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A stochastic control approach to public debt management

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Abstract
We discuss a class of debt management problems in a stochastic environment model. We propose a model for the debt-to-GDP (Gross Domestic Product) ratio where the government interventions via fiscal policies affect the public debt and the GDP growth rate at the same time. We allow for stochastic interest rate and possible correlation with the GDP growth rate through the dependence of both the processes (interest rate and GDP growth rate) on a stochastic factor which may represent any relevant macroeconomic variable, such as the state of economy. We tackle the problem of a government whose goal is to determine the fiscal policy in order to minimize a general functional cost. We prove that the value function is a viscosity solution to the Hamilton-Jacobi-Bellman equation and provide a Verification Theorem based on classical solutions. We investigate the form of the candidate optimal fiscal policy in many cases of interest, providing interesting policy insights. Finally, we discuss two applications to the debt reduction problem and debt smoothing, providing explicit expressions of the value function and the optimal policy in some special cases.

Keywords Optimal stochastic control · Government debt management · Optimal fiscal policy · Hamilton-Jacobi-Bellman equation

JEL Classification C02 · C61 · H63 · E62

1 Introduction

Public debt management is a wide and complex topic in Economics. On the one hand, it is recognized the important role played by public debt in welfare improving, for example as a tax smoothing tool (starting from the seminal paper [4]) or as savings absorber (see [11] and [16] among others). On the other hand, the current high levels of debt in some developed

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countries has drawn the attention of many economists, especially because of possible effects on future taxation levels (see the quoted papers [11] and [16]).

The question of debt management, in broad sense, is up for discussion essentially because "we simply do not have a theory of the optimum debt level" (see [19]). However, in the very recent years some authors gave a rigorous mathematical formulation of a debt management problem, namely the optimal debt ceiling (that is the debt-to-GDP (Gross Domestic Product) ratio level at which the government should intervene in order to reduce it), see [6–8, 13, 14].

These works focus especially on the debt reduction problem of developed countries, where the government aims at reducing the debt-to-GDP ratio through the minimization of two costs: the cost of holding debt and the intervention cost (reducing public spending or increasing taxes). This study is motivated by the fact that concurrently with the financial crisis started in 2007, the debt-to-GDP ratios exploded.

In [6–8, 13] the possibility for the government to increase the level of debt ratio is precluded (fiscal deficit is not allowed) and any possible benefit resulting from holding debt is neglected. In practice, sometimes debt reduction policies might not be appropriate, since public investments in infrastructure, healthcare, education and research induce social and financial benefits (see [17]). In [14] policies of debt reduction and debt expansion are accounted by modeling controls of bounded variation and introducing, in addition to the cost of reduction policies, a benefit associated to expansionary policies.

Moreover, it is a well known evidence (see e.g. the recent studies [2] and [1]) that policies of debt expansion (deficit) induce also an increase of the GDP growth rate of the country (which in turn could imply a reduction of the debt ratio) and this phenomenon is not exploited in the existing mathematical literature on the topic. This paper wishes to be a first effort to fill this lack.

We provide a rigorous mathematical formulation for a class of debt management problems, which are modeled as stochastic control problems. In particular, we tackle the problem of a government whose goal is to determine the fiscal policy in order to minimize a general functional cost, which depends on the debt-to-GDP ratio, an external driver (e.g. the state of economy) and the fiscal policy itself.

The main improvement of our debt-to-GDP model is that the GDP growth rate depends on the fiscal policy. In classical models, the GDP growth rate is assumed to be constant, see for instance [6, 7] and references therein. In [8, 13, 14] it is allowed to be a stochastic process, which may be modulated by an unobservable continuous time Markov chain as in [8]. However, in all these models the government's interventions via fiscal measures do not affect the GDP growth rate, any policy of surplus decreases the debt-to-GDP ratio (see [6–8, 13]), whereas any deficit policy increases it (as in [14]). In the reality, the effects of the fiscal policy on the debt-to-GDP ratio is more complex.

For example, we might assume that the GDP growth rate decreases when the government increases its surplus (which is mostly the case in normal conditions). In this scenario, the gross debt will decrease as well, but the final effect on debt-to-GDP ratio is not unidirectional, contrary to classical models and the above quoted papers, where any surplus translates into debt-to-GDP ratio reduction. Indeed, the economics literature recognizes the possibility of achieving debt-to-GDP reduction via deficit policies, see for instance [10–12].

Moreover, we assume that both the interest rate on debt and the GDP growth rate are affected by a stochastic factor Z, which may represent the state of economy, the economic outlook, or any other macroeconomic variable. The presence of this external driver Z induces a correlation between the GDP growth rate and the interest rate, which is a well known phenomenon. Furthermore, we assume that Z and the GDP are driven by two correlated

Brownian motion, so that we introduce an additional type of dependence. In practice, the macroeconomic conditions described by Z and the GDP have a common source of uncertainty.

We also assume bounded intervention, so that the government is not able to generate infinite surplus/deficit. As a matter of fact, there are many structural constraints on the surplus generation, for example loss of popularity, impossibility of erasing some welfare spending and so on. Analogously, many constraints can be found for deficit policies. In [7] the authors assumed bounded (surplus) intervention, improving the results of [6] and obtaining a bang-bang strategy (either no intervention or maximum surplus is optimal). Differently from this work, since we have a different model and a general objective function, we obtain a solution with a complex structure, so that extreme policies (maximum surplus/deficit) are optimal only in some scenarios while, in general, the government tries to balance the effects on debt and GDP at the same time.

In this framework we solve the problem of minimizing the expected total cost over an infinite time horizon. As mentioned above, the functional cost depends on the dynamics of the debt-to-GDP ratio, the exogenous factor Z and the fiscal policy. This is a very flexible and general problem formulation, which includes many debt management problems as special cases. For instance, we explore the applications to debt reduction and, in a simplified framework, to debt smoothing, which is the minimization of the distance between the current debt-to-GDP and a given threshold. We highlight that our model formulation contains a trade-off between costs and benefits in the debt-to-GDP equation, not only in the cost functional.

The mathematical contributions of the paper are the following. We rigorously derive the debt-to-GDP ratio dynamics, which is controlled by the government's interventions (which include surplus and deficit strategies) and formulate the arising stochastic control problem. Under general assumptions on the functional cost, we prove that the value function is a continuous viscosity solution to the Hamilton-Jacobi-Bellman (HJB) equation and provide a Verification Theorem which applies whenever the HJB equation has a classical solution. Next, we investigate the structure of the candidate optimal fiscal policy, discussing some cases of interest. In particular, when the functional cost is increasing on the debt-to-GDP ratio and does not explicitly depend by the control, we find that the optimal strategy depends only on the effect of the fiscal policy on the GDP growth rate, not by the current level of debt-to-GDP. A similar result was obtained in a different context by [3]. Thus it becomes crucial understanding how the government's interventions via fiscal policies can influence the GDP growth rate of the country. To find more explicit solutions, we consider the example of a linear impact. Finally, we discuss two applications, namely debt reduction and debt smoothing and provide a numerical simulation for the latter in a simplified framework.

The paper is organized as follows. In Sect. 2 we propose and motivate our model formulation. Then we illustrate the optimization problem. In Sect. 3 we provide some properties of the value function. In Sect. 4 we prove that the value function is a solution in the viscosity sense to the Hamilton-Jacobi-Bellman equation and we provide a Verification Theorem based on classical solution. In Sect. 5 we discuss the minimization problem involved in the HJB equation. Finally, some special cases of interest are discussed in Sect. 6.

2 Model formulation

We propose a stochastic model for the gross public debt and the gross domestic product (GDP) in presence of correlation between the interest rate on debt and the economic growth.

Moreover, the government interventions through fiscal policies affect the GDP growth rate and the public debt at the same time.

Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{Q}, \mathbb{F})$ endowed with a complete and right continuous filtration $\mathbb{F} := \{\mathcal{F}_t\}_{t\geq 0}$. Let $D = \{D_t\}_{t\geq 0}$ denote the gross public debt process and $Y = \{Y_t\}_{t\geq 0}$ the gross domestic product process of a country. According to classical models in economics literature (see [5] among others), the sovereign debt stock evolves as:

$$dD_t = r D_t\, dt - d\xi_t\ , \quad D_0 > 0\ ,$$

where $r > 0$ denotes the real interest rate on debt and ξ is the fiscal policy, with the convention that positive values correspond to primary surplus, while negative values represent deficit. We extend the model by introducing a stochastic interest rate of the form $r_t = r(Z^z_t)$, with r a positive measurable function and $Z = \{Z^z_t\}_{t\geq 0}$ a stochastic factor described by the following stochastic differential equation (SDE):

$$dZ^z_t = b_Z(Z^z_t)\, dt + \sigma_Z(Z^z_t)\, dW^Z_t\ , \qquad Z^z_0 = z \in \mathbb{R}\ , \tag{2.1}$$

where $W^Z = \{W^Z_t\}_{t\geq 0}$ is a standard Brownian motion. We assume existence and strong uniqueness of the solution to the SDE (2.1). The process Z describes any stochastic factor, such as underlying macroeconomic conditions, which affects the interest rate on debt of the country.

Let $u = \{u_t\}_{t\geq 0}$ be the rate of the primary balance expressed in terms of the debt, that is $d\xi_t = u_t D_t dt$. We assume that the real GDP growth rate at time t is of the form $g(Z^z_t, u_t)$, with $g(z, u)$ being a measurable function of its arguments. Precisely, the pair (D, Y) follows

$$\begin{cases} dD_t = D_t(r(Z^z_t) - u_t)\, dt\ , \quad D_0 > 0\ , \\ dY_t = Y_t(g(Z^z_t, u_t)\, dt + \sigma\, d\widetilde{W}_t)\ , \quad Y_0 > 0\ , \end{cases} \tag{2.2}$$

where $\widetilde{W} = \{\widetilde{W}_t\}_{t>0}$ a standard Brownian motion correlated with W^Z and $\sigma > 0$ is the GDP volatility.

The debt-to-GDP ratio $X = \{X_t = \frac{D_t}{Y_t}\}_{t\geq 0}$ dynamics can be derived by Itô's formula:

$$\begin{aligned} dX_t &= \frac{1}{Y_t}\big(r(Z^z_t) - u_t\big) D_t dt - \frac{D_t}{Y_t^2} Y_t (g(Z^z_t, u_t)\, dt + \sigma\, d\widetilde{W}_t) + \frac{D_t}{Y_t^3}\sigma^2 Y_t^2\, dt \\ &= X_t\big(r(Z^z_t) - g(Z^z_t, u_t) - u_t\big)\, dt + X_t \sigma(\sigma\, dt - d\widetilde{W}_t)\ , \qquad X_0 = \frac{D_0}{Y_0}\ . \end{aligned}$$

Now we can introduce a new measure $\mathbb{P}$, equivalent to $\mathbb{Q}$, such that $W = \{W_t\}_{t\geq 0} = \{\sigma t - \widetilde{W}_t\}_{t\geq 0}$ is a $\mathbb{P}$-Brownian motion and W^Z remains a $\mathbb{P}$-Brownian motion.

Hence, under $\mathbb{P}$, the debt-to-GDP ratio $X^{u,x} = \{X^{u,x}_t\}_{t\geq 0}$ is a controlled process which solves the following SDE:

$$dX^{u,x}_t = X^{u,x}_t [(r(Z^z_t) - g(Z^z_t, u_t) - u_t)\, dt + \sigma\, dW_t]\ , \qquad X^{u,x}_0 = x\ , \tag{2.3}$$

where $x > 0$ is the initial debt-to-GDP ratio and the control $u = \{u_t\}_{t\geq 0}$ denotes the fiscal policy (i.e. the ratio of primary surplus to gross debt).

It turns out that our state process is the couple $(X^{u,x}, Z^z)$:

$$\begin{cases} dX^{u,x}_t = X^{u,x}_t [(r(Z^z_t) - g(Z^z_t, u_t) - u_t)\, dt + \sigma\, dW_t]\ , & X^{u,x}_0 = x\ , \\ dZ^z_t = b_Z(Z^z_t)\, dt + \sigma_Z(Z^z_t)\, dW^Z_t\ , & Z^z_0 = z \in \mathbb{R}\ . \end{cases}$$

We denote by $\rho \in [-1, 1]$ the correlation coefficient between W and W^Z.

We consider the problem of a government which wants to choose the fiscal policy in order to optimally manage the sovereign debt-to-GDP. For this purpose we introduce the class of admissible fiscal strategies.

Definition 2.1 (**Admissible fiscal policies**) We denote by $\mathcal{U}$ the family of all the $\mathbb{F}$-predictable and $[-U_1, U_2]$-valued processes $u = \{u_t\}_{t \geq 0}$.

The main goal will be to minimize the following objective function:

$$J(x, z, u) = \mathbb{E}\left[\int_0^{+\infty} e^{-\lambda t} f(X_t^{u,x}, Z_t^z, u_t)\, dt\right], \qquad (x, z) \in (0, +\infty) \times \mathbb{R}, u \in \mathcal{U}\ , \quad (2.4)$$

where $\lambda > 0$ is the government discounting factor and $f\colon (0, +\infty) \times \mathbb{R} \times [-U_1, U_2] \to [0, +\infty)$ is a cost function satisfying suitable hypotheses (see Assumption 2.2 below).

Let us remark some aspects of our model, which is different from those usually introduced in the existing literature.

1. This model extends the one considered in [6] and [7] where r and g are constant. Our main improvement is that the GDP growth rate is now affected by the fiscal policy. For instance, if we take a function g decreasing in u, we capture a well known effect: when the government generates surplus, the debt stock is reduced and, at the same time, the GDP is so. Hence the final effect on the debt-to-GDP ratio is not unidirectional as in the models proposed in [6, 7, 13] and [8], where any surplus translates to debt-to-GDP reduction.
2. As a consequence, our model formulation contains a trade-off between costs and benefits in the state equations, not only in the cost functional as in the previous literature.
3. Another property of our model is that the interest rate and the GDP growth rate have a common source of uncertainty, which is modeled through the external driver Z. In many applications Z could represent the economic environment, the state of the economy, the economic outlook. The pathwise measurement of covariance between the interest rate and the GDP growth rate is given by the covariation between the two processes. When the functions r and g are sufficiently regular w.r.t. $z \in \mathbb{R}$, from Itô's formula it can be computed for any fixed u:
$$< r(Z_t^z), g(Z_t^z, u) >= \int_0^t \frac{\partial g}{\partial z}(Z_s^Z, u) r'(Z_s^z) \sigma_Z^2(Z_s^z) ds\ .$$
In particular, this implies that
$$\mathbb{E}[r(Z_t^z) g(Z_t^z, u)] = \mathbb{E}\left[\int_0^t \frac{\partial g}{\partial z}(Z_s^z, u) r'(Z_s^z) \sigma_Z^2(Z_s^z) ds\right].$$
4. Moreover, our model takes into account possible correlation between the Brownian motions W^Z and W driving the dynamics of the environmental stochastic factor process Z and the debt-to-GDP process X, respectively. For instance, there could be a common source of uncertainty between debt-to-GDP and macroeconomics conditions.

We observe that the problem formulation in Eq. (2.4) is very general and flexible. The cost function depends on the debt-to-GDP ratio, which has to be controlled, and the government can take into account fluctuations of the stochastic factor Z. For instance, when Z represents the state of economy, countercyclical policies are allowed. In addition to this, the fiscal policy level can be explicitly controlled as well. Clearly, many operational problems can be addressed in this framework, depending on the configuration that the government assigns to the function f. We will investigate some applications in Sect. 6.

In the sequel we assume the following hypotheses.

Assumption 2.1 The following assumptions are required in the sequel:

- $r : \mathbb{R} \to (0, +\infty)$, i.e. the interest rate on debt, is such that $r(z) \leq R \; \forall z \in \mathbb{R}$ for a given constant $R > 0$;
- u_t, i.e. the surplus (or deficit)-to-debt ratio at time $t > 0$, takes values in a compact set $[-U_1, U_2]$, where $U_1 > 0$ denotes the maximum allowed deficit-to-debt ratio and $U_2 > 0$ is the maximum surplus-to-debt;
- $g : \mathbb{R} \times [-U_1, U_2] \to \mathbb{R}$, which represents the GDP growth rate, is bounded by

$$\bar{g}_1 \leq g(z, u) \leq \bar{g}_2 \qquad \forall (z, u) \in \mathbb{R} \times [-U_1, U_2] \, , \tag{2.5}$$

for some suitable constants $\bar{g}_1 < 0 < \bar{g}_2$.

Example 2.1 We consider the case where the fiscal policy has a linear impact on the GDP growth rate, precisely

$$g(z, u) = g_0(z) - \alpha(z)u \, , \tag{2.6}$$

with $g_0 : \mathbb{R} \to \mathbb{R}$, $\alpha : \mathbb{R} \to (0, +\infty)$ measurable and bounded functions. The process $\{g_0(Z_t^z)\}_{t \geq 0}$ is the GDP growth rate when no intervention is considered and depends on the environment stochastic factor Z. Any positive intervention $u_t > 0$ leads to a reduction of the GDP growth rate $g(Z_t^z, u_t)$. Conversely, any negative intervention $u_t < 0$ leads to an increase of the GDP growth rate $g(Z_t^z, u_t)$. Both the effects are modulated by the environment stochastic factor Z via the coefficient $\alpha(Z_t^z)$. A simplified model can be obtained with $\alpha(z) = \alpha \neq 1 \; \forall z \in \mathbb{R}$, that is

$$g(z, u) = g_0(z) - \alpha u \, . \tag{2.7}$$

In this special case, Eq. (2.3) reduces to

$$dX_t^{u,x} = X_t^{u,x}[(r(Z_t^z) - g_0(Z_t^z) - (1 - \alpha)u_t) \, dt + \sigma \, dW_t] \, , \qquad X_0^{u,x} = x \, . \tag{2.8}$$

Clearly, $0 < \alpha < 1$ means that the effect of the government fiscal policy with $u_t > 0$ (surplus) leads to a reduction of the instantaneous debt-to-GDP growth rate, while $\alpha > 1$ leads to a reduction of the instantaneous debt-to-GDP growth rate when $u_t < 0$ (deficit). This occurs because, for $0 < \alpha < 1$, the government fiscal policy has a smaller effect on the GDP growth rate w.r.t. the debt growth rate (see the first equation in (2.2)), while for $\alpha > 1$ the fiscal policy has a larger effect on the GDP growth rate than on debt growth rate.

Remark 2.1 Let us observe that the SDE (2.3) admits an explicit solution:

$$X_t^{u,x} = x e^{\int_0^t (r(Z_s^z) - g(Z_s^z, u_s) - u_s) \, ds - \frac{1}{2}\sigma^2 t} e^{\sigma W_t} \quad \forall t \geq 0, \; \mathbb{P} - a.s. \, . \tag{2.9}$$

Clearly, $X_t^{u,x} > 0 \; \forall t \geq 0$ for any admissible strategy. Moreover, by Assumption 2.1, for any $m > 0$ we have that

$$\begin{aligned} & x^m e^{-m(\bar{g}_2 + U_2)t} e^{-\frac{m}{2}\sigma^2 t + m\sigma W_t} \\ & \quad \leq (X_t^{u,x})^m \leq x^m e^{m(R - \bar{g}_1 + U_1)t} e^{-\frac{m}{2}\sigma^2 t + m\sigma W_t} \quad \forall t \geq 0, \; \mathbb{P} - a.s. \end{aligned}$$

from which we get this estimation:

$$\mathbb{E}[(X_t^{u,x})^m] \leq x^m e^{\lambda_m t} \quad \forall t \geq 0 \, , \tag{2.10}$$

where

$$\lambda_m \doteq m(R - \bar{g}_1 + U_1) + m(m-1)\frac{\sigma^2}{2} \, . \tag{2.11}$$

Remark 2.2 We can show that the condition $r(z) - g(z, 0) \geq G\ \forall z \in \mathbb{R}$, for some constant $G > 0$, implies an explosive debt-to-GDP ratio when no intervention is considered. Indeed, by Eq. (2.9) we have that

$$X_t^{0,x} = xe^{\int_0^t (r(Z_s^z) - g(Z_s^z, 0))\, ds} e^{-\frac{1}{2}\sigma^2 t + \sigma W_t} \geq xe^{Gt} M_t \tag{2.12}$$

where $M_t = e^{-\frac{1}{2}\sigma^2 t + \sigma W_t}$ denotes the well known exponential martingale with $\mathbb{E}[M_t] = 1$ $\forall t \geq 0$. Hence

$$\lim_{t \to +\infty} \mathbb{E}[X_t^{0,x}] \geq \lim_{t \to +\infty} xe^{Gt} = +\infty ,$$

which implies $\lim_{t \to +\infty} X_t^{0,x} = +\infty\ \mathbb{P}-a.s.$. However, when r and g are constant functions, the debt-to-GDP explodes if $r - g > 0$, which is a popular result in economics literature.

An important feature for a country is to apply fiscal policies which are sustainable. An explosive debt-to-GDP ratio is not a problem for a country if the discounted debt-to-GDP ratio w.r.t. the interest rate on debt converges to 0. The following definition is standard in Economics (see e.g. [15]).

Definition 2.2 A fiscal policy u is called sustainable if it realizes

$$\lim_{t \to +\infty} e^{-\int_0^t r(Z_s^z)\, ds} X_t^{u,x} = 0 \qquad \mathbb{P} - a.s. , \tag{2.13}$$

see e.g. [5].

Remark 2.3 The bounds U_i, $i = 1, 2$, depend by structural economic and political characteristics of the country, in general. It seems reasonable to choose the maximum level of deficit-to-debt and surplus-to-debt, U_i, $i = 1, 2$, respectively, by imposing that the fiscal policies identically equal to these maximum levels (i.e. $u_t^1 = -U_1$ and $u_t^2 = U_2\ \forall t \geq 0$) turn out to be sustainable for the country, i.e.

$$\lim_{t \to +\infty} e^{-\int_0^t r(Z_s^z)\, ds} X_t^{-U_1,x} = 0 , \quad \lim_{t \to +\infty} e^{-\int_0^t r(Z_s^z)\, ds} X_t^{U_2,x} = 0 \quad \mathbb{P} - a.s. . \tag{2.14}$$

For instance, in case of linear GDP growth rate as in (2.6), we get that

$$\mathbb{E}[e^{-\int_0^t r(Z_s^z)\, ds} X_t^{-U_1,x}] = x\mathbb{E}[e^{-\int_0^t (g_0(Z_s^z) - U_1(1-\alpha))\, ds} M_t]$$

and

$$\mathbb{E}[e^{-\int_0^t r(Z_s^z)\, ds} X_t^{U_2,x}] = x\mathbb{E}[e^{-\int_0^t (g_0(Z_s^z) + U_2(1-\alpha))\, ds} M_t]$$

(here we recall that $M_t = e^{-\frac{1}{2}\sigma^2 t + \sigma W_t}$). Let $\underline{g}_0 = \min_{z \in \mathbb{R}} g_0(z)$ we get

$$\mathbb{E}[e^{-\int_0^t r(Z_s^z)\, ds} X_t^{-U_1,x}] \leq xe^{-(\underline{g}_0 - U_1(1-\alpha))t} ,$$
$$\mathbb{E}[e^{-\int_0^t r(Z_s^z)\, ds} X_t^{U_2,x}] \leq xe^{-(\underline{g}_0 + U_2(1-\alpha))t} ,$$

which imply that if U_1 and U_2 satisfy

$$U_1(1-\alpha) < \underline{g}_0, \quad U_2(\alpha - 1) < \underline{g}_0 ,$$

both the fiscal policies $u_t^1 = -U_1\ \forall t \geq 0$ and $u_t^2 = U_2\ \forall t \geq 0$ are sustainable. The economic interpretation is clear: when the maximum deficit-to-debt positive impact on GDP growth is greater than the effect on debt, then $-U_1$ is sustainable. Similarly, when the negative effect of the maximum surplus-to-debt on GDP is surpassed by the positive effect on debt reduction, then U_2 becomes sustainable.

We impose some assumptions on the cost function in Eq. (2.4).

Assumption 2.2 We assume that

- f is nonnegative;
- $\exists C > 0$ and $m > 0$ such that

$$f(x, z, u) \leq C(1 + x^m) \qquad \forall (x, z, u) \in (0, +\infty) \times \mathbb{R} \times [-U_1, U_2] ,$$

- $\lambda > \lambda_m$ (see Eq. (2.11)).

As announced, the government problem can be formalized in this way:

$$v(x, z) = \inf_{u \in \mathcal{U}} J(x, z, u) , \qquad (x, z) \in (0, +\infty) \times \mathbb{R} . \tag{2.15}$$

Proposition 2.1 *Every admissible strategy $u \in \mathcal{U}$ is such that $J(x, z, u) < +\infty$, $\forall (x, z) \in (0, +\infty) \times \mathbb{R}$.*

Proof Under Assumptions 2.1 and 2.2 and recalling (2.10), we get that $\forall u \in \mathcal{U}$, and $(x, z) \in (0, +\infty) \times \mathbb{R}$

$$\begin{aligned} J(x, z, u) &= \mathbb{E}\left[\int_0^{+\infty} e^{-\lambda t} f(X_t^{u,x}, Z_t^z, u_t)\, dt\right] \\ &\leq C\mathbb{E}\left[\int_0^{+\infty} e^{-\lambda t} (X_t^{u,x})^m\, dt + \int_0^{+\infty} e^{-\lambda t}\, dt\right] \\ &\leq C x^m \int_0^{+\infty} e^{(\lambda_m - \lambda)t}\, dt + \frac{C}{\lambda} \\ &= \frac{C x^m}{\lambda - \lambda_m} + \frac{C}{\lambda} < +\infty . \end{aligned} \tag{2.16}$$

□

3 Properties of the value function

In this section we explore some properties of the value function.

Proposition 3.1 *The value function given in (2.15) satisfies the following properties:*

- $v(x, z) \geq 0\ \forall (x, z) \in (0, +\infty) \times \mathbb{R}$;
- $\exists M > 0$ *such that* $v(x, z) \leq M(1 + x^m)\ \forall (x, z) \in (0, +\infty) \times \mathbb{R}$.

If in addition $\exists \tilde{C} > 0$ *such that* $f(x, z, 0) \leq \tilde{C} x^m\ \forall (x, z) \in (0, +\infty) \times \mathbb{R}$, *then*

- $\exists \tilde{M} > 0$ *such that* $v(x, z) \leq \tilde{M} x^m\ \forall (x, z) \in (0, +\infty) \times \mathbb{R}$;
- $v(0^+, z) = 0\ \forall z \in \mathbb{R}$.

Proof Using Assumption 2.2, we easily obtain that v is nonnegative. Now manipulating Eq. (2.16) we obtain that $\forall u \in \mathcal{U}$, and $(x, z) \in (0, +\infty) \times \mathbb{R}$

$$\begin{aligned} J(x, z, u) &\leq \frac{C}{\lambda - \lambda_m}\left(x^m + \frac{\lambda - \lambda_m}{\lambda}\right) \\ &\leq \frac{C}{\lambda - \lambda_m}(1 + x^m) , \end{aligned}$$

hence $v(x,z) \le M(1+x^m)\ \forall (x,z) \in (0,+\infty)\times\mathbb{R}$ with $M = \frac{C}{\lambda-\lambda_m}$. Now we assume that $f(x,z,0) \le \tilde{C}x^m\ \forall (x,z) \in (0,+\infty)\times\mathbb{R}$. Then

$$v(x,z) \le J(x,z,0) \le \tilde{M}x^m\ ,$$

choosing $\tilde{M} = \frac{\tilde{C}}{\lambda-\lambda_m}$ (by imitation of the proof of Proposition 2.1). This in turn implies that

$$0 \le v(x,z) \le \tilde{M}x^m \quad \forall (x,z) \in (0,+\infty)\times\mathbb{R} \quad \Rightarrow \quad v(0^+,z) = 0 \quad \forall z \in \mathbb{R}\ .$$

□

Proposition 3.2 *Suppose that f is increasing in $x \in (0,+\infty)$. Then v is increasing in $x > 0$, i.e. $0 < x \le x' \Rightarrow v(x,z) \le v(x',z)\ \forall z \in \mathbb{R}$.*

Proof Let us take $0 < x \le x'$. By Eq. (2.9) we see that $\forall u \in \mathcal{U}$, $X_t^{u,x} \le X_t^{u,x'}\ \forall t \ge 0$ $\mathbb{P}-a.s.$. Using the monotonicity of f we get that

$$\mathbb{E}\left[\int_0^{+\infty} e^{-\lambda t} f(X_t^{u,x}, Z_t^z, u_t)\,dt\right] \le \mathbb{E}\left[\int_0^{+\infty} e^{-\lambda t} f(X_t^{u,x'}, Z_t^z, u_t)\,dt\right] \qquad \forall z \in \mathbb{R}, u \in \mathcal{U}\ .$$

Taking the infimum over $\mathcal{U}$ of both sides, we obtain our statement. □

The following proposition is also useful when infinite horizon problems are studied.

Proposition 3.3 *This result hold true, $\forall u \in \mathcal{U}$, and $(x,z) \in (0,+\infty)\times\mathbb{R}$*

$$\lim_{T\to+\infty} e^{-\lambda T}\mathbb{E}[v(X_T^{u,x}, z)] = 0\ .$$

Proof Using Proposition 3.1 and Eq. (2.10) we find that

$$\begin{aligned} e^{-\lambda T}\mathbb{E}[v(X_T^{u,x}, z)] &\le e^{-\lambda T}M(1+\mathbb{E}[(X_T^{u,x})^m]) \\ &\le e^{-\lambda T}M(1+x^m e^{\lambda_m T})\ . \end{aligned}$$

Taking $T \to +\infty$, this quantity converges to 0 since $\lambda > \lambda_m$. □

Proposition 3.4 *Assume that f is independent of $u \in [-U_1, U_2]$, convex in $x > 0$ and g is continuous in $u \in [-U_1, U_2]$ uniformly in $z \in \mathbb{R}$. Then the value function in (2.15) is convex in $x \in (0,+\infty)$.*

Proof Let us take $z \in \mathbb{R}$, $x, x' > 0$, $u^1, u^2 \in \mathcal{U}$ and $k \in [0,1]$. Defining $x_k = kx + (1-k)x'$, we will show that there exists an admissible strategy $u \in \mathcal{U}$ such that $X_t^{u,x_k} = kX_t^{u^1,x} + (1-k)X_t^{u^2,x'}$ solves Eq. (2.3). Precisely, we wish to find $u \in \mathcal{U}$ such that

$$\begin{aligned} &X_t^{u,x_k}[(r(Z_t^z) - g(Z_t^z, u_t) - u_t)\,dt + \sigma\,dW_t] \\ &\quad = kX_t^{u^1,x}[(r(Z_t^z) - g(Z_t^z, u_t^1) - u_t^1)\,dt + \sigma\,dW_t] \\ &\quad\quad + (1-k)X_t^{u^2,x'}[(r(Z_t^z) - g(Z_t^z, u_t^2) - u_t^2)\,dt + \sigma\,dW_t]\ . \end{aligned}$$

In other words, $u \in \mathcal{U}$ must satisfy

$$\begin{aligned} &(kX_t^{u^1,x} + (1-k)X_t^{u^2,x'})(g(Z_t^z, u_t) + u_t) \\ &\quad = kX_t^{u^1,x}(g(Z_t^z, u_t^1) + u_t^1) + (1-k)X_t^{u^2,x'}(g(Z_t^z, u_t^2) + u_t^2)\ , \end{aligned}$$

i.e.

$$g(Z_t^z, u_t) + u_t = \frac{kX_t^{u^1,x}(g(Z_t^z, u_t^1) + u_t^1) + (1-k)X_t^{u^2,x'}(g(Z_t^z, u_t^2) + u_t^2)}{kX_t^{u^1,x} + (1-k)X_t^{u^2,x'}} .$$

Clearly, when g is null or linear we can explicitly calculate u, see e.g. [18, Section 3.6]. In our case we notice that, denoting $\hat{g}(z, u) = g(z, u) + u$, the equation above reduces to

$$\hat{g}(Z_t^z, u_t) = \frac{kX_t^{u^1,x}\hat{g}(Z_t^z, u_t^1) + (1-k)X_t^{u^2,x'}\hat{g}(Z_t^z, u_t^2)}{kX_t^{u^1,x} + (1-k)X_t^{u^2,x'}} .$$

This is a convex combination, hence $\hat{g}(Z_t^z, u_t)$ falls between $\hat{g}(Z_t^z, u_t^1)$ and $\hat{g}(Z_t^z, u_t^2)$. Since $\hat{g}$ is continuous in $u \in [-U_1, U_2]$, uniformly in $z \in \mathbb{R}$, we can find an $\mathbb{F}$-predictable, $[-U_1, U_2]$-valued process $u = \{u_t\}_{t\geq 0}$ satisfying the equation above and it turns out to be admissible.

Hence X_t^{u,x_k} is a public debt process starting from x_k with control u. The convexity of $f(x, z)$ w.r.t. the first variable $x \in \mathbb{R}$ implies that

$$\begin{aligned} f(X_t^{u,x_k}, Z_t^z) &= f(kX_t^{u^1,x} + (1-k)X_t^{u^2,x'}, Z_t^z) \\ &\leq kf(X_t^{u^1,x}, Z_t^z) + (1-k)f(X_t^{u^2,x'}, Z_t^z) \qquad \forall t \geq 0, \mathbb{P} - a.s. , \end{aligned}$$

so that

$$v(x_k, z) \leq kJ(x, z, u^1) + (1-k)J(x', z, u^2) .$$

Since this is true for any $u^1, u^2 \in \mathcal{U}$, we can conclude that

$$v(x_k, z) \leq kv(x, z) + (1-k)v(x', z) .$$

□

Remark 3.1 Combining Propositions 3.2 and 3.4, when f is increasing and convex in $x > 0$, independent of u, and g is continuous in $u \in [-U_1, U_2]$ uniformly in $z \in \mathbb{R}$, the value function turns out to be strictly increasing and convex in $x > 0$, with right and left derivatives satisfying $\frac{\partial v}{\partial x}_+(x, z) \geq \frac{\partial v}{\partial x}_-(x, z) > 0 \; \forall (x, z) \in (0, +\infty) \times \mathbb{R}$.

Proposition 3.5 *Let us assume the following hypotheses:*

- *b_Z and σ_Z are Lipschitz continuous functions on $z \in \mathbb{R}$;*
- *f is continuous in $(x, z) \in (0, +\infty) \times \mathbb{R}$, uniformly in $u \in [-U_1, U_2]$;*
- *r is continuous in $z \in \mathbb{R}$;*
- *g is continuous in $z \in \mathbb{R}$, uniformly in $u \in [-U_1, U_2]$.*

Then the value function is continuous in $(x, z) \in (0, +\infty) \times \mathbb{R}$.

Proof Let us denote by $(X^{u,x,z}, Z^z) = \{(X_t^{u,x,z}, Z_t^z)\}_{t\geq 0}$ the solution to the system of Eqs. (2.1) and (2.3) with initial data $(x, z) \in (0, +\infty) \times \mathbb{R}$. By classical results on SDE, the process Z^z depends continuously on the initial data $z \in \mathbb{R}$, moreover by (2.9) we have that for any $u \in \mathcal{U}$

$$X_t^{u,x,z} = xe^{\int_0^t (r(Z_s^z) - g(Z_s^z, u_s) - u_s)\, ds - \frac{1}{2}\sigma^2 t} e^{\sigma W_t} \qquad \forall t \geq 0, \mathbb{P} - a.s. .$$

Let $\{(x_n, z_n)\}_{n\geq 0}$ be any sequence in $(0, +\infty) \times \mathbb{R}$ converging to $(x, z) \in (0, +\infty) \times \mathbb{R}$ as $n \to +\infty$, then $Z_t^{z_n} \to Z_t^z$ $\forall t \geq 0$, as $n \to +\infty$ and by the dominated convergence theorem we have that $\forall t \geq 0$

$$X_t^{u,x_n,z_n} \to X_t^{u,x,z} \qquad \text{uniformly on } u \in \mathcal{U}\text{ , as } n \to +\infty$$

and

$$J(x_n, z_n, u) \to J(x, z, u) \qquad \text{uniformly on } u \in \mathcal{U}\text{ , as } n \to +\infty\ ,$$

which finally implies continuity of $v(x, z) = \inf_{u\in\mathcal{U}} J(x, z, u)$ in $(x, z) \in (0, +\infty) \times \mathbb{R}$. □

4 Characterization of the value function

In this section we aim to characterize the value function v given in Eq. (2.15). Precisely, we prove that it is a viscosity solution of the Hamilton-Jacobi-Bellman equation associated to our problem (see Eq. (4.2) below). To obtain this result only the continuity of v is required. If, in addition, the HJB equation admits a classical solution, then it turns out to be the value function. In this case v will satisfy some additional regularity conditions.

We begin finding the HJB equation associated to our problem.

Remark 4.1 For any $(x, z) \in (0, +\infty) \times \mathbb{R}$ and any $u \in [-U_1, U_2]$ the Markov generator of (X^u, Z) is given by the following differential operator [1]:

$$\mathcal{L}^u\phi(x, z) = x[r(z) - g(z, u) - u]\frac{\partial\phi}{\partial x}(x, z) + \frac{1}{2}\sigma^2x^2\frac{\partial^2\phi}{\partial x^2}(x, z)$$
$$+\rho\sigma x\sigma_Z(z)\frac{\partial^2\phi}{\partial x\partial z}(x, z) + \mathcal{L}^Z\phi(x, z)\ , \tag{4.1}$$

where $\phi\colon (0, +\infty) \times \mathbb{R} \to \mathbb{R}$ is a function on $\mathcal{C}^{2,2}((0, +\infty) \times \mathbb{R})$ and $\mathcal{L}^Z$ denotes the operator

$$\mathcal{L}^Z\phi(x, z) = b_Z(z)\frac{\partial\phi}{\partial z}(x, z) + \frac{1}{2}\sigma_Z(z)^2\frac{\partial^2\phi}{\partial z^2}(x, z)\ .$$

The value function in Eq. (2.15), if sufficiently regular, is expected to solve the HJB equation, which is given by

$$\inf_{u\in[-U_1,U_2]}\{\mathcal{L}^u v(x, z) + f(x, z, u) - \lambda v(x, z)\} = 0\ . \tag{4.2}$$

Before stating the main result, we briefly recall the definition of viscosity solution to Eq. (4.2). Let us notice that, in general, one would require that a function w is locally bounded in order to be the solution of a PDE in viscosity sense (see for instance [18, Chapter 4]). However, under the hypotheses of Proposition 3.5 we know that v given in Eq. (2.15) is continuous, hence we can directly refer to the special case of continuous functions.

Definition 4.1 Let $w\colon (0, +\infty) \times \mathbb{R} \to [0, +\infty)$ be continuous. We say that

- w is a viscosity subsolution of Eq. (4.2) if

$$\inf_{u\in[-U_1,U_2]}\{\mathcal{L}^u\varphi(\bar{x}, \bar{z}) + f(\bar{x}, \bar{z}, u) - \lambda\varphi(\bar{x}, \bar{z})\} \geq 0\ , \tag{4.3}$$

[1] This is a simple application of Itô's formula.

for all $(\bar{x}, \bar{z}) \in (0, +\infty) \times \mathbb{R}$ and for all $\varphi \in \mathcal{C}^{2,2}((0, +\infty) \times \mathbb{R})$ such that $(\bar{x}, \bar{z})$ is a maximum point of $w - \varphi$;

- w is a viscosity supersolution of Eq. (4.2) if

$$\inf_{u \in [-U_1, U_2]} \{\mathcal{L}^u \varphi(\bar{x}, \bar{z}) + f(\bar{x}, \bar{z}, u) - \lambda \varphi(\bar{x}, \bar{z})\} \leq 0 \,, \tag{4.4}$$

for all $(\bar{x}, \bar{z}) \in (0, +\infty) \times \mathbb{R}$ and for all $\varphi \in \mathcal{C}^{2,2}((0, +\infty) \times \mathbb{R})$ such that $(\bar{x}, \bar{z})$ is a minimum point of $w - \varphi$;

- w is a viscosity solution of Eq. (4.2) if it is a viscosity subsolution and a supersolution.

Theorem 4.1 *Under the hypotheses of Proposition 3.5, the value function v given in Eq. (2.15) is a viscosity solution of the HJB Eq. (4.2).*

Proof See Appendix A. □

Remark 4.2 The uniqueness of viscosity solutions received a lot of interest in the PDE literature. The reader can refer to [18] (especially Sections 4.4.1 and 4.4.2) and the references therein for PDEs arising in stochastic control problems. Moreover, under suitable conditions on the PDE coefficients and the cost function, one should be able to prove additional regularity properties of the value function, such as $\mathcal{C}^2$ with respect to x, see for instance [18, Theorem 4.5.6]. However, these aspects are beyond our focus. Instead, we make use of a verification argument.

Now we provide a Verification Theorem based on classical solutions to the HJB Eq. (4.2).

Theorem 4.2 *Let $w \colon (0, +\infty) \times \mathbb{R} \to [0, +\infty)$ be a function in $\mathcal{C}^{2,2}((0, +\infty) \times \mathbb{R})$ and suppose that there exists a constant $C_1 > 0$ such that*

$$|w(x, z)| \leq C_1(1 + |x|^m) \qquad \forall (x, z) \in (0, +\infty) \times \mathbb{R} \,.$$

If

$$\inf_{u \in [-U_1, U_2]} \{\mathcal{L}^u w(x, z) + f(x, z, u) - \lambda w(x, z)\} \geq 0 \qquad \forall (x, z) \in (0, +\infty) \times \mathbb{R} \,, \tag{4.5}$$

then $w(x, z) \leq v(x, z)$ $\forall (x, z) \in (0, +\infty) \times \mathbb{R}$.

Now suppose that there exists a $[-U_1, U_2]$-valued measurable function $u^*(x, z)$ such that

$$\begin{aligned}&\inf_{u \in [-U_1, U_2]} \{\mathcal{L}^u w(x, z) + f(x, z, u) - \lambda w(x, z)\} \\ &\quad = \mathcal{L}^{u^*} w(x, z) + f(x, z, u^*(x, z)) - \lambda w(x, z) = 0 \quad \forall (x, z) \in (0, +\infty) \times \mathbb{R} \,. \end{aligned} \tag{4.6}$$

Then $w(x, z) = v(x, z)$ $\forall (x, z) \in (0, +\infty) \times \mathbb{R}$ and $u^* = \{u^*(X_t^{u^*,x}, Z_t^z)\}_{t \geq 0}$ is an optimal (Markovian) control.

Proof Let $w \in \mathcal{C}^{2,2}((0, +\infty) \times \mathbb{R})$. Let us introduce a sequence of stopping times defined by

$$\begin{aligned}\tau_n \doteq \inf\{t \geq 0 \mid \int_0^t e^{-2\lambda s} \left| \frac{\partial w}{\partial x}(X_s^{u,x}, Z_s^z) \sigma X_s^{u,x} \right|^2 ds \geq n\} \\ \wedge \inf\{t \geq 0 \mid \int_0^t e^{-2\lambda s} \left| \frac{\partial w}{\partial z}(X_s^{u,x}, Z_s^z) \sigma_Z(Z_s^z) \right|^2 ds \geq n\} \,, \qquad n \in \mathbb{N} \,. \end{aligned} \tag{4.7}$$

Applying Itô's formula to $e^{-\lambda t} w(X_t^{u,x}, Z_t^z)$ for any arbitrary $u \in \mathcal{U}$ we get

$$\begin{aligned} & e^{-\lambda T\wedge\tau_n} w(X_{T\wedge\tau_n}^{u,x}, Z_{T\wedge\tau_n}^z) \\ & = w(x,z) + \int_0^{T\wedge\tau_n} e^{-\lambda t}[\mathcal{L}^u w(X_t^{u,x}, Z_t^z) - \lambda w(X_t^{u,x}, Z_t^z)]\, dt \\ & \quad + \int_0^{T\wedge\tau_n} e^{-\lambda t} \frac{\partial w}{\partial x}(X_t^{u,x}, Z_t^z)\sigma X_t^{u,x}\, dW_t \\ & \quad + \int_0^{T\wedge\tau_n} e^{-\lambda t} \frac{\partial w}{\partial z}(X_t^{u,x}, Z_t^z)\sigma_Z(Z_t^z)\, dW_t^Z \,. \end{aligned}$$

The last integrals are real martingales by definition of τ_n (see Eq. (4.7)), hence taking the expectation and using the inequality (4.5) gives

$$\mathbb{E}[e^{-\lambda T\wedge\tau_n} w(X_{T\wedge\tau_n}^{u,x}, Z_{T\wedge\tau_n}^z)] \geq w(x,z) - \mathbb{E}\left[\int_0^{T\wedge\tau_n} e^{-\lambda t} f(X_t^{u,x}, Z_t^z, u_t)\, dt\right].$$

By the growth condition on w and Proposition 2.1 we can apply the dominated convergence theorem, so that letting $n \to +\infty$ gives [2]

$$\mathbb{E}[e^{-\lambda T} w(X_T^{u,x}, Z_t^z)] \geq w(x,z) - \mathbb{E}\left[\int_0^{T} e^{-\lambda t} f(X_t^{u,x}, Z_t^z, u_t)\, dt\right] \qquad \forall u \in \mathcal{U}\,.$$

Recalling Eq. (2.10), since

$$\begin{aligned} \limsup_{T\to+\infty} e^{-\lambda T}\mathbb{E}[w(X_T^{u,x}, Z_t^z)] & \leq C_1 \limsup_{T\to+\infty} e^{-\lambda T}\mathbb{E}[1 + (X_T^{u,x})^m] \\ & \leq 0 \qquad \forall (x,z) \in (0,+\infty)\times\mathbb{R}, \forall u \in \mathcal{U}\,, \end{aligned}$$

we can send $T \to +\infty$ to obtain that

$$w(x,z) \leq \mathbb{E}\left[\int_0^{+\infty} e^{-\lambda t} f(X_t^{u,x}, Z_t^z, u_t)\, dt\right] \qquad \forall u \in \mathcal{U}\,,$$

which implies the first inequality $w(x,z) \leq v(x,z)\ \forall (x,z) \in (0,+\infty)\times\mathbb{R}$.

Now we can repeat the same argument choosing the control $\{u_t^* = u^*(X_t^{u^*,x}, Z_t^z)\}_{t\geq 0}$, obtaining the equality

$$\mathbb{E}[e^{-\lambda T} w(X_T^{u^*,x}, Z_t^z)] = w(x,z) - \mathbb{E}\left[\int_0^{T} e^{-\lambda t} f(X_t^{u^*,x}, Z_t^z, u_t^*)\, dt\right].$$

Using the fact that

$$\liminf_{T\to+\infty} e^{-\lambda T}\mathbb{E}[w(X_T^{u^*,x}, Z_t^z)] \geq 0 \qquad \forall (x,z) \in (0,+\infty)\times\mathbb{R}\,,$$

sending $T \to +\infty$ we deduce

$$w(x,z) \geq \mathbb{E}\left[\int_0^{+\infty} e^{-\lambda t} f(X_t^{u^*,x}, Z_t^z, u^*(X_t^{u^*,x}, Z_t^z))\, dt\right],$$

which shows that $w(x,z) = v(x,z)\ \forall (x,z) \in (0,+\infty)\times\mathbb{R}$ and that u^* is an optimal control. □

[2] Notice that $\tau_n \to +\infty$ because the integrand functions in Eq. (4.7) are continuous by our assumptions.

5 The optimal fiscal policy

In view of the Verification Theorem in this section we aim at investigating the optimal candidate fiscal policy and its properties. Putting the expression (4.1) into the HJB Eq. (4.2) and gathering the terms which depend on u gives

$$\begin{aligned}&\inf_{u\in[-U_1,U_2]}\{x[-g(z,u)-u]\frac{\partial v}{\partial x}(x,z)+f(x,z,u)\}+xr(z)\frac{\partial v}{\partial x}(x,z)\\&+\frac{1}{2}\sigma^2x^2\frac{\partial^2 v}{\partial x^2}(x,z)+\rho\sigma x\sigma_Z(z)\frac{\partial^2 v}{\partial x\partial z}(x,z)\\&+\mathcal{L}^Z v(x,z)-\lambda v(x,z)=0\,.\end{aligned}\tag{5.1}$$

We look for a minimizer of the following function:

$$H(x,z,u)=-x[g(z,u)+u]\frac{\partial v}{\partial x}(x,z)+f(x,z,u)\,.\tag{5.2}$$

Our candidate optimal strategy is given through a function $u^*\colon(0,+\infty)\times\mathbb{R}\to[-U_1,U_2]$ which solves the following minimization problem:

$$H(x,z,u^*(x,z))=\min_{u\in[-U_1,U_2]}H(x,z,u)\,.\tag{5.3}$$

To this end, we make use of the following assumptions.

Assumption 5.1 We assume that

- g is continuous and differentiable in $u\in[-U_1,U_2]$;
- f is continuous and differentiable in $u\in[-U_1,U_2]$;
- $v\in C^{2,2}((0,+\infty)\times\mathbb{R})$ is a classical solution of Eq. (5.1).

We first state the existence and uniqueness of the minimizer of problem (5.3), then we give a characterization of it.

Proposition 5.1 *The problem* (5.3) *admits a minimizer* $u^*(x,z)\in[-U_1,U_2]$ *for any* $(x,z)\in(0,+\infty)\times\mathbb{R}$. *Moreover, if H is strictly convex, the minimizer is also unique.*

Proof The existence of a minimizer immediately follows by the compactness of the interval $[-U_1,U_2]$ and the Weierstrass Theorem. Moreover, the strictly convexity of H implies the uniqueness of the minimizer by classical arguments. □

Proposition 5.2 *Suppose that* $\frac{\partial^2 H}{\partial u^2}(x,z,u)>0$, $\forall(x,z,u)(0,+\infty)\times\mathbb{R}\in[-U_1,U_2]$. *Then there exists a unique minimizer* $u^*(x,z)$ *for the problem* (5.3). *Moreover,* $u^*(x,z)$ *admits the following expression:*

$$\begin{aligned}&u^*(x,z)\\&=\begin{cases}-U_1 & (x,z)\in\left\{(x,z)\in(0,+\infty)\times\mathbb{R}\mid\frac{\partial f}{\partial u}(x,z,-U_1)\ge x\frac{\partial v}{\partial x}(x,z)\left[\frac{\partial g}{\partial u}(z,-U_1)+1\right]\right\}\\U_2 & (x,z)\in\left\{(x,z)\in(0,+\infty)\times\mathbb{R}\mid\frac{\partial f}{\partial u}(x,z,U_2)\le x\frac{\partial v}{\partial x}(x,z)\left[\frac{\partial g}{\partial u}(z,U_2)+1\right]\right\}\\\hat{u}(x,z) & \text{otherwise}\,,\end{cases}\end{aligned}$$

where $\hat{u}(x,z)$ *denotes the solution to*

$$x\frac{\partial v}{\partial x}(x,z)\left[\frac{\partial g}{\partial u}(z,u)+1\right]=\frac{\partial f}{\partial u}(x,z,u)\quad\forall(x,z)\in(0,+\infty)\times\mathbb{R}\,.\tag{5.4}$$

Proof Existence and uniqueness of the minimizer are guaranteed by classical arguments, since H is continuous and strictly convex in $u \in [-U_1, U_2]$. Observing that

$$\frac{\partial H}{\partial u}(x,z,u) = \frac{\partial f}{\partial u}(x,z,u) - x\frac{\partial v}{\partial x}(x,z)\left[\frac{\partial g}{\partial u}(z,u) + 1\right], \tag{5.5}$$

we have only three cases.

1. If $(x,z) \in (0,+\infty)\times\mathbb{R}$ are such that $\frac{\partial H}{\partial u}(x,z,-U_1) \geq 0$, i.e.
$$\frac{\partial f}{\partial u}(x,z,-U_1) \geq x\frac{\partial v}{\partial x}(x,z)\big[\frac{\partial g}{\partial u}(z,-U_1)+1\big],$$
then we must have
$$\frac{\partial H}{\partial u}(x,z,u) \geq \frac{\partial H}{\partial u}(x,z,-U_1) \geq 0 \qquad \forall u \in [-U_1, U_2],$$
because of the convexity of H. Hence H is increasing on the whole interval $[-U_1, U_2]$ and the minimizer turns out to be $u^*(x,z) = -U_1$.
2. If $(x,z) \in (0,+\infty)\times\mathbb{R}$ are such that $\frac{\partial H}{\partial u}(x,z,U_2) \leq 0$, i.e.
$$\frac{\partial f}{\partial u}(x,z,U_2) \leq x\frac{\partial v}{\partial x}(x,z)\big[\frac{\partial g}{\partial u}(z,U_2)+1\big],$$
then we have that
$$\frac{\partial H}{\partial u}(x,z,u) \leq \frac{\partial H}{\partial u}(x,z,U_2) \leq 0 \qquad \forall u \in [-U_1, U_2],$$
so that H is decreasing in $u \in [-U_1, U_2]$ and therefore $u^*(x,z) = U_2$ is the minimizer.
3. Finally, when $\frac{\partial H}{\partial u}(x,z,-U_1) < 0$ and $\frac{\partial H}{\partial u}(x,z,U_2) > 0$, since $\frac{\partial H}{\partial u}$ is continuous in $u \in [-U_1, U_2]$, there exists $\hat{u}(x,z)$ such that $\frac{\partial H}{\partial u}(x,z,\hat{u}(x,z)) = 0$ (see Eq. (5.4)) and this stationary point coincides with the minimizer.

□

5.1 Some special cases

In this section we investigate some cases of interest. Let us first establish a slightly general result, when the cost function does not depend explicitly on u.

Proposition 5.3 *Suppose that the cost function f does not depend on u, i.e. $f(x,z,u) = f(x,z)$ and $\frac{\partial^2 H}{\partial u^2}(x,z,u) > 0$, $\forall (x,z,u)(0,+\infty)\times\mathbb{R} \in [-U_1, U_2]$. Then the minimizer of* (5.3) *is given by*

$$u^*(x,z) = \begin{cases} -U_1 & (x,z) \in \left\{(x,z) \in (0,+\infty)\times\mathbb{R} \mid 0 \geq \frac{\partial v}{\partial x}(x,z)\big[\frac{\partial g}{\partial u}(z,-U_1)+1\big]\right\} \\ U_2 & (x,z) \in \left\{(x,z) \in (0,+\infty)\times\mathbb{R} \mid 0 \leq \frac{\partial v}{\partial x}(x,z)\big[\frac{\partial g}{\partial u}(z,U_2)+1\big]\right\} \\ \hat{u}(z) & \text{otherwise}, \end{cases} \tag{5.6}$$

where $\hat{u}(z)$ is the unique solution to

$$\frac{\partial g}{\partial u}(z,u) = -1 \quad \forall z \in \mathbb{R}. \tag{5.7}$$

Proof Observing that

$$\frac{\partial H}{\partial u}(x,z,u) = -x\frac{\partial v}{\partial x}(x,z)\left[\frac{\partial g}{\partial u}(z,u)+1\right], \tag{5.8}$$

$$\frac{\partial^2 H}{\partial u^2}(x,z,u) = -x\frac{\partial v}{\partial x}(x,z)\frac{\partial^2 g}{\partial u^2}(z,u) > 0\,, \tag{5.9}$$

hence $\frac{\partial v}{\partial x}(x,z) \neq 0$, $\forall (x,z) \in (0,+\infty)\times\mathbb{R}$ and the proof follows the same lines of Proposition 5.2. $\square$

The result of Proposition 5.3 is interesting. When the fiscal policy has a nonlinear impact on GDP growth, the government will select the primary surplus-to-debt in order that the effect on GDP matches that on debt, in general (see Eq. (5.7)). When this is not possible, the extreme policies are chosen. The choice between $-U_1$ and U_2 and the corresponding economic interpretation crucially depend on the government objective, see the comments below Proposition 5.5.

In the case of Example 2.1, that is when the fiscal policy has a linear impact on the GDP growth rate, (see Eq. (2.6)), Proposition 5.3 does not apply, because the condition $\frac{\partial^2 H}{\partial u^2}(x,z,u) > 0$, $\forall (x,z,u) \in (0,+\infty)\times\mathbb{R}\times[-U_1,U_2]$ is not fulfilled. In the next proposition we discuss a tailor-made result for this special case.

Proposition 5.4 *Suppose that the cost function f does not depend on u, i.e. $f(x,z,u) = f(x,z)$ and let the GDP growth rate given by*

$$g(z,u) = g_0(z) - \alpha(z)u\,,$$

with $g_0\colon \mathbb{R}\to\mathbb{R}$ and $\alpha(z) > 0$, $\alpha(z)\neq 1$, $\forall z\in\mathbb{R}$. Then the minimizer of (5.3) is given by

$$u^*(x,z) = \begin{cases} -U_1 & (x,z) \in \left\{(x,z)\in(0,+\infty)\times\mathbb{R} \mid \frac{\partial v}{\partial x}(x,z)(\alpha(z)-1)\geq 0\right\} \\ U_2 & (x,z) \in \left\{(x,z)\in(0,+\infty)\times\mathbb{R} \mid \frac{\partial v}{\partial x}(x,z)(\alpha(z)-1)< 0\right\}. \end{cases} \tag{5.10}$$

Proof The statement follows by observing that H is a linear function on $u\in[-U_1,U_2]$

$$H(x,z,u) = -x[g_0(z)+(1-\alpha(z))u]\frac{\partial v}{\partial x}(x,z) + f(x,z)\,.$$

$\square$

In the next proposition we discuss the case where the cost function $f(x,z)$ does not depend on u and it is increasing in $x\in(0,+\infty)$. This situation refers to the case where debt-to-GDP generates a disutility for the government of the country and thus it aims to reduce the debt-to-GDP ratio. We refer to this case as the debt reduction problem, see Sect. 6.1.

Proposition 5.5 *Suppose that the cost function f does not depend on u, i.e. $f(x,z,u) = f(x,z)$, with $f(x,z)$ increasing in $x\in(0,+\infty)$. Assuming $\frac{\partial v}{\partial x}(x,z)\neq 0$ for any $(x,z)\in(0,+\infty)\times\mathbb{R}$ and $\frac{\partial^2 g}{\partial u^2}(z,u) < 0$ for any $(z,u)\in\mathbb{R}\times[-U_1,U_2]$, the unique minimizer of (5.3) is given by*

$$u^*(z) = -U_1 \vee \hat{u}(z)\wedge U_2\,,$$

where $\hat{u}(z)$ is the unique solution to

$$\frac{\partial g}{\partial u}(z,u) = -1, \quad \forall z\in\mathbb{R}\,.$$

Precisely, $u^*(z)$ admits the following structure:

$$u^*(z) = \begin{cases} -U_1 & z \in \left\{ z \in \mathbb{R} \mid \frac{\partial g}{\partial u}(z, -U_1) < -1 \right\} \\ U_2 & z \in \left\{ z \in \mathbb{R} \mid \frac{\partial g}{\partial u}(z, U_2) > -1 \right\} \\ \hat{u}(z) & \textit{otherwise} . \end{cases} \tag{5.11}$$

Proof By Proposition 3.2 we have that $v(x, z)$ is increasing in $x \in (0, +\infty)$ for any $z \in \mathbb{R}$, hence $\frac{\partial v}{\partial x}(x, z) \geq 0$, $\forall z \in \mathbb{R}$, which together with the assumptions $\frac{\partial v}{\partial x}(x, z) \neq 0$, for any $(x, z) \in (0, +\infty) \times \mathbb{R}$ and $\frac{\partial^2 g}{\partial u^2}(z, u) < 0$ for any $(z, u) \in \mathbb{R} \times [-U_1, U_2]$ imply

$$\frac{\partial^2 H}{\partial u^2}(x, z, u) = -x \frac{\partial v}{\partial x}(x, z) \frac{\partial^2 g}{\partial u^2}(z, u) > 0 \quad \forall (x, z, u) \in (0, +\infty) \times \mathbb{R} \times [-U_1, U_2] .$$

Recalling that

$$\frac{\partial H}{\partial u}(x, z, u) = -x \frac{\partial v}{\partial x}(x, z) \left[\frac{\partial g}{\partial u}(z, u) + 1 \right] ,$$

the first order condition reads as if and only if

$$\frac{\partial g}{\partial u}(z, u) + 1 = 0 ,$$

and the proof follows the same lines of Proposition 5.2. □

The previous result has an intermediate case as in Proposition 5.3, with the same interpretation. However, as announced, now we can provide a deeper insights on extreme fiscal policies.

The maximum surplus-to-debt will be applied only if the beneficial impact on debt more than compensates the negative effect on GDP growth. Indeed, the marginal impact of U_2 on the GDP growth is measured by $\frac{\partial g}{\partial u}(z, U_2)$, which is negative, while the effect on debt is unitary. When the debt can be decreased more than the GDP growth by means of the maximum surplus-to-debt, U_2 is optimal. Similarly, the maximum deficit-to-debt is chosen when the positive effect on GDP exceeds the negative effect on debt.

Proposition 5.5 also highlights a relevant aspect of our model: the cost-benefit trade-off is implicit in the debt-to-GDP equation, because our model allows for effects of surplus/deficit on GDP. Hence the presence of u in the cost functional is not needed from the mathematical point of view.

In the case of Example 2.1, that is when the fiscal policy has a linear impact on the GDP growth rate, (see Eq. (2.6)) we have an analogous result.

Proposition 5.6 *Let the GDP growth rate given by*

$$g(z, u) = g_0(z) - \alpha(z)u ,$$

with $g_0 \colon \mathbb{R} \to \mathbb{R}$ and $\alpha(z) > 0$, $\alpha(z) \neq 1$, $\forall z \in \mathbb{R}$. Then, for any running cost function $f(x, z)$ increasing in $x \in (0, +\infty)$ $\forall z \in \mathbb{R}$ and assuming $\frac{\partial v}{\partial x}(x, z) \neq 0$ $\forall (x, z) \in (0, +\infty) \times \mathbb{R}$, the unique minimizer of (5.3) is given by

$$u^*(z) = \begin{cases} -U_1 & z \in \{ z \in \mathbb{R} \mid \alpha(z) > 1 \} \\ U_2 & z \in \{ z \in \mathbb{R} \mid \alpha(z) < 1 \} . \end{cases} \tag{5.12}$$

Proof Observing that

$$H(x,z,u) = -x[g_0(z) + (1-\alpha(z))u]\frac{\partial v}{\partial x}(x,z) + f(x,z)\ ,$$

we have only two cases.

1. If $z \in \{z \in \mathbb{R} \mid \alpha(z) > 1\}$ then H is increasing in $u \in [-U_1, U_2]$ thus the minimizer is $u^*(z) = -U_1$.
2. If $z \in \{z \in \mathbb{R} \mid 0 < \alpha(z) < 1\}$ then H is decreasing in $u \in [-U_1, U_2]$ thus the minimizer is $u^*(z) = U_2$.

$\square$

In the semplified model of Eq. (2.7), that is when $\alpha(z) = \alpha\ \forall z \in \mathbb{R}$, we have the following result.

Corollary 5.1 *Let the GDP growth rate be given by*

$$g(z,u) = g_0(z) - \alpha u\ ,$$

with $g_0 \colon \mathbb{R} \to \mathbb{R}$ *and* $\alpha > 0$, $\alpha \neq 1$. *Then for any running cost function* $f(x,z)$ *increasing in* $x \in (0,+\infty)\ \forall z \in \mathbb{R}$, *assuming* $\frac{\partial v}{\partial x}(x,z) \neq 0$ *for any* $(x,z) \in (0,+\infty) \times \mathbb{R}$, *the unique minimizer of* (5.3) *is constant and given by*

$$u^* = \begin{cases} -U_1 & \textit{if}\ \alpha > 1 \\ U_2 & \textit{if}\ 0 < \alpha < 1\ . \end{cases}$$

Proof This is a simple application of Proposition 5.6. $\square$

Remark 5.1 If $\alpha > 1$ we get that the candidate optimal strategy is $u^* = -U_1$, that is the optimal choice for the government is to generate the maximum deficit-to-debt. Indeed, the beneficial effect on GDP exceeds the debt increase. By Remark 2.3 this strategy is sustainable if

$$U_1 > -\frac{\min_{z\in\mathbb{R}} g_0(z)}{\alpha - 1}.$$

It would be natural to assume that $\min_{z\in\mathbb{R}} g_0(z) < 0$, and hence the right hand side is positive. Then $u^* = -U_1$ would be sustainable if the government can produce enough deficit, which is usually the case.
Conversely, if $0 < \alpha < 1$, the candidate optimal strategy is $u^* = U_2$, that is the maximum surplus-to-debt. By Remark 2.3 this strategy is sustainable if

$$U_2 > -\frac{\min_{z\in\mathbb{R}} g_0(z)}{1-\alpha}\ .$$

If $\min_{z\in\mathbb{R}} g_0(z) < 0$, the optimal policy $u^* = U_2$ will be sustainable only if the government has the possibility of increasing taxes and reducing public spending more than a given threshold. In some cases, this could be a challenging task for the government.
Clearly, when $\min_{z\in\mathbb{R}} g_0(z) > 0$, the optimal strategy is always sustainable.

Remark 5.2 Propositions 5.5, 5.6 and Corollary 5.1 show that in the case where $f(x,z,u) = f(x,z)$ is increasing in $x \in (0,+\infty), \forall z \in \mathbb{R}$, the minimizer of (5.3) does not depend on the form of the cost function but only on the function $g(z,u)$ which describes the GDP growth rate and the effect of the government policy on it. Thus, given the unitary impact of the fiscal policy on the public debt, the debt-GDP reduction is driven by the effect on the GDP growth rate, which should be the focus of the government attention.

6 Explicit solutions for some cases of interest

In this section we discuss two examples which can be solved applying the Verification Theorem. In the first example we have in mind a country with debt problems, aiming to reduce its debt-to-GDP ratio. In the second application we discuss the debt smoothing problem, that is, the government wishes to smooth the debt-to-GDP by flattening its deviation from a given threshold. In the first case the running cost is increasing w.r.t. $x \in (0, +\infty)$, while in the second case we do not require a monotonic condition.

6.1 The debt reduction problem

We assume that the cost function is given by

$$f(x, z) = C(z)x^m, \quad m \geq 2 , \tag{6.1}$$

where $C : \mathbb{R} \to (0, +\infty)$ is a bounded function. This disutility function generalizes the quadratic function that is widely used in Economics. The parameter m represents the aversion of the government towards holding debt and the importance of debt for the government is modulated by the function $C(z)$, which is a function of the values taken by the environment stochastic factor Z.

For instance, if Z is an indicator of macroeconomic conditions and higher values correspond to better conditions, assuming that C is increasing enables the government to relax fiscal rules and debt reduction goals when a massive government intervention is needed, as during economic crises. That is, $C(z)$ allows for countercyclical policies.

Denoting by

$$G(z) = \inf_{u \in [-U_1, U_2]} \{-(g(z, u) + u)\} ,$$

the HJB Eq. (5.1) reads as

$$\begin{aligned}
&(G(z) + r(z))x \frac{\partial v}{\partial x}(x, z) + f(x, z) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2}(x, z) \\
&\quad + \rho\sigma x \sigma_Z(z) \frac{\partial^2 v}{\partial x \partial z}(x, z) + b_Z(z) \frac{\partial v}{\partial z}(x, z) \\
&\quad + \frac{1}{2}\sigma_Z(z)^2 \frac{\partial^2 v}{\partial z^2}(x, z) - \lambda v(x, z) = 0 .
\end{aligned}$$

With the ansatz $v(x, z) = \phi(z)x^m$ this equation reduces to the following ordinary differential equation (ODE)

$$\begin{aligned}
&\frac{1}{2}\sigma_Z^2(z)\phi''(z) + \big(b_Z(z) + \rho\sigma\sigma_Z(z)m\big)\phi'(z) \\
&\quad - [\lambda - (G(z) + r(z))m - \frac{\sigma^2}{2} m(m-1)]\phi(z) + C(z) = 0 .
\end{aligned} \tag{6.2}$$

Proposition 6.1 *Let us assume the following hypotheses:*

- $b_Z(z)$ *and* $\sigma_Z(z)$ *are Lipschitz continuous functions;*
- $r(z)$ *is continuous;*
- $g(z, u)$ *is continuous in* $z \in \mathbb{R}$, *uniformly in* $u \in [-U_1, U_2]$*;*
- $C(z)$ *is bounded and continuous.*

Then there exists a bounded classical solution $\phi \in C^2(\mathbb{R})$ to the ODE (6.2). Moreover, the following Feynman-Kac representation holds:

$$\phi(z) = \mathbb{E}\left[\int_0^{+\infty} e^{-\int_0^t \widetilde{\lambda}(\widetilde{Z}_s^z)ds} C(\widetilde{Z}_t^z)dt\right] \quad \forall z \in \mathbb{R}, \tag{6.3}$$

where $\widetilde{\lambda}(z) = \lambda - (G(z) + r(z))m - \frac{\sigma^2}{2}m(m-1)$ and $\{\widetilde{Z}_t^z\}_{t\geq 0}$ denotes the solution to the SDE:

$$d\widetilde{Z}_t^z = \widetilde{b}(\widetilde{Z}_t^z)\, dt + \sigma_Z(\widetilde{Z}_t^z)\, d\widetilde{W}_t \ , \qquad \widetilde{Z}_0^z = z \in \mathbb{R} \ , \tag{6.4}$$

with $\widetilde{W} = \{\widetilde{W}_t\}_{t\geq 0}$ a standard Brownian motion and $\widetilde{b}(z) = b_Z(z) + \rho\sigma\sigma_Z(z)m$.

Proof Since the coefficients of (6.2) are continuous functions and $C(z)$ is bounded, from classical results on linear ODEs there exists a bounded solution $\phi \in C^2(\mathbb{R})$. Let us define the process

$$Y_t = \int_0^t e^{-\int_0^s \widetilde{\lambda}(\widetilde{Z}_v^z)dv} C(\widetilde{Z}_s^z)ds + e^{-\int_0^t \widetilde{\lambda}(\widetilde{Z}_s^z)ds}\phi(\widetilde{Z}_t^z), \quad t \geq 0, \tag{6.5}$$

where $\{\widetilde{Z}_t^z\}_{t\geq 0}$ is the unique solution to SDE (6.4). By Itô's formula and Eq. (6.2), we get that

$$Y_t = \phi(z) + \int_0^t \left(\frac{1}{2}\sigma_Z^2(\widetilde{Z}_s^z)\phi''(\widetilde{Z}_s^z) + \widetilde{b}_Z(\widetilde{Z}_s^z)\phi'(\widetilde{Z}_s^z) - \widetilde{\lambda}(\widetilde{Z}_s^z)\phi(\widetilde{Z}_s^z) + C(\widetilde{Z}_s^z)\right)ds \tag{6.6}$$

$$+ \int_0^t e^{-\int_0^s \widetilde{\lambda}(\widetilde{Z}_v^z)dv}\phi'(\widetilde{Z}_s^z)\sigma_Z(\widetilde{Z}_s^z)dW_s \tag{6.7}$$

$$= \phi(z) + \int_0^t e^{-\int_0^s \widetilde{\lambda}(\widetilde{Z}_v^z)dv}\phi'(\widetilde{Z}_s^z)\sigma_Z(\widetilde{Z}_s^z)dW_s \ . \tag{6.8}$$

Let us introduce a sequence of stopping times $\{\tau_n\}_{n\geq 1}$ as follows

$$\tau_n = \inf\{s \geq 0 \mid \left|\widetilde{Z}_s^z - z\right| > n\} \ ,$$

then $\forall t \geq 0$ we get that

$$\mathbb{E}\left[\int_0^{t\wedge\tau_n} e^{-\int_0^t \widetilde{\lambda}(\widetilde{Z}_s^z)ds}(\phi'(\widetilde{Z}_t^z))^2\sigma_Z^2(\widetilde{Z}_t^z)dt\right] < \infty \qquad \forall n \in \mathbb{N} \ , \tag{6.9}$$

because ϕ' and σ_Z are continuous functions and $\widetilde{\lambda}(z) \geq \lambda - \lambda_m > 0\ \forall z \in \mathbb{R}$ (see Eq. (2.11)). As a consequence, $\{Y_{t\wedge\tau_n}\}_{t\geq 0}$ is a martingale, hence recalling (6.5) for any $n \in \mathbb{N}$

$$\phi(z) = \mathbb{E}\left[\int_0^{T\wedge\tau_n} e^{-\int_0^s \widetilde{\lambda}(\widetilde{Z}_v^z)dv} C(\widetilde{Z}_s^z)ds + e^{-\int_0^{T\wedge\tau_n} \widetilde{\lambda}(\widetilde{Z}_s^z)ds}\phi(\widetilde{Z}_{T\wedge\tau_n}^z)\right] \quad \forall T > 0 \ . \tag{6.10}$$

Letting $n \to +\infty$ and applying the dominated convergence theorem we have that

$$\phi(z) = \mathbb{E}\left[\int_0^{T} e^{-\int_0^s \widetilde{\lambda}(\widetilde{Z}_v^z)dv} C(\widetilde{Z}_s^z)ds + e^{-\int_0^{T} \widetilde{\lambda}(\widetilde{Z}_s^z)ds}\phi(\widetilde{Z}_t^z)\right] \quad \forall T > 0 \ . \tag{6.11}$$

Finally sending $T \to +\infty$ and observing that

$$\lim_{T\to+\infty} \mathbb{E}[e^{-\int_0^T \widetilde{\lambda}(\widetilde{Z}_s^z)ds}\phi(\widetilde{Z}_t^z)] = 0 \ ,$$

because ϕ is bounded, we get Eq. (6.3). □

Remark 6.1 When the Brownian motions W^Z and W driving the dynamics of the environmental stochastic factor Z and the debt-to-GDP process X are uncorrelated (i.e. $\rho = 0$), Eq. (6.3) reduces to

$$\phi(z) = \mathbb{E}\left[\int_0^{+\infty} e^{-\int_0^t \widetilde{\lambda}(Z_s^z)ds} C(Z_t^z)dt\right] \quad \forall z \in \mathbb{R}. \tag{6.12}$$

By applying Proposition 6.1 and the Verification Theorem 4.2 we solve the debt reduction problem in our framework.

Proposition 6.2 *Under the assumptions of Proposition 6.1, the value function is $v(x,z) = \phi(z)x^m \in C^{2,2}((0,+\infty)\times\mathbb{R})$, where ϕ given in Eq. (6.3) is a bounded classical solution to the ODE (6.2). Moreover, assuming $\frac{\partial^2 g}{\partial u^2}(z,u) < 0\ \forall (z,u) \in \mathbb{R}\times[-U_1,U_2]$ (or g satisfying (2.6)), we have that $u^* = \{u^*(Z_t^z)\}_{t\geq 0}$ with $u^*(z)$ given in (5.11) (or (5.12), respectively) is an optimal strategy.*

Proof First, let us observe that by Eq. (6.3) we get that ϕ is a strictly positive function. Then, the statement follows by Proposition 6.1, Theorem 4.2 and Proposition 5.5 (or Proposition 5.6, respectively), observing that $\frac{\partial v}{\partial x}(x,z) = (m-1)\phi(z)x^{m-1} > 0\ \forall (x,z) \in (0,+\infty)\times\mathbb{R}$. $\square$

We discuss in the next example the simplified case without the presence of the stochastic factor.

Example 6.1 Let us assume $r(z) = r$, $g(z,u) = g(u)$, $\sigma_Z = b_Z = 0$, $C(z) = C > 0$, hence $G(z) = G = \inf_{u\in[-U_1,U_2]}\{-(g(u)+u)\}$ and the HJB reads as

$$\frac{1}{2}\sigma^2 x^2 v''(x) + (G+r)xv'(x) + Cx^m - \lambda v(x) = 0\,. \tag{6.13}$$

It is easy to find an explicit solution to this equation and by the Verification Theorem 4.2 we get that the value function is given by $v(x) = kx^m$, with

$$k = \frac{C}{\lambda - (G+r)m - \frac{1}{2}m(m-1)\sigma^2} > 0\,.$$

As observed in Remark 5.2, the optimal strategy depends only on the form of the function $g(u)$. We discuss three cases below.

1. If $g''(u) < 0\ \forall u \in [-U_1,U_2]$, then by Proposition 5.5 the optimal control is $u^* = -U_1 \vee \hat{u} \wedge U_2$ where $\hat{u}$ the unique solution to $g'(u) = -1$ and, precisely, it is given by

$$u^* = \begin{cases} -U_1 & \text{if } g'(-U_1) < -1 \\ U_2 & \text{if } g'(U_2) > -1 \\ \hat{u} & \text{otherwise}\,. \end{cases} \tag{6.14}$$

2. In the case of (2.7), i.e. $g(u) = g_0 - \alpha u$, the optimal strategy is given by

$$u^* = \begin{cases} -U_1 & \text{if } \alpha > 1 \\ U_2 & \text{if } 0 < \alpha < 1\,. \end{cases} \tag{6.15}$$

   The considerations in Remark 5.1 apply and the optimal strategy is sustainable in some circumstances: when $\alpha > 1$ if $U_1 > -\frac{g_0}{\alpha-1}$ and when $0 < \alpha < 1$ if $U_2 > -\frac{g_0}{1-\alpha}$.
3. If $g(u) = g_0$, that is the GDP growth rate is not influenced by the fiscal policy, then the optimal strategy is $u^* = U_2$ (the government applies the maximum level of surplus) and it is sustainable if $U_2 > -g_0$.

6.2 Debt smoothing

In the previous section we assumed that a cost is associated to any increase of debt-to-GDP, because the aim was to reduce debt. However, in some cases an increase of debt could be more beneficial than its reduction. In [17] there is a noteworthy discussion of this topic, focusing on the trade-off between lowering public debt and building public infrastructure. Clearly, the latter should be the greater priority for countries with low debt and big infrastructure needs. Also, there are some unclear cases with high debt, no plausible risk of fiscal distress and some infrastructure needs. In these cases the optimal debt level is unclear and sometimes the benefit from increasing debt could surpass the advantages of debt reduction. This is even more true at the current juncture, given the very low level of real interest rates and the existence of demand shortfalls.

Another reason for the government to not let the public debt fall down to zero is related to the role of debt as savings absorber, see [11]. The government wishes to guarantee at any time a given amount of bonds in order to absorb the private savings in the financial market. Finally, it is well known that sudden and large shocks such as economic crises, wars or pandemics might cause spikes of public debt. “In developed countries [...] the aim of debt management was to smooth as much as possible the impact of such temporary expenditure shocks that were initially financed by raising debt” (see [9]).

Motivated by these considerations, in this section we address the debt smoothing problem. We assume that the government wishes to smooth the public debt-to-GDP by flattening its deviation from a given threshold $\bar{x} > 0$. Hence the cost is represented by the (quadratic) distance between the current debt-to-GDP and the target debt-to-GDP $\bar{x}$. Formally, we assume that the cost function is given by

$$f(x) = (x - \bar{x})^2 \, . \tag{6.16}$$

Recalling Proposition 3.1, we can derive the following properties of the value function $v(x, z)$:

- $v(x, z) \geq 0, \forall (x, z) \in (0, +\infty) \times \mathbb{R}$;
- $v(x, z) \leq \frac{x^2}{\lambda - \lambda_2} + \frac{\bar{x}^2}{\lambda} \; \forall (x, z) \in (0, +\infty) \times \mathbb{R}$;
- $\lim_{x \to 0^+} v(x, z) \leq \frac{\bar{x}^2}{\lambda} \; \forall z \in \mathbb{R}$.

Moreover, assuming that g is continuous in $u \in [-U1, U2]$ uniformly in z ∈ R, by Proposition 3.4, $v(x, z)$ is convex w.r.t $x \in (0, +\infty) \; \forall z \in \mathbb{R}$.

In order to provide an explicit solution to the HJB equation we discuss the simplified case without the stochastic factor. Precisely, we assume $\sigma_Z = b_Z = 0$, $r(z) = r$ and $g(z, u) = g(u)$. Then we denote $G_1 = min_{u \in [-U_1, U_2]} \{-(g(u) + u)\}$ and $G_2 = max_{u \in [-U_1, U_2]} \{-(g(u) + u)\}$, thus the HJB (5.2) reads as

$$\frac{1}{2} \sigma^2 x^2 v''(x) + (G_1 + r) x v'(x) - \lambda v(x) + (x - \bar{x})^2 = 0 \, , \tag{6.17}$$

for $v'(x) \geq 0$, and

$$\frac{1}{2} \sigma^2 x^2 v''(x) + (G_2 + r) x v'(x) - \lambda v(x) + (x - \bar{x})^2 = 0 \, , \tag{6.18}$$

for $v'(x) < 0$. Recalling that v' is increasing (because v is convex), we conjecture that there exists a threshold $\widetilde{x} > 0$ such that $v'(x) > 0 \, \forall x > \widetilde{x}$ and $v'(x) < 0 \, \forall x < \widetilde{x}$, so that we make the following ansatz:

$$v(x) = \begin{cases} a_1x^2 + b_1x + c_1 + d_1x^{\gamma_1}, & \text{if } x \geq \widetilde{x} \\ a_2x^2 + b_2x + c_2 + d_2x^{\gamma_2}, & \text{if } x < \widetilde{x}\ . \end{cases} \tag{6.19}$$

By substituting this expression in the Eqs. (6.17) and (6.18) above, we get that

$$a_i = \frac{1}{\lambda - 2(G_i + r) - \sigma^2},\ b_i = \frac{2\bar{x}}{G_i + r - \lambda},\ c_i = \frac{\bar{x}^2}{\lambda},\ i = 1, 2\ , \tag{6.20}$$

$$\gamma_1 = \frac{-[2(G_1 + r) - \sigma^2] - \sqrt{[2(G_1 + r) - \sigma^2]^2 + 4\lambda\sigma^2}}{2\sigma^2} < 0\ , \tag{6.21}$$

$$\gamma_2 = \frac{-[2(G_2 + r) - \sigma^2] + \sqrt{[2(G_2 + r) - \sigma^2]^2 + 4\lambda\sigma^2}}{2\sigma^2} > 0\ . \tag{6.22}$$

Condition (6.21) guarantees the quadratic growth, that is $v(x) \leq C(1+x^2)\ \forall x \in (0, +\infty)$, while condition (6.22) implies that $\lim_{x\to 0^+} v(x) = \frac{\bar{x}^2}{\lambda}$.

We also conjecture that v is twice continuously differentiable. Then the three constants $d_i, i = 1, 2$ and $\widetilde{x}$ can be found by taking the following conditions into account:

1. $v(\widetilde{x}^+) = v(\widetilde{x}^-)$;
2. $v'(\widetilde{x}^+) = v'(\widetilde{x}^-)$;
3. $v''(\widetilde{x}^+) = v''(\widetilde{x}^-)$.

In view of (6.19) these equations read as:

1. $a_1\widetilde{x}^2 + b_1\widetilde{x} + d_1\widetilde{x}^{\gamma_1} = a_2\widetilde{x}^2 + b_2\widetilde{x} + d_2\widetilde{x}^{\gamma_2}$;
2. $2a_1\widetilde{x} + b_1 + \gamma_1 d_1\widetilde{x}^{\gamma_1 - 1} = 2a_2\widetilde{x} + b_2 + d_2\gamma_2\widetilde{x}^{\gamma_2 - 1} = 0$;
3. $2a_1 + \gamma_1(\gamma_1 - 1)d_1\widetilde{x}^{\gamma_1 - 2} = 2a_2 + d_2\gamma_2(\gamma_2 - 1)\widetilde{x}^{\gamma_2 - 2}$.

Conditions 2 and 3 are equivalent to

$$d_1 = d_1(\widetilde{x}) = -\frac{2a_1\widetilde{x} + b_1}{\gamma_1\widetilde{x}^{\gamma_1 - 1}} \tag{6.23}$$

and

$$d_2 = d_2(\widetilde{x}) = \frac{2(a_1 - a_2)\widetilde{x} - (\gamma_1 - 1)(2a_1\widetilde{x} + b_1)}{\gamma_2(\gamma_2 - 1)\widetilde{x}^{\gamma_2 - 3}}\ , \tag{6.24}$$

respectively. Then the equation of item 1 can be rewritten as

$$a_1\widetilde{x}^2 + b_1\widetilde{x} + d_1(\widetilde{x})\widetilde{x}^{\gamma_1} = a_2\widetilde{x}^2 + b_2\widetilde{x} + d_2(\widetilde{x})\widetilde{x}^{\gamma_2}\ . \tag{6.25}$$

Clearly, we first need to compute $\widetilde{x}$ by solving numerically Eq. (6.25), then we obtain d_1 and d_2 by Eqs. (6.23) and (6.24), respectively.

Applying the Verification Theorem 4.2 we obtain the following result.

Proposition 6.3 *Let $a_i, b_i, c_i, \gamma_i, i = 1, 2$ given in Eqs.* (6.20) - (6.21) - (6.22)*, $\widetilde{x} \in (0, +\infty)$ solution to Eq.* (6.25)*, and d_1 and d_2 two constants determined by* (6.23) *and* (6.24)*, respectively. Define*

$$w(x) = \begin{cases} a_1x^2 + b_1x + c_1 + d_1x^{\gamma_1}, & \textit{if } x \geq \widetilde{x} \\ a_2x^2 + b_2x + c_2 + d_2x^{\gamma_2}, & \textit{if } x < \widetilde{x}\ . \end{cases} \tag{6.26}$$

If $\forall x \in (0, +\infty)$ $w''(x) > 0$ and (i) $g''(u) \neq 0$ $\forall u \in [-U_1, U_2]$, or (ii) g is given by (2.7)*, then w is the value function (i.e. $w = v$) and $u^* = \{u^*(X_t^{u^*})\}_{t\geq 0}$ is an optimal strategy, where the function $u^*(x)$ is given by:*

Case (i)

$$u^*(x) = \begin{cases} -U_1 & x \in \left\{x \in (0, +\infty) \mid 0 \geq w'(x)(g'(-U_1) + 1)\right\} \\ U_2 & x \in \left\{x \in (0, +\infty) \mid 0 \leq w'(x)(g'(U_2) + 1)\right\} \\ \hat{u} & \textit{otherwise} \ , \end{cases} \tag{6.27}$$

where $\hat{u}$ is the unique solution to

$$g'(u) = -1 \quad u \in [-U_1, U_2] \ .$$

Case (ii)

$$u^*(x) = \begin{cases} -U_1 & x \in \left\{x \in (0, +\infty) \mid w'(x)(\alpha - 1) \geq 0\right\} \\ U_2 & x \in \left\{x \in (0, +\infty) \mid w'(x)(\alpha - 1) < 0\right\} \ . \end{cases} \tag{6.28}$$

Proof It is sufficient to show that all the conditions of the Verification Theorem 4.2 are satisfied. First we prove that w is twice continuous differentiable, by construction we have that $w(\widetilde{x}^+) = w(\widetilde{x}^-)$, $w''(\widetilde{x}^+) = w''(\widetilde{x}^-)$ and $w'(\widetilde{x}^+) = 0$. Let us observe that $w'(\widetilde{x}^-) = w'(\widetilde{x}^+) = 0$ because w is twice differentiable on $(0, +\infty)$. Since w' is strictly increasing, we have that $w'(x) \geq 0$ for $x \geq \widetilde{x}$ and $w'(x) < 0$ for $x < \widetilde{x}$. w solves the HJB Eq. (5.2) and satisfies the quadratic growth condition by construction. Then the statement follows by Propositions 5.3, 5.4 and the Verification Theorem 4.2. □

Remark 6.2 Let us observe that (6.28) reads as:

- for $0 < \alpha < 1$:

$$u^*(x) = \begin{cases} U_2 & \text{if } x \geq \widetilde{x} \\ -U_1 & \text{if } x < \widetilde{x} \ ; \end{cases} \tag{6.29}$$

- for $\alpha > 1$

$$u^*(x) = \begin{cases} -U_1 & \text{if } x \geq \widetilde{x} \\ U_2 & \text{if } x < \widetilde{x} \ . \end{cases} \tag{6.30}$$

We can distinguish two cases. When the impact of fiscal policy on GDP growth is low (i.e. $0 < \alpha < 1$), the government applies the maximum surplus-to-debt, increasing taxes and decreasing the public spending, as long as the current debt-to-GDP ratio is over the threshold $\widetilde{x}$. When X_t is below $\widetilde{x}$, the maximum deficit-to-debt is applied. This simple rule is reversed when $\alpha > 1$, that is when the fiscal policy is more effective on GDP growth than on debt.

In the sequel we perform some numerical simulations to further investigate the results of Proposition 6.3. In particular, we refer to the case (ii), when the GDP growth rate takes the form of Eq. (2.7), that is

$$g(z, u) = g_0 - \alpha u \ .$$

We consider the parameters in Table 1 below as a reference scenario, unless otherwise specified.

The main task is to solve numerically (6.25). Then we can investigate how the result is sensitive to the model parameters.

For instance, given the parameters as in Table 1, we can find a solution to Eq. (6.25), that is $\tilde{x} = 0.6194$, so that the value function is well defined and it is illustrated in Fig. 1. The reader can easily notice that it is convex, as expected.

Table 1 Simulation parameters

Parameter	Value
r	0.01
g_0	0.03
σ	0.2
$\bar{x}$	0.6
$U_1 = U_2$	1
α	0.9
λ	5

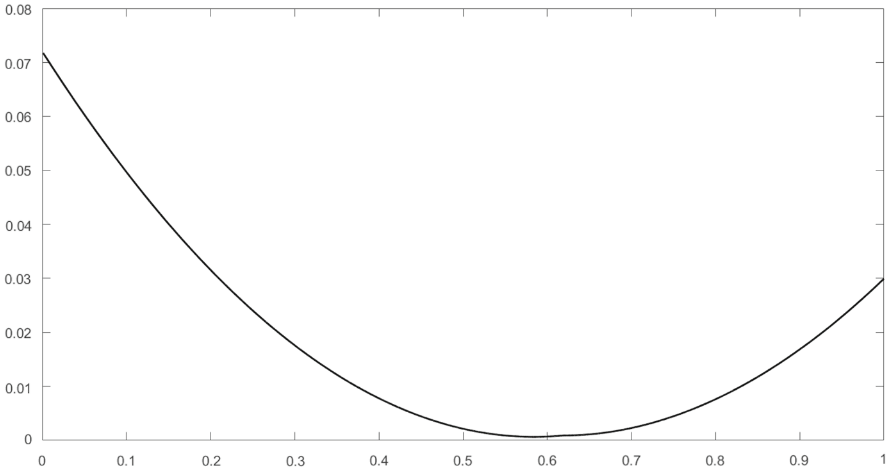

Fig. 1 The value function given by Eq. (6.26)

Table 2 Comparison between strong and weak economies

Parameters	Strong Economy	Weak Economy
$\alpha = 0.9$	$\tilde{x} = 0.6194$	$\tilde{x} = 0.6241$
$\alpha = 0.95$	$\tilde{x} = 0.6052$	$\tilde{x} = 0.5941$
$U_1 = U_2 = 0.8$	$\tilde{x} = 0.6125$	$\tilde{x} = 0.6068$
$U_1 = U_2 = 0.5$	$\tilde{x} = 0.6052$	$\tilde{x} = 0.5941$

Recalling (2.12) we get that the uncontrolled debt-to-GDP ratio is given by

$$X_t^{0,x} = xe^{(r-g_0-\frac{1}{2}\sigma^2)t+\sigma W_t}, \quad t \geq 0 ,$$

hence we call strong economy countries with parameters satisfying $g_0 + \frac{\sigma^2}{2} > r$ and weak economy when the opposite inequality holds, that is $g_0 + \frac{\sigma^2}{2} < r$ (e.g. in [7] the same definition is used).

In Table 2 we perform a comparison between countries with different economy parameters; precisely, we consider the parameters $r = 0.01$, $g_0 = 0.03$, $\sigma = 0.2$ corresponding to a strong economy and $r = 0.07$, $g_0 = 0.015$, $\sigma = 0.3$ for a weak economy.

First, we noticed that the solution $\tilde{x}$ is symmetric with respect to $\alpha = 1$, i.e. $\tilde{x}$ for $\alpha = 0.9$ and $\alpha = 0.95$ is the same as for $\alpha = 1.1$ and $\alpha = 1.05$, respectively. We can observe

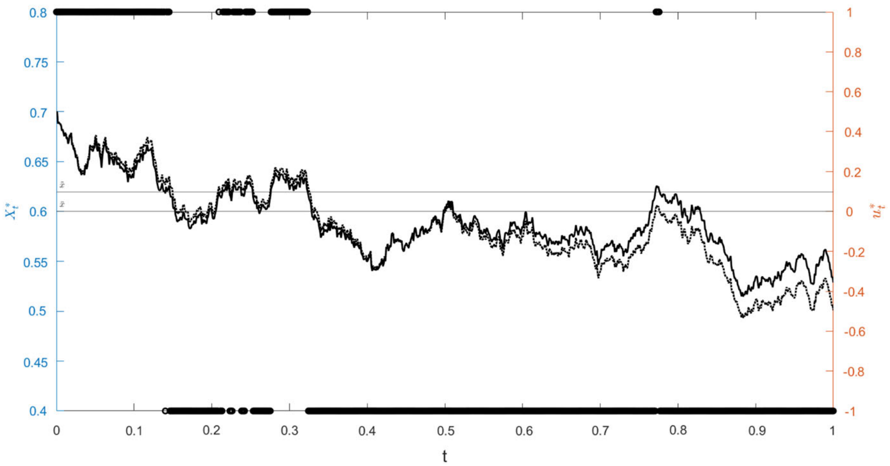

Fig. 2 A simulation of the debt-to-GDP ratio (left y axis) with the corresponding optimal strategy (right y axis). Here $\tilde{x} = 61.94\%$ and $\bar{x} = 0.60\%$. The solid line represents the controlled process, while the dotted line is the debt-to-GDP ratio without intervention

that when the effect of fiscal policy on GDP, which is measured by α, is closer to the effect on debt, which is a unitary, then the threshold $\tilde{x}$ decreases. In particular, for the weak economy this phenomenon is more evident and $\tilde{x}$ becomes lower than the target debt-to-GDP $\bar{x}$. Moreover, when the fiscal margin is reduced (U_1 and U_2 go from 1 to 0.8 and 0.5) we can see that $\tilde{x}$ decreases, becoming closer to $\bar{x}$. Hence, when the fiscal policy is less effective, the government should intervene even when minimal deviations from the target are observed.

To clarify the government behavior and give a deeper insight into Remark 6.2, we simulate a trajectory of debt-to-GDP ratio with the corresponding optimal strategy.

Figure 2 illustrates a trajectory of the (optimally) controlled debt-to-GDP. Recall that in this case $0 < \alpha < 1$, hence the optimal strategy follows Eq. (6.29). Let us recall that here we have $\tilde{x} = 61.94\%$, while our target is $\bar{x} = 60\%$. For instance, let us comment the first two months as illustrated in the picture. Starting from a debt-to-GDP ratio of 70%, the government applies the maximum surplus-to-debt U_2 according to the optimal strategy. We observe the debt-to-GDP decreasing and after 51 days the debt-to-GDP hits the threshold $\tilde{x}$, so that the maximum deficit-to-debt $-U_1$ is applied. Then the debt-to-GDP increases again and the government will stop spending when the threshold is hit again.

Conclusions

The present article introduces a new dynamic stochastic model for the debt-to-GDP ratio, see Eq. (2.3), trying to fill the gap between the economic theory and the mathematical model formulation. In particular, the expansionary/recessionary role of the fiscal policy on the GDP were not fully recognized by some recent research papers, while this phenomenon is well known in Economics. In this general framework, the Debt-to-GDP dynamics has an internal trade-off in addition to that induced by the objective function.

Our results give many interesting insights on the government interventions for managing public debt issues. When the cost function is independent of the fiscal policy u and the surplus/deficit has a nonlinear impact on the GDP growth, the government will select the

primary balance in order that the effect on GDP equals that on debt, as shown in Proposition 5.3. That is, the internal trade-off induced by the Debt-to-GDP dynamics prevails over the cost function, if the latter does not explicitly depend on the fiscal policy.
However, in some circumstances the effect of the fiscal policy on GDP would not match that on debt, because the room for maneuvre is limited, i.e. the interval $[-U_1, U_2]$ does not include the stationary point. In this case, the extreme policies (maximum deficit-to-debt or maximum surplus-to-debt) are chosen. In particular, when the cost functional does not depend on u and it is also increasing in x (e.g. for debt reduction), we proved in Proposition 5.5 that the maximum surplus-to-debt U_2 is applied only when its marginal negative effect on GDP is lower than the positive effect on debt reduction. Similarly, the maximum deficit-to-debt is optimal when its marginal beneficial effect on GDP surpasses the negative effect of increasing debt.
In Propositions 5.4 and 5.6 we show that when the effect of u on GDP is linear, a bang-bang strategy is obtained.

In general, it turns out that the impact of the fiscal policy on GDP is crucial to determine the optimal taxation/spending level and the corresponding optimal debt-to-GDP ratio. In particular, depending on the macroeconomic conditions (described by Z in our model) and the magnitude of the impact of the fiscal policy on GDP, the optimal debt management can be achieved by deficit policies in some circumstances. This is still true when the government goal is debt reduction, as shown in Sect. 6.1.

Since public debt management is a large and urgent topic, we expected an increasing interest on this field in the next years. In particular, we recognize at least two different directions for future researches. On the one hand, the model formulation could be further improved, taking into account the presence of more agents, e.g. a Central Bank, other countries, financial institutions or investors. On the other, starting from the model presented in this article, the study of specific debt management problems is of great interest. As a matter of fact, any specific form of the functional cost has peculiar properties and it can be associated to a specific economic problem, like those investigated in the last section of this paper in simplified frameworks.

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Consent for publication Not applicable.

A Proofs of secondary results

Proof of Theorem 4.1 We adapt the proof of [18, Chapter 4.3] to our framework. Proposition 3.5 ensures the continuity of v. Let $(\bar{x}, \bar{z}) \in (0, +\infty) \times \mathbb{R}$ and take a test function $\varphi \in \mathcal{C}^{2,2}((0, +\infty) \times \mathbb{R})$ such that

$$0 = (v - \varphi)(\bar{x}, \bar{z}) = \max_{(x,z)\in(0,+\infty)\times\mathbb{R}} (v - \varphi)(x, z) \; .$$

Let us observe that $v \leq \varphi$ by construction. Since v is continuous, there exists a sequence $\{(x_n, z_n)\}_{n\geq 1}$ such that

$$(x_n, z_n) \to (\bar{x}, \bar{z}) \qquad \text{and} \qquad v(x_n, z_n) \to v(\bar{x}, \bar{z}) \qquad \text{as } n \to +\infty \; .$$

Correspondingly, we must have that

$$\gamma_n \doteq v(x_n, z_n) - \varphi(x_n, z_n) \to 0 \qquad \text{as } n \to +\infty \; .$$

Now let us consider a control $\bar{u}_t = \bar{u}\ \forall t > 0$, for some arbitrary constant $\bar{u} \in [-U_1, U_2]$. Moreover, let introduce a sequence of stopping times $\{\tau_n\}_{n\geq 0}$ as follows:

$$\tau_n = \inf \left\{ s \geq 0 \mid \max \left\{ \left| X_s^{\bar{u},x_n} - x_n \right|, \left| Z_s^{z_n} - z_n \right| \right\} > \epsilon \right\} \wedge h_n \qquad n \geq 1 \; ,$$

for some fixed $\epsilon > 0$ and $\{h_n\}_{n\geq 1}$ such that

$$h_n \to 0 \; , \qquad \frac{\gamma_n}{h_n} \to 0 \qquad \text{as } n \to +\infty \; .$$

Here $\{Z_t^{z_n}\}_{t\geq 0}$ denotes the solution of the SDE (2.1) with initial condition $Z_0^{z_n} = z_n$. By the dynamic programming principle (see e.g. [18, Theorem 3.3.1]) for any $n \geq 1$ we have that

$$v(x_n, z_n) \leq \mathbb{E}\left[\int_0^{\tau_n} e^{-\lambda t} f(X_t^{\bar{u},x_n}, Z_t^{z_n}, \bar{u})\, dt + e^{-\lambda \tau_n} v(X_{\tau_n}^{\bar{u},x_n}, Z_{\tau_n}^{z_n}) \right] ,$$

hence

$$\varphi(x_n, z_n) + \gamma_n \leq \mathbb{E}\left[\int_0^{\tau_n} e^{-\lambda t} f(X_t^{\bar{u},x_n}, Z_t^{z_n}, \bar{u})\, dt + e^{-\lambda \tau_n} \varphi(X_{\tau_n}^{\bar{u},x_n}, Z_{\tau_n}^{z_n}) \right] .$$

Applying Itô's formula we get that

$$\begin{aligned} e^{-\lambda \tau_n} \varphi(X_{\tau_n}^{\bar{u},x_n}, Z_{\tau_n}^{z_n}) = \varphi(x_n, z_n) + \int_0^{\tau_n} & e^{-\lambda t} [\mathcal{L}^{\bar{u}} \varphi(X_t^{\bar{u},x_n}, Z_t^{z_n}) \\ & - \lambda \varphi(X_t^{\bar{u},x_n}, Z_t^{z_n})]\, dt + M_{\tau_n} \; , \end{aligned} \tag{A.1}$$

where

$$\begin{aligned} M_t = & \int_0^t e^{-\lambda s} \frac{\partial \varphi}{\partial x}(X_s^{\bar{u},x_n}, Z_s^{z_n}) \sigma X_s^{\bar{u},x_n}\, dW_s \\ & + \int_0^t e^{-\lambda s} \frac{\partial \varphi}{\partial z}(X_s^{\bar{u},x_n}, Z_s^{z_n}) \sigma_Z(Z_s^{z_n})\, dW_s^Z \; . \end{aligned}$$

Clearly $\{M_t\}_{t\geq 0}$ is a local martingale (having $\{\tau_n\}_{n\geq 0}$ as localizing sequence of stopping times) because the integrand functions are continuous and hence bounded on the compact sets. Taking expectations in Eq. (A.1) and using the previous inequality yields

$$\begin{aligned}
&\gamma_n + \varphi(x_n, z_n) \\
&\quad = \gamma_n - \mathbb{E}\left[\int_0^{\tau_n} e^{-\lambda t}[\mathcal{L}^{\bar{u}}\varphi(X_t^{\bar{u},x_n}, Z_t^{z_n}) - \lambda\varphi(X_t^{\bar{u},x_n}, Z_t^{z_n})]\,dt\right] \\
&\qquad + \mathbb{E}\left[e^{-\lambda\tau_n}\varphi(X_{\tau_n}^{\bar{u},x_n}, Z_{\tau_n}^{z_n})\right] \\
&\quad \le \mathbb{E}\left[\int_0^{\tau_n} e^{-\lambda t} f(X_t^{\bar{u},x_n}, Z_t^{z_n}, \bar{u})\,dt\right] + \mathbb{E}\left[e^{-\lambda\tau_n}\varphi(X_{\tau_n}^{\bar{u},x_n}, Z_{\tau_n}^{z_n})\right],
\end{aligned}$$

that is, dividing by h_n (using that $\tau_n \le h_n$),

$$\frac{\gamma_n}{h_n} \le \mathbb{E}\left[\frac{1}{h_n}\int_0^{\tau_n} e^{-\lambda t}[\mathcal{L}^{\bar{u}}\varphi(X_t^{\bar{u},x_n}, Z_t^{z_n}) + f(X_t^{\bar{u},x_n}, Z_t^{z_n}, \bar{u}) - \lambda\varphi(X_t^{\bar{u},x_n}, Z_t^{z_n})]\,dt\right].$$

Letting $n \to +\infty$ we have that $X_t^{\bar{u},x_n} \to X_t^{\bar{u},\bar{x}}$ and $Z_t^{z_n} \to Z_t^{\bar{z}}$, $\forall t \ge 0$ $\mathbb{P} - a.s.$ and the right-hand side converges to $\mathcal{L}^{\bar{u}}\varphi(\bar{x}, \bar{z}) + f(\bar{x}, \bar{z}, \bar{u}) - \lambda\varphi(\bar{x}, \bar{z})$ by the mean value theorem for integrals. Hence

$$\mathcal{L}^{\bar{u}}\varphi(\bar{x}, \bar{z}) + f(\bar{x}, \bar{z}, \bar{u}) - \lambda\varphi(\bar{x}, \bar{z}) \ge 0\,.$$

Since $\bar{u}$ is arbitrary, taking the infimum we obtain that v is a viscosity subsolution of Eq. (4.2) (see Eq. (4.3)).

Now we prove that v is a viscosity supersolution. To this end, we take a test function $\varphi \in \mathcal{C}^{2,2}((0, +\infty) \times \mathbb{R})$ such that

$$0 = (v - \varphi)(\bar{x}, \bar{z}) = \min_{(x,z)\in(0,+\infty)\times\mathbb{R}} (v - \varphi)(x, z)\,.$$

By definition of the value function, we can find a strategy $\{\hat{u}_t\}_{t\ge 0} \in \mathcal{U}$ such that

$$v(x_n, z_n) + h_n^2 \ge \mathbb{E}\left[\int_0^{\tau_n} e^{-\lambda t} f(X_t^{\hat{u},x_n}, Z_t^{z_n}, \hat{u}_t)\,dt + e^{-\lambda\tau_n} v(X_{\tau_n}^{\hat{u},x_n}, Z_{\tau_n}^{z_n})\right],$$

and hence

$$\varphi(x_n, z_n) + \gamma_n + h_n^2 \ge \mathbb{E}\left[\int_0^{\tau_n} e^{-\lambda t} f(X_t^{\hat{u},x_n}, Z_t^{z_n}, \hat{u}_t)\,dt + e^{-\lambda\tau_n} v(X_{\tau_n}^{\hat{u},x_n}, Z_{\tau_n}^{z_n})\right].$$

Using Eq. (A.1) and $v \ge \varphi$ we obtain that

$$\begin{aligned}
&\gamma_n + h_n^2 + \mathbb{E}\left[e^{-\lambda\tau_n}\varphi(X_{\tau_n}^{\hat{u},x_n}, Z_{\tau_n}^{z_n})\right] - \mathbb{E}\left[\int_0^{\tau_n} e^{-\lambda t}[\mathcal{L}^{\hat{u}}\varphi(X_t^{\hat{u},x_n}, Z_t^{z_n}) - \lambda\varphi(X_t^{\hat{u},x_n}, Z_t^{z_n})]\,dt\right] \\
&\ge \mathbb{E}\left[\int_0^{\tau_n} e^{-\lambda t} f(X_t^{\hat{u},x_n}, Z_t^{z_n}, \hat{u}_t)\,dt\right] + \mathbb{E}\left[e^{-\lambda\tau_n}\varphi(X_{\tau_n}^{\hat{u},x_n}, Z_{\tau_n}^{z_n})\right],
\end{aligned}$$

and dividing by h_n we get

$$\begin{aligned}
\frac{\gamma_n}{h_n} + h_n &\ge \mathbb{E}\left[\frac{1}{h_n}\int_0^{\tau_n} e^{-\lambda t}[\mathcal{L}^{\hat{u}}\varphi(X_t^{\hat{u},x_n}, Z_t^{z_n}) + f(X_t^{\hat{u},x_n}, Z_t^{z_n}, \hat{u}_t) - \lambda\varphi(X_t^{\hat{u},x_n}, Z_t^{z_n})]\,dt\right] \\
&\ge \mathbb{E}\left[\frac{1}{h_n}\int_0^{\tau_n} \inf_{u\in[-U_1,U_2]}\{\mathcal{L}^{u}\varphi(X_t^{u,x_n}, Z_t^{z_n}) + f(X_t^{u,x_n}, Z_t^{z_n}, u) - \lambda\varphi(X_t^{u,x_n}, Z_t^{z_n})\}\,dt\right].
\end{aligned}$$

Observing that

$$\lim_{n\to+\infty}\frac{\tau_n}{h_n} = \min\left\{\lim_{n\to+\infty}\frac{\inf\left\{s \ge 0 \mid \max\left\{\left|X_s^{\bar{u},x_n} - x_n\right|, \left|Z_s^{z_n} - z_n\right|\right\} \ge \epsilon\right\}}{h_n}, 1\right\} = 1\,,$$

using the mean value theorem for integrals again, we finally get the inequality

$$\inf_{u \in [-U_1, U_2]} \{\mathcal{L}^u \varphi(\bar{x}, \bar{z}) + f(\bar{x}, \bar{z}, u) - \lambda \varphi(\bar{x}, \bar{z})\} \leq 0 ,$$

and hence v is a viscosity supersolution of Eq. (4.2) (see Eq. (4.4)).

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