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A NEW TENSOR PROJECTION METHOD FOR TENSOR VARIATIONAL INEQUALITIES

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Abstract. We obtain the equivalence between a suitable tensor variational inequality and a tensor complementarity problem. Moreover, we present a new tensor projection method for solving tensor variational inequalities by supposing that the involved function is continuous and satisfies a generalization of the monotonicity condition. Finally, we use the method to compute the equilibrium solution of a general oligopolistic market equilibrium problem with demand excesses.

Keywords. Demand excesses; General Cournot-Nash equilibrium; Tensor variational inequalities; Tensor projection method.

1. Introduction

Variational inequalities are a fundamental tool to study many problems arising from the optimization theory and capture various applications, such as partial differential equations, optimal control, and mathematical programming (see, e.g., [1, 2, 3]). The introduction of tensor variational inequalities (i.e. variational inequalities modeled on the space of tensors) have many applications, including some economic problems. Hence, tensor variational inequalities are a relevant mathematical tool. In this paper, we establish the relation between such inequalities and a class of complementarity problems (called tensor complementarity problems). For this reasons, the study of iterative methods for solving such inequalities assumes considerable importance. Historically, two meaningful approaches for solving variational inequalities are projection and descent methods. Recently, many different projection-like methods have been proposed to approximate solutions to variational inequality problems under various types of conditions. Some of them are the projection methods and extragradient methods which require in the stepsize to know the Lipschitz constant or/and the modulus of strong monotonicity of the function considered. In general it is not easy to obtain these constants (see, e.g., [4, 5, 6, 7, 8, 9].

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In [10], the authors extended these kind of methods to tensor variational inequalities. They studied, in particular, a tensor extragradient method and a tensor extragradient method with adaptive steplength (inspired by the Marcotte algorithm presented in [11]).

In this paper, we introduce a new algorithm based on a milder assumption than the monotonicity and even than the pseudomonotonicity. Indeed, we assume only that

$$\langle F(\mathscr{Y}), \mathscr{Y} - \mathscr{X} \rangle \geq 0, \quad \forall \mathscr{Y} \in K,$$

where K is a nonempty, closed, and convex subset of the tensor space $\mathscr{T}_{N,m}$, and $F : K \to \mathscr{T}_{N,m}$ is a tensor function. Moreover, we prove a convergence result for the algorithm.

Tensor variational inequalities are useful to analyze a general oligopolistic market equilibrium problem in which the firms produce several goods and compete in a noncooperative behavior. This means that each player has at his disposal a strategy which he chooses from a set of feasible strategies with the aim of maximizing his utility level, given the decisions of the other players (see, e.g., [12, 13, 14]). Here, we consider a general oligopolistic market equilibrium problem in presence of demand excesses, and we apply the theoretical results to this model. We underline that demand excesses may occur when the supply cannot satisfy the demand especially for fundamental goods. In such an improvement of the model, equilibrium conditions are given by using a generalization of the Cournot-Nash principle. By using variational techniques, the equilibrium distribution can be characterized as the solution to a suitable tensor variational inequality.

The paper is structured as follows. In Section 2, some preliminary results on tensor variational inequalities are recalled. Moreover, we show, under suitable assumptions, the equivalence between tensor variational inequalities and tensor complementarity problems. In Section 3, a numerical method based on the projection operator for solving tensor variational inequalities is presented. Furthermore, we obtain that the sequence of approximation solutions converges to the exact solution. Section 4 concerns the general oligopolistic market equilibrium problem in which the firms produce several goods and demand excesses occur. The equilibrium condition which generalizes the Cournot-Nash principle is presented and expressed by a suitable tensor variational inequality. Then, we show existence results for the equilibrium distribution. In Section 5, a numerical example is provided. At last, Section 6 deals with conclusive remarks.

2. Some preliminaries

2.1. **Notations.** We fix some notations. Let V be a finite-dimensional vector space endowed with an inner product. A N-order tensor is an element of the N-product space $V \times \cdots \times V$, i.e. a multidimensional array. We denote by small letters $v, w, \ldots$ tensors of order one (i.e. vectors), by capital letters $A, B, \ldots$ tensors of order two (i.e. matrices) and by italic capital letters $\mathscr{X}, \mathscr{Y}, \ldots$ tensors of general order.

The space of N-order tensors on the m-dimensional vector space V is indicated by $\mathscr{T}_{N,m}(V)$. In some clear cases, we use the simple notation $\mathscr{T}_{N,m}$ instead of $\mathscr{T}_{N,m}(V)$. A N-order tensor $\mathscr{X}$ on a space of dimension N has m^N entries. The element $(i_1, \ldots, i_N)$ of $\mathscr{X}$ is indicated by $x_{i_1,\ldots,i_N}$. Finally, we denote by $\mathbb{R}^{[m_1 \ldots m_N]}$ the class of $N-$order real tensors made by $\mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_N}$.

We introduce the following inner product on $\mathscr{T}_{N,m}$:

$$\langle \mathscr{X}, \mathscr{Y} \rangle = \sum_{i_1=1}^{m} \cdots \sum_{i_N=1}^{m} x_{i_1,\ldots,i_N} y_{i_1,\ldots,i_N}, \qquad \forall \mathscr{X}, \mathscr{Y} \in \mathscr{T}_{N,m}.$$

Thanks to this definition, $(\mathscr{T}_{N,m}, \langle\cdot,\cdot\rangle)$ is a Hilbert space.

2.2. **Tensor variational inequalities.** Given a nonempty closed and convex subset K of $\mathscr{T}_{N,m}$ and a tensor function $F : K \to \mathscr{T}_{N,m}$, the tensor variational inequality is the problem of finding $\mathscr{X} \in K$ such that

$$\langle F(\mathscr{X}), \mathscr{Y} - \mathscr{X}\rangle \geq 0, \quad \forall \mathscr{Y} \in K. \tag{2.1}$$

In [15] and [16], the authors proved some existence and uniqueness results for tensor variational inequalities. We recall some of them in the bounded case (see Theorem 2.1), in the unbounded case (see Theorem 2.2) and using the monotone approach (see Theorem 2.3).

Theorem 2.1. *If K is a nonempty compact convex subset of $\mathscr{T}_{N,m}$ and $F : K \to \mathscr{T}_{N,m}$ is a continuous tensor function. Then (2.1) admits at least one solution.*

Without the boundedness assumption on the set K, we need to require the coerciveness of the operator F, as the following result states.

Theorem 2.2. *If K is a nonempty closed convex subset of $\mathscr{T}_{N,m}$ and $F : K \to \mathscr{T}_{N,m}$ is a continuous tensor function such that*

$$\lim_{\|\mathscr{X}\|\to+\infty} \frac{\langle F(\mathscr{X}) - F(\mathscr{X}_0), \mathscr{X} - \mathscr{X}_0\rangle}{\|\mathscr{X} - \mathscr{X}_0\|} = +\infty,$$

for some $\mathscr{X}_0 \in K$, then (2.1) admits a solution.

In order to recall the results based on the monotone approach, let us start with the following:

Definition 2.1. Let K be a nonempty subset of $\mathscr{T}_{N,m}$. A tensor function $F : K \to \mathscr{T}_{N,m}$ is said to be

- pseudomonotone on K if, for each $\mathscr{X}, \mathscr{Y} \in \mathscr{T}_{N,m}$,
$$\langle F(\mathscr{Y}), \mathscr{X} - \mathscr{Y}\rangle \geq 0 \;\Rightarrow\; \langle F(\mathscr{X}), \mathscr{X} - \mathscr{Y}\rangle \geq 0;$$
- monotone on K if, for each $\mathscr{X}, \mathscr{Y} \in \mathscr{T}_{N,m}$,
$$\langle F(\mathscr{X}) - F(\mathscr{Y}), \mathscr{X} - \mathscr{Y}\rangle \geq 0;$$
- strictly monotone on K if, for each $\mathscr{X}, \mathscr{Y} \in \mathscr{T}_{N,m}$, with $\mathscr{X} \neq \mathscr{Y}$,
$$\langle F(\mathscr{X}) - F(\mathscr{Y}), \mathscr{X} - \mathscr{Y}\rangle > 0;$$
- strongly monotone on K if, for each $\mathscr{X}, \mathscr{Y} \in \mathscr{T}_{N,m}$, there exists $\nu > 0$ such that
$$\langle F(\mathscr{X}) - F(\mathscr{Y}), \mathscr{X} - \mathscr{Y}\rangle \geq \nu\|\mathscr{X} - \mathscr{Y}\|^2.$$

Theorem 2.3. *Let K be a nonempty closed convex subset of $\mathscr{T}_{N,m}$, and let $F : K \to \mathscr{T}_{N,m}$ be a tensor function.*

a) *If F is continuous and monotone, then the set of solutions, briefly $S(F,K)$, to (2.1) is nonempty, closed, and convex.*
b) *If F is strictly monotone, then if there exists a solution to (2.1), then it is unique.*
c) *If F is continuous and strongly monotone, then there exists a unique solution to (2.1).*

We recall also the following result which has an important role to introduce numerical methods.

Theorem 2.4. *If K is a nonempty compact convex subset of $\mathscr{T}_{N,m}$ and $F : K \to \mathscr{T}_{N,m}$ is a tensor continuous function, then the function F admits a fixed point in K.*

We conclude this subsection with some useful observations for the algorithm presented in the next section. Firstly, from Theorem 2.1, it is easy to deduce that $\mathscr{X}$ is a solution to (2.1) if and only if $\mathscr{X}$ is a fixed point for the operator $\mathscr{X} \mapsto P_K(\mathscr{X} - \alpha F(\mathscr{X}))$, for any $\alpha > 0$, where $P_K(\cdot)$ is the projection operator on K.

Under suitable conditions, as recalled in the previous theorems, the solution set $S(F,K)$ of (2.1) is nonempty. Let $\mathscr{X}$ be any element of the solution set $S(F,K)$. We introduce now the following condition:

$$\langle F(\mathscr{Y}), \mathscr{Y} - \mathscr{X} \rangle \geq 0, \quad \forall \mathscr{Y} \in K, \tag{2.2}$$

which generalizes monotonicity assumptions. More precisely, it is easy to verify that if F is monotone or pseudo-monotone, then (2.2) is satisfied.

For theoretical results, we know that solutions to tensor variational inequality (2.1) coincide with the zeros of the following tensor projection residual function:

$$R(\mathscr{X}) = \mathscr{X} - P_K(\mathscr{X} - F(\mathscr{X})), \tag{2.3}$$

namely, $\mathscr{X} \in S(F,K)$ if and only if $R(\mathscr{X}) = 0$.

2.3. **The tensor complementarity problem via TVI.** The variational inequality problem is strongly connected with the complementarity problem (see, in the vectorial case, for instance [2, 17, 18]. We introduce here a general version of the complementarity problem modeled on tensor spaces.

Definition 2.2. Given $K \subseteq \mathscr{T}_{N,m}$, the polar cone of K is the set defined by

$$K^* = \{\mathscr{X}^* \in (\mathscr{T}_{N,m})^* : \langle \mathscr{X}^*, \mathscr{Y} \rangle \geq 0, \quad \forall \mathscr{Y} \in K\},$$

where $(\mathscr{T}_{N,m})^*$ is the dual space of $\mathscr{T}_{N,m}$, i.e., $(\mathscr{T}_{N,m})^* = \{F : \mathscr{T}_{N,m} \to \mathbb{R} : \ F \text{ is linear}\}$.

The following problem extends the vector complementarity problem, introduced in [17, 18], considering tensor functions.

Problem 2.1. Let K be a closed convex cone of $\mathscr{T}_{N,m}$, and let $F : K \to \mathscr{T}_{N,m}$ be a tensor map. The tensor complementarity problem is the problem of finding $\overline{\mathscr{X}} \in K$ such that

$$F(\overline{\mathscr{X}}) \in K^*, \qquad \langle F(\overline{\mathscr{X}}), \overline{\mathscr{X}} \rangle = 0. \tag{2.4}$$

We can prove the following result.

Proposition 2.1. *Let K be a closed convex cone of $\mathscr{T}_{N,m}$, and let $F : K \to \mathscr{T}_{N,m}$ be a tensor map. The tensor complementarity problem (2.4) is equivalent to the tensor variational inequality (2.1).*

Proof. Let us suppose, at first, that $\overline{\mathscr{X}}$ is a solution to (2.1). Then choosing $\mathscr{Y} = 2\overline{\mathscr{X}}$ and $\mathscr{Y} = 0$ in (2.1) we obtain that $\langle F(\overline{\mathscr{X}}), \overline{\mathscr{X}} \rangle \geq 0$ and $\langle F(\overline{\mathscr{X}}), \overline{\mathscr{X}} \rangle \leq 0$, respectively, which implies $\langle F(\overline{\mathscr{X}}), \overline{\mathscr{X}} \rangle = 0$. Moreover, from (2.1), we obtain that $\langle F(\overline{\mathscr{X}}), \mathscr{Y} \rangle \geq 0$, for every $\mathscr{Y} \in K$, i.e., $F(\overline{\mathscr{X}}) \in K^*$. Then $\overline{\mathscr{X}}$ is a solution of problem (2.4).

Conversely, if $\overline{\mathscr{X}}$ is a solution to Problem 2.1, then

$$\langle F(\overline{\mathscr{X}}), \mathscr{Y} - \overline{\mathscr{X}} \rangle = \langle F(\overline{\mathscr{X}}), \mathscr{Y} \rangle - \langle F(\overline{\mathscr{X}}), \overline{\mathscr{X}} \rangle = \langle F(\overline{\mathscr{X}}), \mathscr{Y} \rangle$$

which is nonnegative since $F(\overline{\mathscr{X}}) \in K^*$. As a consequence, we can conclude that $\overline{\mathscr{X}}$ is a solution to (2.1). $\square$

3. TENSOR NUMERICAL METHOD

This section is devoted to the new algorithm for solving tensor variational inequalities as (2.1) based on the projection operator. More precisely, we extend a numerical method presented by Solodov and Svaiter in [19] for vector variational inequalities. Furthermore, we prove a convergence result under suitable assumptions.

Nevertheless, we only assume the condition (2.2), namely

$$\langle F(\mathscr{Y}), \mathscr{Y} - \mathscr{X} \rangle \geq 0, \quad \forall \mathscr{Y} \in K,$$

and we recall the tensor projection residual function

$$R(\mathscr{X}) = \mathscr{X} - P_K(\mathscr{X} - F(\mathscr{X})).$$

In the following, we describe the numerical method.

Algorithm.

- We choose $\mathscr{X}^0 \in K$, $\gamma \in (0,1)$, and $\sigma \in (0,1)$;
- At each step, we compute $R(\mathscr{X}^i)$. If $R(\mathscr{X}^i) = 0$, then stop.
- If $R(\mathscr{X}^i) \neq 0$, we calculate

$$k_i = \min\left\{k \geq 0,\ k \in \mathbb{Z} :\ \langle F(\mathscr{X}^i - \gamma^k R(\mathscr{X}^i)), R(\mathscr{X}^i)\rangle \geq \sigma \|R(\mathscr{X}^i)\|^2\right\}, \tag{3.1}$$

$$\eta_i = \gamma^{k_i},$$

$$\mathscr{Z}^i = \mathscr{X}^i - \eta_i R(\mathscr{X}^i).$$

- We compute

$$\mathscr{X}^{i+1} = P_{C \cap H_i}(\mathscr{X}^i),$$

where $H_i = \{\mathscr{X} \in \mathscr{T}_{N,m} :\ \langle F(\mathscr{Z}^i), \mathscr{X} - \mathscr{Z}^i \rangle \leq 0\}$.

In order to prove a convergence result, we establish some preliminary lemmas.

Lemma 3.1. *Let K be a nonempty, closed, and convex subset of $\mathscr{T}_{N,m}$. For any $\mathscr{X}, \mathscr{Y} \in \mathscr{T}_{N,m}$ and any $\mathscr{Z} \in K$,*

$$\langle \mathscr{X} - P_K(\mathscr{X}), \mathscr{Z} - P_K(\mathscr{X}) \rangle \leq 0, \tag{3.2}$$

and

$$\|P_K(\mathscr{X}) - P_K(\mathscr{Y})\|^2 \leq \|\mathscr{X} - \mathscr{Y}\|^2 - \|P_K(\mathscr{X}) - \mathscr{X} + \mathscr{Y} - P_K(\mathscr{Y})\|^2. \tag{3.3}$$

Proof. Observe that

$$\begin{aligned} & \|\mathscr{X} - P_K(\mathscr{X})\|^2 - \|\mathscr{X} - \mathscr{Y}\|^2 \\ = \ & \|\mathscr{X} - P_K(\mathscr{X})\|^2 - \|(\mathscr{X} - P_K(\mathscr{X})) + (P_K(\mathscr{X}) - \mathscr{Y})\|^2 \\ = \ & -2\langle \mathscr{X} - P_K(\mathscr{X}), P_K(\mathscr{X}) - \mathscr{Y} \rangle - \|P_K(\mathscr{X}) - \mathscr{Y}\|^2. \end{aligned} \tag{3.4}$$

Rewriting (3.4) with $\mathscr{W} = t\mathscr{Z} + (1-t)P_K(\mathscr{X})$, $0 \leq t \leq 1$, in place of $\mathscr{Y}$, we have

$$\|\mathscr{X} - P_K(\mathscr{X})\|^2 - \|\mathscr{X} - \mathscr{W}\|^2 = -2t\langle \mathscr{X} - P_K(\mathscr{X}), P_K(\mathscr{X}) - \mathscr{Z} \rangle - t^2\|P_K(\mathscr{X}) - \mathscr{Z}\|^2.$$

Then, it results

$$0 \geq -2t\langle \mathscr{X} - P_K(\mathscr{X}), P_K(\mathscr{X}) - \mathscr{Z}\rangle - t^2\|P_K(\mathscr{X}) - \mathscr{Z}\|^2, \quad 0 < t \leq 1.$$

Dividing by t and passing to the limit as $t \to 0^+$, inequality (3.2) follows. In particular, the following inequality holds:

$$\langle \mathscr{X} - P_K(\mathscr{X}), P_K(\mathscr{X}) - P_K(\mathscr{Y})\rangle \geq 0, \quad \forall \mathscr{X}, \mathscr{Y} \in \mathscr{T}_{N,m}. \tag{3.5}$$

Let us prove that P_K satisfies the inequality:

$$\langle (I - P_K)(\mathscr{X}) - (I - P_K)(\mathscr{Y}), P_K(\mathscr{X}) - P_K(\mathscr{Y})\rangle \geq 0, \quad \forall \mathscr{X}, \mathscr{Y} \in \mathscr{T}_{N,m}. \tag{3.6}$$

In fact, rewriting (3.5) as

$$\langle (I - P_K)(\mathscr{X}), P_K(\mathscr{X}) - P_K(\mathscr{Y})\rangle \geq 0, \tag{3.7}$$

and interchanging $\mathscr{X}$ and $\mathscr{Y}$, we have

$$\langle (I - P_K)(\mathscr{Y}), P_K(\mathscr{Y}) - P_K(\mathscr{X})\rangle \geq 0. \tag{3.8}$$

Hence, adding (3.7) and (3.8), we obtain (3.6). Taking into account (3.6), we can derive

$$\|P_K(\mathscr{X}) - P_K(\mathscr{Y})\|^2 \leq \langle \mathscr{X} - \mathscr{Y}, P_K(\mathscr{X}) - P_K(\mathscr{Y})\rangle, \tag{3.9}$$

and

$$\|(I - P_K)(\mathscr{X}) - (I - P_K)(\mathscr{Y})\|^2 \leq \langle \mathscr{X} - \mathscr{Y}, (I - P_K)(\mathscr{X}) - (I - P_K)(\mathscr{Y})\rangle. \tag{3.10}$$

Finally, making use of (3.9) and (3.10), we obtain (3.3) immediately. □

Lemma 3.2. *If the Algorithm is well defined, then* $\mathscr{X}^{i+1} = P_{K\cap H_i}(P_{H_i}(\mathscr{X}^i))$.

Proof. From the Algorithm, we deduce that if

$$\langle F(\mathscr{W}^i), \mathscr{X}^i - \mathscr{W}^i\rangle > 0,$$

then $\mathscr{W}^i \notin H_i$. Since K is convex, it results

$$\mathscr{Z}^i = (1 - \eta_i)\mathscr{X}^i + \eta_i P_K(\mathscr{X}^i - F(\mathscr{X}^i)) \in K.$$

Again, from the numerical procedure, we have $\langle F(\mathscr{Z}^i), \mathscr{X}^* - \mathscr{Z}^i\rangle \leq 0$, for any $\mathscr{X}^* \in S(F,K)$, which implies $\mathscr{X}^* \in H_i$. Moreover, taking into account that $\mathscr{X}^* \in K$, we have $K \cap H_i \neq \emptyset$. Since $K \cap H_i$ is a nonempty closed convex set, we see that $\mathscr{X}^{i+1} = P_{K\cap H^i}(\mathscr{X}^i)$ is well defined. We have

$$\begin{aligned} \overline{\mathscr{X}}^i &= P_{H_i}(\mathscr{X}^i) \\ &= \mathscr{X}^i - \frac{\langle F(\mathscr{Z}^i), \mathscr{X}^i - \mathscr{Z}^i\rangle}{\|F(\mathscr{Z}^i)\|^2} F(\mathscr{Z}^i) \\ &= \mathscr{X}^i - \frac{\eta_i \langle F(\mathscr{Z}^i), R(\mathscr{X}^i)\rangle}{\|F(\mathscr{Z}^i)\|^2} F(\mathscr{Z}^i). \end{aligned}$$

Let $\mathscr{Y} \in K \cap H_i$. Being $\mathscr{X}^i \in K$ but $\mathscr{X}^i \notin H_i$, there exists $\beta \in [0,1]$ such that $\widetilde{\mathscr{X}} = \beta \mathscr{X}^i + (1-\beta)\mathscr{Y} \in K \cap \partial H_i$, with $\partial H_i = \{\mathscr{X} \in \mathscr{T}_{N,m} : \langle F(\mathscr{Z}^i), \mathscr{X} - \mathscr{Z}^i \rangle = 0\}$. It results

$$\begin{aligned} \|\mathscr{Y} - \overline{\mathscr{X}}^i\|^2 &\geq (1-\beta)\|\mathscr{Y} - \overline{\mathscr{X}}^i\|^2 \\ &= \|\widetilde{\mathscr{X}} - \beta \mathscr{X}^i - (1-\beta)\overline{\mathscr{X}}^i\|^2 \\ &= \|\widetilde{\mathscr{X}} - \overline{\mathscr{X}}^i\|^2 + \beta^2 \|\mathscr{X}^i - \overline{\mathscr{X}}^i\|^2 - 2\beta \langle \widetilde{\mathscr{X}} - \overline{\mathscr{X}}^i, \mathscr{X}^i - \overline{\mathscr{X}}^i \rangle \\ &\geq \|\widetilde{\mathscr{X}} - \overline{\mathscr{X}}^i\|^2, \end{aligned} \tag{3.11}$$

where we used Lemma 3.2 with $K = H_i$, $\mathscr{X} = \mathscr{X}^i$ and $\mathscr{Z} = \widetilde{\mathscr{X}} \in H_i$. Moreover, taking into account $\overline{\mathscr{X}}^i = P_{\partial H_i}(\mathscr{X}^i)$, $\widetilde{\mathscr{X}} \in \partial H_i$, and the Pythagoras theorem, we obtain

$$\|\widetilde{\mathscr{X}} - \overline{\mathscr{X}}^i\|^2 = \|\widetilde{\mathscr{X}} - \mathscr{X}^i\|^2 - \|\mathscr{X}^i - \overline{\mathscr{X}}^i\|^2.$$

Hence, by $\widetilde{\mathscr{X}} \in K \cap H_i$ and $\mathscr{X}^{i+1} = P_{K \cap H_i}(\mathscr{X}_i)$, and the Pythagoras theorem again, we have

$$\begin{aligned} \|\widetilde{\mathscr{X}} - \overline{\mathscr{X}}^i\|^2 &\geq \|\mathscr{X}^{i+1} - \mathscr{X}^i\|^2 - \|\mathscr{X}^i - \overline{\mathscr{X}}^i\|^2 \\ &= \|\mathscr{X}^{i+1} - \overline{\mathscr{X}}^i\|^2. \end{aligned} \tag{3.12}$$

By (3.11) and (3.12), we obtain

$$\|\mathscr{Y} - \overline{\mathscr{X}}^i\| \geq \|\mathscr{X}^{i+1} - \overline{\mathscr{X}}^i\|, \quad \forall \mathscr{Y} \in K \cap H_i.$$

Then, it results $\mathscr{X}^{i+1} = P_{K \cap H_i}(\overline{\mathscr{X}}^i)$. □

Now, we are able to establish the convergence result for the numerical procedure.

Theorem 3.1. *Let K be a nonempty, closed, and convex subset of $\mathscr{T}_{N,m}$, and let $F : K \to \mathscr{T}_{N,m}$ be a continuous tensor function. Assume that the solution set $S(F,K)$ is nonempty, and* (2.2) *holds. Then, any sequence $\{\mathscr{X}^i\}$ generated by the Algorithm converges to a solution to tensor variational inequality* (2.1).

Proof. The first step is to prove that the Algorithm is well defined. If $R(\mathscr{X}^i) = 0$, then the method stops at a solution to (2.1). Now, we suppose that $\|R(\mathscr{X}^i)\| > 0$. We remark that $\mathscr{X}^i \in K$, for every i. We assume that, for some i, (3.1) does not hold for any integer k, namely

$$\langle F(\mathscr{X}^i - \gamma^k R(\mathscr{X}^i)), R(\mathscr{X}^i) \rangle < \sigma \|R(\mathscr{X}^i)\|^2, \quad \forall k \in \mathbb{N}. \tag{3.13}$$

Making use of Lemma 3.2 with $K = K$, $\mathscr{X} = \mathscr{X}^i - F(\mathscr{X}^i)$ and $\mathscr{Z} = \mathscr{X}^i \in K$, we deduce

$$\begin{aligned} 0 &\geq \langle \mathscr{X}^i - F(\mathscr{X}^i) - P_K(\mathscr{X}^i - F(\mathscr{X}^i)), \mathscr{X}^i - P_K(\mathscr{X}^i - F(\mathscr{X}^i)) \rangle \\ &= \|R(\mathscr{X}^i)\|^2 - \langle F(\mathscr{X}^i), R(\mathscr{X}^i) \rangle. \end{aligned}$$

Then, it results

$$\langle F(\mathscr{X}^i), R(\mathscr{X}^i) \rangle \geq \|R(\mathscr{X}^i)\|^2. \tag{3.14}$$

Note that $\mathscr{X}^i - \gamma^k R(\mathscr{X}^i) \to \mathscr{X}^i$, as $k \to +\infty$. By using the continuity of F and passing to the limit as $k \to +\infty$ in (3.13), we obtain

$$\langle F(\mathscr{X}^i), R(\mathscr{X}^i) \rangle \leq \sigma \|R(\mathscr{X}^i)\|^2,$$

which contradicts (3.14). Then (3.1) is satisfied for some k_i and the algorithm is well defined. By virtue of Lemma 3.1 with $K = K \cap H_i$, $\mathscr{X} = \overline{\mathscr{X}}^i$, and $\mathscr{Z} = \mathscr{X}^* \in K \cap H_i$, we have

$$\begin{aligned} 0 &\geq \langle \overline{\mathscr{X}}^i - \mathscr{X}^{i+1}, \mathscr{X}^* - \mathscr{X}^{i+1} \rangle \\ &= \|\mathscr{X}^{i+1} - \overline{\mathscr{X}}^i\|^2 + \langle \overline{\mathscr{X}} - \mathscr{X}^{i+1}, \mathscr{X}^* - \overline{\mathscr{X}}^i \rangle. \end{aligned}$$

It follows that

$$\langle \mathscr{X}^* - \overline{\mathscr{X}}^i, \mathscr{X}^{i+1} - \overline{\mathscr{X}}^i \rangle \geq \|\mathscr{X}^{i+1} - \overline{\mathscr{X}}^i\|^2.$$

As a consequence, we have

$$\begin{aligned} & \|\mathscr{X}^{i+1} - \mathscr{X}^*\|^2 \\ = \ & \|\overline{\mathscr{X}}^i - \mathscr{X}^*\|^2 + \|\mathscr{X}^{i+1} - \overline{\mathscr{X}}^i\|^2 + 2\langle \overline{\mathscr{X}}^i - \mathscr{X}^*, \mathscr{X}^{i+1} - \overline{\mathscr{X}}^i \rangle \\ \leq \ & \|\overline{\mathscr{X}}^i - \mathscr{X}^*\|^2 - \|\mathscr{X}^{i+1} - \overline{\mathscr{X}}^i\|^2 \\ = \ & \|\mathscr{X}^i - \mathscr{X}^*\|^2 - \|\mathscr{X}^{i+1} - \overline{\mathscr{X}}^i\|^2 + \left(\frac{\eta_i \langle F(\mathscr{Z}^i), R(\mathscr{X}^i) \rangle}{\|F(\mathscr{Z}^i)\|} \right)^2 \\ & - \frac{2\eta_i \langle F(\mathscr{Z}^i), R(\mathscr{X}^i) \rangle}{\|F(\mathscr{Z}^i)\|^2} \langle F(\mathscr{Z}^i), \mathscr{X}^i - \mathscr{X}^* \rangle \\ = \ & \|\mathscr{X}^i - \mathscr{X}^*\|^2 - \|\mathscr{X}^{i+1} - \overline{\mathscr{X}}^i\|^2 - \left(\frac{\eta_i \langle F(\mathscr{Z}^i), R(\mathscr{X}^i) \rangle}{\|F(\mathscr{Z}^i)\|} \right)^2 \\ & - \frac{2\eta_i \langle F(\mathscr{Z}^i), R(\mathscr{X}^i) \rangle}{\|F(\mathscr{Z}^i)\|^2} \langle F(\mathscr{Z}^i), \mathscr{Z}^i - \mathscr{X}^* \rangle \\ \leq \ & \|\mathscr{X}^i - \mathscr{X}^*\|^2 - \|\mathscr{X}^{i+1} - \overline{\mathscr{X}}^i\|^2 - \left(\frac{\sigma \eta_i}{\|F(\mathscr{Z}^i)\|} \right)^2 \|R(\mathscr{X}^i)\|^4, \end{aligned} \tag{3.15}$$

where we have taken into account (3.1) and (2.2). By (3.15), we deduce that $\{\|\mathscr{X}^* - \mathscr{X}^i\|\}$ is nondecreasing and, then, it is convergent. As a consequence, $\{\mathscr{X}^i\}$ is bounded, and hence $\{\mathscr{Z}^i\}$ is also bounded which means that there exists $M > 0$ such that $\|F(\mathscr{Z}^i)\| \leq M$, for every i. Again from (3.15), we deduce

$$\|\mathscr{X}^{i+1} - \mathscr{X}^*\|^2 \leq \|\mathscr{X}^i - \mathscr{X}^*\|^2 - \|\mathscr{X}^{i+1} - \overline{\mathscr{X}}^i\|^2 - \left(\frac{\sigma \eta_i}{M} \right)^2 \|R(\mathscr{X}^i)\|^4. \tag{3.16}$$

Since $\{\|\mathscr{X}^* - \mathscr{X}^i\|\}$ converges, we have

$$\lim_{i \to +\infty} \eta_i \|R(\mathscr{X}^i)\| = 0. \tag{3.17}$$

If $\limsup_{i \to +\infty} \eta_i > 0$, we obtain from (3.17) that $\lim_{i \to +\infty} \|R(\mathscr{X}^i)\| = 0$. Since R is continuous and $\{\mathscr{X}^i\}$ is bounded, there exists an accumulation point $\hat{\mathscr{X}}$ for $\{\mathscr{X}^i\}$ such that $R(\hat{\mathscr{X}}) = 0$. So it results $\hat{\mathscr{X}} \in S(F, K)$. Choosing $\mathscr{X}^* = \hat{\mathscr{X}}$ in (3.16), we have that $\{\|\mathscr{X}^i - \hat{\mathscr{X}}\|\}$ converges to zero (recall that $\hat{\mathscr{X}}$ is an accumulation point). Then $\{\mathscr{X}^i\}$ converges to $\hat{\mathscr{X}} \in S(F, K)$. If $\lim_{i \to \infty} \eta_i = 0$, by the choice of η_i, we know that (3.15) was not satisfied for $k_i - 1$ (for i large enough), namely

$$\langle F(\mathscr{X}^i - \gamma^{-1} \eta_i R(\mathscr{X}^i)), R(\mathscr{X}^i) \rangle < \sigma \|R(\mathscr{X}^i)\|^2, \quad \forall i \geq i_0. \tag{3.18}$$

Let us denote by $\hat{\mathscr{X}}$ an accumulation point of $\{\mathscr{X}^i\}$ and by $\{\mathscr{X}^{i_j}\}$ the corresponding subsequence converging to $\hat{\mathscr{X}}$. Passing to the limit as $j \to +\infty$ and recalling (3.14), we obtain

$$\sigma \|R(\hat{\mathscr{X}})\|^2 \geq \langle F(\hat{\mathscr{X}}), R(\hat{\mathscr{X}})\rangle \geq \|R(\hat{\mathscr{X}})\|^2.$$

The latter implies that $R(\hat{\mathscr{X}}) = 0$, namely $\hat{\mathscr{X}} \in S(F,K)$. Proceeding as in the precedent case, we obtain that $\{\mathscr{X}^i\}$ converges to $\hat{\mathscr{X}} \in S(F,K)$. Therefore the claim is achieved. □

4. An Oligopolistic Market Model

We want to consider an economic model in which the equilibrium condition is expressed by a tensor variational inequality. More precisely, we present a general market equilibrium model where demand excesses occur, as a generalization of the one introduced in [15]. The economic network is made up of m firms P_i, $i = 1, \ldots, m$, and n demand markets Q_j, $j = 1, \ldots, n$, that are generally spatially separated. Each firm P_i produces l different commodities and sells them to market Q_j, $j = 1, \ldots, n$. We fix the following notations:

- x_{ij}^k is the nonnegative variable expressing the commodity shipment of kind k between the producer P_i and the market Q_j, $i = 1, \ldots, m$, $j = 1, \ldots, n$, $k = 1, \ldots, l$;
- δ_j^k is the nonnegative variable expressing the demand excess for the commodity of kind k of the demand market Q_j, $j = 1, \ldots, n$, $k = 1, \ldots, l$;
- p_i^k is the variable expressing the commodity output of kind k produced by P_i, such that $p_i^k = \sum_{j=1}^n x_{ij}^k$, $i = 1, \ldots, m$, $k = 1, \ldots, l$;
- q_j^k is the variable expressing the demand for the commodity of kind k of demand market Q_j, $j = 1, \ldots, n$, $k = 1, \ldots, l$.

Moreover we make the following assumptions:

- x_{ij}^k, $i = 1, \ldots, m$, $j = 1, \ldots, n$, $k = 1, \ldots, l$, and δ_j^k, $j = 1, \ldots, n$, $k = 1, \ldots, l$, are nonnegative and the following capacity constraints hold:

$$\underline{x}_{ij}^k \leq x_{ij}^k \leq \overline{x}_{ij}^k, \quad \forall i = 1, \ldots, m,\ \forall j = 1, \ldots, n,\ \forall k = 1, \ldots, l,$$

  where $\underline{x}_{ij}^k$, $\overline{x}_{ij}^k$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, $k = 1, \ldots, l$, are nonnegative bounds. We group $\mathscr{X} = (x_{ij}^k)$, $\underline{\mathscr{X}} = (\underline{x}_{ij}^k)$ and $\overline{\mathscr{X}} = (\overline{x}_{ij}^k)$, which belong to $\mathbb{R}^{[mnl]}$ and $\Delta = (\delta_j^k) \in \mathbb{R}^{nl}$;
- q_j^k, $j = 1, \ldots, n$, $k = 1, \ldots, l$, is nonnegative and the following feasibility condition holds:

$$q_j^k = \sum_{i=1}^m x_{ij}^k + \delta_j^k, \quad \forall j = 1, \ldots, n,\ \forall k = 1, \ldots, l, \tag{4.1}$$

  namely the quantity demanded by each demand market Q_j of kind k must be equal to the commodity shipments of such kind from all the firms to that demand market plus the demand excess for such kind of commodity.

As a consequence, the feasible set is

$$\widetilde{\mathbb{K}} = \Bigg\{ (\mathscr{X},\Delta) \in \mathbb{R}^{[mnl]} \times \mathbb{R}^{nl} : \\ \underline{x}_{ij}^k \le x_{ij}^k \le \overline{x}_{ij}^k, \quad \forall i=1,\dots,m,\ \forall j=1,\dots,n,\ \forall k=1,\dots,l, \\ \delta_j^k \ge 0, \quad \forall j=1,\dots,n,\ \forall k=1,\dots,l, \\ q_j^k = \sum_{i=1}^m x_{ij}^k + \delta_j^k, \quad \forall i=1,\dots,m,\ \forall k=1,\dots,l \Bigg\}, \tag{4.2}$$

which is a nonempty convex compact subset of $\mathbb{R}^{[mnl]}$.

We define:

- f_i^k representing the variable expressing the production cost of P_i for each good of type k, $i=1,\dots,m$, $k=1,\dots,l$. We assume that the production cost of a firm P_i may depend upon the entire production pattern, namely, $f_i^k = f_i^k(\mathscr{X})$, $\mathscr{X} \in \mathbb{R}^{[mnl]}$;
- $\widetilde{d}_j^k$ representing the variable expressing the demand price for unity of kind k of the commodity with each demand market Q_j, $j=1,\dots,n$, $k=1,\dots,l$. We assume that the demand price of a demand market Q_j may depend upon the entire consumption pattern, namely, $\widetilde{d}_j^k = \widetilde{d}_j^k(\mathscr{X},\Delta)$, $\mathscr{X} \in \mathbb{R}^{[mnl]}$, $\Delta \in \mathbb{R}^{nl}$;
- $\widetilde{c}_{ij}^k$ representing the variable expressing the transaction cost between firm P_i and demand market Q_j regarding the good of kind k, $i=1,\dots,m$, $j=1,\dots,n,$, $k=1,\dots,l$. We assume also that the transaction cost depends upon the entire shipment pattern, namely, $\widetilde{c}_{ij}^k = \widetilde{c}_{ij}^k(\mathscr{X},\Delta)$, $\mathscr{X} \in \mathbb{R}^{[mnl]}$, $\Delta \in \mathbb{R}^{nl}$.

Then the profit of firm P_i, $i=1,\dots,m$, is

$$\widetilde{v}_i(\mathscr{X},\Delta) = \sum_{k=1}^l \left[\sum_{j=1}^n \widetilde{d}_j^k(\mathscr{X},\Delta) x_{ij}^k - f_i^k(\mathscr{X}) - \sum_{j=1}^n \widetilde{c}_{ij}^k(\mathscr{X},\Delta) x_{ij}^k \right],$$

namely the difference between the price that each demand market P_i is disposed to pay and the sum of the production costs and the transportation costs.

We remark that, by (4.1), we can express the demand excesses in the following way:

$$\delta_j^k = q_j^k - \sum_{i=1}^m x_{ij}^k, \quad \forall j=1,\dots,n,\ \forall k=1,\dots,l \tag{4.3}$$

and hence

$$\sum_{k=1}^l \delta_j^k = q_j - \sum_{k=1}^l \sum_{i=1}^m x_{ij}^k, \quad \forall i=1,\dots,m.$$

Making use of the nonnegativity of the production excesses, we can write an equivalent formulation for the feasible set:

$$\mathbb{K} = \Bigg\{ \mathscr{X} \in \mathbb{R}^{[mnl]} : \quad \underline{x}_{ij}^k \le x_{ij}^k \le \overline{x}_{ij}^k, \quad \forall i=1,\dots,m,\ \forall j=1,\dots,n,\ \forall k=1,\dots,l, \\ \sum_{i=1}^m x_{ij}^k \le q_j^k, \quad \forall j=1,\dots,n, \forall k=1,\dots,l \Bigg\}. \tag{4.4}$$

We stress that $\mathbb{K}$ includes the presence of demand excesses as in $\widetilde{\mathbb{K}}$.

Hence, we can replace the demand prices and the transaction costs as:

$$d_j^k(\mathscr{X}) = \widetilde{d}_j^k(\mathscr{X}, \Delta), \quad \forall j = 1, \dots, n, \ \forall k = 1, \dots, m,$$

$$c_{ij}^k(\mathscr{X}) = \widetilde{c}_{ij}^k(\mathscr{X}, \Delta), \quad \forall i = 1, \dots, m, \ \forall j = 1, \dots, n, \ \forall k = 1, \dots, m.$$

With the same spirit, we rewrite the profit function of firm P_i as

$$v_i(\mathscr{X}) = \widetilde{v}_i(\mathscr{X}, \Delta) = \sum_{k=1}^{l} \left[\sum_{j=1}^{n} d_j^k(\mathscr{X}) x_{ij}^k - f_i^k(\mathscr{X}) - \sum_{j=1}^{n} c_{ij}^k(\mathscr{X}) x_{ij}^k \right],$$

The aim of each firm is to maximize its own profit function considering the optimal distribution pattern of the others, namely it follows a noncooperative behavior. As a consequence, the equilibrium distribution has to satisfy a generalization of the Cournot-Nash equilibrium principle.

Definition 4.1. A tensor distribution $\mathscr{X}^* \in \mathbb{K}$ is a general oligopolistic market equilibrium in presence of demand excesses if and only if

$$v_i(\mathscr{X}^*) \geq v_i(X_i, \mathscr{X}_{-i}^*), \quad \forall i = 1, \dots, m, \tag{4.5}$$

where $\mathscr{X}_{-i}^* = (X_1^*, \dots, X_{i-1}^*, X_{i+1}^*, \dots, X_m^*)$ and $X_i = (x_{ij}^k) \in \mathbb{R}^{nl}$.

In the sequel, we suppose:

(i) $v_i(\mathscr{X})$ is continuously differentiable, for each $i = 1, \dots, m$;

(ii) $v_i(\mathscr{X})$ is pseudoconcave with respect to $X_i \in \mathbb{R}^{nl}$, for each $i = 1, \dots, m$, namely (see [1])

$$\left\langle \frac{\partial v_i}{\partial X_i}(X_1, \dots, X_i, \dots, X_m), X_i - Y_i \right\rangle \geq 0 \Rightarrow v_i(X_1, \dots, X_i, \dots, X_m) \geq v_i(X_1, \dots, Y_i, \dots, X_m).$$

We set

$$\nabla_D v = \left(\frac{\partial v_i}{\partial x_{ij}^k} \right), \quad i = 1, \dots, m, \ j = 1, \dots, n, k = 1, \dots, l,$$

the tensor of partial derivative of v_i with respect to x_{ij}^k, $i = 1, \dots, m, \ j = 1, \dots, n, \ k = 1, \dots, l$.

We now are able to prove the following result.

Theorem 4.1. *Under assumptions (i) and (ii), $x^* \in \mathbb{K}$ is a general oligopolistic market equilibrium distribution in presence of demand excesses if and only if it satisfies*

$$\langle -\nabla_D v(\mathscr{X}^*), \mathscr{X} - \mathscr{X}^* \rangle = -\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{l} \frac{\partial v_i(\mathscr{X}^*)}{\partial x_{ij}^k} (x_{ij}^k - (x_{ij}^k)^*) \geq 0, \quad \forall x \in \mathbb{K}. \tag{4.6}$$

Proof. We start showing that if $\mathscr{X}^*$ is a solution to (4.6), then it is a general oligopolistic market equilibrium distribution in presence of demand excesses. By contradiction, we assume that there exists i^* such that $v_{i^*}(\mathscr{X}^*) < v_{i^*}(X_{i^*}, \mathscr{X}_{-i^*}^*)$. For the pseudoconcavity assumption of $v(\mathscr{X})$, we have $\langle -\nabla_D v(\mathscr{X}^*), \mathscr{X} - \mathscr{X}^* \rangle < 0$. The vice versa clearly follows. □

Taking into account the theoretical results for tensor variational inequalities shown in the previous sections and reminding that the feasible set $\mathbb{K}$ is nonempty convex compact, we obtain the following existence and uniqueness result for the economic model.

Theorem 4.2. *Under assumptions (i) and (ii), if $-\nabla_D v$ is strictly monotone, then there exists a unique general oligopolistic market equilibrium distribution in presence of demand excesses.*

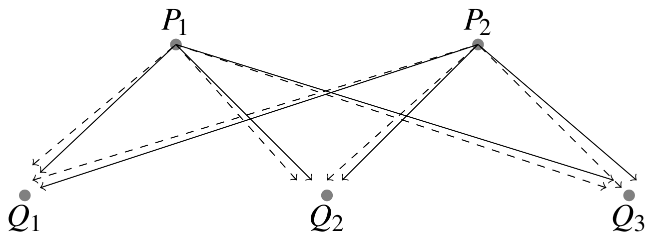

FIGURE 1. Oligopolistic market network

5. NUMERICAL EXAMPLE

We consider a simple oligopolistic market network made up of two firms P_1 and P_2 and three markets Q_1, Q_2 and Q_3. Every firm P_i, $i=1,2$, produces two different kinds of goods. Figure 1 explains the economic network, in particular dashed and continuous lines represent the path of the two kinds of goods. We denote by x_{ij}^k the k-th commodity shipment from P_i to Q_j, $i=1,2$, $j=1,2,3$, $k=1,2$. We also assume that the capacity constraints $0 \leq x_{ij}^k \leq 100$ hold for every $i=1,2$, $j=1,2,3$, $k=1,2$. Since we are studying a general oligopolistic market equilibrium example in presence of demand excesses, we introduce the commodity demand q:

$$q = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 2 & 2 \end{pmatrix},$$

Hence, we have

$$\begin{aligned} \mathbb{K} \;=\; & \Bigg\{ x \in \mathbb{R}^{[12]} : 0 \leq x_{ij}^k \leq 100, \quad \forall i=1,2,\ \forall j=1,2,3,\ \forall k=1,2 \\ & \sum_{i=1}^{2} x_{ij}^k \leq q_j^k, \quad \forall j=1,2,3,\ \forall k=1,2 \Bigg\}. \end{aligned} \tag{5.1}$$

The profit functions v_i, $i=1,2$, are given by:

$$\begin{aligned} v_1 = & -4(x_{11}^1)^2 - 4(x_{12}^1)^2 - 6(x_{13}^1)^2 - 6(x_{11}^2)^2 - 5(x_{21}^1)^2 - 2(x_{22}^1)^2 \\ & - 4x_{11}^1 x_{12}^1 - 6x_{13}^1 x_{11}^2 - 2x_{21}^1 x_{22}^1 + 3x_{11}^1 + 4x_{12}^1 + x_{13}^1 + x_{11}^2 + 2x_{21}^1 + 2x_{22}^1, \end{aligned}$$

and

$$\begin{aligned} v_2 = & -10(x_{22}^2)^2 - 4(x_{23}^2)^2 - 4(x_{13}^2)^2 - 5(x_{12}^2)^2 - 2(x_{23}^1)^2 - 2(x_{21}^2)^2 \\ & - 2x_{11}^1 x_{23}^2 - 2x_{13}^2 x_{12}^2 - 2x_{23}^1 x_{21}^2 + x_{22}^2 + 2x_{23}^2 + 10x_{13}^2 + 3x_{12}^2 + 3x_{23}^1 + 2x_{21}^2. \end{aligned}$$

Hence the nonzero components of $\nabla_D v$ are

$$\frac{\partial v_1}{x_{11}^1} = -8x_{11}^1 - 4x_{12}^1 + 3, \qquad \frac{\partial v_2}{x_{23}^1} = -4x_{23}^1 - 2x_{21}^2 + 3,$$
$$\frac{\partial v_1}{x_{12}^1} = -4x_{11}^1 - 8x_{12}^1 + 4, \qquad \frac{\partial v_2}{x_{21}^2} = -2x_{23}^1 - 4x_{21}^2 + 2,$$
$$\frac{\partial v_1}{x_{13}^1} = -12x_{13}^1 - 6x_{11}^2 + 1, \qquad \frac{\partial v_2}{x_{22}^2} = -20x_{22}^2 - 2x_{12}^1 + 1,$$
$$\frac{\partial v_1}{x_{11}^2} = -6x_{13}^1 - 12x_{11}^2 + 1, \qquad \frac{\partial v_2}{x_{23}^2} = -8x_{23}^2 - 2x_{11}^1 + 2.$$

By Theorem 4.1, the general oligopolistic market equation distribution is a solution to

$$\langle -\nabla_D v(x^*), x - x^* \rangle = -\sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{k=1}^{2} \frac{\partial v_i(x^*)}{\partial x_{ij}^k}(x_{ij}^k - (x_{ij}^k)^*) \geq 0, \quad \forall x \in \mathbb{K}. \tag{5.2}$$

By using the Algorithm, we obtain the following numerical distribution:

$$(x^1)^* = \begin{pmatrix} 0.16667 & 0.41667 & 0.05555 \\ 0.11111 & 0.44444 & 0.66667 \end{pmatrix},$$
$$(x^2)^* = \begin{pmatrix} 0.05555 & 0.05263 & 1.23684 \\ 0.16667 & 0.00833 & 0.20833 \end{pmatrix}.$$

The algorithm was coded making use of Matlab and was run on a PC with 32 GB RAM, Lenovo ThinkPad E570 Intel Core i7-7500U and the stopping criterion used in the numerical computation is $\|\mathscr{X}^{i+1} - \mathscr{X}^i\| < 10^{-5}$.

6. Conclusions

In this paper, we continued the study of tensor variational inequalities. In particular, we established the equivalence between the tensor complementarity problem and a special tensor variational inequality. Moreover, we introduced a projection type method to solve tensor variational inequalities. We proved a convergence result which guarantees that the sequence generated by the algorithm converges to the exact solution. Then, we presented the general oligopolistic market equilibrium problem in which each firm produces several commodities and demand excesses occur. The equilibrium distribution expressed by a generalization of the Cournot-Nash equilibrium principle was characterized by a tensor variational inequality. Thanks to this variational formulation, we obtained an existence result for the model. At last, a numerical example was provided.

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