,y)\phi a_j \quad (2)$$

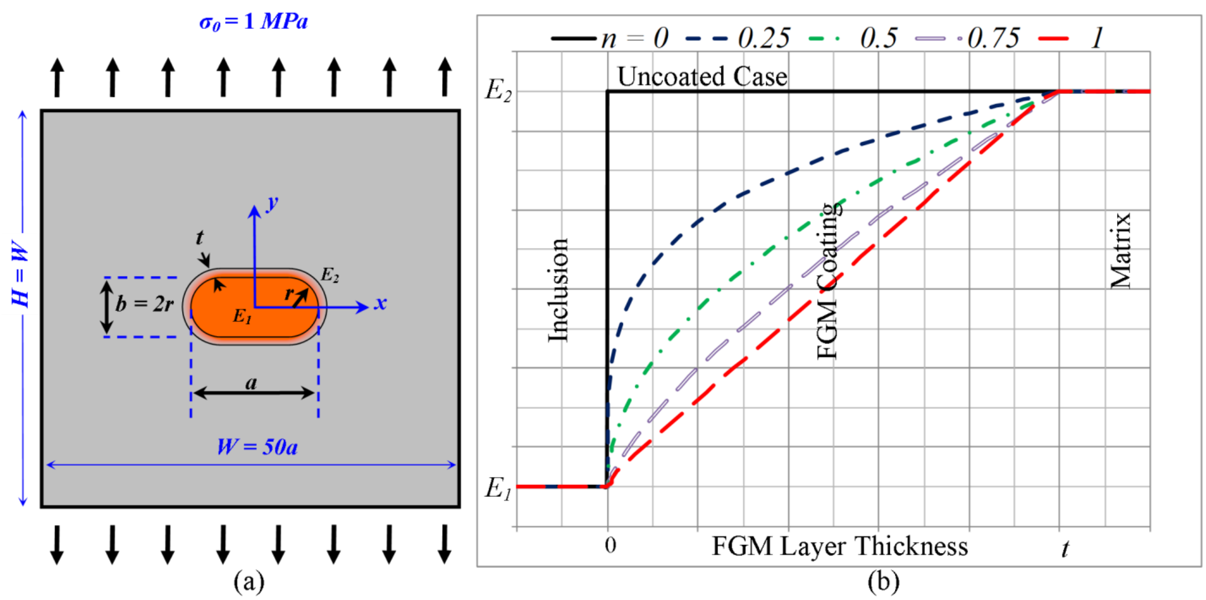

Figure 1. (a) Infinite plate with rounded rectangular inclusion enclosed with FGM layer (b) Variation of *E* in FGM layer with *n*.

The second term of equation (2) is additional if compared with equation (1). It has been added to model interfacial behavior. The ϕ represents the enrichment function, e shows number of enriched nodes in the element and a_j represents the additional degree of freedom. The ϕ can be expressed in terms of shape function and level set function ξ as:

$$\phi = \sum_j N_j(x,y)|\xi_j| - \left|\sum_j N_j(x,y)\xi_j\right| \quad (3)$$

The level set function ξ is the sign distance function in normal direction to the boundary. In order to get the level set function of rounded rectangular inclusion boundary its geometry is prepared by composite geometry of two circles of r radius and one rectangle of sides l and b as shown in Figure 2(a), where $l = a-2r$. The level set function for circle can be obtained by using position vector of point of interest P and position vector of center of circle Q_{ci} and radius of circle r as:

$$\xi_{Ci} = \|P - Q_{ci}\| - r \quad (4)$$

The level set function for rectangle can be calculated as:

$$\xi_R = \|P - Q_{\Gamma R}\| \, sgn(\lambda) \tag{5}$$

$$\lambda = (P - Q_{\Gamma R}).\hat{n} \tag{6}$$

$$sgn(\lambda) = \begin{cases} 1 & if\ \lambda \geq 0 \\ -1 & if\ \lambda < 0 \end{cases} \tag{7}$$

The $Q_{\Gamma R}$ is the position vector of nearest point on the boundary of rectangle whose normal $\hat{n}$ passes through the point of interest P as shown in Figure 2(b). The level set function of rounded rectangular interface can be generated by using the following equation:

$$\xi = \min\,(\xi_{C1}\,, \xi_{C2} \text{ and } \xi_R) \tag{8}$$

The level set function of left circle C_1, right circle C_2, rectangle R and composite geometry i.e. rounded rectangle are shown in Figures 3(a) to 3(d) respectively. The rounded rectangle interface can be traced by the zero level set function in XFEM mesh.

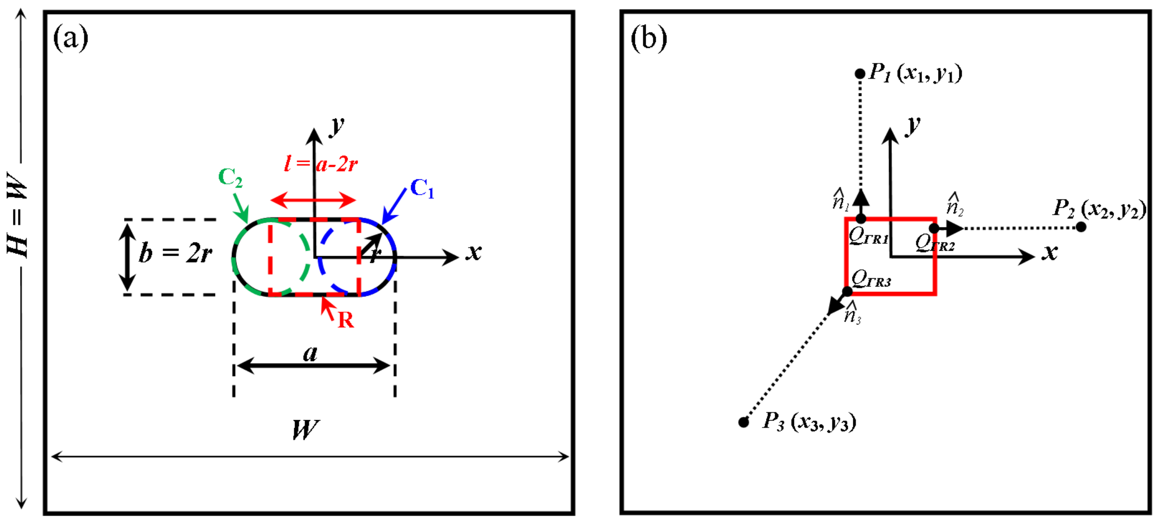

Figure 2. (a) Illustration of composite geometry to model rounded rectangular inclusion with level set functions and (b) Illustration of the normal direction vector $\hat{n}$ used in Equation (6).

The displacement can be obtained using FEM standard discrete equation (9):

$$[k]\{u\} = \{f\} \tag{9}$$

where, stiffness matrix k can be expressed using basis matrix B ($[B] = d[N]$) and constitutive relation matrix C as:

$$[k] = \int_{\Omega e} [B]^T [C][B]\, d\Omega_e \tag{10}$$

The constitutive relation matrix C for current 2D plane stress problem can be expressed as:

$$[C] = \frac{E(x,y)}{(1-v^2(x,y))} \begin{bmatrix} 1 & v(x,y) & 0 \\ v(x,y) & 1 & 0 \\ 0 & 0 & \frac{1-v(x,y)}{2} \end{bmatrix} \tag{11}$$

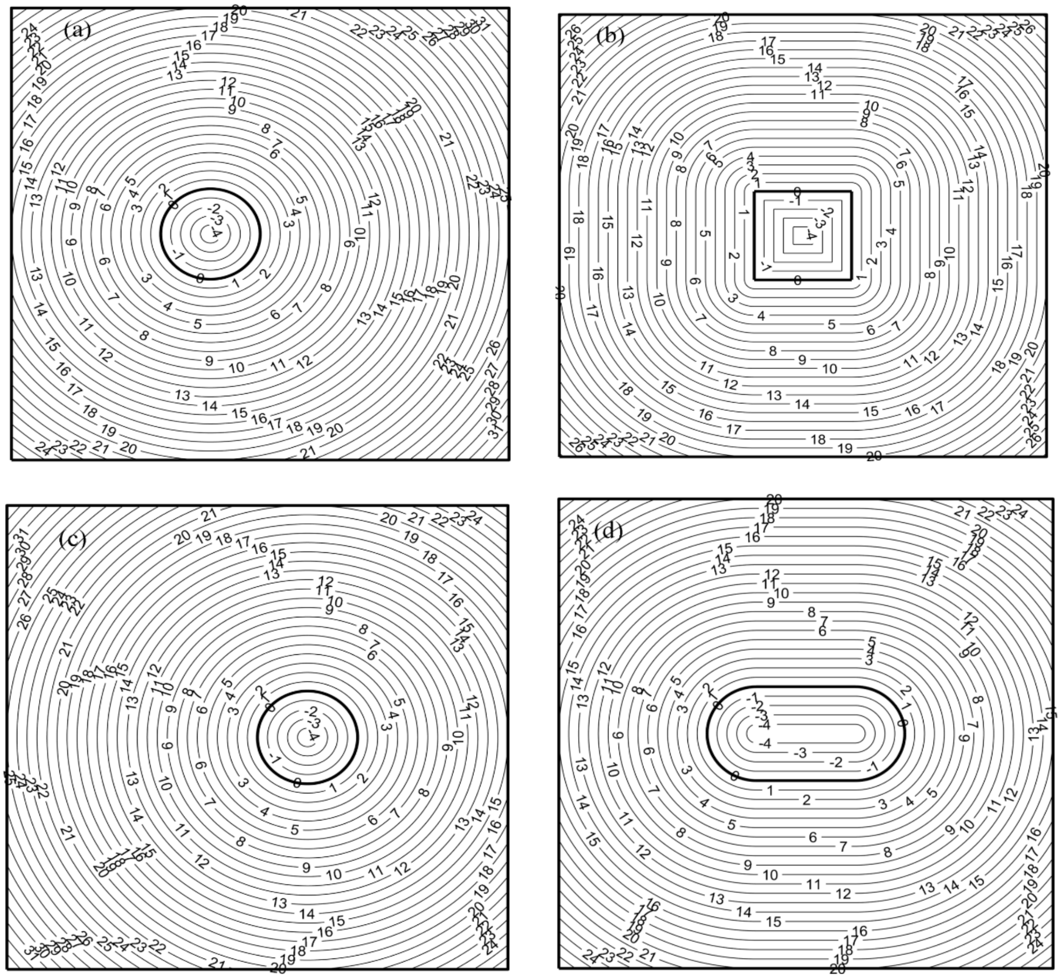

Figure 3. Level set function for (a) left circle C_1, (b) rectangle R, (c) right circle C_2 and (d) rounded rectangle.

To deal with varying material properties Young's Modulus $E(x, y)$ and Poisson's ratio $v(x, y)$ are expressed as varying with x and y. Further, the stress σ and strain ε are calculated by following equations.

$$\{\varepsilon\} = [B]\{u\} \tag{12}$$

$$\{\sigma\} = [C]\{\varepsilon\} \tag{13}$$

3. Material Modeling

For the current problem Young's modulus has been assumed to be varying with the power law function in the normal direction to inclusion. The following expression has been used to vary Young's modulus in FGM layer.

$$E(x, y) = E_1 + (E_2 - E_1)\left(\frac{\xi}{t}\right)^n \tag{14}$$

E_1 denotes Young's modulus of inclusion and E_2 represents Young's modulus of plate. The variation of Youngs' modulus has been controlled by power law index n and layer thickness t. The illustration of equation (14) for different power law index n is shown in Figure 1(b). The ratio of Young's modulus of inclusion E_1 and Young's modulus of plate E_2 is represented by K as $K = E_1/E_2$. Equation (14) can be rewritten in terms of K as:

$$E(x, y) = E_1 + E_2(1 - K)\left(\frac{\xi}{t}\right)^n \quad (15)$$

The literature shows that Poisson's ratio has an insignificant effect on stress behavior of inclusion therefore, in the present work, it has been considered as constant for inclusion, plate and FGM layer as (Goyat et al., 2019):

$$\upsilon(x, y) = \upsilon = 0.3 \quad (16)$$

4. Result and Analysis

Based on mathematical model of XFEM, a computer program in MATLAB language was prepared for stress distribution around the inclusion and SCF analysis. The problem of present work is illustrated in Figure 1(a). Before obtaining results for the current problem, the program has been tested for accuracy by comparing obtained results with existing literature. The program has been tuned for the non-FGM case of Luo and Gao (2009) and FGM case of Yang et al. (2018). Their work was about circular inclusion under far-field uniaxial tension. To make rounded rectangular inclusion circular a is considered as $2r$. For Yang et al. (2018) case program parameters are $a = 2r, t = 0.2r, n = 1$ *and* $K = 0.5$. The program parameters for Luo and Gao (2009) cases are $a = 2r, t = 0, n = 0$ *and* $K = 0.1, 0.5$. The stress distribution of hoop stress has been obtained at the inclusion interface for Luo and Gao (2009) cases and at the FGM layer for Yang et al. (2018) case. The obtained results of hoop stress distribution with angle θ have been calculated and depicted with results taken from the literature in Figure 4. It can be stated from Figure 4 that the results obtained from the prepared program are consistent with the results of literature. It has been confirmed from this comparison that the prepared program is accurate, reliable and efficient for the current analysis.

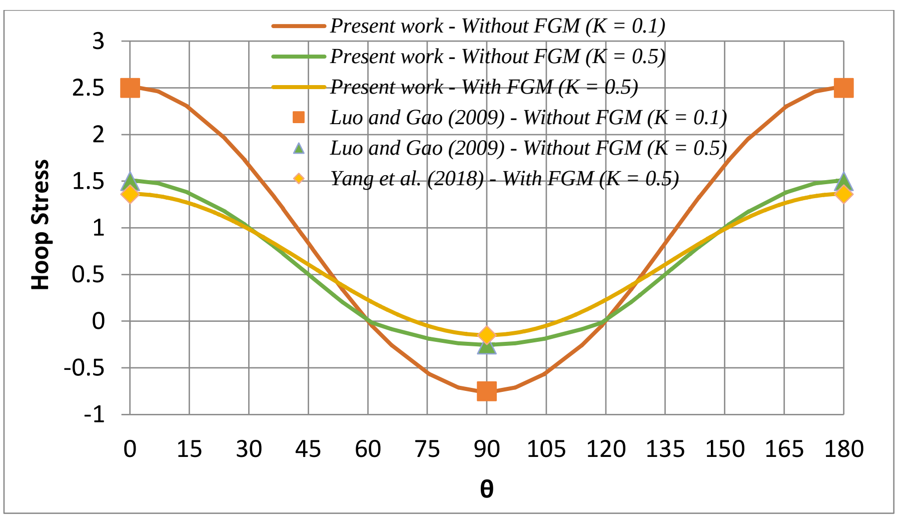

Figure 4. Validation of computer code with Luo and Gao (2009) and Yang et al. (2018).

The stress distribution around the inclusion with and without FGM layer has been obtained for $a/2r = 2, t = 0.5r$ and $n = 0.5$. It has been found that the stress distribution around the inclusion is symmetric about x and y axis hence, the quarter plots of small region around the inclusion are depicted in Figure 5 for the above-said parameters. Figure 5(a) and 5(c) show the stress distribution of inclusion without FGM layer for K = 0.1 and 0.5 respectively. Further, Figure 5(b) and 5(d) depicts the stress distribution for FGM cases. Figure 5(a) and 5(b) share the same color scale depicted at the bottom of these figures and same for Figure 5(c) and 5(d). From Figure 5, it has been observed that the hoop stress reduces significantly in the FGM layer cases if compared with non-FGM layer cases. The maximum stress has been observed at the interface with $\theta = 0$ for without FGM layer cases however, by applying FGM layer it reduces in amount and shifts away from the inclusion interface towards FGM layer interface.

In order to capture the effect of power law index n, the SCF has been calculated for different values of $a/2r$ by considering K = 0.1 and t = 0.5r. The relations of SCF with $a/2r$ for different values of n are depicted in Figure 6. The SCF has been calculated as σ_{max}/σ_0. The n = 0 represents the inclusion without FGM layer. If a comparison has been made between n = 0 and other values of n, it has been noticed that n > 0 has significantly lower SCF values. With a particular value of $a/2r$, the SCF first decreases to a minimum value and then starts increasing with increase in n. The n = 0.5 has been observed to provide lower values of SCF for full range of $a/2r$. Therefore, n = 0.5 is selected for further analysis.

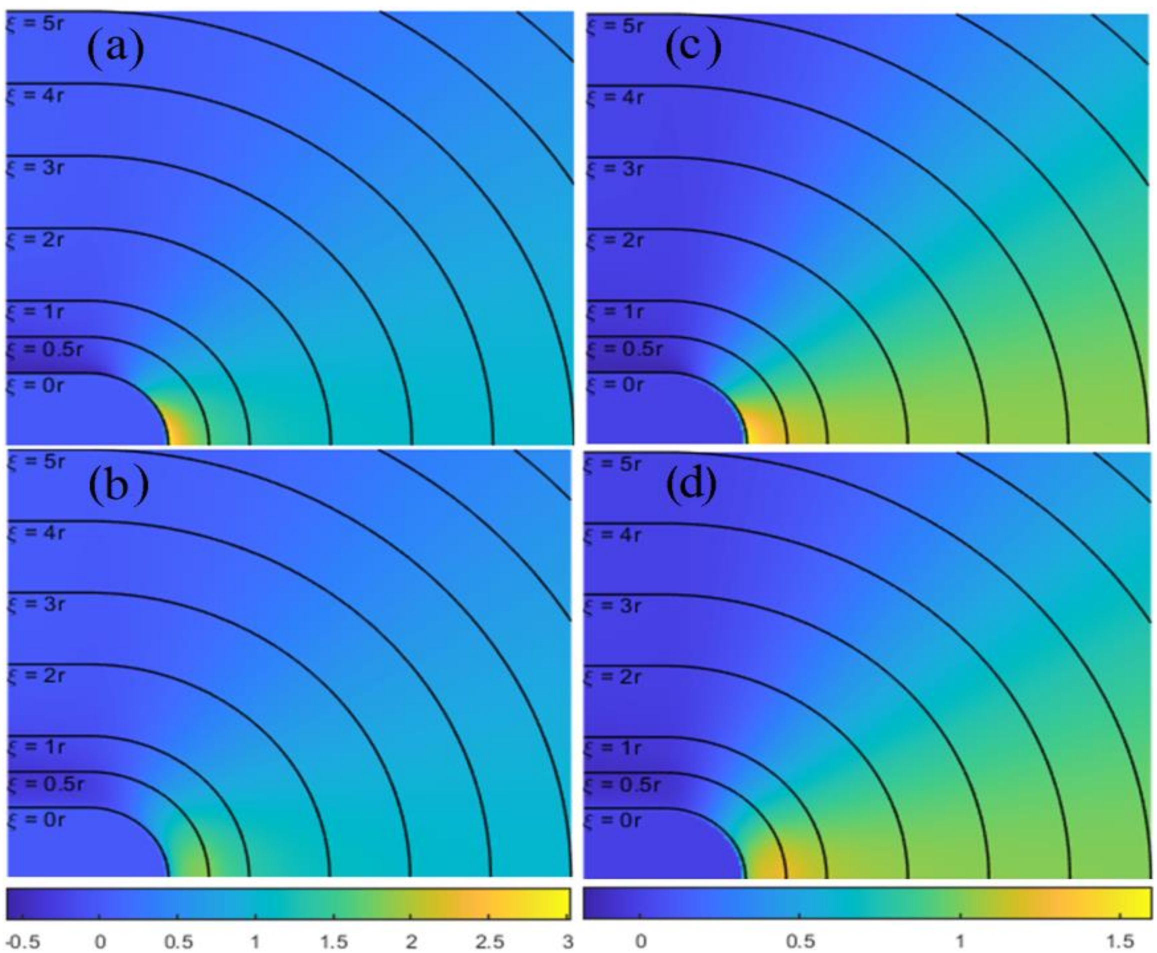

Figure 5. Stress distribution around rounded rectangular inclusion for (a) K = 0.1 (without FGM layer), (b) K = 0.1 (enclosed with FGM layer of n = 0.5, t = 0.5r), (c) K = 0.5 (without FGM layer) and (d) K = 0.5 (enclosed with FGM layer of n = 0.5, t = 0.5r).

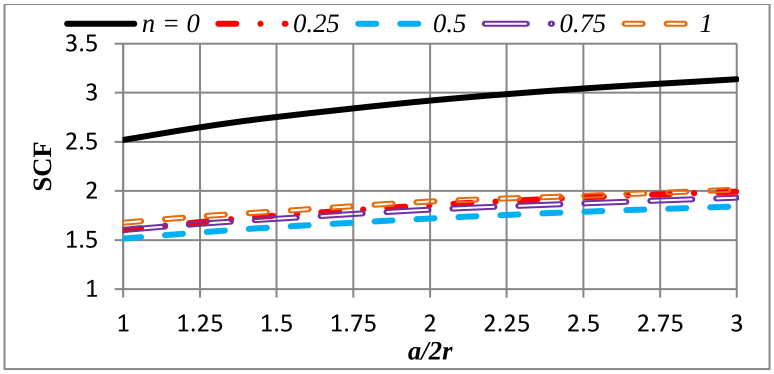

Figure 6. Effect of *n* on SCF for *K* = 0.1 and *t* = 0.5*r*.

To analyze the thickness effect of enclosed FGM layer on SCF the results were obtained for *K* = 0.1 at different values of *t* and *a/2r*. These results are depicted in Figure 7. It has been noted that *t* has a significant effect on SCF. For a particular *a/2r,* SCF decreases with increase in *t*. However, the rate of decrement in SCF noticed decreases with increase in *t*. It has also been observed that the effect of *a/2r* reduces with increase in *t*. It can be concluded that the thicker FGM layer shows the least SCF therefore, *t = r* can be selected for further analysis.

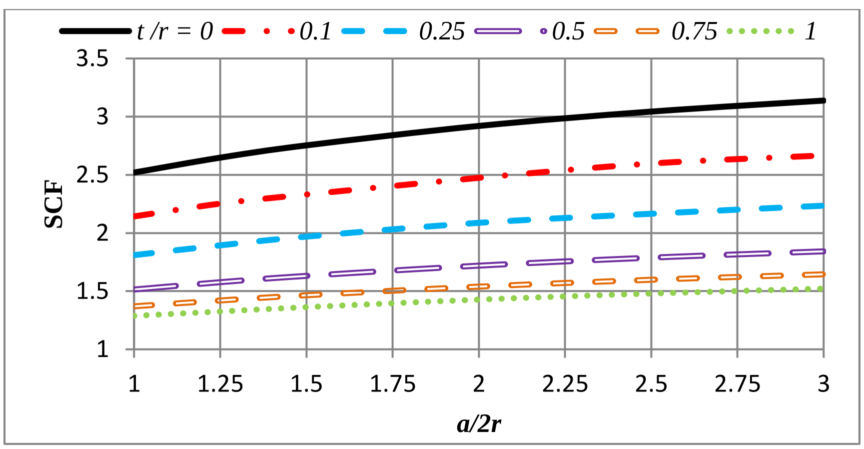

Figure 7. Effect of *t* on SCF for *K* = 0.1 and *n* = 0.5.

The effect of *K* on SCF is depicted in Figure 8. A comparison between with and without FGM layer cases for different values of *K* have been made and it can be stated that FGM layer cases have reduced value of SCF except *K* = 1 cases. However, the amount of reduction is more with lower values of *K*. The case of *K* = 1 shows the perfect bonded inclusion of plate material therefore, no stress concentration. The effect of *a/2r* is observed less in the FGM layer cases and further found diminishing with increase in *K*.

A regression model has been prepared for the rounded rectangular inclusion enclosed with FGM layer of parameters *n* = 0.5 and *t* = *r*. The regression equation to predict the relation of SCF with *K* and *a/2r* is as follows:

$$SCF = 1.272 + 0.142\frac{a}{2r} - 0.838K - 0.402\frac{a}{2r}K + 1.392K^2 + 0.264\frac{a}{2r}K^2 - 0.835K^3 \quad (17)$$

The goodness of fit (R^2) of the regression model has been observed as 0.994 and the percentage root mean square error has been found 1.12%.

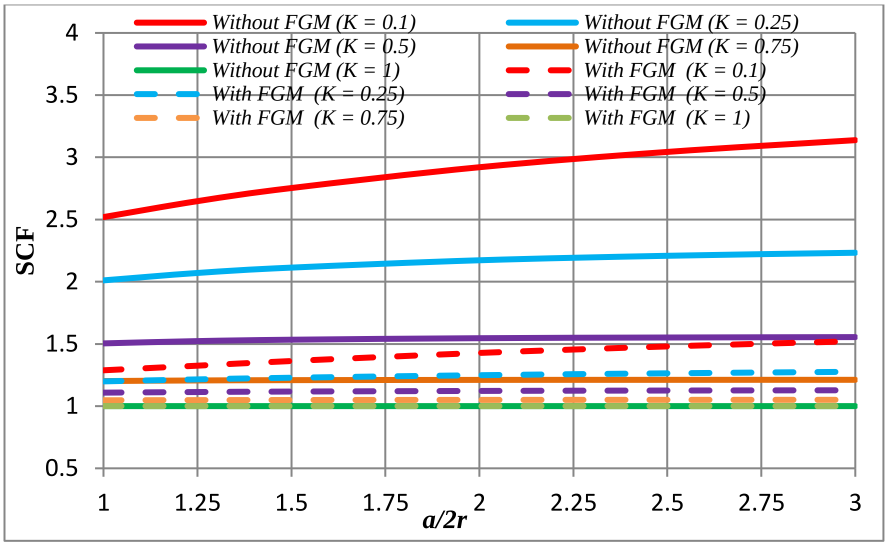

Figure 8. Effect of K on SCF for n = 0.5 and $t = r$.

In summary, the effect of length of rounded rectangular inclusion on SCF has been observed as highly significant for uncoated inclusion but less significant for FGM coated inclusion. The SCF is higher with lower values of K and 1 at K = 1. Enclosing inclusion with FGM, lowered the SCF significantly. The amount of SCF reduction has been found to be highest with the lowest values of K *i.e. 0.1*, which decreases to zero at K = 1 in comparison with respective uncoated cases. The optimal value of power law index n has been noted as 0.5, which ensures the least SCF for a complete range of *a/2r*. The thicker FGM layer shows the least SCF but the rate of SCF reduction in comparison to non-coated inclusion decrease with increase in layer thickness. The power law function-based FGM layer of n =0.5 and thickness $t = r$ has been recommended for rounded rectangular inclusion in a plate subjected to uniaxial tensile far-field stress.

5. Conclusions

Stress distribution and SCF analysis of rounded rectangular inclusion enclosed with FGM layer in an infinite plate of homogeneous material infinite plate subjected to unit far-field tensile stress have been presented. XFEM along with level set method has been used for modeling of inclusion without conformal meshing. Validation analysis confirms the high accuracy of XFEM. The level set method is highly efficient in modeling complicated interface boundary in non-conformal XFEM mesh. The level set method also offered a convenient way to grade FGM along the normal direction to inclusion interface.

Enclosing FGM layer around rounded rectangular inclusion lowered the SCF value significantly. The distribution of hoop stress around FGM enclosed inclusion is smoother and has a lower value of maximum stress compared to uncoated inclusion. The position of maximum stress shifts towards FGM layer interface along x-axis. The optimal value of power law index n is 0.5, which shows the least SCF for the complete considered range of normalized inclusion length. The thicker FGM layer offers least SCF therefore, FGM layer thickness t should be equal to the radius r of rounded rectangular inclusion. A regression model has been proposed to predict the SCF of rounded rectangular inclusion enclosed by FGM layer of parameters n = 0.5 and $t = r$ for different values Young's modulus ratio K and normalized inclusion length $a/2r$.

Conflict of Interest

The authors confirm that there is no conflict of interest to declare for this publication.

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