STRATIFIED CATEGORIES AND A GEOMETRIC APPROACH TO REPRESENTATIONS OF THE SCHUR ALGEBRA

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This is to certify that to the best of my knowledge, the content of this thesis is my own work. This thesis has not been submitted for any degree or other purposes.

I certify that the intellectual content of this thesis is the product of my own work and that all the assistance received in preparing this thesis and sources have been acknowledged.

Signed:

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Abstract

In this thesis we present three new results:

(i) We define a *stratification of abelian categories* as an iterated system of recollements of abelian categories. This definition generalises the definitions of categories of (equivariant) perverse sheaves as well as ε-stratified categories (and in particular highest weight categories) in the sense of Brundan-Stroppel [BS18]. We give necessary and sufficient conditions for a stratification of abelian categories to be equivalent to a category of finite dimensional modules of a finite dimensional algebra - this generalises the main result of Cipriani-Woolf [CW22].

(ii) We define a product of Schur algebra modules that corresponds under Schur-Weyl duality to the Kronecker product of symmetric group modules. This new product is a Schur algebra module theoretic version of Krause's internal product on the category of homogeneous strict polynomial functors defined in [Kra13].

(iii) We give a characteristic-free version of Ginzburg's [CG97, Proposition 4.2.5] construction of the Schur algebra via the convolution product on the Borel-Moore homology of smooth varieties related to the nilpotent cone $\mathcal{N} \subset \mathfrak{gl}_n(\mathbb{C})$. As an application, we give a new proof of Mautner's [Mau14] equivalence of categories between GL_n-equivariant perverse sheaves on $\mathcal{N}$ and a category of Schur algebra modules.

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In memory of my mother Caterina Giuliano
and grandfather Domenico Giuliano

Contents

Chapter 1

Introduction

1.1 Historical background

In his dissertation, Schur [Sch1901] establishes a connection between the polynomial representations of $\mathrm{GL}_n(\mathbb{C})$ of degree d and representations of the group algebra, $\mathbb{C}\mathfrak{S}_d$, of the symmetric group. This conceptual bridge allowed Schur to study the representation theory of $\mathrm{GL}_n(\mathbb{C})$ using known results in the representation theory of $\mathfrak{S}_d$ (see e.g. [Gre80, Introduction] for a more detailed survey of Schur's work and it's influence on the development of the theory of Lie groups).

Green [Gre80], following the work of Carter and Lusztig [CL74], extended the characteristic-zero results in Schur's thesis to results that hold for any infinite field $\Bbbk$. In particular, let $\mathrm{Pol}^d_{\Bbbk}(\mathrm{GL}_n)$ be the category of polynomial representations of $\mathrm{GL}_n(\Bbbk)$ of degree d (where $n \geq d$) and let mod-$\Bbbk\mathfrak{S}_d$ be the category of finite dimensional right $\Bbbk\mathfrak{S}_d$-modules. Green [Gre80, Section 6.2] shows that the *Schur-Weyl duality functor*

$$\mathcal{F}_{SW} := \mathrm{Hom}_{\mathrm{Pol}^d_{\Bbbk}(\mathrm{GL}_n)}(\bigotimes^d \Bbbk^n, -) : \mathrm{Pol}^d_{\Bbbk}(\mathrm{GL}_n) \to \text{mod-}\Bbbk\mathfrak{S}_d$$

is an exact essentially surjective functor - thus establishing an explicit connection between the modular representation theory of $\mathfrak{S}_d$ and the representation theory of GL_n.

An important ingredient in Green's work is the use of an equivalence of categories between $\mathrm{Pol}^d_{\Bbbk}(\mathrm{GL}_n)$ and the category of finite dimensional modules of the

Schur algebra

$$\mathcal{S}_{\Bbbk}(n,d) := \mathrm{End}_{\Bbbk\mathfrak{S}_d}(\bigotimes^d \Bbbk^n).$$

Using the Schur algebra, one can extend the extend the work of Green to the case that $\Bbbk$ is any commutative ring. Indeed, arguing in the same way as Green, if $\Bbbk$ is any commutative ring and $n \geq d$ then the functor

$$\mathcal{F}_{SW} = \mathrm{Hom}(\bigotimes^d \Bbbk^n, -) : \mathcal{S}_{\Bbbk}(n,d)\text{-mod} \to \text{mod-}\Bbbk\mathfrak{S}_d$$

is an exact essentially surjective functor.

In [FS97], Friedlander and Suslin define the category, $\mathrm{Rep}\,\Gamma_d^{\Bbbk}$, of strict polynomial functors of degree d (the definition of this category is recalled in Section 3.7). Moreover they show that, if $n \geq d$, there is an equivalence of categories between $\mathrm{Rep}\,\Gamma_d^{\Bbbk}$ and $\mathcal{S}_{\Bbbk}(n,d)$-mod. This result allows methods available in the study of functors to be used in the study of $\mathcal{S}_{\Bbbk}(n,d)$ modules (see [Tou14] for a survey of such applications). One particular example, that is important to this thesis, is Krause's [Kra13] use of Day convolution to define an internal product

$$-\underline{\otimes}- : \mathrm{Rep}\,\Gamma_d^{\Bbbk} \times \mathrm{Rep}\,\Gamma_d^{\Bbbk} \to \mathrm{Rep}\,\Gamma_d^{\Bbbk}.$$

The work in this thesis is inspired by the dissertations of Rebecca Reischuk [Rei16] and Carl Mautner [Mau10].

In her dissertation, Reischuk [Rei16, Theorem 3.23] (see [AR17, Theorem 4.4] for the published version) shows that the Schur-Weyl duality functor intertwines Krause's product on $\mathrm{Rep}\,\Gamma_d^{\Bbbk}$ with the Kronecker product $-\otimes_{\Bbbk}-$ on $\Bbbk\mathfrak{S}_d$ modules. That is, the following diagram of functors commutes:

$$\begin{tikzcd}
\mathrm{Rep}\,\Gamma_d^{\Bbbk} \times \mathrm{Rep}\,\Gamma_d^{\Bbbk} \arrow[r, "-\underline{\otimes}-"] \arrow[d, "\mathcal{F}_{SW}\times\mathcal{F}_{SW}"'] & \mathrm{Rep}\,\Gamma_d^{\Bbbk} \arrow[d, "\mathcal{F}_{SW}"] \\
\text{mod-}\Bbbk\mathfrak{S}_d \times \text{mod-}\Bbbk\mathfrak{S}_d \arrow[r, "-\otimes_{\Bbbk}-"] & \text{mod-}\Bbbk\mathfrak{S}_d
\end{tikzcd}$$

This result suggests a new approach to the Kronecker product of symmetric group modules - in particular the open problem of calculating Kronecker coefficients[1].

[1] The Kronecker coefficients are the multiplicities of simple modules appearing in the Kronecker product of simple modules.

Unfortunately, the abstract nature of the definition of $\underline{\otimes}$ suggests that this ideal is still beyond reach.

A result in Mautner's dissertation [Mau10, Theorem 1.3.1] (see [Mau14, Theorem 1.1] for the published version) is that the category, $P_{\mathrm{GL}_d}(\mathcal{N}, \Bbbk)$, of $\mathrm{GL}_d(\mathbb{C})$-equivariant perverse sheaves on the nilpotent cone $\mathcal{N} \subset \mathfrak{gl}_d(\mathbb{C})$ is equivalent to the category, $\mathcal{S}_{\Bbbk}(n,d)$-mod, of finite dimensional modules of the Schur algebra $\mathcal{S}_{\Bbbk}(n,d)$ if $n \geq d$.

A natural question to ask is for a geometric definition of a product on $P_{\mathrm{GL}_d}(\mathcal{N}, \Bbbk)$ that corresponds to Krause's product on $\operatorname{Rep} \Gamma^{\Bbbk}_d$. This thesis contains a collection of results discovered whilst searching for (or daydreaming about) an answer to this question. We believe these results are of independent interest.

1.2 Outline

The body of this thesis is partitioned into three chapters, corresponding to three major results.

Chapter 2 studies iterated systems of recollements of abelian categories, that we call *stratifications of abelian categories*. The main result of this chapter is to give sufficient conditions for a stratification of abelian categories to have enough projective objects (Theorem 2.3.9).

Chapter 3 recounts the theory of Schur algebra modules and constructs a Schur algebra theoretic analogue of Krause's product (Theorem 3.6.1).

Chapter 4 uses a characteristic-free version of Ginzburg's construction of the Schur algebra (Theorem 4.3.6) to define an equivalence of categories between $P_{\mathrm{GL}_d}(\mathcal{N}, \Bbbk)$ and $\mathcal{S}(n,d)$-mod for $n \geq d$ (Theorem 4.4.1). Our approach differs from Mautner's approach [Mau14, Theorem 1.1] in that we do not require the geometric Satake correspondence (as defined in [MV07, Theorem 14.1]), and instead argue using the geometry of the partial Springer resolutions.

A complete description of the original results in this thesis are given in Sections 1.3, 1.4, 1.5, which detail the results in Chapters 2, 3, 4 respectively. Chapters 2 and 3 are self-contained, while Chapter 4 relies on results in the preceding chapters.

1.3 Main results from Chapter 2

A *recollement* of abelian categories is a short exact sequence of abelian categories

$$0 \longrightarrow \mathcal{A}_Z \xrightarrow{\ i_*\ } \mathcal{A} \xrightarrow{\ j^*\ } \mathcal{A}_U \longrightarrow 0 \tag{1.1}$$

in which $\mathcal{A}_Z$ is a Serre subcategory of $\mathcal{A}$ (with Serre quotient $\mathcal{A}_U$), i_* has both a left and right adjoint, and j^* has fully-faithful left and right adjoints.

Recollements of abelian (and triangulated) categories arise in many geometric and representation theoretic contexts. This includes the construction of perverse sheaves in [BBD82] (see also [MV86]), and the definition of highest weight categories [CPS88] and their generalisations (see e.g. [CPS96], [BS18]). In these applications, one needs an iterated series of recollements. We formalise the idea of iterated series of recollements by a construction we call a *stratification* of an abelian category.

Definition 2.1.4. A *stratification* of an abelian category $\mathcal{A}$ by a non-empty finite poset Λ consists of

(i) Abelian categories $\mathcal{A}_\lambda$, for each $\lambda \in \Lambda$ (which we call *strata categories*).

(ii) For each closed-downwards subposet $\Lambda' \subset \Lambda$, there is a Serre subcategory $\mathcal{A}_{\Lambda'} \hookrightarrow \mathcal{A}$, in which $\mathcal{A}_\emptyset = 0$ and $\mathcal{A}_\Lambda = \mathcal{A}$. Moreover, for each pair of closed-downwards subposets $\Lambda'_1 \subset \Lambda'_2 \subset \Lambda$, there are inclusions of Serre subcategories $\mathcal{A}_{\Lambda'_1} \hookrightarrow \mathcal{A}_{\Lambda'_2}$, and for each maximal $\lambda' \in \Lambda'$ there is a recollement

$$0 \longrightarrow \mathcal{A}_{\Lambda' \setminus \{\lambda'\}} \longrightarrow \mathcal{A}_{\Lambda'} \longrightarrow \mathcal{A}_\lambda \longrightarrow 0$$

This definition of a stratification of an abelian category is original, however the idea is implicitly used in work dating back to [BBD82] (examples are given in Section 2.1).

In Chapter 2 we develop the theory of stratifications of abelian categories. We list here the main original results from Chapter 2.

Definition 2.2.1 defines the *intermediate extension functor* $j_{!*} : \mathcal{A}_U \to \mathcal{A}$ for a recollement as in (1.1). This definition imitates the definition of the intermediate extension functor in the theory of perverse sheaves. If $\mathcal{A}$ has a stratification by a finite poset Λ, then for each $\lambda \in \Lambda$, there is a functor $j^{\lambda}_{!*} : \mathcal{A}_\lambda \to \mathcal{A}$ defined by the composition

$$\mathcal{A}_\lambda \xrightarrow{j_{!*}} \mathcal{A}_{\{\mu\in\Lambda \mid \mu\leq\lambda\}} \hookrightarrow \mathcal{A}$$

The following two results are proven using the intermediate-extension functor.

Proposition 2.2.4. *Let $\mathcal{A}$ be an abelian category with a stratification by a finite poset Λ. Every simple object $L \in \mathcal{A}$ is of the form $j^{\lambda}_{!*}L_\lambda$, for a unique (up to isomorphism) simple object $L_\lambda \in \mathcal{A}_\lambda$ and unique $\lambda \in \Lambda$.*

Proposition 2.2.6. *If $\mathcal{A}$ is an abelian category with a stratification by a finite poset, then every object in $\mathcal{A}$ has a finite filtration by simple objects if and only if the same is true of all the strata categories.*

The following original result gives sufficient conditions for a category, $\mathcal{A}$, appearing in a recollement as in (1.1) to have enough projectives.

Theorem 2.3.9. *Consider a recollement:*

$$0 \longrightarrow \mathcal{A}_Z \xrightarrow{i_*} \mathcal{A} \xrightarrow{j^*} \mathcal{A}_U \longrightarrow 0$$

Suppose $\mathcal{A}_U$ and $\mathcal{A}_Z$ have finitely many simple objects and every object has a finite filtration by simple objects. Suppose moreover that for any simple objects A, B in $\mathcal{A}$,

$$\dim_{\mathrm{End}_{\mathcal{A}}(B)} \mathrm{Ext}^1_{\mathcal{A}}(A, B) < \infty.$$

Then $\mathcal{A}$ has enough projectives if and only if both $\mathcal{A}_U$ and $\mathcal{A}_Z$ have enough projectives.[2]

As an application we obtain the following important Corollary.

[2]Since $\mathrm{End}_{\mathcal{A}}(B)$ is a division ring, any $\mathrm{End}_{\mathcal{A}}(B)$-module is free. For an $\mathrm{End}_{\mathcal{A}}(B)$-module M, $\dim_{\mathrm{End}_{\mathcal{A}}(B)} M$ is the rank of M as a free $\mathrm{End}_{\mathcal{A}}(B)$-module.

Corollary 2.3.11. *For any field $\Bbbk$, a $\Bbbk$-linear abelian category with a stratification by a finite poset is equivalent to a category of finite dimensional modules of a finite dimensional $\Bbbk$-algebra if and only if the same is true for all strata categories.*

As a special case we recover a result of Cipriani and Woolf [CW22, Theorem 4.6] (see Corollary 2.3.12) that says that a category of perverse sheaves (with coefficients in a field) on a space stratified by finitely many strata is equivalent to a category of finite dimensional modules of a finite dimensional $\Bbbk$-algebra if and only if the same is true for each category of finite type local systems on each stratum.

For the remainder of this section fix an abelian category $\mathcal{A}$ with finitely many simple objects, enough projectives and injectives, and admitting a stratification by a poset Λ. Suppose furthermore that each object in $\mathcal{A}$ has a finite filtration by simple objects. Let B be a set indexing the simple objects in $\mathcal{A}$ (up to isomorphism) and write $L(b)$ for the simple object corresponding to $b \in B$. Define the *stratification function*

$$\rho : B \to \Lambda$$

that maps each $b \in B$ to the corresponding $\lambda \in \Lambda$ in which $L(b) = j^{\lambda}_{!*} L_{\lambda}(b)$ for some simple object $L_{\lambda}(b) \in \mathcal{A}_{\lambda}$.

For each $\lambda \in \Lambda$, define the Serre quotient functor $j^{\lambda} : \mathcal{A}_{\{\mu \in \Lambda \mid \mu \leq \lambda\}} \to \mathcal{A}_{\lambda}$, and let $j^{\lambda}_{!} : \mathcal{A}_{\lambda} \to \mathcal{A}_{\{\mu \in \Lambda \mid \mu \leq \lambda\}}$ and $j^{\lambda}_{*} : \mathcal{A}_{\lambda} \to \mathcal{A}_{\{\mu \in \Lambda \mid \mu \leq \lambda\}}$ be the left and right adjoints of j^{λ}. By a slight abuse of notation, write $j^{\lambda}_{!} : \mathcal{A}_{\lambda} \to \mathcal{A}$ and $j^{\lambda}_{*} : \mathcal{A}_{\lambda} \to \mathcal{A}$ for the functors obtained by postcomposing with the inclusion functor $\mathcal{A}_{\{\mu \in \Lambda \mid \mu \leq \lambda\}} \hookrightarrow \mathcal{A}$.

For $b \in B$ and $\lambda = \rho(b)$, define the *standard* and *costandard* objects

$$\Delta(b) := j^{\lambda}_{!} P_{\lambda}(b), \qquad \nabla(b) := j^{\lambda}_{*} I_{\lambda}(b).$$

The following result follows from the proof of Theorem 2.3.9.

Porism 2.4.1. *For each $b \in B$:*

(i) The projective cover, $P(b)$, of $L(b)$ fits into a short exact sequence

$$0 \to Q(b) \to P(b) \to \Delta(b) \to 0$$

in which $Q(b)$ has a filtration by quotients of $\Delta(b')$ satisfying $\rho(b') > \rho(b)$.

(ii) The injective envelope, $I(b)$, of $L(b)$ fits into a short exact sequence

$$0 \to \nabla(b) \to I(b) \to Q'(b) \to 0$$

in which $Q'(b)$ has a filtration by subobjects of $\nabla(b')$ satisfying $\rho(b') > \rho(b)$.

Theorem 2.5.2 gives a definition of highest weight category using the language of stratifications of abelian categories. This result is essentially a rephrasing of [Kra17, Theorem 3.4] using our terminology.

Theorem 2.5.2. *If $\Bbbk$ is a field, then a $\Bbbk$-linear abelian category $\mathcal{A}$ is a highest weight category with respect to a finite poset Λ if and only if $\mathcal{A}$ has a stratification with respect to Λ in which*

(i) For each closed-downwards subset $\Lambda' \subset \Lambda$, and objects X, Y in the Serre subcategory $\mathcal{A}_{\Lambda'} \hookrightarrow \mathcal{A}$,

$$\mathrm{Ext}^2_{\mathcal{A}_{\Lambda'}}(X, Y) \simeq \mathrm{Ext}^2_{\mathcal{A}}(X, Y).$$

(ii) Every strata category is equivalent to mod-Γ_λ *for some finite dimensional division algebra Γ_λ.*

Brundan and Stroppel [BS18] define a generalization of highest weight categories called an ε-stratified category (this definition is recalled in Definition 2.4.2). It would interesting to have necessary and sufficient conditions for an abelian category $\mathcal{A}$ with a stratification by a poset Λ to be an ε-stratified category. An answer to this problem is suggested in Conjecture 2.4.4.

1.4 Main results from Chapter 3

In Chapter 3 we recall the basic theory of the Schur algebra $\mathcal{S}_\Bbbk(n,d) = \mathrm{End}_{\mathfrak{S}_d}(\bigotimes^d \Bbbk^n)$, for a field $\Bbbk$, and define a new product of Schur algebra modules corresponding to Krause's internal product of polynomial functors defined in [Kra13]. More precisely, we define a product

$$-\underline{\boxtimes}- : \mathcal{S}_\Bbbk(m,d)\text{-mod} \times \mathcal{S}_\Bbbk(n,d)\text{-mod} \to \mathcal{S}_\Bbbk(mn,d)\text{-mod}$$

in which the following diagram of functors commutes.

$$
\begin{tikzcd}
\mathcal{S}_{\Bbbk}(m,d)\text{-mod} \times \mathcal{S}_{\Bbbk}(n,d)\text{-mod} \arrow[r, "-\underline{\boxtimes}-"] \arrow[d, "\mathcal{F}_{SW}\times\mathcal{F}_{SW}"'] & \mathcal{S}_{\Bbbk}(mn,d)\text{-mod} \arrow[d, "\mathcal{F}_{SW}"] \\
\text{mod-}\Bbbk\mathfrak{S}_d \times \text{mod-}\Bbbk\mathfrak{S}_d \arrow[r, "-\otimes_{\Bbbk}-"] & \text{mod-}\Bbbk\mathfrak{S}_d
\end{tikzcd}
\tag{1.2}
$$

To state the definition of $\underline{\boxtimes}$ we need some preliminary definitions. Firstly, denote the standard basis of $\Bbbk^n$ by $\{v_1, \dots, v_n\}$, and denote the standard basis of $\Bbbk^{mn}$ by $\{v_{ij} \mid i \in [m], j \in [n]\}$, where $[m] = \{1, \dots, m\}$. Define the $\Bbbk\mathfrak{S}_d$-equivariant isomorphism

$$\theta : \bigotimes^d \Bbbk^m \otimes_{\Bbbk} \bigotimes^d \Bbbk^n \to \bigotimes^d \Bbbk^{mn}$$

by

$$(v_{i_1} \otimes \cdots \otimes v_{i_d}) \otimes (v_{j_1} \otimes \cdots \otimes v_{j_d}) \mapsto v_{i_1 j_1} \otimes \cdots \otimes v_{i_d j_d}.$$

Define the algebra embedding

$$\Theta : \mathrm{End}_{\Bbbk\mathfrak{S}_d}(\bigotimes^d \Bbbk^m) \otimes_{\Bbbk} \mathrm{End}_{\Bbbk\mathfrak{S}_d}(\bigotimes^d \Bbbk^n) \hookrightarrow \mathrm{End}_{\Bbbk\mathfrak{S}_d}(\bigotimes^d \Bbbk^{mn})$$

by

$$\Theta(f \otimes g)(v_{i_1 j_1} \otimes \cdots \otimes v_{i_d j_d}) = \theta(f(v_{i_1} \otimes \cdots \otimes v_{i_d}) \otimes g(v_{j_1} \otimes \cdots \otimes v_{j_d})).$$

The *homogeneous external product*

$$-\underline{\boxtimes}- : \mathcal{S}_{\Bbbk}(m,d)\text{-mod} \times \mathcal{S}_{\Bbbk}(n,d)\text{-mod} \to \mathcal{S}_{\Bbbk}(mn,d)\text{-mod}$$

is defined by induction along Θ. That is

$$
\begin{aligned}
M\underline{\boxtimes}N &:= \mathcal{S}_{\Bbbk}(mn,d) \otimes_{\mathcal{S}_{\Bbbk}(m,d)\otimes\mathcal{S}_{\Bbbk}(n,d)} (M \otimes_{\Bbbk} N) \\
&= \mathcal{S}_{\Bbbk}(mn,d) \otimes_{\Bbbk} M \otimes_{\Bbbk} N / \langle \Theta(f \otimes g) \otimes m \otimes n - 1 \otimes f \cdot m \otimes g \cdot n \rangle
\end{aligned}
$$

Theorem 3.6.1 says that diagram (1.2) commutes. Theorem 3.7.5 says that, under the equivalence between $\mathcal{S}_{\Bbbk}(n,d)$-mod and $\mathrm{Rep}\,\Gamma^{\Bbbk}_d$ (for $n \geq d$), the homogeneous external product corresponds to Krause's product on strict polynomial functors.

To complement the construction of the homogeneous external product, we define the embedding $\Theta : \mathcal{S}_{\Bbbk}(m, d) \times \mathcal{S}_{\Bbbk}(n, d) \to \mathcal{S}_{\Bbbk}(mn, d)$ in terms of generators of the Schur algebra. We recall the definition of these generators now.

Let $E_i, F_i : \bigotimes^d \mathbb{C}^n \to \bigotimes^d \mathbb{C}^n$ be the endomorphisms given by the actions of the Chevalley generators e_i, f_i on $\bigotimes^d \mathbb{C}^n$ under the natural $\mathfrak{gl}_n(\mathbb{C})$-action. Define the set

$$\Lambda(n, d) := \{(\lambda_1, \dots, \lambda_n) \in \mathbb{N}^n \mid \sum_i \lambda_i = d\}.$$

For $\lambda \in \Lambda(n, d)$, let $1_\lambda : \bigotimes^d \mathbb{C}^n \to \bigotimes^d \mathbb{C}^n$ be the projection onto the $\Bbbk\mathfrak{S}_d$-invariant subspace of $\bigotimes^d \mathbb{C}^n$ generated by $v_1^{\otimes \lambda_1} \otimes \cdots \otimes v_n^{\otimes \lambda_n}$.

The Schur algebra $\mathcal{S}_{\mathbb{C}}(n, d)$ is generated by the E_i, F_i $(i = 1, \dots, n-1)$ and the 1_λ $(\lambda \in \Lambda(n, d))$. The Schur algebra $\mathcal{S}_{\mathbb{Z}}(n, d)$ is isomorphic to the $\mathbb{Z}$-subalgebra of $\mathcal{S}_{\mathbb{C}}(n, d)$ generated by the 1_λ together with the elements

$$E_i^{(r)} := \frac{E_i^r}{r!}, \qquad F_i^{(r)} := \frac{F_i^r}{r!}.$$

Moreover, for a commutative ring $\Bbbk$, there is an isomorphism

$$\mathcal{S}_{\Bbbk}(n, d) \simeq \Bbbk \otimes_{\mathbb{Z}} \mathcal{S}_{\mathbb{Z}}(n, d)$$

and so each Schur algebra $\mathcal{S}_{\Bbbk}(n, d)$ is generated by elements 1_λ, $E_i^{(r)}$, and $F_i^{(r)}$.

Proposition 3.6.4 defines the embedding $\Theta : \mathcal{S}_{\Bbbk}(m, d) \times \mathcal{S}_{\Bbbk}(n, d) \to \mathcal{S}_{\Bbbk}(mn, d)$ in terms of these generators.

1.5 Main results from Chapter 4

In Chapter 4 we give a new construction of Mautner's [Mau14, Theorem 1.1] equivalence of categories $P_{\mathrm{GL}_d}(\mathcal{N}_{\mathrm{GL}_d}, \Bbbk) \to \mathcal{S}_{\Bbbk}(n, d)$-mod, when $n \geq d$. To state the main result we introduce some notation.

Let $G = \mathrm{GL}_d(\mathbb{C})$ and let $B \subset G$ be the Borel subalgebra consisting of upper triangular invertible matrices. For each $n \in \mathbb{N}$ and $\lambda \in \Lambda(n, d)$, let $P_\lambda = L_\lambda \ltimes U_\lambda \subset G$ be the parabolic subgroup containing B and with Levi factor $L_\lambda \simeq \mathrm{GL}_{\lambda_1} \times \cdots \times \mathrm{GL}_{\lambda_n}$.

Use lowercase fraktur letters to denote the Lie algebra of a Lie group denoted by the corresponding uppercase letter.

Write $\mathcal{N}_H \subset \mathfrak{h}$ for the nilpotent cone in a Lie algebra $\mathfrak{h}$, and set $\mathcal{N} := \mathcal{N}_G$. For a partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ of d, let $\mathcal{O}_\lambda \subset \mathcal{N}$ be the G-orbit of the Jordan matrix with Jordan blocks of sizes $\lambda_1, \ldots, \lambda_m$. Let $\lambda^\vee$ be the dual partition of λ.

Define the partial Grothendieck resolution

$$m_\lambda : G \times^{P_\lambda} \mathfrak{p}_\lambda \to \mathfrak{g}; \quad (g, x) \mapsto gxg^{-1}.$$

Consider also the following diagram in which the squares are Cartesian.

$$\begin{tikzcd}
G \times^{P_\lambda} \mathfrak{p}_\lambda \arrow[r, "m_\lambda"] & \mathfrak{g} \\
\check{\mathcal{N}}_\lambda := G \times^{P_\lambda} \mathcal{N}_{P_\lambda} \arrow[r, "\check{m}_\lambda"] \arrow[u, hook] & \mathcal{N} \arrow[u, hook] \\
\tilde{\mathcal{N}}_\lambda := G \times^{P_\lambda} \mathfrak{u}_\lambda \arrow[r, "\tilde{m}_\lambda"] \arrow[u, hook] & \overline{\mathcal{O}_{\lambda^\vee}} \arrow[u, hook]
\end{tikzcd}$$

Define the perverse sheaves

$$\Gamma^\lambda := \check{m}_{\lambda !}\underline{\Bbbk}_{\check{\mathcal{N}}_\lambda}[\dim \mathcal{N}],$$
$$\Lambda^\lambda := \tilde{m}_{\lambda !}\underline{\Bbbk}_{\tilde{\mathcal{N}}_\mu}[2 \dim G/P_\mu],$$

in $P_G(\mathcal{N}, \Bbbk)$.

The following is the main result of Chapter 4.

Theorem 4.4.1. *If $n \geq d$, then the perverse sheaf*

$$\Gamma_{n,d} := \bigoplus_{\lambda \in \Lambda(n,d)} \Gamma^\lambda$$

is a projective generator of $P_G(\mathcal{N}, \Bbbk)$, and $\mathcal{S}_\Bbbk(n, d)^{op} \simeq \mathrm{End}_{P_G(\mathcal{N},\Bbbk)}(\Gamma_{n,d})$.

In particular, the functor

$$\mathrm{Hom}_{P_G(\mathcal{N},\Bbbk)}(\Gamma_{n,d}, -) : P_G(\mathcal{N}, \Bbbk) \to \mathcal{S}_\Bbbk(n, d)\text{-mod}$$

is an equivalence of categories.

Lemma 4.4.9 says that $\Gamma_{n,d}$ is a projective generator. To prove Theorem 4.4.1 we use the geometric Ringel duality functor (as defined in [AM15]) to show (Lemma 4.4.8) that

$$\mathrm{End}_{P_G(\mathcal{N},\Bbbk)}(\bigoplus_{\lambda\in\Lambda(n,d)} \Gamma^\lambda) \simeq \mathrm{End}_{P_G(\mathcal{N},\Bbbk)}(\bigoplus_{\lambda\in\Lambda(n,d)} \Lambda^\lambda).$$

It follows from a result of Ginzburg [CG97, Theorem 8.6.7] that there is a natural isomorphism

$$\mathrm{Hom}_{P_G(\mathcal{N},\Bbbk)}(\Lambda^\mu, \Lambda^\lambda) \simeq H^{BM}_{\dim\tilde{\mathcal{N}}_\lambda+\dim\tilde{\mathcal{N}}_\mu}(\tilde{\mathcal{N}}_\lambda \times_{\mathcal{N}} \tilde{\mathcal{N}}_\mu, \Bbbk),$$

where composition of morphisms corresponds to the convolution product on Borel-Moore cycles. Theorem 4.4.1 follows from the following result.

Theorem 4.3.6. *There is an algebra isomorphism*

$$\mathcal{S}_\Bbbk(n,d)^{op} \simeq \bigoplus_{\lambda,\mu\in\Lambda(n,d)} H^{BM}_{\dim\tilde{\mathcal{N}}_\lambda+\dim\tilde{\mathcal{N}}_\mu}(\tilde{\mathcal{N}}_\lambda \times_{\mathcal{N}} \tilde{\mathcal{N}}_\mu, \Bbbk),$$

where the algebra product on the right hand side is the convolution product.

Theorem 4.3.6 is shown in the case that $\Bbbk$ has characteristic zero by Ginzburg [CG97, Proposition 4.2.5]. Our result is derived using Ginzburg's result, together with the observation that the isomorphism

$$\mathcal{S}_\mathbb{C}(n,d)^{op} \to \bigoplus_{\lambda,\mu\in\Lambda(n,d)} H^{BM}_{\dim\tilde{\mathcal{N}}_\lambda+\dim\tilde{\mathcal{N}}_\mu}(\tilde{\mathcal{N}}_\lambda \times_{\mathcal{N}} \tilde{\mathcal{N}}_\mu, \mathbb{C})$$

sends the generators of $\mathcal{S}_\mathbb{Z}(n,d)^{op}$ to fundamental classes of irreducible components. The result follows by a simple argument using the dimensions of these algebras.

Chapter 2

Stratifications of abelian categories

A *recollement* of abelian categories is a short exact sequence of abelian categories

$$0 \longrightarrow \mathcal{A}_Z \xrightarrow{\quad i_* \quad} \mathcal{A} \xrightarrow{\quad j^* \quad} \mathcal{A}_U \longrightarrow 0$$

in which $\mathcal{A}_Z$ is a Serre subcategory of $\mathcal{A}$ (with Serre quotient $\mathcal{A}_U$), i_* has both a left and right adjoint, and j^* has fully-faithful left and right adjoints. In this case we say that $\mathcal{A}$ is a *gluing* of $\mathcal{A}_Z$ and $\mathcal{A}_U$.

A *stratification* of an abelian category $\mathcal{A}$ by a non-empty finite poset Λ consists of

(i) Abelian categories $\mathcal{A}_\lambda$, for each $\lambda \in \Lambda$.

(ii) For each closed-downwards subposet $\Lambda' \subset \Lambda$, there is a Serre subcategory $\mathcal{A}_{\Lambda'} \hookrightarrow \mathcal{A}$, in which $\mathcal{A}_\emptyset = 0$ and $\mathcal{A}_\Lambda = \mathcal{A}$. Moreover, for each pair of closed-downwards subposets $\Lambda'_1 \subset \Lambda'_2 \subset \Lambda$, there are inclusions of Serre categories $\mathcal{A}_{\Lambda'_1} \hookrightarrow \mathcal{A}_{\Lambda'_2}$, and for each maximal $\lambda' \in \Lambda'$ there is a recollement

$$0 \longrightarrow \mathcal{A}_{\Lambda' \backslash \{\lambda'\}} \longrightarrow \mathcal{A}_{\Lambda'} \longrightarrow \mathcal{A}_\lambda \longrightarrow 0 \ .$$

As an example, consider what it means for a category $\mathcal{A}$ to have a stratification by the poset $1 \leq 2 \leq 3$. Such a stratification consists of the following categories, in which each row and column is a recollement:

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
 & \mathcal{A}_{\{1\}} \simeq \mathcal{A}_1 \arrow[d] & & & \\
0 \arrow[r] & \mathcal{A}_{\{1,2\}} \arrow[r] \arrow[d, "j_2^*"] & \mathcal{A} \arrow[r, "j_3^*"] & \mathcal{A}_3 \arrow[r] & 0 \\
 & \mathcal{A}_2 \arrow[d] & & & \\
 & 0 & & &
\end{tikzcd}$$

For example, the category, $\mathcal{A}$, of representations of the quiver $\bullet \to \bullet \to \bullet$ over a field $\Bbbk$, has a filtration by the poset $1 \leq 2 \leq 3$. Indeed, define the quiver representations:

$$\Delta_1 = 0 \to 0 \to \Bbbk, \qquad \Delta_2 = 0 \to \Bbbk \to \Bbbk, \qquad \Delta_3 = \Bbbk \to \Bbbk \to \Bbbk,$$

and

$$\nabla_1 = 0 \to 0 \to \Bbbk, \qquad \nabla_2 = 0 \to \Bbbk \to 0, \qquad \nabla_3 = \Bbbk \to 0 \to 0.$$

The stratification on $\mathcal{A}$ is defined by setting the categories $\mathcal{A}_{\Lambda'}$ to be the Serre subcategory of $\mathcal{A}$ generated by the Δ_i in which $i \in \Lambda'$, and defining each $\mathcal{A}_i \simeq$ mod-$\operatorname{End}_{\mathcal{A}}(\Delta_i) \simeq$ mod-$\Bbbk$. The Serre quotient functors are defined $j_3^* = \operatorname{Hom}_{\mathcal{A}}(\Delta_3, -)$ and $j_2^* = \operatorname{Hom}_{\mathcal{A}}(\Delta_2, -)$. Note that the functors $j_i^* : \mathcal{A}_{\{1,\ldots,i\}} \to$ mod-$\Bbbk$ have fully-faithful left adjoint $j_!^i$: mod-$\Bbbk \to \mathcal{A}_{\{1,\ldots,i\}}$ mapping $\Bbbk \mapsto \Delta_i$, and fully-faithful right adjoint j_*^i : mod-$\Bbbk \to \mathcal{A}_{\{1,\ldots,i\}}$ mapping $\Bbbk \mapsto \nabla_i$.

Our definition of a stratification of abelian categories is original, however the idea is implicitly used in work dating back to [BBD82]. Examples of stratifications of abelian categories arise in the theory of perverse sheaves (see Example 2.1.7) and in certain categories of modules (see Example 2.1.9). In particular, we'll prove that every highest weight category (or more generally every ε-stratified category in the sense of Brundan-Stroppel [BS18]) with respect to a poset Λ has a stratification by the poset Λ (Theorem 2.5.2). A long list of further examples of recollements of abelian categories can be found in [Psa14, Section 2.1].

In Section 2.2 we define the intermediate extension functor $j_{!*} : \mathcal{A}_U \to \mathcal{A}$ and develop some basic theory about recollements of abelian categories. We show that the simple objects of $\mathcal{A}$ are all either simple objects in $\mathcal{A}_Z$ or the intermediate extension of simple objects in $\mathcal{A}_U$ (Proposition 2.2.4). Moreover we show that every object of $\mathcal{A}$ has finite length if and only if the same is true for $\mathcal{A}_Z$ and $\mathcal{A}_U$ (Proposition 2.2.6). These results are well known in the theory of perverse sheaves and the proofs here are almost identical to the standard proofs of these results in the theory of perverse sheaves (see e.g. [Ach21, Chapter 3]).

In Section 2.3 we give new conditions for when the gluing of two abelian categories with enough projectives has enough projectives (Theorem 2.3.9). This generalises a result of Cipriani-Woolf [CW22, Theorem 4.6], which says that a category of perverse sheaves (with coefficients in a field) on a space stratified by finitely many strata has enough projectives if and only if the same is true for each category of finite type local systems on each stratum.

In Section 2.4 we continue our study of abelian categories equipped with a stratification and enough projectives (as well as some finiteness conditions). In this setting, we define a set of *standard* and *costandard* objects of $\mathcal{A}$ indexed by the simple objects of $\mathcal{A}$. These do not satisfy properties as nice as those of standard/costandard objects in a highest weight category. For example, projective indecomposable objects do not always have a filtration by standard objects. Instead they have filtrations by quotients of standard objects (Porism 2.4.1). Brundan and Stroppel [BS18] define a framework for categories with a 'nice' theory of standard and costandard objects - these they call ε-stratified categories. We conclude Section 2.4 by reviewing their definition and conjecture when an abelian category with a stratification is an ε-stratified category.

In Section 2.5 we show that a category $\mathcal{A}$ is a highest weight category with respect to a poset Λ if and only if $\mathcal{A}$ has a stratification by Λ in which each strata category has one simple object, and $\mathrm{Ext}^2_{\mathcal{A}_{\Lambda'}}(X,Y) \simeq \mathrm{Ext}^2_{\mathcal{A}}(X,Y)$ for each closed-downwards subposet $\Lambda' \subset \Lambda$ and objects X, Y in the Serre subcategory $\mathcal{A}_{\Lambda'} \subset \mathcal{A}$. This result is essentially a rephrasing of a known result (see e.g. [Kra17, Theorem 3.4]), and our proof does not differ from that of [Kra17, Theorem 3.4] in a significant way.

Historically, the concept of a recollement of abelian categories was preceded by the definition of a recollement of triangulated categories due to Beilinson, Bernstein and Deligne [BBD82]. This is a generalisation of Grothendieck's six functors relating the constructible derived category, $\mathcal{D}(X)$, of sheaves on a variety X with the constructible derived categories, $\mathcal{D}(U)$ and $\mathcal{D}(Z)$, of sheaves on an open subvariety $U \subset X$ and the closed complement $Z := X \backslash U$. The conditions defining a recollement of abelian categories are (possibly first) used in [BBD82, Proposition 1.4.16]. This statement says that given a recollement of triangulated categories with t-structure, one obtains a recollement of abelian categories on the hearts of the t-structures by taking zero-th cohomology. We explain and prove this result in Section 2.6.

2.1 Preliminaries

We begin with an axiomatic definition of recollement. The notation used in this definition will be used throughout the paper.

Definition 2.1.1. A *recollement of abelian categories* consists of three abelian categories $\mathcal{A}_Z$, $\mathcal{A}$ and $\mathcal{A}_U$ and functors:

$$\begin{tikzcd} \mathcal{A}_Z \arrow[r, "i_*=i_!" description] & \mathcal{A} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^!=j^*" description] & \mathcal{A}_U \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd} \tag{2.1}$$

satisfying the conditions:

(R1) $(i^*, i_* = i_!, i^!)$ and $(j_!, j^! = j^*, j_*)$ are adjoint triples.

(R2) The functors $i_* = i_!$, $j_!$, j_* are fully-faithful. Equivalently the adjunction maps $i^* i_* \to \mathrm{Id} \to i^! i_!$ and $j^* j_* \to \mathrm{Id} \to j^! j_!$ are isomorphisms.

(R3) The functors satisfy $j^* i_* = 0$ (and so by adjunction $i^* j_! = 0 = i^! j_*$).

(R4) The adjunction maps produce exact sequences for each object $X \in \mathcal{A}$:

$$j_! j^! X \to X \to i_* i^* X \to 0 \tag{2.2}$$

$$0 \to i_! i^! X \to X \to j_* j^* X \tag{2.3}$$

Alternatively, condition (R4) can be replaced by the condition

(R4') For any object $X \in \mathcal{A}$, if $j^* X = 0$ then X is in the essential image of i_*.

A *recollement of triangulated categories* is defined in the same way as a recollement of abelian categories except that condition (R4) is replaced by the existence of the triangles:

$$j_! j^! X \to X \to i_* i^* X \to \tag{2.4}$$

$$i_! i^! X \to X \to j_* j^* X \to \tag{2.5}$$

for each object X.

Remark 2.1.2. The interchangeability of (R4) and (R4') follows from the following argument. If $j^* X = 0$ then (R4) implies that $i_! i^! X \simeq X \simeq i_* i^* X$ and so X is in the essential image of i. Conversely let $\mu : j_! j^! \to \mathrm{Id}$ and $\eta : \mathrm{Id} \to i_* i^*$ be the adjunction natural transformations. Then there is a commutative diagram

$$\begin{tikzcd}
j_! j^! X \arrow[r, "\mu_X"] \arrow[d, "\eta_{j_! j^! X}"'] & X \arrow[r] \arrow[d, "\eta_X"'] & \operatorname{cok} \mu_X \arrow[r] \arrow[d, "\eta_{\operatorname{cok} \mu_X}"] & 0 \\
0 \arrow[r] & i_* i^* X \arrow[r] & i_* i^* (\operatorname{cok} \mu_X) \arrow[r] & 0
\end{tikzcd}$$

in which the rows are exact. By applying j^* to the top row we see that $j^* \operatorname{cok} \mu_X = 0$ and so (R4') implies that $\operatorname{cok} \mu_X \simeq i_* i^* (\operatorname{cok} \mu_X) \simeq i_* i^* X$. Equation (2.3) holds by a similar argument.

Write $\mathcal{A}^Z$ for the essential image of i_*. To reconcile Definition 2.1.1 with the initial definition of recollement note that by (R2), $\mathcal{A}^Z \simeq \mathcal{A}_Z$, and by (R4'), $\mathcal{A}^Z$ is the kernel of the exact functor j^* and is hence a Serre subcategory of $\mathcal{A}$. A particular consequence is that

$$\operatorname{Ext}^1_{\mathcal{A}_Z}(X, Y) \simeq \operatorname{Ext}^1_{\mathcal{A}}(i_* X, i_* Y) \tag{2.6}$$

for any $X, Y \in \mathcal{A}_Z$.

For $k \in \mathbb{N}$, we call a recollement of abelian categories *k-homological* if for all $n \leq k$ and $X, Y \in \mathcal{A}_Z$,

$$\mathrm{Ext}^n_{\mathcal{A}_Z}(X, Y) \simeq \mathrm{Ext}^n_{\mathcal{A}}(i_*X, i_*Y).$$

Say that a recollement of abelian categories is *homological* if it is k-homological for all $k \in \mathbb{N}$. A study of homological recollements is given in [Psa14].

It will be useful to note that if we extend the sequences (2.2) and (2.3) to exact sequences

$$0 \to K \to j_!j^!X \to X \to i_*i^*X \to 0 \tag{2.7}$$

$$0 \to i_!i^!X \to X \to j_*j^*X \to K' \to 0 \tag{2.8}$$

then K and K' are in $\mathcal{A}^Z$. Indeed, by applying the exact functor $j^!$ to (2.7) we get that $j^!K = 0$ and so $i_!i^!K \simeq K \simeq i_*i^*K$. Likewise by applying j^* to (2.8) we get that $K' \in \mathcal{A}^Z$.

Given a recollement of abelian or triangulated categories with objects and morphisms as in (2.1), the opposite categories form the following recollement

$$\begin{tikzcd}[column sep=huge]
\mathcal{A}_Z^{op} \arrow[r, "i_*=i_!"] & \mathcal{A}^{op} \arrow[l, bend right, "i^!"'] \arrow[l, bend left, "i^*"] \arrow[r, "j^!=j^*"] & \mathcal{A}_U^{op} \arrow[l, bend right, "j_*"'] \arrow[l, bend left, "j_!"]
\end{tikzcd}$$

which we call the *opposite recollement*.

The following proposition describes a useful way to characterise the functors i^* and $i^!$ in any recollement.

Proposition 2.1.3. *Suppose we have a recollement of abelian categories with objects and morphisms as in (2.1). Then for any object $X \in \mathcal{A}$:*

(i) i_*i^*X is the largest quotient object of X in $\mathcal{A}^Z$.

(ii) $i_!i^!X$ is the largest subobject of X in $\mathcal{A}^Z$.

Proof. By the adjunction (i_*, i^*) and since i_* is fully-faithful we have natural isomorphisms for $X \in \mathcal{A}$, $Y \in \mathcal{A}_Z$:

$$\mathrm{Hom}_{\mathcal{A}}(i_*i^*X, i_*Y) \simeq \mathrm{Hom}_{\mathcal{A}_Z}(i^*X, Y) \simeq \mathrm{Hom}_{\mathcal{A}}(X, i_*Y)$$

sending f to $f \circ \eta$ where $\eta : X \to i_* i^* X$ is the adjunction unit. In particular any morphism $X \to i_* Y$ factors through $i_* i^* X$. Statement (i) follows. Statement (ii) follows by taking the opposite recollement. $\square$

Definition 2.1.4. A *stratification* of an abelian/triangulated category $\mathcal{A}$ by a non-empty finite poset Λ consists of the following data:

(i) An assignment of an abelian/triangulated category $\mathcal{A}_Z$ to every closed-downwards subset $Z \subset \Lambda$. Moreover, for each pair of closed-downwards subsets $Z_1 \subset Z_2 \subset \Lambda$, there is a corresponding embedding $i_{Z_1,Z_2*} : \mathcal{A}_{Z_1} \hookrightarrow \mathcal{A}_{Z_2}$.

(ii) For each $\lambda \in \Lambda$ an abelian/triangulated category $\mathcal{A}_\lambda$. We call these *strata categories.*

This data must satisfy the following conditions

(S1) $\mathcal{A}_\emptyset = 0$ and $\mathcal{A}_\Lambda = \mathcal{A}$.

(S2) For each $\lambda \in \Lambda$ and closed-downwards subset $Z \subset \Lambda$ in which $\lambda \in Z$ is maximal, the functor $i_* = i_{Z\setminus\{\lambda\},Z*}$ fits into a recollement

$$\begin{tikzcd}[column sep=large]
\mathcal{A}_{Z\setminus\{\lambda\}} \arrow[r, "i_*=i_!" description] & \mathcal{A}_Z \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^!=j^*" description] & \mathcal{A}_\lambda \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"]
\end{tikzcd}$$

Say that a stratification of an abelian category is *k-homological* (respectively *homological*) if each of the recollements described in Condition (S2) are *k-homological* (respectively *homological*).

We proceed with some important examples of recollements and stratifications. We will often use without mentioning a result of Beilinson-Bernstein-Deligne [BBD82, Proposition 1.4.16] that says that given a recollement of triangulated categories with a t-structure one obtains a recollement on the hearts of the t-structure by applying zero-th cohomology. A proof of this result is given in Section 2.6.

Example 2.1.5 (Direct sum category)**.** The simplest example of a recollement is the direct sum of abelian categories:

$$\begin{tikzcd}[column sep=large]
\mathcal{A} \arrow[r, "i_*=i_!"] & \mathcal{A}\oplus\mathcal{B} \arrow[l, shift right=2ex, "i^*"'] \arrow[l, shift left=2ex, "i^!"] \arrow[r, "j^!=j^*"] & \mathcal{B} \arrow[l, shift right=2ex, "j_!"'] \arrow[l, shift left=2ex, "j_*"]
\end{tikzcd}$$

where $i_* = i_!$ and $j_! = j_*$ are the inclusion functors, and $i^* = i^!$ and $j^! = j^*$ are projection functors.

Example 2.1.6 (Grothendieck's derived functors)**.** Let X be a variety. Let $i : Z \hookrightarrow X$ be the inclusion of a closed subvariety and let $j : U \hookrightarrow X$ be the inclusion of the open complement. Write $\mathcal{D}^b(X, \Bbbk)$ for the bounded derived category of sheaves on X with coefficients in a Noetherian ring $\Bbbk$ of finite global dimension. The direct image $i_* : \mathcal{D}^b(Z, \Bbbk) \hookrightarrow \mathcal{D}^b(X, \Bbbk)$ fits into a recollement of triangulated categories

$$\begin{tikzcd}[column sep=large]
\mathcal{D}^b(Z,\Bbbk) \arrow[r, "i_*=i_!"] & \mathcal{D}^b(X,\Bbbk) \arrow[l, shift right=2ex, "i^*"'] \arrow[l, shift left=2ex, "i^!"] \arrow[r, "j^!=j^*"] & \mathcal{D}^b(U,\Bbbk) \arrow[l, shift right=2ex, "j_!"'] \arrow[l, shift left=2ex, "j_*"]
\end{tikzcd}$$

Indeed this is the original and motivating example of a recollement of triangulated categories. After taking zero-th cohomology we get a recollement of abelian categories of sheaves:

$$\begin{tikzcd}[column sep=large]
\mathcal{S}h(Z,\Bbbk) \arrow[r, "i_*=i_!"] & \mathcal{S}h(X,\Bbbk) \arrow[l, shift right=2ex, "\mathcal{H}^0 i^*"'] \arrow[l, shift left=2ex, "\mathcal{H}^0 i^!"] \arrow[r, "j^!=j^*"] & \mathcal{S}h(U,\Bbbk) \arrow[l, shift right=2ex, "\mathcal{H}^0 j_!"'] \arrow[l, shift left=2ex, "\mathcal{H}^0 j_*"]
\end{tikzcd}$$

Likewise if we take zero-th perverse cohomology for any perversity function, we get a recollement of the abelian categories of perverse sheaves.

Example 2.1.7 (Constructible sheaves with respect to a stratification)**.** A *stratification* of a quasiprojective complex variety X is a finite collection, $\{X_\lambda\}_{\lambda\in\Lambda}$ of disjoint, smooth, connected, locally closed subvarieties, called *strata*, in which $X = \bigcup_{\lambda\in\Lambda} X_\lambda$ and for each $\lambda \in \Lambda$, $\overline{X_\lambda}$ is a union of strata. In this case we equip Λ with the partial order

$$\mu \leq \lambda \text{ if } X_\mu \subset \overline{X_\lambda}.$$

We will use Λ to refer to the stratification of X.

For a variety X, let $\mathrm{Loc}^{ft}(X, \Bbbk)$ be the category of local systems on X of finite type with coefficients in a field $\Bbbk$. Recall that, by taking monodromy, $\mathrm{Loc}^{ft}(X, \Bbbk)$ is equivalent to the category, $\Bbbk[\pi_1(X_\lambda)]\text{-mod}_{fg}$, of finitely generated $\Bbbk[\pi_1(X_\lambda)]$-modules (see e.g. [Ach21, Theorem 1.7.9]).

Say that a sheaf $\mathcal{F}$ on X is *constructible* with respect to a stratification, Λ, of X if $\mathcal{F}|_{X_\lambda}$ is a local system of finite type for each $\lambda \in \Lambda$. Write $\mathcal{D}^b_\Lambda(X, \Bbbk)$ for the full triangulated subcategory of $\mathcal{D}^b(X, \Bbbk)$ consisting of objects $\mathcal{F}$ in which $H^k(\mathcal{F})$ is constructible with respect to Λ.

Say that a stratification, Λ, of X is *good* if for any $\lambda \in \Lambda$ and any object $\mathcal{L} \in \mathrm{Loc}^{ft}(X_\lambda, \Bbbk)$, we have $j_{\lambda *}\mathcal{L} \in \mathcal{D}^b_\Lambda(X, \Bbbk)$, where $j_\lambda : X_\lambda \hookrightarrow X$ is the embedding, and $j_{\lambda *}$ is the derived pushforward. It is difficult to tell in general whether a stratification is good (see [Ach21, Remark 2.3.21] for a discussion of these difficulties). A stratification satisfying the *Whitney regularity conditions* [Wit65] is good. In particular, if an algebraic group G acts on X with finitely many orbits (each connected), then the stratification of X by G-orbits is good (see e.g. [Ach21, Exercise 6.5.2]).

Let $cl(\Lambda)$ be the set of closed-downwards subsets of a poset Λ. Given a good stratification Λ on X, the triangulated category $\mathcal{D}^b_\Lambda(X, \Bbbk)$ has a stratification by Λ with strata categories $\mathcal{D}_\lambda := \mathcal{D}^b(\mathrm{Loc}^{ft}(X_\lambda, \Bbbk)) \simeq \mathcal{D}^b(\Bbbk[\pi_1(X_\lambda)]\text{-mod}_{fg})$ and Serre subcategories $\mathcal{D}_{\Lambda'} := \mathcal{D}^b_{\Lambda'}(\bigcup_{\lambda \in \Lambda'} X_\lambda)$ for each closed-downwards subposet $\Lambda' \subset \Lambda$.

For a perversity function $p : \Lambda \to \mathbb{Z}$, the category ${}^pP_\Lambda(X, \Bbbk)$ of perverse sheaves on X with respect to the stratification Λ (and perversity function p) is the full subcategory of $\mathcal{D}^b_\Lambda(X, \Bbbk)$ consisting of complexes $\mathcal{F}$ in which for any strata $h_\lambda : X_\lambda \hookrightarrow X$:

(i) $\mathcal{H}^d(h_\lambda^* \mathcal{F}) = 0$ if $d > p(\lambda)$,

(ii) $\mathcal{H}^d(h_\lambda^! \mathcal{F}) = 0$ if $d < p(\lambda)$,

where $\mathcal{H}^d(\mathcal{F})$ refers to the d-th cohomology sheaf of $\mathcal{F}$. The category $\mathcal{A} = {}^pP_\Lambda(X, \Bbbk)$ is abelian and has a stratification by Λ, with strata categories $\mathcal{A}_\lambda = \mathrm{Loc}^{ft}(X_\lambda, \Bbbk)[p(\lambda)] \simeq \Bbbk[\pi_1(X_\lambda)]\text{-mod}_{fg}$ (see e.g. Theorem 2.6.3).

Example 2.1.8 (G-equivariant perverse sheaves)**.** Another example of a stratification arises in the theory of equivariant perverse sheaves as defined in [BL94]. We will briefly review this theory. We recommend the reader consult [Ach21, Chapter 6] for more details.

For a complex algebraic group G and quasiprojective complex G-variety X, a *G-equivariant perverse sheaf* on X is, roughly speaking, a perverse sheaf on X with a G-action compatible with the G-action on X (see e.g. [Ach21, Definition 6.2.3] for a precise definition). The category, $P_G(X, \Bbbk)$ of G-equivariant perverse sheaves is the heart of a t-structure on the *G-equivariant derived category*, $\mathcal{D}_G(X, \Bbbk)$ defined by Bernstein-Lunts [BL94]. For a G-equivariant map of G-varieties $h : X \to Y$, there are equivariant versions of the (proper) pushforward and (proper) pullback functors: $h_*, h_!, h^!, h^*$. If $i : Z \hookrightarrow X$ is the inclusion of a G-invariant closed subvariety with open complement $j : U \hookrightarrow X$, then there is a recollement of triangulated categories

$$\begin{tikzcd} \mathcal{D}^b_G(Z,\Bbbk) \arrow[r, "i_*=i_!"] & \mathcal{D}^b_G(X,\Bbbk) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^!=j^*"] & \mathcal{D}^b_G(U,\Bbbk) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

If X is a homogeneous G-variety, then every G-equivariant perverse sheaf is a finite type local system (shifted by $\dim_{\mathbb{C}} X$). Moreover, in this case,

$$P_G(X, \Bbbk) \simeq \Bbbk[G^x/(G^x)^\circ]\text{-mod}_{fg}, \tag{2.9}$$

where $G^x \subset G$ is the stabilizer of a point $x \in X$, and $(G^x)^\circ$ is the connected component of G^x containing the identity element (see e.g. [Ach21, Proposition 6.2.13] for a proof of this statement).

Suppose G acts on X with finitely many orbits (each connected). Let Λ be a set indexing the set of G-orbits in X, and write $\mathcal{O}_\lambda$ for the orbit corresponding to $\lambda \in \Lambda$. Consider Λ as a poset with the closure order: $\lambda \leq \mu$ if $\mathcal{O}_\lambda \subset \overline{\mathcal{O}_\mu}$. Then the category $\mathcal{A} = P_G(X, \Bbbk)$ has a stratification with strata categories $\mathcal{A}_\lambda = P_G(\mathcal{O}_\lambda, \Bbbk) \simeq \Bbbk[G^x/(G^x)^\circ]\text{-mod}_{fg}$ (where $x \in \mathcal{O}_\lambda$).

Example 2.1.9 (Modules with idempotents)**.** For a ring A, let Mod-A be the category of all right A-modules, and mod-A be the category of finitely presented right A-modules. Let e be an idempotent in A, and define the inclusion functor $i_* : \text{Mod-}A/AeA \to \text{Mod-}A$. Note that Mod-$A/AeA$ is equivalent to the Serre subcategory of Mod-A consisting of modules annihilated by e. There is a corresponding Serre quotient $j^* : \text{Mod-}A \to \text{Mod-}eAe$ defined

$$j^* := \text{Hom}_A(eA, -) \simeq - \otimes_A Ae.$$

i.e. $j^*M = Me$ for any object $M \in \text{Mod-}A$. These functors fit into a recollement of abelian categories:

$$\begin{tikzcd} \text{Mod-}A/AeA \arrow[r, "i_*=i_!" description] & \text{Mod-}A \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^!=j^*" description] & \text{Mod-}eAe \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

where for any right A-module M:

(i) i^*M is the largest quotient, N, of M in which $Ne = 0$.

(ii) $i^!M$ is the largest subobject, N, of M in which $Ne = 0$.

Moreover $j_! := - \otimes_{eAe} eA$ and $j_* := \text{Hom}_{eAe}(Ae, -)$.

If A is right artinian and has enough injectives then the inclusion $i_* : \text{mod-}A/AeA \to \text{mod-}A$ fits into a recollement

$$\begin{tikzcd} \text{mod-}A/AeA \arrow[r, "i_*=i_!" description] & \text{mod-}A \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^!=j^*" description] & \text{mod-}eAe \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

in which j^* has left adjoint $j_! = - \otimes_{eAe} eA$ (see e.g [Kra17, Lemma 2.5]). This recollement is homological if and only if $A/AeA \otimes^{L}_{A} A/AeA = A/AeA$ [GL91, Theorem 4.4]. This holds in particular if AeA is a projective A-module (see e.g. [Kra17, Lemma 2.7]).

2.2 The intermediate-extension functor

Consider again the recollement:

$$\begin{tikzcd} \mathcal{A}_Z \arrow[r, "i_*=i_!" description] & \mathcal{A} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^!=j^*" description] & \mathcal{A}_U \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd} \tag{2.10}$$

In this section we study the full subcategory, $\mathcal{A}^U \hookrightarrow \mathcal{A}$, whose objects have no subobjects or quotients in $\mathcal{A}^Z := \operatorname{im} i_*$. The main result of this section (Proposition 2.2.3(ii)) is that the restricted functor $j^* : \mathcal{A}^U \to \mathcal{A}_U$ is an equivalence of categories. The quasi-inverse $j_{!*} : \mathcal{A}_U \to \mathcal{A}^U$ is defined as follows.

Definition 2.2.1 ($j_{!*} : \mathcal{A}_U \to \mathcal{A}^U$)**.** For an object $X \in \mathcal{A}_U$, let $\overline{1_X} : j_!X \to j_*X$ be the morphism corresponding to the identity on X under the isomorphism

$$\operatorname{Hom}_{\mathcal{A}}(j_!X, j_*X) \simeq \operatorname{Hom}_{\mathcal{A}_U}(X, j^*j_*X) \simeq \operatorname{Hom}_{\mathcal{A}_U}(X, X).$$

Define

$$j_{!*}X := \operatorname{im}(\overline{1_X} : j_!X \to j_*X) \in \mathcal{A}.$$

It is easy to check that if $X \in \mathcal{A}_U$ then $j_{!*}X \in \mathcal{A}^U$. Indeed as $i^!j_*X = 0$, j_*X has no subobjects in $\mathcal{A}^Z$. In particular, as $j_{!*}X$ is a subobject of j_*X it cannot have any subobjects in $\mathcal{A}^Z$. Likewise as $j_{!*}X$ is a quotient of $j_!X$ it cannot have any quotients in $\mathcal{A}^Z$. We call the functor $j_{!*} : \mathcal{A}_U \to \mathcal{A}$, the *intermediate-extension functor*.

Remark 2.2.2. Not every subquotient of an object in $\mathcal{A}^U$ need be in $\mathcal{A}^U$. In particular, an object in $\mathcal{A}^U$ may still have simple composition factors in $\mathcal{A}^Z$.

Proposition 2.2.3. *Suppose we have a recollement of abelian categories with objects and morphisms as in (2.10). Then*

(i) If $X \in \mathcal{A}$ has no nonzero quotient objects in $\mathcal{A}^Z$, and $Y \in \mathcal{A}$ has no nonzero subobjects in $\mathcal{A}^Z$ (i.e. $i^*X = 0$ and $i^!Y = 0$), then:

$$\operatorname{Hom}_{\mathcal{A}}(X, Y) \simeq \operatorname{Hom}_{\mathcal{A}_U}(j^*X, j^*Y).$$

(ii) $j^* : \mathcal{A}^U \to \mathcal{A}_U$ is an equivalence of categories with quasi-inverse $j_{!*} : \mathcal{A}_U \to \mathcal{A}^U$.

Proof. If $i^*X = 0$ then (2.7) gives an exact sequence

$$0 \to K \to j_!j^!X \to X \to 0$$

in which $K \simeq i_! i^! K$. So applying $\mathrm{Hom}(-, Y)$ we get the exact sequence

$$0 \to \mathrm{Hom}(X, Y) \to \mathrm{Hom}(j_! j^! X, Y) \to \mathrm{Hom}(i_! i^! K, Y)$$

Applying adjunctions gives the exact sequence

$$0 \to \mathrm{Hom}(X, Y) \to \mathrm{Hom}(j^! X, j^! Y) \to \mathrm{Hom}(i^! K, i^! Y)$$

Statement (i) follows as $i^! Y = 0$.

A corollary of statement (i) is that $j^* : \mathcal{A}^U \to \mathcal{A}_U$ is fully-faithful. To show that j^* is essentially surjective it suffices to show that for any object $X \in \mathcal{A}_U$, $j^* j_{!*} X \simeq X$. Now, as j^* is exact:

$$j^* j_{!*} X = j^* \operatorname{im}(j_! X \to j_* X) \simeq \operatorname{im}(j^* j_! X \to j^* j_* X) \simeq \operatorname{im}(\mathrm{Id} : X \to X) = X$$

and so (ii) follows. $\square$

The following result follows immediately from the previous proposition.

Proposition 2.2.4. *Suppose we have a recollement of abelian categories as in (2.10). If $L \in \mathcal{A}_U$ is a simple object, then $j_{!*} L$ is a simple object in $\mathcal{A}$. Moreover all the simple objects of $\mathcal{A}$ are either of the forms:*

(i) $i_* L$ for a simple object $L \in \mathcal{A}_Z$.

(ii) $j_{!*} L$ for a simple object $L \in \mathcal{A}_U$.

The following properties of the intermediate-extension functor will be useful.

Proposition 2.2.5. *Suppose we have a recollement of abelian categories as in (2.10). Then*

(i) The functor $j_{!*} : \mathcal{A}_U \to \mathcal{A}$ maps injective morphisms to injective morphisms and surjective morphisms to surjective morphisms.

(ii) If $X \in \mathcal{A}$ has no nonzero quotient objects in $\mathcal{A}^Z$ then there is a canonical short exact sequence:

$$0 \to i_! i^! X \to X \to j_{!*} j^* X \to 0$$

(iii) If $X \in \mathcal{A}$ has no nonzero subobjects in $\mathcal{A}^Z$ then there is a canonical short exact sequence:

$$0 \to j_{!*}j^*X \to X \to i_*i^*X \to 0$$

Proof. Let $f : X \to Y$ be a map in $\mathcal{A}_U$ and define objects K_1, K_2 in $\mathcal{A}$ by the exact sequence:

$$0 \to K_1 \to j_{!*}X \to j_{!*}Y \to K_2 \to 0$$

To prove statement (i) it suffices to show that if $j^*K_i = 0$ then $K_i = 0$. If $j^*K_i = 0$ then by (R4'), $K_1 \simeq i_*i^*K_1$ and $K_2 \simeq i_!i^!K_2$. Then each $K_i = 0$ since $j_{!*}X$ and $j_{!*}Y$ are in $\mathcal{A}^U$

To prove statement (ii), let $X \in \mathcal{A}$ have no nonzero quotients in $\mathcal{A}^Z$ and consider the short exact sequence

$$0 \to i_!i^!X \to X \to K \to 0$$

Applying $i^!$ to the sequence we see that $i^!K = 0$ and so $K \in \mathcal{A}^U$. So $K \simeq j_{!*}j^*K$ and (by applying j^* to this sequence) $j^*X \simeq K$. Statement (ii) follows immediately. The proof of statement (iii) is similar. □

Say that an abelian category is a *length category* if every object has a finite filtration by simple objects.

Proposition 2.2.6. *Suppose we have a recollement of abelian categories as in (2.10). Then $\mathcal{A}$ is a length category if and only if both $\mathcal{A}_Z$ and $\mathcal{A}_U$ are length categories. In particular if $\mathcal{A}$ has a stratification by a finite poset Λ, then $\mathcal{A}$ is a length category if and only if all the strata categories are length categories.*

Proof. Let X be an object in $\mathcal{A}$ and let K be defined by the short exact sequence:

$$0 \to i_!i^!X \to X \to K \to 0$$

Then $i^!K = 0$ and so applying Proposition 2.2.5(iii) we get the short exact sequence

$$0 \to j_{!*}j^*K \to K \to i_*i^*K \to 0$$

In particular if every object in $\mathcal{A}_Z$ and every object in $\mathcal{A}_U$ has a finite filtration by simple objects, then so does K and hence so does X. The converse statement is obvious.

The last statement in the proposition follows by Noetherian induction. $\square$

2.3 Recollements with enough projectives/injectives

In this section we study the relationship between projective covers of objects in the different categories making up a recollement. More precisely, let $\mathcal{A}$ be a category fitting into a recollement as in (2.10). Proposition 2.3.5 says that if $\mathcal{A}$ has enough projectives/injectives then so does $\mathcal{A}_U$. Proposition 2.3.6 says that if $\mathcal{A}$ is a Krull-Schmidt category then if $\mathcal{A}$ has enough projectives/injectives then so does $\mathcal{A}_Z$. Proposition 2.3.7 says that if $X \in \mathcal{A}_U$ has a projective cover P in $\mathcal{A}_U$ then $j_! P$ is a projective cover in $\mathcal{A}$ of $j_{!*}X$.

Unfortunately it is not easy to find a projective cover in $\mathcal{A}$ of an object $i_* X \in \mathcal{A}^Z$, even if a projective cover of X exists in $\mathcal{A}_Z$. Theorem 2.3.9 gives sufficient conditions for when such a projective cover exists.

2.3.1 Projective covers

Recall that a surjection $\phi : X \to Y$ is *essential* if for any morphism $\alpha : X' \to X$, if $\phi \circ \alpha$ is surjective then α is surjective. Equivalently $\phi : X \to Y$ is essential if for any subobject $U \subset X$, if $U + \ker \phi = X$ then $U = X$. If $P \to X$ is an essential surjection and P is projective then we call P (or more accurately the morphism $P \to X$) a *projective cover* of X. The projective cover of X (if it exists) factors through every other essential cover of X, and is unique up to isomorphism.

If $L \in \mathcal{A}$ is a simple object and P is projective then $\phi : P \to L$ is a projective cover if and only if the following equivalent conditions hold:

(i) $\ker \phi$ is the unique maximal subobject of P.

(ii) The endomorphism ring of P is local.

See e.g. [Kra15, Lemma 3.6] for a proof of these facts.

The dual concept of an essential surjection is called an *essential extension*. If $X \to I$ is an essential extension and I is injective then this extension is called the *injective envelope* of X.

An abelian category has *enough projectives* (resp. *enough injectives*) if every object has a projective cover (resp. injective envelope).

An abelian category $\mathcal{A}$ is a *Krull-Schmidt category* if every object is a finite direct sum of objects with local endomorphism rings. For example, any abelian length category is a Krull-Schmidt category.

In a Krull-Schmidt category, the projective covers of simple objects are exactly the projective indecomposable objects. Moreover a Krull-Schmidt category $\mathcal{A}$ has enough projectives if and only if every simple object has a projective cover. We will need the following characterisation of projective covers of simple objects in Krull-Schmidt categories.

Proposition 2.3.1. *Let $\mathcal{A}$ be a Krull-Schmidt category. Let $P \in \mathcal{A}$ be a projective object and $L \in \mathcal{A}$ be a simple object. A map $P \to L$ is a projective cover if and only if for any simple object L':*

$$\dim_{\mathrm{End}_{\mathcal{A}}(L')} \mathrm{Hom}_{\mathcal{A}}(P, L') = \begin{cases} 1 & \text{if } L = L', \\ 0 & \text{otherwise.} \end{cases} \tag{2.11}$$

Remark 2.3.2. Recall that any module of a division ring is free. For a module M, of a division ring D, the *dimension* $\dim_D M$ is the rank of M as a free D-module.

Proof of Proposition 2.3.1. Suppose we have a projective cover $\phi : P \to L$. Since $\ker \phi$ is the unique maximal subobject of ϕ, $\mathrm{Hom}_{\mathcal{A}}(P, L') = 0$ whenever $L \neq L'$. To show equation (2.11), it remains to show that the $\mathrm{End}_{\mathcal{A}}(L)$-equivariant map

$$- \circ \phi : \mathrm{End}_{\mathcal{A}}(L) \to \mathrm{Hom}_{\mathcal{A}}(P, L)$$

is an isomorphism. Since ϕ is a surjection this map is injective. To show surjectivity, let $f \in \mathrm{Hom}_{\mathcal{A}}(P, L)$ be nonzero. Then as $\ker f$ is a maximal subobject of P, $\ker \phi \subset \ker f$, and so f factors through ϕ.

Conversely, if (2.11) holds, then if $P = P_1 \oplus P_2$, only one P_i can have a simple quotient and the other must be zero. In particular, P is indecomposable. $\square$

2.3.2 Ext-finiteness

To state the main result of this section (Theorem 2.3.9) we need the concept of Ext-*finiteness*. In this section we recall this definition and give two propositions about Ext-finiteness (Propositions 2.3.3 and 2.3.4) that will be needed in the discussion following Theorem 2.3.9.

For $k \in \mathbb{N}$, say that an abelian category $\mathcal{A}$ is Ext^k-*finite* (or Hom-*finite* in the case $k = 0$) if for any simple objects A, B in $\mathcal{A}$,

$$\dim_{\mathrm{End}_{\mathcal{A}}(B)} \mathrm{Ext}^k_{\mathcal{A}}(A, B) < \infty.$$

Note that if $\mathcal{A}$ is a $\Bbbk$-linear category, for some field $\Bbbk$, then

$$\dim_{\mathrm{End}_{\mathcal{A}}(B)} \mathrm{Ext}^k_{\mathcal{A}}(A, B) = \frac{\dim_{\Bbbk} \mathrm{Ext}^k_{\mathcal{A}}(A, B)}{\dim_{\Bbbk} \mathrm{End}_{\mathcal{A}}(B)}.$$

So $\mathcal{A}$ is Ext^k-finite whenever $\dim_{\Bbbk} \mathrm{Ext}^k_{\mathcal{A}}(A, B) < \infty$ for every simple object A, B. The converse is true if the endomorphism ring of every simple object has finite $\Bbbk$-dimension (e.g. if $\Bbbk$ is algebraically closed).

The following two propositions give useful criteria for when a category has finite Ext^k-spaces.

Proposition 2.3.3. *Any* Hom-*finite abelian category with enough projectives is* Ext^k*-finite for every* $k \in \mathbb{N}$.

Proof. Let X, Y be simple objects in $\mathcal{A}$. Consider a projective presentation

$$0 \to K \to P \to X \to 0$$

Then as $\mathrm{Ext}^k_{\mathcal{A}}(P, Y) = 0$, we get that $\mathrm{Ext}^k_{\mathcal{A}}(X, Y)$ embeds into (or is isomorphic to) $\mathrm{Ext}^{k-1}_{\mathcal{A}}(K, Y)$ for each $k > 0$. The result follows by induction. □

Say that a $\Bbbk$-linear abelian category is *finite over* $\Bbbk$ if $\mathcal{A}$ is a Hom-finite length category with enough projectives and finitely many simple objects. It is well-known that $\mathcal{A}$ is finite over $\Bbbk$ if and only if there is a finite-dimensional $\Bbbk$-algebra A in which $\mathcal{A}$ is equivalent to the category, A-mod, of modules that are finite dimensional as $\Bbbk$-vector spaces. Indeed, if $\{P_\lambda\}_{\lambda \in \Lambda}$ are the projective indecomposables in $\mathcal{A}$ (up to isomorphism), then $A = \mathrm{End}_{\mathcal{A}}(\bigoplus_{\lambda\in\Lambda} P_\lambda)^{op}$ and $\mathrm{Hom}_{\mathcal{A}}(\bigoplus_{\lambda\in\Lambda} P_\lambda, -) : \mathcal{A} \simeq A\text{-mod}$. Note that there is a contravariant equivalence $\mathrm{Hom}_{\Bbbk}(-, \Bbbk) : A\text{-mod} \to A^{op}\text{-mod}$. In particular, any finite abelian category has enough injectives.

Proposition 2.3.4. *Let $\Bbbk$ be a field and let $\mathcal{A}$ be a $\Bbbk$-linear abelian category with a stratification in which every strata category is a finite abelian category. Then $\mathcal{A}$ is* Ext^1*-finite.*

Proof. By the assumptions on the strata categories, $\mathcal{A}$ has finite dimensional Hom-spaces. Suppose $\mathcal{A}$ has a recollement with objects and morphisms as in (2.10). Suppose $\mathcal{A}_U$ and $\mathcal{A}_Z$ have enough projectives, and $\mathcal{A}_U$ has enough injectives. By Proposition 2.3.3, both $\mathcal{A}_Z$ and $\mathcal{A}_U$ are Ext^1-finite. We just need to show that $\mathcal{A}$ is Ext^1-finite. It suffices to show that $\dim_{\Bbbk} \mathrm{Ext}^1_{\mathcal{A}}(X, Y) < \infty$ for all simple objects X, Y.

Since $\mathcal{A}^Z$ is a Serre subcategory of $\mathcal{A}$, this is true whenever X and Y are both in $\mathcal{A}^Z$. Let $L \in \mathcal{A}_U$ be simple and let $j_{!*}L$ have projective and injective presentations:

$$0 \to K \to j_! P \to j_{!*}L \to 0$$
$$0 \to j_{!*}L \to j_* I \to K' \to 0$$

The projective presentation implies that $\mathrm{Hom}_{\mathcal{A}}(K, Y)$ surjects onto $\mathrm{Ext}^1_{\mathcal{A}}(j_{!*}L, Y)$. The injective presentation implies that $\mathrm{Hom}_{\mathcal{A}}(Y, K')$ surjects onto $\mathrm{Ext}^1_{\mathcal{A}}(Y, j_{!*}L)$. The result follows. $\square$

2.3.3 Main results

This section contains our original results concerning recollements and projective covers.

Proposition 2.3.5. *Suppose $\mathcal{A}$ is an abelian category with a recollement as in (2.10). Then*

(i) If $X \to Y$ is an essential surjection in $\mathcal{A}$ and $i^*Y = 0$ then $i^*X = 0$ and $j^!X \to j^!Y$ is an essential surjection.

(ii) If $P \in \mathcal{A}$ is projective and $i^*P = 0$ then $j^!P \in \mathcal{A}_U$ is projective. In particular if $P \to X$ is a projective cover in $\mathcal{A}$ and $i^*X = 0$ then $j^!P \to j^!X$ is a projective cover in $\mathcal{A}_U$.

In particular, if $\mathcal{A}$ has enough projectives then so does $\mathcal{A}_U$.

Proof. Let $\phi : X \to Y$ be an essential surjection in $\mathcal{A}$ and suppose $i^*Y = 0$. To show that $i^*X = 0$ it suffices to show that the canonical map $\epsilon_X : j_!j^!X \to X$ is surjective. This follows from the following commutative diagram since ϕ is essential.

$$\begin{tikzcd}
X \arrow[r, two heads, "\phi"] & Y \\
j_!j^!X \arrow[u, "\epsilon_X"] \arrow[r, two heads, "j_!j^!\phi"'] & j_!j^!Y \arrow[u, two heads, "\epsilon_Y"']
\end{tikzcd}$$

Let $\alpha : X' \to j^!X$ be a morphism in $\mathcal{A}_U$, in which $j^!(\phi) \circ \alpha : X' \to j^!Y$ is surjective. Then $\epsilon_Y \circ j_!j^!(\phi) \circ j_!\alpha : j_!X' \to Y$ is surjective and so (since ϕ is essential) $\epsilon_X \circ j_!\alpha : j_!X' \to X$ is surjective. Hence $j^!(\epsilon_X \circ j_!\alpha) \simeq \alpha : X' \to j^!X$ is surjective. This proves (i).

If $P \in \mathcal{A}$ is projective and $i^*P = 0$ then the functor

$$\mathrm{Hom}_{\mathcal{A}_U}(j^!P, -) \simeq \mathrm{Hom}_{\mathcal{A}}(j_!j^!P, j_!(-)) \simeq \mathrm{Hom}_{\mathcal{A}}(P, j_!(-)) : \mathcal{A}_U \to \mathbb{Z}\text{-mod}$$

is exact. Here the last isomorphism follows from the sequence (2.7). It follows that $j^!P$ is projective. Statement (ii) follows.

□

Proposition 2.3.6. *Suppose $\mathcal{A}$ is a Krull-Schmidt category with a recollement of abelian categories as in (2.10). If $P \to L$ is a projective cover in $\mathcal{A}$ of a simple object $L \in \mathcal{A}^Z$, then $i^*P \to i^*L$ is a projective cover in $\mathcal{A}_Z$. In particular, if $\mathcal{A}$ has enough projectives then so does $\mathcal{A}_Z$.*

Proof. Since i^* is the left adjoint of an exact functor it preserves projective objects. For any simple object $L' \in \mathcal{A}^Z$, $\mathrm{Hom}_{\mathcal{A}_Z}(i^*P, i^*L') = \mathrm{Hom}_{\mathcal{A}}(P, i_*i^*L') = \mathrm{Hom}_{\mathcal{A}}(P, L')$. The result follows. □

Proposition 2.3.7. *Suppose $\mathcal{A}$ is an abelian category with a recollement as in (2.10). Let X and Y be objects in $\mathcal{A}_U$. If $X \to Y$ is an essential surjection in $\mathcal{A}_U$ then the composition $j_!X \to j_{!*}X \to j_{!*}Y$ is an essential surjection in $\mathcal{A}$. In particular:*

(i) The canonical surjection $j_!X \to j_{!*}X$ is essential.

(ii) If $P \to X$ is a projective cover of X in $\mathcal{A}_U$ then $j_!P \to j_!X \to j_{!*}X$ is a projective cover of $j_{!*}X$ in $\mathcal{A}$.

Proof. Let $\phi : X \to Y$ be an essential surjection in $\mathcal{A}_U$. The map $\phi' : j_!X \to j_{!*}X \to j_{!*}Y$ is surjective by Proposition 2.2.5(i). Let $\alpha : X' \to X$ be a morphism in which $\phi' \circ \alpha$ is surjective. Now, $j^!(\phi') = \phi : X \to Y$ and since $j^!$ is exact, $j^!(\phi' \circ \alpha) = \phi \circ j^!(\alpha) : j^!X' \to Y$ is surjective. Since ϕ is essential it follows that $j^!(\alpha) : j^!X' \to X$ is surjective in $\mathcal{A}_U$ and so $j_!j^!(\alpha) : j_!j^!X' \to j_!X$ is surjective in $\mathcal{A}$. The surjectivity of α follows from the commutative triangle

$$\begin{tikzcd} j_!j^!X' \arrow[r, two heads, "j_!j^!(\alpha)"] \arrow[d] & j_!X \\ X' \arrow[ur, "\alpha"'] & \end{tikzcd}$$

in which the downward arrow is the adjunction counit. $\square$

The following result holds by an almost identical argument.

Proposition 2.3.8. *The intermediate-extension functor preserves essential surjections and essential extensions.*

The following is the main result of this section.

Theorem 2.3.9. *Let $\mathcal{A}$ be an abelian length category with finitely many simple objects, and a recollement of abelian categories as in (2.10). If $\mathcal{A}$ is Ext^1-finite then $\mathcal{A}$ has enough projectives if and only if both $\mathcal{A}_U$ and $\mathcal{A}_Z$ have enough projectives. Dually if $\mathcal{A}^{op}$ is Ext^1-finite then $\mathcal{A}$ has enough injectives if and only if both $\mathcal{A}_U$ and $\mathcal{A}_Z$ have enough injectives.*

Before giving the proof of this theorem we will explain one important ingredient: the *universal extension*. Let A, B be objects in an abelian category $\mathcal{A}$ in which $\mathrm{End}_{\mathcal{A}}(B)$ is a division ring and $d := \dim_{\mathrm{End}_{\mathcal{A}}(B)} \mathrm{Ext}^1_{\mathcal{A}}(A, B) < \infty$. We form the *universal extension* $\mathcal{E} \in \mathrm{Ext}^1_{\mathcal{A}}(A, B^{\oplus d})$ by the following process. First let $E_1, \ldots, E_d \in \mathrm{Ext}^1_{\mathcal{A}}(A, B)$ be an $\mathrm{End}_{\mathcal{A}}(B)$-basis. The diagonal map $\Delta : A \to A^{\oplus d}$ induces a map $\mathrm{Ext}^1_{\mathcal{A}}(A^{\oplus d}, B^{\oplus d}) \to \mathrm{Ext}^1_{\mathcal{A}}(A, B^{\oplus d})$. Let $\mathcal{E}$ be the image of $E_1 \oplus \cdots \oplus E_d$ under this map. That is we have the commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & B^{\oplus d} \arrow[r] & \bigoplus_i E_i \arrow[r] & A^{\oplus d} \arrow[r] & 0 \\
0 \arrow[r] & B^{\oplus d} \arrow[r] \arrow[u, "\mathrm{Id}"] & \mathcal{E} \arrow[r] \arrow[u] \arrow[ur, phantom, "\llcorner", very near start] & A \arrow[r] \arrow[u, "\Delta"'] & 0
\end{tikzcd}$$

Note that the $\mathrm{End}_{\mathcal{A}}(B)$-equivariant map $\mathrm{Hom}_{\mathcal{A}}(B^{\oplus d}, B) \to \mathrm{Ext}^1_{\mathcal{A}}(A, B)$ induced by the short exact sequence

$$0 \to B^{\oplus d} \to \mathcal{E} \to A \to 0$$

is surjective (this is easy to check on the basis of $\mathrm{Ext}^1_{\mathcal{A}}(A, B)$).

When $B_1, \ldots, B_n$ are objects in $\mathcal{A}$ in which each ring $\mathrm{End}_{\mathcal{A}}(B_i)$ is a division ring and $d_i := \dim_{\mathrm{End}_{\mathcal{A}}(B_i)} \mathrm{Ext}^1_{\mathcal{A}}(A, B_i) < \infty$, we also talk about a *universal extension* $\mathcal{E} \in \mathrm{Ext}^1_{\mathcal{A}}(A, \bigoplus_i B_i^{\oplus d_i})$ constructed in the following way. Let $\mathcal{E}_i \in \mathrm{Ext}^1_{\mathcal{A}}(A, B_i^{\oplus d_i})$ be a universal extension (as defined in the previous paragraph) and define $\mathcal{E}$ to be the image of $\mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_n$ under the map $\mathrm{Ext}^1_{\mathcal{A}}(\bigoplus_i A, \bigoplus_i B^{\oplus d_i}) \to \mathrm{Ext}^1_{\mathcal{A}}(A, \bigoplus_i B^{\oplus d_i})$ induced by the diagonal map $\Delta : A \to A^{\oplus d}$. Then $\mathcal{E}$ has the property that the short exact sequence

$$0 \to \bigoplus_i B^{\oplus d_i} \to \mathcal{E} \to A \to 0$$

induces a surjection $\mathrm{Hom}_{\mathcal{A}}(\bigoplus_i B_i^{\oplus d_i}, B_j) \to \mathrm{Ext}^1_{\mathcal{A}}(A, B_j)$ for each $j = 1, \ldots, n$.

Dually, if $\mathrm{End}_{\mathcal{A}}(A)^{op}$ is a division ring and $\dim_{\mathrm{End}_{\mathcal{A}}(A)^{op}} \mathrm{Ext}^1(A, B) < \infty$ then one can form a universal extension $\mathcal{E}' \in \mathrm{Ext}^1_{\mathcal{A}}(A^{\otimes d}, B)$ using the codiagonal map $\delta : B^{\oplus d} \to B$ instead of the diagonal map.

Proof of Theorem 2.3.9. Suppose that $\mathcal{A}_U$ and $\mathcal{A}_Z$ have enough projectives.

Suppose $\mathcal{A}^Z$ has simple objects $L_1, \ldots, L_m$ with projective covers $\bar{P}_1, \ldots, \bar{P}_m$ in $\mathcal{A}^Z$. Suppose $\mathcal{A}^U$ has simple objects $L_{m+1}, \ldots, L_{m+n}$. By Proposition 2.3.7 every simple object in $\mathcal{A}^U$ has a projective cover in $\mathcal{A}$. It suffices to construct a projective cover in $\mathcal{A}$ of each simple object in $\mathcal{A}^Z$. This amounts to finding, for each $1 \leq t \leq m$, a projective object, P_t, whose unique simple quotient is L_t.

Fix $1 \leq t \leq m$.

Step 1. Define P_t. For simple object $L_{m+k} \in \mathcal{A}^U$, let P_{m+k} denote its projective cover in $\mathcal{A}$. Define Q to be a maximal length quotient of

$$P := \bigoplus_{k=1}^{n} P_{m+k}^{\oplus \dim_{\mathrm{End}_{\mathcal{A}}(L_{m+k})} \mathrm{Ext}^1_{\mathcal{A}}(\bar{P}_t, L_{m+k})}$$

in which there is an extension

$$0 \to Q \to \mathcal{E} \to \bar{P}_t \to 0 \tag{2.12}$$

inducing an isomorphism $\mathrm{Hom}_{\mathcal{A}}(Q, L) \simeq \mathrm{Ext}^1_{\mathcal{A}}(\bar{P}_t, L)$ for each $L \in \mathcal{A}^U$. That is $0 = \mathrm{Hom}_{\mathcal{A}}(\bar{P}_t, L) \simeq \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, L)$ and $\mathrm{Ext}^1_{\mathcal{A}}(\mathcal{E}, L)$ injects into $\mathrm{Ext}^1_{\mathcal{A}}(Q, L)$.

Let P_t be any choice of such $\mathcal{E}$.

Step 2. P_t is well-defined. To show that the maximal quotient Q exists, we just need to find one quotient of P with the required property. Then since $\mathcal{A}$ is a length category there exists a maximal length quotient with the required property. Since $\mathcal{A}$ has finite Ext^1-spaces, we can let

$$R = \bigoplus_{k=1}^{n} L_{m+k}^{\oplus \dim_{\mathrm{End}_{\mathcal{A}}(L_{m+k})} \mathrm{Ext}^1_{\mathcal{A}}(\bar{P}_t, L_{m+k})}$$

and form the universal extension

$$0 \to R \to \mathcal{E} \to \bar{P}_t \to 0$$

Since this is a universal extension it induces a surjection $\mathrm{Hom}_{\mathcal{A}}(R, L_{m+k}) \to \mathrm{Ext}^1_{\mathcal{A}}(\bar{P}_t, L_{m+k})$ for each $L_{m+k} \in \mathcal{A}^U$. This map is an isomorphism since

$$\dim_{\mathrm{End}_{\mathcal{A}}(L_{m+k})} \mathrm{Hom}_{\mathcal{A}}(R, L_{m+k}) = \dim_{\mathrm{End}_{\mathcal{A}}(L_{m+k})} \mathrm{Ext}^1_{\mathcal{A}}(\bar{P}_t, L_{m+k}).$$

Step 3. P_t has unique simple quotient L_t. By definition of P_t and by the (i^*, i_*)-adjunction, for any simple module $L \in \mathcal{A}$:

$$\mathrm{Hom}_{\mathcal{A}}(P_t, L) \simeq \mathrm{Hom}_{\mathcal{A}}(\overline{P}_t, L) \simeq \mathrm{Hom}_{\mathcal{A}}(\overline{P}_t, i_* i^* L)$$

and so the only simple quotient of P_t is L_t with multiplicity one.

Step 4. P_t is projective. We show that $\mathrm{Ext}^1_{\mathcal{A}}(P_t, L) = 0$ for each simple $L \in \mathcal{A}$. For any simple $L \in \mathcal{A}$ there is an exact sequence

$$0 \to \mathrm{Ext}^1_{\mathcal{A}}(P_t, L) \to \mathrm{Ext}^1_{\mathcal{A}}(Q, L) \to \mathrm{Ext}^2_{\mathcal{A}}(\bar{P}_t, L) \tag{2.13}$$

Indeed if $L \in \mathcal{A}^U$ then this holds because $\mathrm{Hom}_{\mathcal{A}}(Q, L) \simeq \mathrm{Ext}^1_{\mathcal{A}}(\bar{P}_t, L)$. If $L \in \mathcal{A}^Z$ then (2.13) holds since $\mathcal{A}^Z$ is a Serre subcategory of $\mathcal{A}$ and so $\mathrm{Ext}^1_{\mathcal{A}}(\bar{P}_t, L) = 0$.

It suffices to show that the third map in (2.13) is injective for any $L \in \mathcal{A}$. Suppose for contradiction that there is a nontrivial extension

$$0 \to L \to Q' \to Q \to 0 \tag{2.14}$$

in the kernel of this map. Then there is an object $\mathcal{E} \in \mathcal{A}$ fitting into the following diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & L \arrow[r] \arrow[d] & Q' \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L \arrow[r] \arrow[d] & \mathcal{E} \arrow[r] \arrow[d] & P_t \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \bar{P}_t \arrow[r] \arrow[d] & \bar{P}_t \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{2.15}$$

in which each row and column is exact. For each $L' \in \mathcal{A}^U$ the sequence (2.14) induces an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(Q, L') \to \mathrm{Hom}_{\mathcal{A}}(Q', L') \to \mathrm{Hom}_{\mathcal{A}}(L, L')$$

Of course, $\mathrm{Hom}_{\mathcal{A}}(L, L') = 0$ if $L \neq L'$. If $L = L'$ the third map must be zero. Indeed if $f : L \to L$ factors through the inclusion $\iota : L \to Q'$ via a map $g : Q' \to L$, then $f^{-1} \circ g : Q' \to L$ is a retraction of ι. This contradicts the assumption that (2.14) does not split. Hence, for any $L' \in \mathcal{A}^U$, the middle column of (2.15) induces an isomorphism

$$\mathrm{Hom}_{\mathcal{A}}(Q', L') \simeq \mathrm{Hom}_{\mathcal{A}}(Q, L') \simeq \mathrm{Ext}^1_{\mathcal{A}}(\bar{P}_t, L'). \tag{2.16}$$

Since P is projective the quotient $P \to Q$ fits into the diagram

$$\begin{tikzcd} & P \arrow[dl, "\varphi"'] \arrow[d, two heads] \\ Q' \arrow[r, two heads] & Q \end{tikzcd}$$

Now φ cannot be surjective, as, by (2.16), this would contradict the maximality of Q. Thus the image of φ is isomorphic to Q and so the sequence (2.14) splits. This is a contradiction. Hence P_t is projective. $\square$

Corollary 2.3.10. *Let $\mathcal{A}$ be an abelian category with a stratification in which every strata category is a length category, and has finitely many simple objects.*

If $\mathcal{A}$ is Ext^1*-finite (respectively $\mathcal{A}^{op}$ is* Ext^1*-finite) then $\mathcal{A}$ has enough projectives (respectively injectives) if and only if every strata category has enough projectives (respectively injectives).*

Proof. For $\lambda \in \Lambda$, let $j^\lambda_{!*} : \mathcal{A}_\lambda \to \mathcal{A}$ be the composition

$$\mathcal{A}_\lambda \xrightarrow{j_{!*}} \mathcal{A}_{\overline{\{\lambda\}}} \hookrightarrow \mathcal{A}$$

where $\overline{\{\lambda\}} = \{\mu \in \Lambda \mid \mu \leq \lambda\}$.

By Proposition 2.2.6, every category $\mathcal{A}_{\Lambda'}$ (for closed $\Lambda' \subset \Lambda$) satisfies the conditions of Theorem 2.3.9. So we can obtain a projective cover in $\mathcal{A}$ of any simple object $j^\lambda_{!*}L$ by repeatedly applying the construction in the proof of Theorem 2.3.9 to larger and larger Serre subcategories of $\mathcal{A}$. $\square$

The following corollary follows immediately from Proposition 2.3.4 and Corollary 2.3.10.

Corollary 2.3.11. *Let $\mathcal{A}$ be a $\Bbbk$-linear abelian category with a stratification. Then $\mathcal{A}$ is finite over $\Bbbk$ if and only if the same is true of every strata category.*

From this result we recover the following result of Cipriani-Woolf.

Corollary 2.3.12 (Corollary 5.2 of [CW22])**.** *Let X be a variety with a good stratification $X = \bigcup_{\lambda\in\Lambda} X_\lambda$, and $\Bbbk$ be a field. Then the category $P(X, \Bbbk)$ of perverse sheaves (for any perversity function) is finite over $\Bbbk$ if and only if each category $\Bbbk[\pi_1(X_\lambda)]\text{-mod}_{fg}$ is finite over $\Bbbk$.*

For example, if X has a stratification $X = \bigcup_{\lambda\in\Lambda} X_\lambda$ in which each X_λ has finite fundamental group, then the category $P(X, \Bbbk)$ is finite over $\Bbbk$.

Corollary 2.3.13. *Let G be an algebraic group and let X be a G-variety with finitely many orbits, each connected. Let $\Bbbk$ be a field. The category, $P_G(X, \Bbbk)$, of G-equivariant perverse sheaves is finite over $\Bbbk$ if and only if for each G-orbit $\mathcal{O}_\lambda$ and $x \in \mathcal{O}_\lambda$, the category $\Bbbk[G^x/(G^x)^\circ]\text{-mod}_{fg}$ is finite over $\Bbbk$.*

2.4 Standard and costandard objects

In this section we focus our attention on abelian length categories $\mathcal{A}$ with finitely many simples, enough projectives and injectives, and admitting a stratification by a poset Λ. For such a category, let B be a set indexing the simple objects up to isomorphism. Let $L(b)$ be the simple object corresponding to B. Let $P(b)$ and $I(b)$ be the projective cover and injective envelope of $L(b)$.

For each $\lambda \in \Lambda$, write $\mathcal{A}_{\leq\lambda} := \mathcal{A}_{\{\mu\in\Lambda \mid \mu\leq\lambda\}}$ and let $j_!^\lambda : \mathcal{A}_\lambda \to \mathcal{A}$ be the composition

$$\mathcal{A}_\lambda \xrightarrow{j_!} \mathcal{A}_{\leq\lambda} \hookrightarrow \mathcal{A}$$

Define $j_*^\lambda : \mathcal{A}_\lambda \to \mathcal{A}$ and $j_{!*}^\lambda : \mathcal{A}_\lambda \to \mathcal{A}$ likewise. Let $j^\lambda : \mathcal{A}_{\leq\lambda} \to \mathcal{A}_\lambda$ denote the Serre quotient functor.

By Proposition 2.2.4, for every simple object, $L(b) \in \mathcal{A}$, there is an $\lambda \in \Lambda$ in which $L(b) = j_{!*}^\lambda L_\lambda(b)$ for a simple object $L_\lambda(b) \in \mathcal{A}_\lambda$. Define the *stratification function* $\rho : B \to \Lambda$ that assigns to each $b \in B$ the $\lambda \in \Lambda$ in which $L(b) = j_{!*}^\lambda L_\lambda(b)$. Let $P_\lambda(b)$ and $I_\lambda(b)$ be the projective cover and injective envelope of the simple object $L_\lambda(b)$ in $\mathcal{A}_\lambda$.

Define the *standard* and *costandard* objects:

$$\Delta(b) := j_!^\lambda P_\lambda(b), \qquad \nabla(b) := j_*^\lambda I_\lambda(b),$$

where $\lambda = \rho(b)$. Note that since $j_!^\lambda$ and j_*^λ are fully-faithful, these objects have local endomorphism rings and are hence irreducible. Note also that if $\rho(b) > \rho(b')$ then

$$\mathrm{Hom}_{\mathcal{A}}(\Delta(b), \Delta(b')) = 0 = \mathrm{Hom}_{\mathcal{A}}(\nabla(b'), \nabla(b)) \tag{2.17}$$

Indeed the only simple quotient of $\Delta(b)$ is $L(b)$, and all simple subobjects, $L(b'')$, of $\Delta(b')$ satisfy $\rho(b') \geq \rho(b'')$. Likewise the only simple subobject of $\nabla(b)$ is $L(b)$, and all simple quotients, $L(b'')$, of $\nabla(b')$ satisfy $\rho(b') \geq \rho(b'')$.

The following original result follows from the proofs of Theorem 2.3.9 and Corollary 2.3.10.

Porism 2.4.1. *Suppose $\mathcal{A}$ is an abelian category with a stratification by a finite poset Λ, in which every strata category is a length category with finitely many simple objects. Let $\rho : B \to \Lambda$ be the stratification function for $\mathcal{A}$.*

If $\mathcal{A}$ has enough projectives then for each $\lambda \in \Lambda$ and $b \in \rho^{-1}(\lambda)$, the projective indecomposable object $P(b) \in \mathcal{A}$ fits into a short exact sequence

$$0 \to Q(b) \to P(b) \to \Delta(b) \to 0$$

in which $Q(b)$ has a filtration by quotients of $\Delta(b')$ satisfying $\rho(b') > \rho(b)$.

If $\mathcal{A}$ has enough injectives then for each $\lambda \in \Lambda$ and $b \in \rho^{-1}(\lambda)$, the injective indecomposable object $I(b) \in \mathcal{A}$ fits into a short exact sequence

$$0 \to \nabla(b) \to I(b) \to Q'(b) \to 0$$

in which $Q'(b)$ has a filtration by subobjects of $\nabla(b')$ satisfying $\rho(b') > \rho(b)$.

Proof. We just prove the first statement by induction on the size of Λ. The base case $\Lambda = \emptyset$ is vacuously true.

Consider the projective cover, $P(b)$, of simple object $L(b)$ in $\mathcal{A}$. If $\rho(b)$ is maximal then $P(b) \simeq \Delta(b)$ and the result holds. Suppose $\rho(b)$ is not maximal, and let $\mu \in \Lambda$ be a maximal element. Consider the recollement

$$\begin{tikzcd} \mathcal{A}_{<\mu} \arrow[r, "i_*=i_!"] & \mathcal{A} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^!=j^*"] & \mathcal{A}_\mu \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

Let $P_{<\mu}(b)$ and $\Delta_{<\mu}(b)$ be the projective indecomposable and standard object in $\mathcal{A}_{<\mu}$ corresponding to the simple object $i^*L(b) \in \mathcal{A}_{<\mu}$. By induction there is a short exact sequence

$$0 \to Q_{<\mu}(b) \to P_{<\mu}(b) \to \Delta_{<\mu}(b) \to 0$$

in which $Q_{<\mu}(b)$ has a filtration by quotients of standard objects $\Delta_{<\mu}(b')$ satisfying $\rho(b') > \rho(b)$. Since i_* is exact we get the following short exact sequence in $\mathcal{A}$:

$$0 \to i_*Q_{<\mu}(b) \to i_*P_{<\mu}(b) \to \Delta(b) \to 0 \tag{2.18}$$

and $i_*Q_{<\mu}(b)$ has a filtration by quotients of standard objects $\Delta(b')$ satisfying $\mu > \rho(b') > \rho(b)$. By applying the construction in step 1 of the proof of Theorem 2.3.9, $P(b)$ fits into the short exact sequence in $\mathcal{A}$:

$$0 \to Q_\mu(b) \to P(b) \to i_*P_{<\mu}(b) \to 0 \tag{2.19}$$

and $Q_\mu(b)$ is a quotient of a direct sum of standard objects of the form $\Delta(b')$ in which $\rho(b') = \mu$. Combining (2.18) and (2.19) gives the following diagram with exact rows and columns:

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & Q_\mu(b) \arrow[r] \arrow[d] & Q_\mu(b) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & Q(b) \arrow[r] \arrow[d] & P(b) \arrow[r] \arrow[d] & \Delta(b) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & i_*Q_{<\mu}(b) \arrow[r] \arrow[d] & i_*P_{<\mu}(b) \arrow[r] \arrow[d] & \Delta(b) \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

The result follows.

$\square$

Two natural questions to ask at this point are: Given an abelian category $\mathcal{A}$ as in Porism 2.4.1,

- Under what conditions do projective indecomposables in $\mathcal{A}$ have a filtration by standard objects?
- Under what conditions do injective indecomposables in $\mathcal{A}$ have a filtration by costandard objects?

One sufficient condition for both of these statements to hold is the condition that the stratification is 2-homological and all strata categories are semisimple (Lemma 2.5.3). In this case $\mathcal{A}$ is a highest weight category (see Theorem 2.5.2).

Categories in which projective and/or injective indecomposable objects have filtrations by standard and/or costandard objects have been widely studied, beginning with Cline-Parshall-Scott's definition of highest weight category in [CPS88]. Categories whose projective indecomposables have filtrations by standard objects where originally studied by Dlab [Dla96] and by Cline, Parshall and Scott [CPS96], where these are called *standardly stratified categories*. Categories in which both projective objects have a filtration by standard objects and injective objects have a filtration by costandard objects have been studied by various authors (see e.g.[Fri07], [CZ19], [LW15]). These situations all fit into the framework of ε-stratified categories (due to Brundan and Stroppel [BS18]). For the remainder of this section we recall the definition of an ε-stratified category and conjecture when a recollement of abelian categories is an ε-stratified category.

For $b \in B$ and $\lambda = \rho(b) \in \Lambda$, define *proper standard* and *proper costandard* objects

$$\overline{\Delta}(b) := j_!^\lambda L_\lambda(b), \qquad \overline{\nabla}(b) := j_*^\lambda L_\lambda(b).$$

For a *sign function* $\varepsilon : \Lambda \to \{\pm\}$, define the *$\varepsilon$-standard* and *$\varepsilon$-costandard objects*

$$\Delta_\varepsilon(b) := \begin{cases} \Delta(b) & \text{if } \varepsilon(\rho(b)) = + \\ \overline{\Delta}(b) & \text{if } \varepsilon(\rho(b)) = - \end{cases}, \qquad \nabla_\varepsilon(b) := \begin{cases} \overline{\nabla}(b) & \text{if } \varepsilon(\rho(b)) = + \\ \nabla(b) & \text{if } \varepsilon(\rho(b)) = - \end{cases}.$$

The following definition is due to Brundan and Stroppel [BS18].

Definition 2.4.2. Let $\mathcal{A}$ be a finite abelian category enriched over an algebraically closed field $\Bbbk$. Suppose $\mathcal{A}$ has a stratification by a poset Λ and stratification function $\rho : B \to \Lambda$. Let $\varepsilon : \Lambda \to \{+,-\}$ be a function. Say that $\mathcal{A}$ is an *ε-stratified category* if it satisfies the following equivalent conditions:

(ε-S1) For every $b \in B$, the projective indecomposable $P(b)$ fits into an exact sequence

$$U(b) \to P(b) \to \Delta_\varepsilon(b)$$

in which $U(b)$ has a filtration by objects of the form $\Delta_\varepsilon(b')$, where $\rho(b') \geq \rho(b)$.

(ε-S2) For every $b \in B$, the injective indecomposable $I(b)$ fits into an exact sequence

$$\nabla_\varepsilon(b) \to I(b) \to U'(b)$$

in which $U'(b)$ has a filtration by objects of the form $\nabla_\varepsilon(b')$, where $\rho(b') \geq \rho(b)$.

The equivalence of statements (ε-S1) and (ε-S2) is shown in [ADL98, Theorem 2.2]. A proof of this fact can also be found in [BS18, Theorem 3.5].

This definition leads to the following question.

Open Question 2.4.3. Let $\mathcal{A}$ be an abelian category with a stratification by a finite poset Λ. Find necessary and sufficient conditions for statements (ε-S1) and (ε-S2) to hold.

To state our conjectured answer to this question we need the following definition. For a function $\varepsilon : \Lambda \to \{\pm\}$, say that a stratification of an abelian category $\mathcal{A}$ by a poset Λ is an *ε-stratification* if for all $\lambda \in \Lambda$ the following hold:

(i) If $\varepsilon(\lambda) = +$ then the functor $j_*^\lambda : \mathcal{A}_\lambda \to \mathcal{A}$ is exact.

(ii) If $\varepsilon(\lambda) = -$ then the functor $j_!^\lambda : \mathcal{A}_\lambda \to \mathcal{A}$ is exact.

Brundan and Stroppel show [BS18, Theorem 3.5] that if $\mathcal{A}$ is an ε-stratified category then the stratification of $\mathcal{A}$ by Λ is an ε-stratification. We conjecture that the converse is true when the stratification is 2-homological.

Conjecture 2.4.4. *Let $\mathcal{A}$ be an abelian length category with finitely many simples, enough projectives and injectives, and admits a 2-homological stratification by a finite poset Λ. Then, for any $\varepsilon : \Lambda \to \{\pm\}$, conditions ($\varepsilon$-S1) and ($\varepsilon$-S2) hold if and only if the stratification of $\mathcal{A}$ is an ε-stratification.*

One case in which Conjecture 2.4.4 is known to be true is if $\mathcal{A}$ has a 2-homological stratification in which every strata category is semisimple. In this case, $\mathcal{A}$ has an ε-stratification for every function $\varepsilon : \Lambda \to \{\pm\}$. Moreover, $\mathcal{A}$ is ε-stratified for every function $\varepsilon : \Lambda \to \{\pm\}$ (see Lemma 2.5.3 below).

2.5 Highest weight categories

In this section we give necessary and sufficient conditions for a finite abelian category with a stratification by a poset Λ to be a highest weight category with respect to the poset Λ. This result is not original. Indeed, a version of this result (stated without the terminology of stratifications of abelian categories) is shown in [Kra17, Theorem 3.4].

The following definition is due to [CPS88][1].

Definition 2.5.1 (Highest weight category)**.** Let $\Bbbk$ be a field. Say that a $\Bbbk$-linear abelian category $\mathcal{A}$ is a *highest weight category* with respect to a finite poset Λ if $\mathcal{A}$ is finite over $\Bbbk$, and for every $\lambda \in \Lambda$ there is a projective indecomposable, P_λ, that fits into a short exact sequence in $\mathcal{A}$:

$$0 \to U_\lambda \to P_\lambda \to \Delta_\lambda \to 0$$

in which:

(HW1) $\mathrm{End}_{\mathcal{A}}(\Delta_\lambda)$ is a division ring for all $\lambda \in \Lambda$.

(HW2) $\mathrm{Hom}_{\mathcal{A}}(\Delta_\lambda, \Delta_\mu) = 0$ whenever $\lambda > \mu$.

(HW3) U_λ has a filtration by standard objects Δ_μ in which $\lambda < \mu$.

[1] The definition of highest weight category used here is stronger that that used in [CPS88]. The paper [CPS88] allows highest weight categories to be locally artinian, whereas we only consider highest weight categories that are finite over a field $\Bbbk$.

(HW4) $\bigoplus_{\lambda\in\Lambda} P_\lambda$ is a projective generator of $\mathcal{A}$.

Theorem 2.5.2. *Let $\Bbbk$ be a field, and let $\mathcal{A}$ be a $\Bbbk$-linear abelian category. The following are equivalent:*

(i) $\mathcal{A}$ is a highest weight category.

(ii) $\mathcal{A}$ has a homological stratification with respect to Λ in which every strata category is equivalent to mod-Γ_λ *for some finite dimensional division algebra Γ_λ.*

(iii) $\mathcal{A}$ has a 2-homological stratification with respect to Λ in which every strata category is equivalent to mod-Γ_λ *for some finite dimensional division algebra Γ_λ.*

We begin with a lemma.

Lemma 2.5.3. *Suppose $\mathcal{A}$ is an* Ext^1*-finite abelian category with a 2-homological stratification with respect to a finite poset Λ, in which every strata category is semisimple with finitely many simple objects. Let $\rho : B \to \Lambda$ be the stratification function for $\mathcal{A}$. Then for each $b \in B$, there is a projective object $P(b)$ that fits into a short exact sequence*

$$0 \to U(b) \to P(b) \to \Delta(b) \to 0$$

in which $U(b)$ has a filtration by standard objects $\Delta(b')$ in which $\rho(b) < \rho(b')$.

Proof. Let $\mathcal{A}$ be an Ext^1-finite abelian category. Suppose $\mathcal{A}$ fits into a 2-homological recollement as in (2.10), and $\mathcal{A}_U$ is semisimple.

Suppose $\mathcal{A}^Z$ has simple objects $L_1, \ldots, L_m$ with projective covers $\bar{P}_1, \ldots, \bar{P}_m$ in $\mathcal{A}^Z$. Suppose $\mathcal{A}_U$ has simple objects $\tilde{L}_{m+1}, \ldots, \tilde{L}_{m+n}$. Then if $\mathcal{A}_U$ is semisimple the simple objects $L_{m+k} := j_{!*}\tilde{L}_{m+k} \in \mathcal{A}^U$ have projective covers $P_{m+k} := j_!\tilde{L}_{m+k}$ in $\mathcal{A}$. For $L_t \in \mathcal{A}^Z$, the projective cover, P_t, of L_t in $\mathcal{A}$ is constructed by the following process (originally due to Krause [Kra17, Theorem 3.4]).

Fix $1 \le t \le m$. Note that each endomorphism ring, $\mathrm{End}_{\mathcal{A}}(P_{m+k})$, is a division ring since $j_!$ is fully-faithful. Hence we can define the universal extension

$$0 \to \bigoplus_k P_{m+k}^{\oplus d_k} \to P_t \to \bar{P}_t \to 0$$

where $d_k := \dim_{\mathrm{End}_{\mathcal{A}}(P_{m+k})} \mathrm{Ext}^1_{\mathcal{A}}(\bar{P}_t, P_{m+k})$. Now, P_t has a filtration by standard objects. We show that P_t is projective in steps.

Step 1. $\mathrm{Ext}^{\ell}_{\mathcal{A}}(P_t, -)$ vanishes on $\mathcal{A}^Z$ for all $1 \leq \ell \leq 2$. Since the recollement is 2-homological we have that $\mathrm{Ext}^{\ell}_{\mathcal{A}}(\bar{P}_t, X) = 0 = \mathrm{Ext}^{\ell}_{\mathcal{A}}(\bigoplus_k P^{\oplus d_k}_{m+k}, X)$ for all $X \in \mathcal{A}^Z$. The result follows.

Step 2. $\mathrm{Ext}^1_{\mathcal{A}}(P_t, -)$ vanishes on the essential image of $j_!$. Since $\mathcal{A}_U$ is semisimple it suffices to show that $\mathrm{Ext}^1_{\mathcal{A}}(P_t, P_{m+l}) = 0$ for all $1 \leq l \leq n$. This holds since the first map in the following exact sequence is surjective:

$$\mathrm{Hom}_{\mathcal{A}}(\bigoplus_k P^{\oplus d_k}_{m+k}, P_{m+l}) \to \mathrm{Ext}^1_{\mathcal{A}}(\bar{P}_t, P_{m+l}) \to \mathrm{Ext}^1_{\mathcal{A}}(P_t, P_{m+l}) \to 0$$

Step 3. $\mathrm{Ext}^1_{\mathcal{A}}(P_t, X) = 0$ for all $X \in \mathcal{A}$. Consider the exact sequence

$$0 \to K \to j_! j^! X \to X \to i_* i^* X \to 0$$

with $K \in \mathcal{A}^Z$. Split this sequence into two short exact sequences:

$$0 \to K \to j_! j^! X \to X' \to 0$$
$$0 \to X' \to X \to i_* i^* X \to 0$$

By the previous steps we have exact sequences

$$0 = \mathrm{Ext}^1_{\mathcal{A}}(P_t, j_! j^! X) \to \mathrm{Ext}^1_{\mathcal{A}}(P_t, X') \to \mathrm{Ext}^2_{\mathcal{A}}(P_t, K) = 0$$

and

$$0 = \mathrm{Ext}^1_{\mathcal{A}}(P_t, X') \to \mathrm{Ext}^1_{\mathcal{A}}(P_t, X) \to \mathrm{Ext}^1_{\mathcal{A}}(P_t, i_* i^* X) = 0$$

It follows that P_t is the projective..

By using an induction argument similar to that used in the proof of Porism 2.4.1, the result follows.

□

Proof of Theorem 2.5.2. We proceed in steps.

Step 1. (i) implies (ii). If $\mathcal{A}$ is a highest weight category with respect to Λ, then a homological stratification of $\mathcal{A}$ by Λ is constructed as follows: For a closed subcategory $\Lambda' \subset \Lambda$ define $\mathcal{A}_{\Lambda'}$ to be the Serre subcategory of $\mathcal{A}$ generated by the standard objects Δ_λ in which $\lambda \in \Lambda'$. Then for any maximal $\mu \in \Lambda'$ there is a homological recollement of abelian categories

$$\begin{tikzcd}
\mathcal{A}_{\Lambda' \setminus \{\mu\}} \arrow[r, "i_*=i_!" description] & \mathcal{A}_{\Lambda'} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^!=j^*" description] & \text{mod-}\operatorname{End}_{\mathcal{A}}(\Delta_\lambda) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"]
\end{tikzcd}$$

in which $j^! = \operatorname{Hom}_{\mathcal{A}_{\Lambda'}}(\Delta_\mu, -)$ (see the proof of [Kra17, Theorem 3.4]).

Step 2. (ii) implies (iii). This is obvious.

Step 3. (iii) implies (i). Suppose $\mathcal{A}$ has a 2-homological stratification with strata categories of the form $\mathcal{A}_\lambda = \text{mod-}\Gamma_\lambda$ for a finite dimensional division ring Γ_λ. Then $\mathcal{A}$ is finite (either by Corollary 2.3.11, or by Lemma 2.5.3 and Propositions 2.2.6 and 2.3.4). Let L_λ denote the unique simple object in $\mathcal{A}_\lambda$. Define $\Delta_\lambda := j_!^\lambda L_\lambda$ and let P_λ be the projective cover of $j_{!*}^\lambda L_\lambda$ in $\mathcal{A}$. Statement (HW1) holds since $j_!$ is fully-faithful, statement (HW2) is exactly equation (2.17), statement (HW3) follows from Lemma 2.5.3, and statement (HW4) is obvious. $\square$

Example 2.5.4. The category of perverse sheaves over a space stratified by finitely many affine spaces is always a highest weight category (see e.g. [BGS96, Theorem 3.3.1]). For example, for a Borel subgroup B of a complex reductive Lie group G, the perverse sheaves on the flag variety G/B with respect to the stratification into B-orbits is a highest weight category (this is the category of B-equivariant perverse sheaves on G/B)[2].

Example 2.5.5. Consider the closed subvariety $\mathcal{N} \subset \mathfrak{gl}_n(\mathbb{C})$ consisting of nilpotent elements. The group $G := GL_n(\mathbb{C})$ acts on $\mathcal{N}$ by conjugation. For a partition $\lambda = (\lambda_1, \dots, \lambda_m)$ of n, let $\mathcal{O}_\lambda$ be the G-orbit in $\mathcal{N}$ consisting of nilpotent matrices whose Jordan form consists of Jordan blocks of sizes $\lambda_1, \dots, \lambda_m$. The closure order for the strata is given by the dominance order on partitions [Ger59] i.e. $\mathcal{O}_\lambda \subset \overline{\mathcal{O}_\mu}$ iff $\lambda_1 + \dots + \lambda_k \leq \mu_1 + \cdots \mu_k$ for all $k \geq 1$ (where partitions are extended by zeros at the end if necessary). Mautner [Mau14, Theorem 1.1] shows that this category is equivalent to the category of finite dimensional modules of the Schur algebra

[2] This category is equivalent to the principal block of the BBG-category $\mathcal{O}$ (see e.g. [BGS96, Proposition 3.5.2])

$\mathcal{S}_{\Bbbk}(n,n)$. A consequence of this equivalence is that the category $\mathrm{Perv}_G(\mathcal{N},\Bbbk)$ is a highest weight category with standard objects

$$\Delta(\lambda) := {}^pH^0(h_{\lambda!}\underline{\Bbbk}_{\mathcal{O}_\lambda}[\dim \mathcal{O}_\lambda])$$

and costandard objects

$$\nabla(\lambda) := {}^pH^0(h_{\lambda*}\underline{\Bbbk}_{\mathcal{O}_\lambda}[\dim \mathcal{O}_\lambda]),$$

where $h_\lambda : \mathcal{O}_\lambda \hookrightarrow \mathcal{N}$ is the inclusion (see e.g. Proposition 4.5.1).

More recent examples in which a stratified abelian category is shown to be a highest weight category include [BM19, Theorem 6.8], [BR22, Theorem 7.2], and [Gou22, Theorem 5.2].

2.6 Appendix: Recollements and t-structures

In this appendix we prove a result of Beilinson-Bernstein-Deligne [BBD82, Proposition 1.4.16]. This statement says that given a recollement of triangulated categories with t-structure, one obtains a recollement of abelian categories on the hearts of the t-structures by taking zero-th cohomology.

Recall that a *t-structure* on a triangulated category $\mathcal{D}$ is a pair $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ of full subcategories (stable under isomorphism) satisfying the conditions:

(T1) If $X \in \mathcal{D}^{\leq 0}$, $Y \in \mathcal{D}^{\geq 0}$ then $\mathrm{Hom}_{\mathcal{D}}(X, Y[-1]) = 0$.

(T2) If $X \in \mathcal{D}^{\leq 0}$ then $X[1] \in \mathcal{D}^{\leq 0}$. If $Y \in \mathcal{D}^{\geq 0}$ then $Y[-1] \in \mathcal{D}^{\geq 0}$.

(T3) If $X \in \mathcal{D}$ then there is a unique triangle

$$\tau^{\leq 0}X \to X \to (\tau^{\geq 0}X)[-1] \to$$

where $\tau^{\leq 0}X \in \mathcal{D}^{\leq 0}$ and $\tau^{\geq 0}X \in \mathcal{D}^{\geq 0}$.

The definition of t-structure is originally from [BBD82]. We refer the reader to [HTT08, Chapter 8] for a good summary of the basic properties of t-structures.

Remark 2.6.1. The uniqueness of the triangle in (T3) is redundant in this definition (see e.g. [HTT08, Proposition 8.1.5]).

Recall that the assignments $X \mapsto \tau^{\leq 0}X$ and $X \mapsto \tau^{\geq 0}X$ extend to functors $\tau^{\leq 0} : \mathcal{D} \to \mathcal{D}^{\leq 0}$ and $\tau^{\geq 0} : \mathcal{D} \to \mathcal{D}^{\geq 0}$ in which:

(i) $\tau^{\leq 0}$ is right adjoint to the inclusion functor $\iota^{\leq 0} : \mathcal{D}^{\leq 0} \to \mathcal{D}$.

(ii) $\tau^{\geq 0}$ is left adjoint to the inclusion functor $\iota^{\geq 0} : \mathcal{D}^{\geq 0} \to \mathcal{D}$.

(see e.g. [HTT08, Proposition 8.1.4]) In fact the morphisms in (T3) are the unit and counit of (a shifted version of) these adjoint functors. Write $\mathcal{D}^{\heartsuit} := \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ for the heart of the t-structure and define the n-th cohomology functor $H^n := \tau^{\leq 0}\tau^{\geq 0}[n] : \mathcal{D} \to \mathcal{D}^{\heartsuit}$.

Fix a recollement of triangulated categories

$$\begin{tikzcd}[column sep=huge] \mathcal{D}_Z \arrow[r, "i_* = i_!"] & \mathcal{D} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^! = j^*"] & \mathcal{D}_U \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd} \tag{2.20}$$

with fixed t-structures on $\mathcal{D}_Z$ and $\mathcal{D}_U$. Then there is a t-structure on $\mathcal{D}$ defined:

$$\begin{aligned} \mathcal{D}^{\leq 0} &= \{X \in \mathcal{D} \mid i^*X \in \mathcal{D}_Z^{\leq 0}, \quad j^*X \in \mathcal{D}_U^{\leq 0}\} \\ \mathcal{D}^{\geq 0} &= \{X \in \mathcal{D} \mid i^!X \in \mathcal{D}_Z^{\geq 0}, \quad j^!X \in \mathcal{D}_U^{\geq 0}\}. \end{aligned}$$

We call this the *BBD t-structure* on $\mathcal{D}$.

Say that a functor $f : \mathcal{D}_1 \to \mathcal{D}_2$ is

- *left t-exact* if $f(\mathcal{D}_1^{\geq 0}) \subset \mathcal{D}_2^{\geq 0}$.
- *right t-exact* if $f(\mathcal{D}_1^{\leq 0}) \subset \mathcal{D}_2^{\leq 0}$.
- *t-exact* if it is both left and right t-exact.

The following proposition follows immediately from the definition of the BBD t-structure.

Proposition 2.6.2. *If a category $\mathcal{D}$ has the BBD t-structure then* *(i)* $i_* = i_!$ *and* $j^! = j^*$ *are* t*-exact.*

(ii) i^* *and* $j_!$ *are right* t*-exact.*

(iii) $i^!$ *and* j_* *are left* t*-exact.*

Theorem 2.6.3 (Proposition 1.4.16 of [BBD82])**.** *If* $\mathcal{D}$ *has the BBD* t*-structure then the following is a recollement of abelian categories:*

$$\begin{tikzcd}[column sep=huge] \mathcal{D}_Z^\heartsuit \arrow[r, "i_*=i_!" description] & \mathcal{D}^\heartsuit \arrow[l, bend right, "H^0(i^*)=\tau^{\geq 0}i^*"'] \arrow[l, bend left, "H^0(i^!)=\tau^{\leq 0}i^!"] \arrow[r, "j^!=j^*" description] & \mathcal{D}_U^\heartsuit \arrow[l, bend right, "H^0(j_!)=\tau^{\geq 0}j_!"'] \arrow[l, bend left, "H^0(j_*)=\tau^{\leq 0}j_*"] \end{tikzcd}$$

Proof. Axioms (R2), (R3), (R4) are obvious. The adjunction pair $(H^0(i^*), i_*)$ follows immediately from the adjunction pair $(\tau^{\geq 0} \circ i^*, i_* \circ \iota^{\geq 0})$. The other adjunctions follow similarly.

$\square$

Remark 2.6.4. As a partial converse to this theorem, Psaroudakis [Psa14, Theorem 7.2] gives conditions for when recollements of abelian categories induce recollements of triangulated categories.

Chapter 3

Polynomial representations and Schur algebras

In this chapter we summarise the basic theory of the Schur algebra $\mathcal{S}_{\Bbbk}(n,d) = \mathrm{End}_{\Bbbk\mathfrak{S}_d}(\bigotimes^d \Bbbk^n)$, for any field $\Bbbk$, and define a new product of Schur algebra modules corresponding to Krause's internal product of polynomial functors defined in [Kra13].

The origin story of Schur algebras begins with the work of Schur [Sch1901] relating polynomial representations of $GL_n(\mathbb{C})$ with representations of the symmetric group $\mathfrak{S}_d$. Green [Gre80] extended this work to the case that $\Bbbk$ is an infinite field, taking an approach that emphasised the role of the Schur algebra.

In the next chapter we construct the Schur algebra via the Borel-Moore homology of varieties related to the nilpotent cone of $\mathfrak{gl}_n(\mathbb{C})$. This is a characteristic-free version of Ginzburg's construction [CG97, Proposition 4.2.5]. Using this geometric version of the Schur algebra we give a new proof of Mautner's [Mau14, Theorem 1.1] equivalence of categories between $\mathcal{S}_{\Bbbk}(n,d)$-mod and the category of perverse sheaves on the nilpotent cone of $\mathfrak{gl}_n(\mathbb{C})$.

The purpose of this chapter is two-fold. Our first aim is to set up the basic theory about Schur algebras that will be needed in our geometric application. This occupies Section 3.1-3.5. Our second aim is to define a product $-\underline{\boxtimes}- : \mathcal{S}_{\Bbbk}(m,d)\text{-mod} \times \mathcal{S}_{\Bbbk}(n,d)\text{-mod} \to \mathcal{S}_{\Bbbk}(mn,d)\text{-mod}$ that corresponds, under

Schur-Weyl duality, to the Kronecker product on $\Bbbk\mathfrak{S}_d$ modules (and corresponds, under the equivalence between the category of Schur algebra modules and strict polynomial functors, to a version of Krause's internal product defined in [Kra13]).

The fact that Krause's product and the Kronecker product commute with Schur-Weyl duality has been shown in [AR17]. The construction of our product and it's relation to Krause's product occupies Sections 3.6 and Section 3.7 respectively, but depends on previous results introduced in Sections 3.1-3.5.

In Section 3.1 we recall the definition of Lusztig's idempotented form of a complex Lie algebra $\mathfrak{g}$, $\dot{\mathrm{U}}_{\Bbbk}\mathfrak{g}$ (originally defined in [Lus93]). We recall Doty's [Dot03, Corollary 6.13] result that identifies generalized Schur algebras (in the sense of [Don86, Section 3.2]) with quotients of $\dot{\mathrm{U}}_{\Bbbk}\mathfrak{g}$.

In Section 3.2 we define the Schur algebra $\mathcal{S}_{\Bbbk}(n, d)$ as a quotient of $\dot{\mathrm{U}}_{\Bbbk}\mathfrak{gl}_n$ and recount basic facts about its representation theory. In particular we construct the Schur-Weyl duality functor $\mathcal{F}_{SW} : \mathcal{S}_{\Bbbk}(n, d)\text{-mod} \to \text{mod-}\Bbbk\mathfrak{S}_d$.

In Sections 3.3 and 3.4 we describe the contravariant duality functor

$$(-)^{\circ} : \mathcal{S}_{\Bbbk}(n, d)\text{-mod} \to \mathcal{S}_{\Bbbk}(n, d)\text{-mod}^{op}$$

and external product

$$- \otimes - : \mathcal{S}_{\Bbbk}(n, d)\text{-mod} \times \mathcal{S}_{\Bbbk}(n, e)\text{-mod} \to \mathcal{S}_{\Bbbk}(n, d + e)\text{-mod}$$

respectively.

In Section 3.5 we recall some facts about double cosets of the symmetric group by Young subgroups.

In Section 3.6 we define the *homogeneous product*

$$-\underline{\boxtimes}- : \mathcal{S}_{\Bbbk}(m, d)\text{-mod} \times \mathcal{S}_{\Bbbk}(n, d)\text{-mod} \to \mathcal{S}_{\Bbbk}(mn, d)\text{-mod}$$

and show that the following diagram of functors commutes:

$$\begin{tikzcd}
\mathcal{S}_{\Bbbk}(m, d)\text{-mod} \times \mathcal{S}_{\Bbbk}(n, d)\text{-mod} \arrow[r, "-\underline{\boxtimes}-"] \arrow[d, "\mathcal{F}_{SW}\times\mathcal{F}_{SW}"'] & \mathcal{S}_{\Bbbk}(mn, d)\text{-mod} \arrow[d, "\mathcal{F}_{SW}"] \\
\text{mod-}\Bbbk\mathfrak{S}_d \times \text{mod-}\Bbbk\mathfrak{S}_d \arrow[r, "-\otimes_{\Bbbk}-"] & \text{mod-}\Bbbk\mathfrak{S}_d
\end{tikzcd}$$

In Section 3.7 we relate the homogeneous product to Krause's internal product of strict polynomial functors.

We include an appendix (Section 3.8) that recalls a basis of the Schur algebra (due to Schur [Sch1901]) and shows that this basis is equal to another basis of the Schur algebra constructed in [Tot97] and used in [Kra13] and [AR17]. This result is not new (see e.g. [Rei16, Appendix]).

In a second appendix (Section 3.9) we recount a diagrammatic approach to studying the Schur algebra that is described in [W19] and builds on the work in [CKM14].

Throughout this chapter, we write $[n] := \{1, \ldots, n\}$. Write $v_1, \ldots, v_n$ for the standard basis of the vector space $\mathbb{k}^n$.

3.1 Integral Lie algebra representations

In this chapter we regard the Schur algebra as a quotient of Lusztig's idempotented enveloping algebra $\dot{\mathrm{U}}\mathfrak{gl}_n$. This definition of the Schur algebra is due to Doty and Giaquinto [DG02, Theorem 1.4] (we recall this definition of the Schur algebra in Section 3.2).

In this section we motivate and recall the definition of $\dot{\mathrm{U}}_{\mathbb{k}}\mathfrak{g}$, for a complex reductive Lie algebra $\mathfrak{g}$. We recall a result of Doty [Dot03, Corollary 6.13] that says that any generalized Schur algebra (in the sense of [Don86, Section 3.2]) is a quotient of an idempotented enveloping algebra.

This result leads to a presentation of generalized Schur algebras by generators and relations (Proposition 3.1.3). This presentation is a refinement of the presentation of generalized Schur algebras given in [Dot03]. This presentation is known but is unpublished (it was told to me by Stephen Doty, but is essentially proven in [CG97, Corollary 4.3.2] using different terminology). We will refer to this presentation in Chapter 4.

Let $\mathfrak{g}$ be a complex reductive Lie algebra of rank n, with decomposition $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{a}$ into semisimple and abelian parts. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{s}$ and for

$\lambda \in \mathfrak{h}^*$ define

$$\mathfrak{g}_\lambda := \{x \in \mathfrak{g} \mid [h, x] = \lambda(h)x \text{ for all } h \in \mathfrak{h}\}.$$

Define the root system $\Phi = \{\lambda \in \mathfrak{h}^* \mid \mathfrak{g}_\lambda \neq 0\} \subset \mathfrak{h}^*$.

Let $\alpha_1, \ldots, \alpha_{\dim \mathfrak{h}}$ be a choice of simple roots in Φ and let $\{e_i, f_i, h_i\}_{i\in[\dim \mathfrak{h}]}$ be Chevalley generators of $\mathfrak{s}$ with respect to this choice of simple roots i.e. choose $e_i \in \mathfrak{g}_{\alpha_i}$, $f_i \in \mathfrak{g}_{-\alpha_i}$ so that $h_i := [e_i, f_i] \in \mathfrak{h}$ satisfies $\alpha_i(h_i) = 2$. Let (a_{ij}) be the Cartan matrix of $\mathfrak{s}$ i.e. $a_{ij} = \alpha_i(h_j)$.

The Lie algebra $\mathfrak{s}$ has a presentation by generators $\{e_i, f_i, h_i\}_{i\in[\dim \mathfrak{h}]}$ and the Chevalley-Serre relations (see e.g. [Ser87, Chapter IV, Appendix]):

$$[h_i, h_j] = 0, \quad [e_i, f_j] = \delta_{ij} h_i, \tag{3.1}$$

$$[h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j, \tag{3.2}$$

$$(\operatorname{ad} e_i)^{1-a_{ij}} e_j = 0 = (\operatorname{ad} f_i)^{1-a_{ij}} f_j \quad (i \neq j). \tag{3.3}$$

Here $\operatorname{ad}(e_i)$ refers to the adjoint action $[e_i, -] : \mathfrak{g} \to \mathfrak{g}$.

Define the Cartan subalgebra $\mathfrak{t} = \mathfrak{h} \oplus \mathfrak{a} \subset \mathfrak{g}$ and let $h_{\dim \mathfrak{h}+1}, \ldots, h_n$ be a basis of $\mathfrak{a}$. Let

$$\Lambda := \{\lambda \in \mathfrak{t}^* \mid \lambda(h_i) \in \mathbb{Z} \text{ for all } i = 1, \ldots, n\}$$

be the weight lattice of $\mathfrak{g}$. Let

$$\Lambda^+ := \{\lambda \in \Lambda \mid \lambda(h_i) \geq 0 \text{ for all } i = 1, \ldots, \dim \mathfrak{h}\}$$

be the set of dominant weights. Define the *dominance order* on Λ: $\lambda \leq \mu$ if $\mu - \lambda$ is a positive root i.e. $\mu - \lambda \in \mathbb{N}\alpha_1 + \cdots \mathbb{N}\alpha_{\dim \mathfrak{h}}$.

The Weyl group, W, of $\mathfrak{g}$ is the group of automorphisms $\mathfrak{t}^* \to \mathfrak{t}^*$ generated by the reflections $s_i : \lambda \mapsto \lambda - \lambda(h_i)\alpha_i$ for $i \in [\dim \mathfrak{h}]$ (here we think of α_i as an element of $\mathfrak{t}^*$ by setting $\alpha_i(x) = 0$ for all $x \in \mathfrak{a}$). The weight lattice $\Lambda \subset \mathfrak{t}^*$ is invariant under W and every $\lambda \in \Lambda$ is in the W-orbit of a unique $\lambda^+ \in \Lambda^+$. In particular $\Lambda = \bigcup_{\lambda\in\Lambda^+} W\lambda$.

We restrict our attention to $\mathfrak{g}$-modules M with a weight space decomposition, $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$, where

$$M_\lambda = \{m \in M \mid x \cdot m = \lambda(x)m \text{ for all } x \in \mathfrak{t}\}$$

is the λ-weight space of M. We call such modules *integral* $\mathfrak{g}$-modules. Integral $\mathfrak{g}$-modules can be viewed as modules over a related algebra, $\dot{\mathrm{U}}_{\mathbb{C}}\mathfrak{g}$, first studied by Lusztig [Lus93, Part IV]. We recall the definition of this algebra now.

Definition 3.1.1 (Lusztig's idempotented enveloping algebra $\dot{\mathrm{U}}_{\mathbb{C}}\mathfrak{g}$)**.** Let $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{a}$ be a complex reductive Lie algebra of rank n. Let Λ be the integral weight lattice of $\mathfrak{g}$ and let (a_{ij}) be the Cartan matrix of $\mathfrak{s}$. The $\mathbb{C}$-algebra $\dot{\mathrm{U}}_{\mathbb{C}}\mathfrak{g}$ is generated by elements E_i, F_i for $i \in [\operatorname{rank} \mathfrak{s}]$, and 1_λ for $\lambda \in \Lambda$. These satisfy the relations

$$1_\lambda 1_\mu = \delta_{\lambda,\mu} 1_\lambda, \tag{3.4}$$

$$E_i 1_\lambda = 1_{\lambda+\alpha_i} E_i, \quad F_i 1_\lambda = 1_{\lambda-\alpha_i} F_i, \tag{3.5}$$

$$E_i F_i 1_\lambda = F_i E_i 1_\lambda + \lambda(h_i) 1_\lambda, \tag{3.6}$$

$$E_i F_j = F_j E_i \quad (i \neq j), \tag{3.7}$$

$$\mathrm{ad}(E_j)^{1-a_{ij}} E_i = 0 = \mathrm{ad}(F_j)^{1-a_{ij}} F_i \quad (i \neq j). \tag{3.8}$$

There is an obvious equivalence of categories between the category of integral $\mathfrak{g}$-modules and the category of modules over $\dot{\mathrm{U}}_{\mathbb{C}}\mathfrak{g}$. More precisely, each integral $\mathfrak{g}$-module M corresponds to the $\dot{\mathrm{U}}_{\mathbb{C}}\mathfrak{g}$-module M, in which 1_λ acts by projection onto the λ-weight space (and the action E_i, F_i on M is the same as the action of the Chevalley generators e_i, f_i on M).

Define the $\mathbb{Z}$-algebra, $\dot{\mathrm{U}}_{\mathbb{Z}}\mathfrak{g}$, to be the $\mathbb{Z}$-subalgebra of $\dot{\mathrm{U}}_{\mathbb{C}}\mathfrak{g}$ generated by the elements

$$E_i^{(r)} := \frac{E_i^r}{r!}, \qquad F_i^{(r)} := \frac{F_i^r}{r!}$$

and all the idempotent generators 1_λ. For a field $\Bbbk$, let

$$\dot{\mathrm{U}}_{\Bbbk}\mathfrak{g} := \Bbbk \otimes_{\mathbb{Z}} \dot{\mathrm{U}}_{\mathbb{Z}}\mathfrak{g}.$$

Example 3.1.2 ($\dot{\mathrm{U}}_{\mathbb{C}}\mathfrak{gl}_n$)**.** Let $\mathfrak{g} = \mathfrak{gl}_n$ and $\mathfrak{t} \subset \mathfrak{gl}_n$ be the space of diagonal matrices. Let $\varepsilon_i \in \mathfrak{t}^*$ be the dual of the matrix e_{ii}, and $\alpha_i = \varepsilon_i - \varepsilon_{i+1} \in \mathfrak{t}^*$. The α_i form a complete set of simple roots. The weight lattice, Λ, of $\mathfrak{gl}_n$ is the $\mathbb{Z}$-span of the ε_i. We identify elements of Λ by their coordinates in $\mathbb{Z}^n$ with respect to this basis. For example, we write $\alpha_i = (0, \dots, 0, 1, -1, 0, \dots, 0) \in \mathbb{Z}^n$, where the 1 is in the i-th position.

The algebra $\dot{\mathrm{U}}_{\mathbb{C}}\mathfrak{gl}_n$ is generated by elements E_i, F_i, for $i \in [n-1]$, and idempotents 1_λ for $\lambda \in \mathbb{Z}^n$. These satisfy the relations

$$1_\lambda 1_\mu = \delta_{\lambda,\mu} 1_\lambda, \tag{3.9}$$

$$E_i 1_\lambda = 1_{\lambda+\alpha_i} E_i, \quad F_i 1_\lambda = 1_{\lambda-\alpha_i} F_i, \tag{3.10}$$

$$E_i F_i 1_\lambda = F_i E_i 1_\lambda + (\lambda_i - \lambda_{i+1}) 1_\lambda, \tag{3.11}$$

$$E_i F_j = F_j E_i \quad (i \neq j), \tag{3.12}$$

$$E_i E_j = E_j E_i, \quad F_i F_j = F_j F_i \quad (|i-j| > 1), \tag{3.13}$$

$$E_i E_j E_i = E_i^{(2)} E_j + E_j E_i^{(2)} \quad (|i-j| = 1), \tag{3.14}$$

$$F_i F_j F_i = F_i^{(2)} F_j + F_j F_i^{(2)} \quad (|i-j| = 1). \tag{3.15}$$

For the remainder of this section fix a reductive Lie algebra $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{a}$ with set, Λ^+, of dominant weights. Let π denote a finite closed-downwards subposet of Λ^+.

The category of finite dimensional $\dot{\mathrm{U}}_{\mathbb{C}}\mathfrak{g}$-modules, $\dot{\mathrm{U}}_{\mathbb{C}}\mathfrak{g}$-mod, is semisimple, and the simple objects are (up to isomorphism) in bijection with the set of dominant weights. More precisely, each $\lambda \in \Lambda^+$, corresponds to the simple module, L_λ, with highest weight λ.

Doty [Dot03, Corollary 6.13] shows that for a finite closed-downwards subposet $\pi \subset \Lambda^+$, the two-sided ideal $\langle 1_\lambda \mid \lambda \in \Lambda^+ \setminus \pi \rangle \subset \dot{\mathrm{U}}_{\mathbb{C}}\mathfrak{g}$ is the ideal of all elements of $\dot{\mathrm{U}}_{\mathbb{C}}\mathfrak{g}$ that annihilate every simple module L_λ for $\lambda \notin \pi$.

In particular the following categories are equivalent:

1. The category of modules $M \in \dot{\mathrm{U}}_{\mathbb{C}}\mathfrak{g}$-mod whose simple composition factors are of the form L_λ, for $\lambda \in \pi$.

2. The category of modules $M \in \dot{\mathrm{U}}_{\mathbb{C}}\mathfrak{g}$-mod in which $1_\lambda M = 0$ whenever $\lambda \in \Lambda^+ \setminus \pi$.

3. The category of finite dimensional modules of the algebra

$$\mathcal{S}_{\mathbb{C}}(\pi) := \dot{\mathrm{U}}_{\mathbb{C}}\mathfrak{g} / \langle 1_\lambda \mid \lambda \in \Lambda^+ \setminus \pi \rangle.$$

For a field $\Bbbk$, define the algebra

$$\mathcal{S}_{\Bbbk}(\pi) := \dot{\mathrm{U}}_{\Bbbk}\mathfrak{g}/\langle 1_\lambda \mid \lambda \in \Lambda^+ \setminus \pi\rangle.$$

For any $\dot{\mathrm{U}}\mathfrak{g}$-module M, there is a linear isomorphism $1_\lambda M \simeq 1_\mu M$ whenever the weights λ and μ are in the same Weyl group orbit (see e.g. [Ser87, Chapter VII, Section 4, Remarks]). In particular $1_\lambda M = 0$ if and only if $1_\mu M = 0$ for every $\mu \in W\lambda$. It follows that

$$\mathcal{S}_{\Bbbk}(\pi) = \dot{\mathrm{U}}_{\Bbbk}\mathfrak{g}/\langle 1_\lambda \mid \lambda \notin W\pi\rangle.$$

Two immediate properties of $\mathcal{S}_{\Bbbk}(\pi)$ worth mentioning are the following:

(i) The algebra $\mathcal{S}_{\Bbbk}(\pi)$ has an identity element $1 = \sum_{\lambda \in W\pi} 1_\lambda$.

(ii) The elements $E_i^{(r)}$ and $F_i^{(r)}$ of $\mathcal{S}_{\Bbbk}(\pi)$ are nilpotent. Moreover these elements are equal to 0 for large enough r. To see this, note for instance that

$$E_i^{(r)} = \sum_{\lambda \in \Lambda(n,d)} E_i^{(r)} 1_\lambda = \sum_{\lambda \in \Lambda(n,d)} 1_{\lambda + r\alpha_i} E_i^{(r)}$$

is equal to zero for large enough r (since $\pi \subset \Lambda^+$ is finite).

Donkin [Don86, Section 3.2] defines the *generalized Schur algebra* $\mathcal{S}'_{\mathbb{C}}(\pi)$ as the quotient of the universal enveloping algebra, $\mathrm{U}\,\mathfrak{g}$, by the ideal of all elements that annihilate every simple module, L_λ, in which $\lambda \notin \pi$. This algebra is generated by the Chevalley generators e_i, f_i, h_i (for $i \in [\dim \mathfrak{h}]$) of $\mathfrak{s}$, together with a basis, $h_{\dim \mathfrak{h}+1}, \ldots, h_n$, of $\mathfrak{a}$. Doty [Dot03, Corollary 6.13] shows that the algebra map $\mathcal{S}'_{\mathbb{C}}(\pi) \to \mathcal{S}_{\mathbb{C}}(\pi)$ defined by

$$e_i \mapsto E_i, \qquad f_i \mapsto F_i, \qquad h_i \mapsto \sum_{\lambda \in W\pi} \lambda(h_i) 1_\lambda. \tag{3.16}$$

is an isomorphism. In particular, for any field $\Bbbk$, the category $\mathcal{S}_{\Bbbk}(\pi)$-mod is a highest weight category with respect to the poset π (see e.g. [Dot03, Theorem 5.4]).

The following unpublished result was told to me by Stephen Doty. The proof of this result follows closely the proof of a similar result of Ginzburg [CG97, Corollary 4.3.2].

Proposition 3.1.3 (Doty's Presentation of Generalized Schur Algebras)**.** *The generalized Schur algebra $\mathcal{S}_{\mathbb{C}}(\pi)$ is the unital $\mathbb{C}$-algebra generated by E_i, F_i for $i \in$ $[\operatorname{rank}\mathfrak{s}]$, and 1_λ for $\lambda \in \Lambda$, and satisfying relations (3.4)-(3.7) together with the relations*

$$1_\lambda = 0 \quad (\lambda \notin W\pi) \tag{3.17}$$

$$\sum_{\lambda \in W\pi} 1_\lambda = 1 \tag{3.18}$$

Proof. By definition, $\mathcal{S}_{\mathbb{C}}(\pi)$ is generated by the E_i, F_i, 1_λ, and satisfies the relations (3.4)-(3.8) that define $\dot{\mathrm{U}}_{\mathbb{C}}\mathfrak{g}$, together with the additional relations (3.17) and (3.18). It remains to show that Relation (3.8) is redundant in this presentation.

For each $j \in [\operatorname{rank}\mathfrak{s}]$, there is a $\mathrm{U}\,\mathfrak{sl}_2$-action on $\mathcal{S}'_{\mathbb{C}}(\pi)$ defined by

$$e \cdot x = \operatorname{ad}(e_j)x, \qquad f \cdot x = \operatorname{ad}(f_j)x, \qquad h \cdot x = \operatorname{ad}(h_j)x.$$

If $i \neq j$, then $f \cdot e_i = 0$ and $h \cdot e_i = a_{ij}e_i$. In particular (and since $a_{ij} < 0$), the submodule of $\mathcal{S}'_{\mathbb{C}}(\pi)$ generated by e_i is the simple module with highest weight $-a_{ij}$. In particular, $e^{1-a_{ij}} \cdot e_i = 0$. By a similar argument, $f^{1-a_{ij}} \cdot f_i = 0$. The result follows. $\square$

3.2 Schur algebras and Schur-Weyl duality

In this section we recall the definition of the classical Schur algebra and recall the statement of Schur-Weyl duality.

Let $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ (where $n \in \mathbb{N}$) and let $\Bbbk$ be a field. Say that a $\dot{\mathrm{U}}_{\Bbbk}\mathfrak{g}$-module M is *polynomial of homogeneous degree d* if M has a weight space decomposition

$$M = \bigoplus_{\lambda \in \Lambda(n,d)} M_\lambda,$$

where

$$\Lambda(n,d) := \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n \mid \sum_i \lambda_i = d\}$$

is the set of *weak compositions* of d of length n. Some examples include

(1) The module $E := \bigotimes^d \Bbbk^n$. Indeed consider the right action of $\mathfrak{S}_d$ on E given by permuting tensor factors i.e.

$$v_1 \otimes \cdots \otimes v_d \cdot \sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}.$$

For $\lambda \in \Lambda(n,d)$ define $v_\lambda \in E$ by

$$v_\lambda := \underbrace{v_1 \otimes \cdots \otimes v_1}_{\lambda_1 \text{ times}} \otimes \underbrace{v_2 \otimes \cdots \otimes v_2}_{\lambda_2 \text{ times}} \otimes \cdots \otimes \underbrace{v_n \otimes \cdots \otimes v_n}_{\lambda_n \text{ times}}$$

Then $E_\lambda = \Bbbk[v_\lambda \cdot \mathfrak{S}_n]$ and $E = \bigoplus_{\lambda \in \lambda(n,d)} E_\lambda$.

(2) The *d-th divided power* $\Gamma^d \Bbbk^n$. This is the submodule of E consisting of $\mathfrak{S}_n$-invariants.

(3) The *d-th symmetric power* $S^d \Bbbk^n$. This is the space of $\mathfrak{S}_n$-coinvariants of E. i.e. $S^d \Bbbk^n$ is the largest quotient of E in which $\mathfrak{S}_n$ acts trivially.

(4) The *d-th exterior power* $\Lambda^d \Bbbk^n$. This is the largest quotient of E in which $\mathfrak{S}_n$ acts via the sign representation.

Write $\mathcal{P}^{\Bbbk}_{n,d}$ for the category of left finite dimensional polynomial representations of $\dot{\mathrm{U}}_{\Bbbk}\mathfrak{g}$ of homogeneous degree d. The category $\mathcal{P}^{\Bbbk}_{n,d}$ is equivalent to the category of finite dimensional left modules for the *Schur algebra*:

$$\mathcal{S}_{\Bbbk}(n,d) := \dot{\mathrm{U}}_{\Bbbk}\mathfrak{g}/\langle 1_\lambda \mid \lambda \notin \Lambda(n,d)\rangle.$$

Note that $\mathcal{S}_{\Bbbk}(n,d)$ is the generalized Schur algebra corresponding to the set,

$$\Lambda^+(n,d) := \{(\lambda_1, \ldots, \lambda_n) \in \Lambda(n,d) \mid \lambda_1 \geq \ldots \geq \lambda_n\},$$

of dominant weights in $\Lambda(n,d)$.

Remark 3.2.1. Schur algebras were first studied by Schur [Sch1901] in the case $\Bbbk = \mathbb{C}$, and later by Green [Gre80] in the case that $\Bbbk$ is an infinite field. Neither of these sources construct Schur algebras in the same way as they are defined here (we recall the original definition in Section 3.8). The equivalence between our definition of $\mathcal{S}_{\Bbbk}(n,d)$ and the classical definition is due to Doty and Giaquinto

[DG02, Theorem 1.4] (see also [DGS09]). If $\Bbbk$ is an infinite field then the polynomial representations of $\dot{\mathrm{U}}_{\Bbbk}\mathfrak{g}$ correspond to polynomial representations of $\mathrm{GL}_n(\Bbbk)$ in the classical sense (see e.g. [Gre80, Section 3.2]).

The Schur algebras fit into a sequence of algebra embeddings

$$\mathcal{S}_{\Bbbk}(1,d) \hookrightarrow \mathcal{S}_{\Bbbk}(2,d) \hookrightarrow \cdots \hookrightarrow \mathcal{S}_{\Bbbk}(n,d) \hookrightarrow \mathcal{S}_{\Bbbk}(n+1,d) \hookrightarrow \cdots$$

in which each embedding maps $1_\lambda \mapsto 1_{(\lambda,0)}$, $E_i^{(r)} \mapsto E_i^{(r)}$, and $F_i^{(r)} \mapsto F_i^{(r)}$. The corresponding induction functors $\mathcal{P}^{\Bbbk}_{n,d} \to \mathcal{P}^{\Bbbk}_{n+1,d}$ are fully-faithful and equivalences of categories when $n \geq d$.

Define the category of *degree d polynomial representations*, $\mathcal{P}^{\Bbbk}_d$, as the colimit of categories

$$\mathcal{P}^{\Bbbk}_d = \mathcal{P}^{\Bbbk}_{\infty,d} := \lim_{n\to\infty} \mathcal{P}^{\Bbbk}_{n,d}.$$

Define the *infinite Schur algebra*

$$\mathcal{S}_{\Bbbk}(\infty,d) := \lim_{n\to\infty} \mathcal{S}_{\Bbbk}(n,d).$$

The algebra $\mathcal{S}_{\Bbbk}(\infty,d)$ is generated by elements of the form $E_i^{(r)}$, $F_i^{(r)}$, for $i \geq 1$, together with idempotents 1_λ, for λ in the set, $\Lambda(\infty,d)$, of infinite sequences of non-negative integers with finitely many nonzero entries and whose entries sum to d.

The category $\mathcal{P}^{\Bbbk}_d$ is equivalent to the full subcategory of $\mathcal{S}_{\Bbbk}(\infty,d)$-Mod consisting of modules, M, in which the $\mathcal{S}_{\Bbbk}(n,d)$-module $M(n) := \bigoplus_{\lambda\in\Lambda(n,d)} 1_\lambda M$ is finite dimensional for all $n \in \mathbb{N}$. Here we think of $\Lambda(n,d) \subset \Lambda(\infty,d)$ by appending zeroes to the end of sequences. Examples of such modules include $\bigotimes^d \Bbbk^\infty$, $\Gamma^d\Bbbk^\infty$, $S^d\Bbbk^\infty$, $\Lambda^d\Bbbk^\infty$.

The algebra $\mathcal{S}_{\Bbbk}(n,d)$ is semisimple if and only if $\operatorname{char}\Bbbk = 0$ or $\operatorname{char}\Bbbk > d$ (see e.g. [Gre80, Corollary 2.6e]). The following is well-known and we include it here without proof.

Theorem 3.2.2 (Schur-Weyl Duality)**.** *Let $n \in \mathbb{N}$ or $n = \infty$. The actions $\mathcal{S}_{\Bbbk}(n,d) \curvearrowright \bigotimes^d \Bbbk^n \curvearrowleft \Bbbk\mathfrak{S}_d$ commute. Moreover the representation map*

$$\Phi : \mathcal{S}_{\Bbbk}(n,d) \to \mathrm{End}_{\Bbbk\mathfrak{S}_d}(\textstyle\bigotimes^d \Bbbk^n)$$

is an algebra isomorphism. The representation map

$$\Psi : \Bbbk\mathfrak{S}_d \to \mathrm{End}_{\mathcal{S}_\Bbbk(n,d)}(\textstyle\bigotimes^d \Bbbk^n)$$

is surjective in general, and an isomorphism when $n \geq d$.

For a proof that Φ is an isomorphism see [DG02, Theorem 1.4]. For a proof that Ψ is an isomorphism when $n \geq d$ see [Bry09, Lemma 2.4].

Schur-Weyl duality has the following well-known corollary (see e.g. [Gre80, Section 6.2] for a proof).

Corollary 3.2.3. *Let $n \geq d$. The functor*

$$\mathcal{F}_{SW} := \mathrm{Hom}_{S_\Bbbk(n,d)}(\textstyle\bigotimes^d \Bbbk^n, -) : \mathcal{P}^\Bbbk_{n,d} \to \text{Mod-}\Bbbk\mathfrak{S}_d$$

is exact, full, and essentially surjective. Moreover if $\mathrm{char}\,\Bbbk = 0$ or $\mathrm{char}\,\Bbbk > d$, then $\mathcal{F}_{SW}$ is an equivalence of categories.

We call $\mathcal{F}_{SW}$ the *Schur-Weyl duality functor*.

3.3 The contravariant duality functor $(-)^\circ : \mathcal{P}^\Bbbk_{n,d} \to (\mathcal{P}^\Bbbk_{n,d})^{op}$

Consider the involutory algebra isomorphism $(-)^t : \mathcal{S}_\Bbbk(n,d) \to \mathcal{S}_\Bbbk(n,d)^{op}$ that interchanges $E_i^{(r)}$ and $F_i^{(r)}$ and maps the idempotent $1_{(\lambda_1,\ldots,\lambda_n)} \mapsto 1_{(\lambda_n,\ldots,\lambda_1)}$. We call this the *transpose map*.

There is an equivalence of categories $(-)^\circ : \mathcal{S}_\Bbbk(n,d)\text{-mod} \to \mathcal{S}_\Bbbk(n,d)\text{-mod}^{op}$ that maps a module M to the module, M°, whose underlying vector space is the linear dual $M^* = \mathrm{Hom}_\Bbbk(M,\Bbbk)$ and with action

$$(x \cdot f)(m) = f(x^t \cdot m) \qquad \text{for } f \in M^*,\ x \in \mathcal{S}_\Bbbk(n,d),\ m \in M.$$

The module M° is called the *contravariant dual* of M.

The following proposition is useful for calculating contravariant duals.

Proposition 3.3.1. *Let M, N be objects in $\mathcal{P}^\Bbbk_{n,d}$. There is a bijective correspondence between morphisms $f : M \to N^\circ$ and $\Bbbk$-bilinear forms $(-,-) : M \times N \to \Bbbk$ satisfying the property*

$$(x \cdot m, n) = (m, x^t \cdot n) \qquad \textit{for all } x \in \mathcal{S}_\Bbbk(n,d),\ m \in M,\ n \in N.$$

Proof. Given a morphism $f : M \to N^\circ$ the corresponding bilinear form is defined $(m, n) = f(m)(n)$. $\square$

Example 3.3.2. The $\mathcal{S}_\Bbbk(n, d)$ module $E = \bigotimes^d \Bbbk^n$ is equal to it's contravariant dual. Indeed there is a non-degenerate bilinear form $(-,-) : E \otimes E \to \Bbbk$ defined by

$$(v_{i_1} \otimes \cdots \otimes v_{i_d}, v_{j_1} \otimes \cdots \otimes v_{j_d}) = \delta_{i_1 j_1} \cdots \delta_{i_d j_d}.$$

Likewise, $(\Gamma^d \Bbbk^n)^\circ \simeq S^d \Bbbk^n$ and $(\Lambda^d \Bbbk^n)^\circ \simeq \Lambda^d \Bbbk^n$.

The following proposition follows immediately from the definitions.

Proposition 3.3.3. *There is an equivalence of categories* $(\mathcal{P}^\Bbbk_{n,d})^{op} \simeq \text{mod-}\mathcal{S}_\Bbbk(n, d)$ *defined by sending a left* $\mathcal{S}_\Bbbk(n, d)$*-module* M *to the right* $\mathcal{S}_\Bbbk(n, d)$*-module with the same underlying vector space and with* $\mathcal{S}_\Bbbk(n, d)$*-action*

$$m \cdot x = x^t \cdot m \qquad \textit{for } x \in \mathcal{S}_\Bbbk(n, d), m \in M.$$

3.4 The external product $- \otimes - : \mathcal{P}^\Bbbk_{n,d} \times \mathcal{P}^\Bbbk_{n,e} \to \mathcal{P}^\Bbbk_{n,d+e}$

It is well known that for any $\mathfrak{gl}_n$-modules M, N, the space $M \otimes_\Bbbk N$ carries a natural $\mathfrak{gl}_n$-module structure. In this section we construct the Schur algebra analogue of this construction. We will then use this product to characterise the projective and injective modules in $\mathcal{P}^\Bbbk_{n,d}$, and give an alternate description of the Schur-Weyl duality functor.

First define the algebra map $\Delta^{d,e} : \mathcal{S}_\Bbbk(n, d + e) \to \mathcal{S}_\Bbbk(n, d) \otimes \mathcal{S}_\Bbbk(n, e)$ by:

$$E_i^{(r)} \mapsto \sum_{k=0}^{r} E_i^{(k)} \otimes E_i^{(r-k)}, \qquad F_i^{(r)} \mapsto \sum_{k=0}^{r} F_i^{(k)} \otimes F_i^{(r-k)},$$
$$1_\lambda \mapsto \sum_{\substack{\mu \in \Lambda(n,d) \\ \nu \in \Lambda(n,e) \\ \mu + \nu = \lambda}} 1_\mu \otimes 1_\nu.$$

We call $\Delta^{d,e}$ the *external comultiplication map* due to the fact that the following diagram commutes:

$$\begin{tikzcd}
S_{\Bbbk}(n,d+e+f) \arrow[r, "\Delta^{d,e+f}"] \arrow[d, "\Delta^{d+e,f}"'] & S_{\Bbbk}(n,d)\otimes S_{\Bbbk}(n,e+f) \arrow[d, "\mathrm{Id}\otimes\Delta^{e,f}"] \\
S_{\Bbbk}(n,d+e)\otimes S_{\Bbbk}(n,f) \arrow[r, "\Delta^{d,e}\otimes\mathrm{Id}"'] & S_{\Bbbk}(n,d)\otimes S_{\Bbbk}(n,e)\otimes S_{\Bbbk}(n,f)
\end{tikzcd}$$

Remark 3.4.1. To see that $\Delta^{d,e}$ is indeed an algebra map use the presentation of $\dot{\mathrm{U}}_{\mathbb{C}}\mathfrak{gl}_n$ to check that $\Delta^{d,e} : \mathcal{S}_{\mathbb{C}}(n,d+e) \to \mathcal{S}_{\mathbb{C}}(n,d) \otimes \mathcal{S}_{\mathbb{C}}(n,e)$ is a $\mathbb{C}$-algebra map. This map sends $\mathcal{S}_{\mathbb{Z}}(n,d+e)$ to $\mathcal{S}_{\mathbb{Z}}(n,d) \otimes \mathcal{S}_{\mathbb{Z}}(n,e)$ and so the result holds for general $\Bbbk$.

Given an $\mathcal{S}_{\Bbbk}(n,d)$-module M and an $\mathcal{S}_{\Bbbk}(n,e)$-module N, the space $M \otimes_{\Bbbk} N$ carries a natural $\mathcal{S}_{\Bbbk}(n,d+e)$-module structure inherited from the external comultiplication map. That is, for $x \in \mathcal{S}_{\Bbbk}(n,d+e)$ and $m \otimes n \in M \otimes N$, if $\Delta(x) = \sum_i x_{i,1} \otimes x_{i,2}$ then

$$x \cdot (m \otimes n) = \sum_i x_{i,1} \cdot m \otimes x_{i,2} \cdot n.$$

This procedure defines the *external product* $- \otimes - : \mathcal{P}^{\Bbbk}_{n,d} \times \mathcal{P}^{\Bbbk}_{n,e} \to \mathcal{P}^{\Bbbk}_{n,d+e}$.

For $\lambda \in \Lambda(n,d)$, define the $\mathcal{S}_{\Bbbk}(n,d)$ modules

$$\Gamma^{\lambda}\Bbbk^n := \Gamma^{\lambda_1}\Bbbk^n \otimes \cdots \otimes \Gamma^{\lambda_n}\Bbbk^n,$$
$$S^{\lambda}\Bbbk^n := S^{\lambda_1}\Bbbk^n \otimes \cdots \otimes S^{\lambda_n}\Bbbk^n,$$
$$\Lambda^{\lambda}\Bbbk^n := \Lambda^{\lambda_1}\Bbbk^n \otimes \cdots \otimes \Lambda^{\lambda_n}\Bbbk^n.$$

For example, $\bigotimes^d \Bbbk^n = \Gamma^{1,\ldots,1}\Bbbk^n = S^{1,\ldots,1}\Bbbk^n = \Lambda^{1,\ldots,1}\Bbbk^n$.

The following result is well-known (see e.g. [AB88, p. 177]).

Proposition 3.4.2. *There is an isomorphism of $S_{\Bbbk}(n,d)$-modules*

$$\mathcal{S}_{\Bbbk}(n,d)1_{\lambda} \to \Gamma^{\lambda}\Bbbk^n$$

mapping 1_{λ} to $v_{\lambda} := v_1^{\otimes\lambda_1} \otimes v_2^{\otimes\lambda_2} \otimes \cdots \otimes v_n^{\otimes\lambda_n}$. In particular the projective objects of $\mathcal{P}^{\Bbbk}_{n,d}$ are isomorphic to direct sums of direct summands of modules of the form $\Gamma^{\lambda}\Bbbk^n$.

A consequence of Proposition 3.4.2 is that for any $\lambda \in \Lambda(n,d)$, and $M \in \mathcal{P}^{\Bbbk}_{n,d}$, there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{P}^{\Bbbk}_{n,d}}(\Gamma^\lambda \Bbbk^n, M) \simeq 1_\lambda M$$

sending a morphism $f : \Gamma^\lambda \Bbbk^n \to M$ to $f(v_\lambda)$. In particular, if $n \geq d$, the Schur-Weyl duality functor,

$$\mathcal{F}_{SW} := \mathrm{Hom}_{\mathcal{P}^{\Bbbk}_{n,d}}(\bigotimes^d \Bbbk^n, -) : \mathcal{P}^{\Bbbk}_{n,d} \to \text{mod-}\Bbbk\mathfrak{S}_d,$$

sends an $\mathcal{S}_{\Bbbk}(n,d)$-module M to it's $(1,\dots,1,0,\dots 0)$-weight space.

For objects $M \in \mathcal{P}^{\Bbbk}_{n,r}$ and $N \in \mathcal{P}^{\Bbbk}_{n,s}$, there is an isomorphism $M^\circ \otimes N^\circ \simeq (M \otimes N)^\circ$ coming from the obvious bilinear form $(M^\circ \otimes N^\circ) \otimes (M \otimes N) \to \Bbbk$. In particular $(\Gamma^\lambda \Bbbk^n)^\circ \simeq S^\lambda \Bbbk^n$. The next proposition follows immediately.

Proposition 3.4.3. *The injective objects of $\mathcal{P}^{\Bbbk}_{n,d}$ are isomorphic to direct sums of direct summands of modules of the form $S^\lambda \Bbbk^n$.*

Define the *dominance order*, $\leq$, on

$$\Lambda^+(n,d) := \{(\lambda_1,\dots,\lambda_n) \in \Lambda(n,d) \mid \lambda_1 \geq \ldots \geq \lambda_n\}$$

by

$$\lambda \leq \mu \text{ if } \lambda_1 + \cdots + \lambda_r \leq \mu_1 + \cdots + \mu_r \text{ for all } r = 1,\dots,n.$$

The decomposition of $\Gamma^\lambda \Bbbk^n$ and $S^\lambda \Bbbk^n$ into indecomposable objects is described by the following proposition. This result is known (see e.g. [Don93, Lemma 3.4]).

Proposition 3.4.4. *For $\lambda \in \Lambda^+(n,d)$, let P_λ (respectively I_λ) be the indecomposable projective cover (respectively injective envelope) of the simple $\mathcal{S}_{\Bbbk}(n,d)$-module, L^n_λ, with highest weight λ. Then*

$$\Gamma^\lambda \Bbbk^n \simeq P_\lambda \oplus \bigoplus_{\mu > \lambda} P_\mu^{\oplus d_{\lambda\mu}},$$
$$S^\lambda \Bbbk^n \simeq I_\lambda \oplus \bigoplus_{\mu > \lambda} I_\mu^{\oplus d_{\lambda\mu}},$$

where $d_{\lambda\mu}$ is the dimension of the λ-weight space in L^n_μ.

Proof. Since $\Gamma^\lambda \Bbbk^n$ is projective, there is a decomposition

$$\Gamma^\lambda \Bbbk^n = \bigoplus_{\mu \in \Lambda^+(n,d)} P_\mu^{m_{\lambda\mu}},$$

for some numbers $m_{\lambda\mu} \in \mathbb{N}$. By Proposition 3.4.2, for partitions λ, μ of d, there is a bijection

$$\mathrm{Hom}_{\mathcal{P}^{\Bbbk}_{n,d}}(\Gamma^\lambda \Bbbk^n, L^n_\mu) \simeq 1_\lambda L^n_\mu.$$

In particular, $m_{\lambda\mu} = \dim 1_\lambda L^n_\mu$. This proves the first isomorphism. The second isomorphism holds by the dual argument. $\square$

For the remainder of this section we derive an alternative description of the Schur-Weyl duality functor (Proposition 3.4.6) that will be needed in the discussion of the homogeneous product of Schur algebra modules.

By Proposition 3.4.2, there are isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathcal{P}^{\Bbbk}_{n,d}}(\Gamma^\lambda \Bbbk^n, M)^* &\simeq (1_\lambda M)^* \\ &\simeq 1_\lambda M^\circ \\ &\simeq \mathrm{Hom}_{\mathcal{P}^{\Bbbk}_{n,d}}(\Gamma^\lambda \Bbbk^n, M^\circ) \\ &\simeq \mathrm{Hom}_{\mathcal{P}^{\Bbbk}_{n,d}}(M, S^\lambda \Bbbk^n) \end{aligned} \tag{3.19}$$

for each $\lambda \in \Lambda(n,d)$ and $M \in \mathcal{P}^{\Bbbk}_{n,d}$.

Remark 3.4.5. The isomorphism $\mathrm{Hom}_{\mathcal{P}^{\Bbbk}_{n,d}}(\Gamma^\lambda \Bbbk^n, M)^* \simeq \mathrm{Hom}_{\mathcal{P}^{\Bbbk}_{n,d}}(M, S^\lambda \Bbbk^n)$ is a special case of Serre duality in $\mathcal{P}^{\Bbbk}_{n,d}$ (see [MS08, Section 4.5] or [Kra13, Proposition 5.4] for a precise description of Serre duality in $\mathcal{P}^{\Bbbk}_{n,d}$).

The following proposition follows immediately from (3.19) in the case $\lambda = (1, \ldots, 1)$.

Proposition 3.4.6. *Let $n \geq d$. The following diagram of functors commutes.*

$$\begin{tikzcd} (\mathcal{P}^{\Bbbk}_{n,d})^{op} \arrow[r, "(-)^\circ"] \arrow[d, "\mathcal{F}_{SW}"'] & \mathcal{P}^{\Bbbk}_{n,d} \arrow[d, "\mathcal{F}_{SW}"] \\ \Bbbk\mathfrak{S}_d\text{-mod} \simeq (\text{mod-}\Bbbk\mathfrak{S}_d)^{op} \arrow[r, "(-)^*"'] & \text{mod-}\Bbbk\mathfrak{S}_d \end{tikzcd}$$

Moreover, there is a natural isomorphism of functors

$$\mathcal{F}_{SW} \simeq \mathrm{Hom}_{\mathcal{P}^{\Bbbk}_{n,d}}(-, \textstyle\bigotimes^d \Bbbk^n)^* : \mathcal{P}^{\Bbbk}_{n,d} \to \text{mod-}\Bbbk\mathfrak{S}_d.$$

3.5 Double cosets of the symmetric group by parabolic subgroups

In this section we define some combinatorial tools that will be needed in our discussion of the homogeneous product of Schur algebra modules.

Every sequence $\lambda \in \Lambda(m,d)$ determines a set partition $[d] = \overline{\lambda}_1 \cup \cdots \cup \overline{\lambda}_m$ in which $\overline{\lambda}_i = \{\lambda_0 + \cdots + \lambda_{i-1}, \ldots, \lambda_0 + \cdots + \lambda_i\}$, where $\lambda_0 = 0$. For each $\lambda \in \Lambda(m,d)$ define the corresponding *parabolic subgroup* of $\mathfrak{S}_d$ by:

$$\mathfrak{S}_\lambda := \mathfrak{S}_{\overline{\lambda}_1} \times \cdots \times \mathfrak{S}_{\overline{\lambda}_m} \subset \mathfrak{S}_d.$$

These are alternatively known as *Young subgroups* of $\mathfrak{S}_d$ in the literature.

Given a second sequence $\mu \in \Lambda(n,d)$, define A^λ_μ to be the set of $m \times n$ matrices, (a_{ij}), with entries in $\mathbb{N}$, with row sum $\lambda_i = \sum_j a_{ij}$ and column sum $\mu_j = \sum_i a_{ij}$. For example,

$$A^{(3,3)}_{(2,2,2)} = \{ \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \end{pmatrix} \}.$$

The following result is due to James and Kerber [JK81, Corollary 1.3.11].

Definition/Theorem 3.5.1. *There is a bijection $\Theta^\lambda_\mu : \mathfrak{S}_\lambda \backslash \mathfrak{S}_d / \mathfrak{S}_\mu \to A^\lambda_\mu$ sending $\mathfrak{S}_\lambda \sigma \mathfrak{S}_\mu$ to the matrix $a_{ij} = |\overline{\lambda}_i \cap \sigma(\overline{\mu}_j)|$.*

Proof. Consider the left action of $\mathfrak{S}_d$ on $\mathfrak{S}_\lambda \backslash \mathfrak{S}_d \times \mathfrak{S}_d / \mathfrak{S}_\mu$ defined:

$$\sigma \cdot (\mathfrak{S}_\lambda g, h\mathfrak{S}_\mu) = (\mathfrak{S}_\lambda g \sigma^{-1}, \sigma h \mathfrak{S}_\mu).$$

Note that the set of $\mathfrak{S}_d$-orbits, $(\mathfrak{S}_\lambda \backslash \mathfrak{S}_d \times \mathfrak{S}_d / \mathfrak{S}_\mu)/\mathfrak{S}_d$, is in bijection with $\mathfrak{S}_\lambda \backslash \mathfrak{S}_d / \mathfrak{S}_\mu$. Specifically the pair $(\mathfrak{S}_\lambda g, h\mathfrak{S}_\mu)$ corresponds to the double coset $\mathfrak{S}_\lambda g h \mathfrak{S}_\mu$.

Consider the map $\theta : \mathfrak{S}_\lambda \backslash \mathfrak{S}_d \times \mathfrak{S}_d / \mathfrak{S}_\mu \to A^\lambda_\mu$ that sends $(\mathfrak{S}_\lambda g, h\mathfrak{S}_\mu)$ to the matrix $a_{ij} = |g^{-1}(\overline{\lambda}_i) \cap h(\overline{\mu}_j)|$. This map is clearly well-defined and $\theta(x) = \theta(y)$ if and only if x and y are in the same $\mathfrak{S}_d$-orbit. The result follows. $\square$

Remark 3.5.2. A survey of results about the matrices in A_μ^λ can be found in [DiG95] where these matrices are called rectangular arrays with fixed margins. In statistics these matrices are alternatively called contingency tables with fixed margins or fixed-margin matrices.

We often regard the matrix $A \in A_\mu^\lambda$ as a sequence $(A_{11}, \ldots, A_{1n}, A_{21}, \ldots, A_{mn})$. In particular we define a corresponding set partition

$$[d] = \overline{A}_{11} \cup \ldots \cup \overline{A}_{1n} \cup \overline{A}_{21} \cup \ldots \cup \overline{A}_{mn}.$$

Note that for $A \in A_\mu^\lambda$,

$$\overline{\lambda}_i = \bigcup_j \overline{A}_{ij} \qquad \text{and} \qquad \overline{\mu}_j = \bigcup_i \overline{A^t}_{ji},$$

where A^t is the transpose of A.

Definition 3.5.3. Say that a permutation $\sigma \in \mathfrak{S}_d$ *twists* $A \in A_\mu^\lambda$ if

$$\sigma(\overline{A^t}_{ji}) = \overline{A}_{ij}.$$

For example, if $A = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ then

$$\overline{A}_{11} = \emptyset, \qquad \overline{A}_{12} = \{1, 2\}, \qquad \overline{A}_{21} = \{3, 4\}, \qquad \overline{A}_{22} = \emptyset,$$

and the permutation $(13)(24)$ twists A.

3.6 Homogeneous product of Schur algebra modules

Recall that if M, N are right $\Bbbk\mathfrak{S}_d$-modules then $M \otimes_\Bbbk N$ is also a right $\Bbbk\mathfrak{S}_d$-module with diagonal action $(m \otimes n) \cdot \sigma = m \cdot \sigma \otimes n \cdot \sigma$. In this section we define a product $-\underline{\otimes}- : \mathcal{P}_d^\Bbbk \times \mathcal{P}_d^\Bbbk \to \mathcal{P}_d^\Bbbk$ (right exact in each variable) that makes the following diagram commute:

$$\begin{tikzcd} \mathcal{P}_d^\Bbbk \times \mathcal{P}_d^\Bbbk \arrow[r, "-\underline{\otimes}-"] \arrow[d, "\mathcal{F}_{SW} \times \mathcal{F}_{SW}"'] & \mathcal{P}_d^\Bbbk \arrow[d, "\mathcal{F}_{SW}"] \\ \text{mod-}\Bbbk\mathfrak{S}_d \times \text{mod-}\Bbbk\mathfrak{S}_d \arrow[r, "-\otimes_\Bbbk-"] & \text{mod-}\Bbbk\mathfrak{S}_d \end{tikzcd}$$

Our definition of $-\underline{\otimes}-$ is original. To state it we need some preliminary definitions. We identify the weight lattice of $\mathfrak{gl}_{mn}$ with the set of $m \times n$ matrices with entries in $\mathbb{Z}$. The algebra $\mathcal{S}_{\Bbbk}(mn, d)$ has generating idempotents, 1_ν, indexed by the set, $\Lambda(m \times n, d)$, of $m \times n$ matrices with entries in $\mathbb{N}$, whose entries sum to d. Denote the standard basis of $\Bbbk^{mn}$ by

$$\{v_{ij} \mid i \in [m], j \in [n]\},$$

and for a matrix $\nu \in \Lambda(m \times n, d)$ define

$$v_\nu = v_{11}^{\otimes \nu_{11}} \otimes v_{12}^{\otimes \nu_{12}} \otimes \cdots \otimes v_{1n}^{\otimes \nu_{1n}} \otimes v_{21}^{\otimes \nu_{21}} \otimes \cdots \otimes v_{mn}^{\otimes \nu_{mn}}.$$

Consider the $\Bbbk\mathfrak{S}_d$-equivariant isomorphism

$$\theta : \bigotimes^d \Bbbk^m \otimes_{\Bbbk} \bigotimes^d \Bbbk^n \to \bigotimes^d \Bbbk^{mn}$$

defined

$$v_{i_1} \otimes \cdots \otimes v_{i_d} \otimes v_{j_1} \otimes \cdots \otimes v_{j_d} \mapsto v_{i_1 j_1} \otimes \cdots \otimes v_{i_d j_d}.$$

Define an algebra embedding $\Theta : S_{\Bbbk}(m, d) \otimes_{\Bbbk} S_{\Bbbk}(n, d) \hookrightarrow S_{\Bbbk}(mn, d)$ by the composition:

$$\mathrm{End}_{\Bbbk\mathfrak{S}_d}(\bigotimes^d \Bbbk^m) \otimes \mathrm{End}_{\Bbbk\mathfrak{S}_d}(\bigotimes^d \Bbbk^n) \hookrightarrow \mathrm{End}_{\Bbbk\mathfrak{S}_d}(\bigotimes^d \Bbbk^m \otimes \bigotimes^d \Bbbk^n) \xrightarrow{\sim} \mathrm{End}_{\Bbbk\mathfrak{S}_d}(\bigotimes^d \Bbbk^{mn}).$$

Here the first map applies the $\otimes$-product to two morphisms, and the second map is defined via the isomorphism θ. That is, if $f \in \mathrm{End}_{\Bbbk\mathfrak{S}_d}(\bigotimes^d \Bbbk^m)$ and $g \in \mathrm{End}_{\Bbbk\mathfrak{S}_d}(\bigotimes^d \Bbbk^n)$, then the endomorphism $\Theta(f \otimes g) \in \mathrm{End}_{\Bbbk\mathfrak{S}_d}(\bigotimes^d \Bbbk^{mn})$ is defined:

$$\begin{aligned}\Theta(f \otimes g)(v_{i_1 j_1} \otimes \cdots \otimes v_{i_d j_d}) &= \theta(f \otimes g)\theta^{-1}(v_{i_1 j_1} \otimes \cdots \otimes v_{i_d j_d}) \\ &= \theta(f(v_{i_1} \otimes \cdots \otimes v_{i_d}) \otimes g(v_{j_1} \otimes \cdots \otimes v_{j_d})).\end{aligned}$$

In particular, if $\nu \in A_\mu^\lambda$ then for $f \in \mathrm{End}_{\Bbbk\mathfrak{S}_d}(\bigotimes^d \Bbbk^m)$ and $g \in \mathrm{End}_{\Bbbk\mathfrak{S}_d}(\bigotimes^d \Bbbk^n)$:

$$\Theta(f \otimes g)(v_\nu) = \theta(f(v_\lambda) \otimes g(v_\mu \cdot \sigma)), \tag{3.20}$$

where $\sigma \in \mathfrak{S}_d$ twists ν (in the sense of Definition 3.5.3). A description of Θ in terms of the generators of the Schur algebra is given in Proposition 3.6.4.

From the embedding $\Theta : S_{\Bbbk}(m,d) \otimes_{\Bbbk} S_{\Bbbk}(n,d) \hookrightarrow S_{\Bbbk}(mn,d)$ we define the *homogeneous external product* $-\underline{\boxtimes}- : \mathcal{P}^{\Bbbk}_{m,d} \times \mathcal{P}^{\Bbbk}_{n,d} \to \mathcal{P}^{\Bbbk}_{mn,d}$ by:

$$M \underline{\boxtimes} N := \mathcal{S}_{\Bbbk}(mn,d) \otimes_{\mathcal{S}_{\Bbbk}(m,d) \otimes \mathcal{S}_{\Bbbk}(n,d)} (M \otimes_{\Bbbk} N)$$

By passing to the colimit $\mathcal{P}^{\Bbbk}_d \simeq \lim_{n \to \infty} \mathcal{P}^{\Bbbk}_{n,d}$ we obtain the product $-\underline{\otimes}- : \mathcal{P}^{\Bbbk}_d \times \mathcal{P}^{\Bbbk}_d \to \mathcal{P}^{\Bbbk}_d$.

Theorem 3.6.1. *The following diagram commutes:*

$$\begin{tikzcd}
\mathcal{P}^{\Bbbk}_d \times \mathcal{P}^{\Bbbk}_d \arrow[r, "-\underline{\otimes}-"] \arrow[d, "\mathcal{F}_{SW} \times \mathcal{F}_{SW}"'] & \mathcal{P}^{\Bbbk}_d \arrow[d, "\mathcal{F}_{SW}"] \\
\text{mod-}\Bbbk\mathfrak{S}_d \times \text{mod-}\Bbbk\mathfrak{S}_d \arrow[r, "-\otimes_{\Bbbk}-"] & \text{mod-}\Bbbk\mathfrak{S}_d
\end{tikzcd}$$

To show this, we begin with a lemma.

Lemma 3.6.2. *There is an isomorphism of $\mathcal{S}_{\Bbbk}(mn,d)$-modules:*

$$\Gamma^{\lambda}\Bbbk^m \underline{\boxtimes} \Gamma^{\mu}\Bbbk^n \simeq \bigoplus_{\nu \in A^{\lambda}_{\mu}} \Gamma^{\nu}\Bbbk^{mn}.$$

Proof. The idempotent $1_{\lambda} \in \operatorname{End}_{\Bbbk\mathfrak{S}_d}(\bigotimes^d \Bbbk^m)$ is the projection onto the λ-weight space. By equation (3.20),

$$\Theta(1_{\lambda} \otimes 1_{\mu}) = \sum_{\nu \in A^{\lambda}_{\mu}} 1_{\nu}.$$

Hence:

$$\begin{aligned}
\mathcal{S}_{\Bbbk}(m,d)1_{\lambda} \underline{\boxtimes} \mathcal{S}_{\Bbbk}(n,d)1_{\mu} &\simeq \mathcal{S}_{\Bbbk}(mn,d) \otimes_{\mathcal{S}_{\Bbbk}(m,d) \otimes \mathcal{S}_{\Bbbk}(n,d)} \mathcal{S}_{\Bbbk}(m,d)1_{\lambda} \otimes \mathcal{S}_{\Bbbk}(n,d)1_{\mu} \\
&\simeq \mathcal{S}_{\Bbbk}(mn,d) \otimes_{\mathcal{S}_{\Bbbk}(m,d) \otimes \mathcal{S}_{\Bbbk}(n,d)} 1_{\lambda} \otimes 1_{\mu} \\
&\simeq \mathcal{S}_{\Bbbk}(mn,d) \cdot \sum_{\nu \in A^{\lambda}_{\mu}} 1_{\nu} \\
&\simeq \bigoplus_{\nu \in A^{\lambda}_{\mu}} \mathcal{S}_{\Bbbk}(mn,d)1_{\nu}.
\end{aligned}$$

The result follows. $\square$

Lemma 3.6.3. *The map* $\theta : \bigotimes^d \Bbbk^m \otimes \bigotimes^d \Bbbk^n \to \bigotimes^d \Bbbk^{mn}$ *restricts to an isomorphism of* $\Bbbk\mathfrak{S}_d$*-modules:*

$$\Bbbk[v_\lambda \cdot \mathfrak{S}_d] \otimes_{\Bbbk} \Bbbk[v_\mu \cdot \mathfrak{S}_d] \simeq \bigoplus_{\nu \in A^\lambda_\mu} \Bbbk[v_\nu \cdot \mathfrak{S}_d].$$

Proof. For $g, h \in \mathfrak{S}_d$, two vectors $v_\lambda \otimes (v_\mu \cdot g)$ and $v_\lambda \otimes (v_\mu \cdot h)$ in $\Bbbk[v_\lambda \cdot \mathfrak{S}_d] \otimes_{\Bbbk} \Bbbk[w_\mu \cdot \mathfrak{S}_d]$ are in the same $\mathfrak{S}_d$ orbit if and only if $\Theta^\lambda_\mu(g) = \Theta^\lambda_\mu(h)$. Moreover, for $\nu \in A^\lambda_\mu$, if $\sigma_\nu \in \mathfrak{S}_d$ twists ν then $\Theta^\lambda_\mu(\sigma_\nu) = \nu$ and so:

$$\Bbbk[v_\lambda \cdot \mathfrak{S}_d] \otimes \Bbbk[v_\mu \cdot \mathfrak{S}_d] \simeq \bigoplus_{\nu \in A^\lambda_\mu} \Bbbk[(v_\lambda \otimes (v_\mu \cdot \sigma_\nu)) \cdot \mathfrak{S}_d].$$

The result follows by the equation

$$\theta(v_\lambda \otimes (v_\mu \cdot \sigma_\nu)) = v_\nu.$$

□

Proof of Theorem 3.6.1. Any object, $X \in \mathcal{P}^{\Bbbk}_d$, can be expressed as the cokernel of a map

$$P^{-1}_X \to P^0_X \to X \to 0$$

in which P^{-1}_X, P^0_X are projective. As $\underline{\otimes}$ is right exact in each variable, for objects $X, Y \in \mathcal{P}^{\Bbbk}_d$,

$$X \underline{\otimes} Y = \operatorname{cok}(X \underline{\otimes} P^{-1}_Y \to X \underline{\otimes} P^0_Y)$$

and

$$X \underline{\otimes} P^i_Y = \operatorname{cok}(P^{-1}_X \underline{\otimes} P^i_Y \to P^0_X \underline{\otimes} P^i_Y).$$

Hence it suffices to show that the diagram

$$\begin{tikzcd} \operatorname{Proj}(\mathcal{P}^{\Bbbk}_d) \times \operatorname{Proj}(\mathcal{P}^{\Bbbk}_d) \arrow[r, "-\underline{\otimes}-"] \arrow[d, "\mathcal{F}_{SW} \times \mathcal{F}_{SW}"'] & \operatorname{Proj}(\mathcal{P}^{\Bbbk}_d) \arrow[d, "\mathcal{F}_{SW}"] \\ \text{mod-}\Bbbk\mathfrak{S}_d \times \text{mod-}\Bbbk\mathfrak{S}_d \arrow[r, "-\otimes_{\Bbbk}-"] & \text{mod-}\Bbbk\mathfrak{S}_d \end{tikzcd} \tag{3.21}$$

commutes, where $\operatorname{Proj}(\mathcal{P}^{\Bbbk}_d) \hookrightarrow \mathcal{P}^{\Bbbk}_d$ is the full subcategory of projective objects.

By Proposition 3.4.6, there are natural isomorphisms of right $\Bbbk\mathfrak{S}_d$-modules

$$\mathcal{F}_{SW}(\Gamma^\lambda \Bbbk^m) \simeq (\bigotimes^d \Bbbk^m)^*_\lambda \simeq \Bbbk[v_\lambda \cdot \mathfrak{S}_d]$$

The commutativity of diagram (3.21) follows from Lemmas 3.6.2 and 3.6.3. Indeed, commutativity on objects follows directly from these lemmas. To see that the functors agree on morphisms, observe that under the isomorphism

$$\mathrm{Hom}_{\mathcal{P}^{\Bbbk}_{n,d}}(\Gamma^{\lambda}\Bbbk^n, \Gamma^{\mu}\Bbbk^n) \simeq 1_{\lambda}\mathcal{S}_{\Bbbk}(n,d)1_{\mu},$$

the product map

$$-\underline{\boxtimes}- : \mathrm{Hom}(\Gamma^{\lambda}\Bbbk^m, \Gamma^{\lambda'}\Bbbk^m)\otimes\mathrm{Hom}(\Gamma^{\mu}\Bbbk^n, \Gamma^{\mu'}\Bbbk^n) \to \mathrm{Hom}(\bigoplus_{\nu\in A^{\lambda}_{\mu}}\Gamma^{\nu}\Bbbk^{mn}, \bigoplus_{\nu'\in A^{\lambda'}_{\mu'}}\Gamma^{\nu'}\Bbbk^{mn})$$

corresponds to the map

$$\Theta : 1_{\lambda}\mathcal{S}_{\Bbbk}(m,d)1_{\lambda'} \otimes 1_{\mu}\mathcal{S}_{\Bbbk}(n,d)1_{\mu'} \to \sum_{\nu\in A^{\lambda}_{\mu}} 1_{\nu}\cdot\mathcal{S}_{\Bbbk}(mn,d)\cdot\sum_{\nu'\in A^{\lambda'}_{\mu'}} 1_{\nu'}.$$

By definition,

$$\Theta \simeq \theta\circ(-\otimes_{\Bbbk}-)\circ\theta^{-1} : \mathrm{End}_{\Bbbk\mathfrak{S}_d}(\bigotimes^d \Bbbk^m)\otimes\mathrm{End}_{\Bbbk\mathfrak{S}_d}(\bigotimes^d \Bbbk^n) \to \mathrm{End}_{\Bbbk\mathfrak{S}_d}(\bigotimes^d \Bbbk^{mn}).$$

The result follows from Lemma 3.6.3.

□

For the remainder of this section we define the map Θ in terms of the generators of the Schur algebras. For this we need to introduce some convenient notation. For $i\in[m], j\in[n], k\in[m-1], l\in[n-1]$ define the $m\times n$ matrices

$$\alpha_{il} := e_{il} - e_{i,l+1}, \qquad \alpha_{kn} := e_{kn} - e_{k+1,1},$$
$$\check{\alpha}_{kj} := e_{kj} - e_{k+1,j}, \qquad \check{\alpha}_{ml} := e_{ml} - e_{1,l+1}.$$

For example, if $m=n=2$ then

$$\alpha_{11} = \begin{pmatrix}1 & -1\\ 0 & 0\end{pmatrix}, \qquad \alpha_{12} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \qquad \alpha_{21} = \begin{pmatrix}0 & 0\\ 1 & -1\end{pmatrix},$$
$$\check{\alpha}_{11} = \begin{pmatrix}1 & 0\\ -1 & 0\end{pmatrix}, \qquad \check{\alpha}_{21} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \qquad \check{\alpha}_{12} = \begin{pmatrix}0 & 1\\ 0 & -1\end{pmatrix}.$$

The sets $\{\alpha_{ij}\}$ and $\{\check{\alpha}_{ij}\}$ are two different sets of simple roots of $\mathfrak{gl}_{mn}$ and so correspond to two different sets of Chevalley generators of $\mathfrak{gl}_{mn}$. Thus the Schur algebra $S_{\Bbbk}(mn, d)$ is generated by idempotents 1_λ for $\lambda \in \Lambda(m \times n, d)$, and either one of the following sets

1. Elements $E_{ij}^{(r)}$ and $F_{ij}^{(r)}$ satisfying e.g.
$$E_{ij}^{(r)} 1_\lambda = 1_{\lambda + r\alpha_{ij}} E_{ij}^{(r)} \qquad \text{and} \qquad F_{ij}^{(r)} 1_\lambda = 1_{\lambda - r\alpha_{ij}} F_{ij}^{(r)}.$$
2. Elements $\check{E}_{ij}^{(r)}$ and $\check{F}_{ij}^{(r)}$ satisfying e.g.
$$\check{E}_{ij}^{(r)} 1_\lambda = 1_{\lambda + r\check{\alpha}_{ij}} \check{E}_{ij}^{(r)} \qquad \text{and} \qquad \check{F}_{ij}^{(r)} 1_\lambda = 1_{\lambda - r\check{\alpha}_{ij}} \check{F}_{ij}^{(r)}.$$

For $\lambda \in \Lambda(m, d)$ and $\mu \in \Lambda(n, d)$ define sets
$$row(\lambda) := \{\nu \in \Lambda(m \times n, d) \mid \lambda_i = \sum_{j=1}^{n} \nu_{ij} \text{ for all } i \in [m]\},$$
$$col(\mu) := \{\nu \in \Lambda(m \times n, d) \mid \mu_j = \sum_{i=1}^{m} \nu_{ij} \text{ for all } j \in [n]\}.$$

For a set X and $r \in \mathbb{N}$ define the set of *multisubsets of X of size r* by
$$\left(\!\!\binom{X}{r}\!\!\right) := \left\{ f : X \to \mathbb{N} \mid \sum_{x \in X} f(x) = r \right\}.$$

For any $K \in \left(\!\binom{[m]}{r}\!\right)$ and $K' \in \left(\!\binom{[n]}{r}\!\right)$ define the following elements of $S_{\Bbbk}(mn, d)$:
$$E_j^{(K)} := E_{1j}^{(K(1))} \cdots E_{mj}^{(K(m))}, \qquad \check{E}_i^{(K')} := \check{E}_{i1}^{(K'(1))} \cdots \check{E}_{in}^{(K'(n))}$$
where $i \in [m]$ and $j \in [n]$. Define $F_j^{(K)}$ and $\check{F}_i^{(K')}$ similarly.

The following result is new.

Proposition 3.6.4. *The map $\Theta : S_{\Bbbk}(m, d) \otimes_{\Bbbk} S_{\mathbb{Z}}(n, d) \hookrightarrow S_{\Bbbk}(mn, d)$ is defined on generators by*
$$1_\lambda \otimes 1 \mapsto \sum_{\nu \in row(\lambda)} 1_\nu, \qquad 1 \otimes 1_\mu \mapsto \sum_{\nu \in col(\mu)} 1_\nu, \tag{3.22}$$
$$E_i^{(r)} \otimes 1 \mapsto \sum_{K \in \left(\!\binom{[n]}{r}\!\right)} \check{E}_i^{(K)}, \qquad 1 \otimes E_j^{(r)} \mapsto \sum_{K \in \left(\!\binom{[m]}{r}\!\right)} E_j^{(K)}, \tag{3.23}$$
$$F_i^{(r)} \otimes 1 \mapsto \sum_{K \in \left(\!\binom{[n]}{r}\!\right)} \check{F}_i^{(K)}, \qquad 1 \otimes F_j^{(r)} \mapsto \sum_{K \in \left(\!\binom{[m]}{r}\!\right)} F_j^{(K)}. \tag{3.24}$$

Proof. Equations (3.22) follow from (3.20). To complete this proof we just show the second equation of (3.23). The other equations follow similarly.

Given a subset $R \subset [d]$, an integer $j \in [n]$, and vector $v = v_{j_1} \otimes \cdots \otimes v_{j_d} \in \bigotimes^d \Bbbk^n$, we define $c_{j,R} \cdot v \in \bigotimes^d \Bbbk^n$ to be the vector obtained from v by replacing, for each $x \in R$, the tensor factor v_{j_x} with v_j. For example

$$c_{1,\{2,3\}} \cdot v_1 \otimes v_2 \otimes v_2 = v_1 \otimes v_1 \otimes v_1.$$

For a subset $R \subset [d]$, a pair $(i,j) \in [m] \times [n]$ and a pure tensor $v = v_{i_1 j_1} \otimes \cdots \otimes v_{i_d j_d} \in \bigotimes^d \Bbbk^{mn}$, we define $c_{ij,R} \cdot v$ likewise. Moreover we define $c_{*j,R} \cdot v$ to be the vector obtained from v by replacing, for each $x \in R$, the tensor factor $v_{i_x j_x}$ with $v_{i_x j}$. Define $c_{i*,R} \cdot v$ similarly.

For a pure tensor $v = v_{j_1} \otimes \cdots \otimes v_{j_d} \in \bigotimes^d \Bbbk^n$, and $j \in [n]$, let

$$\bar{v}_j := \{x \in [d] \mid j_x = j\}.$$

For a pure tensor $v \in \bigotimes^d \Bbbk^{mn}$ and pair $(i,j) \in [m] \times [n]$ we define $\bar{v}_{ij}$ likewise. Moreover define

$$\bar{v}_{*j} := \{x \in [d] \mid j_x = j\} \qquad \text{and} \qquad \bar{v}_{i*} := \{x \in [d] \mid i_x = i\}.$$

For $r \in \mathbb{N}$, the endomorphism $E_j^{(r)} \in \mathrm{End}_{\Bbbk \mathfrak{S}_d}(\bigotimes^d \Bbbk^n)$ is defined on a pure tensor v by:

$$E_j^{(r)}(v) = \sum_{R \in \binom{\bar{v}_{j+1}}{r}} c_{j,R} \cdot v,$$

where $\binom{\bar{v}_{j+1}}{r}$ is the set of subsets of $\bar{v}_{j+1}$ of size r. That is, $E_j^{(r)}(v)$ sums over all the ways to replace r tensor factors of the form v_{j+1} with v_j. Likewise if $j \neq n$ then:

$$E_{ij}^{(r)}(v) = \sum_{R \in \binom{\bar{v}_{i,j+1}}{r}} c_{ij,R} \cdot v \qquad \text{and} \qquad E_{in}^{(r)}(v) = \sum_{R \in \binom{\bar{v}_{i+1,1}}{r}} c_{ij,R} \cdot v.$$

For $v = v_{i_1 j_1} \otimes \cdots \otimes v_{i_d j_d}$ we have:

$$\begin{aligned} \Theta(1 \otimes E_j^{(r)})(v) &= \theta(v_{i_1} \otimes \cdots \otimes v_{i_d} \otimes E_j^{(r)}(v_{j_1} \otimes \cdots \otimes v_{j_d})) \\ &= \sum_{R \in \binom{\bar{v}_{(*,j+1)}}{r}} c_{*j,R} \cdot v \end{aligned}$$

That is, $\Theta(1 \otimes E_j^{(r)})(v)$ sums over all the ways to replace r tensor factors of the form $v_{i,j+1}$ (for some $i \in [m]$) with v_{ij}. Compare this with $E_j^{(K)}(v)$, where $K \in \left(\!\!\binom{[m]}{r}\!\!\right)$: This vector sums over all the ways to replace, for each $i \in [m]$, $K(i)$ tensor factors of the form $v_{i,j+1}$ with v_{ij}. It follows that

$$\Theta(1 \otimes E_j^{(r)}) = \sum_{K \in \left(\!\binom{[m]}{r}\!\right)} E_j^{(K)}.$$

□

Remark 3.6.5. Consider the affine Grassmannian

$$\mathcal{G}r_{\mathrm{GL}_n} := \mathrm{GL}_n(\mathbb{C}(\!(t)\!))/\,\mathrm{GL}_n(\mathbb{C}[\![t]\!]),$$

where $\mathbb{C}(\!(t)\!)$ is the field of formal Laurent series and $\mathbb{C}[\![t]\!]$ is the ring of formal power series (with coefficients in $\mathbb{C}$). Consider the left multiplication action of $\mathrm{GL}_n(\mathbb{C}[\![t]\!])$ on $\mathcal{G}r_{\mathrm{GL}_n}$. For $\lambda \in \mathbb{Z}^n$ in which $\lambda_1 \geq \cdots \geq \lambda_n$, let $\mathcal{G}r^\lambda$ be the $\mathrm{GL}_n(\mathbb{C}[\![t]\!])$-orbit of the point

$$\begin{pmatrix} t^{\lambda_1} & & & \\ & t^{\lambda_2} & & \\ & & \ddots & \\ & & & t^{\lambda_n} \end{pmatrix} \mathrm{GL}_n(\mathbb{C}[\![t]\!]) \in \mathcal{G}r_{\mathrm{GL}_n}.$$

The affine Grassmannian has a stratification by $\mathrm{GL}_n(\mathbb{C}[\![t]\!])$-orbits, with the closure order equal to the dominance order. In particular, consider the closed subspace

$$\mathcal{G}r_{n,d} := \overline{\mathcal{G}r^{(d,0,\ldots,0)}} = \bigcup_{\lambda \in \Lambda^+(n,d)} \mathcal{G}^\lambda \subset \mathcal{G}r_{\mathrm{GL}_n}.$$

By Mirković-Vilonen's geometric Satake correspondence [MV07, Theorem 14.1], there is an equivalence of categories between $\mathcal{P}_{n,d}^{\Bbbk}$ and the category $P_{\mathrm{GL}_n(\mathbb{C}[\![t]\!])}(\mathcal{G}r_{n,d}, \Bbbk)$ of $\mathrm{GL}_n(\mathbb{C}[\![t]\!])$-equivariant perverse sheaves on $\mathcal{G}r_{n,d}$. A natural question to ask is: Is there a geometric definition of the product

$$- \boxtimes - : P_{\mathrm{GL}_m(\mathbb{C}[\![t]\!])}(\mathcal{G}r_{m,d}, \Bbbk) \times P_{\mathrm{GL}_n(\mathbb{C}[\![t]\!])}(\mathcal{G}r_{n,d}, \Bbbk) \to P_{\mathrm{GL}_{m+n}(\mathbb{C}[\![t]\!])}(\mathcal{G}r_{m+n,d}, \Bbbk)$$

that corresponds to the homogeneous external product under the geometric Satake correspondence.

3.7 Strict polynomial functors

There is an equivalence of categories between $\mathcal{P}_d^{\Bbbk}$ and the category of strict polynomial functors of degree d [FS97, Theorem 3.2]. In this section we recall the definition of such functors and show that, under this equivalence, the homogeneous product of Schur algebra modules corresponds to Krause's product of strict polynomial functors defined in [Kra13].

Remark 3.7.1. Strict polynomial functors were defined by Friedlander and Suslin [FS97] in order to show that the cohomology of a finite group scheme over a field $\Bbbk$ is a finitely generated $\Bbbk$-algebra. An excellent survey of this and related topics is given in [Tou14]. We use the definition and notation for strict polynomial functor given in [Kra13]. This is easily shown to be equivalent to the definition of Friedlander-Suslin.

Let $\Gamma_d^{\Bbbk}$ be the category whose objects are $\Bbbk$-vector spaces and with morphisms:

$$\begin{aligned}\operatorname{Hom}_{\Gamma_d^{\Bbbk}}(V,W) &:= \Gamma^d \operatorname{Hom}_{\Bbbk}(V,W) \cong \operatorname{Hom}_{\Bbbk}(V^{\otimes d}, W^{\otimes d})^{\mathfrak{S}_d} \\ &\cong \operatorname{Hom}_{\Bbbk\mathfrak{S}_d}(V^{\otimes d}, W^{\otimes d}),\end{aligned}$$

where $\mathfrak{S}_d$ acts to the right of $\operatorname{Hom}_{\Bbbk}(V^{\otimes d}, W^{\otimes d})$ via the action $(f\sigma)(v) = f(v\sigma^{-1})\sigma$. Define the category, $\operatorname{Rep}\Gamma_d^{\Bbbk}$, of *strict polynomial functors* of degree d to be the category of $\Bbbk$-linear functors from $\Gamma_d^{\Bbbk}$ to mod-$\Bbbk$.

Remark 3.7.2. Note that $\operatorname{Hom}_{\Bbbk}(\Gamma^d V, W) = (\Gamma^d V)^* \otimes W = S^d V^* \otimes W$ is the space of regular maps from V to W. So polynomial functors are equivalent to functors $X :$ mod-$\Bbbk \to$ mod-$\Bbbk$ in which for every pair of objects $V, W \in$ mod-$\Bbbk$, the map $X : \operatorname{Hom}_{\Bbbk}(V,W) \to \operatorname{Hom}_{\Bbbk}(X(V), X(W))$ is a polynomial map of degree d. This is the original definition of polynomial functor in [FS97].

Important examples of strict polynomial functors include the divided power functor $\Gamma^d(-)$, the symmetric power functor $S^d(-)$, the exterior power functor $\Lambda^d(-)$, and the tensor power functor $\bigotimes^d(-)$. For $V \in$ mod-$\Bbbk$, we denote the corresponding representable functor in $\operatorname{Rep}\Gamma_d^{\Bbbk}$ by $\Gamma^{d,V}$ i.e.

$$\Gamma^{d,V}(W) = \Gamma^d \operatorname{Hom}_{\Bbbk}(V,W).$$

There is an equivalence of categories $(-)^\circ : (\operatorname{Rep}\Gamma_d^{\Bbbk})^{op} \to \operatorname{Rep}\Gamma_d^{\Bbbk}$ defined on objects by $X^\circ(V) = X(V^*)^*$, where $(-)^*$ refers to the linear dual. For example $(\Gamma^d)^\circ = S^d$ and $(\Lambda^d)^\circ = (\Lambda^d)^\circ$. The functor $(\Gamma^{d,V})^\circ$ is isomorphic to the functor $S^{d,V} \in \operatorname{Rep}\Gamma_d^{\Bbbk}$ defined on objects by

$$S^{d,V}(W) = S^d(V \otimes W).$$

Define the *external product functors*

$$- \otimes - : \operatorname{Rep}\Gamma_d^{\Bbbk} \times \operatorname{Rep}\Gamma_e^{\Bbbk} \to \operatorname{Rep}\Gamma_{d+e}^{\Bbbk}$$

by $(X \otimes Y)(V) = X(V) \otimes_{\Bbbk} Y(V)$. Given $\lambda \in \Lambda(n,d)$, define the degree-d strict polynomial functors $\Gamma^\lambda := \Gamma^{\lambda_1} \otimes \cdots \otimes \Gamma^{\lambda_n}$, $S^\lambda := S^{\lambda_1} \otimes \cdots \otimes S^{\lambda_n}$, and $\Lambda^\lambda := \Lambda^{\lambda_1} \otimes \cdots \otimes \Lambda^{\lambda_n}$. For example, $\bigotimes^d = \Gamma^{(1,\ldots,1)} = S^{(1,\ldots,1)} = \Lambda^{(1,\ldots,1)}$.

By the Yoneda lemma, the Schur algebra $\mathcal{S}_{\Bbbk}(n,d) = \Gamma^d \operatorname{End}_{\Bbbk}(\Bbbk^n)$ is isomorphic to the algebra $\operatorname{End}_{\operatorname{Rep}\Gamma_d^{\Bbbk}}(\Gamma^{d,\Bbbk^n})^{op}$. Hence, for any object $X \in \operatorname{Rep}\Gamma_d^{\Bbbk}$ there is a right action of $\mathcal{S}_{\Bbbk}(n,d)^{op}$ on $X(\Bbbk^n) \simeq \operatorname{Hom}_{\mathcal{P}_d}(\Gamma^{d,\Bbbk^n}, X)$ given by precomposition. For $n \geq d$ we have equivalences of categories

$$\operatorname{eval}_{\Bbbk^n}(-) := \operatorname{Hom}_{\operatorname{Rep}\Gamma_d^{\Bbbk}}(\Gamma^{d,\Bbbk^n}, -) : \operatorname{Rep}\Gamma_d^{\Bbbk} \to \text{mod-}\mathcal{S}_{\Bbbk}(n,d)^{op} \simeq \mathcal{P}_{n,d}^{\Bbbk} \tag{3.25}$$

[FS97, Lemma 3.4] (this is also proved using Krause's notation in [Kra13, Theorem 2.10]). It follows that that there is an equivalence $\operatorname{Rep}\Gamma_d^{\Bbbk} \simeq \mathcal{P}_d^{\Bbbk}$. This equivalence commutes with the external products labelled $\otimes$ and the contravariant autoequivalences labelled $(-)^\circ$.

For the remainder of this section we define Krauses internal product $-\underline{\otimes}- :$ $\operatorname{Rep}\Gamma_d^{\Bbbk} \times \operatorname{Rep}\Gamma_d^{\Bbbk} \to \operatorname{Rep}\Gamma_d^{\Bbbk}$, and relate this product to the homogeneous product of Schur algebra modules under the equivalence $\operatorname{Rep}\Gamma_d^{\Bbbk} \simeq \mathcal{P}_d^{\Bbbk}$.

To define Krause's product, first note that the usual tensor product on vector spaces defines an exact tensor product $- \otimes_{\Bbbk} - : \Gamma_d^{\Bbbk} \times \Gamma_d^{\Bbbk} \to \Gamma_d^{\Bbbk}$. By the Yoneda embedding $(\Gamma_d^{\Bbbk})^{op} \hookrightarrow \operatorname{Rep}\Gamma_d^{\Bbbk} : V \mapsto \Gamma^{d,V}$ we can define a tensor product on representable functors in $\operatorname{Rep}\Gamma_d^{\Bbbk}$ by:

$$\Gamma^{d,V} \underline{\otimes} \Gamma^{d,W} := \Gamma^{d,V \otimes W}.$$

Krause's internal product extends this product on representable functors to a right exact functor on the whole of $\operatorname{Rep}\Gamma_d^{\Bbbk}$. To define this extension, first note that every object in $\operatorname{Rep}\Gamma_d^{\Bbbk}$ is a colimit of representable functors. Indeed, let $\mathcal{C}_X$ be the category whose objects are natural transformations $\Gamma^{d,V} \to X$ for $V \in \text{mod-}\Bbbk$, and with morphisms from an object $x_v : \Gamma^{d,V} \to X$ to an object $x_w : \Gamma^{d,W} \to X$ being the natural transformations $\phi : \Gamma^{d,V} \to \Gamma^{d,W}$ in which $x_w \circ \phi = x_v$. Then X is the colimit of the diagram functor $\mathcal{F}_X : \mathcal{C}_X \to \operatorname{Rep}\Gamma_d^{\Bbbk}$ that sends an object $\Gamma^{d,V} \to X$ to it's domain $\Gamma^{d,V}$. Extend the definition of $\underline{\otimes}$ to non-representable functors in $\operatorname{Rep}\Gamma_d^{\Bbbk}$ by defining

$$
\begin{aligned}
X\underline{\otimes}\Gamma^{d,W} &:= \operatorname{colim}(\mathcal{F}_X(-)\underline{\otimes}\Gamma^{d,W}), \\
X\underline{\otimes}Y &:= \operatorname{colim}(X\underline{\otimes}\mathcal{F}_Y(-)).
\end{aligned}
$$

for all $X, Y \in \operatorname{Rep}\Gamma_d^{\Bbbk}$.

Remark 3.7.3. Krause's internal product is an example of a construction on categories of functors known as *Day convolution*. This construction is first studied in [Day70].

Remark 3.7.4. There is a left exact version of Krause's product $-\underline{\otimes}^!- : \operatorname{Rep}\Gamma_d^{\Bbbk} \times \operatorname{Rep}\Gamma_d^{\Bbbk} \to \operatorname{Rep}\Gamma_d^{\Bbbk}$ defined

$$X\underline{\otimes}^!Y := (X^\circ\underline{\otimes}Y^\circ)^\circ.$$

The product $\underline{\otimes}^!$ can also be defined by using the fact that every object in $\operatorname{Rep}\Gamma_d^{\Bbbk}$ is a limit of the dual representable functors $S^{d,V}$. Indeed, by proceeding as in the definition of $\underline{\otimes}$ but with colimits replaced by limits, we can directly define $-\underline{\otimes}^!- : \operatorname{Rep}\Gamma_d^{\Bbbk} \times \operatorname{Rep}\Gamma_d^{\Bbbk} \to \operatorname{Rep}\Gamma_d^{\Bbbk}$ as the product that is left exact in each variable and satisfies $S^{d,V}\underline{\otimes}^!S^{d,W} = S^{d,V\otimes W}$. Krause [Kra13] also defines a internal Hom functor $\mathcal{H}om(-,-) : (\operatorname{Rep}\Gamma_d^{\Bbbk})^{op} \times \operatorname{Rep}\Gamma_d^{\Bbbk} \to \operatorname{Rep}\Gamma_d^{\Bbbk}$ whose definition is equivalent to $\mathcal{H}om(X,Y) = X^\circ\underline{\otimes}^!Y$.

Theorem 3.7.5. *The following diagram commutes:*

$$\begin{tikzcd}
\operatorname{Rep}\Gamma_d^{\Bbbk} \times \operatorname{Rep}\Gamma_d^{\Bbbk} \arrow[r, "-\underline{\otimes}-"] \arrow[d, "\mathrm{eval}_{\Bbbk^n} \times \mathrm{eval}_{\Bbbk^n}"'] & \operatorname{Rep}\Gamma_d^{\Bbbk} \arrow[d, "\mathrm{eval}_{\Bbbk^n}"] \\
\mathcal{P}_d^{\Bbbk} \times \mathcal{P}_d^{\Bbbk} \arrow[r, "-\underline{\otimes}-"'] & \mathcal{P}_d^{\Bbbk}
\end{tikzcd}$$

Proof. As in the proof of Theorem 3.6.1 it is enough to check that this diagram commutes on projective objects. The projective objects of $\operatorname{Rep}\Gamma_d^{\Bbbk}$ are direct summands of direct sums of the objects Γ^{λ}, for $\lambda \in \Lambda(n,d)$. By [AR17, Corollary 4.5],

$$\Gamma^{\lambda}\underline{\otimes}\Gamma^{\mu} = \bigoplus_{\nu \in A_{\mu}^{\lambda}} \Gamma^{\nu}.$$

The result follows. □

3.8 Appendix: Standard basis of the Schur algebra

In this section we recall the original definition of the Schur algebra. This definition gives rise to a natural characteristic-free basis of the Schur algebra that was identified by Schur in [Sch1901]. We recall the definition of this basis and show that this basis is equal to the basis of the Schur algebra constructed by Totaro [Tot97, pg. 8], and used in [Kra13] and [AR17]. This result is not new (see e.g. [Rei16, Appendix]).

Consider the polynomial algebra $\Bbbk[x_{ij}]$ with n^2 variables ($i,j \in [n]$). We regard $\Bbbk[x_{ij}]$ as a bialgebra with comultiplication defined on generators by $x_{ij} \mapsto \sum_{\ell=1}^{n} x_{i\ell}\otimes x_{\ell j}$ and counit $x_{ij} \mapsto \delta_{ij}$. Let $A_{\Bbbk}(n,d)$ be the subcoalgebra of $\Bbbk[x_{ij}]$ consisting of polynomials of degree d. Then $A_{\Bbbk}(n,d)^*$ has a natural algebra structure. Schur [Sch1901] constructs an algebra isomorphism

$$\chi : A_{\Bbbk}(n,d)^* \to \operatorname{End}_{\Bbbk\mathfrak{S}_d}(\textstyle\bigotimes^d \Bbbk^n)$$

by the following process. For a function $\underline{i} : [d] \to [n]$, let $v_{\underline{i}} = v_{i(1)} \otimes \cdots \otimes v_{i(d)} \in \bigotimes^d \Bbbk^n$. Then for any $\xi \in A_{\Bbbk}(n,d)^*$,

$$\chi(\xi)(v_{\underline{j}}) = \sum_{\underline{i} \in [n]^{[d]}} \xi(x_{\underline{i},\underline{j}}) v_{\underline{i}},$$

where $[n]^{[d]}$ is the set of functions $[d] \to [n]$, and

$$x_{\underline{i},\underline{j}} = x_{\underline{i}(1)\underline{j}(1)} \cdots x_{\underline{i}(d)\underline{j}(d)} \in A_{\Bbbk}(n,d).$$

A proof of this isomorphism can be found in [Gre80, Section 2.6] and this proof works for any commutative ring $\Bbbk$.

There is a right action of $\mathfrak{S}_d$ on $[n]^{[d]}$ defined $(\underline{i} \cdot \sigma)(a) = \underline{i}(\sigma(a))$. Likewise there is a right $\mathfrak{S}_d$-action on $[n]^{[d]} \times [n]^{[d]}$ given by the diagonal action. The algebra $A_{\Bbbk}(n,d)^*$ has a basis consisting of the elements $\xi_{\underline{i},\underline{j}}$ that are dual to $x_{\underline{i},\underline{j}}$. Clearly $\xi_{\underline{i},\underline{j}} = \xi_{\underline{k},\underline{l}}$ if and only if $(\underline{k},\underline{l}) = (\underline{i}\sigma, \underline{j}\sigma)$ for some $\sigma \in \mathfrak{S}_d$. Note in particular that $\chi(\xi_{\underline{i},\underline{j}})(v_{\underline{l}})$ is the sum over all vectors $v_{\underline{k}}$ in which $(\underline{k},\underline{l})$ is in the $\mathfrak{S}_d$-orbit of $(\underline{i},\underline{j})$.

For $\lambda \in \Lambda(n,d)$, let $\underline{i}_\lambda \in [n]^{[d]}$ be the function sending any element in $\overline{\lambda}_i$ to i i.e. so that $v_{\underline{i}_\lambda} = v_\lambda$. Then every basis element is of the form $\xi_{\underline{i}_\mu\sigma,\underline{i}_\lambda}$ for some $\sigma \in \mathfrak{S}_d$, and $\xi_{\underline{i}_\mu\sigma_1,\underline{i}_\lambda} = \xi_{\underline{i}_\mu\sigma_2,\underline{i}_\lambda}$ if and only if $\mathfrak{S}_\mu\sigma_1\mathfrak{S}_\lambda = \mathfrak{S}_\mu\sigma_2\mathfrak{S}_\lambda$.

Proposition 3.8.1. *The $\mathfrak{S}_d$-equivariant endomorphism $\chi(\xi_{\underline{i}_\mu\sigma,\underline{i}_\lambda}) \in \operatorname{End}_{\Bbbk\mathfrak{S}_d}(\bigotimes^d \Bbbk^n)$ is defined*

$$\chi(\xi_{\underline{i}_\mu\sigma,\underline{i}_\lambda})(v_\nu) = \begin{cases} \sum_{\underline{i}\in\underline{i}_\mu\sigma\mathfrak{S}_\lambda} v_{\underline{i}} & \text{if } \nu = \lambda, \\ 0 & \text{otherwise,} \end{cases}$$

for all $\nu \in \Lambda(n,d)$.

Proof. The value of $\xi_{\underline{i}_\mu\sigma,\underline{i}_\lambda}(x_{\underline{i},\underline{i}_\nu})$ is nonzero only if $\nu = \lambda$. Moreover $\xi_{\underline{i}_\mu\sigma,\underline{i}_\lambda}(x_{\underline{i},\underline{i}_\lambda}) = 1$ if and only if there is an element $w \in \mathfrak{S}_d$ in which $(\underline{i},\underline{i}_\lambda) = (\underline{i}_\mu\sigma w, \underline{i}_\lambda w)$, otherwise $\xi_{\underline{i}_\mu\sigma,\underline{i}_\lambda}(x_{\underline{i},\underline{i}_\lambda}) = 0$. As $\mathfrak{S}_\lambda$ is the stabilizer of $\underline{i}_\lambda$, it follows that $\xi_{\underline{i}_\mu\sigma,\underline{i}_\lambda}(x_{\underline{i},\underline{i}_\lambda}) = 1$ if and only if $\underline{i}$ is in the $\mathfrak{S}_\lambda$-orbit of $\underline{i}_\mu\sigma$. The result follows. □

The following equation holds by an argument similar to the proof of Proposition 3.8.1:

$$\chi(\xi_{\underline{i},\underline{j}})(v_{\underline{j}}) = \sum_{\underline{k}\in\underline{i}\cdot\mathrm{stab}(\underline{j})} v_{\underline{k}}. \tag{3.26}$$

Example 3.8.2. We calculate explicitly the values of $\chi(\xi_{\underline{i}_\mu\sigma,\underline{i}_\lambda})(v_\lambda)$ in the case $\lambda = \mu = (2,2)$. We denote a function $\underline{i} \in [n]^{[d]}$ by the sequence $(i(1),\ldots,i(d))$, and write $v_{i_1 i_2\cdots i_d}$ for the vector $v_{i_1} \otimes \cdots \otimes v_{i_d}$. Then,

$$\chi(\xi_{(1,1,2,2),(1,1,2,2)})(v_{1,1,2,2}) = v_{1,1,2,2},$$

$$\chi(\xi_{(2,2,1,1),(1,1,2,2)})(v_{1,1,2,2}) = v_{2,2,1,1},$$

$$\chi(\xi_{(1,2,1,2),(1,1,2,2)})(v_{1,1,2,2}) = v_{1,2,1,2} + v_{1,2,2,1} + v_{2,1,1,2} + v_{2,1,2,1}.$$

Write $\xi^{\sigma}_{\mu,\lambda}$ for the basis element in $\mathcal{S}_{\Bbbk}(n,d)$ corresponding to $\xi_{\underline{i}_{\mu}\sigma,\underline{i}_{\lambda}} \in A_{\Bbbk}(n,d)^*$. We call $\{\xi^{\sigma}_{\mu,\lambda} \mid \lambda,\mu \in \Lambda(n,d), \sigma \in \mathfrak{S}_d\}$, the *standard basis* of $\mathcal{S}_{\Bbbk}(n,d)$. The generators of $\mathcal{S}_{\Bbbk}(n,d)$ are examples of standard basis elements. More precisely,

$$1_{\lambda} = \xi^{e}_{\lambda,\lambda}, \qquad E_i^{(r)} 1_{\lambda} = \xi^{e}_{\lambda+r\alpha_i,\lambda}, \qquad F_i^{(r)} 1_{\lambda} = \xi^{e}_{\lambda-r\alpha_i,\lambda},$$

where e denotes the identity element in $\mathfrak{S}_d$.

By Proposition 3.8.1, it is clear that $\xi^{\sigma_2}_{\nu,\eta}\xi^{\sigma_1}_{\mu,\lambda} = 0$ whenever $\eta \neq \mu$. The following proposition describes the other structure constants for the Schur algebra.

Proposition 3.8.3. *In $\mathcal{S}_{\Bbbk}(n,d)$:*

$$\xi^{\sigma_2}_{\nu,\mu}\xi^{\sigma_1}_{\mu,\lambda} = \sum_{\sigma} c^{\sigma}_{\sigma_2,\sigma_1,\nu,\mu,\lambda}\xi^{\sigma}_{\nu,\lambda},$$

where the sum ranges over representatives of $\mathfrak{S}_{\nu}\backslash\mathfrak{S}_d/\mathfrak{S}_{\lambda}$, and $c^{\sigma}_{\sigma_2,\sigma_1,\nu,\mu,\lambda}$ is the number of $\underline{i} \in \underline{i}_{\mu}\sigma_1\mathfrak{S}_{\lambda}$ in which $(\underline{i}_{\nu}\sigma, \underline{i})$ is in the $\mathfrak{S}_d$-orbit of $(\underline{i}_{\nu}\sigma_2, \underline{i}_{\mu})$.

Proof. In this proof we will treat the elements $\xi^{\sigma}_{\mu,\lambda}$ as endomorphisms of $\bigotimes^d \Bbbk^n$. The coefficient $c^{\sigma}_{\sigma_2,\sigma_1,\nu,\mu,\lambda}$ is equal to the number of times in which $v_{\underline{i}_{\nu}\sigma}$ appears as a summand in

$$\xi^{\sigma_2}_{\nu,\mu}\xi^{\sigma_1}_{\mu,\lambda}(v_{\lambda}) = \sum_{\underline{i}\in\underline{i}_{\mu}\sigma_1\mathfrak{S}_{\lambda}} \xi^{\sigma_2}_{\nu,\mu}(v_{\underline{i}}).$$

The vector $v_{\underline{i}_{\nu}\sigma}$ appears as a summand of $\xi^{\sigma_2}_{\nu,\mu}(v_{\underline{i}})$ if and only if $(\underline{i}_{\nu}\sigma, \underline{i})$ is in the $\mathfrak{S}_d$-orbit of $(\underline{i}_{\nu}\sigma_2, \underline{i}_{\mu})$. The result follows. ☐

Remark 3.8.4. Schur [Sch1901, pg. 40] identifies a multiplication rule for standard basis elements, and this rule is described by Green in [Gre80, pg. 13]. It is not hard the equate their multiplication rule with ours. A graph theoretic approach to defining these coefficients is given in [Mén01].

By Proposition 3.4.2,

$$1_{\lambda}\mathcal{S}_{\Bbbk}(n,d)1_{\mu} \simeq (\Gamma^{\mu}\Bbbk^n)_{\lambda} \simeq \mathrm{Hom}_{\mathcal{P}^{\Bbbk}_{n,d}}(\Gamma^{\lambda}\Bbbk^n, \Gamma^{\mu}\Bbbk^n)$$

and so there is an algebra isomorphism

$$\mathcal{S}_{\Bbbk}(n,d)^{op} \simeq \bigoplus_{\lambda,\mu\in\Lambda(n,d)} \mathrm{Hom}_{\mathcal{P}^{\Bbbk}_{n,d}}(\Gamma^{\lambda}\Bbbk^n, \Gamma^{\mu}\Bbbk^n),$$

where the multiplication on the right is given by composition where this is defined, and evaluates to zero if the composition is not defined.

For the remainder of this section we recall Totaro's basis of $\mathrm{Hom}_{\mathcal{P}^{\Bbbk}_{n,d}}(\Gamma^{\lambda}\Bbbk^n, \Gamma^{\mu}\Bbbk^n)$ and show that this basis corresponds to the standard basis of the Schur algebra under the above isomorphism.

Given two weak compositions, λ, μ of d (of any length), say that $\lambda \preceq \mu$ if there is an inclusion of the parabolic subgroups $\mathfrak{S}_{\lambda} \subset \mathfrak{S}_{\mu} \subset \mathfrak{S}_d$. We call this the *parabolic ordering* on weak compositions of d.

Note that $\Gamma^{\lambda}\Bbbk^n$ is the space of $\mathfrak{S}_{\lambda}$-invariants of $\bigotimes^d \Bbbk^n$. Hence if $\lambda \preceq \mu$ then $\Gamma^{\mu}\Bbbk^n$ is a submodule of $\Gamma^{\lambda}\Bbbk^n$. We denote the inclusion morphism by $\Delta : \Gamma^{\mu}\Bbbk^n \to \Gamma^{\lambda}\Bbbk^n$. Additionally define the map $\nabla : \Gamma^{\lambda}\Bbbk^n \to \Gamma^{\mu}\Bbbk^n$ by

$$v \mapsto \sum_{g\in\mathfrak{S}_{\lambda}\backslash\mathfrak{S}_{\mu}} v \cdot g$$

As a special case of these maps define:

$$\begin{aligned} \Delta_{r,s} : \Gamma^{r+s}\Bbbk^n \to \Gamma^{r,s}\Bbbk^n; \qquad & v \mapsto v \\ \nabla_{r,s} : \Gamma^{r,s}\Bbbk^n \to \Gamma^{r+s}\Bbbk^n; \qquad & x \otimes y \mapsto \sum_{g\in\mathfrak{S}_{r,s}\backslash\mathfrak{S}_{r+s}} (x\otimes y)\cdot g \end{aligned}$$

Define the isomorphism:

$$\tau : \Gamma^{r,s}\Bbbk^n \to \Gamma^{s,r}\Bbbk^n; \qquad v\otimes w \mapsto w \otimes v$$

If λ is in the $\mathfrak{S}_n$ orbit of μ, let $\tau : \Gamma^{\lambda}\Bbbk^n \to \Gamma^{\mu}\Bbbk^n$ be the unique isomorphism built from the maps of the form $\tau_{r,s}$ by composition and tensoring with identity morphisms.

For $A \in A^{\lambda}_{\mu}$ define the morphism $\phi_A : \Gamma^{\lambda}\Bbbk^n \to \Gamma^{\mu}\Bbbk^n$ as the composition

$$\Gamma^{\lambda}\Bbbk^n = \bigotimes_j \Gamma^{\lambda_j}\Bbbk^n \xrightarrow{\otimes_j\Delta} \bigotimes_j(\bigotimes_i \Gamma^{a_{ij}}\Bbbk^n) \xrightarrow{\tau} \bigotimes_i(\bigotimes_j \Gamma^{a_{ij}}\Bbbk^n) \xrightarrow{\otimes_i\nabla} \bigotimes_i \Gamma^{\mu_i}\Bbbk^n = \Gamma^{\mu}\Bbbk^n$$

Recall from Theorem 3.5.1 that there is a bijection $\Theta_\mu^\lambda : \mathfrak{S}_\lambda\backslash\mathfrak{S}_d/\mathfrak{S}_\mu \to A_\mu^\lambda$ sending $\mathfrak{S}_\lambda\sigma\mathfrak{S}_\mu$ to the matrix $a_{ij} = |\overline{\lambda}_i \cap \sigma(\overline{\mu}_j)|$. For $\sigma \in \mathfrak{S}_d$ we write $\Theta_\mu^\lambda(\sigma)$ for $\Theta_\mu^\lambda(\mathfrak{S}_\lambda\sigma\mathfrak{S}_\mu)$. The morphisms $\phi_A : \Gamma^\lambda\Bbbk^n \to \Gamma^\mu\Bbbk^n$ have the following explicit description.

Proposition 3.8.5. *If $\sigma \in \mathfrak{S}_d$ and $A = \Theta_\mu^\lambda(\sigma)$ then,*

$$\phi_A(v_\lambda) = \sum_{\underline{i}\in\underline{i}_\lambda\sigma\mathfrak{S}_\mu} v_{\underline{i}}.$$

Proof. Applying the map $\otimes_j\Delta : \Gamma^\lambda\Bbbk^n \to \bigotimes_j(\bigotimes_i \Gamma^{a_{ij}}\Bbbk^n)$ to v_λ gives the vector $v_\lambda \in \bigotimes_j(\bigotimes_i \Gamma^{a_{ij}}\Bbbk^n)$. The map $\tau : \bigotimes_j(\bigotimes_i \Gamma^{a_{ij}}\Bbbk^n) \to \bigotimes_i(\bigotimes_j \Gamma^{a_{ij}}\Bbbk^n)$ permutes tensor factors. In particular, if $\sigma_A \in \mathfrak{S}_d$ twists A, then $\tau(v_\lambda) = v_\lambda\sigma_A$. Applying the map $\otimes_i\nabla : \bigotimes_i(\bigotimes_j \Gamma^{a_{ij}}\Bbbk^n) \to \Gamma^\mu\Bbbk^n$ to $v_\lambda\sigma_A$ gives $\sum_{\underline{i}\in\underline{i}_\lambda\sigma_A\mathfrak{S}_\mu} v_{\underline{i}}$. The result follows since

$$\sum_{\underline{i}\in\underline{i}_\lambda\sigma_A\mathfrak{S}_\mu} v_{\underline{i}} = \sum_{\underline{i}\in\underline{i}_\lambda\sigma\mathfrak{S}_\mu} v_{\underline{i}}$$

whenever $\mathfrak{S}_\lambda\sigma_A\mathfrak{S}_\mu = \mathfrak{S}_\lambda\sigma\mathfrak{S}_\mu$.

□

The set $\{\phi_A(v_\lambda) \mid A \in A_\mu^\lambda\}$ is a basis of $(\Gamma^\mu\Bbbk^n)_\lambda$, and so $\{\phi_A \mid A \in A_\mu^\lambda\}$ is a basis of $\mathrm{Hom}_{\mathcal{P}_{n,d}^{\Bbbk}}(\Gamma^\lambda\Bbbk^n, \Gamma^\mu\Bbbk^n)$.

Theorem 3.8.6. *There is an algebra isomorphism*

$$\mathcal{S}_\Bbbk(n,d)^{op} \simeq \bigoplus_{\lambda,\mu\in\Lambda(n,d)} \mathrm{Hom}_{\mathcal{P}_{n,d}^{\Bbbk}}(\Gamma^\lambda\Bbbk^n, \Gamma^\mu\Bbbk^n),$$

sending $\xi_{\mu,\lambda}^\sigma$ to the morphism $\phi_A : \Gamma^\mu\Bbbk^n \to \Gamma^\lambda\Bbbk^n$ in which $\Theta_\lambda^\mu(\sigma) = A$.

Proof. Let $\sigma_1, \sigma_2 \in \mathfrak{S}_d$ be such that $\Theta_\lambda^\mu(\sigma_1) = A_1$ and $\Theta_\mu^\nu(\sigma_2) = A_2$. By Proposition 3.8.5, and using a proof similar to Proposition 3.8.3,

$$\phi_{A_1} \circ \phi_{A_2} = \sum_\sigma c_{\sigma_2,\sigma_1,\nu,\mu,\lambda}^\sigma \phi_{\Theta_\lambda^\nu(\sigma)},$$

where the sum ranges over representatives of $\mathfrak{S}_\nu\backslash\mathfrak{S}_d/\mathfrak{S}_\lambda$. □

Remark 3.8.7. We can also define the standard basis of $\mathcal{S}_{\Bbbk}(n,d)$ using the algebra isomorphism

$$\mathcal{S}_{\Bbbk}(n,d) \simeq \bigoplus_{\lambda,\mu\in\Lambda(n,d)} \mathrm{Hom}_{\mathcal{P}^{\Bbbk}_{n,d}}(S^{\lambda}\Bbbk^n, S^{\mu}\Bbbk^n).$$

Indeed, note that $S^{\lambda}\Bbbk^n$ is the space of $\mathfrak{S}_{\lambda}$-coinvariants of $\bigotimes^d \Bbbk^n$. Hence if $\lambda \preceq \mu$ then there is a surjection $\nabla : S^{\lambda}\Bbbk^n \to S^{\mu}\Bbbk^n$. Additionally define the map $\Delta : S^{\mu}\Bbbk^n \to S^{\lambda}\Bbbk^n$ by

$$\Delta(x_1 \otimes \cdots \otimes x_m) = \sum_{r=0}^{m} (x_0 \otimes x_1 \otimes \cdots \otimes x_r) \sqcup\!\sqcup (x_{r+1} \otimes \cdots \otimes x_m)$$

where $x_0 = 1$ and $\sqcup\!\sqcup$ refers to the shuffle product. For $\sigma \in \mathfrak{S}_d$ in which $A = \Theta^{\lambda}_{\mu}(\sigma)$, the basis element $\xi^{\sigma}_{\mu,\lambda}$ corresponds to the morphism $S^{\mu}\Bbbk^n \to S^{\lambda}\Bbbk^n$ defined as the composition:

$$S^{\mu} = \bigotimes_i S^{\mu_i} \xrightarrow{\otimes_i \Delta} \bigotimes_i (\bigotimes_j S^{a_{ij}}) \simeq \bigotimes_j (\bigotimes_i S^{a_{ij}}) \xrightarrow{\otimes_j \nabla} \bigotimes_j S^{\lambda_j} = S^{\lambda}$$

3.9 Appendix: Definition of the Schur algebra via web diagrams

In this appendix we define the Schur algebra using a diagrammatic approach developed in [CKM14]. A more detailed account of this approach can be found in [W19].

The main idea is to construct a category, $\mathcal{S}_{\Bbbk}$, whose objects are finite sequences of positive integers, and in which for all $n, d \in \mathbb{N}$:

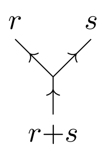
$$\mathcal{S}_{\Bbbk}(n,d) \simeq \bigoplus_{\lambda,\mu\in\Lambda(n,d)} \mathrm{Hom}_{\mathcal{S}_{\Bbbk}}(\kappa(\lambda), \kappa(\mu)),$$

where e.g. $\kappa(\lambda)$ is the sequence obtained from λ by removing all zero terms.

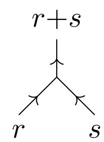
The following definition is due to [CKM14].

Definition 3.9.1 (Free spider category $\mathcal{F}_{\Bbbk}$)**.** Let $\Bbbk$ be a commutative ring. The *free spider category*, $\mathcal{F}_{\Bbbk}$, is a strict monoidal $\Bbbk$-linear category whose objects are finite sequences of positive integers. The monoidal product on objects is given by concatenation, with the empty sequence $\emptyset$ as the monoidal unit.

The morphisms of $\mathcal{F}_{\Bbbk}$ are freely generated (by taking $\Bbbk$-linear combinations, compositions, and monoidal product) by two morphisms depicted diagrammatically:

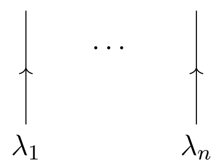


$: (r+s) \to (r,s),$



$: (r,s) \to (r+s).$

The identity morphism on an object $(\lambda_1, \ldots, \lambda_n)$ is depicted by the diagram



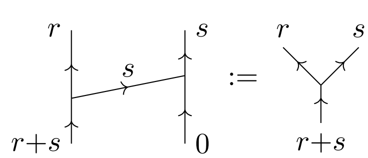
The composition and monoidal operations on morphisms are depicted diagrammatically by the following rules:

- If $f : \lambda \to \mu$ and $g : \mu \to \nu$ are morphisms depicted by string diagrams, then the composition $g \circ f$ is depicted by the diagram obtained by placing g on top of f and connecting strands at the top of f with strands at the bottom of g.

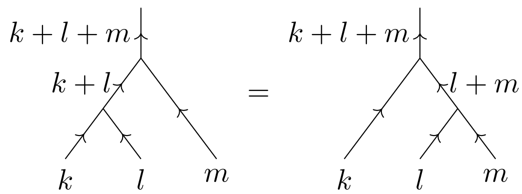
- If $f : \lambda \to \mu$ and $g : \lambda' \to \mu'$ are morphisms depicted by string diagrams, then $f \otimes g$ is depicted by the diagram obtained by placing f to the left of g.

The morphisms in $\mathcal{F}_{\Bbbk}$ are identified up to any planar isotopy that preserves the upwards direction of arrows.

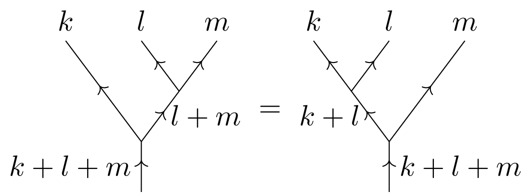
We call the diagrams depicting morphisms in $\mathcal{F}_{\Bbbk}$, *web diagrams*. It is convenient to draw web diagrams with strands labelled by any integer. These are interpreted as follows: strands labelled by negative integers are interpreted as zero morphisms, and strands labelled by 0 are to be removed from the diagram e.g.



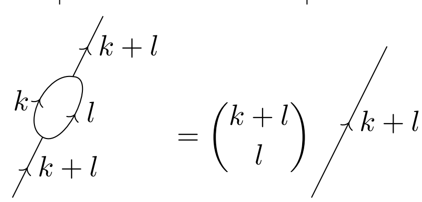
Definition 3.9.2 (Complex Schur category $\mathcal{S}_{\mathbb{C}}$)**.** The *complex Schur category* $\mathcal{S}_{\mathbb{C}}$ is the $\mathbb{C}$-linear strict monoidal category obtained from $\mathcal{F}_{\mathbb{C}}$ by enforcing the following relations among morphisms:



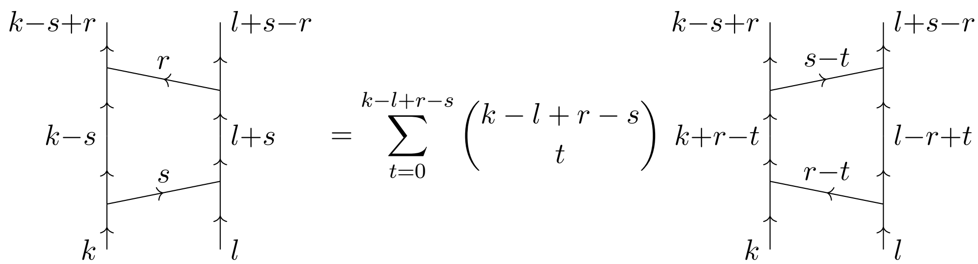
(3.27)



(3.28)



(3.29)



(3.30)

where $k, l, r, s \in \mathbb{Z}_{\geq 0}$.

Let $\mathcal{S}_{\mathbb{Z}}$ be the strict monoidal $\mathbb{Z}$-linear subcategory of $\mathcal{S}_{\mathbb{C}}$ with the same objects as $\mathcal{S}_{\mathbb{C}}$, and whose morphisms are the $\mathbb{Z}$-span of morphisms depicted by web diagrams.

For any commutative ring, $\Bbbk$, let $\mathcal{S}_{\Bbbk}$ be the strict monoidal $\Bbbk$-linear category whose objects are finite sequences of positive integers, and with morphisms

$$\mathrm{Hom}_{\mathcal{S}_{\Bbbk}}(\lambda, \mu) := \Bbbk \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathcal{S}_{\mathbb{Z}}}(\lambda, \mu).$$

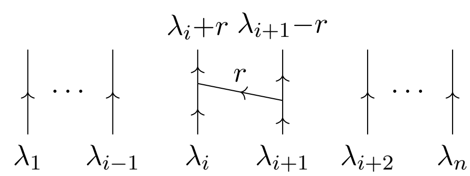
Conjecture 3.9.3. *For any commutative ring $\Bbbk$, the category $\mathcal{S}_\Bbbk$ is equivalent to the category obtained from $\mathcal{F}_\Bbbk$ by enforcing the relations (3.27)-(3.30).*

The following result is shown in [W19, Theorem 3.1.1] in the case $\Bbbk = \mathbb{C}$. The general case follows from the case $\Bbbk = \mathbb{C}$ by the definition of $\mathcal{S}_\Bbbk$.

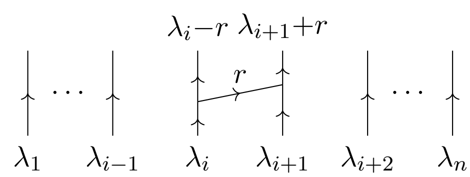
Theorem 3.9.4. *Let $\Bbbk$ be a commutative ring. For a sequence λ of positive integers, let $\kappa(\lambda)$ be the sequence obtained from λ by removing all zero terms. For all $n, d \in \mathbb{N}$, there is an algebra isomorphism:*

$$\mathcal{S}_\Bbbk(n,d) \simeq \bigoplus_{\lambda,\mu \in \Lambda(n,d)} \mathrm{Hom}_{\mathcal{S}_\Bbbk}(\kappa(\lambda), \kappa(\mu))$$

defined

$E_i^{(r)} 1_\lambda \mapsto$ 

,

$F_i^{(r)} 1_\lambda \mapsto$ 

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For the remainder of this section we detail how the Schur category can be used to describe the morphisms between divided and exterior powers in $\mathcal{P}_d^\Bbbk$.

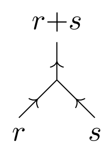
Let $\Gamma_\Bbbk$ be the category whose objects are finite sequences of positive integers, and with morphisms

$$\mathrm{Hom}_{\Gamma_\Bbbk}(\lambda, \mu) := \mathrm{Hom}_{\mathcal{P}_d^\Bbbk}(\Gamma^\lambda \Bbbk^\infty, \Gamma^\mu \Bbbk^\infty).$$

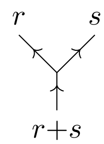
The category $\Gamma_\mathbb{C}$ is a strict monoidal functor with the monoidal product defined by the external tensor product on Schur algebra modules.

The following result follows immediately from Theorem 3.9.4 and Theorem 3.8.6.

Corollary 3.9.5. *There is a monoidal equivalence of categories $\mathcal{S}_{\mathbb{C}}^{op} \to \Gamma_{\mathbb{C}}$ defined*



$$\mapsto \Delta_{r,s} : \Gamma^{r+s}\Bbbk^\infty \to \Gamma^{r,s}\Bbbk^\infty,$$



$$\mapsto \nabla_{r,s} : \Gamma^{r,s}\Bbbk^\infty \to \Gamma^{r+s}\Bbbk^\infty.$$

Let $\Lambda_{\Bbbk}$ be the category whose objects are finite sequences of positive integers, and with morphisms

$$\mathrm{Hom}_{\Lambda_{\Bbbk}}(\lambda, \mu) := \mathrm{Hom}_{\mathcal{P}_d^{\Bbbk}}(\Lambda^\lambda \Bbbk^\infty, \Lambda^\mu \Bbbk^\infty).$$

The category $\Lambda_{\mathbb{C}}$ has a monoidal product defined by the external tensor product on Schur algebra modules.

For any set $S = \{i_1 < i_2 < \cdots < i_d\} \subset \mathbb{N}_{\geq 1}$, define $v_S \in \Lambda^d \Bbbk^\infty$ by

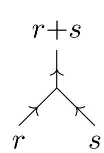
$$v_S := v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_d}.$$

Let $\ell(S,T) = |\{(i,j) \in S \times T \mid i < j\}|$. For a set S, write $\binom{S}{r}$ for the set of subsets of S of size r. Define the morphisms:

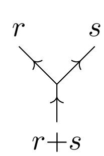
$$\Delta_{r,s} : \Lambda^{r+s}\Bbbk^\infty \to \Lambda^{r,s}\Bbbk^\infty; \qquad v_S \mapsto (-1)^{rs} \sum_{T \in \binom{S}{r}} (-1)^{\ell(S\setminus T, T)} v_T \otimes v_{S \setminus T}$$

$$\nabla_{r,s} : \Lambda^{r,s}\Bbbk^\infty \to \Lambda^{r+s}\Bbbk^\infty; \qquad v_S \otimes v_T \mapsto \begin{cases} (-1)^{\ell(S,T)} v_{S \cup T} & \text{if } S \cap T = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

A consequence of [CKM14, Theorem 3.2.1] is that there is a monoidal equivalence of categories $\mathcal{S}_{\Bbbk} \to \Lambda_{\Bbbk}$ defined



$$\mapsto \nabla_{r,s} : \Lambda^{r,s}\Bbbk^\infty \to \Lambda^{r+s}\Bbbk^\infty,$$



$$\mapsto \Delta_{r,s} : \Lambda^{r+s}\Bbbk^\infty \to \Lambda^{r,s}\Bbbk^\infty.$$

This result is essentially a rephrasing of Donkin's [Don93, Proposition 3.7] algebra isomorphism

$$\mathcal{S}_{\Bbbk}(n,d) \simeq \mathrm{End}_{\mathcal{P}_d^{\Bbbk}}\Big(\bigoplus_{\lambda\in\Lambda(n,d)} \Lambda^{\lambda}\Bbbk^n\Big).$$

Remark 3.9.6. By Proposition 3.4.2 (respectively Proposition 4.5.2 below), the category of projective (respectively partial tilting) objects in $\mathcal{P}_d^{\Bbbk}$ is equivalent to the additive closure of the Karoubi envelope of $\Gamma_{\Bbbk}$ (respectively $\Lambda_{\Bbbk}$).

Chapter 4

Geometric reconstruction of the Schur algebra

4.1 Motivation and outline

Mautner [Mau14, Theorem 1.1] shows that the category, $P_{\mathrm{GL}_d}(\mathcal{N}_{\mathrm{GL}_d}, \mathbb{k})$ of $\mathrm{GL}_d(\mathbb{C})$-equivariant perverse sheaves on the nilpotent cone $\mathcal{N}_{\mathrm{GL}_d} \subset \mathfrak{gl}_d(\mathbb{C})$ is equivalent to the category, $\mathcal{P}_d^{\mathbb{k}}$, of all polynomial representations of degree d. This proof uses Lusztig's [Lus81] embedding of $\mathcal{N}_{\mathrm{GL}_d}$ into the complex affine Grassmannian, $\mathcal{G}r_{\mathrm{GL}_d}$, to show that $P_{\mathrm{GL}_d}(\mathcal{N}_{\mathrm{GL}_d}, \mathbb{k})$ is equivalent to a subcategory of the category of perverse sheaves on $\mathcal{G}r_{GL_d}$ (equivariant with respect to the loop group). This latter subcategory is equivalent to $\mathcal{P}_d^{\mathbb{k}}$ by Mirković-Vilonen's geometric Satake correspondence [MV07, Theorem 14.1].

In this section we prove the equivalence between $P_{\mathrm{GL}_d}(\mathcal{N}_{\mathrm{GL}_d}, \mathbb{k})$ and $\mathcal{P}_d^{\mathbb{k}}$ using the geometry of the nilpotent cone and without appealing to the geometric Satake correspondence. For this, we define, for each $\lambda \in \Lambda(n, d)$, the varieties

$$\breve{\mathcal{N}}_\lambda := \mathrm{GL}_d \times^{P_\lambda} \mathcal{N}_{P_\lambda} \qquad \text{and} \qquad \tilde{\mathcal{N}}_\lambda := \mathrm{GL}_d \times^{P_\lambda} \mathfrak{u}_\lambda,$$

where $P_\lambda = L_\lambda \ltimes U_\lambda$ is the parabolic subgroup of GL_d with Levi factor $L_\lambda \simeq \mathrm{GL}_{\lambda_1} \times \cdots \times \mathrm{GL}_{\lambda_n}$. Define the multiplication maps

$$\breve{m}_\lambda : \breve{\mathcal{N}}_\lambda \to \mathcal{N}_{\mathrm{GL}_d} \qquad \text{and} \qquad \tilde{m}_\lambda : \tilde{\mathcal{N}}_\lambda \to \mathcal{N}_{\mathrm{GL}_d}$$

by $(g,x) \mapsto gxg^{-1}$. Define the perverse sheaves

$$\Gamma^\lambda := \check{m}_{\lambda !}\underline{\Bbbk}_{\tilde{\mathcal{N}}_\lambda}[\dim \mathcal{N}_{\mathrm{GL}_d}] \qquad \text{and} \qquad \Lambda^\lambda := \tilde{m}_{\lambda !}\underline{\Bbbk}_{\tilde{\mathcal{N}}_\lambda}[2\dim \mathrm{GL}_d / P_\lambda].$$

Theorem 4.4.1 states that if $n \geq d$ then $\bigoplus_{\lambda\in\Lambda(n,d)} \Gamma^\lambda$ is a projective generator of $P_{\mathrm{GL}_d}(\mathcal{N}_{\mathrm{GL}_d}, \Bbbk)$ with endomorphism ring

$$\mathrm{End}_{P_{\mathrm{GL}_d}(\mathcal{N}_{\mathrm{GL}_d},\Bbbk)}(\bigoplus_{\lambda\in\Lambda(n,d)} \Gamma^\lambda) \simeq \mathcal{S}_\Bbbk(n,d)^{op}.$$

In particular, the functor

$$\Phi_{n,d} := \mathrm{Hom}_{P_{\mathrm{GL}_d}(\mathcal{N}_{\mathrm{GL}_d},\Bbbk)}(\bigoplus_{\lambda\in\Lambda(n,d)} \Gamma^\lambda, -) : P_{\mathrm{GL}_d}(\mathcal{N}_{\mathrm{GL}_d}, \Bbbk) \to \mathcal{P}^\Bbbk_{n,d}$$

is an equivalence of categories.

To prove this result we first give a characteristic-free version of Ginzburg's construction of the Schur algebra (Theorem 4.3.6). More precisely, Theorem 4.3.6 states that there is an algebra isomorphism

$$\mathcal{S}_\Bbbk(n,d)^{op} \simeq \bigoplus_{\lambda,\mu\in\Lambda(n,d)} H^{BM}_{\dim\tilde{\mathcal{N}}_\lambda+\dim\tilde{\mathcal{N}}_\mu}(\tilde{\mathcal{N}}_\lambda \times_{\mathcal{N}_{\mathrm{GL}_d}} \tilde{\mathcal{N}}_\mu, \Bbbk),$$

where $H^{BM}_*(-,\Bbbk)$ refers to the Borel-Moore homology with coefficients in $\Bbbk$. The algebra product on the right hand side is the convolution product. This result is shown in characteristic zero by Ginzburg [CG97, Proposition 4.2.5]. By a result of Ginzburg [CG97, Theorem 8.6.7] (see Proposition 4.4.11), there is an algebra isomorphism

$$\bigoplus_{\lambda,\mu\in\Lambda(n,d)} H^{BM}_{\dim\tilde{\mathcal{N}}_\lambda+\dim\tilde{\mathcal{N}}_\mu}(\tilde{\mathcal{N}}_\lambda \times_{\mathcal{N}_{\mathrm{GL}_d}} \tilde{\mathcal{N}}_\mu, \Bbbk) \simeq \mathrm{End}_{P_{\mathrm{GL}_d}(\mathcal{N}_{\mathrm{GL}_d},\Bbbk)}(\bigoplus_{\lambda\in\Lambda(n,d)} \Lambda^\lambda).$$

Achar and Mautner [AM15] define an equivalence of categories

$$\mathcal{R} : \mathcal{D}_{\mathrm{GL}_d}(\mathcal{N}_{\mathrm{GL}_d}, \Bbbk) \to \mathcal{D}_{\mathrm{GL}_d}(\mathcal{N}_{\mathrm{GL}_d}, \Bbbk)$$

that satisfies $\mathcal{R}(\Lambda^\lambda) \simeq \Gamma^\lambda$ (Lemma 4.4.8). In particular, there is an algebra isomorphism

$$\mathcal{S}_\Bbbk(n,d)^{op} \simeq \mathrm{End}_{P_{\mathrm{GL}_d}(\mathcal{N}_{\mathrm{GL}_d},\Bbbk)}(\bigoplus_{\lambda\in\Lambda(n,d)} \Lambda^\lambda) \simeq \mathrm{End}_{P_{\mathrm{GL}_d}(\mathcal{N}_{\mathrm{GL}_d},\Bbbk)}(\bigoplus_{\lambda\in\Lambda(n,d)} \Gamma^\lambda).$$

The evaluation of $\Phi_{n,d}$ on simple objects is described in Propositions 4.4.12. More precisely, for a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ of d, let $h_\lambda : \mathcal{O}_\lambda \hookrightarrow \mathcal{N}_{\mathrm{GL}_d}$ be the inclusion of the GL_d-orbit of the Jordan matrix with Jordan blocks of sizes $\lambda_1, \ldots, \lambda_n$. The functor $\Phi_{n,d} : P_{\mathrm{GL}_d}(\mathcal{N}_{\mathrm{GL}_d}, \Bbbk) \to \mathcal{P}^{\Bbbk}_{n,d}$ maps the simple perverse sheaf $h_{\lambda!*}\underline{\Bbbk}_{\mathcal{O}_\lambda}[\dim \mathcal{O}_\lambda]$ to the simple $\mathcal{S}_\Bbbk(n,d)$-module with highest weight λ. Moreover (Propositions 4.4.12 and 4.5.4),

$$\Phi_{n,d}(\Gamma^\lambda) \simeq \Gamma^\lambda \Bbbk^n \qquad \text{and} \qquad \Phi_{n,d}(\Lambda^\lambda) \simeq \Lambda^\lambda \Bbbk^n.$$

In Section 4.2 we summarise the results about the geometry of the nilpotent cone and the varieties $\tilde{\mathcal{N}}_\lambda$ and $\breve{\mathcal{N}}_\lambda$ that will be needed in the proofs and discussion of our main result.

In Section 4.3 we review the sheaf-theoretic definition of Borel-Moore homology and describe the basic operations on Borel-Moore homology from this perspective. Using this we prove the characteristic-free version of Ginzburg's construction of the Schur algebra.

In Section 4.4 we use Ginzburg's construction to show that the category $P_{\mathrm{GL}_d}(\mathcal{N}_{\mathrm{GL}_d}, \Bbbk)$ is equivalent to $\mathcal{P}^{\Bbbk}_{n,d}$ if $n \geq d$.

In Section 4.5 we describe the highest weight structure on $P_G(\mathcal{N}, \Bbbk)$ and evaluate the equivalence $P_G(\mathcal{N}, \Bbbk) \simeq \mathcal{P}^{\Bbbk}_{n,d}$ on partial tilting objects.

In an appendix (Section 4.6) we list some properties that should be satisfied by a geometric version of the homogeneous external product of Schur algebra modules.

Throughout this chapter, by a *variety* we mean a quasiprojective complex algebraic variety i.e. a subset of $\mathbb{CP}^n$ that is locally closed in the Zariski topology. Topological concepts (open, closed, etc.) will usually be with respect to the Zariski topology unless stated otherwise. The only exception being the dimension of a variety, which will always mean dimension as a complex variety.

The following two facts about varieties are important and we will sometimes apply these without mentioning them:

- Every variety X has a smooth Zariski-dense open subset (all non-singular points form an open dense subset).

- Every variety is locally compact, locally path connected, and locally contractible.

Let $\mathcal{D}^b(X, \Bbbk)$ be the bounded constructible derived category of sheaves on a variety X with coefficients in $\Bbbk$. Write $\underline{\Bbbk}_X$ for the constant sheaf on X (concentrated in degree zero). Define the canonical surjection $a_X : X \to pt$, and write $\mathbb{D}_X := a_X^! \underline{\Bbbk}_{pt}$ for the dualizing sheaf. Write $\mathbb{D} := R\mathcal{H}om_{\mathcal{D}^b(X)}(-, \mathbb{D}_X) : \mathcal{D}^b(X, \Bbbk)^{op} \to \mathcal{D}^b(X, \Bbbk)$ for the Verdier duality functor.

Write $P(X, \Bbbk)$ for the category of perverse sheaves on X with coefficients in $\Bbbk$.

Let $\mathcal{D}^b_G(X, \Bbbk)$ be the G-equivariant derived category and $P_G(X, \Bbbk)$ the category of G-equivariant perverse sheaves. If $f : X \to Y$ is a G-equivariant map of G-varieties then the usual sheaf functors $f_*, f^*, f_!, f^!$, as well as the sheaf products $R\mathcal{H}om, \otimes$ have analogues in the equivariant setting (see e.g. [Ach21, Section 6.5]).

In this chapter we will never need to consider the equivariant structure on objects in $P_G(X, \Bbbk)$. This is due to the following result about perverse sheaves that are equivariant with respect to a connected algebraic group (see e.g. [Ach21, Proposition 6.2.17] for a proof).

Proposition 4.1.1. *Let G be a connected algebraic group acting on a variety X and let $\sigma, pr_2 : G \times X \to X$ be the action map $\sigma(g, x) = g \cdot x$ and projection map $pr_2(g, x) = x$. The category $P_G(X, \Bbbk)$, of G-equivariant perverse sheaves on X is equivalent to the full subcategory of $P(X, \Bbbk)$ consisting of objects $\mathcal{F}$ in which $pr_2^*\mathcal{F}[\dim G] \simeq \sigma^*\mathcal{F}[\dim G]$ in $P(G \times X, \Bbbk)$.*

Likewise, if G is a connected algebraic group, then the sheaf operations f_*, f^*, $f_!$, $f^!$, $R\mathcal{H}om$, $\otimes^L$ on G-equivariant perverse sheaves are just the usual sheaf operations on perverse sheaves.

If $h : Y \hookrightarrow X$ is the inclusion of a locally closed subvariety, and $\mathcal{F}$ is an object in $\mathcal{D}^b(Y, \Bbbk)$, then we also write $\mathcal{F}$ for the object $h_!\mathcal{F} \in \mathcal{D}^b(X, \Bbbk)$. For example, the skyscraper sheaf in $\mathcal{D}^b(X, \Bbbk)$ supported on a point $x \in X$ (and concentrated at degree 0) is simply denoted $\underline{\Bbbk}_{\{x\}}$. This notation is primarily used when h is one of the closed embeddings

$$\overline{\mathcal{O}_\lambda} \hookrightarrow \mathcal{N}_{\mathrm{GL}_d} \hookrightarrow \mathfrak{gl}_d,$$

where $\mathcal{O}_\lambda \subset \mathcal{N}_{\mathrm{GL}_d}$ is a GL_d-orbit. For example, perverse sheaves in $P_{\mathrm{GL}_d}(\overline{\mathcal{O}_\lambda}, \Bbbk)$ will often be thought of as perverse sheaves in $P_{\mathrm{GL}_d}(\mathcal{N}_{\mathrm{GL}_d}, \Bbbk)$ (and sometimes as perverse sheaves in $P_{\mathrm{GL}_d}(\mathfrak{gl}_d, \Bbbk)$) that are supported on $\overline{\mathcal{O}_\lambda}$.

Most of the results in sheaf theory that we need follow from the discussion of recollements in Chapter 2. We recall an additional result called *proper base change*: For a Cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y \end{array}$$

there are natural isomorphisms

$$g^* f_! \simeq f'_! g'^* \qquad \text{and} \qquad f^! g_* \simeq g'_* f'^!.$$

See e.g. [Ach21, Theorem 1.2.12] for a proof of this.

Throughout this chapter we use lowercase fraktur letters to denote the Lie algebra of a Lie group denoted by the corresponding uppercase letter.

For a survey on the use of perverse sheaves in modular representation theory that includes examples of calculating of table of stalks for perverse sheaves on the nilpotent cone, the reader should consult [JMW12].

4.2 The nilpotent cone in type A

For a connected complex reductive Lie group H with Lie algebra $\mathfrak{h}$, write $\mathcal{N}_H$ for the closed subvariety of $\mathfrak{h}$ consisting of nilpotent elements. The variety $\mathcal{N}_H$ is called the *nilpotent cone* on $\mathfrak{h}$. We regard $\mathcal{N}_H$ (and $\mathfrak{h}$) as a H-space under the conjugation action.

Let $G = \mathrm{GL}_d(\mathbb{C})$ and write $\mathcal{N} := \mathcal{N}_G$. We recount here some geometric features of $\mathcal{N}$ that will be needed in later constructions. For a more complete survey of the geometry of nilpotent cones of Lie algebras see e.g. [CM93] and [Hen15].

For a partition $\lambda = (\lambda_1, \dots, \lambda_m)$ of d, let $\mathcal{O}_\lambda$ be the G-orbit in $\mathcal{N}$ consisting of nilpotent matrices whose Jordan form consists of Jordan blocks of sizes $\lambda_1, \dots, \lambda_m$.

Note that if $\lambda^\vee$ is the dual partition of λ, then the orbit $\mathcal{O}_{\lambda^\vee}$ is the set of matrices $x \in \mathfrak{g}$ in which $\dim(\ker x^i / \ker x^{i-1}) = \lambda_i$ for all $1 \le i \le m$. These G-orbits satisfy the dimension formula

$$\dim \mathcal{O}_{\lambda^\vee} = 2 \dim G/P_\lambda = d^2 - \lambda_1^2 - \cdots - \lambda_m^2.$$

We consider $\mathcal{N}$ as a stratified space with respect to the G-orbits. The closure order for the strata is given by the dominance order on partitions [Ger59] i.e. $\mathcal{O}_\lambda \subset \overline{\mathcal{O}_\mu}$ if and only if $\lambda_1 + \cdots + \lambda_k \le \mu_1 + \cdots \mu_k$ for all $k \ge 1$ (where partitions are extended by zeros at the end if necessary). In other words, $x \in \overline{\mathcal{O}_\lambda}$ if and only if $\dim \ker x^i \le \lambda_i^\vee$. In particular, $\mathcal{O}_{(1,\dots,1)} = \{0\}$ is the unique closed orbit in $\mathcal{N}$, and $\mathcal{O}_{(d)}$ is the unique dense orbit in $\mathcal{N}$.

We now define and describe some important properties of spaces related to $\mathcal{N}$. These spaces can be summarised by the following diagram, which we describe below.

$$\begin{tikzcd}
\mathfrak{l}_\lambda & \mathfrak{p}_\lambda \arrow[l, two heads] \arrow[r, hook] & \tilde{\mathfrak{g}}_\lambda := G \times^{P_\lambda} \mathfrak{p}_\lambda \arrow[r, two heads, "m_\lambda"] & \mathfrak{g} \\
\mathcal{N}_{L_\lambda} \arrow[u, hook] & \mathcal{N}_{P_\lambda} \arrow[l, two heads] \arrow[r, hook] \arrow[u, hook] & \breve{\mathcal{N}}_\lambda := G \times^{P_\lambda} \mathcal{N}_{P_\lambda} \arrow[r, two heads, "\breve{m}_\lambda"] \arrow[u, hook] & \mathcal{N} \arrow[u, hook] \\
\{0\} \arrow[u, hook] & \mathfrak{u}_\lambda \arrow[l, two heads] \arrow[r, hook] \arrow[u, hook] & \tilde{\mathcal{N}}_\lambda := G \times^{P_\lambda} \mathfrak{u}_\lambda \arrow[r, two heads, "\tilde{m}_\lambda"] \arrow[u, hook] & \overline{\mathcal{O}_{\lambda^\vee}} \arrow[u, hook]
\end{tikzcd} \tag{4.1}$$

Let $B \subset G$ be the Borel subgroup consisting of upper triangular invertible matrices. For a weak composition $\lambda = (\lambda_1, \dots, \lambda_n)$ of d, let $P_\lambda \supset B$ be the parabolic subgroup of G with Levi factor, L_λ, consisting of invertible diagonal block matrices of sizes $\lambda_1, \dots, \lambda_n$ i.e. $L_\lambda \simeq \mathrm{GL}_{\lambda_1} \times \cdots \times \mathrm{GL}_{\lambda_n}$ and $P_\lambda = L_\lambda \ltimes U_\lambda$, for a unipotent subgroup U_λ. For example, $P_{(1,\dots,1)} = B$ and $P_{(d)} = G$. Recall that the *partial flag varieties* G/P_λ can be identified with the spaces

$$\mathcal{P}_\lambda := \{\text{parabolic subalgebras of } \mathfrak{g} \text{ conjugate to } \mathfrak{p}_\lambda\}$$
$$\mathcal{F}l_\lambda := \{0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_m = \mathbb{C}^n \mid \dim \mathcal{F}_i/\mathcal{F}_{i-1} = \lambda_i\}$$

Indeed, G acts transitively on $\mathcal{P}_\lambda$ by conjugation, with stabiliser P_λ. So there is a bijection $G/P_\lambda \to \mathcal{P}_\lambda$ defined $gP_\lambda \mapsto g\mathfrak{p}_\lambda g^{-1}$. The group G acts transitively on $\mathcal{F}l_\lambda$ by the left natural action on flags. The isomorphism $\mathcal{P}_\lambda \to \mathcal{F}l_\lambda$ is the G-equivariant map sending $\mathfrak{p}_\lambda$ to the flag $\mathcal{E}^\lambda_* = (0 \subset \mathcal{E}^\lambda_1 \subset \cdots \subset \mathcal{E}^\lambda_m = \mathbb{C}^n)$ defined by $\mathcal{E}^\lambda_i = \mathbb{C}\{e_1, \ldots, e_{\lambda_1+\cdots+\lambda_i}\}$. The inverse $\mathcal{F}l_\lambda \to \mathcal{P}_\lambda$ sends a flag $\mathcal{F}_*$ the parabolic subalgebra $\{x \in \mathfrak{g} \mid x(\mathcal{F}_i) \subset \mathcal{F}_i\}$.

Consider the diagonal action of G on $G/P_\lambda \times G/P_\mu$. The G-orbits of $G/P_\lambda \times G/P_\mu$ are in bijection with the set of double cosets $\mathfrak{S}_\lambda \backslash \mathfrak{S}_d / \mathfrak{S}_\mu$. Indeed, for $\sigma \in \mathfrak{S}_d$, let $\dot{\sigma} \in G$ be the corresponding permutation matrix. The map

$$\mathfrak{S}_d \to G\backslash(G/P_\lambda \times G/P_\mu)$$

sending $\sigma \in \mathfrak{S}_d$ to the orbit

$$\mathbb{O}^\sigma_{\lambda,\mu} := G \cdot (P_\lambda, \dot{\sigma} P_\mu)$$

is surjective and $\mathbb{O}^{\sigma_1}_{\lambda,\mu} = \mathbb{O}^{\sigma_2}_{\lambda,\mu}$ if and only if $\mathfrak{S}_\lambda \sigma_1 \mathfrak{S}_\mu = \mathfrak{S}_\lambda \sigma_2 \mathfrak{S}_\mu$. For any weak compositions λ, μ of d, the G-orbit $\mathbb{O}^e_{\lambda\mu} \subset G/P_\lambda \times G/P_\mu$ is a smooth closed subvariety (see e.g. [BLM90, Lemma 3.6]).

Remark 4.2.1. The closure order for the stratification of $G/P_\lambda \times G/P_\mu$ is described in [BLM90, Lemma 3.6]. More precisely, let $\lambda \in \Lambda(m,d)$ and $\mu \in \Lambda(n,d)$. Let $\sigma_1, \sigma_2 \in \mathfrak{S}_d$ and define the matrices $A = \Theta^\lambda_\mu(\sigma_1)$ and $B = \Theta^\lambda_\mu(\sigma_2)$ as in Definition 3.5.1. Then there is an inclusion $\mathbb{O}^{\sigma_1}_{\lambda,\mu} \subseteq \overline{\mathbb{O}^{\sigma_2}_{\lambda,\mu}}$ if and only if for all $i \in [m]$, $j \in [n]$,

$$\sum_{p \le i; q \le j} B_{pq} \le \sum_{p \le i; q \le j} A_{pq}.$$

In the closure order, $\mathbb{O}^e_{\lambda\mu}$ is minimal. The unique maximal orbit is $\mathbb{O}^{w_0}_{\lambda\mu}$, where $w_0 \in \mathfrak{S}_d$ is the longest element. For the dimensions of the orbits see [BLM90, Lemma 2.2].

For an algebraic group H with closed subgroup $P \subset H$, and P-variety X, the *induction space* $H \times^P X$ is the quotient of $H \times P$ by the P-action

$$g \cdot (h, x) = (hg^{-1}, g \cdot x).$$

The map $H \times^P X \to H/P$ mapping $(h,x) \mapsto hP$ makes $H \times^P X$ into a H-equivariant fibre bundle over H/P with fibre X.

Define the *partial Grothendieck resolution* corresponding to a weak composition λ of d:

$$\tilde{\mathfrak{g}}_\lambda := G \times^{P_\lambda} \mathfrak{p}_\lambda.$$

The partial Grothendieck resolutions have the following alternative descriptions.

Proposition 4.2.2. *There are bijections*

$$G \times^{P_\lambda} \mathfrak{p}_\lambda \simeq \{(x, \mathfrak{p}) \in \mathfrak{g} \times \mathcal{P}_\lambda \mid x \in \mathfrak{p}\} \simeq \{(x, \mathcal{F}_*) \in \mathfrak{g} \times \mathcal{F}l_\lambda \mid x(\mathcal{F}_i) \subset \mathcal{F}_i\}.$$

Proof. The second bijection is induced from the bijection $\mathcal{P}_\lambda \simeq \mathcal{F}l_\lambda$. The first bijection is defined $(g, x) \mapsto (gxg^{-1}, g\mathfrak{p}_\lambda g^{-1})$. Indeed it is straightforward to check that this map is well-defined with well-defined inverse $(x, g\mathfrak{p}_\lambda g^{-1}) \mapsto (g, g^{-1}xg)$. $\square$

Define the *multiplication map* $m_\lambda : G \times^{P_\lambda} \mathfrak{p}_\lambda \to \mathfrak{g}$ by $(g, x) \mapsto gxg^{-1}$. In terms of the alternate descriptions of $\tilde{\mathfrak{g}}_\lambda$ in Proposition 4.2.2, the multiplication map $m_\lambda : \tilde{\mathfrak{g}}_\lambda \to \mathfrak{g}$ is simply the projection into the first component: $(x, \mathfrak{p}) \mapsto x$ and $(x, \mathcal{F}_*) \mapsto x$.

Define the *parabolic Springer bundle*:

$$\breve{\mathcal{N}}_\lambda := m_\lambda^{-1}(\mathcal{N}) = G \times^{P_\lambda} \mathcal{N}_{P_\lambda} = G \times^{P_\lambda} (\mathfrak{u}_\lambda + \mathcal{N}_{L_\lambda})$$

For example, $\breve{\mathcal{N}}_{(1,\dots,1)} \simeq G \times^B \mathfrak{u}_{(1,\dots,1)}$ is the usual *Springer resolution*. This is a resolution of singularities of $\mathcal{N}$. At the other extreme is $\breve{\mathcal{N}}_{(d)} \simeq \mathcal{N}$. Each variety $\breve{\mathcal{N}}_\lambda$ has dimension $d^2 - d$, and all except for $\breve{\mathcal{N}}_{(1,\dots,1)}$ are singular varieties.

As in Proposition 4.2.2, there is a bijection

$$\breve{\mathcal{N}}_\lambda \simeq \{(x, \mathfrak{p}) \in \mathcal{N} \times \mathcal{P}_\lambda \mid x \in \mathfrak{p}\} \simeq \{(x, \mathcal{F}_*) \in \mathcal{N} \times \mathcal{F}l_\lambda \mid x(\mathcal{F}_i) \subset \mathcal{F}_i\}.$$

Define the multiplication map $\breve{m}_\lambda : \breve{\mathcal{N}}_\lambda \to \mathcal{N}$ to be the restriction of m_λ. The G-orbits of $\breve{\mathcal{N}}_\lambda$ are the subvarieties $G \times^{P_\lambda} (\mathfrak{u}_\lambda + C)$, where $C \subset \mathcal{N}_{L_\lambda}$ is an L_λ-orbit. For each G-orbit $\mathcal{O} \subset \breve{\mathcal{N}}_\lambda$, the proper G-equivariant map $\breve{m}_\lambda : \breve{\mathcal{N}}_\lambda|_{\mathcal{O}} \to \mathcal{N}$ is semismall (see e.g. [Ach21, Theorem 8.4.10]). A consequence of this is that the pushforward map $\breve{m}_{\lambda!} : \mathcal{D}_G(\breve{\mathcal{N}}_\lambda, \Bbbk) \to \mathcal{D}_G(\mathcal{N}, \Bbbk)$ is t-exact for the equivariant perverse t-structure (see e.g. [Ach21, Theorem 3.8.9]). Further details about the geometry of $\breve{\mathcal{N}}_\lambda$ can be found in [Lus84].

Define the *partial Springer resolution*:

$$\tilde{\mathcal{N}}_\lambda := m_\lambda^{-1}(\overline{\mathcal{O}_{\lambda^\vee}}) \simeq G \times^{P_\lambda} \mathfrak{u}_\lambda.$$

Indeed, $\overline{\mathcal{O}_{\lambda^\vee}} = \{x \in \mathcal{N} \mid \dim\ker x^i \leq \lambda_i\} = G \cdot \mathfrak{u}_\lambda$. As in Proposition 4.2.2, there is a bijection

$$\tilde{\mathcal{N}}_\lambda \simeq \{(x, \mathcal{F}_*) \in \mathcal{N} \times \mathcal{F}l_\lambda \mid x(\mathcal{F}_i) \subset \mathcal{F}_{i-1}\}.$$

The vector bundle $\tilde{\mathcal{N}}_\lambda \to G/P_\lambda; (g, x) \mapsto gP_\lambda$ is isomorphic to the cotangent bundle $T^*(G/P_\lambda) \to G/P_\lambda$. Indeed, the cotangent space of a point $gP_\lambda \in G/P_\lambda$ can be identified with $g\mathfrak{u}_\lambda g^{-1}$ via the isomorphism $\mathfrak{g}^* \simeq \mathfrak{g}$ given by the Killing form (see e.g. [CG97, Proposition 4.1.2] for a more detailed proof).

Of central importance to the geometric construction of the Schur algebra are the *partial Steinberg varieties*:

$$\tilde{\mathcal{N}}_{\lambda\mu} := \tilde{\mathcal{N}}_\lambda \times_{\mathcal{N}} \tilde{\mathcal{N}}_\mu.$$

As in Proposition 4.2.2, there is a bijection

$$\begin{aligned}\tilde{\mathcal{N}}_{\lambda\mu} &\simeq \{(x, \mathfrak{p}_1, \mathfrak{p}_2) \in \overline{\mathcal{O}_{\lambda^\vee}} \cap \overline{\mathcal{O}_{\mu^\vee}} \times \mathcal{P}_\lambda \times \mathcal{P}_\mu \mid x \in \mathfrak{p}_1 \cap \mathfrak{p}_2\} \\ &\simeq \{(x, \mathcal{F}_*^{(1)}, \mathcal{F}_*^{(2)}) \in \mathcal{N} \times \mathcal{F}l_\lambda \times \mathcal{F}l_\mu \mid x(\mathcal{F}_i^{(1)}) \subset \mathcal{F}_{i-1}^{(1)},\ x(\mathcal{F}_i^{(2)}) \subset \mathcal{F}_{i-1}^{(2)}\}.\end{aligned}$$

Let $pr_{23} : \tilde{\mathcal{N}}_{\lambda\mu} \to G/P_\lambda \times G/P_\mu$ be the projection onto the second and third components in either of these descriptions of $\tilde{\mathcal{N}}_{\lambda\mu}$. For $\sigma \in \mathfrak{S}_d$, consider the variety

$$pr_{23}^{-1}(\mathbb{O}_{\lambda\mu}^\sigma) \simeq \{(x, g \cdot \mathfrak{p}_\lambda, g\dot{\sigma} \cdot \mathfrak{p}_\mu) \in \tilde{\mathcal{N}}_{\lambda\mu} \mid g \in G\}.$$

The following result is shown in [CG97, Proposition 4.1.6] and [CG97, Corollary 4.1.8].

Proposition 4.2.3. *There is an isomorphism*

$$\tilde{\mathcal{N}}_\lambda \times \tilde{\mathcal{N}}_\mu \simeq T^*G/P_\lambda \times T^*G/P_\mu \simeq T^*(G/P_\lambda \times G/P_\mu)$$

that identifies $pr_{23}^{-1}(\mathbb{O}^\sigma_{\lambda\mu})$ with the conormal bundle, $T^*_{\mathbb{O}^\sigma_{\lambda\mu}}(G/P_\lambda \times G/P_\mu)$, to $\mathbb{O}^\sigma_{\lambda\mu}$ in $G/P_\lambda \times G/P_\mu$. In particular, $\tilde{\mathcal{N}}_{\lambda\mu}$ is equidimensional and

$$\dim \tilde{\mathcal{N}}_{\lambda\mu} = \frac{1}{2}(\dim \tilde{\mathcal{N}}_\lambda + \dim \tilde{\mathcal{N}}_\mu).$$

Moreover, the irreducible components of $\tilde{\mathcal{N}}_{\lambda\mu}$ are exactly the closed irreducible subvarieties

$$\tilde{\mathcal{N}}^\sigma_{\lambda\mu} := \overline{pr_{23}^{-1}(\mathbb{O}^\sigma_{\lambda\mu})},$$

for $\sigma \in \mathfrak{S}_d$, and $\tilde{\mathcal{N}}^{\sigma_1}_{\lambda\mu} = \tilde{\mathcal{N}}^{\sigma_2}_{\lambda\mu}$ if and only if $\mathfrak{S}_\lambda \sigma_1 \mathfrak{S}_\mu = \mathfrak{S}_\lambda \sigma_2 \mathfrak{S}_\mu$.

Example 4.2.4. Consider the variety $\tilde{\mathcal{N}}_{(1,1),(1,1)}$. This is the disjoint union of the subvarieties

$$pr_{23}^{-1}(\mathbb{O}^e_{(1,1),(1,1)}) \simeq \{(x, \mathfrak{b}_1, \mathfrak{b}_1) \in \mathcal{N} \times \mathcal{P}_{(1,1)} \times \mathcal{P}_{(1,1)} \mid x \in \mathfrak{b}_1\}$$

and

$$pr_{23}^{-1}(\mathbb{O}^{(12)}_{(1,1),(1,1)}) \simeq \{(0, \mathfrak{b}_1, \mathfrak{b}_2) \in \mathcal{N} \times \mathcal{P}_{(1,1)} \times \mathcal{P}_{(1,1)} \mid \mathfrak{b}_1 \neq \mathfrak{b}_2\}.$$

The irreducible components of $\tilde{\mathcal{N}}_{(1,1),(1,1)}$ are

$$\tilde{\mathcal{N}}^e_{(1,1),(1,1)} = pr_{23}^{-1}(\mathbb{O}^e_{(1,1),(1,1)})$$

and

$$\tilde{\mathcal{N}}^{(12)}_{(1,1),(1,1)} \simeq \{(0, \mathfrak{b}_1, \mathfrak{b}_2) \in \mathcal{N} \times \mathcal{P}_{(1,1)} \times \mathcal{P}_{(1,1)}\}.$$

Remark 4.2.5. The variety $\tilde{\mathcal{N}}_{(1,\dots,1),(1,\dots,1)}$ is known as the *Steinberg variety*. For a survey of results in geometric representation theory that are proven using the Steinberg variety see [DR09].

4.3 Geometric reconstruction of the Schur algebra

In this section we define the convolution product on Borel-Moore homology and construct the Schur algebra using the convolution product on the Borel-Moore homology of the varieties $\tilde{\mathcal{N}}_{\lambda\mu} := \tilde{\mathcal{N}}_\lambda \times_{\mathcal{N}} \tilde{\mathcal{N}}_\mu$. Before doing this we recall the construction of Borel-Moore homology using sheaf theory, and define the basis of fundamental classes for the top Borel-Moore homology space.

For $\mathcal{F} \in \mathcal{D}^b(X)$ the *sheaf cohomology* of $\mathcal{F}$ is defined

$$H^*(X, \mathcal{F}) := R\Gamma(\mathcal{F}) = \mathrm{Hom}_{\mathcal{D}^b(X)}(\underline{\Bbbk}_X, \mathcal{F}).$$

Define the k-th *Borel-Moore homology group* by

$$H_k^{BM}(X, \Bbbk) := H^{-k}(X, \mathbb{D}_X) = \mathrm{Hom}_{\mathcal{D}^b(X)}(\underline{\Bbbk}_X[k], \mathbb{D}_X).$$

If X is a smooth variety, then say that a morphism of varieties $f : X \to Y$ is *smooth of relative dimension* d if all fibres of f are d-dimensional.

If $f : X \to Y$ is a smooth morphism of relative dimension d then $f^* \simeq f^![-2d]$. In particular if X is a smooth variety of dimension d, then the canonical surjection $a_X : X \to pt$ is smooth of relative dimension d and so $\mathbb{D}_X \simeq \underline{\Bbbk}_X[2d]$. In particular we recover the well-known isomorphism:

$$H_k^{BM}(X, \Bbbk) \simeq H^{2d-k}(X, \Bbbk).$$

If $f : X \to Y$ is smooth of relative dimension d, then define the *pullback map* $f^\# : H_k^{BM}(Y, \Bbbk) \to H_{k+2d}^{BM}(X, \Bbbk)$ by the natural map

$$f^\# : \mathrm{Hom}(\underline{\Bbbk}_Y[k], \mathbb{D}_Y) \to \mathrm{Hom}(f^*\underline{\Bbbk}_Y[k], f^*\mathbb{D}_Y) \simeq \mathrm{Hom}(\underline{\Bbbk}_X[k+2d], \mathbb{D}_X),$$

where since f is smooth we use the identification $f^*\mathbb{D}_Y \simeq f^!\mathbb{D}_Y[-2d] = \mathbb{D}_X[-2d]$.

If $f : X \to Y$ is proper then $f_! = f_*$ and so we can define a map

$$f_\# : R\,\mathrm{Hom}(f^*\mathcal{F}, f^!\mathcal{G}) \simeq R\,\mathrm{Hom}(\mathcal{F}, f_!f^!\mathcal{G}) \to R\,\mathrm{Hom}(\mathcal{F}, \mathcal{G}).$$

Applying this to $\mathcal{F} = \underline{\Bbbk}_Y$, $\mathcal{G} = \mathbb{D}_Y$ gives the *pushforward map*

$$f_\# : H_k^{BM}(X, \Bbbk) \to H_k^{BM}(Y, \Bbbk).$$

Let $i : Z \hookrightarrow X$ be Zariski closed with open complement $j : U \hookrightarrow X$, and consider the triangle

$$i_*\mathbb{D}_Z \simeq i_*i^!\mathbb{D}_X \to \mathbb{D}_X \to j_*j^*\mathbb{D}_X \simeq j_*\mathbb{D}_U \to$$

and corresponding long exact sequence

$$\cdots \to \mathrm{Hom}(\underline{\Bbbk}_X, i_*\mathbb{D}_Z[-k]) \to \mathrm{Hom}(\underline{\Bbbk}_X, \mathbb{D}_X[-k]) \to \mathrm{Hom}(\underline{\Bbbk}_X, j_*\mathbb{D}_U[-k]) \to \mathrm{Hom}(\underline{\Bbbk}_X, i_*\mathbb{D}_Z[-k+1]) \to \cdots$$

After applying adjunctions we recover the long exact sequence in Borel-Moore homology:

$$\longrightarrow H_k^{BM}(Z,\Bbbk) \xrightarrow{i_\#} H_k^{BM}(X,\Bbbk) \xrightarrow{j^\#} H_k^{BM}(U,\Bbbk) \longrightarrow H_{k-1}^{BM}(Z,\Bbbk) \longrightarrow$$

Proposition 4.3.1. *Let X be an irreducible variety of dimension d, and let $j : U \hookrightarrow X$ be the inclusion of a Zariski open subset. Then $j^\# : H_{2d}^{BM}(X,\Bbbk) \to H_{2d}^{BM}(U,\Bbbk)$ is an isomorphism.*

Proof. Let $Z = X \setminus U$. Since X is irreducible, $\dim Z < d$. The result follows by the long exact sequence in Borel-Moore homology. □

A consequence of Proposition 4.3.1 is that if X is an irreducible variety of dimension d and U is a smooth Zariski open subvariety of X, then there is an isomorphism

$$H_{2d}^{BM}(X,\Bbbk) \simeq H_{2d}^{BM}(U,\Bbbk) \simeq \mathrm{Hom}_{\mathcal{D}^b(U)}(\underline{\Bbbk}_U, \underline{\Bbbk}_U).$$

Let $[X] \in H_{2d}^{BM}(X,\Bbbk)$ be the element corresponding to $1 : \Bbbk_U \to \Bbbk_U$ under this isomorphism. In the case that X is smooth, $[X] : \Bbbk_X[2d] \to \mathbb{D}_X$ is the canonical isomorphism. The element $[X] \in H_{2d}^{BM}(X,\Bbbk)$ is independent of the choice of smooth open subset $U \subset X$ (see e.g. [Ach21, Lemma 2.11.8]).

If X is any (not necessarily irreducible) variety, and $i : Z \hookrightarrow X$ is an irreducible closed subvariety of dimension m, then we write $[Z] := i_\#[Z] \in H_{2m}^{BM}(X,\Bbbk)$ where $i_\# : H_{2m}^{BM}(Z,\Bbbk) \to H_{2m}^{BM}(X,\Bbbk)$.

The element $[Z] \in H_{2m}^{BM}(X,\Bbbk)$ is called the *fundamental class* corresponding to the irreducible closed subvariety $Z \subset X$.

The following result is well-known (see e.g. [Ach21, Proposition 2.11.11] for a proof).

Proposition 4.3.2. *Let X be a variety of dimension d, and let $X_1, \ldots, X_k$ be it's d-dimensional irreducible components. Then $H_{2d}^{BM}(X,\Bbbk)$ is a free $\Bbbk$-module with basis $[X_1], \ldots, [X_k]$.*

A corollary of this result is that

$$H_{2d}^{BM}(X,\Bbbk) \simeq \Bbbk \otimes H_{2d}^{BM}(X,\mathbb{Z})$$

for any commutative algebra $\Bbbk$. By Proposition 4.2.3, it follows from this that $H^{BM}_{\dim \tilde{\mathcal{N}}_\lambda + \dim \tilde{\mathcal{N}}_\mu}(\tilde{\mathcal{N}}_{\lambda\mu}, \Bbbk)$ has a basis of fundamental classes, $[\tilde{\mathcal{N}}^\sigma_{\lambda\mu}]$, for representatives σ of $\mathfrak{S}_\lambda \backslash \mathfrak{S}_d / \mathfrak{S}_\mu$.

If X is smooth of dimension d and A and B are closed subvarieties, then there is an intersection pairing

$$\cap : H^{BM}_i(A, \Bbbk) \times H^{BM}_j(B, \Bbbk) \to H^{BM}_{i+j-2d}(A \cap B, \Bbbk).$$

To avoid developing more theory than we will use, we do not define the intersection pairing here.

Definition 4.3.3 (Convolution product)**.** Let M_1, M_2, M_3 be smooth varieties, and define the projections $p_{ij} : M_1 \times M_2 \times M_3 \to M_i \times M_j$. For closed subvarieties $Z_{12} \subset M_1 \times M_2$, $Z_{23} \subset M_2 \times M_3$ define the closed subvariety, Z_{13}, of $M_1 \times M_3$ by:

$$\begin{aligned} Z_{13} &= Z_{12} \circ Z_{23} \\ &= \{(m_1, m_3) \in M_1 \times M_3 \mid \exists m_2 \in M_2 : (m_1, m_2) \in Z_{12}, (m_2, m_3) \in Z_{23}\}. \end{aligned}$$

Suppose the projection $p_{13} : M_1 \times M_2 \times M_3 \to M_1 \times M_3$ restricts to a proper map:

$$p_{13} : p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}) \to Z_{13}.$$

Define the *convolution product*:

$$H^{BM}_i(Z_{12}, \Bbbk) \times H^{BM}_j(Z_{23}, \Bbbk) \to H^{BM}_{i+j-2\dim M_2}(Z_{13}, \Bbbk)$$

by

$$c * d = (p_{13})_\#(p_{12}^\#(c) \cap p_{23}^\#(d)).$$

We now recall a result about the convolution product on the Borel-Moore homology of cotangent bundles of smooth complex varieties.

Let X_1, X_2, X_3 be smooth varieties and consider the diagram:

$$\begin{tikzcd} & X_1 \times X_2 \times X_3 \arrow[dl, "p_{12}"'] \arrow[d, "p_{13}"'] \arrow[dr, "p_{23}"] & \\ X_1 \times X_2 & X_1 \times X_3 & X_2 \times X_3 \\ Y_{12} \arrow[u, hook] & Y_{13} = Y_{12} \circ Y_{23} \arrow[u, hook] & Y_{23} \arrow[u, hook] \end{tikzcd}$$

in which $Y_{ij} \subset X_i \times X_j$ is a closed embedding. Consider the following diagram:

$$\begin{tikzcd}
 & T^*(X_1 \times X_2 \times X_3) \arrow[dl, "pr_{12}"'] \arrow[d, "pr_{13}"'] \arrow[dr, "pr_{23}"] & \\
T^*(X_1 \times X_2) & T^*(X_1 \times X_3) & T^*(X_2 \times X_3) \\
Z_{12} = T^*_{Y_{12}}(X_1 \times X_2) \arrow[u, hook] & Z_{13} = T^*_{Y_{13}}(X_1 \times X_3) \arrow[u, hook] & Z_{23} = T^*_{Y_{23}}(X_2 \times X_3) \arrow[u, hook]
\end{tikzcd}$$

The following result is proven by Ginzburg [CG97, Theorem 2.7.26].

Proposition 4.3.4. *Assume that Y_{12} and Y_{23} satisfy two conditions:*

(a) The spaces $p_{12}^{-1}(Y_{12})$ and $p_{23}^{-1}(Y_{23})$ are transverse.

(b) The map $p_{13} : p_{12}^{-1}(Y_{12}) \cap p_{23}^{-1}(Y_{23}) \to Y_{13}$ is a smooth locally trivial oriented fibration with smooth base Y_{13} and smooth and compact fibre F.

Then the following holds:

(i) There is a set equality $Z_{13} = Z_{13} \circ Z_{23}$.

(ii) The map $pr_{13} : pr_{12}^{-1}(Z_{12}) \cap pr_{23}^{-1}(Z_{23}) \to Z_{13}$ is a smooth locally trivial oriented fibration with fiber F.

(iii) In $H_*^{BM}(Z_{13}, \Bbbk)$:

$$[Z_{12}] * [Z_{23}] = \chi(F) \cdot [Z_{13}],$$

where $\chi(F)$ is the Euler characteristic of F.

Example 4.3.5. Consider the case

$$X_1 = G/P_\lambda, \qquad X_2 = G/P_{\lambda+r\alpha_i}, \qquad X_3 = G/P_{\lambda+(r+s)\alpha_i}.$$

Let

$$Y_{12} = \mathbb{O}^e_{\lambda,\lambda+r\alpha_i} \simeq \{(\mathcal{F}_*, \mathcal{F}'_*) \in \mathcal{F}l_\lambda \times \mathcal{F}l_{\lambda+r\alpha_i} \mid \mathcal{F}_i \subset \mathcal{F}'_i, \mathcal{F}_j = \mathcal{F}'_j \text{ if } j \neq i\}$$

and

$$\begin{aligned} Y_{23} &= \mathbb{O}^e_{\lambda+r\alpha_i,\lambda+(r+s)\alpha_i} \\ &\simeq \{(\mathcal{F}_*, \mathcal{F}'_*) \in \mathcal{F}l_{\lambda+r\alpha_i} \times \mathcal{F}l_{\lambda+(r+s)\alpha_i} \mid \mathcal{F}_i \subset \mathcal{F}'_i, \mathcal{F}_j = \mathcal{F}'_j \text{ if } j \neq i\}. \end{aligned}$$

Note that for any pair $(\mathcal{F}_*, \mathcal{F}'_*)$ in Y_{12} (respectively Y_{23}), $\dim \mathcal{F}'_i/\mathcal{F}_i = r$ (respectively $\dim \mathcal{F}'_i/\mathcal{F}_i = s$). Then

$$Y_{13} = \mathbb{O}^e_{\lambda,\lambda+(r+s)\alpha_i} \simeq \{(\mathcal{F}_*, \mathcal{F}'_*) \in \mathcal{F}l_\lambda \times \mathcal{F}l_{\lambda+(r+s)\alpha_i} \mid \mathcal{F}_i \subset \mathcal{F}'_i, \mathcal{F}_j = \mathcal{F}'_j \text{ if } j \neq i\}$$

and

$$Z_{12} \simeq \tilde{\mathcal{N}}^e_{\lambda,\lambda+r\alpha_i}, \qquad Z_{23} = \tilde{\mathcal{N}}^e_{\lambda+r\alpha_i,\lambda+(r+s)\alpha_i}, \qquad Z_{13} = \tilde{\mathcal{N}}^e_{\lambda,\lambda+(r+s)\alpha_i}.$$

Proposition 4.3.4 says that in $H^{BM}_*(\tilde{\mathcal{N}}^e_{\lambda,\lambda+(r+s)\alpha_i}, \Bbbk)$:

$$[\tilde{\mathcal{N}}^e_{\lambda,\lambda+r\alpha_i}] * [\tilde{\mathcal{N}}^e_{\lambda+r\alpha_i,\lambda+(r+s)\alpha_i}] = \binom{r+s}{r}[\tilde{\mathcal{N}}^e_{\lambda,\lambda+(r+s)\alpha_i}].$$

Indeed $p_{12}^{-1}(Y_{12}) \cap p_{23}^{-1}(Y_{23})$ is the variety

$$\{(\mathcal{F}_*, \mathcal{F}'_*, \mathcal{F}''_*) \in \mathcal{F}l_\lambda \times \mathcal{F}l_{\lambda+r\alpha_i} \times \mathcal{F}l_{\lambda+(r+s)\alpha_i} \mid \mathcal{F}_i \subset \mathcal{F}'_i \subset \mathcal{F}''_i, \mathcal{F}_j = \mathcal{F}'_j = \mathcal{F}''_j \text{ if } j \neq i\},$$

where $\dim \mathcal{F}''_i/\mathcal{F}'_i = s$ and $\dim \mathcal{F}'_i/\mathcal{F}_i = r$ for each $(\mathcal{F}_*, \mathcal{F}'_*, \mathcal{F}''_*) \in p_{12}^{-1}(Y_{12}) \cap p_{23}^{-1}(Y_{23})$. In particular the fibre of the map $p_{13} : p_{12}^{-1}(Y_{12}) \cap p_{23}^{-1}(Y_{23}) \to Y_{13}$ is homeomorphic to the Grassmannian of r dimensional subspaces of $\mathbb{C}^{r+s}$. This Grassmannian has Euler characteristic $\binom{r+s}{r}$.

Theorem 4.3.6. *There is an algebra isomorphism*

$$\mathcal{S}_\Bbbk(n,d)^{op} \simeq \bigoplus_{\lambda,\mu\in\Lambda(n,d)} H^{BM}_{\dim \tilde{\mathcal{N}}_\lambda + \dim \tilde{\mathcal{N}}_\mu}(\tilde{\mathcal{N}}_{\lambda\mu}, \Bbbk)$$

defined

$$\begin{aligned} 1_\lambda &\mapsto [\tilde{\mathcal{N}}^e_{\lambda,\lambda}] \\ E_i^{(r)} 1_\lambda &\mapsto [\tilde{\mathcal{N}}^e_{\lambda,\lambda+r\alpha_i}] \\ F_i^{(r)} 1_\lambda &\mapsto [\tilde{\mathcal{N}}^e_{\lambda,\lambda-r\alpha_i}] \end{aligned}$$

Proof. Define the algebra

$$\mathcal{H}_\Bbbk(n,d) := \bigoplus_{\lambda,\mu\in\Lambda(n,d)} H^{BM}_{\dim \tilde{\mathcal{N}}_\lambda + \dim \tilde{\mathcal{N}}_\mu}(\tilde{\mathcal{N}}_{\lambda\mu}, \Bbbk).$$

Note that identity in $\mathcal{H}_{\Bbbk}(n,d)$ is the element

$$1 := \sum_{\lambda\in\Lambda(n,d)} [\tilde{\mathcal{N}}^e_{\lambda,\lambda}].$$

For $\lambda \in \mathbb{Z}^n$ in which $\lambda \notin \Lambda(n,d)$, we set

$$[\tilde{\mathcal{N}}^\sigma_{\lambda,\mu}] = [\tilde{\mathcal{N}}^\sigma_{\mu,\lambda}] = 0 \in \mathcal{H}_{\Bbbk}(n,d)$$

for any $\sigma \in \mathfrak{S}_d$ and any $\mu \in \mathbb{Z}^n$.

By Proposition 4.3.2, $\mathcal{H}_{\mathbb{Z}}(n,d)$ is the $\mathbb{Z}$-subalgebra of $\mathcal{H}_{\mathbb{C}}(n,d)$ spanned by the fundamental classes $[\tilde{\mathcal{N}}^\sigma_{\lambda\mu}]$ for all $\lambda, \mu \in \Lambda(n,d)$ and $\sigma \in \mathfrak{S}_d$. Ginzburg [CG97, Proposition 4.2.5] defines an isomorphism $\Psi : \mathcal{S}_{\mathbb{C}}(n,d)^{op} \to \mathcal{H}_{\mathbb{C}}(n,d)$ that sends

$$1_\lambda \mapsto [\tilde{\mathcal{N}}^e_{\lambda,\lambda}], \qquad E_i 1_\lambda \mapsto [\tilde{\mathcal{N}}^e_{\lambda,\lambda+\alpha_i}], \qquad F_i 1_\lambda \mapsto [\tilde{\mathcal{N}}^e_{\lambda,\lambda-\alpha_i}].$$

For completeness we sketch a proof of Ginzburg's result. By Proposition 3.1.3, to show that $\Psi : \mathcal{S}_{\mathbb{C}}(n,d)^{op} \to \mathcal{H}_{\mathbb{C}}(n,d)$ is well-defined it suffices to check that the opposite of relations (3.9)-(3.12) hold in $\mathcal{H}_{\mathbb{C}}(n,d)$. That is,

$$[\tilde{\mathcal{N}}^e_{\mu,\mu}] * [\tilde{\mathcal{N}}^e_{\lambda,\lambda}] = \delta_{\lambda,\mu}[\tilde{\mathcal{N}}^e_{\lambda,\lambda}], \tag{4.2}$$

$$\begin{aligned} [\tilde{\mathcal{N}}^e_{\lambda,\lambda}] * [\tilde{\mathcal{N}}^e_{\lambda,\lambda+\alpha_i}] &= [\tilde{\mathcal{N}}^e_{\lambda,\lambda+\alpha_i}] * [\tilde{\mathcal{N}}^e_{\lambda+\alpha_i,\lambda+\alpha_i}], \\ [\tilde{\mathcal{N}}^e_{\lambda,\lambda}] * [\tilde{\mathcal{N}}^e_{\lambda,\lambda-\alpha_i}] &= [\tilde{\mathcal{N}}^e_{\lambda,\lambda-\alpha_i}] * [\tilde{\mathcal{N}}^e_{\lambda-\alpha_i,\lambda-\alpha_i}], \end{aligned} \tag{4.3}$$

$$[\tilde{\mathcal{N}}^e_{\lambda,\lambda-\alpha_i}] * [\tilde{\mathcal{N}}^e_{\lambda-\alpha_i,\lambda}] = [\tilde{\mathcal{N}}^e_{\lambda,\lambda+\alpha_i}] * [\tilde{\mathcal{N}}^e_{\lambda+\alpha_i,\lambda}] + (\lambda_i - \lambda_{i+1})[\tilde{\mathcal{N}}^e_{\lambda,\lambda}] \tag{4.4}$$

$$[\tilde{\mathcal{N}}^e_{\lambda,\lambda-\alpha_j}] * [\tilde{\mathcal{N}}^e_{\lambda-\alpha_j,\lambda-\alpha_j+\alpha_i}] = [\tilde{\mathcal{N}}^e_{\lambda,\lambda+\alpha_i}] * [\tilde{\mathcal{N}}^e_{\lambda+\alpha_i,\lambda+\alpha_i-\alpha_j}] \quad (i \neq j), \tag{4.5}$$

Equations (4.2) and (4.3) follow from the definition of the convolution product. Equation (4.5) follows from Proposition 4.3.4. A proof of Equation (4.4) is given in [CG97, Equation (4.3.8)].

The surjectivity of $\Psi : \mathcal{S}_{\mathbb{C}}(n,d)^{op} \to \mathcal{H}_{\mathbb{C}}(n,d)$ is shown in [CG97, Proposition 4.3.14]. Since $\dim \mathcal{S}_{\mathbb{C}}(n,d)^{op} = \dim \mathcal{H}_{\mathbb{C}}(n,d)$, the map $\Psi : \mathcal{S}_{\mathbb{C}}(n,d)^{op} \to \mathcal{H}_{\mathbb{C}}(n,d)$ is an isomorphism.

It follows from Example 4.3.5 that $\Psi : \mathcal{S}_{\mathbb{C}}(n,d)^{op} \to \mathcal{H}_{\mathbb{C}}(n,d)$ maps

$$E_i^{(r)} 1_\lambda \mapsto [\tilde{\mathcal{N}}^e_{\lambda,\lambda+r\alpha_i}] \qquad \text{and} \qquad F_i^{(r)} 1_\lambda \mapsto [\tilde{\mathcal{N}}^e_{\lambda,\lambda-r\alpha_i}].$$

Indeed, by induction on r:

$$\begin{aligned}\Psi(E_i^{(r)}1_\lambda) &= \frac{1}{r}\Psi(E_i^{(1)}1_\lambda) * \Psi(E_i^{(r-1)}1_{\lambda+\alpha_i}) \\ &= \frac{1}{r}[\tilde{\mathcal{N}}^e_{\lambda,\lambda+\alpha_i}] * [\tilde{\mathcal{N}}^e_{\lambda+\alpha_i,\lambda+r\alpha_i}] \\ &= [\tilde{\mathcal{N}}^e_{\lambda,\lambda+r\alpha_i}].\end{aligned}$$

Hence Ψ maps $\mathcal{S}_{\mathbb{Z}}(n,d)^{op}$ isomorphically onto the $\mathbb{Z}$-subalgebra of $\mathcal{H}_{\mathbb{Z}}(n,d)$ generated by the fundamental classes $[\tilde{\mathcal{N}}^e_{\lambda,\lambda}]$, $[\tilde{\mathcal{N}}^e_{\lambda,\lambda+r\alpha_i}]$, $[\tilde{\mathcal{N}}^e_{\lambda,\lambda-r\alpha_i}]$ for each $\lambda \in \Lambda(n,d)$, each $i \in [n-1]$, and each $r \in \mathbb{N}$. Since $\mathcal{S}_{\mathbb{Z}}(n,d)$ and $\mathcal{H}_{\mathbb{Z}}(n,d)$ are free $\mathbb{Z}$-modules of equal rank, the embedding $\Psi : \mathcal{S}_{\mathbb{Z}}(n,d)^{op} \hookrightarrow \mathcal{H}_{\mathbb{Z}}(n,d)$ is an isomorphism. The result follows. $\square$

The following question follows naturally from Theorem 4.3.6.

Open Question 4.3.7. Where does the the isomorphism $\mathcal{S}_{\mathbb{C}}(n,d)^{op} \to \mathcal{H}_{\mathbb{C}}(n,d)$ in Theorem 4.3.6 send an arbitrary standard basis element $\xi^\sigma_{\mu,\lambda}$ defined in Section 3.8?

4.4 A new proof of Mautner's equivalence $P_G(\mathcal{N},\Bbbk) \simeq \mathcal{S}_{\Bbbk}(n,d)$-mod

In this section we use the characteristic-free version of Ginzburg's construction of the Schur algebra to prove the following result.

Theorem 4.4.1. *If $n \geq d$, then the perverse sheaf*

$$\Gamma_{n,d} := \bigoplus_{\lambda\in\Lambda(n,d)} \breve{m}_{\lambda !}\underline{\Bbbk}_{\breve{\mathcal{N}}_\lambda}[\dim \mathcal{N}]$$

is a projective generator of $P_G(\mathcal{N},\Bbbk)$, and $\mathcal{S}_{\Bbbk}(n,d)^{op} \simeq \mathrm{End}_{P_G(\mathcal{N},\Bbbk)}(\Gamma_{n,d})$.

In particular, the functor

$$\mathrm{Hom}_{P_G(\mathcal{N},\Bbbk)}(\Gamma_{n,d}, -) : P_G(\mathcal{N},\Bbbk) \to \mathcal{S}_{\Bbbk}(n,d)\text{-mod}$$

is an equivalence of categories.

To prove this theorem we use Achar and Mautner's geometric Ringel duality functor [AM15] to define an algebra isomorphism (Lemma 4.4.8)

$$\mathrm{End}_{P_G(\mathcal{N},\Bbbk)}(\Gamma_{n,d}) \simeq \mathrm{End}_{P_G(\mathcal{N},\Bbbk)}\Big(\bigoplus_{\lambda\in\Lambda(n,d)} \tilde{m}_{\lambda!}\underline{\Bbbk}_{\tilde{\mathcal{N}}_\lambda}[2\dim G/P_\lambda]\Big),$$

and use the Ginzburg construction of the Schur algebra to show (Proposition 4.4.11) that

$$\mathcal{S}_{\Bbbk}(n,d)^{op} \simeq \mathrm{End}_{P_G(\mathcal{N},\Bbbk)}\Big(\bigoplus_{\lambda\in\Lambda(n,d)} \tilde{m}_{\lambda!}\underline{\Bbbk}_{\tilde{\mathcal{N}}_\lambda}[2\dim G/P_\lambda]\Big).$$

Lemma 4.4.3 together with Lemma 4.4.9 show that the object $\Gamma_{n,d}$ is projective.

In Sections 4.4.1 and 4.4.2 we recall basic properties of the parabolic induction and the geometric Ringel duality functors on $P_G(\mathcal{N},\Bbbk)$.

In Section 4.4.3 we prove Theorem 4.4.1 and evaluate the equivalence $P_G(\mathcal{N},\Bbbk) \simeq \mathcal{P}^{\Bbbk}_{n,d}$ on simple, projective, and injective objects.

4.4.1 Parabolic induction functors

Consider the following diagram in which the squares are Cartesian.

$$\begin{tikzcd}
\mathfrak{l}_\lambda & \mathfrak{p}_\lambda \arrow[l, two heads, "p_\lambda"'] \arrow[r, hook, "q_\lambda"] & \tilde{\mathfrak{g}}_\lambda := G\times^{P_\lambda}\mathfrak{p}_\lambda \arrow[r, two heads, "m_\lambda"] & \mathfrak{g} \\
\mathcal{N}_{L_\lambda} \arrow[u, hook] & \mathcal{N}_{P_\lambda} \arrow[u, hook] \arrow[l, two heads, "\breve{p}_\lambda"'] \arrow[r, hook, "\breve{q}_\lambda"] & \breve{\mathcal{N}}_\lambda := G\times^{P_\lambda}\mathcal{N}_{P_\lambda} \arrow[u, hook] \arrow[r, two heads, "\breve{m}_\lambda"] & \mathcal{N} \arrow[u, hook] \\
\{0\} \arrow[u, hook] & \mathfrak{u}_\lambda \arrow[u, hook] \arrow[l, two heads, "\tilde{p}_\lambda"'] \arrow[r, hook, "\tilde{q}_\lambda"] & \tilde{\mathcal{N}}_\lambda := G\times^{P_\lambda}\mathfrak{u}_\lambda \arrow[u, hook] \arrow[r, two heads, "\tilde{m}_\lambda"] & \overline{\mathcal{O}_{\lambda^\vee}} \arrow[u, hook]
\end{tikzcd}$$

Here, $p_\lambda : \mathfrak{p}_\lambda \to \mathfrak{p}_\lambda/\mathfrak{u}_\lambda = \mathfrak{l}_\lambda$ is the quotient map and $q_\lambda : \mathfrak{p}_\lambda \to G\times^{P_\lambda}\mathfrak{p}_\lambda$ is the section $x \mapsto (e,x)$, where $e\in G$ is the identity matrix. Recall the following facts:

- Any L_λ-variety X can be regarded as a P_λ-variety with P_λ action factoring through the quotient $P_\lambda \to L_\lambda$. For such an L_λ-variety X, the forgetful functor $\mathrm{For}^{P_\lambda}_{L_\lambda} : \mathcal{D}^b_{P_\lambda}(X,\Bbbk) \to \mathcal{D}^b_{L_\lambda}(X,\Bbbk)$ is an equivalence of categories that is t-exact for the perverse t-structure [BL94, Theorem 3.7.3] (see also [Ach21, Theorem 6.6.16]).

- If X is a P_λ-variety and $q : X \to G \times^{P_\lambda} X$ is the map $x \mapsto (e, x)$, then the functor

$$Q_X := q_\lambda^* \circ \mathrm{For}_{P_\lambda}^G[-\dim G/P_\lambda] : \mathcal{D}_G^b(G \times^{P_\lambda} X, \Bbbk) \to \mathcal{D}_{P_\lambda}^b(X, \Bbbk)$$

  is an equivalence of categories that is t-exact for the perverse t-structure [BL94, Theorem 2.6.3] (see also [Ach21, Theorem 6.5.11]). Moreover, there is a natural isomorphism $Q_X \simeq q^! \circ \mathrm{For}_{P_\lambda}^G[\dim G/P_\lambda]$.

- The map $p_\lambda : \mathfrak{p}_\lambda \to \mathfrak{l}_\lambda$ is smooth of relative dimension $\dim G/P_\lambda$ and so the functor

$$p_\lambda^*[\dim G/P_\lambda] \simeq p_\lambda^![-\dim G/P_\lambda] : \mathcal{D}_{P_\lambda}^b(\mathfrak{p}_\lambda, \Bbbk) \to \mathcal{D}_{P_\lambda}^b(\mathfrak{l}_\lambda, \Bbbk)$$

  is t-exact for the perverse t-structure (see e.g. [Ach21, Proposition 3.6.1]). Moreover, since $m : \tilde{\mathfrak{g}}_\lambda \to \mathfrak{g}$ is proper, the functor

$$m_{\lambda!} \simeq m_{\lambda*} : \mathcal{D}_G^b(\tilde{\mathfrak{g}}_\lambda, \Bbbk) \to \mathcal{D}_G^b(\mathfrak{g}, \Bbbk)$$

  is t-exact for the perverse t-structure.

For each weak composition λ of d, define the *parabolic induction functor*

$$\mathrm{ind}_\lambda := m_{\lambda!} \circ Q_{\mathfrak{p}_\lambda}^{-1} \circ p_\lambda^*[\dim G/P_\lambda] \circ (\mathrm{For}_{L_\lambda}^{P_\lambda})^{-1}.$$

By the above remarks, this functor is t-exact for the perverse t-structure. Moreover

$$\mathrm{ind}_\lambda \simeq m_{\lambda*} \circ Q_{\mathfrak{p}_\lambda}^{-1} \circ p_\lambda^![-\dim G/P_\lambda] \circ (\mathrm{For}_{L_\lambda}^{P_\lambda})^{-1} : \mathcal{D}_{L_\lambda}^b(\mathfrak{l}_\lambda, \Bbbk) \to \mathcal{D}_G^b(\mathfrak{g}, \Bbbk).$$

Let $i_\lambda = m_\lambda \circ q_\lambda : \mathfrak{p}_\lambda \hookrightarrow \mathfrak{g}$ be the inclusion map. The functor ind_λ has left and right adjoints $\mathrm{res}_\lambda := p_{\lambda!} i_\lambda^*$ and $\mathrm{res}_\lambda^! := p_{\lambda*} i_\lambda^!$ respectively (here we are suppressing the forgetful functors in the notation).

Remark 4.4.2. The parabolic induction functor has an alternative description:

$$\mathrm{ind}_\lambda = \Gamma_{P_\lambda}^G \circ i_{\lambda*} \circ p_\lambda^! \circ (\mathrm{For}_{L_\lambda}^{P_\lambda})^{-1} : \mathcal{D}_{L_\lambda}^b(\mathfrak{l}_\lambda, \Bbbk) \to \mathcal{D}_G^b(\mathfrak{g}, \Bbbk),$$

where $\Gamma_{P_\lambda}^G : \mathcal{D}^b_{P_\lambda}(\mathfrak{g}, \Bbbk) \to \mathcal{D}^b_G(\mathfrak{g}, \Bbbk)$ is the right adjoint to the forgetful functor $\mathcal{D}^b_G(\mathfrak{g}, \Bbbk) \to \mathcal{D}^b_{P_\lambda}(\mathfrak{g}, \Bbbk)$ (this adjoint is defined in [BL94, Theorem 3.7.1]). See [AHJR16, Lemma 2.14] for a proof that the two descriptions of parabolic induction are equivalent.

The functor ind_λ sends objects supported on $\mathcal{N}_{L_\lambda}$ (respectively $\{0\}$) to objects supported on $\mathcal{N}_G$ (respectively $\overline{\mathcal{O}_{\lambda^\vee}}$). Indeed, this can be checked using proper base change. In particular, $\mathrm{ind}_\lambda : \mathcal{D}^b_{L_\lambda}(\mathfrak{l}_\lambda, \Bbbk) \to \mathcal{D}^b_G(\mathfrak{g}, \Bbbk)$ restricts to functors

$$\breve{\mathrm{ind}}_\lambda := \breve{m}_{\lambda!} \circ Q^{-1}_{\mathcal{N}_{P_\lambda}} \circ \breve{p}^*_\lambda[\dim G/P_\lambda] \circ (\mathrm{For}^{P_\lambda}_{L_\lambda})^{-1} : \mathcal{D}^b_{L_\lambda}(\mathcal{N}_{L_\lambda}, \Bbbk) \to \mathcal{D}^b_G(\mathcal{N}, \Bbbk)$$
$$\tilde{\mathrm{ind}}_\lambda := \tilde{m}_{\lambda!} \circ Q^{-1}_{\mathfrak{u}_\lambda} \circ \tilde{p}^*_\lambda[\dim G/P_\lambda] \circ (\mathrm{For}^{P_\lambda}_{L_\lambda})^{-1} : \mathcal{D}^b_{L_\lambda}(\{0\}, \Bbbk) \to \mathcal{D}^b_G(\overline{\mathcal{O}_{\lambda^\vee}}, \Bbbk)$$

Likewise, the restriction functors $\mathrm{res}_\lambda, \mathrm{res}^!_\lambda : \mathcal{D}^b_G(\mathfrak{g}, \Bbbk) \to \mathcal{D}^b_{L_\lambda}(\mathfrak{l}_\lambda, \Bbbk)$ restrict to functors $\breve{\mathrm{res}}_\lambda, \breve{\mathrm{res}}^!_\lambda : \mathcal{D}^b_G(\mathcal{N}, \Bbbk) \to \mathcal{D}^b_{L_\lambda}(\mathcal{N}_{L_\lambda}, \Bbbk)$ and $\tilde{\mathrm{res}}_\lambda, \tilde{\mathrm{res}}^!_\lambda : \mathcal{D}^b_G(\overline{\mathcal{O}_{\lambda^\vee}}, \Bbbk) \to \mathcal{D}^b_{L_\lambda}(\{0\}, \Bbbk)$. Moreover, the functors $\breve{\mathrm{res}}_\lambda$, $\breve{\mathrm{res}}^!_\lambda$, $\tilde{\mathrm{res}}_\lambda$, $\tilde{\mathrm{res}}^!_\lambda$ are t-exact for the perverse t-structure [AHR15, Proposition 4.7].

For a weak composition λ of d, define the perverse sheaves

$$\Gamma^\lambda := \mathrm{ind}_\lambda(\underline{\Bbbk}_{\mathcal{N}_{L_\lambda}}[\dim \mathcal{N}_{L_\lambda}]) \simeq \breve{m}_{\lambda!}\underline{\Bbbk}_{\breve{\mathcal{N}}_\lambda}[\dim \mathcal{N}],$$
$$S^\lambda := \mathrm{ind}_\lambda(\mathbb{D}_{\mathcal{N}_{L_\lambda}}[\dim \mathcal{N}_{L_\lambda}]) \simeq \mathbb{D}\Gamma^\lambda,$$
$$\Lambda^\lambda := \mathrm{ind}_\lambda \underline{\Bbbk}_{\{0\}} \simeq \tilde{m}_{\lambda!}(\underline{\Bbbk}_{\tilde{\mathcal{N}}_\lambda}[2\dim G/P_\lambda]).$$

For example, $\Gamma^{(1,\ldots,1)} \simeq S^{(1,\ldots,1)} \simeq \Lambda^{(1,\ldots,1)} \simeq \tilde{m}_{\lambda!}\underline{\Bbbk}_{\tilde{\mathcal{N}}_{(1,\ldots,1)}}[\dim \mathcal{N}]$ is the *Springer sheaf.* At the other extreme,

$$\Gamma^d \simeq \underline{\Bbbk}_{\mathcal{N}}[\dim \mathcal{N}], \quad S^d \simeq \mathbb{D}_{\mathcal{N}}[\dim \mathcal{N}], \quad \text{and} \quad \Lambda^d \simeq \underline{\Bbbk}_{\{0\}}.$$

Lemma 4.4.3. *For each weak composition λ of d:*

(i) The perverse sheaf $\Gamma^\lambda \in P_G(\mathcal{N}, \Bbbk)$ is projective.

(ii) The perverse sheaf $S^\lambda \in P_G(\mathcal{N}, \Bbbk)$ is injective.

Proof. It follows immediately from [AM15, Proposition 5.1] that for any weak composition λ of d, the perverse sheaf $\Bbbk_{\mathcal{N}_{L_\lambda}}[\dim \mathcal{N}_{L_\lambda}]$ is projective in $P_{L_\lambda}(\mathcal{N}_{L_\lambda}, \Bbbk)$. Since $\breve{\mathrm{ind}}_\lambda : P_{L_\lambda}(\mathcal{N}_{L_\lambda}, \Bbbk) \to P_G(\mathcal{N}, \Bbbk)$ has an exact right adjoint, the perverse sheaf

$\Gamma^\lambda = \operatorname{ind}^{\vee}_\lambda(\underline{\Bbbk}_{\mathcal{N}_{L_\lambda}}[\dim \mathcal{N}_{L_\lambda}])$ is projective in $P_G(\mathcal{N}_G, \Bbbk)$. Statement (ii) follows by the dual argument. $\square$

Remark 4.4.4. Given $\mathcal{F} \in \mathcal{D}_{\mathrm{GL}_m}(\mathcal{N}_{\mathrm{GL}_m}, \Bbbk)$, and $\mathcal{G} \in \mathcal{D}_{\mathrm{GL}_n}(\mathcal{N}_{\mathrm{GL}_n}, \Bbbk)$, the complex $\mathcal{F} \boxtimes \mathcal{G}$ is a $\mathrm{GL}_m \times \mathrm{GL}_n$-equivariant sheaf on $\mathcal{N}_{\mathrm{GL}_{m+n}}$. There is a product

$$-\star- := \mathrm{ind}_{(m,n)}(-\boxtimes-) : \mathcal{D}_{\mathrm{GL}_m}(\mathcal{N}_{\mathrm{GL}_m}, \Bbbk) \times \mathcal{D}_{\mathrm{GL}_n}(\mathcal{N}_{\mathrm{GL}_n}, \Bbbk) \to \mathcal{D}_{\mathrm{GL}_{m+n}}(\mathcal{N}_{\mathrm{GL}_{m+n}}, \Bbbk).$$

Moreover there are natural isomorphisms

$$(\mathcal{F} \star \mathcal{G}) \star \mathcal{H} \cong \mathcal{F} \star (\mathcal{G} \star \mathcal{H}) \qquad \text{and} \qquad \mathbb{D}(\mathcal{F} \star \mathcal{G}) \cong \mathbb{D}\mathcal{F} \star \mathbb{D}\mathcal{G}. \tag{4.6}$$

Indeed see e.g. [Ach21, Lemma 7.2.4, Lemma 7.2.7] for proofs about the analogous statements about the convolution product on equivariant sheaves on the flag variety.

It follows directly from the definition that $\Lambda^\lambda = \Lambda^{\lambda_1} \star \cdots \star \Lambda^{\lambda_n}$, $\Gamma^\lambda = \Gamma^{\lambda_1} \star \cdots \star \Gamma^{\lambda_n}$, and $S^\lambda = S^{\lambda_1} \star \cdots \star S^{\lambda_n}$.

4.4.2 Geometric Ringel duality

We next describe the geometric version of Ringel duality due to Achar and Mautner [AM15]. For this we recall the definition of the Fourier-Sato transform.

Let $\mathfrak{h}$ be a complex Lie algebra. Consider the $\mathbb{C}^\times$-action on $\mathfrak{h}$ given by scaling. Say that an object $\mathcal{F} \in \mathcal{D}^b_H(\mathfrak{h}, \Bbbk)$ is *conic* if for each $x \in \mathfrak{h}$, and $i \in \mathbb{Z}$, the sheaf $\mathcal{H}^i(\mathcal{F})|_{\mathbb{C}^\times \cdot x}$ is locally constant. Let $\mathcal{D}^b_{H,con}(\mathfrak{h}, \Bbbk) \subset \mathcal{D}^b_H(\mathfrak{h}, \Bbbk)$ denote the full subcategory of conic objects.

Define the H-stable subset $P \subset \mathfrak{h} \times \mathfrak{h}$ by:

$$P = \{(x, y) \in \mathfrak{h} \times \mathfrak{h} \mid \Re\kappa(x, y) \leq 0\},$$

where $\Re\kappa(x, y)$ is the real part of the Killing form of x and y. Let $\pi_1, \pi_2 : P \to \mathfrak{h}$ be the projections onto the first and second components.

Define the *Fourier-Sato transform*

$$\mathbb{T}_{\mathfrak{h}} := \pi_{2!}\pi_1^*[\dim \mathfrak{h}] : \mathcal{D}^b_{H,con}(\mathfrak{h}, \Bbbk) \to \mathcal{D}^b_{H,con}(\mathfrak{h}, \Bbbk)$$

This functor is an equivalence of categories with quasi-inverse $\mathbb{T}_{\mathfrak{h}}^{-1} = \pi_{1*}\pi_2^![-\dim \mathfrak{h}]$.

Remark 4.4.5. In this thesis we only consider a specific type of Fourier-Sato transform (in the usual definition, $\mathfrak{h}$ would be replaced with any real vector bundle). For the more general definition see [KS90, Definition 3.7.8]. Note also that the usual definition of the Fourier-Sato transform does not include a shift by $\dim \mathfrak{h}$.

We list two important results about the Fourier-Sato transform.

(i) [KS90, Proposition 10.3.18] The functor $\mathbb{T}_{\mathfrak{h}}$ is t-exact for the perverse t-structure.

(ii) [KS90, Lemma 3.7.10] There are isomorphisms in $\mathcal{D}^b_{H,con}(\mathfrak{h}, \Bbbk)$:

$$\begin{aligned} \mathbb{T}_{\mathfrak{h}}(\underline{\Bbbk}_{\mathfrak{h}}[\dim \mathfrak{h}]) &\simeq \underline{\Bbbk}_{\{0\}}, \\ \mathbb{T}_{\mathfrak{h}}(\underline{\Bbbk}_{\{0\}}) &\simeq \underline{\Bbbk}_{\mathfrak{h}}[\dim \mathfrak{h}]. \end{aligned} \tag{4.7}$$

The following proposition extends the latter result in the case that $\mathfrak{h} = \mathfrak{gl}_d$.

Proposition 4.4.6. *For a weak composition, λ, of d, there are natural isomorphisms in $\mathcal{D}^b_G(\mathfrak{g}, \Bbbk)$:*

$$\begin{aligned} \mathbb{T}_{\mathfrak{g}}(m_{\lambda !}\underline{\Bbbk}_{\tilde{\mathfrak{g}}_\lambda}[\dim \mathfrak{g}]) &\simeq \tilde{m}_{\lambda !}\underline{\Bbbk}_{\tilde{\mathcal{N}}_\lambda}[\dim \tilde{\mathcal{N}}_\lambda], \\ \mathbb{T}_{\mathfrak{g}}(\tilde{m}_{\lambda !}\underline{\Bbbk}_{\tilde{\mathcal{N}}_\lambda}[\dim \tilde{\mathcal{N}}_\lambda]) &\simeq m_{\lambda !}\underline{\Bbbk}_{\tilde{\mathfrak{g}}_\lambda}[\dim \mathfrak{g}]. \end{aligned}$$

Proof. Mircović [Mir04, Lemma 4.2] shows that the Fourier-Sato transform functor commutes with the parabolic induction and restriction functors. Hence:

$$\begin{aligned} \mathbb{T}_{\mathfrak{g}}(m_{\lambda !}\underline{\Bbbk}_{\tilde{\mathfrak{g}}_\lambda}[\dim \mathfrak{g}]) &\simeq \mathbb{T}_{\mathfrak{g}}(\operatorname{ind}_\lambda \Bbbk_{\mathfrak{l}_\lambda}[\dim \mathfrak{l}_\lambda]) \\ &\simeq \operatorname{ind}_\lambda(\mathbb{T}_{\mathfrak{l}_\lambda}\Bbbk_{\mathfrak{l}_\lambda}[\dim \mathfrak{l}_\lambda]) \\ &\simeq \operatorname{ind}_\lambda \Bbbk_{\{0\}} \\ &\simeq \tilde{m}_{\lambda !}\underline{\Bbbk}_{\tilde{\mathcal{N}}_\lambda}[\dim \tilde{\mathcal{N}}_\lambda] \end{aligned}$$

The result follows. □

Remark 4.4.7. For a direct proof of Proposition 4.4.6 that does not use the induction functor one could use the methodology of [AHJR14, Lemma 2.2], which proves Proposition 4.4.6 in the case $\lambda = (1, \ldots, 1)$.

Achar and Mautner [AM15] define the *geometric Ringel duality functor*:

$$\mathcal{R} := i^*\mathbb{T}_{\mathfrak{g}} i_![-d] : \mathcal{D}^b_G(\mathcal{N}, \Bbbk) \to \mathcal{D}^b_G(\mathcal{N}, \Bbbk),$$

where $i : \mathcal{N} \hookrightarrow \mathfrak{g}$ is the natural embedding. Note that any object in $\mathcal{D}^b_G(\mathfrak{g}, \Bbbk)$ that is supported on $\mathcal{N}$ is conic, and so this definition makes sense. They show [AM15, Theorem 4.2] that $\mathcal{R}$ is an equivalence of categories with quasi-inverse

$$\mathcal{R}^{-1} := i^!\mathbb{T}^{-1}_{\mathfrak{g}} i_![d] : \mathcal{D}^b_G(\mathcal{N}, \Bbbk) \to \mathcal{D}^b_G(\mathcal{N}, \Bbbk).$$

Lemma 4.4.8. *For any weak composition, λ, of d, there are natural isomorphisms in $P_G(\mathcal{N}, \Bbbk)$:*

(i) $\mathcal{R}\Lambda^\lambda \simeq \Gamma^\lambda$,

(ii) $\mathcal{R}^{-1}\Lambda^\lambda \simeq S^\lambda$.

Proof. Consider the Cartesian square

$$\begin{tikzcd} \tilde{\mathfrak{g}}_\lambda \arrow[r, "m_\lambda"] & \mathfrak{g} \\ \breve{\mathcal{N}}_\lambda \arrow[u, hook, "\iota_\lambda"] \arrow[r, "\breve{m}_\lambda"'] & \mathcal{N} \arrow[u, hook', "i"'] \end{tikzcd}$$

Statement (i) follows from the following sequence of isomorphisms

$$\begin{aligned} i^*\mathbb{T}_{\mathfrak{g}} i_!\Lambda^\lambda[-d] &\simeq i^* m_{\lambda!}\underline{\Bbbk}_{\tilde{\mathfrak{g}}_\lambda}[d^2-d] \\ &\simeq \breve{m}_{\lambda!}\iota^*_\lambda\underline{\Bbbk}_{\tilde{\mathfrak{g}}_\lambda}[d^2-d] \\ &\simeq \breve{m}_{\lambda!}\underline{\Bbbk}_{\breve{\mathcal{N}}_\lambda}[d^2-d] \\ &= \Gamma^\lambda \end{aligned}$$

where the first isomorphism follows from Proposition 4.4.6 and the second isomorphism is an application of proper base change. Statement (ii) follows from the dual argument. $\square$

Lemma 4.4.9. *The following hold:*

(i) The projective objects of $\mathrm{Perv}_G(\mathcal{N}, \Bbbk)$ *are isomorphic to direct sums of direct summands of objects of the form* Γ^λ.

(ii) The injective objects of $\mathrm{Perv}_G(\mathcal{N}, \Bbbk)$ *are isomorphic to direct sums of direct summands of objects of the form* S^λ*.*

Proof. By Corollary 2.3.11, the category $\mathrm{Perv}_G(\mathcal{N}, \Bbbk)$ is finite and the projective indecomposable objects in $\mathrm{Perv}_G(\mathcal{N}, \Bbbk)$ are in one-to-one correspondence with the set, $\Lambda(d)$, of partitions of d. It is already shown in Lemma 4.4.3 that the objects Γ^λ are projective. To show Statement (i) it suffices to show that the object $\bigoplus_{\lambda \in \Lambda(d)} \Gamma^\lambda$ has (at least) $|\Lambda(d)|$ many indecomposable direct summands (up to isomorphism).

By Lemma 4.4.8, there is an isomorphism $\mathcal{R}^{-1}(\bigoplus_{\lambda \in \Lambda(d)} \Gamma^\lambda) \simeq \bigoplus_{\lambda \in \Lambda(d)} \Lambda^\lambda$. Since Λ^λ is supported on $\overline{\mathcal{O}_{\lambda^\vee}}$, the object Λ^λ has an indecomposable direct summand, $\mathcal{T}_{\lambda^\vee}$, in which $\overline{\operatorname{supp} \mathcal{T}_{\lambda^\vee}} \simeq \overline{\mathcal{O}_{\lambda^\vee}}$. The objects $\mathcal{P}_\lambda := \mathcal{R}(\mathcal{T}_{\lambda^\vee})$ form the required collection of indecomposable summands of $\bigoplus_{\lambda \in \Lambda(d)} \Gamma^\lambda$.

Statement (ii) holds by a similar argument. □

For a partition λ of d, let $h_\lambda : \mathcal{O}_\lambda \hookrightarrow \mathcal{N}$ denote the inclusion map. Define the simple perverse sheaf

$$IC_\lambda := h_{\lambda !*} \underline{\Bbbk}_{\mathcal{O}_\lambda}[\dim \mathcal{O}_\lambda].$$

Let $\mathcal{T}_\lambda$ and $\mathcal{P}_\lambda := \mathcal{R}(\mathcal{T}_{\lambda^\vee})$ be the indecomposable objects of $P_G(\mathcal{N}, \Bbbk)$ defined in the proof of Lemma 4.4.9. Achar and Mautner [AM15, Theorem 6.1] show that $\mathcal{P}_\lambda$ is a projective cover of IC_λ[1].

By definition of the indecomposable objects $\mathcal{T}_\lambda$, the perverse sheaf Λ^λ has a decomposition into irreducible objects of the form

$$\Lambda^\lambda \simeq \mathcal{T}_{\lambda^\vee} \oplus \bigoplus_{\mu \leq \lambda^\vee} \mathcal{T}_\mu^{m_{\lambda\mu}}$$

for some numbers $m_{\lambda\mu} \in \mathbb{N}$. By applying the geometric Ringel duality functor, it follows that the perverse sheaf Γ^λ has a decomposition into indecomposables objects

$$\Gamma^\lambda \simeq \mathcal{P}_\lambda \oplus \bigoplus_{\mu \geq \lambda} \mathcal{P}_\mu^{\oplus m_{\lambda\mu}} \tag{4.8}$$

for some numbers $m_{\lambda\mu} \in \mathbb{N}$.

[1]The proof of this fact does not rely on the equivalence $P_G(\mathcal{N}, \Bbbk) \simeq \mathcal{P}_d^{\Bbbk}$.

Remark 4.4.10. It follows from the equivalence $P_G(\mathcal{N}, \Bbbk) \simeq \mathcal{P}_d^{\Bbbk}$ that $P_G(\mathcal{N}, \Bbbk)$ is a highest weight category. Achar and Mautner [AM15, Lemma 6.2] show that the objects $\mathcal{T}_\lambda$ are the indecomposable partial tilting objects in $P_G(\mathcal{N}, \Bbbk)$. In particular, the functor $\mathcal{R}$ restricts to an equivalence between the full subcategory of partial tilting objects in $P_G(\mathcal{N}, \Bbbk)$ to the full subcategory of projective objects in $P_G(\mathcal{N}, \Bbbk)$. This is the motivation for the term *geometric Ringel duality.*

4.4.3 The functor $\mathrm{Hom}(\Gamma_{n,d}, -) : P_G(\mathcal{N}, \Bbbk) \to \mathcal{S}_{\Bbbk}(n, d)$-mod

In this section we prove Theorem 4.4.1, and show that the equivalence $\mathrm{Hom}(\Gamma_{n,d}, -) : P_G(\mathcal{N}, \Bbbk) \to \mathcal{S}_{\Bbbk}(n, d)$-mod maps the simple perverse sheaf IC_λ to the simple $\mathcal{S}_{\Bbbk}(n, d)$-module with highest weight λ (Proposition 4.4.12).

To prove Theorem 4.4.1 we need the following known result (see e.g. [CG97, Theorem 8.6.7]) relating Borel-Moore homology with morphisms of perverse sheaves.

Proposition 4.4.11. *Consider a Cartesian square of varieties:*

$$\begin{tikzcd} M_1 \times_N M_2 \arrow[r, "f_1"] \arrow[d, "f_2"'] & M_1 \arrow[d, "\mu_1"] \\ M_2 \arrow[r, "\mu_2"'] & N \end{tikzcd}$$

in which M_2 is smooth. Write $d_i := \dim M_i$. Then

$$H_k^{BM}(M_1 \times_N M_2, \Bbbk) \simeq \mathrm{Hom}_{\mathcal{D}^b(N)}(\mu_{1!}\underline{\Bbbk}_{M_1}[k], \mu_{2*}\underline{\Bbbk}_{M_2}[2d_2]).$$

Moreover suppose that $\mu_2 : M_2 \to N$ is proper, M_1, M_2 and M_3 are smooth varieties, and $\mu_3 : M_3 \to N$ is a map of varieties. Write

$$H_k(X) := H_k^{BM}(X, \Bbbk)$$

for any variety X and $k \in \mathbb{N}$. Write

$$(M_i[k], M_j[l]) := \mathrm{Hom}_{\mathcal{D}^b(N)}(\mu_{1!}\underline{\Bbbk}_{M_i}[k], \mu_{2!}\underline{\Bbbk}_{M_j}[l]).$$

Then the following diagram commutes:

$$\begin{tikzcd} H_k(M_1 \times_N M_2) \otimes H_l(M_2 \times_N M_3) \arrow[r, "\sim"] \arrow[d, "-*-"'] & (M_1[k+l-2d_2], M_2[l]) \otimes (M_2[l], M_3[2d_3]) \arrow[d, "-\circ-"] \\ H_{k+l-2d_2}(M_1 \times_N M_3) \arrow[r, "\sim"] & (M_1[k+l-2d_2], M_2[2d_3]) \end{tikzcd}$$

Proof. The first part of the lemma follows from the isomorphisms:

$$\begin{aligned}
\operatorname{Hom}_{\mathcal{D}(M_1\times_N M_2)}(\underline{\Bbbk}_{M_1\times_N M_2}, \mathbb{D}\underline{\Bbbk}_{M_1\times_N M_2}) &\simeq \operatorname{Hom}_{\mathcal{D}(M_1\times_N M_2)}(f_1^*\underline{\Bbbk}_{M_1}, f_2^!\mathbb{D}\underline{\Bbbk}_{M_2})\\
&\simeq \operatorname{Hom}_{\mathcal{D}(M_2)}(\underline{\Bbbk}_{M_1}, f_{1*}f_2^!\mathbb{D}\underline{\Bbbk}_{M_2})\\
&\simeq \operatorname{Hom}_{\mathcal{D}(M_2)}(\underline{\Bbbk}_{M_1}, \mu_1^!\mu_{2*}\mathbb{D}\underline{\Bbbk}_{M_2})\\
&\simeq \operatorname{Hom}_{\mathcal{D}(M_2)}(\underline{\Bbbk}_{M_1}, \mu_1^!\mu_{2*}\underline{\Bbbk}_{M_2}[2\dim M_2])\\
&\simeq \operatorname{Hom}_{\mathcal{D}(M_2)}(\mu_{1!}\underline{\Bbbk}_{M_1}, \mu_{2*}\underline{\Bbbk}_{M_2}[2\dim M_2])
\end{aligned}$$

Note in particular that the third isomorphism is due to proper base change and the fourth isomorphism is due to the smoothness of M_2.

The proof of the second part of the lemma involves more theory than we are prepared to develop here. We refer to [CG97, Theorem 8.6.7] for a complete proof. □

Proof of Theorem 4.4.1. Lemma 4.4.9 shows that if $n \geq d$ then the perverse sheaf

$$\Gamma_{n,d} := \bigoplus_{\lambda\in\Lambda(n,d)} \Gamma^\lambda$$

is a projective generator of $P_G(\mathcal{N}, \Bbbk)$. Since $P_G(\mathcal{N}, \Bbbk)$ is finite (see e.g. Corollary 2.3.11), to show Theorem 4.4.1 it suffices to define an algebra isomorphism

$$\mathcal{S}_\Bbbk(n,d)^{op} \simeq \operatorname{End}_{P_G(\mathcal{N},\Bbbk)}(\Gamma_{n,d}).$$

This isomorphism is defined by the following chain of algebra isomorphisms

$$\begin{aligned}
\mathcal{S}_\Bbbk(n,d)^{op} &\simeq \bigoplus_{\lambda,\mu\in\Lambda(n,d)} H^{BM}_{\dim\tilde{\mathcal{N}}_\lambda+\dim\tilde{\mathcal{N}}_\mu}(\tilde{\mathcal{N}}_\lambda\times_{\mathcal{N}}\tilde{\mathcal{N}}_\mu, \Bbbk)\\
&\simeq \bigoplus_{\lambda,\mu\in\Lambda(n,d)} \operatorname{Hom}_{\mathcal{D}^b(\mathcal{N},\Bbbk)}(m_{\lambda!}\underline{\Bbbk}_{\tilde{\mathcal{N}}_\lambda}[\dim\tilde{\mathcal{N}}_\lambda], m_{\mu!}\underline{\Bbbk}_{\tilde{\mathcal{N}}_\mu}[\dim\tilde{\mathcal{N}}_\mu])\\
&\simeq \operatorname{End}_{P_G(\mathcal{N},\Bbbk)}(\bigoplus_{\lambda\in\Lambda(n,d)}\Lambda^\lambda)\\
&\simeq \operatorname{End}_{P_G(\mathcal{N},\Bbbk)}(\bigoplus_{\lambda\in\Lambda(n,d)}\Gamma^\lambda).
\end{aligned}$$

Here the first isomorphism is the characteristic-free Ginzburg construction of the Schur algebra (Theorem 4.3.6), the second isomorphism is due to Proposition 4.4.11, and the fourth isomorphism is given by the geometric Ringel duality functor ([AM15, Theorem 4.2] and Lemma 4.4.8). $\square$

Let $\Lambda^+(n,d) := \{(\lambda_1, \ldots, \lambda_n) \in \Lambda(n,d) \mid \lambda_1 \geq \ldots \geq \lambda_n\}$ be the set of dominant weights in $\Lambda(n,d)$. For $\lambda \in \Lambda^+(n,d)$, let L_λ^n be the simple $\mathcal{S}_\Bbbk(n,d)$-module with highest weight λ.

Proposition 4.4.12. *For $\lambda \in \Lambda^+(n,d)$, there are isomorphisms of $\mathcal{S}_\Bbbk(n,d)$-modules*

$$\mathrm{Hom}_{P_G(\mathcal{N},\Bbbk)}(\Gamma_{n,d}, IC_\lambda) \simeq L_\lambda^n, \tag{4.9}$$

$$\mathrm{Hom}_{P_G(\mathcal{N},\Bbbk)}(\Gamma_{n,d}, \Gamma^\lambda) \simeq \Gamma^\lambda \Bbbk^n, \tag{4.10}$$

$$\mathrm{Hom}_{P_G(\mathcal{N},\Bbbk)}(\Gamma_{n,d}, S^\lambda) \simeq S^\lambda \Bbbk^n. \tag{4.11}$$

Proof. Equation (4.10) follows from the chain of isomorphisms:

$$\begin{aligned}\mathrm{Hom}_{P_G(\mathcal{N},\Bbbk)}(\Gamma_{n,d}, \Gamma^\lambda) &\simeq \bigoplus_{\mu \in \Lambda(n,d)} \mathrm{Hom}_{P_G(\mathcal{N},\Bbbk)}(\Gamma^\mu, \Gamma^\lambda) \\ &\simeq \mathcal{S}_\Bbbk(n,d) 1_\lambda \\ &\simeq \Gamma^\lambda \Bbbk^n.\end{aligned}$$

Equation (4.11) follows from the dual argument.

By comparing the decompositions of Γ^λ and $\Gamma^\lambda \Bbbk^n$ into projective indecomposable objects (see Proposition 3.4.4 and Equation (4.8)), the functor $\mathrm{Hom}_{P_G(\mathcal{N},\Bbbk)}(\Gamma_{n,d}, -)$ sends the projective cover of IC_λ to the projective cover of L_λ^n. Equation (4.9) follows immediately.

$\square$

4.5 Highest weight structure on $P_G(\mathcal{N}, \Bbbk)$

In this section we describe the highest weight structure on $P_G(\mathcal{N}, \Bbbk)$, and use this structure to show that $\mathrm{Hom}_{P_G(\mathcal{N},\Bbbk)}(\Gamma_{n,d}, \Lambda^\lambda) \simeq \Lambda^\lambda \Bbbk^n$ for all $\lambda \in \Lambda(n,d)$, when $n \geq d$ (Proposition 4.5.4).

It is well known that Schur algebras are quasihereditary [Don86, (2.2h)]. More precisely, the category $\mathcal{P}^{\Bbbk}_{n,d}$ is a highest weight category with respect to the dominance order, $\leq$, on $\Lambda^+(n,d)$. It follows from Theorem 4.4.1 that $P_G(\mathcal{N},\Bbbk)$ is a highest weight category. The following proposition describes the standard and costandard objects in $P_G(\mathcal{N},\Bbbk)$.

Proposition 4.5.1. *The category $P_G(\mathcal{N},\Bbbk)$ is a highest weight category with respect to the poset $\Lambda(d)$ of partitions of d. The standard objects are defined*

$$\Delta(\lambda) := {}^pH^0(h_{\lambda!}\underline{\Bbbk}_{\mathcal{O}_\lambda}[\dim \mathcal{O}_\lambda])$$

and costandard objects are defined

$$\nabla(\lambda) := {}^pH^0(h_{\lambda*}\underline{\Bbbk}_{\mathcal{O}_\lambda}[\dim \mathcal{O}_\lambda]),$$

where $h_\lambda : \mathcal{O}_\lambda \hookrightarrow \mathcal{N}$ is the inclusion map.

Proof. The category $\mathcal{A} := P_G(\mathcal{N},\Bbbk)$ has a stratification by $\Lambda(d)$, given by defining Serre subcategories $\mathcal{A}_{\Lambda'} := P_G(\bigcup_{\lambda\in\Lambda'}\mathcal{O}_\lambda,\Bbbk)$, for each closed-downwards subset $\Lambda' \subset \Lambda$ (see Example 2.1.8). The perverse sheaves $\Delta(\lambda)$ and $\nabla(\lambda)$ are the standard and costandard objects of $P_G(\mathcal{N},\Bbbk)$ defined by this stratification (see Section 2.4).

If $n \geq d$ then $\mathcal{B} := \mathcal{P}^{\Bbbk}_{n,d}$ is a highest weight category with respect to $\Lambda(d)$ [Don86, (2.2h)]. By Theorem 2.5.2, $\mathcal{B}$ has a homological stratification by $\Lambda(d)$ defined by setting, for each closed-downwards subset $\Lambda' \subset \Lambda$, the category $\mathcal{B}_{\Lambda'}$ to be the Serre subcategory of $\mathcal{B}$ generated by simple objects L^n_λ in which $\lambda \in \Lambda'$.

By Proposition 4.4.12, the functor $\mathrm{Hom}_{P_G(\mathcal{N},\Bbbk)}(\Gamma_{n,d},-)$ restricts to an equivalence between $\mathcal{A}_{\Lambda'}$ and $\mathcal{B}_{\Lambda'}$ for each closed-downwards $\Lambda' \subset \Lambda$. The result follows immediately. $\square$

Recall that an object in a highest weight category is called *partial tilting* if it has both a filtration by standard objects and a filtration by costandard objects. Ringel [Rin91, Proposition 2] shows that if $\mathcal{A}$ is a highest weight category with simple objects L_λ indexed by a poset Λ, then for every $\lambda \in \Lambda$ there is a unique (up to isomorphism) indecomposable partial tilting object, T_λ, satisfying the conditions:

(i) All composition factors, L_μ, of T_λ satisfy $\mu \leq \lambda$.

(ii) The simple module L_λ is a composition factor of T_λ with multiplicity 1.

Moreover all indecomposable partial tilting objects in $\mathcal{A}$ are of this form. We call T_λ the *indecomposable partial tilting object associated to λ.*

The following proposition is due to [Don93, Lemma 3.4].

Proposition 4.5.2. *The partial tilting objects of $\mathcal{P}^{\Bbbk}_{n,d}$ are isomorphic to direct sums of direct summands of modules of the form $\Lambda^\lambda \Bbbk^n$.*

The partial tilting objects in $P_G(\mathcal{N}, \Bbbk)$ are characterized by the following proposition. A different proof of this result is given in [AM15, Lemma 6.2].

Proposition 4.5.3. *The partial tilting objects of $P_G(\mathcal{N}, \Bbbk)$ are isomorphic to direct sums of direct summands of objects of the form Λ^λ.*

Proof. The induction functor $\widetilde{\mathrm{ind}}_\lambda : P_{L_\lambda}(\{0\}, \Bbbk) \to P_G(\overline{\mathcal{O}_{\lambda^\vee}}, \Bbbk)$ has exact left and right adjoints. Since $\underline{\Bbbk}_{\{0\}}$ is projective and injective in $P_{L_\lambda}(\{0\}, \Bbbk)$, the perverse sheaf $\Lambda^\lambda := \mathrm{ind}_\lambda \underline{\Bbbk}_{\{0\}}$ is projective and injective in $P_G(\overline{\mathcal{O}_{\lambda^\vee}}, \Bbbk)$. In particular, Λ^λ has a filtration by standard objects and a filtration by costandard objects.

It remains to check that there are $|\Lambda(d)|$-many distinct indecomposable summands of $\bigoplus_\lambda \Lambda^\lambda$ (up to isomorphism). This is shown in the proof of Lemma 4.4.9. □

It follows from Proposition 4.5.3 and the proof of Lemma 4.4.9 that, for any $\lambda \in \Lambda(d)$, the perverse sheaf $\mathcal{T}_\lambda$ (as defined in the proof of Lemma 4.4.9) is the indecomposable partial tilting object in $P_G(\mathcal{N}, \Bbbk)$ associated to λ.

We next summarise the theory of Ringel duality in highest weight categories.

Say that an object T in a finite abelian category $\mathcal{A}$ is a *generalized tilting object* if:

(i) T has finite projective dimension.

(ii) $\mathrm{Ext}^i_{\mathcal{A}}(T, T) = 0$ for all $i > 0$.

(iii) For any projective object $P \in \mathcal{A}$, there is an exact sequence $0 \to P \to T_1 \to \cdots \to T_n \to 0$, where the T_i are objects in the category, $\operatorname{add} T$, of finite direct sums of direct summands of T.

Generalized tilting objects were first studied in [Miy86]. Rickard [Ric89, Theorem 6.4] shows that if T is a generalized tilting object in $\mathcal{A}$ then the functor

$$R\operatorname{Hom}_{\mathcal{A}}(T, -) : \mathcal{D}^b\mathcal{A} \to \mathcal{D}^b \operatorname{End}(T)^{op}\text{-mod}$$

is an equivalence of categories that restricts to an equivalence between $\operatorname{add} T$ and the category of projective objects in $\operatorname{End}(T)^{op}$-mod.

If an object, T, in a highest weight category $\mathcal{A}$ contains each indecomposable partial tilting object as a direct summand, then T is a generalized tilting object in $\mathcal{A}$ [Rin91, Section 5]. Such objects are called *characteristic tilting objects*.

If T is a characteristic tilting object in a highest weight category A-mod, then the equivalence

$$R\operatorname{Hom}_{\mathcal{A}}(T, -) : \mathcal{D}^b A\text{-mod} \to \mathcal{D}^b \operatorname{End}(T)^{op}\text{-mod}$$

is called a *Ringel duality functor*, and the algebra $\operatorname{End}(T)^{op}$ is called a *Ringel dual* of A. If A-mod is a highest weight category with respect to a poset Λ, then $\operatorname{End}(T)^{op}$-mod is a highest weight category with respect to the opposite poset Λ^{op} [Rin91, Theorem 6]. Moreover, for $\lambda \in \Lambda$ and corresponding element $\lambda' \in \Lambda^{op}$, the Ringel duality functor maps the indecomposable partial tilting object T_λ to the projective indecomposable $P_{\lambda'}$ [Rin91, Lemma 7].

Donkin [Don93, Proposition 3.7] shows that if $n \geq d$, then there is an algebra isomorphism

$$\mathcal{S}_{\Bbbk}(n,d)^{op} \simeq \operatorname{End}_{\mathcal{P}^{\Bbbk}_{n,d}}\Big(\bigoplus_{\lambda \in \Lambda(n,d)} \Lambda^\lambda \Bbbk^n\Big). \tag{4.12}$$

By Proposition 4.5.2, the object $\bigoplus_{\lambda \in \Lambda(n,d)} \Lambda^\lambda \Bbbk^n$ is a characteristic tilting object. In particular, if $n \geq d$ then there is an equivalence of categories

$$\mathcal{R}_{n,d} := R\operatorname{Hom}_{\mathcal{P}^{\Bbbk}_{n,d}}\Big(\bigoplus_{\lambda \in \Lambda(n,d)} \Lambda^\lambda \Bbbk^n, -\Big) : \mathcal{D}^b\mathcal{P}^{\Bbbk}_{n,d} \to \mathcal{D}^b\mathcal{P}^{\Bbbk}_{n,d}$$

that restricts to an equivalence between the full subcategory of partial tilting objects in $\mathcal{P}^{\Bbbk}_{n,d}$ and the full subcategory of projective objects in $\mathcal{P}^{\Bbbk}_{n,d}$. Moreover, this functor maps the indecomposable partial tilting object $T_{\lambda^\vee}$ to the projective cover, P_λ, of L^n_λ.

We remark that by the isomorphism in (4.12), $\mathcal{R}_{n,d}(\Lambda^\lambda \Bbbk^n) \simeq \Gamma^\lambda \Bbbk^n$.

Proposition 4.5.4. *Let $n \geq d$. For any $\lambda \in \Lambda(n,d)$, there are isomorphisms of $\mathcal{S}_\Bbbk(n,d)$-modules*

$$\mathrm{Hom}_{P_G(\mathcal{N},\Bbbk)}(\Gamma_{n,d}, \Lambda^\lambda) \simeq \Lambda^\lambda \Bbbk^n.$$

Proof. Let $\mathrm{Tilt}_G(\mathcal{N},\Bbbk)$ (respectively $\mathrm{Proj}_G(\mathcal{N},\Bbbk)$) be the full subcategory of $\mathrm{Perv}_G(\mathcal{N},\Bbbk)$ consisting of partial tilting (respectively projective) objects. Let $\mathrm{Tilt}^{\Bbbk}_{n,d}$ (respectively $\mathrm{Proj}^{\Bbbk}_{n,d}$) be the full subcategory of $\mathcal{P}^{\Bbbk}_{n,d}$ consisting of partial tilting (respectively projective) objects.

Hence the result follows from the commutativity of the following diagram of equivalences of categories.

$$\begin{tikzcd}
\mathrm{Tilt}_G(\mathcal{N},\Bbbk) \arrow[r, "\mathrm{Hom}(\Gamma_{n,d},-)"] \arrow[d, "\mathcal{R}"'] & \mathrm{Tilt}^{\Bbbk}_{n,d} \arrow[d, "\mathcal{R}_{n,d}"] \\
\mathrm{Proj}_G(\mathcal{N},\Bbbk) \arrow[r, "\mathrm{Hom}(\Gamma_{n,d},-)"'] & \mathrm{Proj}^{\Bbbk}_{n,d}
\end{tikzcd}$$

This diagram of functors commutes since both paths of functors map the indecomposable object $\mathcal{T}_{\lambda^\vee}$ to the projective cover, P_λ, of L^n_λ. □

For the remainder of this section we discuss a possible generalization of Theorem 4.4.1 that arises naturally from this discussion.

Let H be a connected complex reductive Lie group, and let X be a H-variety with finitely many orbits $\mathcal{O}_\lambda$, for λ in an indexing set Λ. Suppose also that for each $\lambda \in \Lambda$ and $x \in \mathcal{O}_\lambda$, the stabilizer of x has finitely many connected components.

By Corollary 2.3.13, for any field $\Bbbk$, $P_H(X,\Bbbk)$ is equivalent to a category $\mathcal{A}_X$-mod, for some finite dimensional $\Bbbk$-algebra $\mathcal{A}_X$. The proof of Corollary 2.3.13 does not give an explicit construction of the algebra $\mathcal{A}_X$ - indeed the abstract nature of this proof suggests that a construction of $\mathcal{A}_X$ is difficult in general.

The proof of Theorem 4.4.1 suggests an approach to a weaker version of this problem: *Can Borel-Moore homology be used to construct an algebra $\mathcal{H}_X$ in which $\mathcal{D}^b P_H(X, \Bbbk) \simeq \mathcal{D}^b \mathcal{H}_X$-mod?*

The following makes this question more precise.

Open Question 4.5.5. Let H be a connected complex reductive Lie group, and let X be a H-variety with finitely many orbits $\mathcal{O}_\lambda$, for λ in an indexing set Λ. Suppose also that for each $\lambda \in \Lambda$ and $x \in \mathcal{O}_\lambda$, the stabilizer of x has finitely many connected components.

Suppose moreover that each orbit closure $\overline{\mathcal{O}_\lambda}$ has a proper H-equivariant semismall resolution of singularities

$$m^\lambda : \tilde{X}^\lambda \to \overline{\mathcal{O}_\lambda}.$$

Define the convolution algebra

$$\mathcal{H}_X := \bigoplus_{\lambda,\mu\in\Lambda} H^{BM}_{\dim \tilde{X}^\lambda + \dim \tilde{X}^\mu}(\tilde{X}^\lambda \times_X \tilde{X}^\mu, \Bbbk).$$

Define the perverse sheaf $T_X = \bigoplus_{\lambda\in\Lambda} m^\lambda_! \underline{\Bbbk}_{\tilde{X}^\lambda}[\dim \tilde{X}^\lambda] \in \mathrm{Perv}_H(X, \Bbbk)$.

Find necessary and sufficient conditions in which T_X is a generalized tilting object in $P_H(X, \Bbbk)$ i.e. find necessary and sufficient conditions in which

$$R\operatorname{Hom}_{P_H(X,\Bbbk)}(T_X, -) : \mathcal{D}^b_H(X, \Bbbk) \to \mathcal{D}^b \mathcal{H}_X^{op}\text{-mod}$$

is an equivalence of categories.

An example of a $\mathrm{GL}_n(\mathbb{C})$-variety satisfying the conditions of Open Question 4.5.5 is the enhanced nilpotent cone defined in [AH08].

4.6 Appendix: Towards a geometric version of the homogeneous external product

A consequence of the equivalence $P_G(\mathcal{N}, \Bbbk) \simeq \mathcal{P}^{\Bbbk}_d$ is that there is a functor

$$\mathrm{real} : \mathcal{D}^b \mathcal{P}^{\Bbbk}_d \to \mathcal{D}^b_G(\mathcal{N}, \Bbbk)$$

that restricts to an equivalence of abelian categories real : $\mathcal{P}_d^{\Bbbk} \to P_G(\mathcal{N}, \Bbbk)$ (see e.g. [Ach21, Theorem A.7.16]). Such a functor is usually called a *realization functor*.

It would be interesting to have an answer to the following question.

Open Question 4.6.1. Define a product

$$- \diamond - : \mathcal{D}_G^b(\mathcal{N}, \Bbbk) \times \mathcal{D}_G^b(\mathcal{N}, \Bbbk) \to \mathcal{D}_G^b(\mathcal{N}, \Bbbk)$$

in which the following diagram commutes

$$\begin{tikzcd} \mathcal{D}^b\mathcal{P}_d^{\Bbbk} \times \mathcal{D}^b\mathcal{P}_d^{\Bbbk} \arrow[r, "\mathrm{real} \times \mathrm{real}"] \arrow[d, "-\underline{\otimes}^L-"'] & \mathcal{D}_G^b(\mathcal{N}, \Bbbk) \times \mathcal{D}_G^b(\mathcal{N}, \Bbbk) \arrow[d, "-\diamond-"] \\ \mathcal{D}^b\mathcal{P}_d^{\Bbbk} \arrow[r, "\mathrm{real}"] & \mathcal{D}_G^b(\mathcal{N}, \Bbbk) \end{tikzcd}$$

Suppose that such a product $\diamond$ exists. Then the following two properties of $\diamond$ follow from Proposition 4.4.12:

- The unit of $\diamond$ is $\underline{\Bbbk}_{\mathcal{N}}[\dim \mathcal{N}]$. That is,

  $$\underline{\Bbbk}_{\mathcal{N}}[\dim \mathcal{N}] \diamond \mathcal{F} \simeq \mathcal{F}$$

  for any $\mathcal{F} \in \mathcal{D}_G^b(\mathcal{N}, \Bbbk)$. Indeed, this holds since $\Gamma^d \Bbbk^\infty$ is the unit of $\underline{\otimes}^L$.

- For all compositions λ, μ of d, there are isomorphisms

  $$\check{m}_{\lambda !}\underline{\Bbbk}_{\check{\mathcal{N}}_\lambda}[\dim \mathcal{N}] \ \diamond \ \check{m}_{\mu !}\underline{\Bbbk}_{\check{\mathcal{N}}_\mu}[\dim \mathcal{N}] \simeq \bigoplus_{\nu \in A_\mu^\lambda} \check{m}_{\nu !}\underline{\Bbbk}_{\check{\mathcal{N}}_\nu}[\dim \mathcal{N}]$$

  Indeed, this follows from Lemma 3.6.2.

Krause [Kra13, Proposition 5.4] shows that $S^d \underline{\otimes}^L - : \mathcal{D}^b \operatorname{Rep} \Gamma_d^{\Bbbk} \to \mathcal{D}^b \operatorname{Rep} \Gamma_d^{\Bbbk}$ is a Serre functor in the sense of [BK89]. We expect the same to be true of $\mathbb{D}_{\mathcal{N}}[\dim \mathcal{N}] \diamond - : \mathcal{D}_G^b(\mathcal{N}, \Bbbk) \to \mathcal{D}_G^b(\mathcal{N}, \Bbbk)$ since $\operatorname{Hom}(\Gamma_{n,d}, \mathbb{D}_{\mathcal{N}}[\dim \mathcal{N}]) \simeq S^d \Bbbk^n$. That is, we expect that for $\mathcal{F}, \mathcal{G} \in \mathcal{D}_G^b(\mathcal{N}, \Bbbk)$, there are natural isomorphisms

$$\operatorname{Hom}(\mathcal{F}, \mathcal{G})^* \simeq \operatorname{Hom}(\mathcal{G}, \mathbb{D}_{\mathcal{N}}[\dim \mathcal{N}] \diamond \mathcal{F}).$$

Remark 4.6.2. For discussions of the useful properties and applications of Serre functors see e.g. [BO01] and [MS08].

Krause [Kra13, Theorem 5.1] describes Ringel duality of the Schur algebra using the internal product on strict polynomial functors. More precisely, [Kra13, Theorem 5.1] says that, for $n \geq d$, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{D}^b \operatorname{Rep} \Gamma^{\Bbbk}_d \arrow[r, "\sim"] \arrow[d, "\Lambda^d \underline{\otimes}^L -"'] & \mathcal{D}^b \mathcal{P}^{\Bbbk}_d \arrow[d, "(\bigoplus_{\lambda \in \Lambda(n,d)} \Lambda^\lambda \Bbbk^\infty) \otimes^L -"] \\
\mathcal{D}^b \operatorname{Rep} \Gamma^{\Bbbk}_d \arrow[r, "\sim"] & \mathcal{D}^b \mathcal{P}^{\Bbbk}_d
\end{tikzcd}$$

We expect that if the product $\diamond$ exists then it should be related to the inverse geometric Ringel duality functor $\mathcal{R}^{-1}$ in the same way that the internal product on strict polynomial functors is related to the functor $\bigoplus_{\lambda \in \Lambda(n,d)} \Lambda^\lambda \Bbbk^\infty \otimes^L -$. More precisely, we expect that there are natural isomorphisms

$$\underline{\Bbbk}_{\{0\}} \diamond \mathcal{F} \simeq i^! \mathbb{T}_{\mathfrak{g}}^{-1} i_* \mathcal{F}[d]$$

for all $\mathcal{F} \in \mathcal{D}^b_G(\mathcal{N}, \Bbbk)$.

Because of these properties, we believe that such a product $\diamond$ might prove a useful tool in the study of $P_G(\mathcal{N}, \Bbbk)$ and may invite a novel approach to computing the Kronecker coefficients.

Bibliography

[AB88] Kaan Akin, David A. Buchsbaum. Characteristic-free representation theory of the general linear group, II. Homological considerations. *Adv. Math.* 72:171-210, 1988 60

[Ach21] Pramod N. Achar. *Perverse sheaves and applications to representation theory*. Mathematical Surveys and Monographs, no. 258, *American Mathematical Society*, Providence, RI, 2021 14, 20, 21, 89, 90, 93, 94, 97, 103, 104, 106, 118

[ADL98] I. Ágoston, V. Dlab and E. Lukács. Stratified algebras. *Math. Rep. Acad. Sci. Canada* 20:22-28, 1998 40

[AH08] Pramod N. Achar, Anthony Henderson. Orbit closures in the enhanced nilpotent cone. *Adv. Math.* 219(1):27-62, 2008 117

[AHJR14] Pramod N. Achar, Anthony Henderson, Daniel Juteau, Simon Riche. Weyl group actions on the Springer sheaf. *Proc. Lond. Math. Soc.* 108(6):1501-1528, 2014 107

[AHJR16] Pramod N. Achar, Anthony Henderson, Daniel Juteau, Simon Riche. Modular generalized Springer correspondence I: the general linear group. *J. Eur. Math. Soc.* 18:1405-1436, 2016 105

[AHR15] Pramod N. Achar, Anthony Henderson, Simon Riche. Geometric Satake, Springer correspondence, and small representations II. *Represent. Theory* 19:94-166, 2015 105

[AM15] Pramod N. Achar, Carl Mautner. Sheaves on nilpotent cones, Fourier transform, and a geometric Ringel duality. *Mosc. Math. J.* 15(3):407-423, 2015 11, 87, 103, 105, 106, 108, 109, 110, 112, 114

[AR17] Cosima Aquilino, Rebecca Reischuk. The monoidal structure on strict polynomial functors. *J. Algebra* 485:213-229, 2017 2, 49, 50, 75

[BBD82] Alexander Beilinson, Joseph Bernstein, Pierre Deligne. Faisceaux pervers. *Astérisque* 100, 1982 4, 13, 15, 18, 45, 47

[BGS96] Alexander Beilinson, Victor Ginzburg, Wolfgang Soergel. Koszul duality patterns in representation theory. *J. Amer. Math. Soc.* volume 9, 2:473-527, 1996 44

[BK89] Alexei I. Bondal, Mikhail M. Kapranov. Representable functors, Serre functors, and mutations. *Izv. Akad. Nauk SSSR Ser. Mat.* 53(6):1183-1205, 1989 118

[BL94] Joseph Bernstein, Valery Lunts. *Equivariant sheaves and functors.* Lecture Notes in Mathematics, volume 1578, *Springer-Verlag*, 1994 21, 103, 104, 105

[BLM90] Alexander Beilinson, George Lusztig, Robert MacPherson. A geometric setting for the quantum deformation of GL_n. *Duke Math. J.* 61:655-677, 1990 92

[BM19] Tom Braden, Carl Mautner. Ringel duality for perverse sheaves on hypertoric varieties. *Adv. Math.* 344:35-98, 2019 45

[BO01] Alexei Bondal, Dmitri Orlov. Reconstruction of a variety from the derived category and groups of autoequivalences. *Compositio Mathematica* 125:327-344, 2001 118

[BR22] Adam Brown, Anna Romanov. Contravariant pairings between standard Whittaker modules and Verma modules. *J. Algebra* 609:145-179, 2022 45

[BS18] Jonathan Brundan, Catharina Stroppel. Semi-infinite highest weight categories. To appear in *Memoirs of the AMS* i, 4, 7, 13, 14, 39, 40

[Bry09] R. M. Bryant. Lie powers of infinite-dimensional modules. *Beiträge Algebra Geom.* 50:179-193, 2009 58

[CG97] Neil Chriss, Victor Ginzburg. Representation Theory and complex geometry. *Birkhäuser Boston Inc.*, Boston MA, 1997 i, 11, 48, 50, 54, 87, 94, 99, 101, 110, 111

[CKM14] Sabin Cautis, Joel Kamnitzer, Scott Morrison. Webs and quantum skew Howe duality. *Mathematische Annalen* 360:351-390, 2014 50, 80, 84

[CM93] David H. Collingwood, William M. McGovern. *Nilpotent orbits in semisimple Lie algebras*, Van Nostrand Reinhold Co., New York, 1993 90

[CL74] Roger W. Carter, George Lusztig. On the modular representations of the general linear and symmetric groups. *Math Z.* 136:193-242, 1974 1

[CPS88] Edward Cline, Brian J. Parshall, Leonard L. Scott. Finite-dimensional algebras and highest weight categories. *J. Reine Angew Math* 391:85-99, 1988 4, 39, 41

[CPS96] Edward Cline, Brian J. Parshall, Leonard L. Scott. Stratifying endomorphism algebras. *Mem. Amer. Math. Soc.* 124:1-119, 1996 4, 39

[CW22] Alessio Cipriani, Jon Woolf. When are there enough projective perverse sheaves? *Glasgow Mathematical Journal* 64(1):185-196, 2022 i, 6, 14, 36

[CZ19] Kevin Coulembier, Ruibin Zhang. Borelic pairs for stratified algebras. *Adv. Math.* 345:53-115, 2019 39

[Day70] Brian Day. Construction of biclosed categories. Ph.D. thesis, University of New South Wales, 1970 74

[DG02] Stephen Doty, Anthony Giaquinto. Presenting Schur Algebras. *Int. Matg. Res. Not.* 36:1907-1944, 2002 50, 57, 58

[DiG95] Persi Diaconis, Anil Gangolli. Rectangular arrays with fixed margins, in: *Discrete Probability and Algorithms*, Springer-Verlag, Berlin/New York, 15-41, 1995 64

[DGS09] Stephen Doty, Anthony Giaquinto, John Sullivan. On the defining relations for generalized q-Schur algebras. *Adv. in Math.* 221:955-982, 2009 57

[Dla96] V. Dlab Quasi-hereditary algebras revisited *An. Stiin. Univ. Ovidius Constantza* 4:43-54, 1996 39

[Don86] Stephen Donkin. On Schur algebras and related algebras I. *J. Algebra* 104:310-328, 1986 49, 50, 54, 113

[Don93] Stephen Donkin. On tilting modules for algebraic groups. *Mathematische Zeitschrift* 212:39-60, 1993 61, 85, 114, 115

[Dot03] Stephen Doty. Presenting generalized q-Schur algebras. *Representation Theory* 7:196-213 (electronic), 2003. 49, 50, 53, 54

[DR09] James M. Douglass, Gerhand Röhrle. The Steinberg variety and representations of reductive groups. *J. Algebra* 321(11):3158-3196, 2009. 95

[Fri07] Anders Frisk. Dlab's theorem and tilting modules for stratified algebras. *J. Algebra* 314:507-537, 2007. 39

[FS97] Eric M. Friedlander, Andrei Suslin. Cohomology of finite group schemes over a field. *Inventiones Mathematicae* 127(2):209-270, 1997 2, 72, 73

[Ger59] Murray Gerstenhaber. On nilalgebras and linear varieties of nilpotent matrices, III. *Annals of Mathematics* 70(1):167-205, 1959 44, 91

[GL91] Werner Giegle, Helmut Lenzing. Perpendicular categories with applications to representations and sheaves. *J. Algebra* 144(2):273-343, 1991 22

[Gou22] Valentin Gouttard. Perverse monodromic sheaves. *J. London Math. Soc.* 106(1):388-424, 2022 45

[Gre80] James A. Green. *Polynomial representations of GL_n*, volume 830 of *Lecture notes in mathematics*. Springer, Berlin, 1980 1, 48, 56, 57, 58, 76, 77

[Hen15] Anthony Henderson. Singularities of nilpotent orbit closures. *Revue Roumaine de Mathematiques Pures et Appliquees* 60(4):441-469, 2015 90

[HTT08] Ryoshi Hotta, Kiyoshi Takeuchi, Toshiyuki Tanisaki. *D-Modules, Perverse Sheaves, and Representation Theory*, English Edition, Birkhauser, Boston, 2008 45, 46

[JK81] Gordon Douglas James, Adalbert Kerber. *The Representation Theory of the Symmetric Group*, volume 16 of *Encyclopedia of Mathematics and its Applications*. Addison-Wesley, Reading, MA, 1981 63

[JMW12] Daniel Juteau, Carl Mautner, Geordie Williamson. Perverse sheaves and modular representation theory, in: *Geometric methods in representation theory II*, *Séminares et Congrès* 24:313-350, 2012 90

[Kra13] Henning Krause. Koszul, Ringel and Serre duality for strict polynomial functors. *Compositio Mathematica* 149(6):996-1018, 2013 i, 2, 7, 48, 49, 50, 62, 72, 73, 74, 75, 118, 119

[Kra15] Henning Krause. Krull-Schmidt categories and projective covers. *Expositiones Mathematicae* 33(4): 535-549, 2015 26

[Kra17] Henning Krause. Highest weight categories and recollements. *Annales de l'Institut Fourier* 67(6):2679-2701, 2017 7, 14, 22, 41, 42, 44

[KS90] Masaki Kashiwara, Pierre Schapira. *Sheaves on manifolds*, volume 292 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, 1990 107

[Lus81] George Lusztig. Green polynomials and singularities of unipotent classes. *Adv. Math.* 42:169-178, 1981 86

[Lus84] George Lusztig. Intersection cohomology complexes on a reductive group. *Invent. Math.* 75:205-272, 1984 94

[Lus93] George Lusztig. *Introduction to quantum groups*, volume 110 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1993 49, 52

[LW15] Ivan Losev, Ben Webster. On uniqueness of tensor products of irreducible categorifications. *Selecta Math.* 21:345-377, 2015 39

[Mau10] Carl Mautner. Sheaf theoretic methods in modular representation theory. Ph.D. thesis, University of Texas at Austin, 2010 2, 3

[Mau14] Carl Mautner. A geometric Schur functor. *Selecta Math. (N.S.)* 20(4):961-977, 2014 i, 3, 9, 44, 48, 86

[Mén01] Miguel A. Méndez. Directed graphs and the combinatorics of the polynomial representations of $GL_n(\mathbb{C})$. *Annals of Combinatorics* 5:459-478, 2001 77

[Mir04] Ivan Mircović. Character sheaves on reductive Lie algebras. *Mosc. Math. J.* 4(4):897-910, 2004 107

[Miy86] Yoichi Miyashita. Tilting modules of finite projective dimension. *Math. Zeit.* 193:113-146, 1986 115

[MS08] Volodymyr Mazorchuk, Catharina Stroppel. Projective-injective modules, Serre functors and symmetric algebras. *J. Reine Angew. Math.* 616:131-165, 2008 62, 118

[MV86] Robert MacPherson, Kari Vilonen. Elementary construction of perverse sheaves. *Invent. Math.* 84:403-435, 1986 4

[MV07] Ivan Mircović, Kari Vilonen. Geometric Langlands duality and representations of algebraic groups over commutative rings. *Annals of Mathematics* 166(1):95-143, 2007 3, 71, 86

[Psa14] Chrysostomos Psaroudakis. Homological theory of recollements of abelian categories. *J. Algebra* 398:63-110, 2014 13, 17, 47

[Rei16] Rebecca Reischuk. The monoidal structure on strict polynomial functors. Ph.D. thesis, Universität Bielefeld, 2016 2, 50, 75

[Ric89] Jeremy Rickard. Morita theory for derived categories. *J. London Math. Soc.* s2-39(3):436-456, 1989 115

[Rin91] Claus M. Ringel. The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences. *Math. Zeit.* 208:209-223, 1991 113, 115

[Sch1901] Issai Schur. Über eine Klasse von Matrizen, die sich einer gegebenen Matrix zuordnen lasse. Dissertation, Berlin, 1901 1, 48, 50, 56, 75, 77

[Ser87] Jean-Pierre Serre. Complex Semisimple Lie Algebras. *Springer-Verlag*, New York, 1987 51, 54

[Tot97] B. Totaro. Projective resolutions of representations of $\mathrm{GL}(n)$ *J. Reine Angew. Math*, 482, 1997 50, 75

[Tou14] Antoine Touzé. Applications of functor (co)homology. in: *An alpine expedition through algebraic topology*, volume 617 of *Contemporary Mathematics*: pp. 259-277, American Mathematical Society, Providence, RI, 2014 2014 2, 72

[W19] Giulian Wiggins. Presentations of categories of modules using the Cautis-Kamnitzer-Morrison principle. *Journal of Combinatorial Algebra* 3(1):71-112, 2019 50, 80, 83

[Wit65] Hassler Whitney. Local properties of analytic varieties. in: *Differential and Combinatorial Topology (A Symposium in Honor of Marston*

Morse). Princeton University Press: pp 205-244, Princeton N.J., 1965
20