Well-posedness of mild solutions to the drift-diffusion and the vorticity equations in amalgam spaces

AUTHOR(S):

Suguro, Takeshi

CITATION:

Suguro, Takeshi. Well-posedness of mild solutions to the drift-diffusion and the vorticity equations in amalgam spaces. Journal of Mathematical Analysis and Applications 2023, 520(1): 126843.

ISSUE DATE:

2023-04-01

URL:

http://hdl.handle.net/2433/278780

J. Math. Anal. Appl. 520 (2023) 126843

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Regular Articles

Well-posedness of mild solutions to the drift-diffusion and the vorticity equations in amalgam spaces

Takeshi Suguro

Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan

A R T I C L E   I N F O

Article history:
Received 4 July 2022
Available online 8 November 2022
Submitted by P.G. Lemarié-Rieusset

Keywords:
Well-posedness
Drift-diffusion equation
Keller–Segel system
Amalgam spaces
Vorticity equation

A B S T R A C T

We consider the Cauchy problem of the drift-diffusion and the vorticity equations. Both equations involve the Poisson equation and a nonlocal effect of the Green's function influences the solution to the problem. In this paper, we study the well-posedness of the drift-diffusion and the vorticity equations by using amalgam spaces of Lebesgue spaces. Moreover, we show the unconditional uniqueness of mild solutions to the drift-diffusion equation in amalgam spaces.

1. Introduction

We consider the Cauchy problem of the drift-diffusion equation on n dimensional Euclidean space $\mathbb{R}^n$:

$$\begin{cases} \partial_t u - \Delta u + \nabla \cdot (u \nabla \psi) = 0, & t > 0,\ x \in \mathbb{R}^n, \\ -\Delta \psi = u, & t > 0,\ x \in \mathbb{R}^n, \\ \quad u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $u_0 = u_0(x) : \mathbb{R}^n \to \mathbb{R}$ is a given initial data. The above equation is strongly related to the model of self-interacting particles. The unknown functions $u = u(t,x) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ and $\psi = \psi(t,x) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ represent the density and the self-interacting potential of particles such as gravity, respectively. The solution ψ to the second equation in (1.1) is represented by using the Riesz potential as follows:

E-mail address: suguro@kurims.kyoto-u.ac.jp.

https://doi.org/10.1016/j.jmaa.2022.126843

$$\psi(t,x)=(-\Delta)^{-1}u(t,x)=\begin{cases}-\dfrac{1}{2\pi}\displaystyle\int_{\mathbb{R}^2}u(t,y)\log|x-y|\,dy & \text{if } n=2,\\ \dfrac{1}{(n-2)\omega_{n-1}}\displaystyle\int_{\mathbb{R}^n}\dfrac{u(t,y)}{|x-y|^{n-2}}\,dy & \text{if } n\geq 3,\end{cases} \tag{1.2}$$

where $\omega_{n-1}\equiv 2\pi^{n/2}/\Gamma(n/2)$.

The equation (1.1) possesses a scaling invariant structure and according to the Fujita–Kato principle (Fujita–Kato [14], Cazenave–Weissler [9]), it is important to consider the well-posedness in the scaling critical space. Let (u,ψ) be a solution to (1.1), and for $\mu>0$ we define (u_μ,ψ_μ) by

$$u_\mu(t,x)\equiv\mu^2u(\mu^2t,\mu x),\quad \psi_\mu(t,x)\equiv\psi(\mu^2t,\mu x),$$

then (u_μ,ψ_μ) is also a solution to (1.1) and we have

$$\sup_{t>0}\|u_\mu(t)\|_{\frac{n}{2}}=\sup_{t>0}\|u(t)\|_{\frac{n}{2}}.$$

Hence the exponent $p=n/2$ is critical by the scaling argument for (1.1).

Concerning the well-posedness issue for the scaling critical setting, Biler [3] considered the problem (1.1) in the Morrey space M^p, which contains a singular stationary solution to (1.1). Kurokiba–Ogawa [30,31] studied local and global solutions in Lebesgue spaces via the energy method (see also Corrias–Perthame–Zaag [10], Kozono–Sugiyama [28], Ogawa–Shimizu [36]). In more general cases, Iwabuchi [19], Iwabuchi–Nakamura [20], Iwabuchi–Ogawa [21] considered the well-posedness and ill-posedness of (1.1) in critical Besov and Lizorkin–Triebel spaces (see also Biler–Cannone–Guerra–Karch [4], Lemarié-Rieusset [32]).

As a slightly different system, the author [38] considered the Cauchy problem of the parabolic-elliptic Keller–Segel system:

$$\begin{cases}\partial_t u-\Delta u+\nabla\cdot(u\nabla\psi)=0, & t>0,\ x\in\mathbb{R}^n,\\ -\Delta\psi+\lambda\psi=u, & t>0,\ x\in\mathbb{R}^n,\\ u(0,x)=u_0(x), & x\in\mathbb{R}^n,\end{cases} \tag{1.3}$$

where $\lambda>0$. The problem (1.3) is a simplified Keller–Segel model. Keller and Segel [25] introduced this model for describing a model of chemotactic aggregation of microorganisms (see also Patlak [37] for earlier model). In the chemotaxis model, the unknown functions u and ψ denote the density of microorganisms and the concentration of a chemical-attractant by themselves, respectively. Afterwards, Jäger and Luckhaus [22] gave the parabolic-elliptic system (1.3) (see also Nagai [34], Nagai–Mimura [35]). In this case, ψ can be represented by the following:

$$\psi(t,x)=(\mathcal{B}_\lambda*u)(t,x)\quad\text{where } \mathcal{B}_\lambda(x)\equiv(4\pi)^{-\frac{n}{2}}\int_0^\infty e^{-\frac{|x|^2}{4s}}e^{-\lambda s}s^{-\frac{n-2}{2}}\frac{ds}{s},$$

where $\mathcal{B}_\lambda$ is called the Bessel potential. The Bessel potential behaves like the Riesz potential near the origin and decays exponentially at spatial infinity. In the recent paper [38], the author showed the well-posedness of (1.3) in uniformly local Lebesgue spaces, which was introduced by Kato [23] as follows:

$$L^p_{\mathrm{ul}}\equiv\left\{f\in L^p_{\mathrm{loc}}(\mathbb{R}^n);\ \|f\|_{L^p_{\mathrm{ul}}}\equiv\sup_{x\in\mathbb{R}^n}\left(\int_{B_1(x)}|f(y)|^p\,dy\right)^{\frac{1}{p}}<+\infty\right\} \tag{1.4}$$

for $1 \le p < \infty$ and $B_1(x) \equiv \{y \in \mathbb{R}^n;\ |y-x| < 1\}$ with $x \in \mathbb{R}^n$. Such a space contains non-decaying functions, for example, constant functions belong to L^p_{ul}. A constant function is a solution to the problem (1.3) and belongs to uniformly local Lebesgue spaces. In fact, for $\lambda > 0$, the constant function $(u, \psi) = (1, 1/\lambda)$ solves (1.3). Cygan–Karch–Krawczyk–Wakui [11] studied the time-local existence of a mild solution to (1.3) with $\lambda = 1$ in uniformly local Lebesgue spaces and considered the stability of the constant solution.

In the case of $\lambda = 0$, the integral kernel of the Poisson equation governing the potential ψ is given by the Riesz potential in $\mathbb{R}^n$ and it decays by $O(|x|^{-(n-2)})$ as $|x| \to \infty$ for $n \ge 3$ as is shown in (1.2). Therefore, the nonlocal effect from the potential ψ is stronger and the behavior of u at the spatial infinity reflects itself more than the case of $\lambda > 0$. Then it is natural and interesting to ask if we may solve the problem (1.1) in non-decaying function classes. Unfortunately, it is not straightforward to obtain the well-posedness of a solution in uniformly local spaces such as L^p_{ul} defined above. Hence, we adjust our question into a possible setting of solution class, where the functions decay slower at infinity than the conventional Lebesgue space. Along with such motivation, we introduce amalgam spaces defined by the following: For $1 \le p < \infty$ and $1 \le \nu \le \infty$, let $L^{p,\nu}_{\mathrm{A}}$ denote amalgam spaces

$$L^{p,\nu}_{\mathrm{A}} \equiv \{f \in L^p_{\mathrm{loc}}(\mathbb{R}^n);\ \|f\|_{L^{p,\nu}_{\mathrm{A}}} < +\infty\},$$

where we define

$$\|f\|_{L^{p,\nu}_{\mathrm{A}}} \equiv \begin{cases} \left(\sum_{k \in \mathbb{Z}^n} \|f\|^{\nu}_{L^p(Q(k,\frac{1}{2}))} \right)^{\frac{1}{\nu}} & \text{if } 1 \le \nu < \infty, \\ \sup_{k \in \mathbb{Z}^n} \left(\int_{Q(k,\frac{1}{2})} |f(y)|^p \, dy \right)^{\frac{1}{p}} & \text{if } \nu = \infty \end{cases} \tag{1.5}$$

Here, we set $Q(k,\theta) \equiv \{y \in \mathbb{R}^n;\ \max_{1 \le i \le n} |y_i - k_i| \le \theta,\ k = (k_1, k_2, \ldots, k_n)\}$. By the definition of the norm (1.5), $L^{p,p}_{\mathrm{A}}$ and $L^{p,\infty}_{\mathrm{A}}$ coincide L^p and L^p_{ul}, respectively. We note that if $p < \nu$, then $L^p \subsetneq L^{p,\nu}_{\mathrm{A}}$. In contrast with uniformly local Lebesgue spaces, C_0^∞ is dense in $L^{p,\nu}_{\mathrm{A}}$ for $1 \le p, \nu < \infty$. Holland [18] introduced this space (see Wiener [39,40] for particular cases), and Fournier–Stewart [13] and Feichtinger [12] developed a theory of amalgam spaces. By the definition of norms (1.5), we regard amalgam spaces as an amalgamation of a local L^p with a global L^ν (see Gröchenig–Heil–Okoudjou [17]).

We introduce the mild solution to the problem (1.1) defined as follows.

Definition *(The mild solution).* Let $1 \le p < \infty$ and $1 \le \nu < \infty$. For an initial data $u_0 \in L^{p,\nu}_{\mathrm{A}}$ and $T > 0$, we say that u is *a mild solution* to (1.1) if

$$u(t) = e^{t\Delta} u_0 - \int_0^t \nabla e^{(t-s)\Delta} \cdot (u(s) \nabla \psi(s)) \, ds \tag{1.6}$$

in $C([0,T); L^{p,\nu}_{\mathrm{A}}(\mathbb{R}^n))$, where ψ is defined as (1.2).

We now state our main result for the well-posedness of a mild solution to (1.1) in amalgam spaces.

Theorem 1.1 *(The local well-posedness). Let $n \ge 2$. Assume $p \ge n/2$, $1 \le \nu < n$, and $q = \min\{2np/(2n - p), 2p\}$. Suppose that $1 \le \nu_1 < \infty$ satisfies*

$$\frac{1}{n} < \frac{2}{\nu_1} - \frac{1}{\nu} < 1. \tag{1.7}$$

For any $u_0 \in L^{p,\nu}_{\mathrm{A}}$*, there exist* $T > 0$ *and a unique mild solution* $u \in C([0,T); L^{p,\nu}_{\mathrm{A}}) \cap C((0,T); L^{q,\nu_1}_{\mathrm{A}})$ *to* (1.1)*. Furthermore, the Cauchy problem* (1.1) *is well-posed in* $C([0,T); L^{p,\nu}_{\mathrm{A}}) \cap C((0,T); L^{q,\nu_1}_{\mathrm{A}})$*: For any* $u_0, v_0 \in L^{p,\nu}_{\mathrm{A}}$ *and* $u(t)$ *and* $v(t)$ *be the corresponding solutions to* (1.1)*,*

$$u \to v \quad \text{in } C([0,T); L^{p,\nu}_{\mathrm{A}}) \quad \text{as } u_0 \to v_0 \quad \text{in } L^{p,\nu}_{\mathrm{A}}.$$

As a corollary of Theorem 1.1, we obtain the existence of a time-global mild solution to (1.1).

Corollary 1.2 *(The small data global existence). Let* $n \geq 2$*. There exists a constant* $\varepsilon_0 > 0$ *such that for all* $\|u_0\|_{L^{n/2}} \leq \varepsilon_0$*, there exists a global solution* $u \in C([0,\infty); L^{n/2}) \cap C((0,\infty); L^{2n/3})$ *to* (1.1)*.*

The problem (1.1) is strongly related to the Cauchy problem of the incompressible Navier–Stokes equation:

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla) u + \nabla p = 0, & t > 0,\ x \in \mathbb{R}^n, \\ \operatorname{div} u = 0, & t > 0,\ x \in \mathbb{R}^n, \\ \quad u(0,x) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.8}$$

Koch–Tataru [27] considered the Cauchy problem (1.8) in BMO^{-1} and Maekawa–Terasawa [33] constructed a mild solution to the problem (1.8) in uniformly local spaces. Recently, Bradshaw–Tsai [6] considered local energy solutions to (1.8) in amalgam spaces $L^{2,q}_{\mathrm{A}}$ for $2 \leq q < \infty$ and Bradshaw–Lai–Tsai [5] constructed a mild solution to the problem (1.8) in the amalgam spaces. In the case of $n = 2$ and $n = 3$, the problem (1.8) can be expressed in vorticity formulation (see Giga–Miyakawa [15], Giga–Miyakawa–Osada [16]). If we set

$$\omega \equiv \operatorname{rot} u = \begin{cases} \partial_1 u_2 - \partial_2 u_1 & \text{if } n = 2, \\ (\partial_2 u_3 - \partial_3 u_2, \partial_3 u_1 - \partial_1 u_3, \partial_1 u_2 - \partial_2 u_3) & \text{if } n = 3, \end{cases}$$

then ω satisfies the vorticity equation

$$\begin{cases} \partial_t \omega - \Delta \omega + u \cdot \nabla \omega = 0, & t > 0,\ x \in \mathbb{R}^2, \\ -\Delta u = \nabla^{\perp} \omega, & t > 0,\ x \in \mathbb{R}^2, \\ \quad \omega(0,x) = \omega_0(x) \equiv \operatorname{rot} u_0(x), & x \in \mathbb{R}^2, \end{cases} \tag{1.9}$$

$$\begin{cases} \partial_t \omega - \Delta \omega + (u \cdot \nabla) \omega - (\omega \cdot \nabla) u = 0, & t > 0,\ x \in \mathbb{R}^3, \\ -\Delta u = \operatorname{rot} \omega, & t > 0,\ x \in \mathbb{R}^3, \\ \quad \omega(0,x) = \omega_0(x) \equiv \operatorname{rot} u_0(x), & x \in \mathbb{R}^3. \end{cases} \tag{1.10}$$

By the Biot–Savart law, one can write

$$u(t,x) = \begin{cases} (\tilde{K} * \omega)(t,x) & \text{if } n = 2, \\ (\tilde{K} \times \omega)(t,x) & \text{if } n = 3, \end{cases} \quad \text{where } \tilde{K}(x) \equiv \begin{cases} \dfrac{1}{2\pi} \left(-\dfrac{x_2}{|x|^2}, \dfrac{x_1}{|x|^2} \right) & \text{if } n = 2, \\ -\dfrac{1}{4\pi} \left(\dfrac{x_1}{|x|^3}, \dfrac{x_2}{|x|^3}, \dfrac{x_3}{|x|^3} \right) & \text{if } n = 3. \end{cases} \tag{1.11}$$

We introduce the mild solutions to the problems (1.9) and (1.10) defined as follows.

Definition *(The mild solution).* Let $1 \leq p < \infty$ and $1 \leq \nu < \infty$. For an initial data $\omega_0 \in L^{p,\nu}_{\mathrm{A}}$ and $T > 0$, we say that ω is *a mild solution* to (1.9) ((1.10) when $n = 3$) if

$$\omega(t) = \begin{cases} e^{t\Delta}\omega_0 - \int_0^t \nabla e^{(t-s)\Delta}\cdot(u(s)\omega(s))\,ds & \text{if } n=2,\\ e^{t\Delta}\omega_0 - \int_0^t \nabla e^{(t-s)\Delta}\cdot(\omega(s)\otimes u(s))\,ds + \int_0^t \nabla e^{(t-s)\Delta}\cdot(u(s)\otimes\omega(s))\,ds & \text{if } n=3 \end{cases} \tag{1.12}$$

in $C([0,T);L^{p,\nu}_{\mathrm{A}})$, where u is defined as (1.11).

By (1.11) and (1.12), we see that the vorticity equation possesses the same nonlocal structure as the drift-diffusion equation (1.1). We show the well-posedness of (1.9) and (1.10) in amalgam spaces:

Theorem 1.3 *(The local well-posedness). Let $n=2,3$. Assume that $p\ge n/2$, $1\le\nu<n$, $q=\min\{2np/(2n-p),2p\}$, and $1\le\nu_1\le q$ satisfies (1.7). For any $\omega_0\in L^{p,\nu}_{\mathrm{A}}$ (in addition, suppose that $\operatorname{div}\omega_0=0$ when $n=3$), there exist $T>0$ and a unique mild solution $\omega\in C([0,T);L^{p,\nu}_{\mathrm{A}})\cap C((0,T);L^{q,\nu_1}_{\mathrm{A}})$ to (1.9) ((1.10) when $n=3$). Furthermore, the Cauchy problem (1.9) ((1.10) when $n=3$) is well-posed in $C([0,T);L^{p,\nu}_{\mathrm{A}})\cap C((0,T);L^{q,\nu_1}_{\mathrm{A}})$.*

Ben-Artzi [1] showed the unconditional uniqueness of the mild solution to the two-dimensional Navier–Stokes equation. For the problem (1.1), the unconditional uniqueness holds in the following class (cf. Brezis–Cazenave [7], Kato [24], and see also Kozono–Sugiyama–Yahagi [29]):

Theorem 1.4 *(The unconditional uniqueness). Suppose that $n\ge 2$, $p>n/2$, and $1\le\nu<n$. Then uniqueness for (1.1) holds in the class $C([0,T);L^{p,\nu}_{\mathrm{A}})$.*

This paper is organized as follows. In the next section, we prepare the properties of amalgam spaces and the Riesz potential. Section 3 is devoted to the proof of the well-posedness of the problem (1.1) in amalgam spaces. We consider the well-posedness of the Cauchy problem of the vorticity equation in Section 4. In Section 5, we give proof of the unconditional uniqueness of the mild solution to (1.1) in amalgam spaces.

In what follows, we denote various constants by C unless otherwise stated. We denote the characteristic function for a set $A\subset\mathbb{R}^n$ by χ_A.

2. Preliminaries

In this section, we give some properties and inequalities related to amalgam spaces. For $1<\nu_1<p<\nu_2<\infty$, we see that

$$L^{p,1}_{\mathrm{A}} \subsetneq L^{p,\nu_1}_{\mathrm{A}} \subsetneq L^p = L^{p,p}_{\mathrm{A}} \subsetneq L^{p,\nu_2}_{\mathrm{A}} \subsetneq L^{p,\infty}_{\mathrm{A}} = L^p_{\mathrm{ul}}.$$

If $1\le\nu\le p<\infty$,

$$L^p\cap L^\nu \subset L^{p,\nu}_{\mathrm{A}} \subset L^\nu.$$

If $p_1\le p_2$,

$$L^{p_2,\nu}_{\mathrm{A}} \subset L^{p_1,\nu}_{\mathrm{A}}.$$

For example, $f_0(x)\equiv|x|^{-n/p}$ belongs to $L^{q,\nu}_{\mathrm{A}}$ if $1\le q<p<\nu\le\infty$, but $f_0\notin L^q$ for any $1\le q\le\infty$. If we suppose that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \quad \text{and} \quad \frac{1}{\nu} = \frac{1}{\nu_1} + \frac{1}{\nu_2},$$

then Hölder's inequality holds:

$$\|fg\|_{L^{p,\nu}_{\mathrm{A}}} = \left(\sum_{k\in\mathbb{Z}^n} \|fg\|^{\nu}_{L^p(Q(k))}\right)^{\frac{1}{\nu}} \le \left(\sum_{k\in\mathbb{Z}^n} \|f\|^{\nu}_{L^{p_1}(Q(k))}\|g\|^{\nu}_{L^{p_2}(Q(k))}\right)^{\frac{1}{\nu}} \le \|f\|_{L^{p_1,\nu_1}_{\mathrm{A}}}\|g\|_{L^{p_2,\nu_2}_{\mathrm{A}}},$$

where we denote $Q(k, 1/2)$ by $Q(k)$.

Bertrandias–Datry–Dupuis [2] and Busby–Smith [8] showed the convolution estimate in amalgam spaces:

Proposition 2.1. *Let $n \ge 1$, $1 \le p_1, p_2, p_3 \le \infty$, and $1 \le \nu_1, \nu_2, \nu_3 \le \infty$. Suppose that*

$$1 + \frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2} \quad \text{and} \quad 1 + \frac{1}{\nu_3} = \frac{1}{\nu_1} + \frac{1}{\nu_2}.$$

Then there exists a constant $C > 0$ such that for any $f \in L^{p_1,\nu_1}_{\mathrm{A}}$ and $g \in L^{p_2,\nu_2}_{\mathrm{A}}(\mathbb{R}^n)$, it holds that

$$\|f * g\|_{L^{p_3,\nu_3}_{\mathrm{A}}} \le C\|f\|_{L^{p_1,\nu_1}_{\mathrm{A}}}\|g\|_{L^{p_2,\nu_2}_{\mathrm{A}}}.$$

Let G_t be the Gauss function, i.e., $G_t(x) \equiv (4\pi t)^{-n/2}e^{-|x|^2/4t}$. For $\alpha \in \mathbb{N} \cup \{0\}$, we see that the distributional derivative $D^\alpha G_t \in L^{p,\nu}_{\mathrm{A}}$ for any $1 \le p, \nu \le \infty$. Indeed, since G_t is radially decreasing and integrable over $\mathbb{R}^n$, we have

$$\|G_t\|_{L^{p,1}_{\mathrm{A}}} = \sum_{k\in\mathbb{Z}^n} \|G_t\|_{L^p(Q(k))} = \sum_{|k|\le 1} \|G_t\|_{L^p(Q(k))} + \sum_{|k|\ge 2} \|G_t\|_{L^p(Q(k))} \le C(\|G_t\|_p + \|G_t\|_1)$$

for any $1 \le p \le \infty$ (see Proof of Proposition 2.3 and [33] for the last estimate). Applying Proposition 2.1 with $g \equiv D^\alpha G_t$, we have the heat kernel estimate in amalgam spaces:

Proposition 2.2. *Let $1 \le p_1 \le p_2 \le \infty$, $1 \le \nu_1 \le \nu_2 \le \infty$, and $\alpha \in \mathbb{N} \cup \{0\}$. Then there exists a constant $C_0 = C_0(n, \alpha, p_1, p_2, \nu_1, \nu_2) > 0$ such that for any $f \in L^{p_1,\nu_1}_{\mathrm{A}}$, it holds that*

$$\left\|D^\alpha e^{t\Delta} f\right\|_{L^{p_2,\nu_2}_{\mathrm{A}}} \le C_0\left(t^{-\frac{\alpha}{2}} + t^{-\frac{n}{2}\left(\frac{1}{p_1}-\frac{1}{p_2}\right)-\frac{\alpha}{2}}\right)\|f\|_{L^{p_1,\nu_1}_{\mathrm{A}}}. \tag{2.1}$$

Remark. In particular case that $p_1 = \nu_1$ and $p_2 = \nu_2$, which implies $L^{p_1,\nu_1} = L^{p_1}$ and $L^{p_2,\nu_2} = L^{p_2}$, for $1 \le p_1 \le p_2 \le \infty$ and $f \in L^{p_1}$, it holds that

$$\|e^{t\Delta} f\|_{L^{p_2}} \le Ct^{-\frac{n}{2}\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}\|f\|_{L^{p_1}}, \quad t > 0. \tag{2.2}$$

We recall that

$$\nabla\psi(t, x) = -\frac{1}{\omega_{n-1}}\int_{\mathbb{R}^n} \frac{x-y}{|x-y|^n}u(t, y)\,dy.$$

If we set

$$K(x) \equiv \frac{1}{\omega_{n-1}}|x|^{-(n-1)}, \tag{2.3}$$

then $|\nabla\psi(t, x)| \le (K * |u|)(t, x)$. We note that $K \in L^{n/(n-1)}_{\mathrm{w}}$, which is defined as

$$L^p_{\mathrm{w}} \equiv \left\{ f \text{ measurable; } \|f\|_{p,\mathrm{w}} \equiv \sup_{\alpha>0} \left\{ \alpha\mu\left(\{x \in \mathbb{R}^n;\ |f(x)| > \alpha\}\right)^{\frac{1}{p}} \right\} < +\infty \right\}$$

for $p \geq 1$ and the Lebesgue measure μ. Moreover, $K \in L^\nu(B_1(0)^c)$ for any $n/(n-1) < \nu \leq \infty$.

One can derive the convolution estimate of the integral kernel K in amalgam spaces by using the argument of [33] (see Kikuchi–Nakai–Tomita–Yabuta–Yoneda [26] for the boundedness of more general Calderón–Zygmund operators):

Proposition 2.3. *Let $n \geq 2$, $1 < p_1 \leq p_2 < \infty$, and $1 \leq \nu_1 \leq \nu_2 < \infty$. Suppose that*

$$0 \leq \frac{1}{p_1} - \frac{1}{p_2} \leq \frac{1}{n} \quad \text{and} \quad \frac{1}{n} < \frac{1}{\nu_1} - \frac{1}{\nu_2} < 1$$

Then there exists a constant $\tilde{C}_0 = \tilde{C}_0(n, p_1, p_2, \nu_1, \nu_2) > 0$ such that for any $f \in L^{p_1,\nu_1}_{\mathrm{A}}$, it holds that

$$\|K * f\|_{L^{p_2,\nu_2}_{\mathrm{A}}} \leq \tilde{C}_0 \|f\|_{L^{p_1,\nu_1}_{\mathrm{A}}}. \tag{2.4}$$

Proof of Proposition 2.3. We decompose $\mathbb{R}^n$ into countable cubes whose centers are lattice points, that is,

$$\mathbb{R}^n = \bigcup_{k\in\mathbb{Z}^n} Q\left(k, \frac{1}{2}\right),$$

where $Q(k,\theta) \equiv \{y \in \mathbb{R}^n;\ \max_{1\leq i\leq n} |y_i - k_i| \leq \theta,\ k = (k_1, k_2, \dots, k_n)\}$. Let $1 < p_1 < p_2 < \infty$ and $1 \leq \nu_1 \leq \nu_2 < \infty$. By the generalized Hausdorff–Young inequality, we have for any $k \in \mathbb{Z}^n$:

$$\begin{aligned}
&\|K * f\|^{\nu_2}_{L^{p_2}(Q(k,\frac{1}{2}))} \\
&= \left\| \sum_{k',k''\in\mathbb{Z}^n} (\chi_{Q(k',\frac{1}{2})}K) * (\chi_{Q(k'',\frac{1}{2})}f) \right\|^{\nu_2}_{L^{p_2}(Q(k,\frac{1}{2}))} \\
&\leq \left(\sum_{\max(|(k'_i+k''_i)-k_i|)\leq 1} \|(\chi_{Q(k',\frac{1}{2})}K) * (\chi_{Q(k'',\frac{1}{2})}f)\|_{L^{p_2}(Q(k,\frac{1}{2}))} \right)^{\nu_2} \\
&\leq C \left(\sum_{\max(|(k'_i+k''_i)-k_i|)\leq 1} \|\chi_{Q(k',\frac{1}{2})}K\|_{r,\mathrm{w}} \|\chi_{Q(k'',\frac{1}{2})}f\|_{p_1} \right)^{\nu_2},
\end{aligned}$$

where $r > 1$ satisfies

$$1 + \frac{1}{p_2} = \frac{1}{p_1} + \frac{1}{r} \quad \text{and} \quad 1 < r \leq \frac{n}{n-1}.$$

In the case of $1 < \nu_2 < \infty$, it follows from Hölder's inequality and the covering that

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^n} \|K * f\|^{\nu_2}_{L^{p_2}(Q(k,\frac{1}{2}))} \\
&\leq C3^{n\nu_2} \left(\sum_{k'\in\mathbb{Z}^n} \|\chi_{Q(k',\frac{1}{2})}K\|^{\tilde{\nu}}_{r,\mathrm{w}} \right)^{\frac{\nu_2}{\tilde{\nu}}} \left(\sum_{k''\in\mathbb{Z}^n} \|f\|^{\nu_1}_{L^{p_1}(Q(k'',\frac{1}{2}))} \right)^{\frac{\nu_2}{\nu_1}},
\end{aligned}$$

where $\tilde{\nu}$ satisfies

$$1+\frac{1}{\nu_2}=\frac{1}{\tilde{\nu}}+\frac{1}{\nu_1} \quad \text{and} \quad \frac{n}{n-1}<\tilde{\nu}<\infty$$

(see Busby–Smith [8]). In what follows, we estimate

$$\left(\sum_{k\in\mathbb{Z}^n}\|\chi_{Q(k,\frac{1}{2})}K\|_{r,\mathrm{w}}^{\tilde{\nu}}\right)^{\frac{1}{\tilde{\nu}}}.$$

In the case that $k\in\mathbb{Z}^n$ satisfies $\max|k_i|\le 1$, we have

$$\sum_{\max|k_i|\le 1}\|\chi_{Q(k,\frac{1}{2})}K\|_{r,\mathrm{w}}^{\tilde{\nu}}\le C\|\chi_{Q(0,\frac{1}{2})}K\|_{r,\mathrm{w}}^{\tilde{\nu}}\le C\|K\|_{r,\mathrm{w}}^{\tilde{\nu}}. \tag{2.5}$$

On the other hand, K attains the maximum in $Q(k,1/2)$ at the closest point y_k to the origin, because K is radially decreasing. For $k\in\mathbb{Z}^n$ satisfying $\max|k_i|\ge 2$, we see that

$$\|\chi_{Q(k,\frac{1}{2})}K\|_{r,\mathrm{w}}^{\tilde{\nu}}\le\left(\int_{Q(k,\frac{1}{2})}K(x)^r\,dx\right)^{\frac{\tilde{\nu}}{r}}\le CK(y_k)^{\tilde{\nu}}.$$

We draw a line from the origin to y_k and order cubes intersecting with this line. The first cube is $Q(0,1/2)$, and the second cube is the cube in which the line meets when it goes out of $Q(0,1/2)$, and so on. In this order, we denote $I(Q(k,1/2))$ the second-to-last cube. Then we have

$$K(y_k)^{\tilde{\nu}}\le\int_{I(Q(k,\frac{1}{2}))}K(x)^{\tilde{\nu}}\,dx.$$

Since $\tilde{\nu}>n/(n-1)$, we have

$$\sum_{\max|k_i|\ge 2}\|\chi_{Q(k,\frac{1}{2})}K\|_{r,\mathrm{w}}^{\tilde{\nu}}\le C\sum_{\max|k_i|\ge 2}\int_{Q(k,\frac{1}{2})}K(x)^{\tilde{\nu}}\,dx\le C\int_{|x|\ge 2}K(x)^{\tilde{\nu}}\,dx\le C. \tag{2.6}$$

Therefore, combining (2.5) and (2.6), we obtain the result. In the case of $p_1=p_2$, (2.4) also holds by using the Hausdorff–Young inequality instead of the generalized one. □

In the case of the vorticity equation, we recall that u is represented by (1.11). It follows from the Biot–Savart law (1.11) that

$$|u(t,x)|\le(K*\omega)(t,x),$$

where K is defined as (2.3). By a similar argument, Proposition 2.3 provides the following result:

Corollary 2.4. *Let $n=2,3$, $1<p_1\le p_2<\infty$, and $1\le\nu_1\le\nu_2<\infty$. Suppose that*

$$0\le\frac{1}{p_1}-\frac{1}{p_2}\le\frac{1}{n} \quad \text{and} \quad \frac{1}{n}<\frac{1}{\nu_1}-\frac{1}{\nu_2}<1.$$

For any $\omega\in L_{\mathrm{A}}^{p_1,\nu_1}$, set u by (1.11). Then there exists a constant $C>0$ independent on ω such that

$$\|u\|_{L_{\mathrm{A}}^{p_2,\nu_2}}\le C\|\omega\|_{L_{\mathrm{A}}^{p_1,\nu_1}}.$$

3. Well-posedness of the mild solution to the drift-diffusion equation

Proof of Theorem 1.1. Let $p \geq n/2$, $1 \leq \nu < n$. We choose

$$q \equiv \min\left\{\frac{2np}{2n-p}, 2p\right\},$$

and take $1 \leq \nu_1 < \nu_2 < \infty$ as

$$\frac{1}{\nu} = \frac{1}{\nu_1} + \frac{1}{\nu_2} \quad \text{and} \quad \frac{1}{n} < \frac{1}{\nu_1} - \frac{1}{\nu_2} < 1. \tag{3.1}$$

We note that $q = 2n/3$ when $p = n/2$. Let the initial data $u_0 \in L^{p,\nu}_{\mathrm{A}}$. We take a positive constant $M > 0$ such as $\|u_0\|_{L^{p,\nu}_{\mathrm{A}}} \leq M/(2\max\{C_0(n,1,p,p,\nu,\nu), C_0(n,1,p,q,\nu,\nu_1)\})$, where the constant C_0 appears in Proposition 2.2. For $T, \eta > 0$ chosen later, let X_T be a complete space defined as

$$\begin{aligned} X_T \equiv \Big\{ & u \in L^\infty(0,T; L^{p,\nu}_{\mathrm{A}}) \cap L^\infty(0,T; L^{q,\nu_1}_{\mathrm{A}}); \\ & \sup_{0<t<T} \|u(t)\|_{L^{p,\nu}_{\mathrm{A}}} \leq M, \ \sup_{0<t<T} t^{\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|u(t)\|_{L^{q,\nu_1}_{\mathrm{A}}} \leq \eta \Big\} \end{aligned} \tag{3.2}$$

endowed with the distance

$$d(u,v) \equiv \sup_{0<t<T} t^{\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|u(t) - v(t)\|_{L^{q,\nu_1}_{\mathrm{A}}}$$

for $u, v \in X_T$. For $u \in X_T$ and $0 < t < T$, we set

$$\Phi[u](t) \equiv e^{t\Delta}u_0 - \int_0^t \nabla e^{(t-s)\Delta} \cdot (u\nabla\psi)(s)\, ds,$$

where ψ is defined as (1.2). By (2.1) of Proposition 2.2 and Hölder's inequality, we have

$$\begin{aligned} \|\Phi[u](t)\|_{L^{p,\nu}_{\mathrm{A}}} &\leq C_1\|u_0\|_{L^{p,\nu}_{\mathrm{A}}} + C_1 \int_0^t (t-s)^{-\frac{1}{2}} \|u(s)\nabla\psi(s)\|_{L^{p,\nu}_{\mathrm{A}}}\, ds \\ &\leq C_1\|u_0\|_{L^{p,\nu}_{\mathrm{A}}} + C_1 \int_0^t (t-s)^{-\frac{1}{2}} \|u(s)\|_{L^{q,\nu_1}_{\mathrm{A}}} \|\nabla\psi(s)\|_{L^{\tilde{q},\nu_2}_{\mathrm{A}}}\, ds, \end{aligned}$$

where $C_1 = C_0(n,1,p,p,\nu,\nu)$ and one can take $\tilde{q} \geq 1$ satisfying $1/p = 1/q + 1/\tilde{q}$ since $q \geq p$. The constant $C_0 > 0$ appears in (2.1) of Proposition 2.2. From (2.4) of Proposition 2.3, we have

$$\|\nabla\psi(s)\|_{L^{\tilde{q},\nu_2}_{\mathrm{A}}} \leq C_2\|u(s)\|_{L^{p,\nu}_{\mathrm{A}}} \tag{3.3}$$

since $0 \leq 1/q - 1/\tilde{q} = 2/q - 1/p \leq 1/n$ and (3.1), where $C_2 = \tilde{C}_0(n,p,\tilde{q},\nu,\nu_2)$. Thus, we obtain

$$\|\Phi[u](t)\|_{L^{p,\nu}_{\mathrm{A}}} \leq C_1\|u_0\|_{L^{p,\nu}_{\mathrm{A}}} + C_1C_2 \int_0^t (t-s)^{-\frac{1}{2}} \|u(s)\|^2_{L^{q,\nu_1}_{\mathrm{A}}}\, ds.$$

By (3.2), we see that

$$\begin{aligned}\|\Phi[u](t)\|_{L^{p,\nu}_{\mathrm{A}}} &\le \frac{M}{2} + C_1C_2\eta^2\int_0^t (t-s)^{-\frac{1}{2}} s^{-n\left(\frac{1}{p}-\frac{1}{q}\right)}\,ds \\ &\le \frac{M}{2} + C\eta^2 T^{\frac{1}{2}-n\left(\frac{1}{p}-\frac{1}{q}\right)}\end{aligned}$$

for $0<t<T$. In the case of $p>n/2$, we take $\eta = M$ and $T>0$ such as

$$CT^{\frac{1}{2}-n\left(\frac{1}{p}-\frac{1}{q}\right)} \le \frac{1}{2M}.$$

In the case of $p=n/2$, we take η such that

$$C\eta^2 \le \frac{M}{2}. \tag{3.4}$$

Thus, we obtain

$$\|\Phi[u](t)\|_{L^{p,\nu}_{\mathrm{A}}} \le M. \tag{3.5}$$

On the other hand, by (2.1), Hölder's inequality, and (3.3), we have

$$\begin{aligned}&\|\Phi[u](t)\|_{L^{q,\nu_1}_{\mathrm{A}}} \\ &\le \|e^{t\Delta}u_0\|_{L^{q,\nu_1}_{\mathrm{A}}} + C_3\int_0^t \left((t-s)^{-\frac{1}{2}} + (t-s)^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}}\right)\|u(s)\nabla\psi(s)\|_{L^{p,\nu}_{\mathrm{A}}}\,ds \\ &\le \|e^{t\Delta}u_0\|_{L^{q,\nu_1}_{\mathrm{A}}} + C_2C_3\int_0^t \left((t-s)^{-\frac{1}{2}} + (t-s)^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}}\right)\|u(s)\|^2_{L^{q,\nu_1}_{\mathrm{A}}}\,ds,\end{aligned}$$

where $C_3 = C_0(n,1,p,q,\nu,\nu_1)$. By (3.2), we see that

$$\begin{aligned}\|\Phi[u](t)\|_{L^{q,\nu_1}_{\mathrm{A}}} &\le \|e^{t\Delta}u_0\|_{L^{q,\nu_1}_{\mathrm{A}}} + C_2C_3\eta^2\int_0^t \left((t-s)^{-\frac{1}{2}} + (t-s)^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}}\right)s^{-n\left(\frac{1}{p}-\frac{1}{q}\right)}\,ds \\ &\le \|e^{t\Delta}u_0\|_{L^{q,\nu_1}_{\mathrm{A}}} + C\eta^2\left[t^{\frac{1}{2}-n\left(\frac{1}{p}-\frac{1}{q}\right)} + t^{\frac{1}{2}-\frac{3n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\right].\end{aligned}$$

This gives

$$t^{\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|\Phi[u](t)\|_{L^{q,\nu_1}_{\mathrm{A}}} \le t^{\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|e^{t\Delta}u_0\|_{L^{q,\nu_1}_{\mathrm{A}}} + C\eta^2\left[T^{\frac{1}{2}-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} + T^{\frac{1}{2}-n\left(\frac{1}{p}-\frac{1}{q}\right)}\right]$$

for $0<t<T$. In the case of $p>n/2$, by (2.1), we have

$$t^{\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|e^{t\Delta}u_0\|_{L^{q,\nu_1}_{\mathrm{A}}} \le C_3\left(T^{\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}+1\right)\|u_0\|_{L^{p,\nu}_{\mathrm{A}}}.$$

Since we take $\eta = M$ as above, it suffices to choose T such as

$$T^{\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \le \frac{1}{2} \quad\text{and}\quad C\left[T^{\frac{1}{2}-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} + T^{\frac{1}{2}-n\left(\frac{1}{p}-\frac{1}{q}\right)}\right] \le \frac{1}{4M}.$$

In the case of $p = n/2$, there exists $\{u_{0,k}\}_{k\in\mathbb{N}} \subset C_0^\infty$ such that

$$\|u_0 - u_{0,k}\|_{L^{p,\nu}_{\mathrm{A}}} \to 0 \quad \text{as } k \to \infty.$$

Thus, by (2.1), we have

$$t^{\frac{1}{4}}\|e^{t\Delta}u_0\|_{L^{q,\nu_1}_{\mathrm{A}}} \le C\|u_0 - u_{0,k}\|_{L^{p,\nu}_{\mathrm{A}}} + t^{\frac{1}{4}}\|e^{t\Delta}u_{0,k}\|_{L^{q,\nu_1}_{\mathrm{A}}}.$$

One can take $K \in \mathbb{N}$ such that for any $k > K$, $\|u_0 - u_{0,k}\|_{L^{p,\nu}_{\mathrm{A}}} \le \eta/(4C)$. We fix k_0 sufficiently large such that $k_0 > K$. Since $\{u_{0,k}\} \subset C_0^\infty$, we see that

$$t^{\frac{1}{4}}\|e^{t\Delta}u_{0,k_0}\|_{L^{q,\nu_1}_{\mathrm{A}}} \le C_4 T^{\frac{1}{4}}\|u_{0,k_0}\|_{L^{q,\nu_1}_{\mathrm{A}}},$$

where $C_4 = C_0(n, 0, q, q, \nu_1, \nu_1)$. Choosing $T > 0$ such that

$$C_4 T^{\frac{1}{4}}\|u_{0,k_0}\|_{L^{q,\nu_1}_{\mathrm{A}}} \le \frac{\eta}{4}, \tag{3.6}$$

it follows from (3.4) and (3.6) that

$$C\eta\left(T^{\frac{1}{4}} + 1\right) \le \frac{1}{2} \quad \text{and} \quad \sup_{0<t<T} t^{\frac{1}{4}}\|e^{t\Delta}u_0\|_{L^{q,\nu_1}_{\mathrm{A}}} \le \frac{\eta}{2}. \tag{3.7}$$

Thus, we obtain

$$\sup_{0<t<T} t^{\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|\Phi[u](t)\|_{L^{q,\nu_1}_{\mathrm{A}}} \le \eta. \tag{3.8}$$

Hence, we see from (3.5) and (3.8) that $\Phi : X_T \to X_T$. We show that Φ is a contraction mapping. For $u, v \in X_T$, we set $\psi \equiv (-\Delta)^{-1}u$ and $\phi \equiv (-\Delta)^{-1}v$. Analogously to the derivation of (3.5), we have

$$\begin{aligned}
&\|\Phi[u](t) - \Phi[v](t)\|_{L^{q,\nu_1}_{\mathrm{A}}} \\
&\quad \le C_3 \int_0^t \left((t-s)^{-\frac{1}{2}} + (t-s)^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}}\right)\|u(s)\|_{L^{q,\nu_1}_{\mathrm{A}}}\|\nabla\psi(s) - \nabla\phi(s)\|_{L^{\tilde{q},\nu_2}_{\mathrm{A}}}\,ds \\
&\qquad + C_3 \int_0^t \left((t-s)^{-\frac{1}{2}} + (t-s)^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}}\right)\|u(s) - v(s)\|_{L^{q,\nu_1}_{\mathrm{A}}}\|\nabla\phi(s)\|_{L^{\tilde{q},\nu_2}_{\mathrm{A}}}\,ds.
\end{aligned}$$

By (2.4) and (3.2), we see that

$$\begin{aligned}
\|\Phi[u](t) - \Phi[v](t)\|_{L^{q,\nu_1}_{\mathrm{A}}} &\le C_2C_3\eta d(u,v)\int_0^t \left((t-s)^{-\frac{1}{2}} + (t-s)^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}}\right)s^{-n\left(\frac{1}{p}-\frac{1}{q}\right)}\,ds \\
&\le C\eta d(u,v)\left(t^{\frac{1}{2}-n\left(\frac{1}{p}-\frac{1}{q}\right)} + t^{\frac{1}{2}-\frac{3n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\right).
\end{aligned}$$

This gives

$$t^{\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|\Phi[u](t) - \Phi[v](t)\|_{L^{q,\nu_1}_{\mathrm{A}}} \le C\eta d(u,v)\left(T^{\frac{1}{2}-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} + T^{\frac{1}{2}-n\left(\frac{1}{p}-\frac{1}{q}\right)}\right)$$

for $0 < t < T$. Thus, we obtain

$$\sup_{0<t<T} t^{\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|\Phi[u](t) - \Phi[v](t)\|_{L^{q,\nu_1}_{\mathrm{A}}} \le \frac{1}{2} d(u,v).$$

Therefore, $\Phi : X_T \to X_T$ is a contraction mapping, and by the Banach fixed point theorem, there exists a unique fixed point $u = \Phi[u]$ in X_T.

We show that $u \in C([0,T); L^{p,\nu}_{\mathrm{A}})$. For $0 < t < t+h < T$, we have

$$\begin{aligned} \|u(t+h) - u(t)\|_{L^{p,\nu}_{\mathrm{A}}} &\le \|e^{(t+h)\Delta} u_0 - e^{t\Delta} u_0\|_{L^{p,\nu}_{\mathrm{A}}} \\ &\quad + \left\| \int_0^{t+h} \nabla e^{(t+h-s)\Delta} \cdot (u\nabla\psi)(s)\, ds - \int_0^t \nabla e^{(t-s)\Delta} \cdot (u\nabla\psi)(s)\, ds \right\|_{L^{p,\nu}_{\mathrm{A}}}. \end{aligned} \tag{3.9}$$

Since $e^{t\Delta} u_0 \in L^{p,\nu}_{\mathrm{A}}$, we have

$$\|e^{(t+h)\Delta} u_0 - e^{t\Delta} u_0\|_{L^{p,\nu}_{\mathrm{A}}} \to 0 \quad \text{as } h \to 0.$$

On the second term of the right hand side in (3.9), we see that

$$\begin{aligned} &\left\| \int_0^{t+h} \nabla e^{(t+h-s)\Delta} \cdot (u\nabla\psi)(s)\, ds - \int_0^t \nabla e^{(t-s)\Delta} \cdot (u\nabla\psi)(s)\, ds \right\|_{L^{p,\nu}_{\mathrm{A}}} \\ &\le \int_0^t \|(\nabla e^{(t+h-s)\Delta} - \nabla e^{(t-s)\Delta}) \cdot (u\nabla\psi)(s)\|_{L^{p,\nu}_{\mathrm{A}}}\, ds + \int_t^{t+h} \|\nabla e^{(t-s)\Delta} \cdot (u\nabla\psi)(s)\|_{L^{p,\nu}_{\mathrm{A}}}\, ds. \end{aligned} \tag{3.10}$$

Again, since $\nabla e^{(t-\cdot)\Delta} \cdot (u\nabla\psi)(\cdot) \in L^{p,\nu}_{\mathrm{A}}$, it holds that

$$\|(\nabla e^{(t+h-s)\Delta} - \nabla e^{(t-s)\Delta}) \cdot (u\nabla\psi)(s)\|_{L^{p,\nu}_{\mathrm{A}}} \to 0 \quad \text{as } h \to 0.$$

Moreover, by (2.1) and (2.4), we have

$$\begin{aligned} &\|(\nabla e^{(t+h-s)\Delta} - \nabla e^{(t-s)\Delta}) \cdot (u\nabla\psi)(s)\|_{L^{p,\nu}_{\mathrm{A}}} \\ &\quad \le 2C_1C_2(t-s)^{-\frac{1}{2}} \|u(s)\|^2_{L^{q,\nu_1}_{\mathrm{A}}} \\ &\quad \le 2C_1C_2(t-s)^{-\frac{1}{2}} s^{-n\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\sup_{0<s<t} s^{\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|u(s)\|_{L^{q,\nu_1}_{\mathrm{A}}} \right)^2. \end{aligned}$$

By the Lebesgue convergence theorem, we see that the first term of the right hand side in (3.10) converges to 0 as $h \to 0$. Similarly, the second term in the right hand side (3.10) vanishes as $h \to 0$. Furthermore, we have

$$\|u(t) - u_0\|_{L^{p,\nu}_{\mathrm{A}}} \le \|e^{t\Delta} u_0 - u_0\|_{L^{p,\nu}_{\mathrm{A}}} + C_1C_2 t^{\frac{1}{2}-n\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\sup_{0<s<t} s^{\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|u(s)\|_{L^{q,\nu_1}_{\mathrm{A}}} \right)^2.$$

Since the initial data u_0 belongs to $L^{p,\nu}_{\mathrm{A}}$, $u(t)$ converges to u_0 in $L^{p,\nu}_{\mathrm{A}}$ as $t \to 0$.

Finally, we show that for any $u_0, v_0 \in L^{p,\nu}_{\mathrm{A}}$ and corresponding solutions $u(t)$, $v(t)$ to (1.1), it holds that

$$u \to v \quad \text{in } C([0,T); L^{p,\nu}_{\mathrm{A}}) \quad \text{as } u_0 \to v_0 \quad \text{in } L^{p,\nu}_{\mathrm{A}}.$$

We set $\psi \equiv (-\Delta)^{-1}u$ and $\phi \equiv (-\Delta)^{-1}v$. It follows from (2.1) and (2.4) that

$$\begin{aligned}&\|u(t)-v(t)\|_{L^{p,\nu}_{\mathrm{A}}}\\&\le C\|u_0-v_0\|_{L^{p,\nu}_{\mathrm{A}}} + C_1\int_0^t (t-s)^{-\frac12}\left(\|u(s)\|_{L^{q,\nu_1}_{\mathrm{A}}}+\|v(s)\|_{L^{q,\nu_1}_{\mathrm{A}}}\right)\|u(s)-v(s)\|_{L^{q,\nu_1}_{\mathrm{A}}}\,ds.\end{aligned}$$

On the other hand, we see that

$$\begin{aligned}&t^{\frac{n}{2}\left(\frac1p-\frac1q\right)}\|u(t)-v(t)\|_{L^{q,\nu_1}_{\mathrm{A}}}\\&\le C_5\left(t^{\frac{n}{2}\left(\frac1p-\frac1q\right)}+1\right)\|u_0-v_0\|_{L^{p,\nu}_{\mathrm{A}}} + C\eta d(u,v)\left(T^{\frac12-\frac{n}{2}\left(\frac1p-\frac1q\right)}+T^{\frac12-n\left(\frac1p-\frac1q\right)}\right)\end{aligned}$$

for $0<t<T$, where $C_5 = C_0(n,0,p,q,\nu,\nu_1)$. By the choice of η, we have

$$\sup_{0<t<T} t^{\frac{n}{2}\left(\frac1p-\frac1q\right)}\|u(t)-v(t)\|_{L^{q,\nu_1}_{\mathrm{A}}} \le C\|u_0-v_0\|_{L^{p,\nu}_{\mathrm{A}}}.$$

Thus, we obtain

$$\sup_{0\le t<T}\|u(t)-v(t)\|_{L^{p,\nu}_{\mathrm{A}}} \le C(1+\eta)\|u_0-v_0\|_{L^{p,\nu}_{\mathrm{A}}},$$

which proves the continuous dependence of initial data. □

Proof of Corollary 1.2. The heat semigroup estimate (2.2) derives

$$\sup_{0<t<\infty} t^{\frac14}\|e^{t\Delta}u_0\|_{L^{\frac{2n}{3}}} \le C\|u_0\|_{L^{\frac{n}{2}}} \le \frac{\eta}{2}$$

for sufficiently small data $u_0 \in L^{n/2}$, where the constant η appears in the proof of Theorem 1.1. On the other hand, the inequality $\|f\|_{L^{n/2,\nu}_{\mathrm{A}}} \le \|f\|_{L^{n/2}}$ holds for $n/2 \le \nu < n$. This inequality and the estimate (3.7) in the proof of Theorem 1.1 imply that the existence time T is estimated by

$$T^{\frac14}+1 \ge \frac{C}{\|u_0\|_{L^{\frac{n}{2}}}}.$$

Therefore, one can find a small constant $\varepsilon_0>0$ such that there exists a time-global solution to (1.1) for any initial data $u_0\in L^{n/2}$ with $\|u_0\|_{L^{n/2}} \le \varepsilon_0$. □

4. Well-posedness of the mild solution to the vorticity equation

Proof of Theorem 1.3. Let $n=2,3$. Suppose that $p\ge n/2$ and $1\le \nu<n$. We choose

$$q \equiv \min\left\{\frac{2np}{2n-p}, 2p\right\},$$

and take $1\le \nu_1<\nu_2<\infty$ as

$$\frac1\nu = \frac1{\nu_1}+\frac1{\nu_2} \quad\text{and}\quad \frac1n < \frac1{\nu_1}-\frac1{\nu_2} < 1. \tag{4.1}$$

Let the initial data $\omega_0 \in L^{p,\nu}_{\mathrm{A}}$, in addition, suppose that ω_0 satisfies $\operatorname{div}\omega_0 = 0$ when $n = 3$. We take a positive constant $M > 0$ such as $\|\omega_0\|_{L^{p,\nu}_{\mathrm{A}}} \le M/(2\max\{C_0(n,1,p,p,\nu,\nu), C_0(n,1,p,q,\nu,\nu_1)\})$, where the constant C_0 appears in Proposition 2.2. For $T, \eta > 0$ chosen later, let X_T be a complete space defined as

$$X_T \equiv \Big\{\omega \in L^\infty(0,T; L^{p,\nu}_{\mathrm{A}}) \cap L^\infty(0,T; L^{q,\nu_1}_{\mathrm{A}}); \\ \sup_{0<t<T} \|\omega(t)\|_{L^{p,\nu}_{\mathrm{A}}} \le M, \ \sup_{0<t<T} t^{\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|\omega(t)\|_{L^{q,\nu_1}_{\mathrm{A}}} \le \eta \Big\} \tag{4.2}$$

endowed with the distance

$$d(\omega, \tilde{\omega}) \equiv \sup_{0<t<T} t^{\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|\omega(t) - \tilde{\omega}(t)\|_{L^{q,\nu_1}_{\mathrm{A}}}$$

for $\omega, \tilde{\omega} \in X_T$. For $\omega \in X_T$ and $0 < t < T$, we set

$$\Phi[\omega](t) \equiv e^{t\Delta}\omega_0 + B[\omega, u](t)$$

where we denote the bilinear term by

$$B[\omega,u](t) \equiv \begin{cases} -\displaystyle\int_0^t \nabla e^{(t-s)\Delta} \cdot (\omega(s)u(s))\, ds & \text{if } n = 2, \\ -\displaystyle\int_0^t \nabla e^{(t-s)\Delta} \cdot (\omega(s) \otimes u(s))\, ds + \int_0^t \nabla e^{(t-s)\Delta} \cdot (u(s) \otimes \omega(s))\, ds & \text{if } n = 3 \end{cases}$$

and u is defined as (1.11).

We consider the estimate of the bilinear term only because one can apply the similar argument of Proof of Theorem 1.1 to the initial data ω_0. By the symmetry of the bilinear term, Proposition 2.2, and Hölder's inequality, we have

$$\|B[\omega,u](t)\|_{L^{p,\nu}_{\mathrm{A}}} \le 2C_1 \int_0^t (t-s)^{-\frac{1}{2}} \|\omega(s)\|_{L^{q,\nu_1}_{\mathrm{A}}} \|u(s)\|_{L^{\tilde{q},\nu_2}_{\mathrm{A}}}\, ds,$$

where $C_1 = C_0(n,1,p,p,\nu,\nu)$ and $\tilde{q} \ge 1$ satisfies $1/p = 1/q + 1/\tilde{q}$. It follows from Corollary 2.4 that

$$\|B[\omega,u](t)\|_{L^{p,\nu}_{\mathrm{A}}} \le 2C_1C_2 \int_0^t (t-s)^{-\frac{1}{2}} \|\omega(s)\|^2_{L^{q,\nu_1}_{\mathrm{A}}}\, ds$$

since $0 \le 1/q - 1/\tilde{q} \le 1/2$ and (4.1), where $C_2 = \tilde{C}_0(n,p,\tilde{q},\nu,\nu_2)$. By (4.2), we see that

$$\|B[\omega,u](t)\|_{L^{p,\nu}_{\mathrm{A}}} \le C\eta^2 T^{\frac{1}{2}-n\left(\frac{1}{p}-\frac{1}{q}\right)}$$

for $0 < t < T$. On the other hand, we have

$$\|B[\omega,u](t)\|_{L^{q,\nu_1}_{\mathrm{A}}} \le 2C_2C_3 \int_0^t \left((t-s)^{-\frac{1}{2}} + (t-s)^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}}\right) \|\omega(s)\|^2_{L^{q,\nu_1}_{\mathrm{A}}}\, ds,$$

where $C_3 = C_0(n, 1, p, q, \nu, \nu_1)$. By (4.2), we see that

$$\|B[\omega, u](t)\|_{L_A^{q,\nu_1}} \le C\eta^2 \left[t^{\frac{1}{2}-n\left(\frac{1}{p}-\frac{1}{q}\right)} + t^{\frac{1}{2}-\frac{3n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \right].$$

This gives

$$t^{\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|B[\omega, u](t)\|_{L_A^{q,\nu_1}} \le C\eta^2 \left[T^{\frac{1}{2}-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} + T^{\frac{1}{2}-n\left(\frac{1}{p}-\frac{1}{q}\right)} \right]$$

for $0 < t < T$. Taking η and T in the same way as in the proof of Theorem 1.1, we obtain $\Phi : X_T \to X_T$. Also, arguing similarly to the proof of Theorem 1.1, there exists a unique solution ω to (1.9). □

5. Unconditional uniqueness of the mild solution

In this section, we prove the unconditional uniqueness of the mild solution to (1.1) in the subcritical case $p > n/2$.

Proof of Theorem 1.4. Let $n \ge 2$, $p > n/2$, and $1 \le \nu < n$. Suppose that u and v are mild solutions to (1.1) with $u, v \in C([0, T); L_A^{p,\nu})$. We set

$$M \equiv \sup_{0 \le t < T} \left(\|u(t)\|_{L_A^{p,\nu}} + \|v(t)\|_{L_A^{p,\nu}} \right) \quad \text{and} \quad d(u, v) \equiv \sup_{0 \le t < T} \|u(t) - v(t)\|_{L_A^{p,\nu}}.$$

We also set $\psi \equiv (-\Delta)^{-1}u$ and $\phi \equiv (-\Delta)^{-1}v$. By (1.6), we have

$$u(t) - v(t) = -\int_0^t \nabla e^{(t-s)\Delta} \cdot (u(s)\nabla\psi(s) - v(s)\nabla\phi(s))\, ds.$$

It follows from (2.1) and (2.4) that

$$\begin{aligned}
\|u(t) - v(t)\|_{L_A^{p,\nu}} \le\ & C_1 \int_0^t \left[(t-s)^{-\frac{1}{2}} + (t-s)^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{2}} \right] \|u(s)(\nabla\psi(s) - \nabla\phi(s))\|_{L_A^{q,\nu_1}}\, ds \\
& + C_1 \int_0^t \left[(t-s)^{-\frac{1}{2}} + (t-s)^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{2}} \right] \|(u(s) - v(s))\nabla\phi(s)\|_{L_A^{q,\nu_1}}\, ds \\
\le\ & C_1 \int_0^t \left[(t-s)^{-\frac{1}{2}} + (t-s)^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{2}} \right] \|u(s)\|_{L_A^{p,\nu}} \|\nabla\psi(s) - \nabla\phi(s)\|_{L_A^{\tilde{q},\nu_2}}\, ds \\
& + C_1 \int_0^t \left[(t-s)^{-\frac{1}{2}} + (t-s)^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{2}} \right] \|u(s) - v(s)\|_{L_A^{p,\nu}} \|\nabla\phi(s)\|_{L_A^{\tilde{q},\nu_2}}\, ds,
\end{aligned}$$

where $C_1 = C_0(n, p, q, \nu, \nu_1)$ appearing in Proposition 2.2 and we take

$$p \ge q \ge 1, \quad \nu \ge \nu_1 \ge 1, \quad \frac{1}{\tilde{q}} = \frac{1}{q} - \frac{1}{p} \quad \frac{1}{\nu_1} = \frac{1}{\nu} + \frac{1}{\nu_2}.$$

If we choose q such that

$$0 \le \frac{2}{p} - \frac{1}{q} \le \frac{1}{n},$$

then we have

$$\begin{aligned}\|u(t)-v(t)\|_{L^{p,\nu}_{\mathrm{A}}} &\le CMd(u,v)\int_0^t \left[(t-s)^{-\frac{1}{2}} + (t-s)^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{2}}\right] ds \\ &\le CMd(u,v)\left(T^{\frac{1}{2}} + T^{\frac{1}{2}-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\right)\end{aligned}$$

since $p > n/2$ satisfies

$$\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)+\frac{1}{2} < 1, \quad \text{i.e.,} \quad \frac{1}{q}-\frac{1}{p} < \frac{1}{n}.$$

Hence, $d(u,v) = 0$ for T small enough. Repeating the same argument, we obtain $d(u,v) = 0$ for $t \in [0,T)$. □

Acknowledgments

The author would like to thank the editor and the referee for their kind suggestions and comments. The author is grateful to Professor Takayoshi Ogawa for valuable advice. The author's work is supported by JSPS Grant-in-Aid for JSPS Fellows #19J20763 and Research Activity Start-up #22K20336.

References

[1] M. Ben-Artzi, Global solutions of two-dimensional Navier-Stokes and Euler equations, Arch. Ration. Mech. Anal. 128 (1994) 329–358.
[2] J.-P. Bertrandias, C. Datry, C. Dupuis, Unions et intersections d'espaces L^p invariantes par translation ou convolution, Ann. Inst. Fourier (Grenoble) 28 (1978) v, 53–84.
[3] P. Biler, The Cauchy problem and self-similar solutions for a nonlinear parabolic equation, Stud. Math. 114 (1995) 181–205.
[4] P. Biler, M. Cannone, I.A. Guerra, G. Karch, Global regular and singular solutions for a model of gravitating particles, Math. Ann. 330 (2004) 693–708.
[5] Z. Bradshaw, C.-C. Lai, T.-P. Tsai, Mild solutions and spacetime integral bounds for Stokes and Navier-Stokes flows in Wiener amalgam spaces, arXiv:2207.04298.
[6] Z. Bradshaw, T.-P. Tsai, Local energy solutions to the Navier-Stokes equations in Wiener amalgam spaces, SIAM J. Math. Anal. 53 (2021) 1993–2026.
[7] H. Brezis, T. Cazenave, A nonlinear heat equation with singular initial data, J. Anal. Math. 68 (1996) 277–304.
[8] R.C. Busby, H.A. Smith, Product-convolution operators and mixed-norm spaces, Trans. Am. Math. Soc. 263 (1981) 309–341.
[9] T. Cazenave, F.B. Weissler, Asymptotically self-similar global solutions of the nonlinear Schrödinger and heat equations, Math. Z. 228 (1998) 83–120.
[10] L. Corrias, B. Perthame, H. Zaag, Global solutions of some chemotaxis and angiogenesis systems in high space dimensions, Milan J. Math. 72 (2004) 1–28.
[11] S. Cygan, G. Karch, K. Krawczyk, H. Wakui, Stability of constant steady states of a chemotaxis model, J. Evol. Equ. 21 (2021) 4873–4896.
[12] H.G. Feichtinger, Generalized amalgams, with applications to Fourier transform, Can. J. Math. 42 (1990) 395–409.
[13] J.J.F. Fournier, J. Stewart, Amalgams of L^p and l^q, Bull. Am. Math. Soc. (N.S.) 13 (1985) 1–21.
[14] H. Fujita, T. Kato, On the Navier-Stokes initial value problem. I, Arch. Ration. Mech. Anal. 16 (1964) 269–315.
[15] Y. Giga, T. Miyakawa, Navier-Stokes flow in $\mathbf{R}^3$ with measures as initial vorticity and Morrey spaces, Commun. Partial Differ. Equ. 14 (1989) 577–618.
[16] Y. Giga, T. Miyakawa, H. Osada, Two-dimensional Navier-Stokes flow with measures as initial vorticity, Arch. Ration. Mech. Anal. 104 (1988) 223–250.
[17] K. Gröchenig, C. Heil, K. Okoudjou, Gabor analysis in weighted amalgam spaces, Sampl. Theory Signal Image Process. 1 (2002) 225–259.
[18] F. Holland, Harmonic analysis on amalgams of L^p and l^q, J. Lond. Math. Soc. (2) 10 (1975) 295–305.
[19] T. Iwabuchi, Global well-posedness for Keller-Segel system in Besov type spaces, J. Math. Anal. Appl. 379 (2011) 930–948.
[20] T. Iwabuchi, M. Nakamura, Small solutions for nonlinear heat equations, the Navier-Stokes equation, and the Keller-Segel system in Besov and Triebel-Lizorkin spaces, Adv. Differ. Equ. 18 (2013) 687–736.

[21] T. Iwabuchi, T. Ogawa, Ill-posedness issue for the drift diffusion system in the homogeneous Besov spaces, Osaka J. Math. 53 (2016) 919–939.
[22] W. Jäger, S. Luckhaus, On explosions of solutions to a system of partial differential equations modelling chemotaxis, Trans. Am. Math. Soc. 329 (1992) 819–824.
[23] T. Kato, The Cauchy problem for quasi-linear symmetric hyperbolic systems, Arch. Ration. Mech. Anal. 58 (1975) 181–205.
[24] T. Kato, Strong L^p-solutions of the Navier-Stokes equation in $\mathbf{R}^m$, with applications to weak solutions, Math. Z. 187 (1984) 471–480.
[25] E.F. Keller, L.A. Segel, Initiation of slime mold aggregation viewed as an instability, J. Theor. Biol. 26 (1970) 399–415.
[26] N. Kikuchi, E. Nakai, N. Tomita, K. Yabuta, T. Yoneda, Calderón-Zygmund operators on amalgam spaces and in the discrete case, J. Math. Anal. Appl. 335 (2007) 198–212.
[27] H. Koch, D. Tataru, Well-posedness for the Navier-Stokes equations, Adv. Math. 157 (2001) 22–35.
[28] H. Kozono, Y. Sugiyama, Local existence and finite time blow-up of solutions in the 2-d Keller-Segel system, J. Evol. Equ. 8 (2008) 353–378.
[29] H. Kozono, Y. Sugiyama, Y. Yahagi, Existence and uniqueness theorem on weak solutions to the parabolic-elliptic Keller-Segel system, J. Differ. Equ. 253 (2012) 2295–2313.
[30] M. Kurokiba, T. Ogawa, Finite time blow-up of the solution for a nonlinear parabolic equation of drift-diffusion type, Differ. Integral Equ. 16 (2003) 427–452.
[31] M. Kurokiba, T. Ogawa, Well-posedness for the drift-diffusion system in L^p arising from the semiconductor device simulation, J. Math. Anal. Appl. 342 (2008) 1052–1067.
[32] P.G. Lemarié-Rieusset, Small data in an optimal Banach space for the parabolic-parabolic and parabolic-elliptic Keller-Segel equations in the whole space, Adv. Differ. Equ. 18 (2013) 1189–1208.
[33] Y. Maekawa, Y. Terasawa, The Navier-Stokes equations with initial data in uniformly local L^p spaces, Differ. Integral Equ. 19 (2006) 369–400.
[34] T. Nagai, Blow-up of radially symmetric solutions to a chemotaxis system, Adv. Math. Sci. Appl. 5 (1995) 581–601.
[35] T. Nagai, M. Mimura, Asymptotic behavior for a nonlinear degenerate diffusion equation in population dynamics, SIAM J. Appl. Math. 43 (1983) 449–464.
[36] T. Ogawa, S. Shimizu, The drift-diffusion system in two-dimensional critical Hardy space, J. Funct. Anal. 255 (2008) 1107–1138.
[37] C.S. Patlak, Random walk with persistence and external bias, Bull. Math. Biophys. 15 (1953) 311–338.
[38] T. Suguro, Well-posedness and unconditional uniqueness of mild solutions to the Keller-Segel system in uniformly local spaces, J. Evol. Equ. 21 (2021) 4599–4618.
[39] N. Wiener, On the representation of functions by trigonometrical integrals, Math. Z. 24 (1926) 575–616.
[40] N. Wiener, Tauberian theorems, Ann. Math. (2) 33 (1932) 1–100.