Stoch PDE: Anal Comp
https://doi.org/10.1007/s40072-022-00284-4

Degenerate Kolmogorov equations and ergodicity for the stochastic Allen–Cahn equation with logarithmic potential

Luca Scarpa[1] · **Margherita Zanella**[1]

Received: 21 July 2022 / Revised: 5 December 2022 / Accepted: 16 December 2022

Abstract
Well-posedness à la Friedrichs is proved for a class of degenerate Kolmogorov equations associated to stochastic Allen–Cahn equations with logarithmic potential. The thermodynamical consistency of the model requires the potential to be singular and the multiplicative noise coefficient to vanish at the respective potential barriers, making thus the corresponding Kolmogorov equation not uniformly elliptic in space. First, existence and uniqueness of invariant measures and ergodicity are discussed. Then, classical solutions to some regularised Kolmogorov equations are explicitly constructed. Eventually, a sharp analysis of the blow-up rates of the regularised solutions and a passage to the limit with a specific scaling yield existence à la Friedrichs for the original Kolmogorov equation.

Keywords Stochastic Allen–Cahn equation · Invariant measures · Ergodicity · Kolmogorov equations · Degenerate elliptic equations

Mathematics Subject Classification 35J70 · 37A25 · 37L40 · 47D07 · 47H06 · 60H15

Contents

✉ Luca Scarpa
luca.scarpa@polimi.it
https://sites.google.com/view/lucascarpa

Margherita Zanella
margherita.zanella@polimi.it

1 Department of Mathematics, Politecnico di Milano, Via E. Bonardi 9, 20133 Milan, Italy

Published online: 11 January 2023

1 Introduction

Modelling the evolution of multiphase materials - e.g. binary fluid mixtures, metallic alloys, heterogenous human tissues - has become fundamental in the last decades in numerous fields such as Material Science, Biology, and Engineering. One of the well-established mathematical ways of describing phase-separation is the so-called *diffuse interface*, or *phase-field*, approach. This consists in introducing a phase-variable u, or order parameter, with values in $[-1, 1]$: the regions $\{u = 1\}$ and $\{u = -1\}$ represent the pure phases, and it is assumed that there is a narrow blurred interfacial layer in between, where u can take also the intermediate values $(-1, 1)$. Such description has been firstly proposed by Cahn and Hilliard [15] to model conserved dynamics of spinodal decomposition in metallic alloys, and since then has been extensively employed in several contexts.

One of the classical phase-field models for non-conserved phase-separation is the Allen–Cahn equation: this has been originally introduced in the context of Van oder Waals theory of phase transition and has then been employed by Allen and Cahn in [3] for describing growth of grains in crystalline materials close to their melting points. In its classical form, the deterministic Allen–Cahn equation reads

$$\partial_t u - \nu \Delta u + F'(u) = f \qquad \text{in } (0, T) \times D, \tag{1.1}$$

where D is a smooth bounded domain in $\mathbb{R}^d$ $(d = 2, 3)$, $T > 0$ is a given reference time, f is a suitable forcing term, and $\nu > 0$ is a given constant depending on the structural data such as the thickness of the separation layer. The equation is usually complemented with a given initial datum, and homogeneous boundary conditions of Neumann or Dirichlet type. The nonlinearity F' represents the derivative of a double-well potential F, which is required to be singular at ± 1 by the thermodynamical consistency of the model: the relevant choice for F is indeed the so-called Flory-Huggins logarithmic potential [32] given by

$$F_{log}(r) := \frac{\theta}{2}\left[(1+r)\ln(1+r) + (1-r)\ln(1-r)\right] - \frac{\theta_0}{2} r^2, \quad r \in (-1, 1), \tag{1.2}$$

where $0 < \theta < \theta_0$ are fixed constant related to the critical temperature of the material in consideration. Note that $F_{\log}$ is continuous on $[-1, 1]$, with two global minima in $(-1, 1)$, while F'_{log} blows up at the potential barriers ± 1. This is coherent with the physical interpretation of diffuse-interface modelling in which only the values of the variable $u \in [-1, 1]$ are meaningful. The Allen–Cahn equation can also be seen as the gradient flow with respect to the $L^2(D)$-metric of the associated free-energy functional

$$\mathcal{E}(u) := \int_D \left(\frac{\nu}{2} |\nabla u|^2 + F(u) \right), \tag{1.3}$$

where the former energy contribution penalises for high oscillations of u while the latter takes into account the typical mixing/demixing effects.

Due to the singularity of the derivative F', for mathematical simplicity the double-well potential F is often approximated by a smooth one in polynomial form. Let us stress that although this may be useful in the mathematical treatment of the equation, it is a severe drawback on the modelling side: for example, such choice does not even ensure the preservation of the physically relevant bound $u \in [-1, 1]$ in general. For this reason, throughout the paper we deal only with thermodynamically relevant potentials such as the logarithmic one (1.2), as required by the model.

The deterministic Allen–Cahn equation provides a good description of the evolution of the phase separation. Nonetheless, it presents some disadvantages. Indeed, it is not general enough to capture possible unpredictable effects which may affect phase-separation, such as thermal fluctuations, magnetic disturbances, or microscopic configurational phenomena. These can be taken into account by adding a Wiener noise in the equation, as suggested originally in the well celebrated stochastic Cook model for phase-separation [17] and then confirmed in several contributions (see e.g. [10, 11]). By allowing for a stochastic Wiener-type forcing in (1.1), we deal with the stochastic Allen–Cahn equation in the general form

$$\begin{cases} \mathrm{d}u - \nu \Delta u \,\mathrm{d}t + F'(u)\,\mathrm{d}t = B(u)\,\mathrm{d}W & \text{in } (0, T) \times D, \\ \alpha_d u + \alpha_n \partial_{\mathbf{n}} u = 0 & \text{in } (0, T) \times \partial D, \\ u(0) = u_0 & \text{in } D, \end{cases} \tag{1.4}$$

where W is a cylindrical Wiener process defined on a certain separable Hilbert space and B is a suitable stochastically integrable operator with respect to W. The parameters $\alpha_d, \alpha_n \in \{0, 1\}$ are such that $\alpha_d + \alpha_n = 1$ and are thus responsible for the choice of Dirichlet or Neumann boundary conditions.

In the case of a logarithmic relevant potential (1.2), well-poseness for the stochastic Allen–Cahn equation with Neumann boundary conditions has been addressed for the first time in the very recent contribution [7]. Qualitative studies on the associated random separation principle have then been analysed in [8]. Roughly speaking, the novel idea to overcome the singularity of F' was to employ a degenerate noise coefficient B that vanishes at the potential barriers ± 1 in such a way to compensate the blow up of F'': existence of analytically strong solutions (see Definition 2.2 below) is obtained for initial data satisfying

$$u_0 \in V \cap \mathcal{A}, \quad \mathcal{A} := \left\{ v \in L^2(D) : |v(\mathbf{x})| \le 1 \quad \text{for a.e. } \mathbf{x} \in D \right\},$$

where V is either $H^1(D)$ or $H_0^1(D)$, depending on the boundary condition. The method is quite robust, in the sense that it has been applied also to different singular phase-field type equations: let us mention, above all, the contributions [25, 31] on the stochastic thin-film equation, [49] on the stochastic Cahn-Hilliard equation with degenerate mobility, and [6] on the stochastic Allen–Cahn equation with single obstacle potential.

In general, the mathematical literature on stochastic phase-field models is becoming increasingly popular, both in the analytical and probabilistic communities. We refer, for example, to the works [35, 44] on the stochastic Allen–Cahn equation, and to [20, 47, 48] on the stochastic Cahn–Hilliard equation, as well as to the references therein.

The aim and novelty of the present paper is to investigate the elliptic Kolmogorov equations associated to the stochastic dynamics given by (1.4) on the Hilbert space $H := L^2(D)$. The motivations are numerous. In particular, the Kolmogorov equation is intrinsically connected with the long-time behaviour of solutions and ergodicity of the stochastic system (1.4). Indeed, provided to prove existence of invariant measures for the associated transition semigroup, the Kolmogorov operator is the natural candidate to be its respective infinitesimal generator.

For the stochastic Allen–Cahn Eq. (1.4), setting $Q := BB^*$ the Kolmogorov equation reads

$$\alpha\varphi(x) - \frac{1}{2}\operatorname{Tr}\left[Q(x)D^2\varphi(x)\right] + \left(-\Delta x + F'(x), D\varphi(x)\right)_H = g(x), \qquad x \in \mathcal{A}_{str}, \tag{1.5}$$

where α is a fixed positive constant, $g \in C_b^0(H)$ is a given forcing, and

$$\mathcal{A}_{str} := \left\{ v \in V \cap \mathcal{A} : -\Delta v + F'(v) \in H \right\}.$$

Note that the nonlinear condition on $x \in \mathcal{A}_{str}$ is necessary. Indeed, the singularity of the derivative F' in (1.2) forces the solution u to take values in $(-1, 1)$: consequently, the respective Kolmogorov Eq. (1.5) only makes sense on the bounded subset $\mathcal{A}_{str}$ of H.

The main severely pathological behaviour of Eq. (1.5) is that the second-order diffusion operator is *not* uniformly elliptic in space: this is due to the degeneracy of B at the boundary $\partial\mathcal{A}$, which is needed in order to solve the SPDE (1.4), as pointed out above. Of course, such degeneracy has important consequences on the mathematical analysis of (1.5), as in general one cannot expect to obtain solutions with some reasonable space-regularity. This inevitably calls for the introduction of weaker notions of solutions which are better suited to incorporate such lack of control: the idea is to employ so-called solutions à la Friedrichs (see Proposition 4.4 below), which are defined, *roughly speaking*, as limits of classical solutions in suitable topologies. Clearly, one needs to properly identify which is the natural functional setting in order to pass to the limit. In this direction, a preliminary study on long-time behaviour and ergodicity for the transition semigroup associated to (1.4) reveals that every invariant measure is concentrated on the bounded subset $\mathcal{A}_{str}$. This suggests that the natural functional setting that allows to pass to the limit in the sense of Friedrichs is the one of Lebesgue spaces associated to some invariant measure for the SPDE (1.4), since invariant measures for (1.4) basically "ignore" the behaviour of φ outside $\mathcal{A}_{str}$.

The second difficulty that comes in play concerns the multiplicative nature of the covariance operator Q. Indeed, in order to pass to the limit in the sense of Friedrichs, one has to sharply balance the convergence of some regularised operators $Q_{\lambda,n}$ to Q with the explosion of the second derivatives of the respective classical solutions $\varphi_{\lambda,n}$, as the regularisation parameters λ and n vanish. Intuitively speaking, if one is able to show that the convergence rate of $Q_{\lambda,n} - Q$ dominates the explosion rate of $D^2\varphi_{\lambda,n}$, a passage to the limit yields existence of solutions for the limit Kolmogorov Eq. (1.5) à la Friedrichs. Of course, this calls for a sharp analysis on the explosion and convergence rates of the approximating classical solutions with respect to their respective regularising parameters.

The literature on long-time behaviour and ergodicity for stochastic systems is extremely developed. A very general study on ergodicity and Kolmogorov equations for stochastic evolution equations in variational form with additive noise was carried out by Barbu and Da Prato [5] in a very general setting. Still in the framework of variational approach to ergodicity of SDPEs, we can mention the contributions [41] on Poisson-type noise and [38] in the case of semilinear equations with singular drift. An extensive literature on ergodicity and Kolmogorov equations in the mild setting has been growing in the last decades, for which we refer to the works [16, 18, 22, 51]. In particular, in the context of semilinear reaction-diffusion equations existence and uniqueness of invariant measures, as well as moment estimates, are obtained in Ref. [29, 30]. For stochastic porous media equations we refer to the recent contribution [26]. Ergodicity for stochastic damped Schrödinger equation has been studied in [13, 14, 27], while long-time behaviour for Euler- and Navier-Stokes-type equations has been addressed, among many others, in [9, 21, 33, 34, 46].

Concerning Kolmogorov equations with degenerate covariance operator Q, well-posedness results are significantly less developed. To the best of our knowledge, the main available contributions so far concern the parabolic Kolmogorov equations associated to semilinear stochastic equations: through the notion of generalised solutions existence is obtained "by hand" via regular dependence of the SPDE on the initial datum, by exploiting some suitable smoothness assumptions on the nonlinearities in play. For further detail we refer the reader to [23, Sect. 7.5]. In the same spirit, parabolic Kolmogorov equations associated to stochastic PDEs with multiplicative noise are dealt with in Ref. [19], still under appropriate smoothness requirements on the coefficients or nondegeneracy conditions on the covariance. More generally, the study of Kolmogorov equations associated to stochastic PDEs has become crucial in the last years in the direction of uniqueness and regularisation by noise. Let us point out, above all, the recent contributions [42] on non-explosion for SDEs via Stratonovich noise and [1] on a BSDE approach to uniqueness by noise.

Let us conclude by briefly summarise the content of the paper. In Sect. 2 we introduce the mathematical setting, state the main assumptions, and recall the available well-posedness results. Section 3 is devoted then to the study of invariant measures and ergodicity for Eq. (1.4): in particular, we show existence of (possibly ergodic and strongly mixing) invariant measures, we provide sufficient conditions for uniqueness, and we characterise their support. In Sect. 4 we focus on the Kolmogorov equation associated to (1.4). In particular, we first introduce the Kolmogorov operator, as well as some suitable regularised Kolmogorov equations, depending on two approximating parameters. Secondly, we construct classical solutions to such regularised equations "by hand", by exploiting appropriate regular dependence on the initial data for the corresponding regularised SPDEs. Eventually, we obtain uniform estimates on the approximated solutions and sharp blow-up rates on their derivatives, allowing us to prove existence of solution for the original Eq. (1.5) through a passage to the limit on a specific scaling of the parameters. This shows well-posedness à la Friedrichs for the Kolmogorov equation, and characterises the Kolmogorov operator as the infinitesimal generator of the transition semigroup in some Lebesgue space associated to some invariant measure. Eventually, Appendices A–B contain useful estimates on the stochastic Allen–Cahn Eq. (1.4) and a density result used in the proofs, respectively.

2 Mathematical framework

2.1 Notation and setting

For any real Banach space E, we denote its dual by E^*. The duality pairing between E and E^* will be indicated by $\langle \cdot, \cdot \rangle_E$. For any real Hilbert space H we denote by $\|\cdot\|_H$ and $(\cdot, \cdot)_H$ the norm and the scalar product respectively. Given any two Banach spaces E and F, we use the symbol $\mathcal{L}(E, F)$ for the space of all linear bounded operators form E to F. Furthermore, we write $E \hookrightarrow F$, if E is continuously embedded in F. If H and K are separable Hilbert spaces, we employ the symbol $\mathcal{L}_{HS}(H, K)$ for the space of Hilbert-Schmidt operators from H to K. For any topological space E, the Borel σ-algebra on E is denoted by $\mathcal{B}(E)$. All measures on E are intended to be defined on its Borel σ-algebra. The spaces of bounded Borel-measurable and bounded continuous functions on E will be denoted by $B_b(E)$ and $C_b^0(E)$ respectively.

If $(A, \mathcal{A}, \mu)$ is a finite measure space, we denote by $L^p(A; E)$ the space of p-Bochner integrable functions, for any $p \in [1, \infty)$. For a fixed $T > 0$, we denote by $C^0([0, T]; E)$ the space of strongly continuous functions from $[0, T]$ to E.

If quantities $a, b \geq 0$ satisfy the inequality $a \leq C(A)b$ with a constant $C(A) > 0$ depending on the expression A, we write $a \lesssim_A b$; for a generic constant we put no subscript. If we have $a \lesssim_A b$ and $b \lesssim_A a$, we write $a \simeq_A b$.

Throughout the paper, $D \subset \mathbb{R}^d$, $d = 2, 3$, is a bounded domain with Lipschitz boundary Γ and Lebesgue measure denoted by $|D|$. The coefficients $\alpha_d, \alpha_n \in \{0, 1\}$ are such that $\alpha_d + \alpha_n = 1$: the case $(\alpha_d, \alpha_n) = (1, 0)$ corresponds to Dirichlet boundary conditions, while $(\alpha_d, \alpha_n) = (0, 1)$ yields Neumann boundary conditions. We introduce the functional spaces

$$H := L^2(D)$$

and

$$V := \begin{cases} H_0^1(D) & \text{if } (\alpha_d, \alpha_n) = (1, 0), \\ H^1(D) & \text{if } (\alpha_d, \alpha_n) = (0, 1), \end{cases}$$

$$Z := \begin{cases} H^2(D) \cap H_0^1(D) & \text{if } (\alpha_d, \alpha_n) = (1, 0), \\ \left\{ v \in H^2(D) : \partial_{\mathbf{n}} v = 0 \text{ a.e. on } \Gamma \right\} & \text{if } (\alpha_d, \alpha_n) = (0, 1). \end{cases}$$

all endowed with their natural respective norms $\|\cdot\|_H$, $\|\cdot\|_V$, and $\|\cdot\|_Z$. Identifying the Hilbert space H with its dual through the Riesz isomorphism, we have the following continuous, dense and compact inclusions

$$Z \hookrightarrow V \hookrightarrow H \simeq H^* \hookrightarrow V^* \hookrightarrow Z^*.$$

In particular, (V, H, V^*) constitutes a Gelfand triple. The norm of the continuous inclusion $V \hookrightarrow H$ will be denoted by K_0: note that K_0 can be estimated by means of Poincaré-type inequalities in terms of the first positive eigenvalue of the Laplacian.

We recall that the Laplace operator with homogeneous (Dirichlet or Neumann) conditions can be seen either as a variational operator

$$-\Delta \in \mathcal{L}(V, V^*), \qquad \langle -\Delta u, v \rangle_V := \int_D \nabla u \cdot \nabla v, \quad u, v \in V,$$

or as an unbounded linear operator on H with effective domain Z. In the sequel we will use the same symbol $-\Delta$ to denote the Laplace operator intended both as a variational operator and as an operator defined from Z with values in H.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, U a separable real Hilbert space, with a given orthonormal basis $(e_k)_{k\in\mathbb{N}}$, and W a canonical cylindrical Wiener processes taking values in U and adapted to a filtration $\mathbb{F}$ satisfying the usual conditions. Given $p, q \in [1, +\infty)$, $T > 0$, and a Banach space E, we denote by the symbol $L^p(\Omega; L^q(0, T; E))$ the space of E-valued progressively measurable processes $X : \Omega \times (0, T) \to E$ such that $\mathbb{E}\left(\int_0^T \|X(s)\|_E^q \,\mathrm{d}s\right)^{p/q} < +\infty$. When E is a separable Hilbert space, $p \in (1, +\infty)$, and $q = +\infty$, the symbol $L^p(\Omega; L^\infty(0, T; E^*))$ denotes the space of weak star measurable random variables $X : \Omega \to L^\infty(0, T; E^*)$ such that $\mathbb{E}\,\|X\|_{L^\infty(0,T;E^*)}^p < +\infty$, which by [28, Thm. 8.20.3] is isomorphic to the dual of $L^{\frac{p}{p-1}}\left(\Omega; L^1(0, T; E)\right)$.

2.2 Assumptions

Let us state the set of Assumptions that will be used throughout the paper. We work in an analogous framework as the one of [7].

H1 The potential $F : [-1, 1] \to [0, \infty)$ satisfies the following conditions:

(i) $F \in C^0([-1, 1]) \cap C^3(-1, 1)$ and $F'(0) = 0$,
(ii) there exists $K > 0$ such that $F''(r) \geq -K$ for all $r \in (-1, 1)$,

(iii) it holds that

$$\lim_{r\to(\pm1)^{\mp}} F'(r) = \pm\infty.$$

In this setting, note that conditions **(i)–(iii)** ensure the existence of constants $C_0, C_1 > 0$ such that

$$F'(r)r \geq C_0 r^2 - C_1. \tag{2.1}$$

It is straightforward to see that the logarithmic potential (1.2) (up to some additive constant) satisfies conditions **(i)–(iii)**.

H2 Let $\{h_k\}_{k\in\mathbb{N}} \subset C^1([-1,1])$ satisfy for every $k \in \mathbb{N}$ that $h_k(\pm1) = 0$ and

$$C_B := \sum_{k\in\mathbb{N}} \left(\|h_k\|^2_{C^1([-1,1])} + \left\| h_k^2 F'' \right\|_{L^\infty(-1,1)} \right) < \infty. \tag{2.2}$$

Setting $\mathcal{A} := \{v \in H : |v(\mathbf{x})| \leq 1 \text{ for a.e. } \mathbf{x} \in D\}$, condition (2.2) implies that the operator

$$B : \mathcal{A} \to \mathcal{L}_{HS}(U, H), \qquad B(x)e_k := h_k(x), \quad x \in \mathcal{A}, \quad k \in \mathbb{N}, \tag{2.3}$$

is well-defined and Lipschitz-continuous. Indeed, this amounts to saying that

$$B(x)e := \sum_{k\in\mathbb{N}} (e, e_k)_U h_k(x), \quad x \in \mathcal{A}, \ e \in U,$$

and (2.2) yields by a direct computation (see e.g. [7, Sect. 2]) that

$$\|B(x)\|^2_{\mathcal{L}_{HS}(U,H)} \leq C_B|D| \qquad \forall x \in \mathcal{A}, \tag{2.4}$$

$$\|B(x) - B(y)\|^2_{\mathcal{L}_{HS}(U,H)} \leq C_B|D| \|x - y\|^2_H \qquad \forall x, y \in \mathcal{A}. \tag{2.5}$$

2.3 Well posedness results

The existence and uniqueness of solutions to problem (1.4) is proved in Ref. [7] in the case of Neumann boundary conditions and exclusively for relevant case of logarithmic potential (1.2). One can easily check that the same results hold true in the case of Dirichlet boundary conditions and under the more general assumption **H1** for F, by using (2.2) (see e.g. [49] for details). We recall here the main well posedness results.

Definition 2.1 Let

$$u_0 \in L^2(\Omega, \mathscr{F}_0; H), \qquad \mathbb{P}\{u_0 \in \mathcal{A}\} = 1. \tag{2.6}$$

A variational solution to problem (1.4) is a process u such that, for every $T > 0$,

$$u \in L^2(\Omega; C([0,T]; H)) \cap L^2\left(\Omega; L^2(0,T; V)\right), \tag{2.7}$$

$$F'(u) \in L^2\left(\Omega; L^2(0,T; H)\right), \tag{2.8}$$

and for all $\psi \in V$ it holds that, for every $t \geq 0$, $\mathbb{P}$-a.s.,

$$\begin{aligned}\int_D u(t)\psi + \nu \int_0^t \int_D \nabla u(s) \cdot \nabla \psi(s)\, \mathrm{d}s + \int_0^t \int_D F'(u(s))\psi\, \mathrm{d}s \\ = \int_D u_0\psi + \int_D \left(\int_0^t B(u(s))\, \mathrm{d}W(s)\right)\psi.\end{aligned} \tag{2.9}$$

Definition 2.2 Let

$$u_0 \in L^2(\Omega, \mathscr{F}_0; V), \qquad \mathbb{P}\{u_0 \in \mathcal{A}\} = 1. \tag{2.10}$$

An analitically strong solution to problem (1.4) is a process u such that, for every $T > 0$,

$$u \in L^2(\Omega; C([0,T]; H)) \cap L^2\left(\Omega; L^\infty(0,T; V)\right) \cap L^2\left(\Omega; L^2(0,T; Z)\right), \tag{2.11}$$

$$F'(u) \in L^2(\Omega; L^2(0,T; H)), \tag{2.12}$$

and it holds that, for every $t \geq 0$, $\mathbb{P}$-a.s.,

$$u(t) - \nu \int_0^t \Delta u(s)\, \mathrm{d}s + \int_0^t F'(u(s))\, \mathrm{d}s = u_0 + \int_0^t B(u(s))\, \mathrm{d}W(s). \tag{2.13}$$

The well-posedness result following from [7, Thm. 2.1] is the following.

Theorem 2.3 *Assume* **H1–H2**. *For every u_0 satisfying (2.6) there exists a unique variational solution to (1.4) in the sense of Definition 2.1. Furthermore, for every $T > 0$ there exists a positive constant C_T such that, for every initial data u_0^1, u_0^2 satisfying (2.6) the respective variational solutions u_1, u_2 of (1.4) satisfy*

$$\|u_1 - u_2\|_{L^2(\Omega; C([0,T], H)) \cap L^2(\Omega; L^2(0,T,V))} \leq C_T \|u_0^1 - u_0^2\|_{L^2(\Omega; H)}. \tag{2.14}$$

Moreover, for every u_0 satisfying (2.10) there exists a unique analytically strong solution to (1.4) in the sense of Definition 2.2.

3 Invariant measures

This section is devoted to the long-time analysis of the stochastic Eq. (1.4), in terms of existence-uniqueness of invariant measures and ergodicity. Before moving on, we recall some general definitions that will be used in the sequel.

For every $x \in \mathcal{A}$, the unique variational solution to Eq. (1.4) as given in Theorem 2.3 will be denoted by u^x, and for every $t \in [0, T]$ we set $u(t; x) := u^x(t)$ for its value at time t. Note that for every $t \in [0, T]$ $u(t; x) : \Omega \to H$ is a random variable in $L^2(\mathscr{F}_t; H)$.

The main issue in defining the concept of invariant measure in our framework is that Eq. (1.4) can be solved only if the initial datum satisfies a nonlinear type condition (see (2.6)). In this direction, it is useful to extend F to $+\infty$ outside $[-1, 1]$, and obtain a proper convex lower semicontinuous function $F : \mathbb{R} \to [0, +\infty]$. With this notation, we have the characterisation (see again **H2**)

$$\mathcal{A} = \left\{x \in H : \|x\|_{L^\infty(D)} \le 1\right\} = \left\{x \in H : F(x) \in L^1(D)\right\}. \tag{3.1}$$

We claim that $\mathcal{A}$ is a Borel subset of H. Indeed, one has that

$$\mathcal{A} = \bigcup_{n\in\mathbb{N}} \left\{x \in H : \int_D F(x) \le n\right\},$$

where the right-hand side is a countable union of closed sets in H by lower semicontinuity of F, hence is a Borel subset of H. The equality in (3.1) follows from the fact that the domain of F is exactly $[-1, 1]$.

We consider on $\mathcal{A}$ the metric d given by the restriction to $\mathcal{A}$ of the metric on H induced by the H-norm. Being $\mathcal{A}$ a closed subspace of the complete separable metric space $(H, \|\cdot\|_H)$, $(\mathcal{A}, \mathsf{d})$ is also complete and separable.

The space $(\mathcal{A}, \mathsf{d})$ is therefore a separable complete metric space. We denote by $\mathscr{B}(\mathcal{A})$ the σ-algebra of all Borel subsets of $\mathcal{A}$ and by $\mathscr{P}(\mathcal{A})$ the set of all probability measures on $(\mathcal{A}, \mathscr{B}(\mathcal{A}))$. Also, the symbol $\mathscr{B}_b(\mathcal{A})$ denotes the space of Borel measurable bounded functions from $\mathcal{A}$ to $\mathbb{R}$. If $A \in \mathscr{B}(\mathcal{A})$, we denote by A^C its complement.

With this notation and by virtue of Theorem 2.3, we can introduce the family of operators $P := (P_t)_{t\ge0}$ associated to Eq. (1.4) as

$$(P_t\varphi)(x) := \mathbb{E}[\varphi(u(t; x))], \quad x \in \mathcal{A}, \quad \varphi \in \mathscr{B}_b(\mathcal{A}). \tag{3.2}$$

Remark 3.1 Let us point out once more that, due to the nonlinear nature of the problem, the solution of Eq. (1.4) exists on $\mathcal{A}$, hence the transition semigroup can only make sense as a family of operators acting on $\mathscr{B}_b(\mathcal{A})$, and not on $\mathscr{B}_b(H)$ as in more classical cases.

It is clear that $P_t\varphi$ is bounded for every $\varphi \in \mathscr{B}_b(\mathcal{A})$. We know from [43, Cor. 23] that the transition function is jointly measurable, that is for any $A \in \mathscr{B}(\mathcal{A})$ the map

$\mathcal{A} \times [0, \infty) \ni (x, t) \mapsto \mathbb{P}\{u(t; x) \in A\} \in \mathbb{R}$ is measurable. So $P_t\varphi$ is also measurable for every $\varphi \in \mathcal{B}_b(\mathcal{A})$, hence P_t maps $\mathscr{B}_b(\mathcal{A})$ into itself for every $t \geq 0$. Furthermore, since the unique solution of (1.4) is an H-valued continuous process, then it is also a Markov process, see [43, Theorem 27]. Therefore we deduce that the family of operators $\{P_t\}_{t\geq 0}$ is a Markov semigroup, namely $P_{t+s} = P_t P_s$ for any $s, t \geq 0$.

We are ready to give the precise definition of invariant measure.

Definition 3.2 An invariant measure for the transition semigroup P is a probability measure $\mu \in \mathscr{P}(\mathcal{A})$ such that

$$\int_{\mathcal{A}} \varphi(x)\, \mu(\mathrm{d}x) = \int_{\mathcal{A}} P_t\varphi(x)\, \mu(\mathrm{d}x) \quad \forall\, t \geq 0, \quad \forall\, \varphi \in C_b(\mathcal{A}).$$

3.1 Existence of an invariant measure

We focus here on showing that P admits at least an invariant measure. The main idea is to use an adaptation of the Krylov-Bogoliubov theorem to the case of complete separable metric spaces, which we prove here for clarity. The proof is an adaptation of the one in the more classical Hilbert space setting, which can be found in [24, Thm. 11.7].

Theorem 3.3 (Krylov-Bogoliubov) *Let $R := \{R_t\}_{t\geq 0}$ be a time-homogeneous Markov semigroup on the complete separable metric space $(\mathcal{A}, \mathsf{d})$. Assume that*

(i) *the semigroup $\{R_t\}_{t\geq 0}$ is Feller in $\mathcal{A}$;*
(ii) *for some $x_0 \in \mathcal{A}$, the set $(\mu_t)_{t>0} \subset \mathscr{P}(\mathcal{A})$ given by*

$$\mu_t(A) := \frac{1}{t}\int_0^t (R_s \mathbf{1}_A)(x_0)\, \mathrm{d}s, \qquad A \in \mathscr{B}(\mathcal{A}),\ t > 0, \tag{3.3}$$

is tight. Then there exists at least one invariant measure for R.

Proof By the Prokhorov Theorem (see e.g. [12, Vol. II, Thm 8.6.2]) there exists a subsequence $\{t_n\}_n$ with $\lim_{n\to\infty} t_n = \infty$ and a probability measure $\mu \in \mathscr{P}(\mathcal{A})$ such that

$$\lim_{n\to\infty} \int_{\mathcal{A}} \varphi(x)\, \mu_{t_n}(\mathrm{d}x) = \int_{\mathcal{A}} \varphi(x)\, \mu(\mathrm{d}x) \qquad \forall\, \varphi \in C_b(\mathcal{A}).$$

By (3.3) and the Fubini Theorem the above expression is equivalent to

$$\lim_{n\to\infty} \frac{1}{t_n}\int_0^{t_n} (R_t\varphi)(x_0)\, \mathrm{d}t = \int_{\mathcal{A}} \varphi(x)\, \mu(\mathrm{d}x) \qquad \forall\, \varphi \in C_b(\mathcal{A}). \tag{3.4}$$

Given $s \geq 0$ and $\psi \in C_b(\mathcal{A})$, we have that $R_s\psi \in C_b(\mathcal{A})$ by the Feller property. Hence, we can choose $\varphi = R_s\psi$ in (3.4) and infer that

$$\lim_{n\to\infty} \frac{1}{t_n}\int_0^{t_n} (R_{t+s}\psi)(x_0)\, \mathrm{d}t = \int_{\mathcal{A}} R_s\psi(x)\, \mu(\mathrm{d}x) \qquad \forall\, \psi \in C_b(\mathcal{A}). \tag{3.5}$$

Now, bearing in mind equality (3.4) we have

$$\frac{1}{t_n}\int_0^{t_n}(R_{t+s}\psi)(x_0)\,\mathrm{d}t = \frac{1}{t_n}\int_s^{s+t_n}(R_t\psi)(x_0)\,\mathrm{d}t$$
$$= \frac{1}{t_n}\int_0^{t_n}(R_t\psi)(x_0)\,\mathrm{d}t + \frac{1}{t_n}\int_{t_n}^{s+t_n}(R_t\psi)(x_0)\,\mathrm{d}t - \frac{1}{t_n}\int_0^{s}(R_t\psi)(x_0)\,\mathrm{d}t$$
$$\longrightarrow \int_{\mathcal{A}}\psi(x)\,\mu(\mathrm{d}x) \qquad \text{as } t_n\to\infty.$$

Taking this into account in the left had side of (3.5) shows that μ is invariant. □

We are now ready to show that the transition semigroup P of Eq. (1.4) admits invariant measures.

Theorem 3.4 *Assume* **H1–H2**. *Then, the transition semigroup P is Feller and admits at least one invariant measure.*

Proof The result is a consequence of the Krylov-Bougoliuov Theorem 3.3, provided that we check that P is Feller and the tightness property.
(i). Let us show first that P is Feller: this follows directly the continuous dependence of the solution on the initial data. Indeed, let $t > 0$ and $\varphi \in C_b(\mathcal{A})$ be fixed. We have to prove that, given a sequence $(x_n)_n \subset \mathcal{A}$ which converges in $\mathcal{A}$ to $x \in \mathcal{A}$ as $n\to\infty$, the sequence $P_t\varphi(x_n)$ converges to $P_t\varphi(x)$ as $n\to\infty$. As a consequence of the continuous dependence property w.r.t. the initial datum (2.14), we have that

$$\|u(t;x_n)-u(t;x)\|_{L^2(\Omega;H)} \le \|u^{x_n}-u^x\|_{L^2(\Omega;C([0,t];H))} \le C_t\|x_n-x\|_H$$

It follows that, as $n\to\infty$, $u(t;x_n)\to u(t;x)$ in $L^2(\Omega;H)$, hence also in probability. This in turn implies that $\varphi(u(t;x_n))\to\varphi(u(t;x))$ in probability by the continuity of φ. The boundedness of φ and the Vitali Theorem yield in particular that $\varphi(u(t;x_n))\to\varphi(u(t;x))$ in $L^1(\Omega)$, and thus

$$|(P_t\varphi)(x_n)-(P_t\varphi)(x)| \le \mathbb{E}\left[|\varphi(u(t;x_n))-\varphi(u(t;x))|\right]\to 0,$$

as $n\to\infty$. This shows that P is Feller.
(ii). We prove now that P satisfies the tightness property of Theorem 3.3. To this end, let $x_0 = 0\in\mathcal{A}$ and let u^{x_0} be the corresponding variational solution of problem (1.4). We are going to show that the family of measures $(\mu_t)_{t>0}\subset\mathscr{P}(\mathcal{A})$ defined by

$$\mu_t : A\mapsto \frac{1}{t}\int_0^t(P_s\mathbf{1}_A)(0)\,\mathrm{d}s = \frac{1}{t}\int_0^t\mathbb{P}\{u(t;0)\in A\}\,\mathrm{d}s, \qquad A\in\mathscr{B}(\mathcal{A}),\ t>0,$$

is tight. Let B_n be the closed ball in V of radius $n\in\mathbb{N}$, and set $\bar{B}_n := B_n\cap\mathcal{A}$. Then $\bar{B}_n$ is a compact subset of $\mathcal{A}$ since the embedding $V\hookrightarrow H$ is compact. Hence, Lemma A.1 and the Chebychev inequality yield, for any $t > 0$,

$$\mu_t(\bar{B}_n^C) = \frac{1}{t}\int_0^t (P_s \mathbf{1}_{\bar{B}_n^C})(0)\,\mathrm{d}s = \frac{1}{t}\int_0^t \mathbb{P}\left\{\|u(s;0)\|_V^2 \geq n^2\right\}\mathrm{d}s$$
$$\leq \frac{1}{tn^2}\int_0^t \mathbb{E}\|u(s;0)\|_V^2\,\mathrm{d}s \lesssim_{C_1,C_B,|D|,\nu} \frac{1}{n^2},$$

from which

$$\sup_{t>0} \mu_t(B_n^C) \lesssim_{C_1,C_B,|D|,\nu} \frac{1}{n^2} \to 0 \qquad \text{as } n \to \infty,$$

and the thesis follows. □

3.2 Support of the invariant measures

Once existence of invariant measures is establishes, we focus here on some qualitative properties of the invariant measures concerning their support. In particular, we show that every invariant measure is supported in a more regular set than just $\mathcal{A}$. To this end, we introduce the set

$$\mathcal{A}_{str} := \left\{x \in \mathcal{A} \cap Z : \ F'(x) \in H\right\}. \tag{3.6}$$

Proceeding as for (3.1) and exploiting the lower semicontinuity of $|F'|$, one can show that $\mathcal{A}_{str}$ is a Borel subset of H, hence of $\mathcal{A}$.

Proposition 3.5 *Assume* **H1–H2**. *Then, there exists a constant $C > 0$, only depending on C_0, C_1, C_B, $|D|$, ν, K, and K_0, such that every invariant measure $\mu \in \mathscr{P}(\mathcal{A})$ for the transition semigroup P satisfies*

$$\int_{\mathcal{A}} \left(\|x\|_Z^2 + \|F'(x)\|_H^2\right) \mu(\mathrm{d}x) \leq C. \tag{3.7}$$

In particular, every invariant measure μ is supported in $\mathcal{A}_{str}$, i.e. $\mu(\mathcal{A}_{str}) = 1$.

Proof Let $\mu \in \mathscr{P}(\mathcal{A})$ be an invariant measure for the transition semigroup P.
Step 1. First we note that the definition of $\mathcal{A}$ itself trivially implies that μ has finite moments of any order on H. More precisely, it holds that

$$\|x\|_{L^\infty(D)} \leq 1 \quad \forall x \in \mathcal{A},$$

which readily ensures by the embedding $L^\infty(D) \hookrightarrow H$ that

$$\int_{\mathcal{A}} \|x\|_H^2\,\mathrm{d}\mu(x) \leq |D|. \tag{3.8}$$

STEP 2. Now we show that

$$\int_{\mathcal{A}} \|x\|_V^2 \,\mu(\mathrm{d}x) \le C. \tag{3.9}$$

To this end, we consider the mapping $\Phi : \mathcal{A} \to [0, +\infty]$ defined as

$$\Phi : x \mapsto \|x\|_V^2 \mathbf{1}_{\mathcal{A}\cap V}(x) + \infty\, \mathbf{1}_{\mathcal{A}\cap V^c}(x), \quad x \in \mathcal{A},$$

and its approximations $\{\Phi_n\}_{n\in\mathbb{N}}$, where for every $n \in \mathbb{N}$ $\Phi_n : \mathcal{A} \to [0, n^2]$ is defined (setting B_n^V as the closed ball of radius n in V) as

$$\Phi_n : x \mapsto \begin{cases} \|x\|_V^2 & \text{if } x \in B_n^V \cap \mathcal{A}, \\ n^2 & \text{otherwise,} \end{cases} \qquad x \in \mathcal{A}.$$

It is not difficult to check that actually $\Phi_n \in \mathscr{B}_b(\mathcal{A})$ for every $n \in \mathbb{N}$. Hence, exploiting the invariance of μ, the boundedness of Φ_n, the Definition (3.2) of P, and the Fubini-Tonelli Theorem we have that

$$\begin{aligned} \int_{\mathcal{A}} \Phi_n(x)\,\mu(\mathrm{d}x) &= \int_0^1 \int_{\mathcal{A}} \Phi_n(x)\,\mu(\mathrm{d}x)\,\mathrm{d}s = \int_0^1 \int_{\mathcal{A}} P_s\Phi_n(x)\,\mu(\mathrm{d}x)\,\mathrm{d}s \\ &= \int_0^1 \int_{\mathcal{A}} \mathbb{E}\left[\Phi_n(u(s;x))\right]\,\mu(\mathrm{d}x)\,\mathrm{d}s = \int_{\mathcal{A}} \int_0^1 \mathbb{E}\left[\Phi_n(u(s;x))\right]\,\mathrm{d}s\,\mu(\mathrm{d}x). \end{aligned}$$

Since

$$\Phi_n(\cdot) = \|\cdot\|_V^2 \wedge n^2 \le \|\cdot\|_V^2,$$

by Lemma A.1 and (3.8) we infer that

$$\begin{aligned} \int_{\mathcal{A}} \Phi_n(x)\,\mu(\mathrm{d}x) &\le \int_{\mathcal{A}} \int_0^1 \mathbb{E}\left[\|u(s;x)\|_V^2\right]\,\mathrm{d}s\,\mu(\mathrm{d}x) \\ &\lesssim \int_{\mathcal{A}} \|x\|_H^2\,\mu(\mathrm{d}x) + 1 \le C. \end{aligned}$$

Since Φ_n converges pointwise and monotonically from below to Φ, the Monotone Convergence Theorem yields (3.9).

STEP 3. We prove now that

$$\int_{\mathcal{A}} \|x\|_Z^2 \,\mu(\mathrm{d}x) \le C. \tag{3.10}$$

To this end, we argue as in STEP 2, considering the map $\Psi : \mathcal{A} \to [0, +\infty]$ defined as

$$\Psi : x \mapsto \|x\|_Z^2 \mathbf{1}_{\mathcal{A}\cap Z}(x) + \infty\, \mathbf{1}_{\mathcal{A}\cap Z^c}(x), \quad x \in \mathcal{A},$$

and its approximations $\{\Psi_n\}_{n\in\mathbb{N}}$, where for every $n \in \mathbb{N}$ $\Psi_n : \mathcal{A} \to [0, n^2]$ is defined (setting B_n^Z as the closed ball of radius n in Z) as

$$\Psi_n : x \mapsto \begin{cases} \|x\|_Z^2 & \text{if } x \in B_n^Z \cap \mathcal{A}, \\ n^2 & \text{otherwise,} \end{cases} \qquad x \in \mathcal{A}.$$

Again, one has that $\Psi_n \in \mathscr{B}_b(\mathcal{A})$ for every $n \in \mathbb{N}$. Hence, arguing as above by using the invariance of μ, the boundedness of Ψ_n, the definition of P, and the Fubini-Tonelli Theorem, exploiting the fact that μ is concentrated on $\mathcal{A} \cap V$ by (3.9) yields

$$\begin{aligned} \int_{\mathcal{A}} \Psi_n(x)\,\mu(\mathrm{d}x) = \int_{\mathcal{A}\cap V} \Psi_n(x)\,\mu(\mathrm{d}x) &= \int_{\mathcal{A}\cap V} \int_0^1 \mathbb{E}\left[\Psi_n(u(s;x))\right] \mathrm{d}s\,\mu(\mathrm{d}x) \\ &\le \int_{\mathcal{A}\cap V} \int_0^1 \mathbb{E}\left[\|u(s;x)\|_Z^2\right] \mathrm{d}s\,\mu(\mathrm{d}x). \end{aligned}$$

Lemma A.2 together with the estimate (3.9) entail then

$$\int_{\mathcal{A}} \Psi_n(x)\,\mu(\mathrm{d}x) \lesssim \int_{\mathcal{A}} \|x\|_V^2\,\mu(\mathrm{d}x) + 1 \le C.$$

The Monotone Convergence Theorem establish then (3.10).

STEP 4. Eventually, we show here that

$$\int_{\mathcal{A}} \|F'(x)\|_H^2\,\mu(\mathrm{d}x) \le C \tag{3.11}$$

by arguing as above. Define $\Lambda : \mathcal{A} \to [0, +\infty]$ as

$$\Lambda : x \mapsto \begin{cases} \|F'(x)\|_H^2 & \text{if } F'(x) \in H, \\ +\infty & \text{otherwise,} \end{cases} \qquad x \in \mathcal{A},$$

and its approximations $\{\Lambda_n\}_{n\in\mathbb{N}}$, where for every $n \in \mathbb{N}$ $\Lambda_n : \mathcal{A} \to [0, n^2]$ is defined as

$$\Lambda_n : x \mapsto \begin{cases} \|F'(x)\|_H^2 & \text{if } \|F'(x)\|_H \le n, \\ n^2 & \text{otherwise,} \end{cases} \qquad x \in \mathcal{A}.$$

As above, it holds that $\Lambda_n \in \mathscr{B}_b(\mathcal{A})$ for every $n \in \mathbb{N}$. Using the the invariance of μ, the boundedness of Λ_n, the definition of P, the Fubini-Tonelli Theorem, and the fact that μ is concentrated on $\mathcal{A} \cap V$, we infer that

$$\begin{aligned} \int_{\mathcal{A}} \Lambda_n(x)\,\mu(\mathrm{d}x) = \int_{\mathcal{A}\cap V} \Lambda_n(x)\,\mu(\mathrm{d}x) &= \int_{\mathcal{A}\cap V} \int_0^1 \mathbb{E}\left[\Lambda_n(u(s;x))\right] \mathrm{d}s\,\mu(\mathrm{d}x) \\ &\le \int_{\mathcal{A}\cap V} \int_0^1 \mathbb{E}\left[\|F'(u(s;x))\|_H^2\right] \mathrm{d}s\,\mu(\mathrm{d}x). \end{aligned}$$

At this point, Lemma A.3 implies directly that

$$\int_{\mathcal{A}} \Lambda_n(x)\, \mu(\mathrm{d}x) \le C,$$

and (3.11) follows from the Monotone Convergence Theorem. □

3.3 Existence of an ergodic invariant measure

Let us recall first the definition of ergodicity for the transition semigroup P. In this direction, note that for every invariant measure μ, by density and by definition of invariance the semigroup P can be extended (with the same symbol for brevity) to a strongly continuous linear semigroup of contractions on $L^p(\mathcal{A}, \mu)$ for every $p \in [1, +\infty)$.

Definition 3.6 An invariant measure $\mu \in \mathscr{P}(\mathcal{A})$ for the semigroup P is said to be ergodic if

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t P_s\varphi \, \mathrm{d}s = \int_{\mathcal{A}} \varphi(x)\, \mu(\mathrm{d}x) \qquad \text{in } L^2(\mathcal{A}, \mu) \quad \forall \varphi \in L^2(\mathcal{A}, \mu).$$

The estimate (3.7) implies that the set of ergodic invariant measures is not empty. More precisely, we have the following result.

Proposition 3.7 *Assume* **H1–H2**. *Then, there exists an ergodic invariant measure for the transition semigroup* P.

Proof It is well known (see e.g. [2, Theorem 19.25]) that for an arbitrary Markov transition semigroup $\{P_t\}_{t\ge 0}$, the ergodic measures are precisely the extreme points of the (possibly empty) convex set of its invariant measures. On the other hand, the Krein-Milman Theorem (see e.g. [2, Theorem 7.68]) characterizes the convex compact sets, in locally convex Hausdorff spaces, as closed convex hull of its extreme points. Let us denote by $\Pi \subset \mathscr{P}(\mathcal{A})$ the convex set of all invariant measures for the Markov semigroup $\{P_t\}_{t\ge 0}$. In Theorem 3.4 we proved that Π is non empty, thus, in view of the above discussion, it only remains to show that its closure is compact or equivalently that Π is tight. By estimate (3.7) in Proposition 3.5 we know that there exists a constant C, depending on the structural data, such that

$$\int_{\mathcal{A}} \|x\|_V^2 \, \mu(\mathrm{d}x) \le C \qquad \forall \mu \in \Pi.$$

Therefore, using the same notation of the proof of Theorem 3.4, by the Markov inequality we infer that

$$\sup_{\mu\in\Pi} \mu\left(\bar{B}_n^C\right) = \sup_{\mu\in\Pi} \mu(\{x \in \mathcal{A} : \|x\|_V > n\}) \le \frac{1}{n^2} \sup_{\mu\in\Pi} \int_{\mathcal{A}} \|x\|_V^2 \, \mu(\mathrm{d}x) \le \frac{C}{n^2} \to 0,$$

as $n \to \infty$. Hence Π is tight and admits extreme points, which are ergodic invariant measures for P. □

3.4 Uniqueness of the invariant measure

Intuitively, uniqueness of invariant measures depends on how dissipative the stochastic Eq. (1.4) really is. Here, we show that for a "large enough" diffusion coefficient ν, the invariant measure is unique and strongly mixing, according to the following definition.

Definition 3.8 An invariant measure $\mu \in \mathscr{P}(\mathcal{A})$ for the semigroup P is said to be strongly mixing if

$$\lim_{t\to\infty} P_t\varphi = \int_{\mathcal{A}} \varphi(x)\,\mu(\mathrm{d}x) \qquad \text{in } L^2(\mathcal{A},\mu) \quad \forall\,\varphi \in L^2(\mathcal{A},\mu).$$

Theorem 3.9 *Assume* **H1–H2** *and suppose that*

$$\alpha_0 := \nu\left(\frac{1}{K_0^2} - 1\right) - \frac{C_B}{2} - K > 0. \tag{3.12}$$

Then, there exists a unique invariant measure μ for the transition semigroup P. Moreover, μ is ergodic and strongly mixing.

Remark 3.10 Note that condition (3.12) is relevant since $K_0 \in (0,1)$ by definition of K_0 itself. Roughly speaking, the dissipativity inequality (3.12) is satisfied either when the diffusion coefficient ν is large enough or when the structural coefficient K_0 is small enough. For the latter case, we recall that K_0 depends exclusively on the domain D, and can be estimated in terms of the first positive eigenvalue of the Laplacian operator on D (according to the boundary conditions).

Proof of Theorem 3.9 Let $x, y \in \mathcal{A}$ and let u^x, u^y be the respective variational solutions to problem (1.4). Setting $w := u^x - u^y$, the Itô formula for the square of the H norm of w yields for every $t \geq 0$, $\mathbb{P}$-almost surely, that

$$\begin{aligned}
&\frac{1}{2}\|w(t)\|_H^2 + \nu\int_0^t \|\nabla w(s)\|_H^2\,\mathrm{d}s + \int_0^t \left(F'(u^x(s)) - F'\left(u^y(s)\right), w(s)\right)_H \mathrm{d}s \\
&= \frac{1}{2}\|x-y\|_H^2 + \frac{1}{2}\int_0^t \left\|B\left(u^x(s)\right) - B\left(u^y(s)\right)\right\|_{\mathcal{L}_{HS}(U,H)}^2 \mathrm{d}s \\
&\quad + \int_0^t \left(w(s), \left(B(u^x(s)) - B(u^y(s))\right)\mathrm{d}W(s)\right)_H.
\end{aligned} \tag{3.13}$$

Using the Lipschitz continuity of the operator B in **H2** we estimate

$$\frac{1}{2}\int_0^t \|B(u(s)) - B(v(s))\|_{\mathcal{L}_{HS}(U,H)}^2\,\mathrm{d}s \leq \frac{C_B}{2}\int_0^t \|w(s)\|_H^2\,\mathrm{d}s,$$

while exploiting assumption **H1** we have that

$$\int_0^t \left(F'(u^x(s)) - F'\left(u^y(s)\right), w(s)\right)_H \mathrm{d}s \geq -K \int_0^t \|w(s)\|_H^2 \,\mathrm{d}s.$$

Noting that the last term in (3.13) is a square integrable martingale thanks to (2.4) and the regularity of w, taking expectations we infer that

$$\begin{aligned}&\frac{1}{2}\,\mathbb{E}\,\|w(t)\|_H^2 + \nu\,\mathbb{E}\int_0^t \|\nabla w(s)\|_H^2\,\mathrm{d}s \\ &\quad\leq \frac{1}{2}\|x-y\|_H^2 + \left(\frac{C_B}{2} + K\right)\mathbb{E}\int_0^t \|w(s)\|_H^2\,\mathrm{d}s.\end{aligned}$$

It follows that

$$\begin{aligned}&\frac{1}{2}\,\mathbb{E}\,\|w(t)\|_H^2 + \nu\,\mathbb{E}\int_0^t \|w(s)\|_V^2\,\mathrm{d}s \\ &\quad\leq \frac{1}{2}\|x-y\|_H^2 + \left(\frac{C_B}{2} + K + \nu\right)\mathbb{E}\int_0^t \|w(s)\|_H^2\,\mathrm{d}s,\end{aligned}$$

hence also, thanks to the continuous inclusion $V \hookrightarrow H$, that

$$\frac{1}{2}\,\mathbb{E}\,\|w(t)\|_H^2 + \alpha_0\,\mathbb{E}\int_0^t \|w(s)\|_H^2\,\mathrm{d}s \leq \frac{1}{2}\|x-y\|_H^2.$$

Now, exploiting the fact that $x, y \in \mathcal{A}$, hence in particular $\|x-y\|_{L^\infty} \leq 2$, by the Gronwall lemma we obtain

$$\mathbb{E}\,\|(u^x - u^y)(t)\|_H^2 \leq e^{-\alpha_0 t}\|x-y\|_H^2 \leq 4|D|e^{-\alpha_0 t} \quad \forall\, t \geq 0, \quad \forall\, x, y \in H. \tag{3.14}$$

Consequently, let μ be an invariant measure for $\{P_t\}_{t\geq 0}$. For any $\varphi \in C_b^1(H)$ and $x \in \mathcal{A}$, by definition of invariance and the estimate (3.14) we have

$$\begin{aligned}\left|P_t(\varphi_{|\mathcal{A}})(x) - \int_{\mathcal{A}} \varphi(y)\,\mu(\mathrm{d}y)\right|^2 &\leq \|D\varphi\|_\infty^2 \int_{\mathcal{A}} \mathbb{E}\left[\|u^x(t) - u^y(t)\|_H^2\right]\mu(\mathrm{d}y) \\ &\leq 2|D|\|D\varphi\|_\infty^2 e^{-\alpha_0 t}\end{aligned}$$

uniformly in x. Since $C_b^1(H)_{|\mathcal{A}}$ is dense in $L^2(\mathcal{A}, \mu)$, we deduce that

$$\left|P_t\varphi(x) - \int_{\mathcal{A}} \varphi(y)\,\mu(\mathrm{d}y)\right| \to 0 \quad \text{as } t \to \infty, \quad \forall \varphi \in L^2(\mathcal{A}, \mu),$$

that is the strong mixing property holds true. Notice that the above computation easily implies also the uniqueness of the invariant measure. Indeed, let π be another invariant measure, then for all $\varphi \in C_b^1(H)$ we have

$$\begin{aligned}
&\left|\int_{\mathcal{A}} \varphi(y)\,\mu(\mathrm{d}y) - \int_{\mathcal{A}} \varphi(x)\,\pi(\mathrm{d}x)\right| \\
&\quad = \left|\int_{\mathcal{A}}\int_{\mathcal{A}} \big(P_t(\varphi_{|\mathcal{A}})(y) - P_t(\varphi_{|\mathcal{A}})(x)\big)\,\pi(\mathrm{d}x)\mu(\mathrm{d}y)\right| \\
&\quad \le 2|D|\|D\varphi\|_\infty^2 e^{-\alpha_0 t} \to 0 \qquad \text{as } t \to \infty.
\end{aligned}$$

The fact that the unique invariant measure is also ergodic follows from Proposition 3.7, and this concludes the proof. ◻

4 Analysis of the Kolmogorov equation

In this section we focus on the Kolmogorov operator associated to the stochastic Eq. (1.4). We aim at characterising the infinitesimal generator of the transition semigroup P in terms of the closure of the Kolmogorov operator associated to (1.4) in the space $L^2(\mathcal{A}, \mu)$, where μ is an invariant measure for P.

Throughout the section, we assume **H1–H2** and μ is an invariant measure for the semigroup P. We have already pointed out that P extends by density to a strongly continuous linear semigroup of contractions on $L^2(\mathcal{A}, \mu)$, which will be denoted by the same symbol P for convenience. As such, for the semigroup P on $L^2(\mathcal{A}, \mu)$ it is well defined the infinitesimal generator $(L, D(L))$, namely

$$D(L) := \left\{\varphi \in L^2(\mathcal{A}, \mu) : \quad \lim_{t\to 0^+} \frac{P_t\varphi - \varphi}{t} \text{ exists in } L^2(\mathcal{A}, \mu)\right\}$$

and

$$-L\varphi := \lim_{t\to 0^+} \frac{P_t\varphi - \varphi}{t} \quad \text{in } L^2(H, \mu), \quad \varphi \in D(L).$$

The main issue that we address is to characterise the infinitesimal generator $(L, D(L))$ in terms of the Kolmogorov operator associated to (1.4).

4.1 The Kolmogorov operator

We define the Kolmogorov operator $(L_0, D(L_0))$ associated to the stochastic Eq. (1.4) as follows. We set

$$D(L_0) := C_b^2(H)_{|\mathcal{A}} = \left\{\varphi \in C_b(\mathcal{A}) : \exists\, \psi \in C_b^2(H) : \varphi(x) = \psi(x) \quad \forall\, x \in \mathcal{A}\right\}.$$

and

$$L_0\varphi(x) := -\frac{1}{2}\,\mathrm{Tr}[B(x)^* D^2\varphi(x)B(x)] + (-\Delta x + F'(x), D\varphi(x))_H,$$
$$x \in \mathcal{A}_{str}, \quad \varphi \in D(L_0).$$

Note that the definition of the domain of L_0 through restrictions on $\mathcal{A}$ is essential, as the operators B and F' are not defined on the whole H. More specifically, let us stress that not even considering $x \in \mathcal{A}$ is enough: this is because $F'(x)$ makes sense in H only for $x \in \mathcal{A}_{str}$, and not for any $x \in \mathcal{A}$.

We note that for every $\varphi \in D(L_0)$, with this definition the element $L_0\varphi$ is actually well defined as an element in $L^2(\mathcal{A}, \mu)$. Indeed, thanks to the estimate (2.4) one has, for every $y \in \mathcal{A}_{str}$, that

$$\begin{aligned}|L_0\varphi(y)| &\le \|\varphi\|_{C_b^2(H)}\left(\|B(y)\|^2_{\mathcal{L}_{HS}(U,H)} + \left\|F'(y)\right\|_H + \|\Delta y\|_H\right)\\ &\le \|\varphi\|_{C_b^2(H)}\left(C_B + \left\|F'(y)\right\|_H + \|\Delta y\|_H\right),\end{aligned}$$

so that the estimate (3.7) yields that

$$L_0\varphi \in L^2(\mathcal{A}, \mu) \quad \forall\, \varphi \in D(L_0).$$

The fact that $L_0\varphi$ is explicitly defined only on $\mathcal{A}_{str}$, and not on $\mathcal{A}$, is irrelevant when working in $L^2(\mathcal{A}, \mu)$ since $\mu(\mathcal{A}_{str}) = 1$. It follows then that $(D(L_0), L_0)$ is a linear unbounded operator on the Hilbert space $L^2(\mathcal{A}, \mu)$.

The elliptic Kolmogorov equation associated to (1.4) reads

$$\alpha\varphi(x) + L_0\varphi(x) = g(x), \quad x \in \mathcal{A}_{str}, \tag{4.1}$$

where $\alpha > 0$ is a given coefficient and $g : \mathcal{A} \to \mathbb{R}$ is a given datum.

The first natural result is the following.

Lemma 4.1 *In this setting, it holds that $D(L_0) \subset D(L)$ and*

$$L\varphi(x) = L_0\varphi(x) \ \ \textit{for } \mu\textit{-a.e. } x \in \mathcal{A}, \quad \forall\, \varphi \in D(L_0).$$

Remark 4.2 As we have already point out above, notice that in this identity the expression $L_0\varphi(x)$ makes sense for μ-almost every $x \in \mathcal{A}$ (and not just in $\mathcal{A}_{str}$) by virtue of Proposition 3.5, which ensures indeed that $\mu(\mathcal{A}_{str}) = 1$.

Proof of Lemma 4.1 Let $x \in \mathcal{A} \cap V$ and let $u := u^x$ be the respective unique analytically strong solution to (1.4). Then, for every $\varphi \in D(L_0)$ the Itô formula yields directly, for every $t \geq 0$,

$$\mathbb{E}\,\varphi(u(t)) + \mathbb{E}\int_0^t (-\nu\Delta u(s) + F'(u(s)), D\varphi(u(s)))_H \,\mathrm{d}s \\ = \mathbb{E}\,\varphi(x) + \frac{1}{2}\mathbb{E}\int_0^t \mathrm{Tr}\left[B(u(s))^* D^2\varphi(u(s))B(u(s))\right]\mathrm{d}s.$$

Since $u(t) \in \mathcal{A}$ $\mathbb{P}$-almost surely for every $t \geq 0$, this yields

$$P_t\varphi(x) - \varphi(x) + \int_0^t P_s(L_0\varphi)(x)\,\mathrm{d}s = 0 \quad \forall\, x \in \mathcal{A}. \tag{4.2}$$

Since $L_0\varphi \in L^2(\mathcal{A}, \mu)$ and P is strongly continuous on $L^2(\mathcal{A}, \mu)$, this implies

$$s \mapsto P_s(L_0\varphi) \in C([0, t]; L^2(\mathcal{A}, \mu)),$$

from which it follows that

$$\lim_{t\to 0^+} \frac{1}{t}\int_0^t P_s(L_0\varphi)\,\mathrm{d}s = L_0\varphi \quad \text{in } L^2(\mathcal{A}, \mu).$$

This shows by comparison in (4.2) that $\varphi \in D(L)$. Moreover, dividing by t and taking the limit in $L^2(\mathcal{A}, \mu)$ as $t \to 0^+$ in the identity (4.2), we get that

$$-L\varphi(x) + L_0\varphi(x) = 0 \quad \text{for } \mu\text{-a.e. } x \in \mathcal{A},$$

and we conclude. □

4.2 A regularised Kolmogorov equation

A first main issue that we aim at addressing is to investigate existence and uniqueness of strong solutions to the Kolmogorov Eq. (4.1) in $L^2(\mathcal{A}, \mu)$. However, some preliminary preparations are necessary. In particular, we construct here a family of regularised Kolmogorov equations that approximate (4.1) and for which we are actually able to show existence of *classical* solutions on the whole space H. This will be done by using a double approximation of the operators: one in the parameter $\lambda > 0$, which basically removes the singularity of F' and allows to work on H rather than just $\mathcal{A}$, and one in the parameter $n \in \mathbb{N}$, which conveys enough smoothness to the operators themselves.

Due to the presence of the multiplicative noise, in order to tackle the Kolmogorov equation, in the current Sect. 4.2 we shall need the following reinforcement of assumption **H2**, namely:

H2' The sequence $\{h_k\}_{k\in\mathbb{N}}$ is included also in $C^2([-1,1])$ and satisfies for every $k \in \mathbb{N}$ that $h_k'(\pm 1) = 0$. Moreover, it holds that

$$C_B' := \sum_{k\in\mathbb{N}} \|h_k''\|^2_{C([-1,1])} < \infty. \tag{4.3}$$

4.2.1 First approximation

Thanks to the assumption **H1**, the function

$$\beta : (-1,1) \to \mathbb{R}, \qquad \beta(r) := F'(r) + Kr, \quad r \in (-1,1), \tag{4.4}$$

is continuous non-decreasing, hence can be identified to a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$. In particular, for every $\lambda > 0$ it is well defined its resolvent operator $J_\lambda := (I + \lambda\beta)^{-1} : \mathbb{R} \to (-1,1)$, i.e. for every $r \in \mathbb{R}$, $J_\lambda(r)$ is the unique element in $(-1,1)$ such that $J_\lambda(r) + \lambda\beta(J_\lambda(r)) = r$. The Yosida approximation of β is defined as $\beta_\lambda : \mathbb{R} \to \mathbb{R}$, $\beta_\lambda(r) := \beta(J_\lambda(r))$, $r \in \mathbb{R}$. We recall that β_λ is Lipschitz-continuous and non-decreasing: for further properties on monotone and convex analysis we refer to [4].
Let also $\rho \in C_c^\infty(\mathbb{R})$ with $\sup(\rho) = [-1,1]$, $\rho \geq 0$, $\|\rho\|_{L^1(\mathbb{R})} = 1$, and set

$$\rho_\lambda : \mathbb{R} \to \mathbb{R}, \qquad \rho_\lambda(r) := \lambda^{-1}\rho(\lambda^{-1}r), \quad r \in \mathbb{R}, \qquad \lambda > 0,$$

so that $(\rho_\lambda)_{\lambda>0}$ is a usual sequence of mollifiers on $\mathbb{R}$. Let us set for convenience

$$c_\rho := \left\|\rho'\right\|_{L^1(\mathbb{R})}.$$

Let us construct the approximated operators. First of all, for $\lambda > 0$ we define the λ-regularised potential $F_\lambda : \mathbb{R} \to [0,+\infty)$ as

$$F_\lambda(x) := F(0) - \frac{K}{2}|x|^2 + \int_0^x (\rho_{\lambda^2} \star \beta_\lambda)(y)\,\mathrm{d}y, \quad x \in \mathbb{R}.$$

Note that one has

$$F_\lambda'(x) = (\rho_{\lambda^2} \star \beta_\lambda)(x) - Kx, \quad x \in \mathbb{R}, \tag{4.5}$$

and from the properties of the Yosida approximation and convolutions it is not difficult to show that

$$|F_\lambda''(x)| \leq K + \frac{1}{\lambda} \quad \forall x \in \mathbb{R}, \qquad |F_\lambda'''(x)| \leq \frac{c_\rho}{\lambda^3} \quad \forall x \in \mathbb{R}. \tag{4.6}$$

For clarity, let us use a separate notation for the superposition operator induced by the Lipschitz real function F_λ' on the Hilbert space H, namely we set

$$\mathcal{F}_\lambda : H \to H, \qquad (\mathcal{F}_\lambda(v))(\mathbf{x}) := F_\lambda'(v(\mathbf{x})) \quad \text{for a.e. } \mathbf{x} \in D, \quad v \in H.$$

Since $F'_\lambda \in C^\infty(\mathbb{R})$ by definition and has bounded derivatives of any order, thanks to the continuous embedding $V \hookrightarrow L^6(D)$ and the dominated convergence theorem, $\mathcal{F}_\lambda$ can be shown to be twice Fréchet differentiable in H along directions of V: more precisely, this means that for every $x \in H$ there exist two operators

$$D\mathcal{F}_\lambda(x) \in \mathcal{L}(V, H), \qquad D^2\mathcal{F}_\lambda(x) \in \mathcal{L}(V; \mathcal{L}(V, H)) \cong \mathcal{L}_2(V \times V; H)$$

such that

$$\lim_{\|h\|_V \to 0} \frac{\|\mathcal{F}_\lambda(x+h) - \mathcal{F}_\lambda(x) - D\mathcal{F}_\lambda(x)[h]\|_H}{\|h\|_V} = 0,$$
$$\lim_{\|h\|_V \to 0} \frac{\left\| D\mathcal{F}_\lambda(x+h) - D\mathcal{F}_\lambda(x) - D^2\mathcal{F}_\lambda(x)[h, \cdot]\right\|_{\mathcal{L}(V,H)}}{\|h\|_V} = 0.$$

In particular, one has that

$$D\mathcal{F}_\lambda(x)[h] = F''_\lambda(x)h, \qquad x \in H, \quad h \in V,$$
$$D^2\mathcal{F}_\lambda(x)[h_1, h_2] = F'''_\lambda(x)h_1h_2, \qquad x \in H, \quad h_1, h_2 \in V,$$

so that (4.6) yields, for a constant $c > 0$ only depending on ρ, K, and D,

$$\|D\mathcal{F}_\lambda(x)\|_{\mathcal{L}(V,H)} \leq c\left(1 + \frac{1}{\lambda}\right) \qquad \forall x \in H, \tag{4.7}$$
$$\left\|D^2\mathcal{F}_\lambda(x)\right\|_{\mathcal{L}(V;\mathcal{L}(V,H))} \leq \frac{c}{\lambda^3} \qquad \forall x \in H. \tag{4.8}$$

As far as the operator B is concerned, for every $k \in \mathbb{N}$ we first extend h_k to $\tilde{h}_k : \mathbb{R} \to \mathbb{R}$ by setting

$$\tilde{h}_k(x) := \begin{cases} h_k(x) & \text{if } x \in [-1, 1], \\ 0 & \text{otherwise,} \end{cases} \qquad k \in \mathbb{N}.$$

In this way, by assumptions **H2** and **H2'** it is clear that $\{\tilde{h}_k\}_{k\in\mathbb{N}} \subset W^{2,\infty}(\mathbb{R})$, and we define then

$$h_{k,\lambda} := \rho_{\lambda^\gamma} \star \tilde{h}_k, \quad k \in \mathbb{N}, \quad \lambda > 0,$$

where $\gamma > 0$ is a prescribed fixed rate coefficient that will be chosen later. The reason of introducing γ here may sound not intuitive at this level, and will be clarified in the following sections: roughly speaking, γ is needed in order to suitable compensate for the blow-up in (4.8). With these definitions, we set then

$$B_\lambda : H \to \mathcal{L}_{HS}(U, H), \qquad B_\lambda(x)e_k := h_{k,\lambda}(x), \quad x \in H, \quad k \in \mathbb{N}. \tag{4.9}$$

Clearly, for every λ it holds that $\{h_{k,\lambda}\}_{k\in\mathbb{N}} \subset C_c^\infty(\mathbb{R})$ and, by the properties of convolutions and assumptions **H2** and **H2'**, for every $\lambda > 0$ it holds that

$$\sum_{k\in\mathbb{N}} \|h_{k,\lambda}\|^2_{C^2(\mathbb{R})} \leq \sum_{k\in\mathbb{N}} \|\tilde{h}_k\|^2_{W^{2,\infty}(\mathbb{R})} = \sum_{k\in\mathbb{N}} \|h_k\|^2_{C^2([-1,1])} \leq C_B + C'_B. \tag{4.10}$$

It follows in particular that for every $\lambda > 0$ the operator B_λ constructed above is $\sqrt{C_B}$-Lipschitz continuous and bounded. Also, similarly as above one can check that B_λ is twice Fréchet differentiable along the directions of V, in the sense that for every $x \in H$ there exist two operators

$$\begin{aligned} &DB_\lambda(x) \in \mathcal{L}(V, \mathcal{L}_{HS}(U, H)),\\ &D^2B_\lambda(x) \in \mathcal{L}(V; \mathcal{L}(V, \mathcal{L}_{HS}(U, H))) \cong \mathcal{L}_2(V \times V; \mathcal{L}_{HS}(U, H)) \end{aligned}$$

such that

$$\begin{aligned} &\lim_{\|h\|_V \to 0} \frac{\|B_\lambda(x+h) - B_\lambda(x) - DB_\lambda(x)[h]\|_{\mathcal{L}_{HS}(U,H)}}{\|h\|_V} = 0,\\ &\lim_{\|h\|_V \to 0} \frac{\left\|DB_\lambda(x+h) - DB_\lambda(x) - D^2B_\lambda(x)[h,\cdot]\right\|_{\mathcal{L}(V,\mathcal{L}_{HS}(U,H))}}{\|h\|_V} = 0. \end{aligned}$$

More precisely, it holds that

$$\begin{aligned} DB_\lambda(x)[z]e_k &= h'_{k,\lambda}((x))z, && x \in H, \quad z \in V, \quad k \in \mathbb{N},\\ D^2B_\lambda(x)[z_1, z_2]e_k &= h''_{k,\lambda}((x))z_1z_2, && x \in H, \quad z_1, z_2 \in V, \quad k \in \mathbb{N}. \end{aligned}$$

From the continuous embedding $V \hookrightarrow L^6(D)$, condition (4.10) ensures the existence of a constant c independent of λ such that

$$\|DB_\lambda(x)\|^2_{\mathcal{L}(V,\mathcal{L}_{HS}(U,H))} \leq C_B \qquad \forall x \in H, \tag{4.11}$$

$$\left\|D^2B_\lambda(x)\right\|^2_{\mathcal{L}(V,\mathcal{L}(V,\mathcal{L}_{HS}(U,H)))} \leq c \qquad \forall x \in H. \tag{4.12}$$

Furthermore, it is not difficult to see that actually B_λ is also Gâteaux differentiable from the whole H to $\mathcal{L}_{HS}(U, H)$, and DB_λ is Gâteaux differentiable along the directions of $L^4(D)$, so that for every $x \in H$ $DB_\lambda(x)$ and $D^2B_\lambda(x)$ extend to well defined operators in the spaces $\mathcal{L}(H, \mathcal{L}_{HS}(U, H))$ and $\mathcal{L}(L^4(D), \mathcal{L}(L^4(D), \mathcal{L}_{HS}(U, H)))$, respectively. Again by (4.10) we also have then

$$\|DB_\lambda(x)\|^2_{\mathcal{L}(H,\mathcal{L}_{HS}(U,H))} \leq C_B \qquad \forall x \in H, \tag{4.13}$$

$$\left\|D^2B_\lambda(x)\right\|^2_{\mathcal{L}(L^4(D),\mathcal{L}(L^4(D),\mathcal{L}_{HS}(U,H)))} \leq c \qquad \forall x \in H. \tag{4.14}$$

4.2.2 Second approximation

While the approximation in λ is enough for proving well posedness of the stochastic Eq. (1.4), in order to approximate the Kolmogorov Eq. (4.1) we need more smoothness on the coefficients. In this direction, we shall rely on some smoothing operators in infinite dimensions. Let now $\lambda > 0$ be fixed.

Let $\mathcal{C}$ be the unbounded linear operator $(I - \Delta)$ on H with effective domain Z (note that the definition of Z includes either Dirichlet or Neumann boundary conditions, according to α_d and α_n). Then, $\mathcal{C}$ is linear maximal monotone, coercive on V, and $-\mathcal{C}$ generates a strongly continuous semigroup of contractions $(e^{-t\mathcal{C}})_{t\geq 0}$ on H. Furthermore, since we are working in dimension $d = 2, 3$, it is possible to show that $\mathcal{C}^{-1} \in \mathcal{L}_{HS}(H, H)$: this follows from the fact that the eigenvalues $\{\lambda_k\}_k$ of $\mathcal{C}$ satisfy $\lambda_k \approx 1 + k^{2/d}$ as $k \to \infty$. We consider the Ornstein-Uhlenbeck transition semigroup $R := (R_t)_{t\geq 0}$ given by

$$R_t\varphi(x) := \int_H \varphi\left(e^{-t\mathcal{C}}x + y\right) N_{Q_t}(\mathrm{d}y), \quad \varphi \in \mathscr{B}_b(H),$$

where

$$Q_t := \int_0^t e^{-s\mathcal{C}}\mathcal{C}^{-2}e^{-s\mathcal{C}}\,\mathrm{d}s = \frac{1}{2}\mathcal{C}^{-3}\left(I - e^{-2t\mathcal{C}}\right), \quad t \geq 0. \tag{4.15}$$

Note that Q_t is trace class on H for every $t \geq 0$ and

$$\mathrm{Tr}(Q_t) \leq \left\|\mathcal{C}^{-1}\right\|^2_{\mathcal{L}_{HS}(U,H)} t \quad \forall t \geq 0. \tag{4.16}$$

Moreover, if $\{c_k\}_k$ is a complete orthonormal system of H made of eigenfunctions of $\mathcal{C}$, with eigenvalues $\{\lambda_k\}_k$, one has

$$\left\|Q_t^{-1/2}e^{-t\mathcal{C}}c_k\right\|_H = \sqrt{2}\lambda_k^{3/2}\frac{e^{-\lambda_k t}}{(1 - e^{-2\lambda_k t})^{1/2}} \leq \frac{\sqrt{2}}{t^{3/2}}\max_{r>0}\frac{r^{3/2}e^{-r}}{(1 - e^{-2r})^{1/2}},$$

from which it follows, thanks to the characterisation of null-controllability in [23, Prop. B.2.1], that

$$e^{-t\mathcal{C}}(H) \subset Q_t^{1/2}(H) \qquad \text{and} \qquad \left\|Q_t^{-1/2}e^{-t\mathcal{C}}\right\|_{\mathcal{L}(H,H)} \leq \frac{C}{t^{3/2}}, \quad \forall t > 0. \tag{4.17}$$

Thanks to (4.17), it is well known (see [23, Thm. 6.2.2]) that R is strong Feller, in the sense that for every $\varphi \in \mathscr{B}(H)$ and $t > 0$ it holds that $R_t\varphi \in UC_b^\infty(H)$, as well as $R_t\varphi(x) \to \varphi(x)$ for every $x \in H$ as $t \to 0^+$. It is natural then to introduce, for every $n \in \mathbb{N}$, the regularisations

$$\mathcal{F}_{\lambda,n} : H \to H, \qquad B_{\lambda,n} : H \to \mathcal{L}_{HS}(U, H),$$

as

$$\mathcal{F}_{\lambda,n}(x) := \int_H e^{-\frac{\mathcal{C}}{n}} \mathcal{F}_\lambda \left(e^{-\frac{\mathcal{C}}{n}} x + y\right) N_{Q_{1/n}}(\mathrm{d}y), \quad x \in H, \tag{4.18}$$

$$B_{\lambda,n}(x) := \int_H e^{-\frac{\mathcal{C}}{n^\delta}} B_\lambda \left(e^{-\frac{\mathcal{C}}{n^\delta}} x + y\right) N_{Q_{1/n^\delta}}(\mathrm{d}y), \quad x \in H, \tag{4.19}$$

where $\delta > 0$ is a positive rate coefficient that will be chosen later on. Again, as for the case of γ in the λ-approximation, the need of allowing for a general rate δ will be needed to suitably compensate the blow-up of $\mathcal{F}_{\lambda,n}$.

It follows for every $n \in \mathbb{N}$ that the approximated operators satisfy $\mathcal{F}_{\lambda,n} \in C^\infty(H; H)$, $B_{\lambda,n} \in C^\infty(H; \mathcal{L}_{HS}(U, H))$ and have bounded derivatives of any order. Moreover, since $e^{-\mathcal{C}/n}(H) \subset Z \subset V$ and $\mathcal{F}_\lambda$ and B_λ are twice differentiable along directions of V, for every $n \in \mathbb{N}$ and for every $x, z, z_1, z_2 \in H$ it holds that

$$D\mathcal{F}_{\lambda,n}(x)[z] := \int_H e^{-\frac{\mathcal{C}}{n}} D\mathcal{F}_\lambda \left(e^{-\frac{\mathcal{C}}{n}} x + y\right) \left[e^{-\frac{\mathcal{C}}{n}} z\right] N_{Q_{1/n}}(\mathrm{d}y),$$

$$DB_{\lambda,n}(x)[z] := \int_H e^{-\frac{\mathcal{C}}{n^\delta}} DB_\lambda \left(e^{-\frac{\mathcal{C}}{n^\delta}} x + y\right) \left[e^{-\frac{\mathcal{C}}{n^\delta}} z\right] N_{Q_{1/n^\delta}}(\mathrm{d}y),$$

and

$$D^2\mathcal{F}_{\lambda,n}(x)[z_1, z_2] := \int_H e^{-\frac{\mathcal{C}}{n}} D^2\mathcal{F}_\lambda \left(e^{-\frac{\mathcal{C}}{n}} x + y\right) \left[e^{-\frac{\mathcal{C}}{n}} z_1, e^{-\frac{\mathcal{C}}{n}} z_2\right] N_{Q_{1/n}}(\mathrm{d}y),$$

$$D^2B_{\lambda,n}(x)[z_1, z_2] := \int_H e^{-\frac{\mathcal{C}}{n^\delta}} D^2B_\lambda \left(e^{-\frac{\mathcal{C}}{n^\delta}} x + y\right) \left[e^{-\frac{\mathcal{C}}{n^\delta}} z_1, e^{-\frac{\mathcal{C}}{n^\delta}} z_2\right] N_{Q_{1/n^\delta}}(\mathrm{d}y).$$

In particular, from assumption **H1**, the non-expansivity of $e^{-\mathcal{C}/n}$ and the estimates (4.13)–(4.14) we have that

$$(D\mathcal{F}_{\lambda,n}(x)[z], z)_H \geq -K \|z\|_H^2 \qquad \forall x, z \in H, \tag{4.20}$$

$$\left\| DB_{\lambda,n}(x)\right\|^2_{\mathcal{L}(H,\mathcal{L}_{HS}(U,H))} \leq C_B \qquad \forall x \in H, \tag{4.21}$$

$$\left\| D^2B_{\lambda,n}(x)\right\|^2_{\mathcal{L}(L^4(D),\mathcal{L}(L^4(D),\mathcal{L}_{HS}(U,H)))} \leq c \qquad \forall x \in H, \tag{4.22}$$

where we note that all constants K, C_B, and c are independent of λ and n. Furthermore, exploiting the estimate (4.8) and the fact that $V = D(\mathcal{C}^{1/2})$, for every $x, z_1, z_2 \in H$

we infer that

$$
\begin{aligned}
&\left\| D^2\mathcal{F}_{\lambda,n}(x)[z_1, z_2] \right\|_H \\
&\quad \leq \int_H \left\| D^2\mathcal{F}_\lambda(e^{-\frac{\mathcal{C}}{n}}x + y) \right\|_{\mathcal{L}(V,\mathcal{L}(V,H))} \left\| e^{-\frac{\mathcal{C}}{n}} z_1 \right\|_V \left\| e^{-\frac{\mathcal{C}}{n}} z_1 \right\|_V \, N_{Q_{1/n}}(\mathrm{d}y) \\
&\quad \leq \frac{c}{\lambda^3} \left\| e^{-\frac{\mathcal{C}}{n}} z_1 \right\|_V \left\| e^{-\frac{\mathcal{C}}{n}} z_1 \right\|_V \\
&\quad \lesssim \frac{1}{\lambda^3} n^{1/2} \|z_1\|_H \, n^{1/2} \|z_2\|_H .
\end{aligned}
$$

It follows that there exists a positive constant $c > 0$, independent of n and λ, such that

$$
\left\| D^2\mathcal{F}_{\lambda,n}(x) \right\|_{\mathcal{L}(H,\mathcal{L}(H,H))} \leq c\frac{n}{\lambda^3} \qquad \forall\, x \in H. \tag{4.23}
$$

Analogously, thanks to the continuous embedding $H^{3/4}(D) \hookrightarrow L^4(D)$ in dimensions $d = 2, 3$, one has that $D(\mathcal{C}^{3/8}) \hookrightarrow L^4(D)$: hence, proceeding as above and using (4.14) instead, one gets for every $x, z_1, z_2 \in H$ that

$$
\begin{aligned}
&\left\| D^2 B_{\lambda,n}(x)[z_1, z_2] \right\|_H \\
&\quad \leq \int_H \left\| D^2 B_\lambda(e^{-\frac{\mathcal{C}}{n^\delta}}x + y) \right\|_{\mathcal{L}(L^4(D),\mathcal{L}(L^4(D),\mathcal{L}_{HS}(U,H)))} \\
&\qquad \times \left\| e^{-\frac{\mathcal{C}}{n^\delta}} z_1 \right\|_{L^4(D)} \left\| e^{-\frac{\mathcal{C}}{n^\delta}} z_1 \right\|_{L^4(D)} N_{Q_{1/n^\delta}}(\mathrm{d}y) \\
&\quad \leq c \left\| e^{-\frac{\mathcal{C}}{n^\delta}} z_1 \right\|_{L^4(D)} \left\| e^{-\frac{\mathcal{C}}{n^\delta}} z_1 \right\|_{L^4(D)} \\
&\quad \lesssim n^{\frac{3}{8}\delta} \|z_1\|_H \, n^{\frac{3}{8}\delta} \|z_2\|_H .
\end{aligned}
$$

It follows that there exists a positive constant $c > 0$, independent of n and λ, such that

$$
\left\| D^2 B_{\lambda,n}(x) \right\|_{\mathcal{L}(H,\mathcal{L}(H,\mathcal{L}_{HS}(U,H)))} \leq cn^{\frac{3}{4}\delta} \qquad \forall\, x \in H. \tag{4.24}
$$

4.2.3 Construction of classical solutions

Let now $\lambda > 0$ and $n \in \mathbb{N}$ be fixed. For every $x \in H$ the doubly approximated stochastic equation

$$
\begin{cases}
\mathrm{d}u_{\lambda,n}(t) - \nu\Delta u_{\lambda,n}(t)\,\mathrm{d}t + \mathcal{F}_{\lambda,n}(u_{\lambda,n}(t))\,\mathrm{d}t & \\
\quad = B_{\lambda,n}(u_{\lambda,n}(t))\,\mathrm{d}W(t) & \text{in } (0,T) \times D, \\
\alpha_d u_{\lambda,n} + \alpha_n \partial_{\mathbf{n}} u_{\lambda,n} = 0 & \text{in } (0,T) \times \Gamma, \\
u_{\lambda,n}(0) = x & \text{in } D,
\end{cases} \tag{4.25}
$$

admits a unique solution $u_{\lambda,n} = u^x_{\lambda,n} \in L^p(\Omega; C([0,T]; H) \cap L^2(0,T;V))$, for all $p \geq 2$ and $T > 0$. Moreover, due to the smoothness of the coefficients $\mathcal{F}_{\lambda,n}$ and $B_{\lambda,n}$, by the regular dependence results in [36] (see also [39]) one can infer in particular that the solution map satisfies, for all $T > 0$,

$$S_{\lambda,n} : x \mapsto u^x_{\lambda,n} \in C^2_b(H; L^2(\Omega; C([0,T]; H))). \tag{4.26}$$

Furthermore, for $x, z, z_1, z_2 \in H$, the derivatives of $S_{\lambda,n}$ are given by

$$DS_{\lambda,n}(x)[z] = v^z_{\lambda,n}, \qquad D^2S_{\lambda,n}(x)[z_1, z_2] = w^{z_1,z_2}_{\lambda,n},$$

where

$$v^z_{\lambda,n}, w^{z_1,z_2}_{\lambda,n} \in L^2\left(\Omega; C([0,T]; H) \cap L^2(0,T;V)\right) \quad \forall T > 0$$

are the unique solutions the stochastic equations

$$\begin{cases} \mathrm{d}v^z_{\lambda,n}(t) - \nu\Delta v^z_{\lambda,n}(t)\,\mathrm{d}t + D\mathcal{F}_{\lambda,n}(u_{\lambda,n}(t))v^z_{\lambda,n}(t)\,\mathrm{d}t & \\ \quad = DB_{\lambda,n}(u_{\lambda,n}(t))v^z_{\lambda,n}(t)\,\mathrm{d}W(t) & \text{in } (0,T)\times D, \\ \alpha_d v^z_{\lambda,n} + \alpha_n \partial_{\mathbf{n}} v^z_{\lambda,n} = 0 & \text{in } (0,T)\times \Gamma, \\ v^z_{\lambda,n}(0) = z & \text{in } D, \end{cases} \tag{4.27}$$

and

$$\begin{cases} \mathrm{d}w^{z_1,z_2}_{\lambda,n}(t) - \nu\Delta w^{z_1,z_2}_{\lambda,n}(t)\,\mathrm{d}t & \\ \quad + D\mathcal{F}_{\lambda,n}(u_{\lambda,n}(t))w^{z_1,z_2}_{\lambda,n}(t)\,\mathrm{d}t & \\ \quad + D^2\mathcal{F}_{\lambda,n}(u_{\lambda,n})[v^{z_1}_{\lambda,n}(t), v^{z_2}_{\lambda,n}(t)]\,\mathrm{d}t & \\ = DB_{\lambda,n}(u_{\lambda,n}(t))w^{z_1,z_2}_{\lambda,n}(t)\,\mathrm{d}W(t) & \\ \quad + D^2B_{\lambda,n}(u_{\lambda,n}(t))[v^{z_1}_{\lambda,n}(t), v^{z_2}_{\lambda,n}(t)]\,\mathrm{d}W(t) & \text{in } (0,T)\times D, \\ \alpha_d w^{z_1,z_2}_{\lambda,n} + \alpha_n \partial_{\mathbf{n}} w^{z_1,z_2}_{\lambda,n} = 0 & \text{in } (0,T)\times \Gamma, \\ w^{z_1,z_2}_{\lambda,n}(0) = 0 & \text{in } D. \end{cases} \tag{4.28}$$

We are ready now to consider the approximated Kolmogorov equation, which is the actual Kolmogorov equation associated to the doubly regularised problem (4.25). We define the regularised Kolmogorov operator $\left(D\left(L_0^{\lambda,n}\right), L_0^{\lambda,n}\right)$ by setting

$$D\left(L_0^{\lambda,n}\right) := C^2_b(H)$$

and

$$L_0^{\lambda,n}\varphi(x) := -\frac{1}{2}\operatorname{Tr}\left[B_{\lambda,n}(x)^* D^2\varphi(x) B_{\lambda,n}(x)\right] + \left(-\Delta x + F'_{\lambda,n}(x), D\varphi(x)\right)_H,$$
$$x \in Z, \quad \varphi \in D\left(L_0^{\lambda,n}\right).$$

Arguing exactly as in Lemma 4.1 but for the stochastic Eq. (4.25), it is straightforward to see that on $C_b^2(H)$ the regularised operator $-L_0^{\lambda,n}$ coincides with the infinitesimal generator of the transition semigroup $P^{\lambda,n} = \left(P_t^{\lambda,n}\right)_{t\geq 0}$ associated to (4.25), i.e.

$$P_t^{\lambda,n}\varphi(x) := \mathbb{E}\,\varphi(u_{\lambda,n}(t;x)), \quad x \in H, \quad \varphi \in C_b^2(H).$$

Thanks to the regular dependence on the initial datum, one is able to obtain well posedness in the classical sense at λ and n fixed. We collect these results in the following statement.

Lemma 4.3 *In the current setting, there exists a positive constant $\bar{\alpha}$, independent of λ and n, such that for every $\alpha > \bar{\alpha}$ there exists $C = C(\alpha) > 0$ such that the following holds: for every $g \in C_b^1(H)$, the function*

$$\varphi_{\lambda,n}(x) := \int_0^{+\infty} e^{-\alpha t}\,\mathbb{E}\left[g(u_{\lambda,n}(t;x))\right]\mathrm{d}t, \quad x \in H, \quad n \in \mathbb{N}, \quad \lambda > 0, \tag{4.29}$$

satisfies $\varphi_{\lambda,n} \in C_b^1(H)$ and

$$\left\|\varphi_{\lambda,n}\right\|_{C_b^1(H)} \leq C\,\|g\|_{C_b^1(H)} \quad \forall\,\lambda > 0, \quad \forall n \in \mathbb{N}. \tag{4.30}$$

Moreover, if the dissipativity condition (3.12) holds and $g \in C_b^2(H)$, then for every $\lambda > 0$ and $n \in \mathbb{N}$ it also holds that $\varphi_{\lambda,n} \in C_b^2(H)$ with

$$\left\|\varphi_{\lambda,n}\right\|_{C_b^2(H)} \leq C\left(1 + \frac{n}{\lambda^3} + n^{\frac{3}{4}\delta}\right)\|g\|_{C_b^2(H)} \quad \forall\,\lambda > 0, \quad \forall n \in \mathbb{N}, \tag{4.31}$$

and

$$\alpha\varphi_{\lambda,n}(x) + L_0^{\lambda,n}\varphi_{\lambda,n}(x) = g(x) \quad \forall x \in Z. \tag{4.32}$$

Proof of Lemma 4.3 It is obvious that $\varphi_{\lambda,n} \in C_b(H)$ and

$$\left\|\varphi_{\lambda,n}\right\|_{C_b(H)} \leq \frac{1}{\alpha}\,\|g\|_{C_b(H)}.$$

Let $x \in H$ and $u_{\lambda,n} := u^x_{\lambda,n}$. The Itô formula for the square of the H-norm in (4.27) yields, exploiting (4.20)–(4.21), that

$$\begin{aligned}
&\frac{1}{2}\left\|v^z_{\lambda,n}(t)\right\|_H^2 + \nu\int_0^t \left\|\nabla v^z_{\lambda,n}(s)\right\|_H^2 \mathrm{d}s \\
&\quad\leq \frac{1}{2}\,\|z\|_H^2 + \left(K + \frac{C_B}{2}\right)\int_0^t \left\|v^z_{\lambda,n}(s)\right\|_H^2 \mathrm{d}s \\
&\quad+ \int_0^t \left(v^z_{\lambda,n}(s), DB_{\lambda,n}(u_{\lambda,n}(s))v^z_{\lambda,n}(s)\,\mathrm{d}W(s)\right)_H.
\end{aligned} \tag{4.33}$$

Taking expectations, we readily deduce by the Gronwall lemma that

$$\left\| v^z_{\lambda,n}(t) \right\|^2_{L^2(\Omega;H)} \leq e^{(2K+C_B)t} \|z\|^2_H \quad \forall t \geq 0. \tag{4.34}$$

If $g \in C^1_b(H)$, recalling (4.26), by the chain rule one has, for every $t \geq 0$, that

$$x \mapsto g(u_{\lambda,n}(t;x)) \in C^1_b\left(H; L^2(\Omega)\right),$$

with

$$D\left(x \mapsto g(u_{\lambda,n}(t;x))\right)[z] = Dg(u_{\lambda,n}(t;x))v^z_{\lambda,n}(t;x), \quad x, z \in H.$$

It follows, thanks to (4.34) that

$$\left\| D\left(x \mapsto g(u_{\lambda,n}(t;x))\right) \right\|_{\mathcal{L}(H;L^2(\Omega))} \leq \|g\|_{C^1_b(H)} e^{(K+C_B/2)t} \quad \forall t \geq 0.$$

As soon as $\alpha > K + \frac{C_B}{2}$ (recall that K and C_B are independent of λ and n), the dominated convergence theorem implies that $\varphi_{\lambda,n} \in C^1(H)$ with

$$D\varphi_{\lambda_n}(x)[z] = \int_0^{+\infty} e^{-\alpha t} \, \mathbb{E}\left[Dg(u_{\lambda,n}(t;x))v^z_{\lambda,n}(t;x) \right] \mathrm{d}t, \quad z \in H,$$

and

$$\left\| D\varphi_{\lambda,n}(x) \right\|_H \leq \|g\|_{C^1_b(H)} \int_0^{+\infty} e^{-(\alpha-K-C_B/2)t} \, \mathrm{d}t \quad \forall x \in H.$$

Choosing then $\alpha > K + \frac{C_B}{2}$, this proves that actually $\varphi_{\lambda,n} \in C^1_b(H)$, as well as the estimate (4.30).

Let us now further assume (3.12) and that $g \in C^2_b(H)$. From the Itô formula (4.33), exploiting the continuous embedding $V \hookrightarrow H$ and the assumption (3.12), one gets that

$$\begin{aligned} &\frac{1}{2} \left\| v^z_{\lambda,n}(t) \right\|^2_H + \alpha_0 \int_0^t \left\| v^z_{\lambda,n}(s) \right\|^2_H \mathrm{d}s \\ &\quad \leq \frac{1}{2} \|z\|^2_H + \int_0^t \left(v^z_{\lambda,n}(s), DB_{\lambda,n}(u_{\lambda,n}(s))v^z_{\lambda,n}(s) \, \mathrm{d}W(s) \right)_H. \end{aligned}$$

Noting that the Burkholder-Davis-Gundy inequality together with (4.21) yield

$$\begin{aligned} &\mathbb{E} \sup_{r\in[0,t]} \left| \int_0^r \left(v^z_{\lambda,n}(s), DB_{\lambda,n}(u_{\lambda,n}(s))v^z_{\lambda,n}(s) \, \mathrm{d}W(s) \right)_H \right|^2 \\ &\quad \leq 4C_B \, \mathbb{E} \int_0^t \left\| v^z_{\lambda,n}(s) \right\|^4_H \mathrm{d}s, \end{aligned}$$

by raising to the square power in Itô's formula and taking expectations we obtain

$$\mathbb{E}\sup_{r\in[0,t]}\left\|v^z_{\lambda,n}(t)\right\|^4_H \le 2\left\|z\right\|^4_H + 32C_B\int_0^t \mathbb{E}\left\|v^z_{\lambda,n}(s)\right\|^4_H\,\mathrm{d}s.$$

The Gronwall lemma ensures then that

$$\left\|v^z_{\lambda,n}\right\|^4_{L^4(\Omega;C([0,t];H)} \le 2e^{32C_Bt}\left\|z\right\|^4_H \quad \forall t\ge 0. \tag{4.35}$$

Similarly, using the Itô formula for the square of the H-norm in (4.28), and exploiting now (4.20)–(4.24) and the Young inequality, we get for a constant C independent of λ, n that

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\left\|w^{z_1,z_2}_{\lambda,n}(t)\right\|^2_H + \nu\,\mathbb{E}\int_0^t\left\|\nabla w^{z_1,z_2}_{\lambda,n}(s)\right\|^2_H\,\mathrm{d}s\\
&\quad\le\left(K+\frac{C_B}{2}+\frac{1}{2}\right)\int_0^t\mathbb{E}\left\|w^{z_1,z_2}_{\lambda,n}(s)\right\|^2_H\,\mathrm{d}s\\
&\qquad + C\left(\frac{n^2}{\lambda^6}+n^{\frac{3}{2}\delta}\right)\mathbb{E}\int_0^t\left\|v^{z_1}_{\lambda,n}(s)\right\|^2_H\left\|v^{z_2}_{\lambda,n}(s)\right\|^2_H\,\mathrm{d}s.
\end{aligned}$$

Exploiting the Hölder inequality together with (4.35), it follows that

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\left\|w^{z_1,z_2}_{\lambda,n}(t)\right\|^2_H + \nu\,\mathbb{E}\int_0^t\left\|\nabla w^{z_1,z_2}_{\lambda,n}(s)\right\|^2_H\,\mathrm{d}s\\
&\quad\le\left(K+\frac{C_B}{2}+\frac{1}{2}\right)\int_0^t\mathbb{E}\left\|w^{z_1,z_2}_{\lambda,n}(s)\right\|^2_H\,\mathrm{d}s\\
&\qquad + 2C\left(\frac{n^2}{\lambda^6}+n^{\frac{3}{2}\delta}\right)\left\|z_1\right\|^2_H\left\|z_2\right\|^2_H\int_0^t e^{32C_Bs}\,\mathrm{d}s,
\end{aligned}$$

and the Gronwall lemma ensures that, for every $t\ge 0$,

$$\left\|w_{\lambda,n}(t)\right\|^2_{L^2(\Omega;H)} \le 4C\left(\frac{n^2}{\lambda^6}+n^{\frac{3}{2}\delta}\right)te^{(2K+33C_B+1)t}\left\|z_1\right\|^2_H\left\|z_2\right\|^2_H. \tag{4.36}$$

Since $g\in C^2_b(H)$, condition (4.26) and the chain rule give, for every $t\ge 0$, that

$$x\mapsto g(u_{\lambda,n}(t;x))\in C^2_b\left(H;L^2(\Omega)\right),$$

with

$$\begin{aligned} &D^2\left(x \mapsto g(u_{\lambda,n}(t;x))\right)[z_1,z_2] \\ &\quad = Dg\left(u_{\lambda,n}(t;x)\right) w^{z_1,z_2}_{\lambda,n}(t;x) \\ &\qquad + D^2 g\left(u_{\lambda,n}(t;x)\right)\left[v^{z_1}_{\lambda,n}(t;x), v^{z_2}_{\lambda,n}(t;x)\right], \quad x, z_1, z_2 \in H. \end{aligned}$$

The estimates (4.35) and (4.36) imply then, possibly renominating the constant C independently of λ and n, that

$$\begin{aligned} &\left\| D^2\left(x \mapsto g(u_{\lambda,n}(t;x))\right)\right\|_{\mathcal{L}_2(H\times H; L^2(\Omega))} \\ &\quad \le C\,\|g\|_{C^2_b(H)}\left[\left(\frac{n}{\lambda^3}+n^{\frac{3}{4}\delta}\right)\sqrt{t}e^{\frac{1}{2}(2K+33C_B+1)t}+e^{16C_Bt}\right] \quad \forall t \ge 0, \end{aligned}$$

Choosing then

$$\bar{\alpha} := \frac{1}{2}(2K+33C_B+1) \vee 16C_B, \tag{4.37}$$

(which is independent of λ and n), the dominated convergence theorem implies that $\varphi_{\lambda,n} \in C^2_b(H)$ and, for $z_1, z_2 \in H$,

$$\begin{aligned} D^2\varphi_{\lambda_n}(x)[z_1,z_2] = \int_0^{+\infty} e^{-\alpha t}\,\mathbb{E}\Big[&Dg(u_{\lambda,n}(t;x))w^{z_1,z_2}_{\lambda,n}(t;x) \\ &+ D^2 g\left(u_{\lambda,n}(t;x)\right)\left[v^{z_1,z_2}_{\lambda,n}(t;x), v^{z_2}_{\lambda,n}(t;x)\right]\Big]\,\mathrm{d}t. \end{aligned}$$

It follows for every $x \in H$ that

$$\left\|D^2\varphi_{\lambda,n}(x)\right\|_{\mathcal{L}(H,H)} \le C\left(\frac{n}{\lambda^3}+n^{\frac{3}{4}\delta}\right)\|g\|_{C^2_b(H)}\int_0^{+\infty}(\sqrt{t}+1)e^{-(\alpha-\bar{\alpha})t}\,\mathrm{d}t,$$

hence for $\alpha > \bar{\alpha}$ this shows that $\varphi_{\lambda,n} \in C^2_b(H)$, as well as the estimate (4.31). Eventually, let us show that (4.32) holds. To this end, we readily note from the stochastic Eq. (4.25) that for every $x \in Z$ we have in particular that $u_{\lambda,n} \in L^2\left(\Omega; L^2(0,T;H)\right)$. Hence, the fact that $g \in C^2_b(H)$, Itô's formula, and the definition of $L^{\lambda,n}_0$ readily give for every $t \ge 0$ that

$$\mathbb{E}\,g\left(u_{\lambda,n}(t)\right) + \int_0^t \mathbb{E}\left(L^{\lambda,n}_0 g\right)\left(u_{\lambda,n}(s)\right)\,\mathrm{d}s = g(x).$$

Since $-L^{\lambda,n}_0$ coincides on $C^2_b(H)$ with the infinitesimal generator of the transition semigroup $P^{\lambda,n}$ associated to (4.25) (so in particular $L^{\lambda,n}_0$ and $P^{\lambda,n}$ commute on

$C_b^2(H)$), it holds in particular that

$$\int_0^t \mathbb{E}\left(L_0^{\lambda,n} g\right)\left(u_{\lambda,n}(s)\right) \mathrm{d}s = \int_0^t P_s^{\lambda,n}\left(L_0^{\lambda,n} g\right)(x)\, \mathrm{d}s = \int_0^t L_0^{\lambda,n}\left(P_s^{\lambda,n} g\right)(x)\, \mathrm{d}s$$
$$= L_0^{\lambda,n} \int_0^t P_s^{\lambda,n} g(x)\, \mathrm{d}s = L_0^{\lambda,n} \int_0^t \mathbb{E}\, g\left(u_{\lambda,n}(s)\right) \mathrm{d}s,$$

from which we get that

$$e^{-\alpha t}\, \mathbb{E}\, g\left(u_{\lambda,n}(t)\right) + \alpha \int_0^t e^{-\alpha s}\, \mathbb{E}\, g\left(u_{\lambda,n}(s)\right) \mathrm{d}s$$
$$+ L_0^{\lambda,n} \int_0^t e^{-\alpha s}\, \mathbb{E}\, g\left(u_{\lambda,n}(s)\right) \mathrm{d}s = g(x).$$

By boundedness of g, letting $t \to +\infty$ yields (4.32). This concludes the proof. □

4.3 Well posedness à la Friedrichs

We are now ready to show that the Kolmogorv Eq. (4.1) is well posed in the sense of Friedrichs, as rigorously specified in Proposition 4.4 below. This will allow to fully characterise the infinitesimal generator of the transition semigroup P on $L^2(\mathcal{A}, \mu)$ in terms of the Kolmogorov operator.

Proposition 4.4 *In the current setting, assume the dissipativity condition* (3.12)*, let* $\bar{\alpha}$ *be as in* (4.37)*, and let* $\alpha > \bar{\alpha}$*. Then, for every* $g \in L^2(\mathcal{A}, \mu)$ *there exist a unique* $\varphi \in L^2(\mathcal{A}, \mu)$ *and two sequences* $\{g_m\}_{m\in\mathbb{N}} \subset L^2(\mathcal{A}, \mu)$ *and* $\{\varphi_m\}_{m\in\mathbb{N}} \subset D(L_0)$ *such that*

$$\alpha\varphi_m + L_0\varphi_m = g_m \quad \mu\text{-a.s. in } \mathcal{A}, \quad \forall\, m \in \mathbb{N},$$

and, as $m \to \infty$,

$$\varphi_m \to \varphi \quad \text{in } L^2(\mathcal{A}, \mu), \qquad g_m \to g \quad \text{in } L^2(\mathcal{A}, \mu).$$

In particular, the range of $\alpha I + L_0$ *is dense in* $L^2(\mathcal{A}, \mu)$.

Proof of Proposition 4.4 Given $g \in L^2(\mathcal{A}, \mu)$, we define $\tilde{g} : H \to \mathbb{R}$ by extending g to zero outside $\mathcal{A}$, namely

$$\tilde{g}(x) := \begin{cases} g(x) & \text{if } x \in \mathcal{A}, \\ 0 & \text{if } x \in H \setminus \mathcal{A}. \end{cases}$$

Analogously, note that the probability measure $\mu \in \mathscr{P}(\mathcal{A})$ extends (uniquely) to a probability measure $\tilde{\mu} \in \mathscr{P}(H)$, by setting

$$\tilde{\mu}(E) := \mu(E \cap \mathcal{A}), \quad E \in \mathscr{B}(H).$$

With this notation, it is clear that $\tilde{g} \in L^2(H, \tilde{\mu})$: by density of $C_b^2(H)$ in $L^2(H, \tilde{\mu})$, there exists a sequence $\{\tilde{f}_j\}_j \subset C_b^2(H)$ such that (see Lemma B.1)

$$\lim_{j\to\infty} \|\tilde{f}_j - \tilde{g}\|_{L^2(H,\tilde{\mu})} = 0. \tag{4.38}$$

Clearly, setting $f_j := \left(\tilde{f}_j\right)_{|\mathcal{A}}$ for every $j \in \mathbb{N}$, one has that $\{f_j\}_{j\in\mathbb{N}} \subset D(L_0)$ by definition of $D(L_0)$, and also, thanks to the definition of $\tilde{\mu}$,

$$\lim_{j\to\infty} \left\| f_j - g \right\|_{L^2(\mathcal{A},\mu)} = 0. \tag{4.39}$$

Let $j \in \mathbb{N}$ be fixed: for every $\lambda > 0$ and $n \in \mathbb{N}$ we set

$$\tilde{\varphi}_{\lambda,n,j}(x) := \int_0^{+\infty} e^{-\alpha t}\, \mathbb{E}\left[\tilde{f}_j\left(u_{\lambda,n}(t;x)\right)\right] \mathrm{d}t, \quad x \in H,$$

and

$$\varphi_{\lambda,n,j} := \left(\tilde{\varphi}_{\lambda,n,j}\right)_{|\mathcal{A}}.$$

Lemma 4.3 ensures, for all $\lambda > 0$ and $n \in \mathbb{N}$, that $\tilde{\varphi}_{\lambda,n,j} \in C_b^2(H)$, hence in particular that $\varphi_{\lambda,n,j} \in D(L_0)$, and

$$\alpha\tilde{\varphi}_{\lambda,n,j}(x) + L_0^{\lambda,n}\tilde{\varphi}_{\lambda,n,j}(x) = \tilde{f}_j(x) \quad \forall\, x \in H.$$

It follows that, for every $x \in \mathcal{A}_{str}$,

$$\alpha\varphi_{\lambda,n,j}(x) + L_0\varphi_{\lambda,n,j}(x) = f_j(x) + L_0\varphi_{\lambda,n,j}(x) - L_0^{\lambda,n}\tilde{\varphi}_{\lambda,n,j}(x). \tag{4.40}$$

Let now $j \in \mathbb{N}$ be fixed. For every $x \in \mathcal{A}_{str}$ one has that

$$\begin{aligned}
&L_0\varphi_{\lambda,n,j}(x) - L_0^{\lambda,n}\tilde{\varphi}_{\lambda,n,j}(x)\\
&\quad= -\frac{1}{2}\,\mathrm{Tr}\left[B(x)^*D^2\varphi_{\lambda,n,j}(x)B(x) - B_{\lambda,n}(x)^*D^2\varphi_{\lambda,n,j}(x)B_{\lambda,n}(x)\right]\\
&\qquad+ \left(F'(x) - \mathcal{F}_{\lambda,n}(x),\, D\varphi_{\lambda,n,j}(x)\right)_H\\
&\quad= -\frac{1}{2}\,\mathrm{Tr}\left[\left(B(x)^* - B_{\lambda,n}(x)^*\right)D^2\varphi_{\lambda,n,j}(x)B(x)\right]\\
&\qquad- \frac{1}{2}\,\mathrm{Tr}\left[B_{\lambda,n}(x)^*D^2\varphi_{\lambda,n,j}(x)\left(B(x) - B_{\lambda,n}(x)\right)\right]\\
&\qquad+ \left(F'(x) - \mathcal{F}_{\lambda,n}(x),\, D\varphi_{\lambda,n,j}(x)\right)_H.
\end{aligned}$$

It follows from Lemma 4.3 that there exists a constant $C > 0$, which is independent of λ, n, and j, such that

$$\begin{aligned}
&|L_0\varphi_{\lambda,n,j}(x) - L_0^{\lambda,n}\tilde{\varphi}_{\lambda,n,j}(x)| \\
&\leq \left\|\varphi_{\lambda,n,j}\right\|_{C_b^2(H)} \left\|B_{\lambda,n}(x) - B(x)\right\|^2_{\mathcal{L}_{HS}(U,H)} \\
&\quad + \left\|\varphi_{\lambda,n,j}\right\|_{C_b^1(H)} \left\|\mathcal{F}_{\lambda,n}(x) - F'(x)\right\|_H \\
&\leq C\|\tilde{f}_j\|_{C_b^2(H)} \left(1 + \frac{n}{\lambda^3} + n^{\frac{3}{4}\delta}\right) \left\|B_{\lambda,n}(x) - B(x)\right\|^2_{\mathcal{L}_{HS}(U,H)} \\
&\quad + C\|\tilde{f}_j\|_{C_b^1(H)} \left\|\mathcal{F}_{\lambda,n}(x) - F'(x)\right\|_H .
\end{aligned} \tag{4.41}$$

Now, let us estimate the two terms on the right hand side. As for the first one, we have that

$$\begin{aligned}
\left\|B_{\lambda,n}(x) - B(x)\right\|^2_{\mathcal{L}_{HS}(U,H)} &\leq 3\left\|B_{\lambda,n}(x) - e^{-\frac{\mathcal{C}}{n^\delta}}B_\lambda(x)\right\|^2_{\mathcal{L}_{HS}(U,H)} &&=: 3I_1 \\
&\quad + 3\left\|e^{-\frac{\mathcal{C}}{n^\delta}}B_\lambda(x) - e^{-\frac{\mathcal{C}}{n^\delta}}B(x)\right\|^2_{\mathcal{L}_{HS}(U,H)} &&=: 3I_2 \\
&\quad + 3\left\|e^{-\frac{\mathcal{C}}{n^\delta}}B(x) - B(x)\right\|^2_{\mathcal{L}_{HS}(U,H)} &&=: 3I_3.
\end{aligned}$$

Exploiting the definition of $B_{\lambda,n}$, the Jensen inequality, and the fact that B_λ is $\sqrt{C_B}$-Lipschitz continuous on H, we have

$$\begin{aligned}
I_1 &= \left\|B_{\lambda,n}(x) - e^{-\frac{\mathcal{C}}{n^\delta}}B_\lambda(x)\right\|^2_{\mathcal{L}_{HS}(U,H)} \\
&= \left\|\int_H e^{-\frac{\mathcal{C}}{n^\delta}}\left(B_\lambda\left(e^{-\frac{\mathcal{C}}{n^\delta}}x + y\right) - B_\lambda(x)\right) N_{Q_{1/n^\delta}}(\mathrm{d}y)\right\|^2_{\mathcal{L}_{HS}(U,H)} \\
&\leq \int_H \left\|B_\lambda\left(e^{-\frac{\mathcal{C}}{n^\delta}}x + y\right) - B_\lambda(x)\right\|^2_{\mathcal{L}_{HS}(U,H)} N_{Q_{1/n^\delta}}(\mathrm{d}y) \\
&\leq C_B\int_H \left\|e^{-\frac{\mathcal{C}}{n^\delta}}x + y - x\right\|^2_H N_{Q_{1/n^\delta}}(\mathrm{d}y) \\
&\leq 2C_B\left\|e^{-\frac{\mathcal{C}}{n^\delta}}x - x\right\|^2_H + 2C_B\int_H \|y\|^2_H\ N_{Q_{1/n^\delta}}(\mathrm{d}y) \\
&\leq 4C_B\|x\|_H\left\|e^{-\frac{\mathcal{C}}{n^\delta}}x - x\right\|_H + 2C_B\int_H \|y\|^2_H\ N_{Q_{1/n^\delta}}(\mathrm{d}y)
\end{aligned}$$

from which we get, using definition (4.15), estimate (4.16), and the fact that $x \in \mathcal{A}_{str} \subset D(\mathcal{C}) = Z$ with $\|x\|_H \le |D|^{1/2}$,

$$
\begin{aligned}
I_1 &\le 4C_B \|x\|_H \int_0^{1/n^\delta} \left\| e^{-s\mathcal{C}} \mathcal{C} x \right\|_H \, \mathrm{d}s + 2C_B \operatorname{Tr}\left(Q_{1/n^\delta}\right) \\
&\le \frac{4C_B |D|^{1/2}}{n^\delta} \|x\|_Z + \frac{2C_B}{n^\delta} \left\| \mathcal{C}^{-1} \right\|^2_{\mathcal{L}_{HS}(H,H)}
\end{aligned}
$$

Possibly renominating C independently of λ, n, and j, this shows that

$$
I_1 \le \frac{C}{n^\delta} \left(1 + \|x\|_Z\right). \tag{4.42}
$$

As far as I_2 is concerned, it is immediate to see, thanks to the non-expansivity of $e^{-t\mathcal{C}}$ and the definition (4.9), that

$$
\begin{aligned}
I_2 &\le \|B_\lambda(x) - B(x)\|^2_{\mathcal{L}_{HS}(U,H)} = \sum_{k\in\mathbb{N}} \left| \left(\tilde{h}_k \star \rho_{\lambda^\gamma}\right)(x) - h_k(x) \right|^2 \\
&= \sum_{k\in\mathbb{N}} \left| \left(\tilde{h}_k \star \rho_{\lambda^\gamma}\right)(x) - \tilde{h}_k(x) \right|^2 \le \lambda^{2\gamma} \sum_{k\in\mathbb{N}} \left\| \tilde{h}_k' \right\|^2_{L^\infty(\mathbb{R})} \le C_B \lambda^{2\gamma}.
\end{aligned} \tag{4.43}
$$

Also, we have by the contraction of $e^{-\frac{\mathcal{C}}{n^\delta}}$ and the Hölder inequality that

$$
\begin{aligned}
I_3 &= \sum_{k\in\mathbb{N}} \left\| e^{-\frac{\mathcal{C}}{n^\delta}} h_k(x) - h_k(x) \right\|^2_H \le 2 \sum_{k\in\mathbb{N}} \|h_k(x)\|_H \left\| e^{-\frac{\mathcal{C}}{n^\delta}} h_k(x) - h_k(x) \right\|_H \\
&\le 2C_B^{1/2} \left(\sum_{k\in\mathbb{N}} \left\| \int_0^{1/n^\delta} e^{-s\mathcal{C}} \mathcal{C} h_k(x) \, \mathrm{d}s \right\|^2_H \right)^{1/2} \\
&\le \frac{2C_B^{1/2}}{n^{\delta/2}} \left(\sum_{k\in\mathbb{N}} \int_0^{1/n^\delta} \left\| e^{-s\mathcal{C}} \mathcal{C} h_k(x) \right\|^2_H \mathrm{d}s \right)^{1/2} \le \frac{2C_B^{1/2}}{n^\delta} \left(\sum_{k\in\mathbb{N}} \|\mathcal{C} h_k(x)\|^2_H \right)^{1/2},
\end{aligned}
$$

where, by definition of $\mathcal{C}$ and the regularity of $\{h_k\}_k$, it holds that

$$
\begin{aligned}
\|\mathcal{C} h_k(x)\|_H &\le \|h_k(x)\|_H + \left\| h_k'(x) \Delta x \right\|_H + \left\| h_k''(x) |\nabla x|^2 \right\|_H \\
&\le \|h_k\|_{C^2([-1,1])} \left(|D|^{1/2} + \|\Delta x\|_H + \left\| |\nabla x|^2 \right\|_H \right).
\end{aligned}
$$

Noting that $H^2(D) \hookrightarrow W^{1,4}(D) \hookrightarrow L^\infty(D)$, the Gagliardo-Nierenberg interpolation inequality yields that

$$
\|\nabla y\|_{L^4(D)} \lesssim_{D,d} \|y\|^{1/2}_{H^2(D)} \|y\|^{1/2}_{L^\infty(D)} + \|y\|_{L^1(D)} \quad \forall\, y \in H^2(D),
$$

hence, recalling that $\|x\|_{L^\infty(D)} \le 1$ for all $x \in \mathcal{A}_{str}$, one has

$$\left\| |\nabla x|^2 \right\|_H = \|\nabla x\|^2_{L^4(D)} \lesssim_D \|x\|_{H^2(D)} \|x\|_{L^\infty(D)} + \|x\|_{L^1(D)} \lesssim \|x\|_{H^2(D)} + 1.$$

Putting this information together, by **H2'** we deduce that there exists a positive constant C independent of λ, n, and j, such that

$$I_3 \le \frac{C}{n^\delta} (1 + \|x\|_Z) \left(\sum_{k\in\mathbb{N}} \|h_k\|^2_{C^2([-1,1])} \right)^{1/2} \le \frac{C}{n^\delta} (1 + \|x\|_Z) . \tag{4.44}$$

Going back to (4.41), we infer that, possibly renominating the constant C independently of λ, n, and j,

$$\begin{aligned} &|L_0\varphi_{\lambda,n,j}(x) - L_0^{\lambda,n}\tilde{\varphi}_{\lambda,n,j}(x)| \\ &\quad\le C\|\tilde{f}_j\|_{C^2_b(H)} \left(1 + \frac{n}{\lambda^3} + n^{\frac{3}{4}\delta}\right)\left(\frac{1}{n^\delta} + \lambda^{2\gamma}\right)(1 + \|x\|_Z) \\ &\qquad + C\|\tilde{f}_j\|_{C^1_b(H)} \left\|\mathcal{F}_{\lambda,n}(x) - F'(x)\right\|_H . \end{aligned}$$

As for the second term on the right hand side, we proceed as above, getting

$$\begin{aligned} \left\|\mathcal{F}_{\lambda,n}(x) - F'(x)\right\|_H \le &\left\|\mathcal{F}_{\lambda,n}(x) - e^{-\frac{\mathcal{C}}{n}}\mathcal{F}_\lambda(x)\right\|_H && =: J_1 \\ &+ \left\|e^{-\frac{\mathcal{C}}{n}}\mathcal{F}_\lambda(x) - e^{-\frac{\mathcal{C}}{n}}F'(x)\right\|_H && =: J_2 \\ &+ \left\|e^{-\frac{\mathcal{C}}{n}}F'(x) - F'(x)\right\|_H && =: J_3. \end{aligned}$$

The analogous computations as the term I_1 above imply, using the definition (4.18), the $\frac{1}{\lambda}$-Lipschitz continuity of $\mathcal{F}_\lambda : H \to H$, and (4.15)–(4.16), that

$$\begin{aligned} J_1 &\le \frac{1}{\lambda}\int_H \left\|e^{-\frac{\mathcal{C}}{n}}x + y - x\right\|_H N_{Q_{1/n}}(\mathrm{d}y) \\ &\le \frac{1}{\lambda}\int_0^{1/n} \left\|e^{-\frac{\mathcal{C}}{n}}\mathcal{C}x\right\|_H \mathrm{d}s + \frac{1}{\lambda}\int_H \|y\|_H N_{Q_{1/n}}(\mathrm{d}y) \\ &\le \frac{1}{\lambda n}\|x\|_Z + \frac{1}{\lambda}\sqrt{\mathrm{Tr}(Q_{1/n})} \le \frac{1}{\lambda n}\|x\|_Z + \frac{1}{\lambda\sqrt{n}}. \end{aligned} \tag{4.45}$$

Furthermore, the definition of Yosida approximation and (4.5), together with the contraction of $e^{-\frac{\mathcal{C}}{n}}$, yield

$$\begin{aligned} J_2 &\le \left\|\mathcal{F}_\lambda(x) - F'(x)\right\|_H = \left\|(\rho_{\lambda^2}\star\beta_\lambda)(x) - \beta(x)\right\|_H \\ &\le \left\|(\rho_{\lambda^2}\star\beta_\lambda)(x) - \beta_\lambda(x)\right\|_H + \|\beta_\lambda(x) - \beta(x)\|_H \\ &\le \lambda^2\frac{1}{\lambda} + \|\beta_\lambda(x) - \beta(x)\|_H \le \lambda + \|\beta_\lambda(x) - \beta(x)\|_H . \end{aligned} \tag{4.46}$$

Hence, exploiting (4.38), (4.45), and (4.46) in (4.45), possibly renominating the constant C independently of λ, n, and j, we obtain that

$$\begin{aligned}
&|L_0\varphi_{\lambda,n,j}(x) - L_0^{\lambda,n}\tilde{\varphi}_{\lambda,n,j}(x)| \\
&\leq C\|\tilde{f}_j\|_{C_b^2(H)}(1+\|x\|_Z)\left(1+\frac{n}{\lambda^3}+n^{\frac{3}{4}\delta}\right)\left(\frac{1}{n^\delta}+\lambda^{2\gamma}\right) \\
&+ C\|\tilde{f}_j\|_{C_b^1(H)}\left[(1+\|x\|_Z)\left(\lambda+\frac{1}{\lambda\sqrt{n}}\right)+\|\beta_\lambda(x)-\beta(x)\|_H+\left\|e^{-\frac{\mathcal{C}}{n}}F'(x)-F'(x)\right\|_H\right],
\end{aligned} \tag{4.47}$$

where the constant C is independent of λ, n, and j. Choosing the specific sequence $\lambda_n := n^{-1/4}$ in (4.47), we deduce for every $j, n \in \mathbb{N}$ that

$$\begin{aligned}
&\left|L_0\varphi_{\lambda_n,n,j}(x) - L_0^{\lambda_n,n}\tilde{\varphi}_{\lambda_n,n,j}(x)\right| \\
&\leq C\|\tilde{f}_j\|_{C_b^2(H)}(1+\|x\|_Z)\left(1+n^{\frac{7}{4}}+n^{\frac{3}{4}\delta}\right)\left(\frac{1}{n^\delta}+\frac{1}{n^{\frac{\gamma}{2}}}\right) \\
&+ C\|\tilde{f}_j\|_{C_b^1(H)}\left[\frac{2}{n^{\frac{1}{4}}}(1+\|x\|_Z)+\left\|\beta_{\lambda_n}(x)-\beta(x)\right\|_H+\left\|e^{-\frac{\mathcal{C}}{n}}F'(x)-F'(x)\right\|_H\right].
\end{aligned}$$

At this point, if we choose the rate coefficients γ and δ is such a way that

$$\delta > \frac{7}{4}, \quad \gamma > \frac{7}{2}, \quad \gamma > \frac{3}{2}\delta,$$

by setting for example

$$\gamma := 4, \qquad \delta := 2,$$

renominating C independently of n and j it easily follows that

$$\begin{aligned}
&\left|L_0\varphi_{\lambda_n,n,j}(x) - L_0^{\lambda_n,n}\tilde{\varphi}_{\lambda_n,n,j}(x)\right| \\
&\leq \frac{C}{n^{\frac{1}{4}}}\|\tilde{f}_j\|_{C_b^2(H)}(1+\|x\|_Z) \\
&+ C\|\tilde{f}_j\|_{C_b^1(H)}\left[\left\|\beta_{\lambda_n}(x)-\beta(x)\right\|_H+\left\|e^{-\frac{\mathcal{C}}{n}}F'(x)-F'(x)\right\|_H\right].
\end{aligned}$$

Since the right-hand side belongs to $L^2(\mathcal{A},\mu)$ thanks to Proposition 3.5, this yields integrating with respect to μ and renominating the constant C as usual that

$$\begin{aligned}
&\left\|L_0\varphi_{\lambda_n,n,j} - L_0^{\lambda_n,n}\tilde{\varphi}_{\lambda_n,n,j}\right\|_{L^2(\mathcal{A},\mu)}^2 \\
&\leq \frac{C}{n^{\frac{1}{2}}}\|\tilde{f}_j\|_{C_b^2(H)}^2 \\
&+ C\|\tilde{f}_j\|_{C_b^1(H)}^2\int_H\left(\left\|\beta_{\lambda_n}(x)-\beta(x)\right\|_H^2+\left\|e^{-\frac{\mathcal{C}}{n}}F'(x)-F'(x)\right\|_H^2\right)\mu(\mathrm{d}x)
\end{aligned} \tag{4.48}$$

We are ready now to construct the sequence $\{g_m\}_{m\in\mathbb{N}}$. Let $m \in \mathbb{N}$ be arbitrary. By virtue of (4.39), we can pick $j_m \in \mathbb{N}$ such that

$$\|f_{j_m} - g\|_{L^2(\mathcal{A},\mu)} \leq \frac{1}{m}.$$

Also, by the properties of the Yosida approximation and of the semigroup generated by $-\mathcal{C}$ one has $\left\|\beta_{\lambda_n}(x) - \beta(x)\right\|_H^2 + \left\|e^{-\frac{\mathcal{C}}{n}}F'(x) - F'(x)\right\|_H^2 \to 0$ as $n \to \infty$ for all $x \in \mathcal{A}_{str}$. Moreover, since also $|\beta_{\lambda_n}| \leq |\beta|$ by the properties of the Yosida approximation (see [4]), one has that $\left\|\beta_{\lambda_n}(x) - \beta(x)\right\|_H^2 + \left\|e^{-\frac{\mathcal{C}}{n}}F'(x) - F'(x)\right\|_H^2 \lesssim \|\beta(x)\|_H^2 + \left\|F'(x)\right\|_H^2 \lesssim_K 1 + \left\|F'(x)\right\|_H^2$. Consequently, the Proposition 3.5 and the dominated convergence theorem imply

$$\lim_{n\to\infty} \int_H \left(\left\|\beta_{\lambda_n}(x) - \beta(x)\right\|_H^2 + \left\|e^{-\frac{\mathcal{C}}{n}}F'(x) - F'(x)\right\|_H^2 \right) \mu(\mathrm{d}x) = 0.$$

Hence, given j_m as above, we can then choose $n = n_m \in \mathbb{N}$ sufficiently large such that

$$\frac{1}{n_m^{\frac{1}{2}}} \left\|\tilde{f}_{j_m}\right\|_{C_b^2(H)}^2 \leq \frac{1}{m^2}$$

and

$$\|\tilde{f}_{j_m}\|_{C_b^1(H)}^2 \int_H \left(\left\|\beta_{\lambda_n}(x) - \beta(x)\right\|_H^2 + \left\|e^{-\frac{\mathcal{C}}{n}}F'(x) - F'(x)\right\|_H^2 \right) \mu(\mathrm{d}x) \leq \frac{1}{m^2}.$$

Setting then

$$\varphi_m := \varphi_{\lambda_{n_m}, n_m, j_m} \in D(L_0),$$
$$g_m := f_{j_m} + L_0\varphi_{\lambda_{n_m}, n_m, j_m} - L_0^{\lambda_{n_m}, n_m} \tilde{\varphi}_{\lambda_{n_m}, n_m, j_m} \in L^2(\mathcal{A}, \mu),$$

thanks to (4.40) one has exactly

$$\alpha\varphi_m + L_0\varphi_m = g_m \quad \mu\text{-a.s. in } \mathcal{A}, \quad \forall m \in \mathbb{N},$$

while the estimate (4.48) yields, by the choices made above,

$$\|g_m - g\|_{L^2(\mathcal{A},\mu)} \leq \frac{C}{m} \longrightarrow 0 \quad \text{as } m \to \infty.$$

Also, we note that L is accretive in $L^2(\mathcal{A}, \mu)$ because it is the infinitesimal generator of the semigroup of contractions P on $L^2(\mathcal{A}, \mu)$: hence, since by Lemma 4.1 we know that $L_0 = L$ on $D(L_0)$, it is immediate to deduce that

$$\alpha \left\| \varphi_{m_1} - \varphi_{m_2} \right\|_{L^2(\mathcal{A},\mu)} \leq \left\| g_{m_1} - g_{m_2} \right\|_{L^2(\mathcal{A},\mu)} \quad \forall m_1, m_2 \in \mathbb{N}.$$

It follows that $\{\varphi_m\}_{m\in\mathbb{N}}$ is Cauchy in $L^2(\mathcal{A}, \mu)$, hence it converges to some $\varphi \in L^2(\mathcal{A}, \mu)$. It is not difficult to see by using again the accretivity that φ is unique, in the sense that it does not depend on the sequences $\{\varphi_m\}_{m\in\mathbb{N}}$ and $\{g_m\}_{m\in\mathbb{N}}$. This finally concludes the proof of Proposition 4.4. □

We are now ready to state the main result of this section, which completely characterises the infinitesimal generator of the transition semigroup P on $L^2(\mathcal{A}, \mu)$ in terms of the Kolmogorov operator L_0.

Theorem 4.5 *In the current setting, assume the dissipativity condition* (3.12). *Then, the Kolmogorov operator L_0 is closable in $L^2(\mathcal{A}, \mu)$, and its closure $\overline{L_0}$ coincides with the infinitesimal generator L of the transition semigroup P on $L^2(\mathcal{A}, \mu)$.*

Proof Since P is a semigroup of contractions in $L^2(\mathcal{A}, \mu)$, its infinitesimal generator L is m-accretive in $L^2(\mathcal{A}, \mu)$. Since by Lemma 4.1, we know that $L_0 = L$ in $D(L_0)$, it follows that L_0 is accretive in $L^2(\mathcal{A}, \mu)$, hence closable. Let $\left(\overline{L_0}, D\left(\overline{L_0}\right)\right)$ denote such closure and let $\alpha > \bar{\alpha}$, where $\bar{\alpha}$ is given as in (4.37). By the Lumer-Philips theorem, the range of $\alpha I + \overline{L_0}$ coincides with the closure in $L^2(\mathcal{A}, \mu)$ of the range of $\alpha I + L_0$. Since the range of $\alpha I + L_0$ is dense in $L^2(\mathcal{A}, \mu)$ by Proposition 4.4, it follows that $\left(\overline{L_0}, D\left(\overline{L_0}\right)\right)$ is m-accretive in $L^2(\mathcal{A}, \mu)$, hence it generates a strongly continuous semigroup of contractions in $L^2(\mathcal{A}, \mu)$. Since $D(L_0)$ is a core for $\overline{L_0}$ and $L = \overline{L_0}$ on $D(L_0)$, it follows that $(L, D(L)) = \left(\overline{L_0}, D\left(\overline{L_0}\right)\right)$. □

Acknowledgements The authors are members of Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA), Istituto Nazionale di Alta Matematica (INdAM), and gratefully acknowledge financial support through the project CUP_E55F22000270001.

Funding Open access funding provided by Politecnico di Milano within the CRUI-CARE Agreement.

Declarations

Conflict of interest There is no conflict of interest.

A A priori estimates

We collect here the needed apriori estimates on the solution process.

Lemma A.1 *Assume* **H1–H2**. *Then, for every initial datum $x \in \mathcal{A}$ the respective variational solution to Eq.* (1.4) *satisfies*

$$\mathbb{E} \sup_{r\in[0,t]} \|u(r;x)\|_H^2 + \int_0^t \mathbb{E}\left[\|u(s;x)\|_V^2\right] \mathrm{d}s \lesssim_{C_1,C_B,|D|,\nu} \left(\|x\|_H^2 + t\right).$$

Proof Let $u := u^x$ be the unique variational solution to Eq. (1.4) starting from x. The Itô formula for the squared H-norm $\|\cdot\|_H^2$ yields, for every $t \geq 0$, $\mathbb{P}$-almost surely,

$$\begin{aligned}\frac{1}{2}\|u(t)\|_H^2 &+ \nu \int_0^t \|\nabla u(s)\|_H^2 \,\mathrm{d}s + \int_0^t \left(F'(u(s)), u(s)\right)_H \mathrm{d}s \\ &= \frac{1}{2}\|x\|_H^2 + \int_0^t (u(s), B(u(s))\mathrm{d}W(s))_H + \frac{1}{2}\int_0^t \|B(u(s))\|_{\mathcal{L}_{HS}(U,H)}^2 \,\mathrm{d}s.\end{aligned} \tag{A.1}$$

By means of (2.1) we estimate

$$\int_0^t \left(F'(u), u\right)_H \mathrm{d}s \geq C_0 \int_0^t \|u(s)\|_H^2 \,\mathrm{d}s - C_1 t,$$

while from (2.4) we immediately get

$$\frac{1}{2}\int_0^t \|B(u(s))\|_{\mathcal{L}_{HS}(U,H)}^2 \,\mathrm{d}s \leq \frac{C_B|D|t}{2}.$$

Hence, taking the supremum in time and expectations in (A.1) we have

$$\begin{aligned}\frac{1}{2}\mathbb{E}\sup_{r\in[0,t]} \|u(r)\|_H^2 &+ \nu \int_0^t \mathbb{E}\,\|\nabla u(s)\|_H^2 \,\mathrm{d}s \\ &\leq \frac{1}{2}\|x\|_H^2 + \left(\frac{C_B|D|}{2} + C_1\right) t + \mathbb{E}\sup_{r\in[0,t]} \int_0^r (u(s), B(u(s))\mathrm{d}W(s))_H.\end{aligned}$$

By means of the Burkholder-Davis-Gundy and Young inequalities (see [37, Lemma 4.3] and [40, Lemma A.1] for details), we estimate the last term in the above expression as

$$\mathbb{E}\sup_{r\in[0,t]} \int_0^r (u(s), B(u(s))\mathrm{d}W(s))_H \leq \frac{1}{4}\mathbb{E}\sup_{r\in[0,t]} \|u(r)\|_H^2 + Ct,$$

where the constant C depends only on C_B and $|D|$ (not on t). Combining the above estimates, the thesis follows. □

Lemma A.2 *Assume* **H1–H2**. *Then, for every initial datum $x \in \mathcal{A} \cap V$ the respective analytically strong solution to Eq.* (1.4) *satisfies*

$$\mathbb{E} \sup_{r\in[0,t]} \|u(r;x)\|_V^2 + \int_0^t \mathbb{E}\|u(s;x)\|_Z^2 \,\mathrm{d}s \lesssim_{K,C_0,C_1,C_B,|D|,\nu} \left(\|x\|_V^2 + t\right).$$

Proof Let $u := u^x$ be the unique analytically strong solution to Eq. (1.4) starting from x. We apply the Itô formula in [45, Theorem 4.2] (see also [50, Prop. 3.3]) to the functional $\frac{1}{2}\|\nabla \cdot\|_H^2$. We obtain, for any $t \geq 0$, $\mathbb{P}$-almost surely,

$$\begin{aligned}
&\frac{1}{2}\|\nabla u(t)\|_H^2 + \nu \int_0^t \|\Delta u(s)\|_H^2 \,\mathrm{d}s + \int_0^t \int_D F''(u(s))|\nabla u(s)|^2 \,\mathrm{d}s \\
&\quad = \frac{1}{2}\|\nabla x\|_H^2 + \frac{1}{2}\int_0^t \sum_{k\in\mathbb{N}} \int_D |h_k'(u(s))\nabla u(s)|^2 \,\mathrm{d}s \\
&\qquad - \int_0^t (\Delta u(s), B(u(s))\,\mathrm{d}W(s))_H \,.
\end{aligned} \tag{A.2}$$

By **H1** we have that

$$\int_0^t \int_D F''(u(s))|\nabla u(s)|^2 \,\mathrm{d}s \geq -K \int_0^t \|\nabla u(s)\|_H^2 \,\mathrm{d}s,$$

while assumption **H2** yields

$$\frac{1}{2}\int_0^t \sum_{k\in\mathbb{N}} \int_D |h_k'(u(s))\nabla u(s)|^2 \,\mathrm{d}s \leq \frac{C_B}{2}\int_0^t \|\nabla(u(s))\|_H^2 \,\mathrm{d}s.$$

Using again the Burkholder-Davis-Gundy and Young inequalities (as in [37, Lem. 4.3]) together with (2.4) we get that

$$\mathbb{E} \sup_{r\in[0,t]} \int_0^r (\Delta u(s), B(u(s))\mathrm{d}W(s))_H \leq \frac{\nu}{2}\mathbb{E}\int_0^t \|\Delta u(s)\|_H^2 \,\mathrm{d}s + Ct,$$

where the constant C depends only on ν, C_B, and $|D|$ (not on t). Hence, taking supremum in time and expectations in (A.2), rearranging the terms we obtain

$$\begin{aligned}
&\frac{1}{2}\mathbb{E} \sup_{r\in[0,t]} \left[\|\nabla u(r)\|_H^2\right] + \frac{\nu}{2}\int_0^t \mathbb{E}\,\|\Delta u(s)\|_H^2 \,\mathrm{d}s \\
&\quad \leq \frac{1}{2}\|\nabla x\|_H^2 + \left(K + \frac{C_B}{2}\right)\int_0^t \mathbb{E}\left[\|\nabla u(s)\|_H^2\right] \mathrm{d}s + Ct,
\end{aligned}$$

and the thesis follows from Lemma A.1. □

Lemma A.3 *Assume* **H1–H2**. *Then, for every initial datum* $x \in \mathcal{A} \cap V$ *the respective analytically strong solution to Eq.* (1.4) *satisfies*

$$\int_0^t \mathbb{E}\|F'(u(s;x))\|_H^2 \,\mathrm{d}s \lesssim_{K,C_0,C_1,C_B,|D|,\nu} (1+t)\,.$$

Proof As a consequence of Itô formula on suitable Yosida-type approximations of F' (see [7, Sec. 4.2] for details) it holds for every $t \geq 0$ that

$$\mathbb{E}\int_D F(u(t;x)) + \nu\,\mathbb{E}\int_0^t \int_D F''(u(s;x))|\nabla u(s;x)|^2\,\mathrm{d}s + \mathbb{E}\int_0^t \left\| F'(u(s;x))\right\|_H^2\,\mathrm{d}s \\ \leq \int_D F(x) + \frac{1}{2}\,\mathbb{E}\int_0^t \sum_{k\in\mathbb{N}} \int_D F''(u(s;x))|h_k(u(s;x))|^2\,\mathrm{d}s.$$

Combining then assumptions **H1–H2** yields

$$E\int_0^t \left\| F'(u(s;x))\right\|_H^2\,\mathrm{d}s \leq |D|\,\|F\|_{C([-1,1])} + K\nu\int_0^t \|\nabla u(s;x)\|_H^2\,\mathrm{d}s + \frac{C_B|D|}{2}t,$$

and the thesis follows from Lemma A.2. □

B A density result

Lemma B.1 *Let $\tilde{\mu} \in \mathscr{P}(H)$ and $\tilde{g} \in L^2(H,\tilde{\mu})$. Then, there exists a sequence $\{\tilde{f}_j\}_{j\in\mathbb{N}} \subset C_b^2(H)$ such that*

$$\lim_{j\to\infty} \left\| \tilde{f}_j - \tilde{g} \right\|_{L^2(H,\tilde{\mu})} = 0.$$

Proof For every $i \in \mathbb{N}$ we define $T_i : \mathbb{R} \to \mathbb{R}$ as $T_i(r) := \max\{-i, \min\{r, i\}\}$, $r \in \mathbb{R}$. Then, for every $\ell \in \mathbb{N}$ we set

$$\tilde{f}_{i,\ell}(x) := R_{1/\ell}T_i(\tilde{g})(x) = \int_H T_i\left(\tilde{g}\left(e^{-\frac{\mathcal{C}}{\ell}}x + y\right)\right) N_{Q_{1/\ell}}(\mathrm{d}y), \quad x \in H.$$

Since $T_i(\tilde{g}) \in \mathscr{B}(H)$ and R is strong Feller, we have that $\tilde{f}_{i,\ell} \in UC_b^\infty(H)$ for every $i,\ell \in \mathbb{N}$. Moreover, since R extends to a strongly continuous semigroup on $L^2(H,\tilde{\mu})$, one has by the dominated convergence theorem that

$$\lim_{\ell\to\infty} \|\tilde{f}_{i,\ell} - T_i(\tilde{g})\|_{L^2(H,\tilde{\mu})} = 0 \quad \forall i \in \mathbb{N}, \qquad \lim_{i\to\infty} \|T_i(\tilde{g}) - \tilde{g}\|_{L^2(H,\tilde{\mu})} = 0,$$

so the conclusion follows trivially. □

References

1. Addona, D., Masiero, F., Priola, E.: A BSDEs approach to pathwise uniqueness for stochastic evolution equations (2021)
2. Aliprantis, C.D., Border, K.C.: Infinite dimensional analysis, 3rd edn. Springer, Berlin (2006)
3. Allen, S.M., Cahn, J.W.: A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening. Acta Metall. **27**(6), 1085–1095 (1979)

4. Barbu, V.: Nonlinear differential equations of monotone types in Banach spaces. Springer, New York (2010)
5. Barbu, V., Da Prato, G.: Ergodicity for nonlinear stochastic equations in variational formulation. Appl. Math. Optim. **53**(2), 121–139 (2006)
6. Bauzet, C., Bonetti, E., Bonfanti, G., Lebon, F., Vallet, G.: A global existence and uniqueness result for a stochastic Allen–Cahn equation with constraint. Math. Methods Appl. Sci. **40**(14), 5241–5261 (2017)
7. Bertacco, F.: Stochastic Allen–Cahn equation with logarithmic potential. Nonlinear Anal. **202**, 22 (2021)
8. Bertacco, F., Orrieri, C., Scarpa, L.: Random separation property for stochastic Allen-Cahn-type equations. Electron. J. Probab. **27**, 1–32 (2022). https://doi.org/10.1214/22-EJP830
9. Bessaih, H., Ferrario, B.: Invariant measures for stochastic damped 2D Euler equations. Commun. Math. Phys. **377**(1), 531–549 (2020)
10. Blömker, D., Gawron, B., Wanner, T.: Nucleation in the one-dimensional stochastic Cahn–Hilliard model. Discrete Contin. Dyn. Syst. **27**(1), 25–52 (2010)
11. Blömker, D., Maier-Paape, S., Wanner, T.: Second phase spinodal decomposition for the Cahn–Hilliard–Cook equation. Trans. Amer. Math. Soc. **360**(1), 449–489 (2008)
12. Bogachev, V.I.: Measure theory. Springer-Verlag, Berlin (2007)
13. Brzeźniak, Z., Ferrario, B., Zanella, M.: Invariant measures for a stochastic nonlinear and damped 2D Schrödinger equation. ArXiv:2106.07043 (2021)
14. Brzeźniak, Z., Ferrario, B., Zanella, M.: Ergodic results for the stochastic nonlinear Schrödinger equation with large damping, ArXiv:2205.13364 (2022)
15. Cahn, J.W., Hilliard, J.E.: Free energy of a nonuniform system. I. Interfacial free energy. J. Chem. Phys. **28**(2), 258–267 (1958)
16. Cerrai, S.: Second order PDE's in finite and infinite dimension. Springer-Verlag, Berlin (2001)
17. Cook, H.: Brownian motion in spinodal decomposition. Acta Metall. **18**(3), 297–306 (1970)
18. Da Prato, G.: Kolmogorov equations for stochastic PDEs. Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser Verlag, Basel (2004)
19. Da Prato, G.: Kolmogorov equations for stochastic PDE's with multiplicative noise. In: Stochastic analysis and applications, volume 2 of Abel Symp., pp. 235–263. Springer, Berlin, (2007)
20. Da Prato, G., Debussche, A.: Stochastic Cahn–Hilliard equation. Nonlinear Anal. **26**(2), 241–263 (1996)
21. Da Prato, G., Debussche, A.: Ergodicity for the 3D stochastic Navier–Stokes equations. J. Math. Pures Appl. **82**(8), 877–947 (2003)
22. Da Prato, G., Zabczyk, J.: Ergodicity for infinite-dimensional systems. Cambridge University Press, Cambridge (1996)
23. Da Prato, G., Zabczyk, J.: Second order partial differential equations in Hilbert spaces. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge (2002)
24. Da Prato, G., Zabczyk, J.: Stochastic equations in infinite dimensions. Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge (2014)
25. Dareiotis, K., Gess, B., Gnann, M.V., Grün, G.: Non-negative Martingale solutions to the stochastic thin-film equation with nonlinear gradient noise. Arch. Ration. Mech. Anal. **242**(1), 179–234 (2021)
26. Dareiotis, K., Gess, B., Tsatsoulis, P.: Ergodicity for stochastic porous media equations with multiplicative noise. SIAM J. Math. Anal. **52**(5), 4524–4564 (2020)
27. Debussche, A., Odasso, C.: Ergodicity for a weakly damped stochastic non-linear schrödinger equation. J. Evolut. Eq. **5**(3), 317–356 (2005)
28. Edwards, R.E.: Functional Analysis. Theory and Applications. Holt Rinehart and Winston, New York-Toronto-London (1965)
29. Es-Sarhir, A., Stannat, W.: Invariant measures for semilinear SPDE's with local Lipschitz drift coefficients and applications. J. Evol. Equ. **8**(1), 129–154 (2008)
30. Es-Sarhir, A., Stannat, W.: Improved moment estimates for invariant measures of semilinear diffusions in Hilbert spaces and applications. J. Funct. Anal. **259**(5), 1248–1272 (2010)
31. Fischer, J., Grün, G.: Existence of positive solutions to stochastic thin-film equations. SIAM J. Math. Anal. **50**(1), 411–455 (2018)
32. Flory, P.J.: Thermodynamics of high polymer solutions. J. Chem. Phys. **10**(1), 51–61 (1942)
33. Glatt-Holtz, N., Mattingly, J.C., Richards, G.: On unique ergodicity in nonlinear stochastic partial differential equations. J. Stat. Phys. **166**(3–4), 618–649 (2017)

34. Hairer, M., Mattingly, J.C.: Ergodicity of the 2D Navier–Stokes equations with degenerate stochastic forcing. Ann. of Math. **164**(3), 993–1032 (2006)
35. Hairer, M., Ryser, M.D., Weber, H.: Triviality of the 2D stochastic Allen–Cahn equation. Electron. J. Probab. **17**, 14 (2012)
36. Marinelli, C., Prévôt, C., Röckner, M.: Regular dependence on initial data for stochastic evolution equations with multiplicative Poisson noise. J. Funct. Anal. **258**(2), 616–649 (2010)
37. Marinelli, C., Scarpa, L.: A variational approach to dissipative SPDEs with singular drift. Ann. Probab. **46**(3), 1455–1497 (2018)
38. Marinelli, C., Scarpa, L.: Ergodicity and Kolmogorov equations for dissipative SPDEs with singular drift: a variational approach. Potential Anal. **52**(1), 69–103 (2020)
39. Marinelli, C., Scarpa, L.: Fréchet differentiability of mild solutions to SPDEs with respect to the initial datum. J. Evol. Equ. **20**(3), 1093–1130 (2020)
40. Marinelli, C., Scarpa, L.: Refined existence and regularity results for a class of semilinear dissipative SPDEs. Infin. Dimens. Anal. Quantum Probab. Relat. Top. **23**(2), 0014 (2020)
41. Marinelli, C., Ziglio, G.: Ergodicity for nonlinear stochastic evolution equations with multiplicative Poisson noise. Dyn. Partial Differ. Equ. **7**(1), 1–23 (2010)
42. Maurelli, M.: Non-explosion by Stratonovich noise for ODEs. Electron. Commun. Probab. **25**, 10 (2020)
43. Ondreját, M.: Brownian representations of cylindrical local martingales, martingale problem and strong Markov property of weak solutions of SPDEs in Banach spaces. Czechoslovak Math. J. **55**, 1003–1039 (2005)
44. Orrieri, C., Scarpa, L.: Singular stochastic Allen–Cahn equations with dynamic boundary conditions. J. Differ. Eq. **266**(8), 4624–4667 (2019)
45. Pardoux, E.: Équations aux dérivées partielles stochastiques nonlinéaires monotones. Ph.D. Thesis. Université Paris XI (1975)
46. Röckner, M., Zhang, X.: Stochastic tamed 3D Navier-Stokes equations: existence, uniqueness and ergodicity. Probab. Theory Related Fields **145**(1–2), 211–267 (2009)
47. Scarpa, L.: On the stochastic Cahn–Hilliard equation with a singular double-well potential. Nonlinear Anal. **171**, 102–133 (2018)
48. Scarpa, L.: Analysis and optimal velocity control of a stochastic convective Cahn–Hilliard equation. J. Nonlinear Sci. **31**(2), 57 (2021)
49. Scarpa, L.: The stochastic Cahn–Hilliard equation with degenerate mobility and logarithmic potential. Nonlinearity **34**(6), 3813–3857 (2021)
50. Scarpa, L., Stefanelli, U.: Doubly nonlinear stochastic evolution equations II. Stoch. Partial Differ. Equ. Anal. Comput. (2022)
51. Stannat, W.: Stochastic partial differential equations: Kolmogorov operators and invariant measures. Jahresber. Dtsch. Math.-Ver. **113**(2), 81–109 (2011)