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ATTENUATION AND LOCALIZATION OF WAVES IN TAUT CABLES WITH A DISCRETE ARRAY OF SCATTER ELEMENTS

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Abstract. This work analyzes structural waves that propagate freely along taut cables, characterized by a discrete array of scatter elements. The outcomes underline the role played by the periodic distribution of such elements, whose presence alters the response of the system when subjected to propagating waves. Namely, when the domain is perfectly periodic, band gaps are found in the spectrum of the problem. It is also shown that the introduction of a defect of periodicity can lead to the appearance of eigenvalues inside band gaps, corresponding to a motion localized around the defect.

1 INTRODUCTION

In the recent years, it has been shown that the mechanical behavior of periodic domains assembled by repeating a specifically designed unit cell can offer peculiar dynamic properties, among which the presence of band gaps in the spectrum of the problem, i.e. intervals of frequencies corresponding to attenuated waves. This behavior was initially studied in solid mechanics by reinterpreting some concepts typical of solid state physics. When the periodicity is of the same order of magnitude of the wavelength of interest, a Bragg scattering phenomenon can lead to the formation of band gaps [1, 2]. To activate band gaps in the subwavelength regime, unit cells with local resonances have been proposed [3, 4].

This ability to forbid the propagation of waves is generally employed for the design of efficient wave shields [5, 6]. Systems of this type are also studied for a variety of new phenomena, such as lensing [7, 8] and cloaking [9, 10]. In order to emphasize their peculiar dynamic properties, materials whose behavior is governed by the periodic microstructure are generally collected under the name of metamaterials (or metastructures).

When a defect of periodicity is introduced, the system response varies, as a displacement field well-localized around such defect can take place for some specific frequencies belonging to a band gap of the underlying periodic structure [11, 12, 13]. Mathematically, this can be explained by looking at the spectrum of the problem. When dealing with perfectly periodic media, the spectrum would only be composed of its essential part. By introducing a compact defect, the essential spectrum is not altered and thus its band gap structure remains unchanged [14]. This also means that, if a particular band gap is considered, the spectrum for the defective system in that band gap can consist at most of the discrete part. The localized displacement field is thus an eigenmode of the problem, generally known as "defect mode". As this eigenmode is an harmonic vibration at a frequency belonging to a band gap, the corresponding displacement field must decay exponentially away from the defect. This justifies the localized response.

In this work, we aim to study the attenuating and localizing phenomena in a taut cable with scatter elements distributed along it. The structural model here investigated is made up of a cable, with a discrete and periodic set of scatter elements, subjected to an external tension which maintains an almost horizontal equilibrium configuration. The system can be of interest as a starting point towards a reinterpretation in terms of metamaterials of the problem of wave propagation in cables typically used in civil engineering applications, such as: suspension bridges, cableways and overhead lines with stockbridge dampers and marker balls. Here we treat the problem by deriving an equivalent mass for the continuous system, that determines the form of the solutions.

The rest of the paper is organized as follows. In section 2, the model under consideration is presented and the associated motion problem is described. In section 3, we highlight the attenuating properties by studying the behavior of the frequency dependent equivalent mass. In section 4, we study how the dynamics of the perfectly periodic system changes as a defect is introduced, leading to a localization phenomenon. Some conclusions are then given in the final section.

2 DESCRIPTION OF THE PROBLEM

Let us consider a 1D perfectly flexible cable of mass per unit of length ρ, in tension between two fixed ends positioned at distance L under the applied force H. Let us assume that the cable is made up of a linear elastic material of constant Young's modulus E. By neglecting the effect of gravity, the static equilibrium configuration can be approximated to be straight, i.e. with zero curvature. This assumption is valid either when the cable is parallel to the gravity field or when the applied tension H is much larger than the total weight of cable. For the sake of simplicity, we fix the equilibrium configuration to be along the horizontal direction. A particle at point s_0 of the undeformed (natural) configuration is moved to point s of the statically deformed configuration. Specifically, s is related to s_0 by $s = s_0 + \int_0^{s_0} \varepsilon_{\text{eq}}(s_0)\, ds_0$, where ε_{eq} denotes the static axial strain. Calling $N_{\text{eq}}(s_0)$ the axial force along the cable, from equilibrium one has

$$N'_{\text{eq}}(s_0) = 0 \quad \longrightarrow \quad N_{\text{eq}}(s_0) = H \quad \text{and} \quad \varepsilon_{\text{eq}}(s_0) = H/EA \quad \forall s_0,$$

where EA is the axial stiffness of the cable with undeformed area A and where we use $(\bullet)'$ for a spatial derivative.

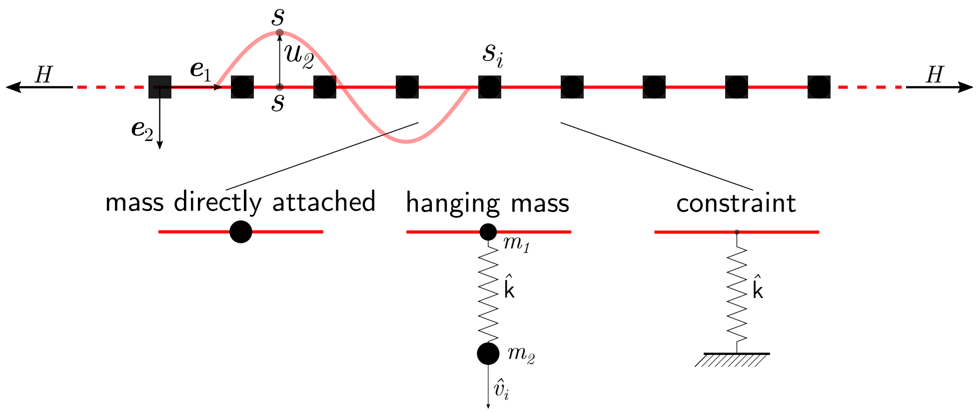

Figure 1: Cable in tension under an axial force H between two fixed ends, with a periodic distribution of generic scatter elements. Direction $\boldsymbol{e}_3$ is out-of-plane.

2.1 Addition of the scatter elements

Let us now add a discrete set of n scatter elements, periodically attached along the pretensioned cable at distance d. In particular, these elements can be: masses directly attached to the cable, masses hanging by means of elastic springs and spring constraints (c.f. figure 1). Let us call $\mathcal{P}$ the set containing the positions $s = s_i$ of the i-th scatter element. We aim to study elastic waves propagating along the pretensioned taut cable with a discrete set of generic scatter elements (c.f. figure 1). Specifically, we are here considering the propagation of waves that have not reached the boundaries yet. By calling $\boldsymbol{x}(s_0, t)$ the position at time t of a particle originally at position s_0, we have

$$\boldsymbol{x}(s_0, t) = (s(s_0) + u_1(s_0, t))\boldsymbol{e}_1 + u_2(s_0, t)\boldsymbol{e}_2 + u_3(s_0, t)\boldsymbol{e}_3,$$

where $\boldsymbol{e}_j$, with $j = \{1, 2, 3\}$, denote the unit vectors along the system of reference depicted in figure 1 (direction $\boldsymbol{e}_3$ is out-of-plane) and u_j are the displacement components.

Updating the static equilibrium configuration to be the new reference configuration and using $(\dot{\bullet})$ for a time derivative, the equations of motion at a point s where a scatter element is not present can be written as:

$$\begin{cases} \rho\ddot{u}_1(s,t) = \alpha \left[\dfrac{EA\varepsilon(s,t)}{1+\varepsilon(s,t)}\,(1+u_1'(s,t))\right]' \\ \rho\ddot{u}_2(s,t) = \alpha \left[\dfrac{EA\varepsilon(s,t)}{1+\varepsilon(s,t)} u_2'(s,t)\right]' \\ \rho\ddot{u}_3(s,t) = \alpha \left[\dfrac{EA\varepsilon(s,t)}{1+\varepsilon(s,t)} u_3'(s,t)\right]' \end{cases} \tag{1}$$

where $\alpha := 1 + \varepsilon_{\text{eq}}$ and $\varepsilon(s, t)$ is the dynamic axial strain given by

$$\varepsilon(s,t) = \|\boldsymbol{x}'(s,t)\| - 1 = \alpha\sqrt{(1+u_1'(s,t))^2 + u_2'(s,t)^{\,2} + u_3'(s,t)^{\,2}} - 1.$$

By assuming u_j, with $j = \{1, 2, 3\}$, (and their derivatives) as infinitesimal at least of order one and by neglecting non-linear terms in relations (1), one obtains the classical (uncoupled) equations governing the linear dynamics of taut cables, such that

$$\begin{cases} \ddot{u}_1(s,t) - c_l^2 u_1''(s,t) = 0 \\ \ddot{u}_2(s,t) - c_t^2 u_2''(s,t) = 0 \\ \ddot{u}_3(s,t) - c_t^2 u_3''(s,t) = 0 \end{cases} \qquad \forall s \setminus s \in \mathcal{P}, \tag{2}$$

with

$$c_l^2 = \frac{EA\alpha^2}{\rho} \quad \text{and} \quad c_t^2 = \frac{N_{\text{eq}}\alpha}{\rho}$$

being respectively the speeds of longitudinal and transverse waves.

We are interested in the propagation of transverse harmonic waves with polarization along direction $\boldsymbol{e}_2$. Accordingly, one can write

$$u_2(s,t) = L\hat{u}_2(s) \exp\{\mathrm{i}\omega t\}, \tag{3}$$

where i denotes the imaginary unit.

The second of equations (2) can thus be rewritten in dimensionless form as follows

$$\hat{u}_2''(\hat{s}) + (n+1)^2\Omega^2\hat{u}_2(\hat{s}) = 0, \quad \forall \hat{s} \in \left(\frac{i-1}{n+1}, \frac{i}{n+1}\right), \ 1 \le i \le n+1 \tag{4}$$

where $\hat{s} = s/L$ and $\Omega = \omega d/c_t$ are respectively dimensionless coordinate and frequency.

2.2 The equivalent governing equation

The above equation (4) can be integrated within each interval i, obtaining

$$\hat{u}_2(\hat{s}) = \hat{u}_{2(i-1)} \cos\left(\Omega(n+1)\hat{s} - i + 1\right) + \frac{\hat{u}_{2(i)} - \hat{u}_{2(i-1)}\cos\Omega}{\sin\Omega} \sin\Omega((n+1)\hat{s} - i + 1), \tag{5}$$

where $\hat{u}_{2(i)}$, with $1 < i < n$, denote the non-dimensional amplitude of the vertical displacement defined as in (3) of the cable points s_i where the i-th scatter element is present. Therefore, solutions to the original problem can be found by imposing the jump conditions at each of these positions s_i. These conditions read

$$[\![EA\varepsilon_{\text{eq}}u_2']\!](s_i,t) = F(s_i,t) \tag{6}$$

where $[\![(\bullet)]\!] = (\bullet)^+ - (\bullet)^-$, with $(\bullet)^+$ (resp. $(\bullet)^-$) denoting the right (resp. left) limit of $(\bullet)$ at s, and F is given by the following relations for the three cases considered in figure 1

$$F(s_i,t) = \begin{cases} m_1\ddot{u}_2(s_i,t) & \text{masses directly attached} \\ m_1\ddot{u}_2(s_i,t) - k(v_i(t) - u_2(s_i,t)) & \text{masses directly attached and hanging} \\ ku_2(s_i,t) & \text{elastic constraint} \end{cases}$$

with m_1 being the mass scatter element directly attached to the cable and k the stiffness of the elastic springs. The quantity $v_i(t)$ in the second expression denotes the vertical displacement of the i-th hanging mass m_2 and can be derived from the following relation

$$m_2 \ddot{v}_i(t) = -k(v_i(t) - u_2(s_i, t)). \tag{7}$$

Note that the first and third relations for $F(s_i, t)$ can be retrieved from the second one by considering, respectively, an infinite stiffness (in this case the mass m_1 would be substituted by $m_1 + m_2$ in the first relation) and an infinite hanging mass m_2, with mass m_1 equal to zero.

Let us assume small harmonic time variations for the displacement of masses m_2, such that $v_i(t) = L\hat{v}_i \exp\{\mathrm{i}\omega t\}$. Using this in equation (7), from relations (5) and (6) one finds an equivalent equation governing the motion of points $s_i \in \mathcal{P}$, such that

$$\Delta_i \hat{u}_2 + \mu(\Omega)\hat{u}_{2(i)} = 0, \quad \text{with} \quad \Delta_i \hat{u}_2 = \hat{u}_{2(i+1)} + \hat{u}_{2(i-1)} - 2\hat{u}_{2(i)} \quad \text{for} \quad 1 \le i \le n. \tag{8}$$

In the above equation (8), $\mu(\Omega)$ can be interpreted as a frequency-dependent equivalent mass for the cable and it is defined as follows for the three configurations

$$\mu(\Omega) = \begin{cases} 2(1-\cos(\Omega)) + \Theta_1 \Omega \sin(\Omega) & \text{masses directly attached} \\ 2(1-\cos(\Omega)) + \left(\Theta_1 + \dfrac{\hat{k}\Theta_2}{\hat{k} - \Theta_2 \Omega^2}\right)\Omega \sin(\Omega) & \text{masses directly attached and hanging} \\ 2(1-\cos(\Omega)) - \hat{k}\dfrac{\sin(\Omega)}{\Omega} & \text{elastic constraint} \end{cases} \tag{9}$$

with

$$\Theta_1 = \frac{m_1}{\rho d}, \quad \Theta_2 = \frac{m_2}{\rho d}, \quad \hat{k} = \frac{kd}{N_{\mathrm{eq}}}.$$

By solving equation (8), using relation (5), the solution of the wave propagation problem can finally be found.

3 WAVE ATTENUATION PHENOMENON

Let us here study problem (8). The solutions depend on the behavior of the frequency varying equivalent mass $\mu(\Omega)$, defined by relations (9) and represented in figure 2 for a specific set of parameters.

One has that:

- For $0 \le \mu(\Omega) \le 4$, the general solution reads:

$$\hat{u}_{2(i)} = a_1 \exp\{-\mathrm{i}\mathsf{K}^* i\} + a_2 \exp\{\mathrm{i}\mathsf{K}^* i\} \quad \text{with} \quad \mathsf{K}^* \in [0, \pi] \tag{10}$$

  and corresponds to a superposition of a right- and left-propagating waves;

- For $\mu(\Omega) < 0$, the general solution reads:

$$\hat{u}_{2(i)} = a_1 \exp\{-\mathsf{K}^* i\} + a_2 \exp\{\mathsf{K}^* i\}. \tag{11}$$

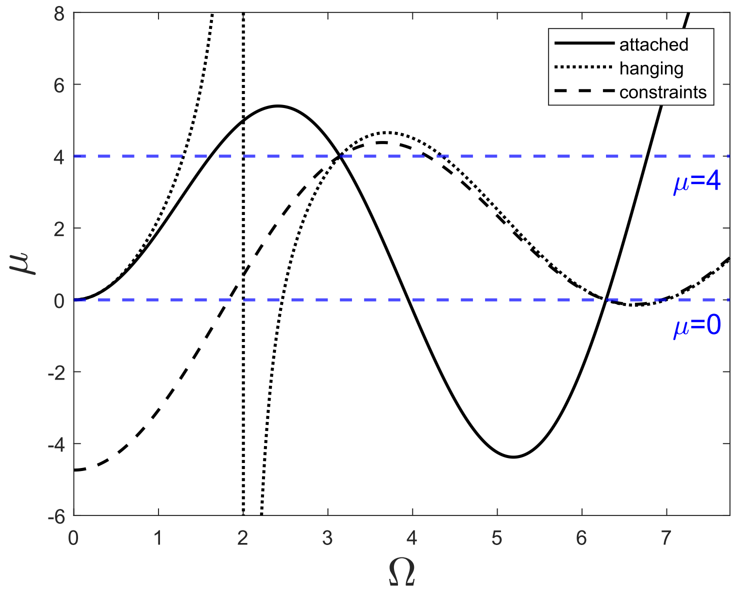

Figure 2: Equivalent mass $\mu(\Omega)$ vs dimensionless frequency Ω for the case of masses directly attached (continuous), hanging masses (dotted) and spring constraints (dashed). The following parameters have been used: $\Theta_1 = \Theta_2 = 1.1842$ and $\hat{k} = 4.7368$. For the case with hanging masses, Θ_1 was fixed to zero.

- For $\mu(\Omega) > 4$, the general solution reads:

$$\hat{u}_{2(i)} = a_1(-1)^i \exp\{-\mathsf{K}^* i\} + a_2(-1)^i \exp\{\mathsf{K}^* i\}. \tag{12}$$

  Note that, for the last two cases, the solutions correspond to a superposition of attenuated waves.

The term K^* represents a wave number normalized with respect to the distance d between two neighboring scatter elements and can be obtained from

$$1 - \frac{\mu(\Omega)}{2} = \begin{cases} \cos \mathsf{K}^* & \text{for } 0 \leq \mu(\Omega) \leq 4 \\ \cosh \mathsf{K}^* & \text{for } \mu(\Omega) < 0 \\ -\cosh \mathsf{K}^* & \text{for } \mu(\Omega) > 4 \end{cases} . \tag{13}$$

We thus have found that the problem can be characterized by the presence of intervals of frequencies corresponding to attenuated waves, i.e. by band gaps in its spectrum. Specifically, they can be visualized in figure 2 any time μ is either < 0 or > 4. Note that these intervals differ for the three typologies of scatter elements under consideration.

4 DEFECT OF PERIODICITY AND WAVE LOCALIZATION

As we stated in the introduction to this work, a defect of periodicity in the system can result in a displacement field well-localized around such defect. To show this, we will limit ourselves to the case when masses are directly attached to the cable. Specifically, we generate a defect by modifying the central mass of the system. We here again neglect the presence of boundary conditions, by taking them to be far away from the defect. We thus hypothetically consider a system of infinite dimensions. This assumption is valid for the case when the displacement is strongly localized around the defect, rapidly decaying away from it, so that it is approximately null at the two ends of the cable.

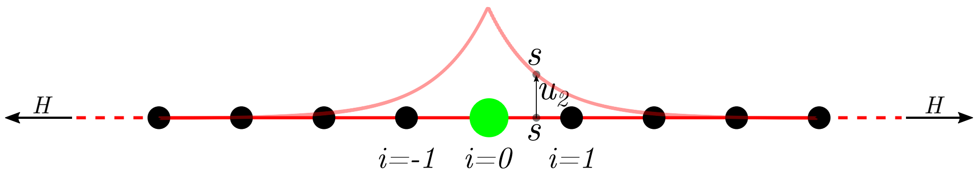

Figure 3: Periodic array of masses m_1 attached to the pre-tensioned cable with a central defect generated by a modified mass M.

Let us analyze the system depicted in figure 3, where we called M the modified central mass. For convenience, the index i can now take any number in $\mathbb{Z}$ and is 0 for the central mass.

Let us denote as δm_1 the perturbation of the mass M with respect to mass m_1, such that

$$M = m_1 + \delta m_1,$$

where $\delta m_1 \in \mathbb{R}$.

Jump conditions (6) can now be rewritten as

$$[\![EA\varepsilon_{\text{eq}} u_2']\!](s_i, t) = m\ddot{u}_2(s_i, t), \tag{14}$$

where

$$m = \begin{cases} m_1 & \forall i \in \mathbb{Z} \setminus 0 \\ M & i = 0 \end{cases}.$$

Using again relation (5) together with jump conditions (14), one obtains the following equivalent equation:

$$\Delta_i \hat{u}_2 + \mu(\Omega)\hat{u}_{2(i)} = -\delta\mu(\Omega)\hat{u}_{2(0)}\delta_{i0} \quad \text{for} \quad i \in \mathbb{Z}, \tag{15}$$

where δ_{ij} is the Kronecker's delta and

$$\delta\mu(\Omega) := \frac{\delta m_1}{\rho d}\Omega \sin\Omega$$

is an additional term coming from the perturbation δm_1 of the central mass. Note that (15) coincides with (8) for $i \neq 0$.

4.1 Characterization of the defective motion

To localize the response of the system around the central defect, the displacement field must decay moving away from it. This behavior can thus be activated only at those frequencies belonging to a band gap of the periodic system. In particular, from the results in section 3, we have this condition whenever the equivalent mass $\mu(\Omega)$ is either < 0 or > 4. Therefore, by choosing a frequency Ω belonging to a band gap, the motion of the scatter masses corresponding to a localized response can be obtained either from relation (11) (for $\mu < 0$) or from relation (12) (for $\mu > 4$), by making use of relation (13) to find K^*.

Specifically, by imposing the continuity of the displacement at $i = 0$, we obtain

$$\hat{u}_{2(i)} = \begin{cases} A \exp\{-\mathsf{K}^* \, |i|\} & \text{for } \mu(\Omega) < 0 \\ A(-1)^i \exp\{-\mathsf{K}^* \, |i|\} & \text{for } \mu(\Omega) > 4 \end{cases} \quad \forall i \in \mathbb{Z}, \tag{16}$$

where A is a constant of integration. Note that, in both cases, the response of the system is localized at the central mass and exponentially decays away from it.

Up to now, the mass M that would generate a localized motion (16) at the specific frequency Ω is still unknown. To fix it, we can make use of the following method. First, let us consider the equation

$$\Delta_i \hat{u}_2 + \mu(\Omega) \hat{u}_{2(i)} = -\delta_{i0} \quad \text{for} \quad i \in \mathbb{Z}, \tag{17}$$

corresponding to the case when the central mass M is not modified from m_1, but instead subjected to an harmonic unitary force at frequency Ω belonging to a band gap of the periodic system and pointing downwards. A solution $\hat{u}_{2(i)}$ to problem (17) can be obtained by using the general solutions (16). Specifically, one finds:

$$\hat{u}_{2(i)} = \begin{cases} \dfrac{-1}{-2\exp\{-\mathsf{K}^*\} - 2 + \mu(\Omega)} (-1)^i \exp\{-\mathsf{K}^* \, |i|\} & \text{for } \mu(\Omega) > 4 \\ \dfrac{-1}{2\exp\{-\mathsf{K}^*\} - 2 + \mu(\Omega)} \exp\{-\mathsf{K}^* \, |i|\} & \text{for } \mu(\Omega) < 0 \end{cases} \quad \forall i \in \mathbb{Z}. \tag{18}$$

Let us then compute the spatial Discrete Fourier Transform (DFT) of equation (17), such that

$$F_G(\hat{u}_{2(i)}) = \frac{-1}{2(\cos \mathsf{K} - 1) + \mu(\Omega)}, \quad \text{for } \mathsf{K} \in (0, \pi), \tag{19}$$

where we use $F_G(\hat{u}_{2(i)})$ to denote the DFT of the Green's function that is solution of problem (17). As Ω belongs to a band gap, imposing a unit force at the central mass M with an harmonic variation Ω generates a localized motion whose DFT is given by relation (19).

Let us now consider the case with a modified mass M and no external forces. From problem (15), the DFT $F(\hat{u}_{2(i)})$ of its solution can now be found by multiplying relation (19) by the term $\delta\mu(\Omega)\hat{u}_{2(0)}$, to obtain

$$F(\hat{u}_{2(i)}) = \frac{-\delta\mu(\Omega)\hat{u}_{2(0)}}{2(\cos \mathsf{K} - 1) + \mu(\Omega)} \quad \text{for } \mathsf{K} \in (0, \pi).$$

The DFT $F(\hat{u}_{2(i)})$ is equal to $F_G(\hat{u}_{2(i)})$ if the following condition is respected

$$\delta\mu(\Omega)\hat{u}_{2(0)} = 1 \longrightarrow \delta m_1 = \frac{1}{\hat{u}_{2(0)}} \frac{\rho d}{\Omega \sin \Omega}. \tag{20}$$

Accordingly, also the motion resulting from the case with the modified mass will be equal to the motion from the case with the external force.

We have thus found that, by using the fixed frequency Ω, the perturbation δm_1 generating the localized displacement field previously computed from relation (16) can be obtained from relation (20). For this, we need to compute the displacement $\hat{u}_{2(0)}$ that, according to the imposed equality between $F(\hat{u}_{2(i)})$ and $F_G(\hat{u}_{2(i)})$, can be found by solving problem (17) and thus by using relations (18) with $i = 0$. In particular, the mass m_1 can be in principle either reduced or increased depending on the chosen frequency Ω. Nevertheless, in the following subsection, we show that for the system under study a defect mode can appear only by decreasing the central mass.

4.2 Considerations regarding the perturbed mass and example

We aim here to make some considerations on the sign of the perturbation δm_1 of the central mass. Let us rewrite condition (20) as follows:

$$\frac{\rho d}{\delta m_1} = \hat{u}_{2(0)} \underbrace{\Omega \sin \Omega}_{\star},$$

with $\hat{u}_{2(0)}$ obtained from one between relations (18). By studying the behavior of the equivalent mass $\mu(\Omega)$, given by the first of relations (9), one can obtain the sign of term $\star$:

$$\begin{cases} \mu(\Omega) > 4 \longrightarrow \star > 0 \\ \mu(\Omega) < 0 \longrightarrow \star < 0 \end{cases}.$$

Accordingly, the sign of the perturbation δm_1 of the central mass will depend on the sign of the displacement $\hat{u}_{2(0)}$. Let us thus study the two cases separately.

When $\mu(\Omega) > 4$, using the third of relations (13), $\hat{u}_{2(0)}$ becomes

$$\hat{u}_{2(0)} = \frac{-1}{2\sqrt{\left(1 - \frac{\mu(\Omega)}{2}\right)^2 - 1}} < 0 \quad \forall \Omega \quad \text{such that} \quad \mu(\Omega) > 4.$$

This means that δm_1 must be necessarily < 0 to get a defect mode when $\mu(\Omega) > 4$.

When $\mu(\Omega) < 0$, using the second of relations (13), $\hat{u}_{2(0)}$ becomes

$$\hat{u}_{2(0)} = \frac{1}{2\sqrt{\left(1 - \frac{\mu(\Omega)}{2}\right)^2 - 1}} > 0 \quad \forall \Omega \quad \text{such that} \quad \mu(\Omega) < 0.$$

This means that δm_1 must be necessarily < 0 to get a defect mode when $\mu(\Omega) < 0$.

These results can be illustrated by an example case. For this, let us use the parameters given in table 1. We first consider a fixed frequency $\Omega_1 = 1.8$ within the first band gap of the system with masses directly attached to the cable (c.f. figure 2). At this frequency, $\mu(\Omega_1) > 4$. Using relation (20), with $\Omega = \Omega_1$, we find $\delta m_1 = -2.688 \times 10^{-4}$ kg, corresponding to $M = 0.912 \times 10^{-4}$ kg. We then consider a fixed frequency $\Omega_2 = 4.8$ within the second band gap for the same system. At this frequency, $\mu(\Omega_2) < 0$. From relation (20), with $\Omega = \Omega_1$, we find $\delta m_1 = -3.487 \times 10^{-4}$

Table 1: Parameters used for the example case.

m_1 [kg]	ρ [kg/m]	d [m]	H [N]	N
3.6×10^{-4}	1.52×10^{-3}	0.2	10	21

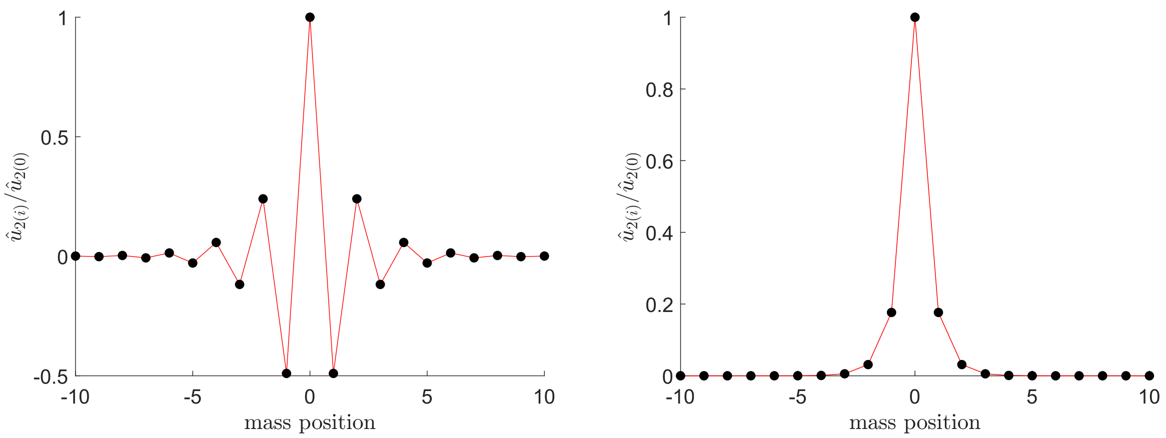

Figure 4: Localized motion computed using $\Omega = \Omega_1$ (left panel) and $\Omega = \Omega_2$ (right panel). Masses are denoted with black dots.

kg, corresponding to $M = 0.113 \times 10^{-4}$ kg. Note that, in both cases, the modified mass must be lighter than mass m_1. Note also that no conditions on δm_1 have been given; accordingly, only those cases for which $|\delta m_1| < m_1$ can be considered.

The resulting localized defect modes are reported in figure 4 for Ω_1 (left panel) and Ω_2 (right panel).

5 CONCLUSIONS

The dynamics of straight cables with a periodic array of scatter elements has been analyzed. We have shown that the problem can be studied by looking at the behavior of an equivalent frequency-dependent mass. In particular, we have found that the spectrum of the operator governing the problem is characterized by the presence of band gaps. We have then shown for the case when the scatter elements are masses directly attached to the cable, that the introduction of a defect by the modification of the central mass of the system can lead to the activation of localized displacement fields at frequencies belonging to a band gap of the problem written for the periodic system. This particular behavior is typical of the so-called metamaterials. For the system considered in this work, it has been possible to derive an analytic expression giving the relation between the frequency of the harmonic localized motion of the scatter elements and the modification of the defective mass.

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