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Name principles, and hierarchies of regular cardinals applied to LST numbers and inner model theory

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# Name Principles, and Hierarchies of Regular Cardinals applied to LST numbers and Inner Model Theory 

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A dissertation submitted to the University of Bristol in accordance with the requirements for award of the degree of Doctor of Philosophy in the Faculty of Science

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#### Abstract

We present several new discoveries in two multiple areas of set theory. First, we introduce a new kind of forcing axiom known as a "name principle". We give a detailed and comprehensive account of how these name principles relate to one another and to the classical forcing axioms. This leads us to consider new variants of the traditional forcing axioms, and we also include these in our account. We then show several examples of uses for these relationships, including improvements to, and substantially simplified proofs of, several well-known theorems.

We then turn to another topic, looking at stratifying the class Reg of $V$ regular cardinals by CantorBendixson rank. Letting $\operatorname{Reg}_{<\alpha}$ be the class of all elements of Reg of rank $<\alpha$, we show (under fairly weak large cardinal assumptions) that $L\left[\mathrm{Reg}_{<\alpha}\right]$ can be expressed as a generic extension of an iteration of a type of model called a "mouse". We also prove a similar result for Regs ${ }^{s}$, the class of strong inaccessibles. Then we go on to show that all the mice we've used exist in $L\left[\mathrm{Reg}^{s}\right]$.

Finally, we generalise a result in [29]. We define two different predicates of second order logic which are related to $\operatorname{Reg}_{<\alpha}$ in much the same way that the famous predicate $I$ is related to Card. We then find a lower bound for the Löwenheim-Skolem-Tarski number of these two predicates (together with $I$ ), and show that this lower bound is optimal.


## Acknowledgements

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The COVID-19 pandemic started in the middle of my degree, and I would like to thank my parents for letting me stay with them over the long months of lockdown. While I still couldn't exactly describe my experience of the pandemic as "pleasant", it was a lot better than it would have been if I'd spent it stuck on my own in Bristol. I would also like to thank my fiancée Deborah and my parents for the emotional support they've given me throughout my degree. In particular, Deborah's support has been invaluable over the last few months of preparing my thesis.

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Finally, and above all, I am grateful to the Lord Jesus Christ. Without his help, in so many different ways, this thesis would not have been possible.

## Author's Declaration

I declare that the work in this dissertation was carried out in accordance with the requirements of the University's Regulations and Code of Practice for Research Degree Programmes and that it has not been submitted for any other academic award. Except where indicated by specific reference in the text, the work is the candidate's own work. Work done in collaboration with, or with the assistance of, others, is indicated as such. Any views expressed in the dissertation are those of the author.

SIGNED:
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## Introduction

Since its invention roughly a century and a half ago, set theory has blossomed into a myriad of a different subfields. Two of the more notable of these fields are Inner Model Theory and Forcing.

Inner Model Theory began early on with Gödel's constructible universe $L$ : he showed that given any model $V$ of ZF, there will be a transitive proper subclass $L$ of $V$ which is also a model of ZF. This $L$ will also believe the axiom of choice and the continuum hypothesis; thus, we can conclude that if ZF is consistent then so is $\mathrm{ZFC}+\mathrm{CH}$.

We call $L$ an "inner model": a class-sized model $M \subset V$ of ZF which is in some sense recognised and understood by $V$. By modifying Gödel's construction in very natural ways, one can build other inner models. Gödel's universe $L$ is the least of these inner models, in the sense that it's both contained in and definable over any other one.

It's not actually possible to prove from ZFC alone that $V$ contains any inner models other than $L$. (To see this, consider what would happen if we happened to choose $L$ itself as our universe $V$.) But if we add certain other common axioms of set theory in, we get a rich tapestry of different inner models. Those which are generated by "mice" are of particular interest. A mouse is a small (i.e. set-size) object which looks suspiciously like part of an inner model, and which satisfies a selection of other requirements. The defining feature of a mouse is that it can be "iterated": we can push the ordinals around and make the mouse grow larger by using a series of elementary embeddings.

Forcing deals with a process invented by Cohen in 1963 to answer the converse to Gödel's result: If ZF is consistent, then are $\mathrm{ZF}+\neg$ Choice and $\mathrm{ZFC}+\neg \mathrm{CH}$ also consistent? To prove that they are, we do the opposite of Gödel: we start with a universe $V$ and add something into it. We take a forcing - a type of partial order in $V$, and use it to define an object known as a generic filter. In nontrivial cases we can show that no generic filter actually exists in $V$, but we can use certain tricks to somehow conjure up one which lives outside $V$.

Having got a generic filter, we then use it to define a minimal model of ZFC which contains it as a set, and contains the whole of $V$ as a subclass. If we start with the right forcing, we can get a model of ZFC for which CH fails; or (with a small additional step) a model of ZF for which Choice fails.

The discovery of forcing revolutionised set theory, because the same process enables us to create universes with many different kinds of sets in them, depending on which forcing (i.e. partial order) we started with. Suddenly, many consistency results which were previously inaccessible became easy to prove.

There is a weakness to forcing, however. By its nature, it deals only with consistency results: in all nontrivial cases, the generic filter won't be a set in our original universe $V$, so we can't really say anything much about that universe. To work around this, set theorists found it natural to introduce forcing axioms, which say that $V$ contains filters which are close (in a certain technical sense) to being generic. This avoids the need to move outside $V$ when we want to use a "generic" filter, so we can prove things about $V$ itself using the methods of forcing. Iconic examples of forcing axioms include Martin's Axiom MA and the Proper Forcing Axiom PFA.

In the first part of this thesis, I will present a new kind of forcing axiom known as a name principle. They also express the idea that $V$ contains filters that are close to being generic, but do so in a completely different way to the classical forcing axioms. Very simple instances of these name principles are often proved ad-hoc in arguments involving forcing axioms. But so far as I am aware, there has not hitherto been any proper study of name principles as axioms in their own right. The first half of this thesis is devoted to conducting such an analysis. It turns out that there is a detailed web of connections and equivalences, far beyond the simple results that are routinely proved ad-hoc. The web also naturally suggests the invention of new forcing axioms in the classical style, which also get included in the study. This work is adapted from [37], which is
the result of collaboration between myself and Philipp Schlicht.
In the second part of this thesis, we will turn to another topic and look at the class Reg of all regular cardinals. We stratify Reg according to Cantor-Bendixson rank, so $\operatorname{Reg}_{0}$ is the class of all successor cardinals, $\operatorname{Reg}_{1}$ is the class of all simple (weak) inaccessibles, $\mathrm{Reg}_{2}$ is the class of all inaccessible simple limits of inaccessibles, and so on. We can also do a similar stratification of $\mathrm{Reg}^{s}$, the class of all strong inaccessibles. Of course, $\operatorname{Reg}_{0}$ is simply the successors of Card, the class of all cardinals. It turns out that known results about Card also tend to hold for $\operatorname{Reg}_{<\alpha}$ for reasonably large $\alpha$. We present two cases of such a phenomenon, in different fields of set theory.

The first of these two results is in inner model theory, based on [46]. We show that - assuming the existence of certain mice - the inner model $L\left[\mathrm{Reg}_{<\alpha}\right]$ can be generated by iterating one such mouse On many times, and then taking a generic extension of the resulting model by (a hyperclass version of) a forcing invented by Magidor. Conversely, we also show that the mice we started with, if they exist, will always be found in $L\left[\mathrm{Reg}^{s}\right]$.

The second of this pair of results is in the intersection of set theory and model theory. Recall that the Löwenheim-Skolem theorem, the foundation of model theory, says that if $\mathcal{L}$ is a first order language, then any $\mathcal{L}$ structure will contain an elementary substructure of cardinality at most $|\mathcal{L}|$ or $\aleph_{0}$, whichever is larger. We can adapt this concept to second order logics: the Löwenheim-Skolem-Tarski number of a second order language $\mathcal{L}$ is the least cardinal $\kappa$ such that any $\mathcal{L}$ structure contains an elementary substructure of cardinality $<\kappa$.

We study two different schemes of second order languages, both giving information about $\operatorname{Reg}_{<\alpha}^{V}$ (for $\alpha$ an ordinal, or $\alpha=\infty$ ). We begin by analysing how the LST numbers of these languages relate to one another, eventually showing that they do exactly what we intuitively expected unless $\alpha$ is a very large ordinal. We then generalise the main result of [29], by finding a lower bound for each of these LST numbers, and proving - assuming the consistency of certain standard "large cardinal" axioms - that these lower bounds are optimal by constructing a model in which the LST number is exactly the lower bound.

The structure of the thesis is as follows.
Chapter 1 is a brief overview of the standard results of forcing which will be used in the thesis. This is mostly well-known, but the final section (on class forcings) is more obscure.

Chapter 2 is about the new results about name principles (including defining both name principles and forcing axioms).

Chapter 3 covers some additional standard concepts we need for the second part of the thesis. It opens by introducing certain "large cardinal" axioms, but the bulk of the chapter is taken up by a detailed account of mice and iterations. No prior knowledge is assumed.

Chapter 4 contains the new results about $L\left[\operatorname{Reg}_{<\alpha}\right]$ and mice.
Chapter 5 covers the new results about LST numbers.

## Chapter 1

## Introduction to Forcing

We open with a refresher of the basic concepts of forcing we will be using. We will not go through these results in full detail, and in particular, shall omit almost all proofs. Readers with no knowledge of forcing at all should refer to [26] for more detailed explanations.

Readers who are already fully comfortable with forcing will probably find little new in this chapter. However, they may still find the final section on class and hyperclass forcings instructive.

### 1.1 The Basics

Forcing is a method of generating new universes of sets, which satisfy desirable axioms. To do this, we start with a universe $V$, and add a new "generic" set to $V$ (together with everything which can be defined from $V$ and that set). Depending on what forcing we use, we can show that the generic set has certain properties.

Definition 1.1.1. A forcing is a partial order $\mathbb{P}$ with a maximal element, usually denoted $\mathbb{1}$. If $p \leqslant q$ then we say $p$ is stronger than $q$. If $p, q \in \mathbb{P}$ are such that there is some $r \leqslant p, q$ then we write $p \| q$ and say that $p$ and $q$ are compatible. Otherwise we write $p \perp q$ and say that they are incompatible.

Unfortunately, there is some disagreement about which way the inequality should go in a forcing. Some authors, particularly those in Israel, prefer the reverse: that $\mathbb{1}$ is the least element, and $p$ is stronger than $q$ if $q \leqslant p$. But here, we shall exclusively use the (more common) notation where $\mathbb{1}$ is at the top. Of course, it makes no difference to the underlying mathematics.

There are a couple of nontriviality conditions which we (almost) always assume when dealing with forcings.
Definition 1.1.2. Let $\mathbb{P}$ be a forcing. $\mathbb{P}$ is atomless if for all $p \in \mathbb{P}$ there exist $q, r \leqslant p$ such that $q \perp r$. The forcing is separative if for all $p, q \in \mathbb{P}$ if $q \leqslant p$ then there exists $r \leqslant q$ such that $r \perp p$.

These properties are easy to verify, and we won't dwell on proving them for specific forcings.
The following two definitions are very standard, even outside the world of forcing.
Definition 1.1.3. Let $\mathbb{P}$ be a forcing. A dense subset $D$ of $\mathbb{P}$ is a subset such that for all $p \in \mathbb{P}$ there exists $q \in D$ such that $q \leqslant p$. The set $D$ is open if it is downwards closed; i.e. if $p \in D \wedge q \leqslant p \rightarrow q \in D$.

A filter on $\mathbb{P}$ is a collection $G$ of elements of $\mathbb{P}$ such that:

- $\mathbb{1} \in G$
- $p, q \in G \Longrightarrow \exists r \leqslant p, q: r \in G$
- $p \geqslant q \in G \Longrightarrow p \in G$

Notice that we have carefully not said that $G$ is a set in the above definition. In general, the convention with forcing is to write $g$ for a filter which is a set in $V$, and $G$ for one which isn't.

The key definition of forcing is as follows:

Definition 1.1.4. Let $\mathbb{P}$ be a forcing. A filter $G$ on $\mathbb{P}$ is generic if for all dense subsets $D$ of $\mathbb{P}$,

$$
G \cap D \neq \varnothing
$$

This definition is relative to the universe we're working in: even if the same forcing $\mathbb{P}$ is a set in two universes, they will normally have different collections of dense subsets of $\mathbb{P}$ so a filter may be generic in the sense of one universe but not the other. If we need to clarify things, we say that $G$ is $V$ generic over $\mathbb{P}$, or $\mathbb{P}$ generic over $V$, or $V$ generic, etc.

Proposition 1.1.5. Let $\mathbb{P}$ be atomless. Then there is no filter $g$ which is a set in $V$ and is $V$ generic.
Proof. Suppose such a filter $g$ exists. Then the set

$$
D:=\{p \in \mathbb{P}: p \notin g\}
$$

exists in $V$. Since $\mathbb{P}$ is atomless, $D$ is dense: given any $p \in \mathbb{P}$ we can find incompatible $q, r \leqslant p$; and then since all elements of $g$ are compatible at least one of $q$ and $r$ is not in $g$. But since $g$ is generic, $g \cap D \neq \varnothing$, which is obviously a contradiction.

So we can't find any generic filters in $V$. However:
Proposition 1.1.6. Suppose that $M$ is a countable transitive model of ZFC in $V$, and that $M$ contains a forcing $\mathbb{P}$. Let $q \in \mathbb{P}$. Then there is a filter $G \in V$ on $\mathbb{P}$ which is $M$ generic and contains $q$.

Proof. Since $M$ is countable, it contains only countably many dense sets. Enumerate them as $\left\{D_{n}: n \in \omega\right\}$. Construct a descending chain $q \geqslant p_{0} \geqslant p_{1} \geqslant p_{2} \geqslant \ldots$ of elements of $\mathbb{P}$, where $p_{n} \in D_{n}$ for all $n$. (This is easy, since each $D_{n}$ is dense.) Let

$$
G=\left\{p \in \mathbb{P}: \exists n \in \omega p_{n} \leqslant p\right\}
$$

It is easy to see that $G$ is a generic filter and contains $q$.
Although it's a useful simplification, we don't actually need to assume that we're working in a countable transitive model: we can discuss filters $G$ that are $V$ generic. This has some metatheory level problems - if $\mathbb{P}$ is atomless, then such a filter would have to be a "set" which was not in the universe of sets, nor definable over it. But with some technical work, we can work inside $V$ and still discover what would happen if we added a (hypothetical) generic filter. See [26, VII.9] for the details of this.

It is therefore standard to speak as though, given any forcing $\mathbb{P}$ in $V$, we can find a generic filter outside $V$, and can manipulate any such filter as if it were a set.

### 1.1.1 Names

So, we now have a filter $G$ which is outside $V$. But we want to get an entire universe containing $G$. For this, we need the concept of names.

Definition 1.1.7. Let $\mathbb{P}$ be a forcing. A $\mathbb{P}$ name is defined (recursively) as a set $\sigma$ whose elements are all of the form $(\tau, p)$ where $\tau$ is a name and $p \in \mathbb{P}$.

The intuition behind a name is that we're taking some set in $V$, and labelling all its elements, the elements of its elements, and so on, with conditions from $\mathbb{P}$. (But do note that this intuitition isn't exactly right: a single name $\tau$ can appear in $\sigma$ multiple times, associated with different elements of $\mathbb{P}$.) Unlike generic filters, names are always elements of $V$; and moreover, $V$ knows what the class of all $\mathbb{P}$ names looks like.

Definition 1.1.8. Let $\mathbb{P}$ be a forcing, let $g$ be a filter (which might be generic, or an element of $V$, or neither), and let $\sigma$ be a name. The interpretation $\sigma^{g}$ of $\sigma$ is defined (again, recursively) as

$$
\sigma^{g}:=\left\{\tau^{g}: \exists p \in g(\tau, p) \in \sigma\right\}
$$

So intuitively, interpreting $\sigma$ using $g$ means throwing away anything which isn't associated with an element of $g$. If a universe contains $\sigma$ and $g$ (and believes ZFC) then it will also contain $\sigma^{g}$.

We can define canonical names for every set in $V$, and for the filter $g$.
Definition 1.1.9. Let $\mathbb{P}$ be a forcing.

1. Let $x \in V$. We recursively define the name

$$
\check{x}:=\{(\check{y}, \mathbb{1}): y \in x\}
$$

2. The name $\dot{G}$ is defined as:

$$
\dot{G}:=\{(\check{p}, p): p \in \mathbb{P}\}
$$

It is trivial to show that these are always interpreted in the way we expect:
Proposition 1.1.10. Let $\mathbb{P}$ be a forcing, and let $g$ be any filter (generic, in $V$, or otherwise). For $x \in V$, $\check{x}^{g}=x$. Also, $\dot{G}^{g}=g$.

### 1.1.2 Generic Extensions

Definition 1.1.11. Let $\mathbb{P}$ be a forcing, and let $G$ be a $V$-generic filter. The generic extension $V[G]$ of $V$ with respect to $G$ is the class of all $G$ interpretations of names:

$$
V[G]=\left\{\sigma^{G}: \sigma \text { is a } \mathbb{P} \text { name }\right\}
$$

By 1.1.10 we know that $V[G]$ contains $V$ (as a class) and $G$ (as a set). But in fact, we can prove a lot more than this.

Theorem 1.1.12. Suppose $V$ believes $Z F$. Let $\mathbb{P}$ be a separative, atomless forcing, and $G$ a generic filter. Then in any model of $Z F$ containing $V$ (as a class) and $G$, the generic extension $V[G]$ is the smallest class which contains $G$ (as an element) and $V$ (as a subclass) and believes $Z F$. Moreover, if $V$ believes the axiom of choice, then so does $V[G]$.

The proof of this theorem is rather in depth and can usually be treated as a black box, so we won't cover it here. In the process of proving it, one obtains another black box result. This one is where the word "forcing" comes from - a testament to how fundamental it is to forcing theory.

Definition 1.1.13. Let $\mathbb{P}$ be a forcing, and let $p \in \mathbb{P}$. Let $\varphi\left(x_{0}, \ldots, x_{n}\right)$ be a formula, and let $\sigma_{0}, \ldots, \sigma_{n}$ be $\mathbb{P}$ names. We say that $p$ forces $\varphi\left(\sigma_{0}, \ldots, \sigma_{n}\right)$, and write

$$
p \Vdash \varphi\left(\sigma_{0}, \ldots, \sigma_{n}\right)
$$

if whenever $G$ is a generic filter which contains $p$

$$
V[G] \vDash \varphi\left(\sigma_{0}^{G}, \ldots, \sigma_{n}^{G}\right)
$$

As we've stated it, this is a definition that quantifies over all generic filters $G$ that could possibly exist, and it's not immediately clear what that means. If $V$ is a countable transitive structure in some larger universe, then there is no problem - we mean all the generic filters $G$ that exist in that larger universe. But note that the definition is using knowledge of that wider universe, so it's not - as written - something that $V$ knows about. If $V$ is class-sized, then the definition has even more problems - how can we quantify over filters that "could possibly" exist? We would have to use some kind of second order logic to express it, which is philosophically difficult to justify and requires us to make several irritating metatheory definitions.

Fortunately, we can work around this by showing that the relation $\Vdash$ isn't fundamentally second order, and is actually definable in $V$.

Theorem 1.1.14. 1. Let $\mathbb{P}$ be a forcing in some countable transitive model $M$ of $Z F$. There is an $M$ definable relation $\Vdash^{\star}$ such that for any $p \in \mathbb{P}$, any formula $\varphi\left(v_{0}, \ldots, v_{n}\right)$ and any names $\sigma_{0}, \ldots, \sigma_{n}$

$$
p \Vdash \varphi\left(\sigma_{0}, \ldots, \sigma_{n}\right) \Longleftrightarrow p \Vdash^{\star}\left({ }^{\ulcorner } \varphi^{\top}, \sigma_{0}, \ldots, \sigma_{n}\right)
$$

where ${ }^{\ulcorner } \varphi$ denotes the Gödel number of $\varphi$. Moreover, $\Vdash^{\star}$ can be defined recursively in a canonical way.
2. Let $\mathbb{P}$ be a forcing in any model $V$ of $Z F$. Let $G$ be a generic filter, let $\varphi\left(v_{0}, \ldots, v_{n}\right)$ be a formula, and let $\sigma_{0}, \ldots, \sigma_{n}$ be names. Then

$$
V[G] \vDash \varphi\left(\sigma_{0}^{G}, \ldots, \sigma_{n}^{G}\right)
$$

if and only if there exists some $p \in G$ such that

$$
p \Vdash^{\star}\left({ }^{\ulcorner } \varphi^{\top}, \sigma_{0}, \ldots, \sigma_{n}\right)
$$

3. Let $\mathbb{P}$ be any forcing in $V$, let $p \in \mathbb{P}$, and let $\varphi, \sigma_{0}, \ldots, \sigma_{n}$ be as above. Then the following are equivalent:

- $p \Vdash^{\star}\left({ }^{\ulcorner } \varphi^{\top}, \sigma_{0}, \ldots, \sigma_{n}\right)$
- For densely many $q \leqslant p, q \Vdash^{\star}\left({ }^{\ulcorner } \varphi^{\top}, \sigma_{0}, \ldots, \sigma_{n}\right)$
- For all $q \leqslant p, q \Vdash^{\star}\left({ }^{\ulcorner } \varphi^{\top}, \sigma_{0}, \ldots, \sigma_{n}\right)$

In particular, if for every condition $p \in \mathbb{P}$ there is some generic filter $G$ containing $p$, then $\Vdash^{\star}$ and $\Vdash$ agree with one another.

This means that whatever metatheory you use, in any context where $\Vdash$ is definable and Proposition 1.1.6, $\Vdash$ will agree with $\Vdash^{\star}$. And even in contexts where $\Vdash$ is not definable, $\Vdash^{\star}$ still behaves in all ways as though it were the same as $\Vdash$. So in practice, we never talk about $\Vdash^{\star}$ : we simply assume that $\Vdash^{\star}$ is the same as $\Vdash$, and that $\Vdash$ is definable not just in the metatheory but within $V$ itself. This means that $V$ can "almost" tell what $V[G]$ will look like. Somehow the only information it's missing is what $G$ actually is - it knows exactly what having any given condition in $G$ will mean for $V[G]$. The reason this result is so critical is that it allows us to express many statements about $V[G]$ inside $V$.

### 1.1.3 Boolean Algebras

There is also another way to formulate forcings, by using complete Boolean algebras instead of partial orders. Recall the following definitions from order theory.

Definition 1.1.15. A partial order $B$ is a Boolean algebra if it satisfies the following:

1. $B$ has a largest element 1 and a smallest element 0 .
2. For any two elements $a, b \in B$, there is a least upper bound $a \vee b$ and a greatest lower bound $a \wedge b$ of $a$ and $b$.
3. $\wedge$ and $\vee$ are distributive over each other: $a \wedge(b \vee c)=(a \wedge b) \vee(a \wedge c)$ and $a \vee(b \wedge c)=(a \vee b) \wedge(a \vee c)$.
4. For any element $b \in B$, there is another element $\neg b \in B$ such that $b \vee \neg b=1$ and $b \wedge \neg b=0$.
$B$ is complete if every subset $S$ of $B$ (not just pair of elements) has a least upper bound $\bigvee S$ and a greatest lower bound $\bigwedge S$.
Lemma 1.1.16. Let $P$ be any partial order. Then there is a minimal complete Boolean algebra $B$ which contains $P$. It is called the Boolean completion of $P$, and $P \backslash\{0\}$ is a dense subset of $B \backslash\{0\}$.

Let $B$ be a complete Boolean algebra. Then $B \backslash\{0\}$ is a partial order with a maximal element. So we can do forcing with $B \backslash\{0\}$ exactly as defined above. This is what is meant by forcing with a (complete) Boolean algebra.

Proposition 1.1.17. Let $\mathbb{P}$ be a forcing, and let $B$ be its Boolean completion. If $G$ is a $\mathbb{P}$ generic filter, then its upwards closure $\tilde{G}:=\{b \in B: \exists p \in G p \leqslant b\}$ is a $B$ generic filter. If $\tilde{G}$ is a $B$ generic filter, then $G:=\tilde{G} \cap \mathbb{P}$ is a $\mathbb{P}$ generic filter. In both cases, $V[G]=V[\tilde{G}]$.

Due to this proposition, we can generally toggle between forcings and their Boolean completions whenever we want to, according to what is convenient. The advantage of using the Boolean completion is that it lets us very easily express certain statements about $\Vdash$. For example, if some sentence $\varphi$ is forced by (precisely) the elements of some set $S \subset B$, then " $p \Vdash \varphi$ " is equivalent to the simple order theory statement " $p \leqslant \bigvee S$ ". (We sometimes simplify this statement still further by writing $\llbracket \varphi \rrbracket$ to denote $\bigvee S$.) On the other hand the advantage of using $\mathbb{P}$ directly is that it's a lot easier to understand what the conditions are and what the forcing actually looks like.

### 1.2 Standard Forcing Results

So now we have met the basic theory of forcing. Next we shall give a small handful of standard results which we will be using later on. The first two are easy tests to establish that $\mathbb{P}$ does not significantly change certain parts of $V$.

### 1.2.1 Chain Conditions

Definition 1.2.1. Let $\mathbb{P}$ be a forcing. A set $A \subset \mathbb{P}$ is an antichain if all its elements are pairwise incompatible.
Definition 1.2.2. Let $\mathbb{P}$ be a forcing, and $\kappa$ be a cardinal. We say that $\mathbb{P}$ meets the $\kappa$ chain condition (the $\kappa$-c.c.) if it contains no antichains of size $\kappa$. We say $\mathbb{P}$ meets the countable chain condition (the c.c.c.) if it meets the $\omega_{1}$ chain condition.

Note that this terminology is a little confusing: it should probably be called the $\kappa$ antichain condition. (A chain is a descending sequence of conditions, which isn't really connected to the $\kappa$-c.c.!) And the countable chain condition is associated with the first uncountable cardinal. Unfortunately, the definitions got stuck this way in the very early days of forcing, and it's now far too late to change them.

Theorem 1.2.3. [26, 6.9] Let $\mathbb{P}$ satisfy the $\kappa$ chain condition, and let $\lambda \geqslant \kappa$ be a cardinal of $V$. If $\lambda>\kappa$ or $\lambda=\kappa$ is regular in $V$, then $\lambda$ is still a cardinal in any generic extension $V[G]$. Moreover, if $\lambda \geqslant \kappa$ is regular in $V$ then it is still a regular cardinal in $V[G]$, and otherwise $\operatorname{Cof}^{V[G]}(\lambda)=\operatorname{Cof}^{V[G]}\left(\operatorname{Cof}^{V}(\lambda)\right)$.

### 1.2.2 Closed and Distributive Forcings

Definition 1.2.4. Let $\mathbb{P}$ be a forcing, and $\kappa$ be a cardinal. We say $\mathbb{P}$ is $\kappa$-distributive if whenever $\left(D_{\alpha}\right)_{\alpha<\kappa}$ is a sequence of dense open subsets of $\mathbb{P}$, the set $\bigcap_{\alpha<\kappa} D_{\alpha}$ is also dense. We say $\mathbb{P}$ is $<\kappa$-distributive if it is $\lambda$ distributive for every $\lambda<\kappa$.

We say $\mathbb{P}$ is $\kappa$-closed if for every descending chain of conditions $\left(p_{\alpha}\right)_{\alpha<\kappa}$, there is a condition $q \in \mathbb{P}$ such that for all $\alpha, q \leqslant p_{\alpha}$. We say $\mathbb{P}$ is $<\kappa$ closed if it is $\lambda$-closed for every $\lambda<\kappa$.

1
Closed-ness is most often used as an easy way to verify distributivity:
Proposition 1.2.5. If a forcing is $\kappa$-closed (resp. $<\kappa$-closed), then it is $\kappa$-distributive (resp. $<\kappa$-distributive).
Theorem 1.2.6. If a forcing is <к-distributive, then it does not add any new bounded subsets of $\kappa$. In particular, if $\lambda<\kappa$ is a cardinal in $V$, then it is still a cardinal in $V[G]$ and has the same cofinality.

Moreover, if a forcing is < $\kappa$-closed, then it does not add any new sequences of ordinals of length less than $\kappa$, and hence does not collapse or change the cofinality of $\kappa$ either.

[^1]
### 1.2.3 The Continuum

We can show that forcings which satisfy smallness or closed-ness conditions do not change certain parts of GCH.

Lemma 1.2.7. Let $\lambda$ be a cardinal of $V$ such that $2^{\lambda}=\lambda^{+}$. Suppose that $\mathbb{P}$ is a forcing which does not collapse $\lambda$ or $\lambda^{+}$, and that either $\mathbb{P}$ is $<\lambda^{+}$distributive; or $2^{|\mathbb{P}|}<\lambda$; or $|\mathbb{P}| \leqslant \lambda$ and $\mathbb{P}$ has the $\lambda$ chain condition and $\beta<\lambda \Longrightarrow \lambda^{\beta}=\lambda$. Then $2^{\lambda}=\lambda^{+}$in the generic extension.
Proof. If $\mathbb{P}$ is $<\lambda^{+}$distributive then it does not add any new bounded subsets of $\lambda^{+}$and hence does not add any new subsets of $\lambda$. So $\mathcal{P}(\lambda)$ is the same in $V$ and in the generic extension $V[G]$.

If one of the other two possibilities holds, then consider a generic filter $G$. If $X \in V[G]$ is a subset of $\lambda$ then let $\sigma$ be a name such that $\sigma^{G}=X$. For $\gamma<\lambda$, let $A_{\gamma} \subset\{p \in \mathbb{P}: p \Vdash \check{\gamma} \in \sigma\}$ be a maximal antichain. Let $\tau$ be the name:

$$
\tau:=\left\{(\check{\gamma}, p): \gamma<\lambda, p \in A_{\gamma}\right\}
$$

Then $\mathbb{1}_{\mathbb{P}} \Vdash \tau=\sigma$ so $\tau^{G}=\sigma^{G}=X$. So any subset of $\lambda$ in $V[G]$ can be named by a name $\tau^{\prime}$ whose elements are all of the form $(\check{\gamma}, p), \gamma<\lambda$, such that for all $\gamma<\lambda,\left\{p \in \mathbb{P}:(\check{\gamma}, p) \in \tau^{\prime}\right\}$ is an antichain. (Such a $\tau^{\prime}$ is sometimes called a nice name.)

If $2^{|\mathbb{P}|} \leqslant \lambda$ then there are only $|\mathbb{P}|^{|\mathbb{P}|} \leqslant \lambda$ many such antichains, and so there are only at most $\lambda^{\lambda}=\lambda^{+}$ many nice names in $V$.

Similarly, if the third condition holds, then there are only at most $\lambda^{<\lambda}=\sum_{\gamma<\lambda} \lambda^{\gamma}=\lambda$ many antichains, and so there are only $\lambda^{\lambda}=\lambda^{+}$many nice names. So $\mathbb{P}$ can only ever add $\left(\lambda^{+}\right)^{V}$ many new subsets of $\lambda$.

### 1.2.4 Iterations of Forcings

Sometimes, it's not enough to perform a single forcing: we may want to do multiple, or even transfinitely many. It is often helpful to turn the whole process into a single forcing over $V$. Here, we present a way of combining two forcings into one.
Definition 1.2.8. Let $\mathbb{P}$ be a forcing, and let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$ name for another forcing. The forcing

$$
\mathbb{P} * \dot{\mathbb{Q}}
$$

consists of pairs $(p, \dot{q})$ where $p \in \mathbb{P}$ and $p \Vdash \dot{q} \in \dot{\mathbb{Q}}$. We define $(p, \dot{q}) \leqslant\left(p^{\prime}, \dot{q}^{\prime}\right)$ if $p \leqslant p^{\prime}$ and $p \Vdash \dot{q} \leqslant \dot{q}^{\prime}$.
Naturally, we can extend this to iterations of arbitrarily large finite numbers of forcings. The relation * is almost associative: the $(\mathbb{P} * \dot{Q}) * \dot{R}$ and $\mathbb{P} *(\mathbb{Q} * \mathbb{Z}$ are different forcings, but they are equivalent to one another. When the context is clear, we sometimes omit the dot and speak informally about the forcing $\mathbb{Q}$ itself, but we must always keep in mind that this is a forcing in the $\mathbb{P}$ generic extension of $V$ and will depend on our choice of generic over $\mathbb{P}$.

What about if we have infinitely many forcings we want to put together? The concept gets a bit technical when written out fully, but is essentially an extension of the above approach. We sketch the construction briefly here. Later on in Chapter 4, we shall meet the Magidor iteration of infinitely many Prikry forcings, and then in Chapter 5 we will encounter another custom-made iteration of Prikry-style forcings; we will define them formally when we get to those chapters.

Say we want to combine the forcings named by $\left(\dot{\mathbb{Q}}_{\gamma}\right)_{\gamma<\alpha}$ say. (Since this is a sketch, we'll ignore the question of what forcing generated these names.) We define the iterated forcing $\mathbb{P}_{\alpha}$ recursively. A condition of $\mathbb{P}_{\alpha}$ will consist of sequences $p:=\left(\dot{q}_{\gamma}\right)_{\gamma<\alpha}$, such that for $\gamma<\alpha$, the sequence $p 1 \gamma:=\left(\dot{q}_{\beta}\right)_{\beta<\gamma} \in \mathbb{P}_{\gamma}$; and $p 1 \gamma \Vdash \dot{q}_{\gamma} \in \dot{\mathbb{Q}}_{\gamma}$. We usually also include some requirement saying that most of the $\dot{q}_{\gamma}$ should be in some way trivial.

We say $p \leqslant p^{\prime}=\left(\dot{q}_{\gamma}^{\prime}\right)$ if for all $\gamma<\alpha, p \upharpoonleft \gamma \Vdash \dot{q}_{\gamma} \leqslant \dot{q}_{\gamma}^{\prime}$.

### 1.2.5 Proper forcings

A forcing is proper if whenever $\kappa$ is regular and uncountable, and $S \in V$ is a stationary subset of $[\kappa]^{\omega}, S$ is still stationary in the generic extension. Proper forcings are generally used in connection with the proper forcing axiom PFA, which we shall meet later.

### 1.3 Some Selected Forcings

In this section, we introduce certain specific forcings which we will be working with later. All of these forcings are well-known and widely used. Some authors muddle the terminology around these forcings, using the names and symbols given here to refer to both the partial orders we will define and their Boolean completions. We will be careful not to do this, because many of the results found in Chapter 2 only work for one or the other.

### 1.3.1 Cohen Forcing

Cohen forcing was the first forcing discovered, and is the simplest nontrivial one possible. It adds a new real number. Cohen invented it, and the process of forcing, as a way to show the consistency of $\neg \mathrm{CH}$.

Definition 1.3.1. The conditions of Cohen forcing are the finite sequences of natural numbers. They are ordered by end extension.

A Cohen-generic filter will contain sequences of arbitrary length, and all the sequences in the filter will be end extensions of each other. Taking unions gives us an $\omega$ long sequence of natural numbers, and it can easily be shown that it will not be an element of $V$. One can informally talk about "adding a Cohen real", and talk about a real number as though it were a filter on the Cohen forcing.

Proposition 1.3.2. Cohen forcing satisfies the countable chain condition, and hence does not collapse or singularise cardinals.

### 1.3.2 Random Forcing

Random forcing also adds a new real. It is a little more complicated than Cohen forcing, since it uses the Lebesgue measure on sets of reals as its conditions. Recall the definition of the Lebesgue measure:

Definition 1.3.3. The product topology on $2^{\omega}$ is induced by the basic open sets $N_{t}=\left\{x \in 2^{\omega}: t \subset x\right\}$ for $t \in 2^{<\omega}$. Lebesgue measure is the unique measure $\mu$ on the Borel subsets of $2^{\omega}$ such that $\mu\left(N_{t}\right)=\frac{1}{2^{n}}$, where $n$ is the length of $t$.

Definition 1.3.4. Let $P$ be the set of Borel subsets of $2^{\omega}$ with positive Lebesgue measure. For $A, B \in P$ let $A \sim B$ if their symmetric difference $A \triangle B:=(A \backslash B) \cup(B \backslash A)$ has Lebesgue measure 0 . Note that $\sim$ is an equivalence relation.

Random forcing $\mathbb{P}$ is the set $P / \sim$. It is ordered by inclusion, i.e. $[A] \leqslant[B]: \Leftrightarrow A \subseteq B$.
To simplify notation, we will leave $\sim$ implicit, and talk about Borel sets of positive measure as if they were conditions in random forcing.

Lemma 1.3.5. Random forcing satisfies the countable chain condition, and hence does not collapse or singularise cardinals.

Proof. Suppose $\mathcal{A} \subset \mathbb{P}$ is an uncountable antichain. By a pigeonhole argument, we can find some rational number $x>0$ such that infinitely many distinct elements $A_{n} \in \mathcal{A}, n<\omega$ satisfy $\mu\left(A_{n}\right)>x$. By incompatibility, $m, n<\omega, m \neq n \Longrightarrow \mu\left(A_{m} \cap A_{n}\right)=0$. So

$$
\mu\left(\bigcup_{m \neq n} A_{m} \cap A_{n}\right)=0
$$

Hence

$$
\mu\left(\bigcup_{n} A_{n}\right)=\sum_{n} \mu\left(A_{n}\right)=\infty
$$

But $\mu\left(2^{\omega}\right)=1$ is finite.

### 1.3.3 Hechler Forcing

Hechler forcing is a third way to add a real. Like Cohen forcing, each condition gives us a finite initial segment of the real, but unlike in Cohen forcing we also get some information about what the rest of the real will look like.

Definition 1.3.6. The conditions of Hechler forcing are pairs $(s, f)$, where $s$ is a finite sequence of natural numbers, and $f:(\omega \backslash \operatorname{length}(s)) \rightarrow \omega$ is a function. We say $\left(s^{\prime}, f^{\prime}\right) \leqslant(s, f)$ if $s^{\prime}$ is an end extension of $s$; and for $n \in \operatorname{length}\left(s^{\prime}\right) \backslash$ length $(s)$ we have $s^{\prime}(n) \geqslant f(n)$, and for $n>$ length $\left(s^{\prime}\right)$ we have $f^{\prime}(n) \geqslant f(n)$.

Essentially, $f$ is a pointwise lower bound for the real being added. The distinguishing feature of Hechler forcing is that the new real added is eventually pointwise greater than any given real number in the ground model.

Proposition 1.3.7. Hechler forcing satisfies the c.c.c.

### 1.3.4 Prikry Forcing

Prikry forcing changes a cardinal $\kappa$ to have cofinality $\omega$. This is a forcing we will be using a lot in this thesis, so we'll go into a bit more detail on it here. It works a little like Hechler forcing. Each condition has two components: an initial segment of the cofinal $\omega$ sequence we're going to add, and a piece of information about the rest of that sequence. In this case, that piece of extraa information is that the rest of the sequence is contained in some "large" subset of $\kappa$.

In order for this to work, we need some consistent concept of "largeness" for a subset of $\kappa$. The notion we need is a normal measure.

Definition 1.3.8. A cardinal $\kappa$ is measurable if there exists $U \subset \mathcal{P}(\kappa)$ which satisfies the following:

1. $U$ is an ultrafilter on $\mathcal{P}(\kappa)$. That is, $U$ is closed under supersets and intersections, and for any $X \subset \kappa$, exactly one of $X$ and $\kappa \backslash X$ is in $U$.
2. $U$ is non-principal: There is no $\alpha \in \kappa$ such that $\{\alpha\} \in U$.
3. $U$ is $\kappa$ complete: it is closed under intersections of $<\kappa$ many of its elements.
4. $U$ is closed under diagonal intersections of length $\kappa$ : If $\left\{X_{\alpha}: \alpha<\kappa\right\}$ is a sequence of elements of $U$, then

$$
\bigwedge_{\alpha<\kappa} X_{\alpha}:=\left\{\alpha<\kappa: \alpha \in \bigcap_{\beta<\alpha} X_{\beta}\right\} \in U
$$

We say a subset of $\kappa$ is measure 1 if it is in $U$, and we call $U$ a normal measure on $\kappa$.
We will meet measurable cardinals again, and will examine them in much more detail, in Chapter 3. For now, we shall simply mention that any measurable cardinal is regular.

Definition 1.3.9. Let $\kappa$ be a measurable cardinal with measure $U$. The conditions of the Prikry forcing on $\kappa$ are pairs $(s, X)$, where $s \in \kappa^{<\omega}$ is a finite increasing sequence of ordinals, $X \in U$ and $(\max (s)+1) \cap X=\varnothing$.
$\left(s^{\prime}, X^{\prime}\right) \leqslant(s, X)$ if:

- $s^{\prime}$ is an end extension of $s$;
- $X^{\prime} \subset X$; and
- $s^{\prime} \backslash s \subset X$

By convention, we write $\left(s^{\prime}, X^{\prime}\right) \leqslant^{*}(s, X)$ if $\left(s^{\prime}, X^{\prime}\right) \leqslant(s, X)$ and $s^{\prime}=s$.
Given a generic filter $G$, we can easily generate a cofinal $\omega$ sequence below $\kappa$, by taking unions of the $s$ components of the conditions in $G$. Conversely, given a cofinal $\omega$ sequence below $\kappa$, we can generate a filter on the Prikry forcing. This filter is not automatically generic, but there is an easy way to test whether it is or not.

Lemma 1.3.10 (Mathias Criterion). Let $\kappa$ be a measurable cardinal, and let $\mathbb{P}$ be the Prikry forcing on $\kappa$. Let $s$ be a cofinal $\omega$ sequence below $\kappa$ (so $s \notin V$, as $\kappa$ is regular in $V$ ). Then the $\mathbb{P}$ filter generated by $s$ is generic if and only if for all $X \in U, s \backslash X$ is finite.

We shall not be using this result as is, but we will be using a generalisation of it to an infinite iteration of Prikry forcings. There is another useful fact about Prikry forcings which we shall also be using.

Lemma 1.3.11. Let $\mathbb{P}$ be the Prikry forcing on $\kappa$. Let $\varphi$ be some sentence (perhaps with parameters) and let $p \in \mathbb{P}$. There is some $q \leqslant^{*} p$ such that either $q \Vdash \varphi$ or $q \Vdash \neg \varphi$.

Notice that if we had $\leqslant$ in place of $\leqslant^{*}$ then there would be nothing to prove: it would be true about any forcing.

Finally, we give the usual preservation properties we've been giving for all the forcings.
Proposition 1.3.12. The Prikry forcing on $\kappa$ satisfies the $\kappa^{+}$c.c.
It does not collapse any cardinals, and does not change the cofinality of any cardinals except those with $V$ cardinality $\kappa$.

Obviously Prikry forcings are not even $\omega$ closed, because there is nothing below a descending sequence of conditions $\left(\left(s_{n}, X_{n}\right)\right)$ where $s_{n}$ has length $n$. But Lemma 1.3 .11 means that somehow this is not a problem. Every sentence will be decided by $\leqslant^{*}$ densely many conditions, and the Prikry forcing is closed under $\leqslant^{*}$ descending sequences of length less than $\kappa$, so it turns out that most of the usual closed/distributive results can be reproved here by just replacing $\leqslant$ with $\leqslant *$.

### 1.3.5 The Collapsing Forcing

This forcing collapses a cardinal $\kappa$, adding a surjection onto it from some smaller cardinal $\lambda$.
Definition 1.3.13. Let $\lambda<\kappa$ be cardinals. The conditions of the collapsing forcing $\operatorname{Col}(\lambda, \kappa)$ are partial functions $f: \lambda \rightarrow \kappa$, whose domains have cardinality $<\lambda$. They are ordered by inclusion.

A generic filter will generate a bijection $f: \lambda \rightarrow \kappa$.
Proposition 1.3.14. Suppose that $\left|\kappa^{<\lambda}\right|=\kappa$. Then $\operatorname{Col}(\lambda, \kappa)$ satisfies the $\kappa$ chain condition, and is $<\lambda$ closed.

So $\operatorname{Col}(\lambda, \kappa)$ doesn't collapse (or singularise) any cardinals other than the ones between $\lambda$ and $\kappa$.
If $\mu:=\left|\kappa^{<\lambda}\right|>\kappa$, then it is possible to show that $\operatorname{Col}(\lambda, \kappa)$ and $\operatorname{Col}(\lambda, \mu)$ are equivalent ${ }^{2}$, so $\operatorname{Col}(\lambda, \kappa)$ collapses all the cardinals up to $\mu$ but nothing higher than that.

We can also collapse all the cardinals below $\kappa$, while keeping $\kappa$ itself as a cardinal.
Definition 1.3.15. Let $\lambda<\kappa$ be cardinals, and suppose that $\kappa$ is a limit cardinal. The conditions of the collapsing forcing $\operatorname{Col}(\lambda,<\kappa)$ are partial functions $f: \lambda \times \kappa \rightarrow \kappa$ whose domains have cardinality $<\lambda$, such that if $\alpha<\lambda, \beta<\kappa$ and $f(\alpha, \beta)$ is defined, then $f(\alpha, \beta)<|\beta|^{+}$. The conditions are ordered by inclusion.

As before, a generic filter generates a full function $f: \lambda \times \kappa \rightarrow \kappa$ such that for $\alpha<\kappa, f(-, \beta): \lambda \rightarrow|\beta|^{+}$ is surjective.

Proposition 1.3.16. $\operatorname{Col}(\lambda,<\kappa)$ collapses all the cardinals in the interval $(\lambda, \kappa)$ to $\lambda$. If $\kappa$ is a regular limit cardinal (a weak inaccessible) and $\forall \alpha<\kappa\left|\alpha^{<\lambda}\right|<\kappa$ then $\operatorname{Col}(\lambda,<\kappa)$ has the $\kappa$ chain condition and is $<\lambda$ closed, so it does not collapse or singularise any other cardinals.

This is slightly more obscure than some other results, and we give a brief proof of it. We use the following definition and theorem, from [26].

Definition 1.3.17. A $\Delta$ system is a family of sets $\mathcal{A}$ such that there is some fixed set $r$ with $a \cap b=r$ for all $a, b \in \mathcal{A}, a \neq b$. We call $r$ the root of $\mathcal{A}$.

[^2]Theorem 1.3.18. [26, 1.6] Let $\lambda$ be an infinite cardinal. Let $\kappa>\lambda$ be regular, and satisfy $\forall \alpha<\kappa\left(\left|\alpha^{<\lambda}\right|<\right.$ $\kappa$ ). Let $\mathcal{A}$ be a family of sets, such that $|\mathcal{A}| \geqslant \kappa$ and $\forall x \in \mathcal{A}(|x|<\lambda)$. Then there is a subfamily $\mathcal{B} \subset \mathcal{A}$, of cardinality exactly $\kappa$ which is a $\Delta$ system.

Proof (proposition). It is trivial to see that $\operatorname{Col}(\lambda,<\kappa)$ is $<\lambda$ closed; the interesting part is showing that it satisfies the $\kappa$ chain condition.

Suppose $A \subset \operatorname{Col}(\lambda,<\kappa)$ is an antichain of cardinality $\kappa$. Let $\mathcal{A}=\{\operatorname{dom}(p): p \in A\}$. Note that $\mathcal{A}$ is a family of subsets of $\lambda \times \kappa$, each of which have cardinality $<\lambda$. We apply the theorem to $\mathcal{A}$, to get $\mathcal{B} \subset \mathcal{A}$ of cardinality $\kappa$ which is a $\Delta$ system. Let $r \subset \lambda \times \kappa$ be the root of $\mathcal{B}$; note that since $r \subset \operatorname{dom}(p)$ for $p \in A$, we know $|r|<\lambda$. Let $\beta=\sup \{\gamma<\kappa: \exists \alpha<\lambda,(\alpha, \gamma) \in r\}$. Then $\beta<\kappa$.

Let $B=\{p \in A: \operatorname{dom} p \in \mathcal{B}\}$. Note that $|B|=\kappa$. If $p, q \in B$ and $p \neq q$ then $p \perp q$ (since $B \subset A$ ). So $(p \uparrow r) \perp(q \upharpoonleft r)$, and in particular, $(p \uparrow r) \neq(q \upharpoonleft r)$. Hence, $C:=\{p \uparrow r: p \in B\}$ is an antichain of cardinality $\kappa$. But $r$ has cardinality less than $\lambda$, and any condition in $C$ is a partial function from $r$ to $\beta^{+}$. Since $\kappa$ is a limit cardinal, $\beta^{+}<\kappa$, and so an easy pigeonhole argument shows that there are only $\left(\beta^{+}\right)^{<\lambda}$ many conditions that could possibly be in $C$. But by assumption, $\left(\beta^{+}\right)^{<\lambda}<\kappa$.

It is easy to see that the requirements are all needed. If $\kappa$ is singular, then it will still be singular in $V[G]$, and so it cannot be the successor of $\lambda$ and hence can't be a cardinal in $V[G]$. And if there is some $\alpha$ such that $\left|\alpha^{<\lambda}\right| \geqslant \kappa$ then $\operatorname{Col}(\lambda,|\alpha|)$ will already collapse $\kappa$, as we saw a moment ago.

### 1.3.6 Tree Forcing

This isn't a single kind of forcing, but an observation. A tree, in set theory, is a partial order ( $T, \leqslant$ ) with a maximal element $r \in T$ called the root, such that for any $s \in T$, the set of all $s^{\prime}>s$ is totally well-ordered by $\leqslant$. If the order type of this set is $\alpha$, then we say $s$ is on level $\alpha$ of $T$. If $s$ is on level $\alpha$, and $s^{\prime} \leqslant s$ is on level $\alpha+1$ then we say $s^{\prime}$ is a direct successor of $\alpha$. The height of $T$ is the $\sup \{\alpha: \exists s \in T \operatorname{level}(s)=\alpha\}$. A branch of $T$ is a maximal subset of $T$ which is totally ordered by $\leqslant$.

Being a partial order, any tree can be used as a forcing. A generic filter on $T$ will be a branch through $T$. If densely many elements of $T$ have multiple direct successors, then $T$ is atomless and the branch given by a generic filter is not in $T$. If $\beta \leqslant \operatorname{ht}(T)$ is such that for all $s \in T$ and for all $\alpha<\beta$ there exists $s^{\prime} \leqslant s$ such that $s^{\prime}$ is on level $\alpha$, then the branch added will have order type at least $\beta$.

### 1.3.7 Club Shooting

The last forcing in this section adds a new club, which destroys the stationarity of a stationary set. We say that it "shoots" a club through the complement of $S$.

Definition 1.3.19. Let $S \subset \kappa$ be a stationary set. The club shooting forcing on $S$ has as its conditions all closed bounded subsets of $\kappa \backslash S$, ordered by end inclusions.

A generic filter of this forcing generates a club which does not meet $S$. Thus $S$ is non-stationary in the generic extension. A special case of the Club Shooting forcing is the non-Mahlo forcing, where we take $S$ to be the set of singular cardinals below $\kappa$.

### 1.4 Class and Hyperclass Forcing

We now turn to a less well-trodden field. Usually, when we force, we are using set forcing: the forcing is a set in $V$, a generic filter meets every dense set, and the generic ends up being a set in the extension. However, occasionally it becomes necessary to work with forcings that are too large to be sets. This is usually because we want to change something about the entire structure of the universe $V$, some property which cannot be encapsulated in a single set. When this happens, we must use class forcing. In this section, we give a brief overview of this process insofar as we will be using it in the thesis. Proofs are mostly omitted. Readers interested in more detail are referred to [4].

As a preliminary attempt, it is possible to naively formulate a kind of class forcing in a model of ZFC, by working with only definable classes.

Definition 1.4.1. [ZFC] A definable class forcing is a definable class $\mathbb{P}$ of $V$ which is a partial order and has a maximal element. (That is, $\mathbb{P}$ is a collection of pairs which define a partial order on some definable class when we interpret $p \leqslant q \Longleftrightarrow(p, q) \in \mathbb{P}$. We will, as usual, also use $\mathbb{P}$ to refer to the "domain" of $\mathbb{P}$.)

A generic filter $G$ on $\mathbb{P}$ is a (perhaps class-sized) collection of elements of $\mathbb{P}$ such that:

- $G$ is upwards closed and nonempty;
- $p, q \in G \rightarrow \exists r \in g(r \leqslant p \wedge r \leqslant q)$;
- For any definable subclass $D$ of $\mathbb{P}$ which is dense, $G \cap D \neq \varnothing$.

In this formulation, we only define set-sized names. We'll be a little more precise than we were when defining names for set forcing.
Definition 1.4.2. [ZFC] Let $\mathbb{P}$ be a class forcing, and let $\alpha \in \mathrm{On}$. A set $\sigma \in V$ is an $\alpha$-level $\mathbb{P}$ set-name if it consists of pairs $(\tau, p)$, with $\tau$ some $<\alpha$-level $\mathbb{P}$ set-name and $p \in \mathbb{P}$.

If $G$ is a generic filter, then the $G$ interpretation $\sigma^{G}$ of $\sigma$ is defined in the usual way:

$$
\sigma^{G}=\left\{\tau^{G}: \exists p \in G(\tau, p) \in \sigma\right\}
$$

Depending on who you listen to, the generic extension $V[G]$ is then defined to be either the collection $W$ of all these interpretations, or the structure $(W, \in, G)$. Unfortunately, these are not equivalent: because $G$ is class-sized, it is not an element of $W$ as it would be when working with set forcing.

This highlights a problem with doing class forcing in ZFC: A class sized forcing should really add a new class, not just new sets, but of course ZFC doesn't allow for classes as objects in their own right, so we can't do that. Because we can only deal with definable classes, we keep finding that structures which somehow "should" exist aren't available to us automatically, and we have to appeal to a rather developed metatheory to prove results which were automatic with set forcing.

It therefore makes more sense to define class forcing in a structure which treats classes as objects in their own right. The standard theory for this is MK, Morse-Kelley set theory. MK is a theory in a language with two types of objects: sets and classes. (To formulate this in first-order logic, we should strictly say that MK has an additional unary predicate symbol $S$ and we say that $x$ is a set if $S(x)$ holds. We then call the elements of a model of MK classes; a class which is not a set is called a proper class. We don't usually bother writing $S$ explicitly, however. Instead, we will write a uppercase letter for any variable which is allowed to range over all classes, and a lowercase letter for any variable which is only allowed to range over sets.)
Definition 1.4.3. The axioms of MK are as follows:

1. Only sets can be contained in other objects: $\forall X \forall Y(Y \in X \rightarrow S(Y))$.
2. Extensionality of classes: $\forall X \forall Y(\forall z z \in X \Longleftrightarrow z \in Y) \Longleftrightarrow X=Y$.
3. Empty set: There is a set which is empty.
4. Pairs: For all sets $x$ and $y$, the collection $\{x, y\}$ is a set.
5. Unions: For every set $x$, the collection $\cup x$ is a set.
6. Class-Comprehension: If $\varphi\left(v_{0}, V_{1}, \ldots, V_{n}\right)$ is a formula of MK (which may quantify over classes as well as sets) and $X_{1}, \ldots, X_{n}$ are classes, then

$$
\left\{x: \varphi\left(x, X_{1}, \ldots, X_{n}\right)\right\}
$$

is a class.
7. Infinity: There is an infinite set.
8. Power set: For every set $x$, the collection $\mathcal{P}(x)$ of subsets of $x$ is a set.
9. Foundation: Every nonempty class has an $\in$ minimal element.
10. Replacement: If $F$ is a class function, then for any set $x$, the collection $\{F(y): y \in x\}$ is a set.
11. Global Choice: There is a class which is a global well ordering of all sets.

It is easy to verify that the collection $V$ of all sets in a model of MK is a proper class, and that it satisfies the axioms of ZFC. It is also easy to verify that every definable class of $V$ is also a class in the sense of MK.

We can now define class forcing in the context of MK. The definition is almost the same as in ZFC, only we drop the word "definable":

Definition 1.4.4. [MK] A class forcing is a class $\mathbb{P}$ of $V$ which is a partial order and has a maximal element.
A generic filter $G$ on $\mathbb{P}$ is a (perhaps class-sized) collection of elements of $\mathbb{P}$ such that:

1. $G$ is upwards closed and nonempty;
2. $p, q \in G \rightarrow \exists r \in g(r \leqslant p \wedge r \leqslant q)$;
3. For any dense subclass $D$ of $\mathbb{P}, G \cap D \neq \varnothing$.

In MK, we can define not only names for sets, but also class-size names for classes.
Definition 1.4.5. [MK] Let $\mathbb{P}$ be a class forcing, and let $\alpha \in \mathrm{On}$. A set $\sigma$ is an $\alpha$-level $\mathbb{P}$ set-name if it consists of pairs $(\tau, p)$, with $\tau$ some $<\alpha$-level $\mathbb{P}$ set-name and $p \in \mathbb{P}$.

A class $\sigma$ is a $\mathbb{P}$ class-name if it consists of pairs $(\tau, p)$ with $\tau$ some $\mathbb{P}$ set-name (of any rank) and $p \in \mathbb{P}$.
If $G$ is a generic filter, then the $G$ interpretation $\sigma^{G}$ of a (set- or class-) name $\sigma$ is defined in the usual way:

$$
\sigma^{G}=\left\{\tau^{G}: \exists p \in G(\tau, p) \in \sigma\right\}
$$

If $M$ is the model of MK we are working in, then the generic extension $M[G]$ of $M$ is the collection of all $G$ interpretations of both set- and class-names. Its sets are the interpretations of the set-names.

This time, $G$ is contained in the domain $M[G]$ as a class, so there is no ambiguity in what structure we mean by $M[G]$.

We can, of course, define check names and a canonical (class) name for $G$ in the usual way. Working in some larger universe, we can define a forcing relation as usual, from an external perspective.

Definition 1.4.6. Let $M \in V$ be a countable transitive model of $M K$, and let $\mathbb{P}$ be a class forcing over $M$. Let $\varphi\left(X_{0}, \ldots, X_{n}\right)$ be a formula in the language of MK , and let $\sigma_{0}, \ldots, \sigma_{n}$ be (set or class) $\mathbb{P}$ names. Let $p \in \mathbb{P}$. We say that $p \Vdash \varphi\left(\sigma_{0}, \ldots, \sigma_{n}\right)$ if for all generic filters $p \in G \in V, M[G] \vDash \varphi\left(\sigma_{0}^{G}, \ldots, \sigma_{n}^{G}\right)$.

Even in this context, we can't quite get all the usual results of forcing automatically. The limitations of working with classes, even in a MK context, mean that we must make a few extra assumptions about the forcing $\mathbb{P}$.

Definition 1.4.7. [MK] A class forcing $\mathbb{P}$ is pretame if for every set-long sequence $\left\langle D_{i}: i \in a\right\rangle$ of dense subclasses of $\mathbb{P}$, and for every $p \in \mathbb{P}$, there is some $q \leqslant p$ and some set size $d_{i} \subset D_{i}$ for each $i \in a$ such that every $d_{i}$ is predense below $q$.
$\mathbb{P}$ is tame if it is pretame and there exists $\alpha \in$ On and densely many $q \in \mathbb{P}$ such that the following holds:
If $r \leqslant q$ and $\vec{D}:=\left\langle\left(D_{i}^{0}, D_{i}^{1}\right): i \in a\right\rangle$ is a sequence of set-many subclasses of $\mathbb{P}$ which are all predense below $r$, and if and for all $i \in a, D_{i}^{0}$ and $D_{i}^{1}$ are pairwise incompatible, and if $C$ is the class of all conditions $s \leqslant r$ such that $\vec{D}$ is equivalent below $s$ to some $\vec{d} \in V_{\alpha}$, then $C$ is dense below $r$.

By "equivalent below $s$ " we mean $\vec{d}$ also partitions $\mathbb{P}$ below $s$ in the same way as $\vec{D}$, and that there are densely many conditions $t \leqslant s$ such that $D_{0}^{i}$ is predense below $t$ if and only if $d_{0}^{i}$ is.

The definition we have given for tameness is a slightly strengthened version found in [15], which fixes a minor issue with the original definition.

Pretameness and tameness are precisely what we need to get the usual results of class forcing.

Theorem 1.4.8. [3] [MK] Let $\mathbb{P}$ be a class forcing. If $\mathbb{P}$ is pretame, then the relation $\Vdash$ is definable in a canonical way within the ground model, and for any generic filter $G$, the extension $M[G] \vDash \mathrm{MK}^{-}$(where $\mathrm{MK}^{-}$denotes the axioms of MK without powerset). If $\mathbb{P}$ is tame, then $M[G] \vDash \mathrm{MK}$.

So much for class forcing. We can also go a level higher up, and will need to do so for the main result of Chapter 4.

In a model of MK, we can consider definable collections of classes, much like we can consider definable classes of ZFC. Obviously, we can't call these collections over MK classes, so instead we use the term "hyperclass" to refer to them.

This gives us the opportunity to define hyperclass forcing, which is defined much like class forcing was defined over ZFC.

Definition 1.4.9. [MK] A definable hyperclass forcing is a definable hyperclass $\mathbb{P}$ which is a partial order and has a maximal element.

A generic filter $G$ on $\mathbb{P}$ is a (perhaps hyperclass-sized) collection of elements of $\mathbb{P}$ such that:

1. $G$ is upwards closed and nonempty;
2. $p, q \in G \rightarrow \exists r \in g(r \leqslant p \wedge r \leqslant q)$;
3. For any definable subclass $D$ of $\mathbb{P}$ which is dense, $G \cap D \neq \varnothing$.

We could go through and define the interpretation of a name like we did previously with class forcings. However, in [4], Antos and Friedman noticed a shortcut. If we add another axiom to MK, then we can turn a model of MK into a special kind of model of ZFC ${ }^{-}$(that is ZFC without the powerset axiom), and vice versa. In the $\mathrm{ZFC}^{-}$model, $\mathbb{P}$ becomes a class forcing. A generic filter on $\mathbb{P}$ will be generic over the $\mathrm{ZFC}^{-}$model if and only if it is generic over the MK model. And in sufficiently nice situations, we can borrow the results about class forcings to show that the axioms of MK are preserved, and the forcing relation is definable. The axioms we need to add to MK are those in the following definition.

Definition 1.4.10. The axiom scheme MK** consists of the axioms of MK together with the following two axioms:
12. Class bounding: $(\forall x \exists A \varphi(x, A)) \rightarrow\left(\exists B \forall x \exists y \varphi\left(x,(B)_{y}\right)\right)$, where for any set $y,(B)_{y}$ denotes the class of all $z$ such that $(z, y) \in B$
13. Dependent choice for classes: if $\varphi$ is such that any set-long sequence of classes $\vec{X}$ can be extended by one set $Y$ such that $\varphi(\vec{X}, Y)$ holds, then for any choice of $X_{0}$, there is an On-long sequence of classes $\vec{X}$ starting with $X_{0}$ such that $\forall \alpha \in \operatorname{On} \varphi\left(\vec{X} \upharpoonleft \alpha, X_{\alpha}\right)$

The double asterisk is an artefact of the way that Antos and Friedman introduce this subject. They start by just adding the class bounding axiom to define the theory $\mathrm{MK}^{*}$, which turns out to be sufficient for everything except one technical issue near the end, then go back and add the dependent choice axiom to resolve this difficulty.

Definition 1.4.11. The axiom scheme Set MK ${ }^{* *}$ consists of ZFC $^{-}$together with the following axioms:

1. Set bounding: $(\forall x \in a \exists y \varphi(x, y)) \rightarrow \exists b \forall x \in a \exists y \in b \varphi(x, y)$
2. There exists a strongly inaccessible cardinal $\kappa$, and every set can be mapped injectively into $\kappa$. (This implies that $\kappa$ is the largest cardinal.)
3. $\kappa$ dependent choice: If $\varphi$ is such that any $<\kappa$ long sequence of sets $\vec{x}$ can be extended by some $y$ such that $\varphi(\vec{x}, y)$ holds, then for any $x_{0}$, there is a $\kappa$ long sequence $\vec{x}$ starting with $x_{0}$ such that for all $\alpha<\kappa$, $\varphi\left(\vec{x} \mid \alpha, x_{\alpha}\right)$.

The "right" intuition to apply to understand this axiom scheme is to think of a model of it as a structure $V_{\kappa+\alpha}$, where $\kappa$ is strongly inaccessible and $\alpha$ is some small ordinal.

Theorem 1.4.12. [4, 20] Let $M$ be a model of $\mathrm{MK}^{* *}$. Then there is a unique transitive structure $M^{+}$ which is a model of Set MK** with largest cardinal $\kappa$, such that:

1. The sets of $M$ are precisely the elements of $V_{\kappa}^{M^{+}}$; and
2. The classes of $M$ are precisely the subsets of $V_{\kappa}^{M^{+}}$which are elements of $M^{+}$.

Conversely, if $M^{+}$is a model of Set $\mathrm{MK}^{* *}$ with largest cardinal $\kappa$, then the structure $M$ of $\mathrm{MK}^{* *}$ whose set-universe is $V_{\kappa}^{M^{+}}$, and whose classes are the subsets of $V_{\kappa}^{M^{+}}$in $M^{+}$, is a model of $\mathrm{MK}^{* *}$.

There's an unstated meta-level assumption here. The process of constructing $M^{+}$from $M$ involves some moderately complex manipulations of sets and classes of $M$, to calculate the new sets which will be added to $M^{+}$. These manipulations can't be carried out inside $M$. So in order for this theorem to work, we must assume either that $M$ is a countable transitive model of $\mathrm{MK}^{* *}$ inside some larger model $V$ of ZFC, or we have to use some kind of trick to pretend that sets exist outside MK ${ }^{* *}$. (Note the similarities between this and our remarks when we first introduced forcing: in both situations, what we're saying only strictly makes sense if we're working with countable transitive models, but we can fiddle things around to pretend it works in the "real" universe as well.)

Theorem 1.4.13. Let $\mathbb{P}$ be a hyperclass forcing in a model $M$ of $\mathrm{MK}^{* *}$. Then in the model $M^{+}$defined above, $\mathbb{P}$ is a definable class forcing. Conversely, if $\mathbb{Q}$ is a definable class forcing in a model $M^{+}$of $S e t M K^{* *}$, then $\mathbb{Q}$ is a hyperclass forcing over the corresponding model $M$ of $\mathrm{MK}^{* *}$.

A filter $G$ on such a forcing is generic in the sense of $M$ if and only if it generic in the sense of $M^{+}$.
Of course Set MK ${ }^{* *}$ does not believe the whole of ZFC, let alone MK. But it believes enough of it for pretame generic extensions to work properly.

Theorem 1.4.14. [4, 18] Let $M^{+}$be a model of Set $\mathrm{MK}^{* *}$. Let $\mathbb{P}$ be a pretame definable class forcing on $M^{+}$which has a definable forcing relation, and which preserves the properties of $\kappa$ given by Set $\mathrm{MK}^{* *}$. If $G$ is $\mathbb{P}$ generic over $M^{+}$, then the version of the class generic extension without the predicate $G$ believes Set $\mathrm{MK}^{* *}$; and the class generic extension with the predicate $G$ believes Set $\mathrm{MK}^{* *}$ relativised to $G$.

This gives us a way to define the generic extension of a pretame hyperclass forcing, provided that the forcing preserves $\kappa$ and has a definable forcing relation. First, though, we will formally state what pretameness means for a hyperclass forcing. The definition is exactly the same as in a class forcing, just with "definable hyperclass" in place of "class" and "class" in place of "set". To make the distinction between classes and hyperclasses clearer in the following definition, we will use $\mathcal{A}, \mathcal{B}$ etc. to denote (definable) hyperclasses, standard uppercase letters to denote classes, and lowercase letters for sets.

Definition 1.4.15. [MK ${ }^{* *}$ ] A hyperclass forcing $\mathbb{P}$ is pretame if for every class-long sequence $\left\langle\mathcal{D}_{i}: i \in A\right\rangle$ of dense definable subhyperclasses of $\mathbb{P}$, and for every $P \in \mathbb{P}$, there is some $Q \leqslant P$ and some class size $D_{i}^{\prime} \subset \mathcal{D}_{i}$ for each $i \in A$ such that every $D_{i}^{\prime}$ is predense below $Q$. (By this, we mean that $D_{i}^{\prime}$ is a class of pairs of sets, and there is some class $C_{i}$ such that if $x \in C$ then $\left(D_{i}^{\prime}\right)_{x}$ is a condition in $\mathcal{D}_{i}$, and otherwise $\left(D_{i}^{\prime}\right)_{x}=\varnothing$.)

We could also define an analogue of tameness, but we don't really need to. It's only used in proving the preservation of powerset, and that doesn't actually hold in this context anyway.

Proposition 1.4.16. Let $M$ be a model of $M^{* *}$ and let $\mathbb{P}$ be a definable hyperclass forcing over it. Then $\mathbb{P}$ is pretame in $M$ if and only if it is pretame in $M^{+}$, viewed as a class forcing.
Definition 1.4.17. Let $M$ be a model of $\mathrm{MK}^{* *}$, let $\mathbb{P}$ be a pretame hyperclass forcing over it, and let $G$ be a generic filter on $\mathbb{P}$ over $M$. Then in the corresponding model $M^{+}$of Set $\mathrm{MK}^{* *}, \mathbb{P}$ is a definable pretame class forcing, and $G$ is still generic. Suppose $\mathbb{P}$ preserves the properties of $\kappa$ and has a definable forcing relation. Then $M^{+}[G] \vDash$ Set MK ${ }^{* *}$. We define $M[G]$ to be the unique model of $\mathrm{MK}^{* *}$ corresponding to $M^{+}[G]$.

For our purposes in this thesis, we will exclusively take the generic extension $M[G]$ to not include the predicate $G$.

Preserving $\kappa$ is relatively easy to check. However, the forcing relation is not automatically definable, even if the forcing is pretame. We need the ground model to have a tiny bit more structure.

Theorem 1.4.18. [4, p.19-20] Let $M^{+}$be a (set-size) model of Set $\mathrm{MK}^{* *}$ of the form $L_{\kappa^{*}}[X]$ where $\kappa^{*}=\mathrm{On} \cap M^{+}$and $X \subset \kappa$ is some predicate. Let $\mathbb{P}$ be a pretame definable class forcing over $M^{+}$. Then the forcing relation on $\mathbb{P}$ is definable, and forcing with $\mathbb{P}$ preserves ZFC $^{-}$.

This is in fact sufficient for our purposes - the only structures we want to hyperclass force over in this thesis are of the form described above. But to complete this account of hyperclass forcing, we finish by briefly sketching how we get around this issue in general models of $\mathrm{MK}^{* *}$.

First, we relativise all the results we have given in this section: instead of proving them in the context of MK ${ }^{* *}$ and Set MK ${ }^{* *}$, we prove them in MK ${ }^{* *}$ and Set MK ${ }^{* *}$ relativised to some arbitrary predicate $X$. The proofs all go through nicely and the results still hold. Then we prove the following theorem:

Theorem 1.4.19. [4, 22] Let $M^{+}$be a transitive model of Set $\mathrm{MK}^{* *}$, with $\kappa^{*}=\mathrm{On} \cap M^{+}[G]$. Then there is a pretame definable class forcing $\mathbb{Q}$ which has a definable forcing relation, such that if $G$ is $\mathbb{Q}$ generic then $M^{+}[G]$ and $M^{+}$have the same elements (so the generic extension is trivial if we don't add $G$ as a predicate), and such that $M^{+}=L_{\kappa^{*}}[G]$.

We have already seen that this $M^{+}[G]$ will be a model of Set MK** relativised to $G$.
So if we want to force over a model $M$ of $\mathrm{MK}^{* *}$ with a pretame hyperclass forcing $\mathbb{P}$ in general, we use the following process:

1. Convert $M$ to a model $M^{+}$of Set $\mathrm{MK}^{* *}$.
2. Do a preliminary forcing $\mathbb{Q}$ of $M^{+}$using the previous theorem, to get a model $M^{+}[H]$ of Set $\mathrm{MK}^{* *}$ of the form $L_{\kappa^{*}}[H]$, such that $M^{+}[H]$ and $M^{+}$have the same elements.
3. Convert $M^{+}[H]$ back to a model $M[H]$ of $\mathrm{MK}^{* *}$ relativised to $H$.
4. In $M[H]$, define the pretame hyperclass forcing $\mathbb{P}$ we want to use, and select a generic filter $G$.
5. Back in $M^{+}[H]$, view $\mathbb{P}$ as a class forcing (which has a definable forcing relation) and take the generic extension $M^{+}[H][G]$ which is a model of Set MK ${ }^{* *}$ relativised to $H$ (and $G$, if we choose to include $G$ as a predicate in the extension).
6. Convert $M^{+}[H][G]$ back into a model $M[H][G]$ of $\mathrm{MK}^{* *}$ relativised to $H$ (and $G$ ).
7. Optionally, forget about $H$ and $G$ in order to drop down to a model of $\mathrm{MK}^{* *}$ without any extra predicates.

This gives us a well defined generic extension of $M$ with respect to the two forcings used in the above process. Note that the overall forcing relation will be definable within $M^{+}$and $M^{+}[H]$, and therefore within $M$ and $M[H]$ as well. If $M$ is already of the form $L_{\kappa^{*}}[X]$ (which it will be in this thesis) then we can take $\mathbb{Q}$ to just be a trivial forcing, and effectively skip the first three steps of this process.

## Chapter 2

## Forcing Axioms and Name Principles

Now that we have a basic background in forcing, we are ready for the first of three chapters of new results in this thesis. The content of this chapter is taken from [37], a paper cowritten by myself and Philipp Schlicht, where we look at forcing axioms. Forcing axioms say the following, for some forcing $\mathbb{P}$ and cardinal $\kappa$ :

Let $\left(D_{\gamma}: \gamma<\kappa\right)$ be a collection of $\kappa$ many dense subsets of $\mathbb{P}$. Then there is a filter $g \subset \mathbb{P}$ in the ground model $V$ such that for all $\gamma<\kappa, g \cap D_{\gamma} \neq \varnothing$.

We denote this forcing axiom as $\mathrm{FA}_{\mathbb{P}, \kappa}$. We can think of $g$ as an approximation of a generic filter, which exists within $V$. Forcing axioms have been in use for many years, and are an active field of study. The first one developed was Martin's Axiom $\mathrm{MA}_{\omega_{1}}$, which says that $\mathrm{FA}_{\mathbb{P}, \omega_{1}}$ holds for every c.c.c. forcing $\mathbb{P}$. Another forcing axiom is the Proper Forcing Axiom PFA, which says the same but with "proper" replaced by "c.c.c."

When a forcing axiom is invoked in a proof, its use generally follows a specific pattern. We define some very simple name $\sigma$, which we can easily see is forced to be equal to some set $A$ in the ground model, or to have some other useful property. We then list all the dense sets which a filter needs to meet for that particular property to be true, and invoke an appropriate forcing axiom to find a filter $g \in V$ which meets all those dense sets. Then $\sigma^{g}$ will be equal to $A$, or will satisfy that other property, and we can use that fact in the rest of the proof.

In effect, the forcing axiom is being used to prove a special case of a more general claim:
Let $\sigma$ be any sufficiently nice $\mathbb{P}$ name, which is forced to have a certain property $P$. Then there is a filter $g \in V$ such that $P$ is true about $\sigma^{g}$.

We will be studying claims of this sort, which we shall call "name principles". Simple name principles have been used ad-hoc in many proofs in the way described above, and are intuitively natural to talk about in the context of forcing axioms. But here we study them as axioms in their own right, which so far as we are aware has never been done systematically before. It turns out that they are intricately connected to forcing axioms: we can prove highly complex name principles from simple forcing axioms. Moreover, the implication goes the other way, too: every forcing axiom is implied by (indeed, equivalent to) a whole class of name principles. On the other hand, there are also many name principles which are not equivalent to any forcing axiom but sit between two different forcing axioms in strength. So name principles can be thought of as a generalisation of forcing axioms.

The following theorem is (a simplification of) the main equivalence result of this section. We will, of course, define the terms used below in a more precise way after this introduction!

Theorem 2.0.1. (see Theorem 2.3.1 ${ }^{1}$ ) Suppose that $\mathbb{P}$ is a forcing and $\kappa$ is a cardinal. Then the following statements are equivalent:
(1) $\mathrm{FA}_{\mathbb{P}, \kappa}$
(2) The name principle $\mathrm{N}_{\mathbb{P}, \kappa}$ for nice names $\sigma$ and the property $P$ being $\sigma=\check{\kappa}$.

[^3](3) The simultaneous name principle $\Sigma_{0}^{(\operatorname{sim)}}-\mathbb{N}_{\mathbb{P}, \kappa}$ for nice names $\sigma$ and $P$ being any first-order formula over the structure $(\kappa, \epsilon, \sigma)$.

In addition to simplifying many proofs, this equivalence has several applications that let us prove new results or improve known ones. We shall meet a few of them later in the chapter. In particular, we shall examine an connection found by Bagaria between so-called "bounded" forcing axioms and generic absoluteness principles $[5,6]$. No only does Theorem 2.3.2 allow Bagaria's characterisation to be proved in a much easier way, but we can see that it is only a special case of a more general result. Here, BFA and BN denote bounded analogues of forcing axioms and name principles respectively; again, we shall meet them properly in the next section.

Theorem 2.0.2. (see Theorems 2.3.17 and 2.3.22) Suppose that $\kappa$ is an uncountable cardinal, $\mathbb{P}$ is a complete Boolean algebra and $\dot{G}$ is a $\mathbb{P}$-name for the generic filter. The following conditions are equivalent:
(1) $\mathrm{BFA}_{\mathbb{P}, \kappa}$
(2) $\Sigma_{0}^{(\text {sim })}-\mathrm{BN}_{\mathbb{P}, \kappa}^{1}$
(3) $\Vdash_{\mathbb{P}} V<_{\Sigma_{1}^{1}(\kappa)} V[\dot{G}]$

If $\operatorname{Cof}(\kappa)>\omega$ or there is no inner model with a Woodin cardinal, then the next condition is equivalent to (1), (2) and (3):
(4) $\Vdash_{\mathbb{P}} H_{\kappa^{+}}^{V}<_{\Sigma_{1}} H_{\kappa^{+}}^{V[\dot{G}]}$

If $\operatorname{Cof}(\kappa)=\omega$ and $2^{<\kappa}=\kappa$, then the next condition is equivalent to (1), (2) and (3):
(5) $1_{\mathbb{P}}$ forces that no new bounded subset of $\kappa$ are added.

There are also name principles which are too weak to be equivalent to conventional forcing axioms. But these sometimes still turn out to be equivalent to weaker forcing axioms, where we only ask to meet a "large" subset of the dense sets. For example, with the right interpretation of "sufficiently nice", the name principle for $P(\sigma)=" \sigma$ contains a club in $\omega_{1}$ " is equivalent to the club forcing axiom:

Let $\left(D_{\gamma}: \gamma<\omega_{1}\right)$ be a collection of dense subsets of $\mathbb{P}$. Then there is a filter $g \subset \mathbb{P}$ in the ground model $V$ such that the class of all $\gamma<\omega_{1}$ such that $g \cap D_{\gamma} \neq \varnothing$ contains a club.

So far as I am aware, this club forcing axiom, and the corresponding stationary and unbounded forcing axioms, are new to the literature. In the second half of this chapter, we shall conduct a detailed survey of them, their corresponding name principles, and the relations between them. Our results are illustrated in the following diagram. We will formally define all the principles shown here over the course of the next section. Solid arrows denote non-reversible implications, dotted arrows stand for implications whose converse remains open, and dashed lines indicate that no implication is provable. The numbers indicate where in this chapter to find the proofs.


Figure 2.1: Forcing axioms and name principles for regular $\kappa$
We also investigate whether similar implications hold for $\lambda$-bounded name principles and forcing axioms, where $\lambda$ is any cardinal. The results about the cases $\kappa \leqslant \lambda, \omega \leqslant \lambda<\kappa$ and $1 \leqslant \lambda<\kappa$ are displayed in the next diagrams. CBA stands for "complete Boolean algebra".


Figure 2.2: $\lambda$-bounded forcing axioms and name principles for regular $\kappa$ and $\lambda \geqslant \kappa$
It is open whether club- $\mathrm{BN}_{\mathbb{P}, \kappa}^{\lambda}$ implies stat- $\mathrm{BN}_{\mathbb{P}, \kappa}^{\lambda}$. The converse is known to be false: there are forcings $\mathbb{P}$ where stat- $\mathrm{BN}_{\mathbb{P}, \kappa}^{\lambda}$ holds for all $\lambda$, but club- $\mathrm{BN}_{\mathbb{P}, \kappa}^{\lambda}$ fails for all $\lambda \geqslant \omega$ (see Section 2.4.1, Lemma 2.4.6 and Remark 2.4.14).


Figure 2.3: $\lambda$-bounded forcing axioms and name principles for regular $\kappa$ and $\omega \leqslant \lambda<\kappa$
Again, it is open whether club- $\mathrm{BN}_{\mathbb{P}, \kappa}^{\lambda}$ implies stat- $\mathrm{BN}_{\mathbb{P}, \kappa}^{\lambda}$, but the converse implication does not hold.


Figure 2.4: $n$-bounded forcing axioms and name principles for regular $\kappa$ and $1 \leqslant n<\omega$
The principles in the bottom row and $\mathrm{BN}_{\kappa}^{n}$ are all provable in ZFC.
Our survey will also involve looking at what happens with most of the specific forcings we introduced in Chapter 1, which is how we prove most of the non-implications of the diagrams. Some highlights include:

Proposition 2.0.3. (see Lemma 2.4.15) Let $\mathbb{P}$ denote random forcing. The following are equivalent:
(1) $F A_{\mathbb{P}, \omega_{1}}$
(2) ub-FA $\mathbb{P}_{\mathbb{P}, \omega_{1}}$
(3) $2^{\omega}$ is not the union of $\omega_{1}$ many null sets

Proposition 2.0.4. (see Corollary 2.4.21) Suppose that a Suslin tree exists. Then there exists a Suslin tree $T$ such that stat- $\mathrm{BN}_{T, \omega_{1}}^{1}$ fails.

For some forcings, most of Figure 2.1 collapses. In particular, if ub-FA $\mathbb{P}_{\mathbb{P}, \kappa}$ implies $\mathrm{FA}_{\mathbb{P}, \kappa}$, then all entries other than stat- $\mathrm{N}_{\mathbb{P}, \kappa}$ are equivalent. We investigate when this implication holds. For instance:

Proposition 2.0.5. (see Lemma 2.4.1) For any $<\kappa$-distributive forcing $\mathbb{P}$, we have ub-FA $\mathbb{P}_{\mathbb{P}} \ldots \Longrightarrow \mathrm{FA}_{\mathbb{P}, \kappa}$.
In a broader range of cases, ub- $\mathrm{FA}_{\mathbb{P}, \kappa}$ implies most of the entries in Figure 2.2:

Proposition 2.0.6. (see Lemma 2.3.24) If $\kappa$ an uncountable cardinal and $\mathbb{P}$ is a complete Boolean algebra that does not add bounded subsets of $\kappa$, then

$$
\left(\forall q \in \mathbb{P} \text { ub-FA } \mathbb{P}_{q}, \kappa\right) \Longrightarrow \mathrm{BFA}_{\mathbb{P}, \kappa}^{\kappa} .
$$

The previous result is a corollary to the proof of Theorem 2.0.2.
The structure of the chapter is as follows. We collect some definitions in Section 2.1. In Section 2.2, we prove the positive implications in Figure 2.1. In Section 2.3, we prove a general correspondence between forcing axioms and name principles. (Theorem 2.0.1 is a special case of this correspondence.) We further derive results about generic absoluteness and other consequences of the correspondence. In Section 2.4, we study the principles in Figures 2.1-2.4 for specific classes of forcings such as $\sigma$-distributive and c.c.c. and for specific forcings such as Cohen and random forcing. We use these results to separate some of the principles in the figures.

### 2.1 Definitions

In this section, we introduce the forcing axioms and name principles formally. We will also define a few pieces of notation that we will want to use repeatedly in the coming sections.

Definition 2.1.1. Let $X$ be a set and $\alpha$ an ordinal. We recursively define $\mathcal{P}^{\alpha}(X)$ and $\mathcal{P}^{<\alpha}(X)$ :
$\mathcal{P}^{0}(X)=X$
$\mathcal{P}^{<\alpha}(X)=\bigcup_{\beta<\alpha} \mathcal{P}^{\beta}(X)$
$\mathcal{P}^{\alpha}(X)=\mathcal{P}\left(\mathcal{P}^{<\alpha}(X)\right)$ for $\alpha>0$.
Throughout this section, assume that $\mathbb{P}$ is a forcing and $\mathcal{C}$ is a class of forcings. $G$ will be a generic filter $($ on $\mathbb{P}) ; g$ will be a filter on $\mathbb{P}$ which is contained in the ground model $V$ (and therefore certainly not generic, if $\mathbb{P}$ is atomless).

### 2.1.1 Forcing axioms

Notation. In the following, $\vec{D}=\left\langle D_{\gamma}: \gamma\langle\kappa\rangle\right.$ always denotes a sequence of dense (or predense) subsets of a forcing $\mathbb{P}$. If $g$ is a subset of $\mathbb{P}$, then its trace with respect to $\vec{D}$ is defined as the set

$$
\operatorname{Tr}_{g, \vec{D}}=\left\{\alpha<\kappa: g \cap D_{\alpha} \neq \varnothing\right\} .
$$

Definition 2.1.2. Let $\kappa$ be a cardinal. The forcing axiom $\mathrm{FA}_{\mathbb{P}, \kappa}$ says:
"For any $\vec{D}$, there exists a filter $g \in V$ with $\operatorname{Tr}_{g, \vec{D}}=\kappa . "$
The forcing axiom $\mathrm{FA}_{\mathcal{C}, \kappa}$ asserts that $\mathrm{FA}_{\mathbb{P}, \kappa}$ holds for all $\mathbb{P} \in \mathcal{C}$.
Of course, we could just as well have written "predense" instead of "dense" in the above definition.
We will suppress the $\mathbb{P}$ or $\mathcal{C}$ in the above notation when it is clear which forcing we are referring to. If $\kappa=\omega_{1}$ we will suppress it too, just writing $\mathrm{FA}_{\mathbb{P}}$ (or just FA if $\mathbb{P}$ is clear as well).

We can weaken this axiom: instead of insisting that $g$ must meet every $D_{\gamma}$, we could insist only that it meets "many" of them in some sense. The following forcing axioms do exactly that, for various senses of "many".

Definition 2.1.3. Suppose that $\kappa$ is a cardinal and $\varphi(x)$ is a formula. The axiom $\varphi$ - $\mathrm{FA}_{\mathbb{P}, \kappa}$ states:
"For any $\vec{D}$, there is a filter $g$ on $\mathbb{P}$ such that $\varphi\left(\operatorname{Tr}_{g, \vec{D}}\right)$ holds."
In particular, we will consider the following formulas:
(1) $\operatorname{club}(x)$ states that $x$ contains a club in $\kappa$. club- $\mathrm{FA}_{\mathbb{P}, \kappa}$ is called the club forcing axiom.
(2) $\operatorname{stat}(x)$ states that $x$ is stationary in $\kappa$. stat- $\mathrm{FA}_{\mathbb{P}, \kappa}$ is called the stationary forcing axiom.
(3) $\mathrm{ub}(x)$ states that $x$ is an unbounded subset of $\kappa$. ub- $\mathrm{FA}_{\mathbb{P}, \kappa}$ is called the unbounded forcing axiom.
(4) $\omega$-ub $(x)$ states that $x$ contains $\omega$ as a subset and is also unbounded in $\kappa$. $\omega$-ub- $\mathrm{FA}_{\mathbb{P}, \kappa}$ is called the $\omega$-unbounded forcing axiom.
We define club-FA $\mathcal{C}_{\mathcal{C}, \kappa}$, stat-FA $\mathcal{C}_{, \kappa}$, ub-FA $\mathcal{C}_{\mathcal{C}, \kappa}$ and $\omega$-ub-FA $\mathcal{C}_{\mathcal{C}, \kappa}$ in the same way as we defined $\mathrm{FA}_{\mathcal{C}, \kappa}$ in Definition 2.1.2.
$\omega$-ub-FA can also be expressed as a combined version of two forcing axioms: that given a $\kappa$ long sequence $\vec{D}$ and a separate $\omega$ long sequence $\vec{E}$ of (pre)dense sets, we can find a filter $g$ such that $\operatorname{Tr}_{g, \vec{D}}$ is unbounded and $\operatorname{Tr}_{g, \vec{E}}=\omega$.

Again, we will suppress $\mathbb{P}$ or $\mathcal{C}$ where they are obvious, and will suppress $\kappa$ when $\kappa=\omega_{1}$.
We can also weaken the axiom by insisting that every dense set $D_{\gamma}$ be bounded in cardinality, by some small cardinal.
Definition 2.1.4. Let $\kappa$ and $\lambda$ be cardinals. The bounded forcing axiom $\mathrm{BFA}_{\mathbb{P}, \kappa}^{\lambda}$ says
"Whenever $\left\langle D_{\gamma}: \gamma\langle\kappa\rangle\right.$ is a sequence of predense subsets of $\mathbb{P}$, and for all $\gamma$ we have $| D_{\gamma} \mid \leqslant \lambda$, then there is a filter $g \in V$ such that for all $\gamma<\kappa, g \cap D_{\gamma} \neq \varnothing$."

We define $\mathrm{BFA}_{\mathcal{C}, \kappa}^{\lambda}$, club- $\mathrm{BFA}_{\mathbb{P}, \kappa}^{\lambda}$ and so forth in the natural way, using definitions analogous to those in 2.1.2 and 2.1.3.

Again, we will suppress notation as described above. We will suppress the $\lambda$ if $\lambda=\kappa$.
Note that we are definitely looking at predense sets here, since actual dense sets are likely to be rather large and the axiom would be likely to be trivial if we had to use dense sets. These bounded forcing axioms are only really of interest when $\mathbb{P}$ is a Boolean algebra, since they always contain (nontrivial) predense sets with as few as two elements so the axiom will not be vacuous.

There is one more forcing axiom we want to introduce, but it requires some additional notation so we will postpone it until later in this section.

### 2.1.2 Name principles

We now need to define name principles, but we need to cover some other terminology first in order to express the definitions. In the motivating work in the introduction to this chapter, we talked about "sufficiently nice" names; it's now time to explain exactly what that means.

Definition 2.1.5. Let $X$ be a set (in $V$ ). We recursively define a name's rank as follows.
$\sigma$ is an $\alpha$ rank $X$ name (or a rank $\alpha$ name for short) if either:

- $\alpha=0$ and $\sigma=\check{x}$ for some $x \in X$; or
- $\sigma$ is not $\operatorname{rank} 0$ and $\alpha=\sup \{\operatorname{rank}(\tau)+1: \exists p \in \mathbb{P}(\tau, p) \in \sigma\}$

We also call a 1 (or 0 ) rank $X$ name a good name. Of course, we will also talk about rank $\leqslant \alpha$ names, meaning names which are either rank $<\alpha$ or rank $\alpha$.

This definition is a name analogue to saying that $\sigma \in \mathcal{P}^{\alpha}(X)$, where $X$ is transitive. Most of the time, we will be interested in the case where $X$ is some cardinal, most often either 0 or $\omega_{1}$. Note that every $\mathbb{P}$ name is an $\alpha$ rank $X$ name for some $\alpha$.

Definition 2.1.6. Let $\sigma$ be a $\mathbb{P}$ name and $\kappa$ be a cardinal. We say $\sigma$ is locally $\kappa$ small if there are at most $\kappa$ many names $\tau$ such that for some $p \in \mathbb{P}$, we have $(\tau, p) \in \sigma$. A name $\sigma$ is $\kappa$ small if it is locally $\kappa$ small, and every name $\tau$ in the above definition is $\kappa$ small.

If being rank $\alpha$ is analogous to being in $\mathcal{P}^{\alpha}\left(\right.$ or $\left.\mathcal{P}^{\alpha}(X)\right)$ then the analogue of being $\kappa$ small would be being in $H_{\kappa^{+}}$. We could also easily define a version of this for $H_{\kappa^{+}}(X)$ if we wanted. However, we don't actually need to: in all the cases we're going to be interested in, $\bar{X}$ will have cardinality $\leqslant \kappa$ and the definition would be equivalent to the above one.

The following proposition says that we only really need to worry about $\kappa$ smallness when we go above rank 1 names.

Proposition 2.1.7. Let $X$ be transitive, and of size at most $\kappa$. Let $\sigma$ be a rank or 1 rank $X$ name. Then $\sigma$ is $\kappa$ small.

On the other hand if $X$ has size greater than $\kappa$ then no interesting rank 1 name will be $\kappa$ small.
The next definition does not have an easy analogue, but is a kind of complement to the previous one and is critical when we work with bounded forcing axioms.
Definition 2.1.8. Let $\sigma$ be a $\mathbb{P}$ name and $\lambda$ be a cardinal. We say $\sigma$ is locally $\lambda$ bounded if it can be written as

$$
\sigma=\left\{(\tau, p): \tau \in T, p \in S_{\tau}\right\}
$$

where $T$ is some set of names, and for $\tau \in T$ the set $S_{\tau}$ is a subset of $\mathbb{P}$ of size at most $\lambda$. A name $\sigma$ is $\lambda$ bounded if it is locally $\lambda$ bounded, and every name $\tau \in T$ in the above definition is $\lambda$ bounded.

A good name which is 1 bounded is known as a very good name. A check name $\check{x}$ has the form $\{(\check{y}, 1)$ : $y \in x\}$ and is therefore guaranteed to be $\lambda$ bounded for any $\lambda>0$.

We will be talking about interpreting names with respect to a filter. Unfortunately, the literature uses two different meanings of the word "interpretation", which only coincide if the filter is generic. For clarity:

Definition 2.1.9. Let $\sigma$ be a name, and $g$ a filter. (Here, $g$ may be inside $V$ or in some larger model.) When we refer to the interpretation $\sigma^{g}$ of $\sigma$, we mean the recursive interpretation:

$$
\sigma^{g}:=\left\{\tau^{g}: \exists p \in g(\tau, p) \in \sigma\right\}
$$

When we refer to the quasi-interpretation $\sigma^{(g)}$, we mean the following set:

$$
\sigma^{(g)}:=\{x \in V: \exists p \in g p \Vdash \check{x} \in \sigma\}
$$

Proposition 2.1.10. $\sigma^{g}=\sigma^{(g)}$ if $\sigma$ is a 1 rank $X$ name (for some $X$ ) and either
(1) $g$ is generic; or
(2) $\sigma$ is 1 bounded.

Proposition 2.1.11. Suppose $\mathbb{P}$ is a complete Boolean algebra, and $\sigma$ is a 1 rank $X$ name. Then we can find a name $\tau$ such that for every filter $g, \tau^{g}=\tau^{(g)}=\sigma^{(g)}$.

Proof. For $x \in X$ let $p_{x}=\sup \{p \in \mathbb{P}:(\check{x}, p) \in \sigma\}$ (so $\left.p_{x} \in \mathbb{P} \cup\{0\}\right)$. Let $\tau=\left\{\left(\check{x}, p_{x}\right): x \in X, p_{x} \neq 0\right\}$.
We can now define our name principles. Here, we take $\mathbb{P}$ to be a forcing, $\mathcal{C}$ a class of forcings, and $X$ an arbitrary set.

Definition 2.1.12. Let $\alpha$ be an ordinal, $\kappa$ a cardinal and $X$ a transitive set of size at most $\kappa$. The name principle $\mathrm{N}_{\mathbb{P}, X, \kappa}(\alpha)$ says the following:
"Whenever $\sigma$ is a $\kappa$ small $\leqslant \alpha$ rank $X$ name, and $A \in H_{\kappa^{+}} \cap \mathcal{P}^{\alpha}(X)$ is a set such that $\mathbb{P} \Vdash \sigma=\check{A}$, there is a filter $g \in V$ such that $\sigma^{g}=A$."
$\mathrm{N}_{\mathcal{C}, X,{ }_{\kappa}}(\alpha)$ is the statement that $\mathrm{N}_{\mathbb{Q}, X, \kappa}(\alpha)$ holds for all $\mathbb{Q} \in \mathcal{C} . \mathrm{N}_{\mathbb{P}, \kappa}(\infty)$ (resp. $\mathrm{N}_{\mathcal{C}, \kappa}(\infty)$ ) is the statement that $\mathrm{N}_{\mathbb{P}, X, \kappa}(\alpha)$ (resp. $\left.\mathrm{N}_{\mathcal{C}, X, \kappa}(\alpha)\right)$ holds for all $\alpha \in \mathrm{On}$ and all $X \in H_{\kappa^{+}}$. (Equivalently, we could just require that it holds for $\alpha \leqslant \kappa^{+}$and all $X \in H_{\kappa^{+}}$.)

Some comments on this definition: It is easy to see that if $\sigma$ is a $\kappa$ small $X$ name, and $g \in V$, then $\sigma^{g} \in H_{\kappa^{+}}$. If $\sigma$ is rank $\leqslant \alpha$, then it is also easy to see that $\sigma^{g} \in \mathcal{P}^{\alpha}(X)$. So if we didn't require that $A \in H_{\kappa^{+}} \cap \mathcal{P}^{\alpha}(X)$, then the principle would fail trivially for most forcings. The only forcings on which it could hold would be those which don't force any names to be equal to such large $A$ anyway.

This argument also shows that the name principle fails trivially if, for some $\lambda<\kappa$, there is a $\lambda$ small $\sigma$ which is forced to be equal to some $A \notin H_{\lambda^{+}}$. So we might think we should exclude such names from the principle as well. But in fact, we shall see in Section 2.3 that it makes little difference: the proof of Theorem 2.3.1 shows that if a name principle fails because of such a name, then it also fails for non-trivial reasons.

We can easily see that if $\sigma$ is a $\kappa$-small 1 rank $X$ name, and is forced to be equal to $A$, then $A \subseteq X$ and $|A| \leqslant \kappa$. Hence, when we're dealing with $\mathrm{N}(1)$, we don't need to worry about checking if the names we're working with are in $H_{\kappa^{+}} \cap \mathcal{P}(X)$, as this is automatically true. On the other hand, once we go above rank 1 , these names can exist, even for small values of $\alpha$ and $\kappa$. For example, [20, Lemma 7.1] has an $\omega$ bounded rank 2 name which is forced to be equal to $\left(2^{\omega}\right)^{V}$.

One might ask why we allowed $X$-names for all $X \in H_{\kappa^{+}}$in the definition of $\mathrm{N}_{\mathbb{P}, \kappa}(\infty)$. This is because any such name can be understood as an $\varnothing$-name of some high rank, so these principles already follow from the conjunction of $\mathbb{N}_{\mathbb{P}, \varnothing, \kappa}(\alpha)$ for all $\alpha \in$ On.

As with the forcing axioms, we will sometimes omit part of this notation. We will drop $\mathbb{P}$ and $\mathcal{C}$ when they are clear from context. We will omit $\alpha$ when $\alpha=1$. While $X$ is formally just some arbitrary set, most of the time it can be thought of as a cardinal; we will omit it in the case that $X=\kappa$, and will then omit $\kappa$ as well if $\kappa=\omega_{1}$.

Most often, these omissions will come up when we're assuming $\alpha=1$ and taking $X$ to be some cardinal. In that situation, $\kappa$ smallness is essentially trivial: if $\kappa<X$ then our class of names is too restrictive to do anything interesting, and if $\kappa \geqslant X$ then every 1 rank $X$ name will be $\kappa$ small, automatically. So when $\alpha=1$ and $X$ is a cardinal we can find out everything we need to know just by looking at the case $X=\kappa$.

We can also define variations analogous to club-FA, stat-FA, etc. However, this only really makes sense when we know $\sigma$ a subset of some cardinal. For this reason, we only define these variations for the case where $\alpha=1$ (also dropping the requirement of $\kappa$-smallness) and where $X$ is a cardinal.

Definition 2.1.13. Let $\kappa$ be a cardinal and $\varphi(x)$ a formula. The axiom $\varphi-\mathrm{N}_{\mathbb{P}, \kappa}$ states:
"For any 1 rank $\kappa$ name $\sigma$, if $\mathbb{P} \Vdash \varphi(\sigma)$ then there is a filter $g$ on $\mathbb{P}$ such that $\varphi\left(\sigma^{g}\right)$ holds in $V$."
In particular, we shall consider the axioms for the formulas $\operatorname{club}(x), \operatorname{stat}(x), \mathrm{ub}(x)$ and $\omega-\mathrm{ub}(x)$ given in Definition 2.1.3:
(1) The club name principle club- $\mathrm{N}_{\mathbb{P}, \kappa}$.
(2) The stationary name principle stat- $\mathbb{N}_{\mathbb{P}, \kappa}$.
(3) The unbounded name principle ub- $\mathbb{N}_{\mathbb{P}, \kappa}$.
(4) The $\omega$-unbounded name principle $\omega$-ub- $\mathbb{N}_{\mathbb{P}, \kappa}$.

As usual, we also define similar axioms with $\mathcal{C}$ in place of $\mathbb{P}$. Note that we could also express $\omega$-ub-N as an axiom about two names, one of which is forced to be an unbounded subset of $\kappa$ while the other is forced to be equal to $\omega$.

Remark 2.1.14. The axioms club- $\mathrm{FA}_{\mathbb{P}, \kappa}$, stat- $\mathrm{FA}_{\mathbb{P}, \kappa}$, ub- $\mathrm{FA}_{\mathbb{P}, \kappa}$ and $\omega$-ub- $\mathrm{FA}_{\mathbb{P}, \kappa}$ in Definition 2.1 .3 can be understood as a more general form of name principles for two formulas $\varphi(x)$ and $\psi(x)$ :
"For any $1 \operatorname{rank} \kappa$ name $\sigma$, if $\mathbb{P} \Vdash \varphi(\sigma)$ then there is a filter $g$ on $\mathbb{P}$ such that $\psi\left(\sigma^{g}\right)$ holds in $V$,"
For instance, stat- $\mathrm{FA}_{\mathbb{P}, \kappa}$ is equivalent to the statement:
"If $\sigma$ is a rank 1 name for $\omega_{1}$, then there is a filter $g \in V$ such that $\sigma^{g}$ is stationary."
We can also generalise the ideas here: rather than simply working with a single statement like " $\sigma$ is unbounded" or " $\sigma$ is some particular set in $V$ ", we could ask to be able to find a filter to correctly interpret every reasonable statement.

In the following definition, we allow bounded quantifiers in our $\Sigma_{0}$ formulas.
Definition 2.1.15. Let $\alpha$ be an ordinal and $\kappa$ a cardinal. The simultaneous name principle $\Sigma_{0}^{(\text {sim })}-\mathbb{N}_{\mathbb{P}, X, \kappa}(\alpha)$ says the following:
"Whenever $\sigma_{0}, \ldots, \sigma_{n}$ are $\kappa$ small $\leqslant \alpha$ rank $X$ names, we can find a filter $g$ in $V$ such that $\varphi\left(\sigma_{0}^{g}, \ldots, \sigma_{n}^{g}\right)$ holds for every $\Sigma_{0}$ formula $\varphi$ such that $\mathbb{P} \Vdash \varphi\left(\sigma_{0}, \ldots, \sigma_{n}\right)$."

Moreover:

- The simultaneous name principle $\Sigma_{0}^{(\operatorname{sim})}-\mathrm{N}_{\mathbb{P}, \kappa}(\infty)$ is the same statement, except that the names are $X$ names for some $X \in H_{\kappa^{+}}$and there is no restriction on their rank.
- $\Sigma_{0}^{(\text {sim })}-\mathrm{N}_{\mathcal{C}, X, \kappa}(\alpha)$ is the statement that $\Sigma_{0}^{(\text {sim })}-\mathrm{N}_{\mathbb{Q}, X, \kappa}(\alpha)$ holds for all $\mathbb{Q} \in \mathcal{C}$.
- $\Sigma_{0}^{(\text {sim })}-\mathrm{N}_{\mathcal{C}, \kappa}(\infty)$ is defined similarly.
- The bounded name principles $\Sigma_{0}^{(\operatorname{sim)}}-\mathrm{BN}_{\mathbb{P}, X, \kappa}^{\lambda}(\alpha)$ are defined similarly.

The $\Sigma_{0}$ requirement on $\varphi$ is necessary, because otherwise the axiom would say that any sentence which is forced to be true by $\mathbb{P}$ is already true in $V$. This would make the axiom trivially false for almost all interesting forcings. Again we will suppress $X, \kappa$ and $\alpha$ as described earlier.

All of these name principles also have bounded variants:
Definition 2.1.16. Let $\alpha$ be an ordinal and $\kappa, \lambda$ cardinals. The bounded name principle $\mathrm{BN}_{\mathbb{P}, X, \kappa}^{\lambda}(\alpha)$ says the following:
"Whenever $\sigma$ is a $\kappa$ small $\lambda$ bounded $\leqslant \alpha$ rank $X$ name, and $A$ is a set such that $\mathbb{P} \Vdash \sigma=A$, we can find a filter $g \in V$ such that $\sigma^{g}=A$."
We define similar bounded forms of all the other name principles we have introduced so far. Again, we will suppress $\lambda$ when $\lambda=\kappa$ and will suppress other notation as described above.

### 2.1.3 Hybrid axioms

There is one more group of axioms which are worth mentioning, because of their frequent use in the literature. They are a hybrid of forcing axiom and name principle. The axioms $\mathrm{MA}^{+}$and PFA ${ }^{+}$were introduced introduced by Baumgartner in [8, Section 8].
Definition 2.1.17. The forcing axiom $\mathrm{FA}_{\mathbb{P}, \kappa}^{+}$says:
Suppose $\vec{D}=\left\langle D_{\gamma}: \gamma\langle\kappa\rangle\right.$ is a sequence of dense subsets of $\mathbb{P}$ and let $\sigma$ be a 1 rank $\kappa$ name such that $\mathbb{P} \Vdash$ " $\sigma$ is stationary". Then there is a filter $g$ such that
(1) For all $\gamma, D_{\gamma} \cap g \neq \varnothing$; and
(2) $\sigma^{g}$ is stationary.

The forcing axiom $\mathrm{FA}_{\mathbb{P}, \kappa}^{++}$says:
Let $\left\langle D_{\gamma}: \gamma\langle\kappa\rangle\right.$ be dense subsets of $\mathbb{P}$ and let $\left\langle\sigma_{\gamma}: \gamma<\kappa\right\rangle$ be 1 rank $\kappa$ names such that $\mathbb{P} \Vdash$ " $\sigma_{\gamma}$ is stationary" for every $\gamma$. Then we can find a filter $g$ such that
(1) For all $\gamma, D_{\gamma} \cap g \neq \varnothing$; and
(2) For all $\gamma, \sigma_{\gamma}^{g}$ is stationary.

As usual, we will also use versions of the above with $\mathcal{C}$ in place of $\mathbb{P}$, and bounded versions.
We have actually gone against convention slightly here: the literature generally uses the quasi-interpretation $\sigma^{(g)}$ when defining $\mathrm{FA}^{+}$and $\mathrm{FA}^{++}$style axioms. However, our version is in fact equivalent, as the following theorem shows:
Theorem 2.1.18. Let $\mathrm{FA}^{(+)}$and $\mathrm{FA}^{(++)}$be defined in the same way as $\mathrm{FA}^{+}$and $\mathrm{FA}^{++}$above, but with $\sigma^{(g)}$ and $\sigma_{\gamma}^{(g)}$ in place of $\sigma^{g}$ and $\sigma_{\gamma}^{g}$ respectively. Then $\mathrm{FA}_{\mathbb{P}, \kappa}^{+} \Longleftrightarrow \mathrm{FA}_{\mathbb{P}, \kappa}^{(+)}$and $\mathrm{FA}_{\mathbb{P}, \kappa}^{++} \Longleftrightarrow \mathrm{FA}_{\mathbb{P}, \kappa}^{(++)}$.
Proof. We will prove the $\mathrm{FA}^{+}$case; the $\mathrm{FA}^{++}$version is similar. The $\Leftarrow$ direction is trivial.
$\Rightarrow$ : Let $\left\langle D_{\gamma}: \gamma<\kappa\right\rangle$ be a collection of $\kappa$ many dense subsets of $\mathbb{P}$. Let $\sigma$ be a rank 1 name with $\mathbb{P} \Vdash$ " $\sigma$ is stationary".

For $\gamma \in \kappa$, let

$$
E_{\gamma}:=\{p \in \mathbb{P}: p \Vdash \check{\gamma} \notin \sigma \text { or } \exists q \geqslant p(\check{\gamma}, q) \in \sigma\}
$$

We can see that $E_{\gamma}$ is dense: given $p \in \mathbb{P}$, either we can find some $q \| p$ with $\langle\check{\gamma}, q\rangle \in \sigma$ and we're done, or $p \Vdash \check{\gamma} \notin \sigma$ since all the elements of $\sigma$ are check names.

Claim 2.1.19. If $g$ is any filter which meets all the $E_{\gamma}$, then $\sigma^{g}=\sigma^{(g)}$
Proof. $\subseteq$ : Let $\gamma \in \sigma^{g}$. Then there is a $q \in g$ with $(\check{\gamma}, q) \in \sigma$. Clearly $q \Vdash \check{\gamma} \in \sigma$, so $\gamma \in \sigma^{(g)}$.
〇: Let $\gamma \in \sigma^{(g)}$. Then we can find $r \in g$ with $r \Vdash \check{\gamma} \in \sigma$. Certainly, then, there is no $p \in g$ with $p \Vdash \check{\gamma} \notin \sigma$. Since nonetheless $g$ meets $E_{\gamma}$, there must be some $q \in g$ with $(\check{\gamma}, q) \in \sigma$. Hence $\gamma \in \sigma^{g}$.

Now we simply use our forcing axiom to take a filter $g$ which meets all the $D_{\gamma}$, all the $E_{\gamma}$, and which is such that $\sigma^{(g)}$ is stationary.

In defining the $E_{\gamma}$ in the above proof, we used a technique which we will be invoking many times. It will save us a lot of time if we give it a name now.

Definition 2.1.20. Let $\tau$ and $\sigma$ be names, and $p \in \mathbb{P}$. We say $p$ strongly forces $\tau \in \sigma$, and write $p \Vdash^{+} \tau \in \sigma$, if there exists $q \geqslant p$ with $(\tau, q) \in \sigma$.

The value of this definition is shown in the following two propositions.
Proposition 2.1.21. Let $\sigma$ and $\tau$ be names, and $p \in \mathbb{P}$.
(1) If $p \Vdash \tau \in \sigma$, then there exist densely many $r \leqslant p$ such that for some name $\tilde{\tau}, r \Vdash \tilde{\tau}=\tau$ and $r \Vdash^{+} \tilde{\tau} \in \sigma$.
(2) If $p \Vdash^{+} \tau \in \sigma$ then $p \Vdash \tau \in \sigma$.

Proof. (1): Assume $p \Vdash \tau \in \sigma$. Let $q \leqslant p$, and let $G$ be a generic filter containing $q$. Then we know that $\tau^{G} \in \sigma^{G}$. Hence there is some pair $(\tilde{\tau}, s) \in \sigma$ such that $s \in G$ and $\tilde{\tau}^{G}=\tau^{G}$. Since $\tilde{\tau}^{G}=\tau^{G}$, there exists some condition $t \in G$ such that $t \Vdash \tilde{\tau}=\tau$. Now choose $r \leqslant q, s, t$, which exists by compatibility of elements of $G$. It is immediate that $r \Vdash \tilde{\tau}=\tau$ and that $r \Vdash^{+} \tilde{\tau} \in \sigma$.
(2): Trivial.

Proposition 2.1.22. Let $\sigma$ and $\tau$ be names, let $p \in \mathbb{P}$ and let $g$ be any filter containing $p$.
(1) If $p \Vdash^{+} \tau \in \sigma$ then $\tau^{g} \in \sigma^{g}$.
(2) If for all $\tilde{\tau}$ with $(\tilde{\tau}, q) \in \sigma$ (for some $q \in \mathbb{P}$ ) we either know $\tau^{g} \neq \tilde{\tau}^{g}$ or have $p \Vdash \tilde{\tau} \notin \sigma$ then $\tau^{g} \notin \sigma^{g}$.

### 2.2 Results for rank 1

We will start by looking at the positive results we can prove in general about forcing axioms and rank 1 name principles. We again take $\mathbb{P}$ to be an arbitrary forcing. We also take $\kappa$ to be an uncountable cardinal, although we're mostly interested in the case where $\kappa=\omega_{1}$. Since $\mathbb{P}$ is arbitrary, we could just as easily replace it with a class $\mathcal{C}$ of forcings in all our results.

### 2.2.1 Basic implications

All the positive results expressed in Figure 2.1 are proved in this section. The negative results will be proved later, when we look at the specific forcings that provide counterexamples. We will not need that $\kappa$ is regular. In the case of $\operatorname{Cof}(\kappa)=\omega$, a club is

Lemma 2.2.1. $\mathrm{FA}_{\mathbb{P}, \kappa} \Longleftrightarrow \mathrm{N}_{\mathbb{P}, \kappa}$
Proof. $\Longrightarrow$ : Assume $\mathrm{FA}_{\kappa}$. (That is, $\mathrm{FA}_{\mathbb{P}, \kappa}$, recall that we said we'd suppress the $\mathbb{P}$ whenever it was clear.) Let $\sigma$ be a rank 1 name for a subset of $\kappa$, and suppose that $1 \Vdash \sigma=A$ for some $A \subseteq \kappa$. For $\gamma \in A$, let

$$
D_{\gamma}=\left\{p \in \mathbb{P}: p \Vdash^{+} \check{\gamma} \in \sigma\right\}
$$

It is clear that $D_{\gamma}$ is dense by Proposition 2.1.21.
For $\gamma \in \kappa \backslash A$, let $D_{\gamma}=\mathbb{P}$.
Using $\mathrm{FA}_{\kappa}$, take a filter $g$ that meets every $D_{\gamma}$. We claim that $\sigma^{g}=A$.

For $\gamma \in A$, we know that some $p \in g$ strongly forces $\check{\gamma} \in \sigma$. By 2.1.22 then, $\gamma \in \sigma^{g}$. Conversely, if $\gamma \notin A$ then $1 \Vdash \check{\gamma} \notin \sigma$ and by the same proposition $\gamma \notin \sigma$.
$\Longleftarrow$ : Assume $\mathbf{N}_{\kappa}$. Let $\left\langle D_{\gamma}, \gamma\langle\kappa\rangle\right.$ be a collection of dense subsets of $\mathbb{P}$.
Let

$$
\sigma=\left\{(\check{\gamma}, p): \gamma<\kappa, p \in D_{\gamma}\right\}
$$

It is easy to see that $1 \Vdash \sigma=\check{\kappa}$. Take a filter $g$ such that $\sigma^{g}=\kappa$, and then for all $\gamma<\kappa D_{\gamma} \cap g \neq \varnothing$.
Lemma 2.2.2. $\mathrm{FA}_{\mathbb{P}, \kappa}$ holds if and only if for every rank 1 name $\sigma$ for a subset of $\kappa$, there is some $g$ with $\sigma^{(g)}=\sigma^{g}$.

Proof. First suppose that $\mathrm{FA}_{\mathbb{P}, \kappa}$ holds and $\sigma$ is a rank $1 \mathbb{P}$-name for a subset of $\kappa$. Note that $\sigma^{g} \subseteq \sigma^{(g)}$ holds for all filters $g$ on $\mathbb{P}$. For each $\alpha<\omega_{1}$,

$$
D_{\alpha}=\left\{p \in \mathbb{P}: p \Vdash \check{\alpha} \notin \sigma \vee p \Vdash^{+} \check{\alpha} \in \sigma\right\}
$$

is dense. By $\mathrm{FA}_{\mathbb{P}, \kappa}$, there is a filter $g$ with $g \cap D_{\alpha}$ for all $\alpha<\omega_{1}$. To see that $\sigma^{(g)} \subseteq \sigma^{g}$ holds, suppose that $\alpha \in \sigma^{(g)}$. Thus there is some $p \in g$ which forces $\check{\alpha} \in \sigma$. Take any $q \in g \cap D_{\alpha}$. Since $p \| q$, we have $p \Vdash^{+} \check{\alpha} \in \sigma$ by the definition of $D_{\alpha}$ and thus $\alpha \in \sigma^{g}$.

On the other hand, $\mathrm{N}_{\mathbb{P}, \kappa}$ and thus $\mathrm{FA}_{\mathbb{P}, \kappa}$ (by Lemma 2.2.1) follows trivially from this principle, since for any rank 1 name $\sigma$ with $\Vdash \sigma=\check{A}$, we have $\sigma^{(g)}=A$ for any filter $g$.

## Lemma 2.2.3.

(1) $\mathrm{FA}_{\mathbb{P}, \kappa} \Longrightarrow$ club-FA $\mathbb{P}_{\mathbb{P}} \Longrightarrow$ ub- $\mathrm{FA}_{\mathbb{P}, \kappa}$
(2) $\mathrm{FA}_{\mathbb{P}, \kappa} \Longrightarrow$ stat $-\mathrm{FA}_{\mathbb{P}, \kappa} \Longrightarrow$ ub- $\mathrm{FA}_{\mathbb{P}, \kappa}$
(3) $\mathrm{FA}_{\mathbb{P}, \kappa} \Longrightarrow \omega-u b-\mathrm{FA}_{\mathbb{P}, \kappa} \Longrightarrow u b-\mathrm{FA}_{\mathbb{P}, \kappa}$
(4) If $\operatorname{Cof}(\kappa)>\omega$, then club- $-\mathrm{FA}_{\mathbb{P}, \kappa} \Longrightarrow$ stat- $\mathrm{FA}_{\mathbb{P}, \kappa}$

Proof. Follows immediately from the definitions of the axioms.
Lemma 2.2.4. club- $\mathrm{FA}_{\mathbb{P}, \kappa} \Longleftrightarrow \mathrm{FA}_{\mathbb{P}, \operatorname{Cof}(\kappa)}$.
Proof. For $\operatorname{Cof}(\kappa)=\omega$, the statements are both provably true. So assume $\operatorname{Cof}(\kappa)>\omega$.
$\Longleftarrow:$ Let $\pi: \operatorname{Cof}(\kappa) \rightarrow \kappa$ be a continuous cofinal function. Let $\vec{D}=\left\langle D_{\alpha}: \alpha<\kappa\right\rangle$ be a sequence of dense open subsets of $\mathbb{P}$. Let $\vec{E}=\left\langle E_{\beta}: \beta<\lambda\right\rangle$, where $E_{\alpha}=D_{\pi(\alpha)}$ for $\alpha<\operatorname{Cof}(\kappa)$. By $\mathrm{FA}_{\mathbb{P}, \operatorname{Cof}(\kappa)}$, there is a filter $g$ with $g \cap E_{\alpha}$ for $\alpha<\operatorname{Cof}(\kappa)$. Thus for all $\beta=\pi(\alpha) \in \operatorname{ran}(\pi), g \cap D_{\alpha}=g \cap E_{\beta} \neq \varnothing$. This suffices since $\operatorname{ran}(\pi)$ is club in $\kappa$.
$\Longrightarrow$ : We first claim that club-FA $\mathbb{P}_{\mathbb{P}, \kappa}$ implies club- $\mathrm{FA}_{\mathbb{P}, \operatorname{Cof}(\kappa)}$. To see this, let $\pi: \operatorname{Cof}(\kappa) \rightarrow \kappa$ be a continuous cofinal function. Let $\vec{D}=\left\langle D_{\alpha}: \alpha<\operatorname{Cof}(\kappa)\right\rangle$ be a sequence of dense open subsets of $\mathbb{P}$. Let $E_{\pi(\alpha)}=D_{\alpha}$ and $E_{\gamma}=\mathbb{P}$ for all $\gamma \notin \operatorname{ran}(\pi)$. Since $C \cap \operatorname{ran}(\pi)$ is club in $\kappa$ and $\pi$ is continuous, $\pi^{-1}(C)$ is club in $\operatorname{Cof}(\kappa)$ and $g \cap D_{\alpha}=g \cap E_{\pi(\alpha)} \neq \varnothing$ for all $\alpha \in \pi^{-1}(C)$ as required.

It now suffices to prove club-FA $\mathbb{P}_{, \lambda} \Longrightarrow \mathrm{FA}_{\mathbb{P}, \lambda}$ for regular $\lambda$. Given a sequence $\vec{D}=\left\langle D_{\alpha}: \alpha<\lambda\right\rangle$ of dense open subsets, partition $\lambda$ into disjoint stationary sets $S_{\alpha}$ for $\alpha<\kappa$. Let $\vec{E}=\left\langle E_{\beta}: \beta<\lambda\right\rangle$, where $E_{\beta}=D_{\alpha}$ for $\beta \in S_{\alpha}$. By club- $\mathrm{FA}_{\lambda}$, there is a filter $g$ and a club $C$ in $\lambda$ with $g \cap E_{\beta}$ for $\beta \in C$. Since $C$ is club, $S_{\alpha} \cap C \neq \varnothing$ for all $\alpha<\lambda$. Thus $g \cap D_{\alpha}=g \cap E_{\beta} \neq \varnothing$.

## Lemma 2.2.5.

(1) $\mathrm{FA}_{\kappa} \Longrightarrow$ club- $\mathrm{N}_{\kappa}$
(2) club- $\mathrm{N}_{\kappa} \Longrightarrow$ club- $\mathrm{FA}_{\kappa}$

Proof. (1): Let $\sigma$ be a rank 1 name such that $1 \Vdash$ " $\sigma$ contains a club in $\kappa$ ". Then we can find a rank 1 name $\tau$ such that $1 \Vdash \tau \subseteq \sigma$ and $1 \Vdash$ " $\tau$ is a club in $\kappa$ ". For $\gamma<\kappa$, let $D_{\gamma}$ denote the set of $p \in \mathbb{P}$ such that either
(a) $p \Vdash^{+} \check{\gamma} \in \tau$, or
(b) for all sufficiently large $\alpha<\gamma, p \Vdash \check{\alpha} \notin \tau$.

We claim $D_{\gamma}$ is dense. Let $p \in \mathbb{P}$. If $p \Vdash \check{\gamma} \in \tau$ then by Proposition 2.1.21 we can find $q \leqslant p$ strongly forcing this, and then $q \in D_{\gamma}$. Otherwise, take $q \leqslant p$ with $q \Vdash \check{\gamma} \notin \tau$. Then $q \Vdash$ " $\tau \cap \gamma$ is bounded in $\gamma$ ". Take $r \leqslant q$ deciding that bound, and then $r$ satisfies condition b above.

For any filter $g$ with $g \cap D_{\gamma} \neq \varnothing, \tau^{g}$ is closed at $\gamma$ by Proposition 2.1.22.
Let $E_{\gamma}$ denote the set of $p \in \mathbb{P}$ such that for some $\delta \geqslant \gamma, p \Vdash^{+} \check{\delta} \in \tau$. Again, this is dense since $\tau$ is forced to be unbounded. For any filter $g$ with $g \cap E_{\gamma} \neq \varnothing$ for all $\gamma<\kappa, \tau^{g}$ is unbounded.

Let $F_{\gamma}$ denote the dense set of $p \in \mathbb{P}$ such that $p \Vdash^{+} \check{\gamma} \in \sigma$ or $p \Vdash \check{\gamma} \notin \tau$. Once again, $F_{\gamma}$ is dense: given $p \in \mathbb{P}$ take $q \leqslant p$ deciding whether $\gamma \in \tau$. If it decides $\gamma \notin \tau$ then we're done; otherwise $q \Vdash \check{\gamma} \in \sigma$ and we can find $r \leqslant q$ with $r \Vdash^{+} \check{\gamma} \in \sigma$

For any filter $g$ with $g \cap F_{\gamma} \neq \varnothing, \gamma \in \tau^{g} \Rightarrow \gamma \in \sigma^{g}$.
Putting things together, if we find a filter $g$ which meets every $D_{\gamma}, E_{\gamma}$ and $F_{\gamma}$ then $\tau^{g}$ will be both a club and a subset of $\sigma^{g}$.
(2): This works much like the proof that $\mathrm{N} \Rightarrow \mathrm{FA}$ above. Let $\left\langle D_{\gamma}: \gamma\langle\kappa\rangle\right.$ be a collection of dense sets. Let

$$
\sigma=\left\{(\check{\gamma}, p): \gamma<\kappa, p \in D_{\gamma}\right\}
$$

Clearly $1 \Vdash \sigma=\check{\kappa}$, and hence that $\sigma$ contains a club. Take a filter $g$ where $\sigma^{g}$ contains a club. Then $\sigma^{g}=\left\{\gamma<\kappa: D_{\gamma} \cap g \neq \varnothing\right\}$ so $g$ meets a club of $D_{\gamma}$.

Putting together the previous results, we complete the top left corner of Figure 2.1.
Corollary 2.2.6. The following are all equivalent for all uncountable regular cardinals $\kappa$ : $\mathrm{FA}_{\kappa}, \mathrm{N}_{\kappa}$, club-FA ${ }_{\kappa}$, club- $\mathrm{N}_{\kappa}$.

The second half of the previous lemma also applies for the other special name principles.
Lemma 2.2.7. stat- $\mathrm{N}_{\kappa} \Longrightarrow$ stat- $\mathrm{FA}_{\kappa}$
Proof. As for the club case, except that we just insist on $\sigma^{g}$ being stationary.
Lemma 2.2.8. ub- $\mathrm{N}_{\kappa} \Longrightarrow$ ub-FA ${ }_{\kappa}$
Proof. As for the club case, except that we insist on $\sigma^{g}$ being unbounded.
Lemma 2.2.9. $\omega$-ub- $\mathrm{N}_{\kappa} \Longrightarrow \omega$-ub-FA ${ }_{\kappa}$
Proof. Define $\sigma$ as in the club case. Define

$$
\tau=\left\{(\check{n}, p): n<\omega, p \in E_{n}\right\}
$$

where we want to meet all of the dense sets $\left\langle E_{n}: n\langle\omega\rangle\right.$ as well as unboundedly many of the dense sets $D_{\gamma}$. Take $g$ such that $\tau^{g}=\omega$ and $\sigma^{g}$ is unbounded.

We can also get converses for these in the case of ub and $\omega$-ub.

## Lemma 2.2.10.

(1) ub-FA ${ }_{\kappa} \Longrightarrow u b-\mathrm{N}_{\kappa}$
(2) $\omega-\mathrm{ub}-\mathrm{FA}_{\kappa} \Longrightarrow \omega-\mathrm{ub}-\mathrm{N}_{\kappa}$

Proof. (1): Assume ub-FA ${ }_{\kappa}$. Let $\sigma$ be a rank 1 name for an unbounded subset of $\kappa$. For $\gamma<\kappa$ let $D_{\gamma}$ be the set of all $p \in \mathbb{P}$ such that for some $\delta>\gamma, p \Vdash^{+} \check{\delta} \in \sigma$. Let $g$ be a filter meeting unboundedly many $D_{\gamma}$; then $\sigma^{g}$ is unbounded.
(2): Let $\sigma$ be a rank 1 name for an unbounded subset of $\kappa$ and $\tau$ be a good name for $\omega$. Define $D_{\gamma}$ as above, and for $n<\omega$ let $E_{n}$ be the set of all $p \in \mathbb{P}$ which strongly force $n \in \tau$. Find $g$ meeting unboundedly many $D_{\gamma}$ and every $E_{n}$; then $\sigma^{g}$ is unbounded and $\tau^{g}=\omega$.

This proves every implication in the left two columns of Figure 2.1.

### 2.2.2 Extremely bounded name principles

Now, we address the right most column of Figure 2.4. These axioms are more interesting if $\mathbb{P}$ is a complete Boolean algebra, since they can be trivial otherwise.
Lemma 2.2.11. $\mathrm{BN}_{\kappa}^{1}$ is provable in ZFC .
Proof. Let $\sigma$ be a 1 -bounded rank 1 name such that $1 \Vdash \sigma=\check{A}$ for some set $A$. Then for $\gamma \in \kappa \backslash A$, there is no $p \in \mathbb{P}$ such that $(\check{\gamma}, p) \in \sigma$. For $\gamma \in A$ there is a unique $p \in \mathbb{P}$ such that $(\check{\gamma}, p) \in \sigma$; and $p$ is contained in every generic filter. Assuming $\mathbb{P}$ is atomless, it follows that $p=1$ and hence that, if we let $g$ be any filter at all, $\sigma^{g}=A$. It is also possible to adjust this proof to work for forcings with atoms; this is left as an exercise for the reader.

All of these results also hold if we work with bounded name principles and forcing axioms, provided that the bound is at least $\kappa$.

For bounds below $\kappa$, we can almost get an equivalence between the different bounds for the stationary and unbounded name principles. A forcing is called well-met if any two compatible conditions $p, q$ have a greatest lower bound $p \wedge q$.

The next result and proof is due to Hamkins for trees (see Corollary 2.2.13). We noticed that his proof shows a more general fact.

Lemma 2.2.12 (with Hamkins). Suppose $\lambda<\kappa$ and $\mathbb{P}$ is well-met.
(1) If stat- $\mathrm{BN}_{\mathbb{P}, \kappa}^{\lambda}$ fails, then there are densely many conditions $p \in \mathbb{P}$ such that stat- $\mathrm{BN}_{\mathbb{P}_{p}, \kappa}^{1}$ fails, where $\mathbb{P}_{p}:=\{q \in \mathbb{P}: q \leqslant p\}$.
(2) The same result holds with ub in place of stat.

Proof. We prove the stat case; the ub case is identical. The key fact the proof uses is that if we partition a stationary/unbounded subset of $\kappa$ into $\lambda<\kappa$ many parts, then one of those parts must be stationary/unbounded.

Let $\sigma$ be a $\lambda$-bounded (rank 1) name for a stationary set, such that there is no $g \in V$ with $\sigma^{g}$ stationary. Then, without loss of generality, we can enumerate the elements of $\sigma$ :

$$
\sigma=\left\{\left(\check{\gamma}, p_{\gamma, \delta}\right): \gamma<\kappa, \delta<\lambda\right\}
$$

For $\delta<\lambda$, we define:

$$
\sigma_{\delta}=\left\{\left(\check{\gamma}, p_{\gamma, \delta}\right): \gamma<\kappa\right\}
$$

Clearly, $\sigma_{\delta}$ is 1-bounded.
For any generic filter $G, \bigcup \sigma_{\delta}^{G}=\sigma^{G}$ is stationary in $V[G]$. Hence, $\mathbb{P}$ forces "There is some $\delta<\lambda$ such that $\sigma_{\delta}$ is stationary." Now, let $p \in \mathbb{P}$ be one of the densely many conditions which decides which $\delta$ this is. Then

$$
\sigma_{\delta, p}=\left\{\left(\check{\gamma}, p_{\gamma, \delta} \wedge p\right): \gamma<\kappa\right\}
$$

is a 1 -bounded $\mathbb{P}_{p}$-name and $\mathbb{P}_{p} \Vdash \sigma_{\delta, p}$ is stationary. If stat- $\mathrm{BN}_{\mathbb{P}_{p}, \kappa}^{1}$ would hold, there would exist a filter $g$ such that $\sigma_{\delta, p}^{g}$ is stationary. Then $g$ generates a filter $h$ in $\mathbb{P}$ such that $\sigma_{\delta, p}^{h} \supseteq \sigma_{\delta, p}^{g}$ is stationary.

Corollary 2.2.13 (Hamkins). Suppose that $T$ is a tree, $\mathbb{P}_{T}$ is $T$ with reversed order and $\lambda<\kappa$.
(1) If stat- $\mathrm{BN}_{\mathbb{P}_{T}, \kappa}^{\lambda}$ fails, then there are densely many conditions $p \in \mathbb{P}$ such that stat- $\mathrm{BN}_{\left(\mathbb{P}_{T}\right)_{p}, \kappa}^{1}$ fails, where $\left(\mathbb{P}_{T}\right)_{p}:=\left\{q \in \mathbb{P}_{T}: q \leqslant p\right\}$.
(2) The same result holds with ub in place of stat.

Corollary 2.2.14. Suppose $\lambda<\kappa$ and $\mathbb{P}$ is a well-met forcing such that for every $p \in \mathbb{P}, \mathbb{P}_{p}$ embeds densely into $\mathbb{P}$. Then

$$
\begin{aligned}
\text { stat }-\mathrm{BN}_{\mathbb{P}, \kappa}^{\lambda} & \Longleftrightarrow \text { stat }-\mathrm{BN}_{\mathbb{P}, \kappa}^{1} \\
\text { ub- } \mathrm{BN}_{\mathbb{P}, \kappa}^{\lambda} & \Longleftrightarrow \text { ub- }-\mathrm{BN}_{\mathbb{P}, \kappa}^{1}
\end{aligned}
$$

Proof. We show that a failure of stat- $\mathrm{BN}_{\mathbb{P}, \kappa}^{\lambda}$ implies the failure of stat- $\mathrm{BN}_{\mathbb{P}, \kappa}^{1}$. The converse direction is clear and the proof for the unbounded name principles is analogous.

By Lemma 2.2.12, there is some $p \in \mathbb{P}$ such that stat- $\mathrm{BN}_{\mathbb{P}_{p}, \kappa}^{1}$ fails. Let $i: \mathbb{P}_{p} \rightarrow \mathbb{P}$ be a dense embedding and $\mathbb{Q}:=i\left(\mathbb{P}_{p}\right)$. Since stat-BN ${ }_{\mathbb{Q}, \kappa}^{1}$ fails, let $\sigma$ be a 1-bounded $\mathbb{Q}$-name witnessing this failure. We claim that there is no filter $g$ on $\mathbb{P}$ such that $\sigma^{g}$ is stationary. Assume otherwise. Using that $\mathbb{Q}$ is well-met, let $h$ denote the set of all $q \geqslant p_{0} \wedge \mathbb{Q} \cdots \wedge \mathbb{Q} p_{n}$ for some $p_{0}, \ldots, p_{n} \in g \cap \mathbb{Q}$. It is easy to check that $h$ is a well-defined filter on $\mathbb{Q}$ and contains $g \cap \mathbb{Q}$. Then $\sigma^{h} \supseteq \sigma^{g}$ is stationary. But this contradicts the choice of $\sigma$.

### 2.2.3 Extremely bounded forcing axioms

We next study forcing axioms for very small predense sets. The next lemmas show that $\mathrm{BFA}_{\mathbb{P}, \omega_{1}}^{\omega}$ has some of the same consequences as BFA.

Lemma 2.2.15. If $\mathbb{P}$ is a complete Boolean algebra such that $\mathrm{BFA}_{\mathbb{P}, \omega_{1}}^{\omega}$ holds, then $1_{\mathbb{P}}$ does not force that $\omega_{1}$ is collapsed.
Proof. Suppose $\Vdash \dot{f}: \omega_{1} \rightarrow \omega$ is injective. Let $A_{\alpha}=\{\llbracket \dot{f}(\alpha)=n \rrbracket \neq 0: n \in \omega\}$. Since each $A_{\alpha}$ is a maximal antichain, there is a filter $g$ with $g \cap A_{\alpha} \neq \varnothing$ for all $\alpha<\omega_{1}$. Define $f^{\prime}: \omega_{1} \rightarrow \omega$ by letting $f^{\prime}(\alpha)=n$ if $\llbracket \dot{f}(\alpha)=n \rrbracket \in g$ for all $\alpha<\omega_{1}$. Since $g$ is a filter, $f^{\prime}: \omega_{1} \rightarrow \omega$ is well-defined and injective.

Lemma 2.2.16. If $\mathbb{P}$ is a complete Boolean algebra such that $\mathrm{BFA}_{\mathbb{P}, \omega_{1}}^{\omega}$ holds and $\mathbb{P}$ adds a real, then CH fails.
Proof. Suppose CH holds and let $\left\langle x_{\alpha}: \alpha<\omega_{1}\right\rangle$ be an enumeration of all reals. Let $\sigma$ be a name for the real added by $\mathbb{P}$. For $\alpha<\omega_{1}$, let

$$
D_{\alpha}=\left\{\llbracket t \_\langle n\rangle \subseteq \sigma \rrbracket: t \in 2^{<\omega}, n \in 2, t \subseteq x_{\alpha}, t^{\wedge}\langle n\rangle \ddagger x_{\alpha}\right\}
$$

For $n<\omega$, let

$$
E_{n}=\{\llbracket \sigma(n)=m \rrbracket: m \in 2\}
$$

Then the $D_{\alpha}$ and $E_{n}$ are all predense and countable. Take a filter $g$ which meets every $D_{\alpha}$ and $E_{n}$. The $E_{n}$ ensure that $g$ defines a real $x$ (by $x(n)=m$ where $\llbracket \sigma(n)=m \rrbracket \in g$ ). But if $x=x_{\alpha}$ then $g \cap D_{\alpha}=\varnothing$.

There exist forcings $\mathbb{P}$ such that the implication $B F A_{\mathbb{P}, \omega_{1}}^{\omega} \Rightarrow \operatorname{BFA}_{\mathbb{P}, \omega_{1}}^{\omega_{1}}$ fails. To see this, suppose that $\mathbb{Q}$ is a forcing such that $\operatorname{BFA} A_{\mathbb{Q}, \omega_{1}}^{\omega_{1}}$ fails. Let $\mathbb{P}$ be a lottery sum of $\omega_{1}$ many copies of $\mathbb{Q}$. Since $\mathrm{BFA}_{\mathbb{Q}, \omega_{1}}^{\omega_{1}}$ fails, $\mathrm{BFA}_{\mathbb{P}}^{\omega_{1} \omega_{1}}$ fails as well. On the other hand, $\mathrm{BFA}_{\mathbb{P}, \omega_{1}}^{\omega}$ holds trivially since any countable predense subset of $\mathbb{P}$ contains $0_{\mathbb{P}}$.

Question 1. Does the implication $B F A_{\mathbb{P}, \omega_{1}}^{\omega} \Rightarrow \operatorname{BFA}_{\mathbb{P}, \omega_{1}}^{\omega_{1}}$ hold for all complete Boolean algebras $\mathbb{P}$ ?
By the previous lemmas, any forcing which is a counterexample cannot force that $\omega_{1}$ is collapsed, and if it adds reals then CH holds.

### 2.2.4 Basic results on ub-FA

In this section, we collect some observations about weak forcing axioms. We aim to prove some consequences of these axioms. We first consider ub-FA and stat-FA. How strong is ub-FA? The next lemmas show that is has some of the same consequences as FA.

Lemma 2.2.17. If $\mathrm{ub}-\mathrm{FA}_{\mathbb{P}, \omega_{1}}$ holds, then $\mathbb{P}$ does not force that $\omega_{1}$ is collapsed.
Proof. Towards a contradiction, suppose $\mathbb{P}$ forces that $\omega_{1}$ is collapsed. Let $\dot{f}$ be a $\mathbb{P}$-name for an injective function $\omega_{1} \rightarrow \omega$. For $\alpha<\omega_{1}$, let $D_{\alpha}=\{p \in \mathbb{P}: \exists n \in \omega p \Vdash \dot{f}(\alpha)=n\}$. By ub-FA $\mathbb{P}_{\mathbb{P}, \omega_{1}}$, there is a filter $g$ and an unbounded subset $A$ of $\omega_{1}$ such that $g \cap D_{\alpha} \neq \varnothing$ for all $\alpha \in A$. Define $f: A \rightarrow \omega$ by letting $f(\alpha)=n$ if there is some $p \in g \cap D_{\alpha}$ with $p \Vdash \dot{f}(\alpha)=n$. Since $g$ is a filter, $f$ is injective.

Lemma 2.2.18. If ub- $\mathrm{FA}_{\mathbb{P}, \omega_{1}}$ holds and $\mathbb{P}$ does not add reals, then for each stationary subset $S$ of $\omega_{1}, \mathbb{P}$ does not force that $S$ is nonstationary.

Proof. Suppose that $\dot{C}$ is a name for a club such that $\Vdash_{\mathbb{P}} S \cap \dot{C}=\varnothing$. Let $\dot{f}$ be a name for the characteristic function of $\dot{C}$. For each $\alpha<\omega_{1}$,

$$
D_{\alpha}=\left\{p \in \mathbb{P}: \exists t \in 2^{\alpha} t \subseteq \dot{f}\right\}
$$

is dense in $\mathbb{P}$, since $\mathbb{P}$ does not add reals. By ub- $\mathrm{FA}_{\mathbb{P}, \omega_{1}}$, there is a filter $g$ and an unbounded subset $A$ of $\omega_{1}$ such that $g \cap D_{\alpha} \neq \varnothing$ for all $\alpha \in A$. Since $g$ is a filter, $C:=\left\{\alpha<\omega_{1}: \exists p \in g p \Vdash \alpha \in \dot{C}\right\}$ is a club in $\omega_{1}$ with $S \cap C \neq \varnothing$.

The previous lemma also follows from Theorem 2.3.17 and Lemma 2.3.24 below via an absoluteness $\operatorname{argument}$, assuming $\mathbb{P}$ is a homogeneous complete Boolean algebra. It is open whether the lemma holds for forcings $\mathbb{P}$ which add reals.

What is the relationship between ub-FA $\mathbb{P}_{\mathbb{P}, \omega_{1}}$ and other forcing axioms? We find two opposite situations. For any $\sigma$-centred forcing, ub-FA $\mathbb{P}_{\mathbb{P}} \omega_{1}$ and stat- $\mathrm{FA}_{\mathbb{P}, \omega_{1}}$ are provable in ZFC by Lemma 2.4 .6 below. For many other forcings though, ub-FA $\mathbb{P}_{\mathbb{P}, \omega_{1}}$ implies nontrivial axioms such as $\mathrm{FA}_{\mathbb{P}, \omega_{1}}$ or $\mathrm{BFA}_{\mathbb{P}, \omega_{1}}^{\omega_{1}}$. For instance, the implication ub-FA $\mathbb{P}_{\mathbb{P}, \omega_{1}} \Rightarrow \mathrm{FA}_{\mathbb{P}, \omega_{1}}$ holds for all $\sigma$-distributive forcing by Lemma 2.4.1 below. We will further see in Lemma 2.3.24 below that for any complete Boolean algebra $\mathbb{P}$ which does not add reals, $\left(\forall q \in \mathbb{P}\right.$ ub-FA $\left.\mathbb{P}_{q}, \omega_{1}\right)$ implies $B F A_{\mathbb{P}, \omega_{1}}^{\omega_{1}}$. Moreover, the implication ub- $F A_{\mathbb{P}, \omega_{1}} \Rightarrow \mathrm{FA}_{\mathbb{P}, \omega_{1}}$ also holds for some forcings that add reals, for instance for random forcing by Lemma 2.4.15.

We do not have any examples of forcings where ub- $\mathrm{FA}_{\mathbb{P}, \omega_{1}}$ and stat- $\mathrm{FA}_{\mathbb{P}, \omega_{1}}$ sit between these two extremes: strictly weaker than $\mathrm{FA}_{\mathbb{P}, \omega_{1}}$, but not provable in ZFC.

In particular, we have not been able to separate the two axioms:
Question 2. Can forcings $\mathbb{P}$ exist such that ub- $\mathrm{FA}_{\mathbb{P}, \kappa}$ holds, but stat- $F A_{\mathbb{P}, \kappa}$ fails?
For instance, we would like to know if these axioms hold for the following forcings:
Question 3. Do Baumgartner's forcing to add a club in $\omega_{1}$ with finite conditions [8, Section 3] and Abraham's and Shelah's forcing for destroying stationary sets with finite conditions [1, Section 2] satisfy ub-FA $\mathbb{P}_{\mathbb{P}, \omega_{1}}$ and stat-FA $\mathbb{P}_{\mathbb{P}, \omega_{1}}$ ?

### 2.2.5 Characterisations of $\mathrm{FA}^{+}$and $\mathrm{FA}^{++}$

The proof of the equivalence of FA and N still goes through fine if we change the axioms slightly, demanding some extra property to be true of the filter $g$ we're looking for. This gives us a nice way to express $\mathrm{FA}^{+}$and $\mathrm{FA}^{++}$.

Lemma 2.2.19. $\mathrm{FA}_{\mathcal{C}, \kappa}^{+}$is equivalent to the following statement:
For all $\mathbb{P} \in \mathcal{C}$, for all rank 1 names $\sigma$ and $\tau$ for subsets of $\kappa$ such that $\mathbb{P}$ forces " $\sigma=\check{A}$ " for some $A$ and " $\tau$ is stationary", there is some filter $g$ with $\sigma^{g}=A$ and $\tau^{g}$ stationary.
Similarly, $\mathrm{FA}_{\mathcal{C}, \kappa}^{++}$is equivalent to being able to correctly interpret $\kappa$ many stationary rank 1 names and a single rank 1 name for a specific set $A$.

Proof. Analogous to the proof of 2.2 .1 in the previous section.
In the case of $\mathrm{FA}^{++}$this result can be sharpened further, getting rid of the name for $A$ :
Lemma 2.2.20. $\mathrm{FA}_{\mathcal{C}, \kappa}^{+}$is equivalent to the statement:
For all collections of $\kappa$ many rank 1 names $\left\langle\sigma_{\gamma}: \gamma<\kappa\right\rangle$ with $\mathbb{P} \Vdash$ " $\sigma_{\gamma}$ is stationary for all $\gamma$ ", there is a filter $g \in V$ such that for all $\gamma, \sigma_{\gamma}^{g}$ is stationary.
Proof. $\Longrightarrow$ : By the previous lemma.
$\Longleftarrow$ : Let $\sigma$ be a rank 1 name, such that $\mathbb{P} \Vdash \sigma=\check{A}$ for some $A \subseteq \kappa$. We claim there is a collection $\left\langle\tau_{\gamma}: \gamma\langle\kappa\rangle\right.$ of rank 1 names, which are forced to be stationary in $\kappa$, such that any filter $g$ which interprets every $\tau_{\gamma}$ as stationary will interpret $\sigma$ as $A$. Once we have proved this claim, the lemma follows immediately from the second part of Lemma 2.2.19. For $\gamma \in A$, let $\tau_{\gamma}=\left\{(\check{\alpha}, p): \alpha \in \kappa, p \Vdash^{+} \check{\gamma} \in \sigma\right\}$. For $\gamma \notin A$, let $\tau_{\gamma}=\check{\kappa}$. We will see that $\mathbb{P} \Vdash " \tau_{\gamma}=\kappa "$ for $\gamma \in A$. Note that $\mathbb{P} \Vdash \sigma=\bar{A}$ by assumption. So for $\gamma \in A$, every generic
filter will contain some $p$ with $p \Vdash^{+} \check{\gamma} \in \sigma$. Hence $\mathbb{P} \Vdash \tau_{\gamma}=\check{\kappa}$. There is a filter $g$ such that $\tau_{\gamma}^{g}$ is stationary for all $\gamma<\kappa$ by assumption. If $\gamma \in A$, then in particular $\tau_{\gamma}^{g} \neq \varnothing$. Hence $\gamma \in \sigma^{g}$. If a filter interprets all the $\tau_{\gamma}$ as stationary sets, then $\sigma^{g} \supseteq A$. If $\gamma \in \sigma^{g} \backslash A$, then there is some $p \in \mathbb{P}$ with $\langle\check{\gamma}, p\rangle \in \sigma$, which is impossible as $\mathbb{P} \Vdash \check{\gamma} \notin \sigma$.

### 2.3 A correspondence for arbitrary ranks

We now move on to discuss higher ranked name principles, including those of the ranked or unranked simultaneous variety. It turns out that even at high ranks, a surprising variety of these are equivalent to one another and to a suitable forcing axiom. These are summarised in the following theorems.

### 2.3.1 The correspondence

(First proved by the author in [37][Section 4].)
Theorem 2.3.1. Let $\mathbb{P}$ be a forcing and let $\kappa$ be a cardinal. The following implications hold, given the assumptions noted at the arrows:

(2) For any ordinal $\alpha>0$, and any transitive set $X$ of size at most $\kappa:{ }^{2}$


As usual, we can generally think of $X$ as being a cardinal.
There is also a bounded version of this theorem.
Theorem 2.3.2. Let $\mathbb{P}$ be a complete Boolean algebra, and let $\kappa, \lambda$ be cardinals. The following implications hold, given the assumptions noted at the arrows:
(1)

(2) For any ordinal $\alpha>0$, and transitive set $X$ of size at most $\kappa$ :


[^4]Remark 2.3.3. For the $\infty$ case it suffices to look only at $\varnothing$ names, as we discussed after Definition 2.1.12. Moreover, for the implication $\mathrm{N}_{\mathbb{P}, \kappa}(\infty) \Rightarrow \mathrm{FA}_{\mathbb{P}, \kappa}$ (and the corresponding ones in the other diagrams), we need only rank $1 \kappa$-names for $\kappa$. These can be understood as rank $\kappa \varnothing$-names for $\kappa$. For $\mathrm{N}_{\mathbb{P}, X, \kappa}(\alpha) \Rightarrow \mathrm{FA}_{\mathbb{P}, \kappa}$, rank $1 Y$-names for a fixed set $Y$ of size $\kappa$ suffice. These can be understood as rank $\leqslant \alpha X$-names. These remarks are also true for the bounded versions. Note that for $\mathrm{N}_{\mathbb{P}, \kappa}(1) \Rightarrow \mathrm{FA}_{\mathbb{P}, \kappa}$, rank $1 \kappa$-names for $\kappa$ suffice by Lemma 2.2.1.

We give some simple instances of Theorem 2.3.1 (2) and postpone the proofs to Section 2.3.2. The variant for bounded forcing axioms has similar consequences. The next result follows by letting $\kappa=X$ and $\alpha=1$.

Corollary 2.3.4. For any forcing $\mathbb{P}, \mathrm{FA}_{\mathbb{P}, \kappa} \Longleftrightarrow \Sigma_{0}^{(\operatorname{sim})}-\mathrm{N}_{\mathbb{P}, \kappa} \Longleftrightarrow \mathrm{N}_{\mathbb{P}, \kappa}$.
To illustrate this, we note how some concrete forcing axioms can be characterized by name principles. For example, we can characterize PFA as follows:

$$
\text { PFA } \Longleftrightarrow \Sigma_{0}^{(\operatorname{sim})}-N_{\text {proper }, \omega_{1}} \Longleftrightarrow N_{\text {proper }, \omega_{1}}
$$

In other words, rank 1 names for $\omega_{1}$ can be interpreted correctly.
For higher ranks, it is useful to choose $\alpha, \kappa$ and $X$ such that $\left|\mathcal{P}^{<\alpha}(X)\right| \geqslant \kappa$ holds to get an equivalence in Theorem 2.3.1 (2). This condition holds for $\kappa \geqslant 2^{\omega}, X=\omega$ and $\alpha=2$.

Corollary 2.3.5. For any cardinal $\kappa \leqslant 2^{\omega}$ and any forcing $\mathbb{P}$, we have $\mathrm{FA}_{\mathbb{P}, \kappa} \Longleftrightarrow \Sigma_{0}^{(\operatorname{sim})}-\mathrm{N}_{\mathbb{P}, \omega, \kappa}(2) \Longleftrightarrow$ $\mathrm{N}_{\mathbb{P}, \omega, \kappa}(2)$.

For example, we can characterize PFA as follows:

$$
\text { PFA } \Longleftrightarrow \Sigma_{0}^{(\operatorname{sim})}-\mathrm{N}_{\text {proper }, \omega, \omega_{1}}(2) \Longleftrightarrow \mathrm{N}_{\text {proper }, \omega, \omega_{1}}(2) .
$$

In other words, rank 2 names for sets of reals can be interpreted correctly. We leave open how to characterise higher rank (e.g. rank 2) principles for names for reals.

### 2.3.2 The proofs

Proof of Theorem 2.3.1. We prove both parts of the theorem simultaneously, by fixing $X$ and $\alpha$ and proving all the implications in the following diagram:


Of these, the first $\mathrm{FA}_{\mathbb{P}, \kappa} \Rightarrow \Sigma_{0}^{(\operatorname{sim})}-\mathrm{N}_{\mathbb{P}, \kappa}(\infty)$ is the hardest to prove, and the main work on the theorem. We'll leave it to the end, and prove the other implications first. Note that $\mathrm{FA}_{\mathbb{P}, \kappa} \Rightarrow \Sigma_{0}^{(\operatorname{sim})}-\mathrm{N}_{\mathbb{P}, X, \kappa}(\alpha)$ follows from the rest of the diagram.

Proof of $\Sigma_{0}^{(\operatorname{sim})}-\mathrm{N}_{\mathbb{P}, \kappa}(\infty) \Rightarrow \Sigma_{0}^{(\operatorname{sim})}-\mathrm{N}_{\mathbb{P}, \kappa, X}(\alpha)$. The latter is a special case of the former.
Proof of $\mathrm{N}_{\mathbb{P}, \kappa}(\infty) \Rightarrow \mathrm{N}_{\mathbb{P}, X, \kappa}(\alpha)$. Again, this is a special case.
Proof of $\Sigma_{0}^{(\operatorname{sim})}-\mathrm{N}_{\mathbb{P}, X, \kappa}(\alpha) \Rightarrow \mathrm{N}_{\mathbb{P}, X, \kappa}(\alpha)$. Given a $\kappa$-small name $\sigma$ of rank $\alpha$ or less, and a set $A$ as called for by $\mathbb{N}_{\mathbb{P}, \kappa}(\alpha)$, we know $A \in \mathcal{P}^{\alpha}(X) \cap H_{\kappa^{+}}$. Hence $\check{A}$ is a $\kappa$ small $\alpha \operatorname{rank} X$ name, so " $\sigma=\check{A}$ " is one of the formulas discussed by the simultaneous name principle.

Proof of $\Sigma_{0}^{(\operatorname{sim})}-\mathrm{N}_{\mathbb{P}, \kappa}(\infty) \Rightarrow \mathrm{N}_{\mathbb{P}, \kappa}(\infty)$. Similar to the previous proof: if $\sigma$ is any $\kappa$-small name, and $A \in H_{\kappa^{+}}$ is such that $\mathbb{P} \Vdash \sigma=\check{A}$, then since $\check{A}$ is $\kappa$-small we know from $\Sigma_{0}^{(\operatorname{sim})}-\mathrm{N}(\infty)$ that we can find a filter $g$ such that $\sigma^{g}=\check{A}^{g}=A$.

Proof of $\mathrm{N}_{\mathbb{P}, X, \kappa}(\alpha) \Rightarrow \mathrm{FA}_{\mathbb{P}, \kappa}$. We assume $\left|P^{<\alpha}(X)\right| \geqslant \kappa$. The idea is similar to the proof of $\mathrm{N}_{\mathbb{P}, \kappa} \Rightarrow \mathrm{FA}_{\mathbb{P}, \kappa}$ from Lemma 2.2.1, but first we must prove a technical claim.

Claim 2.3.6. $\mathcal{P}^{<\alpha}(X)$ contains at least $\kappa$ many elements whose check names are $\kappa$-small $<\alpha$-rank $X$-names.
Proof (Claim). Let $\alpha^{\prime} \leqslant \alpha$ be minimal such that $\left|\mathcal{P}^{<\alpha^{\prime}}(X)\right| \geqslant \kappa$.
Let $A \in \mathcal{P}^{<\alpha^{\prime}}(X)$. Then $A \in \mathcal{P}^{\epsilon}(X)$ for some $\epsilon<\alpha^{\prime}$. We show by induction on $\epsilon$ that $\check{A}$ is in fact a $\kappa$-small $\epsilon$-rank $X$-name. From this and the assumption on the size of $\kappa$, it of course follows that there are at least $\kappa$ many elements of $\mathcal{P}^{<\alpha^{\prime}}(X) \subseteq \mathcal{P}^{<\alpha}(X)$ whose check names are $\kappa$-small $<\alpha$-rank $X$-names.

The case $\epsilon=0$ is trivial. Suppose $\epsilon>0$. By inductive hypothesis, we know that all the names which are contained in $\check{A}$ are $\kappa$-small $<\epsilon$-rank $X$-names. It remains to check that there are at most $\kappa$ many of them; that is, that $|A| \leqslant \kappa$. But this is obvious, since $A \subseteq \mathcal{P}^{<\epsilon}(X)$ and $\left|\mathcal{P}^{<\epsilon}(X)\right|<\kappa$ by our choice of $\alpha^{\prime}$.

Given the claim, we can now take a set of $\kappa$ many distinct sets $A:=\left\{A_{\gamma}: \gamma<\kappa\right\} \subseteq \mathcal{P}^{<\alpha}(X)$, such that for all $\gamma$, the name $\check{A}_{\gamma}$ is a $\kappa$ small $<\alpha$ rank $X$-name.

Let $\left\langle D_{\gamma}\right\rangle_{\gamma<\kappa}$ be a sequence of dense sets in $\mathbb{P}$. We define a name $\sigma$ :

$$
\sigma=\left\{\left\langle\check{A}_{\gamma}, p\right\rangle: \gamma<\kappa, p \in D_{\gamma}\right\}
$$

Then $\sigma$ is a $\kappa$-small $\leqslant \alpha$-rank $X$-name, and $\mathbb{P} \Vdash \sigma=\check{A}$. Hence, if we assume $\mathbb{N}_{\mathbb{P}, X, \kappa}(\alpha)$ we can choose a filter $g$ such that $\sigma^{g}=A$. It is easy to see that $g$ must meet every $D_{\gamma}$.

Proof of $\mathrm{N}_{\mathbb{P}, \kappa}(\infty) \Rightarrow \mathrm{FA}_{\mathbb{P}, \kappa}$. Essentially the same as the previous proof, but since we're no longer required to make sure $\sigma$ has rank $\alpha$ we can omit the technical claim and just take $A_{\gamma}:=\gamma$ for all $\gamma<\kappa$.

Proof of $\mathrm{FA}_{\mathbb{P}, \kappa} \Rightarrow \Sigma_{0}^{(\operatorname{sim})}-\mathrm{N}_{\mathbb{P}, \kappa}(\infty)$. This is the main work of the theorem. By a delicate series of inductions, we will prove the following lemma:

Lemma 2.3.7. Let $\varphi(\vec{\sigma})$ be a $\Sigma_{0}$ formula where $\vec{\sigma}$ is a tuple of $\kappa$-small names. Then there is a collection $\mathcal{D}_{\varphi(\vec{\sigma})}$ of at most $\kappa$ many dense sets, which has the following property: if $g$ is any filter meeting every set in $\mathcal{D}_{\varphi(\vec{\sigma})}$ and $g$ contains some $p$ such that $p \Vdash \varphi(\vec{\sigma})$, then in fact $\varphi\left(\vec{\sigma}^{g}\right)$ holds in $V$.

The result we're trying to show follows easily from this lemma: Fix a tuple $\vec{\sigma}=\left\langle\sigma_{0}, \ldots, \sigma_{n}\right\rangle$ of $\kappa$ small names, and let $\mathcal{D}:=\bigcup\left\{\mathcal{D}_{\varphi(\vec{\sigma})}: \varphi\left(v_{0}, \ldots, v_{n}\right)\right.$ is $\left.\Sigma_{0}\right\} . \mathcal{D}$ is a collection of at most $\kappa$ many dense sets. Using $\mathrm{FA}_{\mathbb{P}, \kappa}$, take a filter $g$ meeting every dense set in $\mathcal{D}$. If $\varphi\left(v_{0}, \ldots, v_{n}\right)$ is a $\Sigma_{0}$ formula and $1 \Vdash \varphi(\vec{\sigma})$ then since $1 \in g$ we know that $\varphi\left(\vec{\sigma}^{g}\right)$ holds.

We will work our way up to proving the lemma, by first proving it in simpler cases. We opt for a direct proof of the name principle $\mathrm{N}_{\mathbb{P}, \kappa}(\infty)$ in the next Claim 2.3.8. This and Claim 2.3 .11 could be replaced by shorter arguments for $\kappa$-small $\varnothing$-names, since it suffices to deal with $\Sigma_{0}^{(\operatorname{sim})}-\mathrm{N}_{\mathbb{P}, \varnothing, \kappa}(\infty)$ as discussed after Definition 2.1.12.

Claim 2.3.8. The lemma holds when $\varphi$ is of the form $\sigma=\check{A}$ for some set $A \in H_{\kappa^{+}}$and ( $\kappa$-small) name $\sigma$.
Note that since $A \in H_{\kappa^{+}}$, we know that $\check{A}$ is a $\kappa$ small name. So the statement in the claim does make sense.

Proof. We use induction on the rank of $\sigma$. If $\sigma$ is rank 0 then it is a check name, and so the lemma is trivial: we can just take $\mathcal{D}_{\sigma=\check{A}}=\varnothing$. So say $\sigma$ is rank $\alpha>0$ and the lemma is proved for all names of rank $<\alpha$. Since $\sigma$ is $\kappa$-small, we can write $\sigma=\left\{\left(\sigma_{\gamma}, p\right): \gamma<\kappa, p \in S_{\gamma}\right\}$ for some $\kappa$-small names $\sigma_{\gamma}$ and sets $S_{\gamma} \subseteq \mathbb{P}$.

First, let $B \in A$. We shall define a set $D_{B}$, whose "job" is to ensure $B$ ends up in $\sigma^{g}$.

$$
D_{B}=\left\{p \in \mathbb{P}:(p \Vdash \sigma \neq \check{A}) \vee\left(\exists \gamma<\kappa\left(p \Vdash \sigma_{\gamma}=\check{B}\right) \wedge\left(p \Vdash^{+} \sigma_{\gamma} \in \sigma\right)\right)\right\}
$$

$D_{B}$ is dense: if we take $p \in \mathbb{P}$ then either we can find $r \leqslant p$ with $r \Vdash \sigma \neq \check{A}$, or else $p \Vdash \sigma=\check{A}$. In the first case, we're done. In the second, given any (truly) generic filter $G$ containing $p$, there will be some $\gamma<\kappa$ and
$q \in G$ such that ${ }^{3} \sigma_{\gamma}^{G}=B$ and $\left(\sigma_{\gamma}, q\right) \in \sigma$, so $q \Vdash^{+} \sigma_{\gamma} \in \sigma$. Take $r \in G$ such that $r \Vdash \sigma_{\gamma}=\check{B}$, and take $s$ below $p, q$ and $r$ by compatibility; then $s \in D_{B}$.

Now let $\gamma<\kappa$. In a similar way, we define a set $E_{\gamma}$, which is designed to ensure that $\sigma_{\gamma}$ ends up in $A$ if it's going to be in $\sigma$.

$$
E_{\gamma}=\left\{p \in \mathbb{P}:(p \Vdash \sigma \neq \check{A}) \vee\left(p \Vdash \sigma_{\gamma} \notin \sigma\right) \vee\left(\exists B \in A, p \Vdash \sigma_{\gamma}=\check{B}\right)\right\}
$$

Again, $E_{\gamma}$ is dense: Let $p \in \mathbb{P}$. We can assume that $p \Vdash \sigma=\check{A}$ and $p \Vdash \sigma_{\gamma} \in \sigma$; otherwise we're done immediately. But now we can strengthen $p$ to some $r \leqslant p$ which forces $\sigma_{\gamma} \in \check{B}$ for some $B \in A$ and again we're done.

We define

$$
\mathcal{D}_{\sigma=\check{A}}:=\left\{D_{B}: B \in A\right\} \cup\left\{E_{\gamma}: \gamma<\kappa\right\} \cup \bigcup_{\gamma<\kappa} \bigcup_{B \in A} \mathcal{D}_{\sigma_{\gamma}=\check{B}}
$$

Every $\sigma_{\gamma}$ is a $\kappa$-small name of rank less than $\alpha$, and every $B \in H_{\kappa^{+}}$, so this is well defined by inductive hypothesis. By assumption, $|A| \leqslant \kappa$. Hence $\mathcal{D}_{\sigma=\check{A}}$ contains at most $\kappa$ many dense sets. Fix a filter $g$ which meets every element of $\mathcal{D}_{\sigma=\check{A}}$, and which contains some $p$ forcing $\sigma=\check{A}$. We must verify that $\sigma^{g}=A$.

First, let $B \in A$. Find $q \in g \cap D_{B}$, and without loss of generality say $q \leqslant p$. Then clearly $q \Vdash \sigma=\check{A}$, so (by definition of $D_{B}$ ) we can find $\gamma$ such that $q \Vdash \sigma_{\gamma}=\check{B}$ and $q \Vdash^{+} \sigma_{\gamma} \in \sigma$. The latter means that $\sigma_{\gamma}^{g} \in \sigma^{g}$. Since $g$ also meets every element of $\mathcal{D}_{\sigma_{\gamma}=\check{B}}$, the fact that $q \in g$ forces $\sigma_{\gamma}=\check{B}$ implies that $\sigma_{\gamma}^{g}=\check{B}^{g}=B$. Hence $B \in \sigma^{g}$.

Now let $B \in \sigma^{g}$. Then we can find $\gamma<\kappa$ such that $B=\sigma_{\gamma}^{g}$ and such that for some $q \in g$ we have $q \Vdash^{+} \sigma_{\gamma} \in \sigma$. Without loss of generality, say $q \leqslant p$. Then $q \Vdash \sigma=\check{A}$. Let $r \in g \cap E_{\gamma}$, and again without loss of generality say $r \leqslant q$. Then for some $B^{\prime} \in A, r \Vdash \sigma_{\gamma}=\check{B}^{\prime}$. Since $g$ meets every element of $\mathcal{D}_{\sigma_{\gamma}=\tilde{B}^{\prime}}$, this tells us that $\sigma_{\gamma}^{g}=B^{\prime}$. But then $B=\sigma_{\gamma}^{g}=B^{\prime} \in A$.

Hence $\sigma^{g}=A$ as required.
Next, we go up one step in complexity, by allowing both sides of the equality to be nontrivial.
Claim 2.3.9. The lemma holds when $\varphi$ has the form $\sigma=\tau$ for two ( $\kappa$-small) names $\sigma$ and $\tau$.
Proof. We use induction on the ranks of $\sigma$ and $\tau$. Without loss of generality, let us assume the rank of $\sigma$ is $\alpha$, and the $\operatorname{rank}$ of $\tau$ is $\leqslant \alpha$. If $\operatorname{rank}(\tau)=0$ then $\tau$ is a check name. Since $\tau$ is $\kappa$-small, it can only be a check name for some $A \in H_{\kappa^{+}}$, so we are already done by the previous claim. So suppose $\operatorname{rank}(\sigma)=\alpha \geqslant \operatorname{rank}(\tau)>0$, and the result is proven for all $\tau^{\prime}, \sigma^{\prime}$ where $\operatorname{rank}\left(\sigma^{\prime}\right)<\operatorname{rank}(\sigma)$ and $\operatorname{rank}\left(\tau^{\prime}\right)<\operatorname{rank}(\tau)$.

Let us write $\sigma=\left\{\left(\sigma_{\gamma}, p\right): \gamma<\kappa, p \in S_{\gamma}\right\}$ and $\tau=\left\{\left(\tau_{\delta}, q\right): \delta<\kappa, q \in T_{\delta}\right\}$.
For $\gamma \in \kappa$, we define a set $D_{\gamma}$, whose job is to ensure that if $\sigma_{\gamma}$ ends up being put in $\sigma$ by $g$, then it will also be equal to some element of $\tau$.

$$
\begin{aligned}
D_{\gamma}=\{p \in \mathbb{P}: & (p \Vdash \sigma \neq \tau) \vee\left(p \Vdash \sigma_{\gamma} \notin \sigma\right) \\
& \left.\vee \exists \delta<\kappa\left(\left(p \Vdash \sigma_{\gamma}=\tau_{\delta}\right) \wedge\left(p \Vdash^{+} \tau_{\delta} \in \tau\right)\right)\right\}
\end{aligned}
$$

 one of these statements, and we are done. If $p \Vdash \sigma_{\gamma} \in \sigma \wedge \sigma=\tau$ then take a generic filter $G$ containing $p$. We know $\sigma_{\gamma}^{G} \in \tau^{G}$, so $\sigma_{\gamma}^{G}=\tau_{\delta}^{G}$ for some $\tau_{\delta}$ which is strongly forced to be in $\tau$ by some $q \in G$. Then take $r \in G$ below $p$ and $q$, and we know $r \Vdash \sigma_{\gamma}=\tau_{\delta}$ and $r \Vdash^{+} \tau_{\delta} \in \tau$. Hence $r \in D_{\gamma}$.

Symmetrically, for $\delta<\kappa$ let

$$
\begin{aligned}
E_{\delta}=\{p \in \mathbb{P}: & (p \Vdash \sigma \neq \tau) \vee\left(p \Vdash \tau_{\delta} \notin \tau\right) \\
& \left.\vee \exists \gamma<\kappa\left(\left(p \Vdash \sigma_{\gamma}=\tau_{\delta}\right) \wedge\left(p \Vdash^{+} \sigma_{\gamma} \in \sigma\right)\right)\right\}
\end{aligned}
$$

[^5]Again, $E_{\delta}$ is dense.
We now let

$$
\mathcal{D}_{\sigma=\tau}:=\left\{D_{\gamma}: \gamma<\kappa\right\} \cup\left\{E_{\delta}: \delta<\kappa\right\} \cup \bigcup_{\gamma, \delta<\kappa} \mathcal{D}_{\sigma_{\gamma}=\tau_{\delta}}
$$

Note that for all $\sigma, \delta<\kappa$, we know $\operatorname{rank}\left(\sigma_{\gamma}\right)<\operatorname{rank}(\sigma)$ and $\operatorname{rank}\left(\tau_{\delta}\right)<\operatorname{rank}(\tau)$, so $\mathcal{D}_{\sigma_{\gamma}=\tau_{\delta}}$ is already defined. Clearly, $\mathcal{D}_{\sigma=\tau}$ contains at most $\kappa$ many dense sets. Let $g$ be a filter meeting every element of it, and let $p \in g$ force $\sigma=\tau$.

Suppose $B \in \sigma^{g}$. Then for some $q \in g$ and $\gamma<\kappa, B=\sigma_{\gamma}^{g}$ and $q \Vdash^{+} \sigma_{\gamma} \in \sigma$ (and hence $q \Vdash \sigma_{\gamma} \in \sigma$ ). We can also find some $r \in g \cap D_{\gamma}$. Without loss of generality, say $r$ is below both $p$ and $q$. Certainly $r$ cannot force $\sigma \neq \tau$, nor that $\sigma_{\gamma} \notin \sigma$. Hence, for some $\delta<\kappa$, we know $r \Vdash \sigma_{\gamma}=\tau_{\delta}$ and $r \Vdash^{+} \tau_{\delta} \in \tau$. But then $\tau_{\delta}^{g} \in \tau^{g}$, and since $g$ meets every element of $\mathcal{D}_{\sigma_{\gamma}=\tau_{\delta}}$, we also know that $B=\sigma_{\gamma}^{g}=\tau_{\delta}^{g}$. Hence $B \in \tau$.

Hence $\sigma^{g} \subseteq \tau^{g}$, and by a symmetrical argument $\tau^{g} \subseteq \sigma^{g}$.
Claim 2.3.10. The lemma holds when $\varphi$ has the form $\tau \in \sigma$.
Proof. Write $\sigma=\left\{\left(\sigma_{\gamma}, p\right): \gamma<\kappa, p \in S_{\gamma}\right\}$ as usual. Let

$$
D=\left\{p \in \mathbb{P}:(p \Vdash \tau \notin \sigma) \vee \exists \gamma<\kappa\left(\left(p \Vdash \tau=\sigma_{\gamma}\right) \wedge\left(p \Vdash^{+} \sigma_{\gamma} \in \sigma\right)\right)\right\}
$$

As usual, $D$ is dense. Let

$$
\mathcal{D}_{\tau \in \sigma}:=\{D\} \cup \bigcup_{\gamma<\kappa} \mathcal{D}_{\tau=\sigma_{\gamma}}
$$

Let $g$ meet every element of $\mathcal{D}_{\tau \in \sigma}$ and contain some $p$ forcing $\tau \in \sigma$. Let $q \in g \cap D$, and assume $q \leqslant p$. Then for some $\gamma, q \Vdash \tau=\sigma_{\gamma}$ and $q \Vdash^{+} \sigma_{\gamma} \in \sigma$, so $\sigma_{\gamma}^{g} \in \sigma^{g}$. Since $g$ meets every element of $\mathcal{D}_{\tau=\sigma_{\gamma}}$ we know $\tau^{g}=\sigma_{\gamma}^{g} \in \sigma^{g}$.

We next need to prove similar claims about the negations of all these formulas.
Claim 2.3.11. The lemma holds when $\varphi$ is of the form $\sigma \neq \check{A}$ for $A \in H_{\kappa}$.
Proof. As before, this is trivial is $\sigma$ is rank 0 . Otherwise, let us write $\sigma=\left\{\left(\sigma_{\gamma}, p\right): \gamma<\kappa, p \in S_{\gamma}\right\}$ and let

$$
\begin{aligned}
& D=\left\{p \in \mathbb{P}:(p \Vdash \sigma=\check{A}) \vee\left(\exists \gamma<\kappa\left(p \Vdash^{+} \sigma_{\gamma} \in \sigma\right) \wedge\left(p \Vdash \sigma_{\gamma} \nsubseteq \check{A}\right)\right)\right. \\
&\vee(\exists B \in A: p \Vdash \check{B} \notin \sigma)\}
\end{aligned}
$$

As usual, $D$ is dense.
We then let

$$
\mathcal{D}_{\sigma \neq \check{A}}:=\{D\} \cup \bigcup_{\gamma<\kappa} \bigcup_{B \in A} \mathcal{D}_{\sigma_{\gamma} \neq \check{B}}
$$

By induction, this is well defined, and since $A$ is in $H_{\kappa^{+}}$it has cardinality at most $\kappa$. Let $g$ be a filter meeting all of $\mathcal{D}_{\sigma \neq \check{A}}$ with $p \in g$ forcing $\sigma \neq \check{A}$. Take $q \in g \cap D$ below $p$. There are two cases to consider.
(1) For some $\gamma, q \Vdash^{+} \sigma_{\gamma} \in \sigma$ and $q \Vdash \sigma_{\gamma} \notin \check{A}$. Then certainly $\sigma_{\gamma}^{g} \in \sigma^{g}$. Let $B \in A$. Then $q \Vdash \sigma_{\gamma} \neq \check{B}$. Since $g$ meets all of $\mathcal{D}_{\sigma_{\gamma} \neq B}$, we know $\sigma_{\gamma}^{g} \neq B$. Hence $\sigma_{\gamma}^{g} \in \sigma^{g} \backslash A$ so $\sigma^{g} \neq A$.
(2) For some $B \in A, q \Vdash \check{B} \notin \sigma$. Let $B^{\prime} \in \sigma^{g}$. Then for some $\gamma<\kappa$ and $r \leqslant q$ in $g, \sigma_{\gamma}^{g}=B^{\prime}$ and $r \Vdash^{+} \sigma_{\gamma} \in \sigma$. Hence $r \Vdash \sigma_{\gamma} \in \sigma$. But also $r \Vdash \check{B} \notin \sigma$ since $r \leqslant q$. Therefore $r \Vdash \sigma_{\gamma} \neq \check{B}$, and so $B^{\prime}=\sigma_{\gamma}^{g} \neq B$ since $g$ meets $\mathcal{D}_{\sigma_{\gamma} \neq \check{B}}$. Hence $B \in A \backslash \sigma^{g}$, so again $\sigma^{g} \neq A$.

Claim 2.3.12. The lemma holds when $\varphi$ is of the form $\sigma \neq \tau$.

Proof. The dense sets we need to use are very similar to the ones in the previous lemma. We assume $\operatorname{rank}(\sigma) \geqslant \operatorname{rank}(\tau)$ and note that if $\operatorname{rank}(\tau)=0$ we're looking at the previous case. So let us assume $\operatorname{rank}(\sigma) \geqslant \operatorname{rank}(\tau)>0$ and that we have proved the statement for all $\sigma^{\prime}$ and $\tau^{\prime}$ with lower ranks than $\sigma$ and $\tau$ respectively. As usual, write $\sigma=\left\{\left(\sigma_{\gamma}, p\right): \gamma<\kappa, p \in S_{\gamma}\right\}$ and $\tau=\left\{\left(\tau_{\delta}, q\right): \delta<\kappa, q \in T_{\gamma}\right\}$.

Let

$$
\begin{gathered}
D=\left\{p \in \mathbb{P}:(p \Vdash \sigma=\tau) \vee\left(\exists \gamma<\kappa\left(p \Vdash^{+} \sigma_{\gamma} \in \sigma\right) \wedge\left(p \Vdash \sigma_{\gamma} \notin \tau\right)\right)\right. \\
\left.\vee\left(\exists \delta<\kappa\left(p \Vdash^{+} \tau_{\delta} \in \tau\right) \wedge\left(p \Vdash \tau_{\delta} \notin \sigma\right)\right)\right\}
\end{gathered}
$$

Once again $D$ is dense. We define

$$
\mathcal{D}_{\sigma \neq \tau}:=\{D\} \cup \bigcup_{\gamma, \delta<\kappa} \mathcal{D}_{\sigma_{\gamma} \neq \tau_{\delta}}
$$

Letting $g$ be our usual filter meeting all of $\mathcal{D}_{\sigma \neq \tau}$ and containing some $p$ forcing $\sigma \neq \tau$, we can find $q \in g \cap D$ below $p$. Without loss of generality, there exists $\gamma<\kappa$ such that $q \Vdash^{+} \sigma_{\gamma} \in \sigma$ and $q \Vdash \sigma_{\gamma} \notin \tau$. As always, the first statement implies $\sigma_{\gamma}^{g} \in \sigma^{g}$. If $\sigma_{\gamma}^{g} \in \tau^{g}$ then for some $\delta<\kappa$ and $r \in g$ (which we can take to be below $q$ ), $\sigma_{\gamma}^{g}=\tau_{\delta}^{g}$ and $r \Vdash^{+} \tau_{\delta} \in \tau$. But then we know $r \Vdash \sigma_{\gamma} \neq \tau_{\delta}$. Since $g$ meets all of $\mathcal{D}_{\sigma_{\gamma} \neq \tau_{\delta}}$ this implies $\sigma_{\gamma}^{g} \neq \tau_{\gamma}^{g}$. Contradiction. Hence $\sigma_{\gamma}^{g} \in \sigma^{g} \backslash \tau^{g}$, so $\sigma^{g} \neq \tau^{g}$.

Claim 2.3.13. The lemma holds when $\varphi$ has the form $\tau \notin \sigma$.
Proof. Write $\sigma=\left\{\left(\sigma_{\gamma}, p\right): \gamma<\kappa, p \in S_{\gamma}\right\}$ as usual. Let

$$
\mathcal{D}_{\tau \notin \sigma}:=\bigcup_{\gamma<\kappa} \mathcal{D}_{\tau \neq \sigma_{\gamma}}
$$

Suppose $g$ meets all of $\mathcal{D}_{\tau \notin \sigma}$ and contains some $p$ forcing $\tau \notin \sigma$. Let $B \in \sigma^{g}$. For some $\gamma<\kappa$ and some $q \in g$ below $p, B=\sigma_{\gamma}^{g}$ and $q \Vdash^{+} \sigma_{\gamma} \in \sigma$. Then $q \Vdash \tau \neq \sigma_{\gamma}$, so $\tau^{g} \neq \sigma_{\gamma}^{g}=B$. Hence $\tau^{g} \notin \sigma^{g}$.

We can now finally prove the full lemma.
Claim 2.3.14. The lemma holds in all cases.
Proof. We use induction on the length of the formula $\varphi$. By rearranging $\varphi$, we can assume that all the $\neg$ 's in $\varphi$ are in front of atomic formulas. Throughout this proof, we will suppress the irrelevant variables $\vec{\sigma}$ of formulas $\psi(\vec{\sigma})$, and will write $\psi^{g}$ to denote $\psi\left(\vec{\sigma}^{g}\right)$.

The base case, where $\varphi$ is either atomic or the negation of an atomic formula, was covered in the previous lemmas.
$\varphi=\psi \wedge \chi$ : We let $\mathcal{D}_{\varphi}:=\mathcal{D}_{\psi} \cup \mathcal{D}_{\chi}$. If $p \in g$ forces $\varphi$ then it also forces $\psi$ and $\chi$, so if also $g$ meets all of $\mathcal{D}_{\varphi}$ then $\psi^{g}$ and $\chi^{g}$ hold.
$\varphi=\psi \vee \chi$ : We let $D=\{p \in \mathbb{P}:(p \Vdash \neg \varphi) \vee(p \Vdash \psi) \vee(p \Vdash \chi)\}$, and let $\mathcal{D}_{\varphi}:=\{D\} \cup \mathcal{D}_{\psi} \cup \mathcal{D}_{\chi}$. If $g$ meets all of $\mathcal{D}_{\varphi}$ and contains some $p$ which forces $\varphi$ then take $q \leqslant p$ in $g \cap D$. Then $q \Vdash \psi$ or $q \Vdash \chi$, and by definition of $\mathcal{D}_{\psi}$ and $\mathcal{D}_{\chi}$ this implies $\psi^{g}$ or $\chi^{g}$ respectively.
$\varphi=\forall x \in \sigma \psi(x):$ Write $\sigma=\left\{\left(\sigma_{\gamma}, p\right): \gamma<\kappa, p \in S_{\gamma}\right\}$, and let $\mathcal{D}_{\varphi}:=\bigcup_{\gamma<\kappa} \mathcal{D}_{\psi\left(\sigma_{\gamma}\right)}$. Suppose, as usual, that $g$ meets all of $\mathcal{D}_{\varphi}$ and contains some $p$ forcing $\varphi$. Let $B \in \sigma^{g}$. Then we have some $\gamma<\kappa$ and $q \in g$ such that $\sigma_{\gamma}^{g}=B$ and $q \Vdash^{+} \sigma_{\gamma} \in \sigma$. Taking (without loss of generality) $q \leqslant p$, we then have that $q \Vdash \psi\left(\sigma_{\gamma}\right)$. Hence $\psi^{g}\left(\sigma_{\gamma}^{g}\right)$ holds. But we know $\sigma_{\gamma}^{g}=B$. Hence $\psi^{g}(B)$ holds for all $B \in \sigma^{g}$, so $\varphi^{g}$ holds.
$\varphi=\exists x \in \sigma \psi(x)$ : Again we write $\sigma=\left\{\left(\sigma_{\gamma}, p\right): \gamma<\kappa, p \in S_{\gamma}\right\}$. Let $D$ be the dense set $\{p \in \mathbb{P}:(p \Vdash$ $\left.\neg \varphi) \vee \exists \gamma<\kappa\left(p \Vdash^{+} \sigma_{\gamma} \in \sigma \wedge p \Vdash \psi\left(\sigma_{\gamma}\right)\right)\right\}$, and let $\mathcal{D}_{\varphi}:=\{D\} \cup \bigcup_{\gamma<\kappa} \mathcal{D}_{\psi\left(\sigma_{\gamma}\right)}$. If $g$ meets all of $\mathcal{D}_{\varphi}$ and contains $p$ forcing $\varphi$ then we can take some element $q$ of $g \cap D$ below $p$. Then for some $\gamma<\kappa$, we know $q \Vdash \psi\left(\sigma_{\gamma}\right)$ and $q \Vdash^{+} \sigma_{\gamma} \in \sigma$. Then $\psi^{g}\left(\sigma_{\gamma}^{g}\right)$ holds, and $\sigma_{\gamma}^{g} \in \sigma^{g}$.

This completes the proof of Lemma 2.3.7. Hence $\mathrm{FA}_{\mathbb{P}, \kappa}$ implies $\Sigma_{0}^{(\operatorname{sim})}-\mathrm{N}_{\mathbb{P}, \kappa}(\infty)$, as discussed earlier.
This completes the proof of Theorem 2.3.1.

In fact, this proof works even if we allow formulas to have conjunctions and disjunctions of $\kappa$ many formulas (and accordingly let formulas have $\kappa$ many variables).

The proof of Theorem 2.3.2 is essentially the same:
Proof of Theorem 2.3.2. We prove all the implications in the following diagram.


Note that $\mathrm{BFA}_{\mathbb{P}, \kappa}^{\lambda} \Rightarrow \Sigma_{0}^{(\operatorname{sim})}-\mathrm{BN}_{\mathbb{P}, X, \kappa}^{\lambda}(\alpha)$ for $\kappa \leqslant \lambda$ follows from the rest of the diagram.
Proof of $\Sigma_{0}^{(\operatorname{sim})}-\mathrm{BN}_{\mathbb{P}, \kappa}^{\lambda}(\infty) \Rightarrow \Sigma_{0}^{(\operatorname{sim})}-\mathrm{BN}_{\mathbb{P}, \kappa}^{\lambda}(\alpha)$ and $\mathrm{BN}_{\mathbb{P}, \kappa}^{\lambda}(\infty) \Rightarrow \mathrm{BN}_{\mathbb{P}, \kappa}^{\lambda}(\alpha)$.
The latter are special cases of the former.
Proof of $\Sigma_{0}^{(\operatorname{sim})}-\mathrm{BN}_{\mathbb{P}, X, \kappa}^{\lambda}(\alpha) \Rightarrow \mathrm{BN}_{\mathbb{P}, X, \kappa}^{\lambda}(\alpha)$ and $\Sigma_{0}^{(\operatorname{sim})}-\mathrm{BN}_{\mathbb{P}, \kappa}^{\lambda}(\infty) \Rightarrow \mathrm{BN}_{\mathbb{P}, \kappa}^{\lambda}(\infty)$.
As before, similar to the proofs in Theorems 2.3.1.
Proof of $\mathrm{BN}_{\mathbb{P}, X, \kappa}^{\lambda}(\alpha) \Rightarrow \mathrm{BFA}_{\mathbb{P}, \kappa}^{\lambda}$ and $\mathrm{BN}_{\mathbb{P}, \kappa}^{\lambda}(\infty) \Rightarrow \mathrm{BFA}_{\mathbb{P}, \kappa}^{\lambda}$. Letting $\left\langle D_{\gamma}: \gamma<\kappa\right\rangle$ be a sequence of predense sets of cardinality at most $\lambda$, we define a name $\sigma$ exactly as in the corresponding proof from Theorem 2.3.1. Since the $D_{\gamma}$ have cardinality at most $\lambda$, and all the names that appear in $\sigma$ are 1 bounded check names, $\sigma$ is $\lambda$-bounded.

As in the earlier proof, a filter $g$ such that $\sigma^{g}=A$ will meet all of the $D_{\gamma}$.
Proof of $\mathrm{BFA}_{\mathbb{P}, \kappa}^{\lambda} \Rightarrow \Sigma_{0}^{(\operatorname{sim})}-\mathrm{BN}_{\mathbb{P}, \kappa}^{\lambda}$. Assume $\lambda \geqslant \kappa$. We prove the following lemma (very similar to Lemma 2.3.7).

Lemma 2.3.15. Let $\varphi(\vec{\sigma})$ be a $\Sigma_{0}$ formula where $\vec{\sigma}$ is a tuple of $\kappa$-small $\lambda$-bounded names. Then there is a collection $\mathcal{D}_{\varphi(\vec{\sigma})}$ of at most $\kappa$ many predense sets each of cardinality at most $\lambda$, which has the following property: if $g$ is any filter meeting every set in $\mathcal{D}_{\varphi(\vec{\sigma})}$ and $g$ contains some $p$ such that $p \Vdash \varphi(\vec{\sigma})$, then in fact $\varphi\left(\overrightarrow{\sigma^{g}}\right)$ holds in $V$.

We use the same proof as in Theorem 2.3.1, adjusting the dense sets we work with. Whenever a dense set appears, we will replace it with a predense set of size at most $\lambda$ which fulfills all the same functions. To obtain these sets, we use a few techniques.

First, whenever the original proof calls for an arbitrary condition which forces some desirable property, we replace it with the supremum of all such conditions (exploiting the fact that we are in a complete Boolean algebra).

For example, in place of

$$
E_{\gamma}=\left\{p \in \mathbb{P}:(p \Vdash \sigma \neq \check{A}) \vee\left(p \Vdash \sigma_{\gamma} \notin A\right) \vee\left(\exists B \in A, p \Vdash \sigma_{\gamma}=\check{B}\right)\right\}
$$

in Claim 2.3.8, we would take the set

$$
E_{\gamma}^{*}:=\left\{q_{0}, q_{1}\right\} \cup\left\{q_{B}: B \in A\right\}
$$

where

$$
\begin{aligned}
& q_{0}=\sup \{p \in \mathbb{P}: p \Vdash \sigma \neq \check{A}\} \\
& q_{1}=\sup \left\{p \in \mathbb{P}: p \Vdash \sigma_{\gamma} \notin \check{A}\right\}
\end{aligned}
$$

and for $B \in A$,

$$
q_{B}=\sup \left\{p \in \mathbb{P}: p \Vdash \sigma_{\gamma}=\check{B}\right\}
$$

$E_{\gamma}^{*}$ has cardinality at most $\lambda$, since $|A| \leqslant \kappa \leqslant \lambda$.
When the original set calls for a condition which strongly forces $\tau \in \sigma$ for some $\tau$ and $\sigma$, simply taking suprema won't work. Instead, we ask for a condition $q$ such that $(\tau, q) \in \sigma$. Since all the names $\sigma$ we deal with in the proof are $\lambda$-bounded, there will be at most $\lambda$ many such conditions.

For example, in the same claim,

$$
D_{B}:=\left\{p \in \mathbb{P}:(p \Vdash \sigma \neq \check{A}) \vee\left(\exists \gamma<\kappa\left(p \Vdash \sigma_{\gamma}=\check{B}\right) \wedge\left(p \Vdash^{+} \sigma_{\gamma} \in \sigma\right)\right)\right\}
$$

will be replaced by

$$
D_{B}^{*}:=\{r\} \cup\left\{r_{\gamma, q}: \gamma<\kappa, q \in \mathbb{P},\left(\sigma_{\gamma}, q\right) \in \sigma, r_{\gamma, q} \neq 0\right\}
$$

where

$$
r=\sup \{p \in \mathbb{P}: p \Vdash \sigma \neq \check{A}\}
$$

and for $\gamma<\kappa, q \in \mathbb{P}$,

$$
r_{\gamma, q}=\sup \left\{p \leqslant q: p \Vdash \sigma_{\gamma}=\check{B}\right\} .
$$

Checking that we can indeed apply these techniques to turn all the dense sets in the proof into predense sets of cardinality at most $\lambda$ is left as an exercise for the particularly thorough reader.

This completes the proof of Theorem 2.3.2.

### 2.3.3 Generic absoluteness

In this section, we derive generic absoluteness principles from the above correspondence.
Fix a cardinal $\kappa$. We start by defining the class of $\Sigma_{1}^{1}(\kappa)$-formulas. To this end, work with a twosorted logic with two types of variables, interpreted as ranging over ordinals below $\kappa$ and over subsets of $\kappa$, respectively. The language contains a binary relation symbol $\epsilon$ and a binary function symbol $p$ for a pairing function $\kappa \times \kappa \rightarrow \kappa$. Thus, atomic formulas are of the form $\alpha=\beta, x=y, \alpha \in x$ and $p(\alpha, \beta)=\gamma$, where $\alpha, \beta, \gamma$ denote ordinals and $x, y$ denote subsets of $\kappa$.

Definition 2.3.16. A $\Sigma_{1}^{1}(\kappa)$ formula is of the form

$$
\exists x_{0}, \ldots, x_{m} \varphi\left(x_{0}, \ldots, x_{m}, y_{0}, \ldots, y_{n}\right),
$$

where the $x_{i}$ are variables for subsets of $\kappa$, the $y_{i}$ are either type of variables, and $\varphi$ is a formula which only quantifies over variables for ordinals.

As a corollary to the results in Section 2.3.1, we obtain Bagaria's characterisation of bounded forcing axioms [6, Theorem 5] as the equivalence (1) $\Leftrightarrow(4)$ of the next theorem. It also shows that the principles $\Sigma_{0}^{(\operatorname{sim})}-\mathrm{BN}_{\mathbb{P}, \kappa}^{\lambda}$ for $\lambda<\kappa$ are all equivalent to $\mathrm{BFA}_{\mathbb{P}, \kappa}^{\kappa}$.
Theorem 2.3.17. Suppose that $\kappa$ is a cardinal with $\operatorname{Cof}(\kappa)>\omega, \mathbb{P}$ is a complete Boolean algebra and $\dot{G}$ is $a \mathbb{P}$-name for the generic filter. Then the following conditions are equivalent: ${ }^{4}$
(1) $\mathrm{BFA}_{\mathbb{P}, \kappa}^{\kappa}$
(2) $\Sigma_{0}^{(\text {sim })}-\mathrm{BN}_{\mathbb{P}, \kappa}^{1}(1)^{5}$
(3) $\Vdash_{\mathbb{P}} V<_{\Sigma_{1}^{1}(\kappa)} V[\dot{G}]$
(4) $\Vdash \Vdash_{\mathbb{P}} H_{\kappa^{+}}^{V}<_{\Sigma_{1}} H_{\kappa^{+}}^{V[\dot{G}]}$

[^6]Proof. The implication $(1) \Rightarrow(2)$ holds since $\mathrm{BFA}_{\mathbb{P}, \kappa}^{\kappa} \Leftrightarrow \Sigma_{0}^{(\operatorname{sim})}-\mathrm{BN}_{\mathbb{P}, \kappa}^{\kappa}(1)$ by Theorem 2.3.2 and $\Sigma_{0}^{(\operatorname{sim})}-\mathrm{BN}_{\mathbb{P}, \kappa}^{\kappa}(1)$ implies $\Sigma_{0}^{(\mathrm{sim})}-\mathrm{BN}_{\mathbb{P}, \kappa}^{1}(1)$.
$(2) \Rightarrow(3)$ : To simplify the notation, we will only work with $\Sigma_{1}^{1}(\kappa)$-formulas of the form $\exists x \varphi(x, y)$, where $x$ and $y$ range over subsets of $\kappa$. Suppose that $y$ is a subset of $\kappa$ and $p \Vdash \exists x \varphi(x, \check{y})$. Let $\sigma$ be a $\mathbb{P}$-name with $p \Vdash_{\mathbb{P}} \varphi(\sigma, \check{y})$. Since the variables of $\varphi$ are interpreted as subsets of $\kappa$, this means that $p \Vdash \sigma \subseteq \check{\kappa}$. Let $\tau$ be defined by

$$
\tau:=\{(\check{\alpha}, \llbracket \check{\alpha} \in \sigma \rrbracket): \alpha<\kappa, \llbracket \check{\alpha} \in \sigma \rrbracket \neq 0\} .
$$

Then $\tau$ is a 1 -bounded 1 rank $\kappa$ name with $p \Vdash_{\mathbb{P}} \sigma=\tau$. Note that $\check{y}$ is a 1 -bounded rank 1 name, too. By $\Sigma_{0}^{(\operatorname{sim})}-\mathrm{BN}_{\mathbb{P}, \kappa}^{\kappa}(1)$, there exists a filter $g \in V$ on $\mathbb{P}$ such that $V \models \varphi\left(\sigma^{g}, y\right)$. Hence $V \models \exists x \varphi(x, y)$.

The implication $(3) \Rightarrow(1)$ works just like in the proof of $[6$, Theorem 5$]$. In short, the existence of the required filter is equivalent to a $\Sigma_{1}^{1}(\kappa)$-statement.

For $(3) \Rightarrow(4)$, suppose that $\psi=\exists x \varphi(x, y)$ is a $\Sigma_{1}$-formula with a parameter $y \in H_{\kappa^{+}}$. Then

$$
H_{\kappa^{+}} \models \psi \Longleftrightarrow H_{\kappa^{+}} \models " \exists M \text { transitive s.t. } y \in M \wedge M \models \psi " .
$$

We express the latter by a $\Sigma_{1}^{1}(\kappa)$-formula $\theta$ with a parameter $A \subseteq \kappa$ which codes $y$ in the sense that $f(0)=y$ for the transitive collapse $f$ of $\left(\kappa, p^{-1}[A]\right)$.
$\theta$ states the existence of a subset $B$ of $\kappa$ such that $\epsilon_{M}:=p^{-1}[B]$ has the following properties:

- $\epsilon_{M}$ is wellfounded and extensional
- For all $\alpha<\beta<\kappa, 2 \cdot \alpha \epsilon_{M} 2 \cdot \beta$ and for all $\alpha, \beta<\kappa, 2 \cdot \alpha+1 \not \ddagger_{M} 2 \cdot \beta$.
- There is some $\hat{\kappa}<\kappa$ with $\left\{\alpha<\kappa: \alpha \in_{M} \hat{\kappa}\right\}=\{2 \cdot \alpha: \alpha<\kappa\}$
- There exists some $\hat{A}<\kappa$ such that for all $\beta<\kappa, \beta \in_{M} \hat{A} \Leftrightarrow \exists \alpha \in A 2 \cdot \alpha=\beta$
- There exists some $\hat{y}<\kappa$ such that in $\left(\kappa, \in_{M}\right), \hat{A}$ codes $\hat{y}$
- $\varphi(\hat{y})$ holds in $\left(\kappa, \in_{M}\right)$

The transitive collapse $f$ of $\left(\kappa, \in_{M}\right)$ to a transitive set $M$ will satisfy $f(2 \cdot \alpha)=\alpha$ for all $\alpha<\kappa, f(\hat{\kappa})=\kappa$, $f(\hat{A})=A, f(\hat{y})=y$ and $M \models \psi(y)$.

All the above conditions apart from wellfoundedness of $\epsilon_{M}$ are first order over $\left(\kappa, \epsilon, p, A, \epsilon_{M}\right)$. It remains to express wellfoundedness of $\epsilon_{M}$ in a $\Sigma_{1}^{1}(\kappa)$ way. ${ }^{6}$ To see that we can do this, suppose that $R$ is a binary relation on $\kappa$. Since $\operatorname{Cof}(\kappa)>\omega, R$ is wellfounded if and only if for all $\gamma<\kappa, R \upharpoonright \gamma$ is wellfounded. Since $\gamma<\kappa, R \upharpoonright \gamma$ is wellfounded if and only if there exists a map $f: \gamma \rightarrow \kappa$ such that for all $\alpha, \beta<\gamma,(\alpha, \beta) \in R \Rightarrow$ $f(\alpha)<f(\beta)$. The existence of such a map $f$ is a $\Sigma_{1}^{1}(\kappa)$ statement.

Finally, $(4) \Rightarrow(3)$ holds since every $\Sigma_{1}^{1}(\kappa)$-formula is equivalent to a $\Sigma_{1}$-formula over $H_{\kappa^{+}}$with parameter $\kappa$.

Remark 2.3.18. Note that for rank $1, \Sigma_{0}^{(\operatorname{sim})}-\mathrm{BN}_{\mathbb{P}, \kappa}^{\lambda}(1)$ implies the simultaneous $\lambda$-bounded rank 1 name principle for all $\Sigma_{1}^{1}(\kappa)$-formulas (see Definition 2.1.15) by picking 1-bounded names for witnesses.

Remark 2.3.19. The previous results cannot be extended to higher complexity. To see this, recall that a $\Pi_{1}^{1}(\kappa)$-formula is the negation of a $\Sigma_{1}^{1}(\kappa)$-formula. We claim that there exists a $\Pi_{1}^{1}\left(\omega_{1}\right)$-formula $\varphi(x)$ such that the 1-bounded rank $1 \Pi_{1}^{1}\left(\omega_{1}\right)$-name principle for the class of c.c.c. forcings fails. Otherwise $\mathrm{MA}_{\omega_{1}}$ would hold by $(2) \Rightarrow(1)$ of Theorem 2.3.17. So in particular, there are no Suslin trees. Since adding a Cohen real adds a Suslin tree, let $\sigma$ be a 1-bounded rank $1 \mathbb{P}$-name for it, where $\mathbb{P}$ denotes the Boolean completion of Cohen forcing, and apply the name principle to the statement " $\sigma$ is a Suslin tree". But then we would have a Suslin tree in $V$.

Remark 2.3.20. Fuchs and Minden show in [17, Theorem 4.21] assuming CH that the bounded subcomplete forcing axiom BSCFA can be characterised by the preservation of ( $\omega_{1}, \leqslant \omega_{1}$ )-Aronszajn trees. The latter can be understood as the 1-bounded name principle for statements of the form " $\sigma$ is an $\omega_{1}$-branch in $T$ ", where $T$ is an $\left(\omega_{1}, \leqslant \omega_{1}\right)$-Aronszajn tree. (See $[17,22]$ for more about subcomplete forcing.)

[^7]We now consider forcing axioms at cardinals $\kappa$ of countable cofinality. To our knowledge, these have not been studied before. $\mathrm{BFA}_{\text {c.c.c., } \kappa}^{\kappa}=\mathrm{MA}_{\kappa}$ is an example of a consistent forcing axiom of this form. We fix some notation. If $\kappa$ is an uncountable cardinals with $\operatorname{Cof}(\kappa)=\mu$, we fix a continuous strictly increasing sequence $\left\langle\kappa_{i}: i \in \mu\right\rangle$ of ordinals with $\kappa_{0}=0$ and $\sup _{i \in \mu} \kappa_{i}=\kappa$. We assume that each $\kappa_{i}$ is closed under the pairing function $p .{ }^{7}$ For each $x \in 2^{\kappa}$, we define a function $f_{x}: \mu \rightarrow 2^{<\kappa}$ by letting $f_{x}(i)=x \upharpoonright \kappa_{i}$.

Lemma 2.3.21. Suppose that $\kappa$ is an uncountable cardinal with $\operatorname{Cof}(\kappa)=\mu$. Suppose that $\varphi(x, y)$ is a formula with quantifiers ranging over $\kappa$ and $y \in 2^{\kappa}$ is fixed. Then there is a subtree $T \in V$ of $\left(\left(2^{<\kappa}\right)^{<\mu}\right)^{2}$ such that in all generic extensions $V[G]$ of $V^{8}$ which do not add new bounded subsets of $\kappa$,

$$
\varphi(x, y) \Longleftrightarrow \exists g \in\left(2^{<\kappa}\right)^{\mu}\left(f_{x}, g\right) \in[T]
$$

holds for all $x \in\left(2^{\kappa}\right)^{V[G]}$. Moreover, for any branch $(\vec{s}, \vec{t}) \in[T]$ in $V[G]$ with $\vec{s}=\left\langle s_{i}: i \in \mu\right\rangle, \bigcup_{i \in \mu} s_{i}=f_{x}$ for some $x \in\left(2^{\kappa}\right)^{V[G]}$.

Proof. We construct the $i$-th levels $\operatorname{Lev}_{i}(T)$ by induction on $i \in \mu$. Let $\operatorname{Lev}_{0}(T)=\{(\varnothing, \varnothing)\}$. If $j \in \mu$ is a limit, let $(\vec{s}, \vec{t}) \in \operatorname{Lev}_{j}(T)$ if $(\vec{s} \upharpoonright i, \vec{t} \upharpoonright i) \in \operatorname{Lev}_{i}(T)$ for all $i<j$.

For the successor step, suppose that $\operatorname{Lev}_{j}(T)$ has been constructed. Write $\vec{s}=\left\langle s_{i}: i \leqslant j\right\rangle$ and $\vec{t}=\left\langle t_{i}\right.$ : $i \leqslant j\rangle$. Let $(\vec{s}, \vec{t}) \in \operatorname{Lev}_{j+1}(T)$ if the following conditions hold:
(1) $(\vec{s} \upharpoonright j, \vec{t} \upharpoonright j) \in \operatorname{Lev}_{j}(T)$.
(2) $s_{j} \in 2^{\kappa_{j}}$ and $\forall i<j s_{j} \upharpoonright \kappa_{i}=s_{i}$.
(3) $t_{j} \in 2^{\kappa_{j}}$ codes the following two objects.
(i) A truth table $p_{j}$ which assigns to each formula $\psi\left(\xi_{0}, \ldots, \xi_{k}\right)$ and parameters $\alpha_{0}, \ldots, \alpha_{k}<\kappa_{j}$ a truth value 0 or 1 .
(ii) A function $q_{j}$ which assigns a value in $\omega$ to each existential formula $\exists \eta \psi\left(\xi_{0}, \ldots, \xi_{k}, \eta\right)$ and associated parameters $\alpha_{0}, \ldots, \alpha_{k}<\kappa_{j}$.

They satisfy $p_{i} \subseteq p_{j}, q_{i} \subseteq q_{j}=q_{i}$ for all $i<j$ and the following conditions:
(a) $p_{j}(\varphi)=1$.
(b) $p_{j}$ satisfies the equality axioms:

$$
p_{j}((\psi(\vec{\xi})), \vec{\alpha})=1 \wedge \vec{\alpha}=\vec{\beta} \Longleftrightarrow p_{j}((\psi(\vec{\xi})), \vec{\beta})=1
$$

(c) $p_{j}$ is correct about atomic formulas $\psi(\vec{\xi})$ which do not mention $\dot{x}$ and $\dot{y}$ :

$$
p_{j}((\psi(\vec{\xi})), \vec{\alpha})=1 \Longleftrightarrow \psi(\vec{\alpha})
$$

(d) The truth in $p_{j}$ of all atomic formulas of the form $\xi \in \dot{x}, \xi \in \dot{y}$ is fixed according to $s_{j}$ and $y$, respectively:

$$
\begin{aligned}
p_{j}((\xi \in \dot{x}), \alpha) & =1 \Longleftrightarrow \alpha \in s_{j} \\
p_{j}((\xi \in \dot{y}), \alpha) & =1 \Longleftrightarrow \alpha \in y
\end{aligned}
$$

(e) $p_{j}$ respects propositional connectives:

$$
\begin{gathered}
p_{j}(\psi \wedge \theta, \vec{\alpha})=1 \Longleftrightarrow p_{j}(\psi, \vec{\alpha})=1 \wedge p_{j}(\theta, \vec{\alpha})=1 \\
p_{j}(\neg \psi, \vec{\alpha})=1 \Longleftrightarrow p_{j}(\psi, \vec{\alpha})=0
\end{gathered}
$$

(f) $p_{j}$ respectes witnesses of existential formulas $\left.\exists \eta \psi(\vec{\xi}, \eta), \vec{\alpha}\right)$ which it has identified:

$$
\exists \beta<\kappa_{j} p_{j}(\psi(\vec{\xi}, \eta), \vec{\alpha}, \beta)=1 \Longrightarrow p_{j}(\exists \eta \psi(\vec{\xi}, \eta), \vec{\alpha})=1
$$

[^8](g) $q_{j}$ promises the existence of existential witnesses: for any existential formula $\exists \eta \psi(\vec{\xi}, \eta)$ and any tuple $\vec{\alpha}$ of parameters, if $p_{j}(\exists \eta \psi(\vec{\xi}, \eta), \vec{\alpha})=1$ and $q_{j}(\exists \eta \psi(\vec{\xi}, \eta), \vec{\alpha}) \leqslant n$, then there exists some $\beta<\kappa_{j}$ such that $p_{j}(\psi(\vec{\xi}, \eta), \vec{\alpha}, \beta)=1$.

Let $V[G]$ be a generic extension of $V$ with no new bounded subsets of $\kappa$. Work in $V[G]$.
$\Rightarrow$ : Suppose that $\varphi(x, y)$ holds. We define $s_{j}=x \upharpoonright \kappa_{j}$ for each $j \in \mu$ and $p_{j}(\psi(\vec{\xi}), \vec{\alpha})=1$ if $(\kappa, \in, p, x, y) \models$ $\psi(\vec{\alpha})$. We further define $q_{j}(\exists \eta \psi(\vec{\xi}, \eta), \vec{\alpha})=0$ if $p_{j}(\exists \eta \psi(\vec{\xi}, \eta), \vec{\alpha})=0$. Otherwise, $q_{j}(\exists \eta \psi(\vec{\xi}, \eta), \vec{\alpha})$ is defined as the least $l \in \mu$ such that for some $\beta<\kappa_{l},(\kappa, \in, p, x, y) \models \psi(\vec{\alpha}, \beta)$. Let $t_{j}$ code $p_{j}$ and $q_{j}$ (via the pairing function $p$ ). Note that $s_{j}, p_{j}$ and $q_{j}$ are in $V$, since $V[G]$ has no new bounded subsets of $\kappa$. Hence $\left\langle\left(s_{j}, t_{j}\right): j \in \mu\right\rangle$ is a branch through $T$.
$\Leftarrow$ : Suppose that $\left\langle\left(s_{j}, t_{j}\right): j \in \mu\right\rangle$ is a branch through $T$. Let $x=\bigcup_{j \in \mu} s_{j}$. By induction on complexity of formulas, $p_{j}$ and $q_{j}$ are correct about $x$ and $y$. Therefore $(\kappa, \epsilon, p, x, y) \models \varphi(x, y)$.

Theorem 2.3.22. Suppose that $\kappa$ is an uncountable cardinal with $\operatorname{Cof}(\kappa)=\omega, \mathbb{P}$ is a complete Boolean algebra and $\dot{G}$ is a $\mathbb{P}$-name for the generic filter. Then the following conditions are equivalent:
(1) $\mathrm{BFA}_{\mathbb{P}, \kappa}^{\kappa}$
(2) $\Sigma_{0}^{(\operatorname{sim})}-\mathrm{BN}_{\mathbb{P}, \kappa}^{1}$
(3) $\Vdash_{\mathbb{P}} V \prec_{\Sigma_{1}^{1}(\kappa)} V[\dot{G}]$

If moreover $2^{<\kappa}=\kappa$ holds, ${ }^{9}$ then the next condition is equivalent to (1), (2) and (3):
(4) $1_{\mathbb{P}}$ forces that no new bounded subset of $\kappa$ are added.

If there exists no inner model with a Woodin cardinal, ${ }^{10}$ then the next condition is equivalent to (1), (2) and (3):
(5) $\Vdash_{-\mathbb{P}} H_{\kappa^{+}}^{V}{<\Sigma_{1}} H_{\kappa^{+}}^{V[\dot{G}]}$

Proof. The proofs of $(1) \Leftrightarrow(2) \Leftrightarrow(3) \Leftarrow(5)$ in Theorem 2.3.17 work for all uncountable cardinals $\kappa$.
$(3) \Rightarrow(4)$ : We assume $2^{<\kappa}=\kappa$. Towards a contradiction, suppose that $V[G]$ is a generic extension that adds a new subset of $\gamma<\kappa$. Note that $2^{\gamma} \leqslant \kappa$. Let $\vec{y}=\left\langle y_{i}: i<2^{\gamma}\right\rangle$ list all subsets of $\gamma$. We define $x \subseteq \gamma \cdot 2^{\gamma} \subseteq \kappa$ by letting $\gamma \cdot i+j \in x \Leftrightarrow j \in y_{i}$. The next formula expresses "there is a new subset of $\gamma<\kappa$ " as a $\Sigma_{1}^{1}(\kappa)$-statement in parameters coding the + and $\cdot$ operations:

$$
\exists z[z \subseteq \gamma \wedge \neg \exists i \forall j<\gamma(j \in z \Leftrightarrow \gamma \cdot i+j \in x)]
$$

This contradicts $\Sigma_{1}^{1}(\kappa)$-absoluteness.
$(4) \Rightarrow(3)$ : Suppose that $\exists x \psi(x, y)$ is a $\Sigma_{1}^{1}(\kappa)$-formula and $y \in\left(2^{\kappa}\right)^{V}$. Let $T$ be a subtree of $\left(\left(2^{<\kappa}\right)^{<\omega}\right)^{2}$ as in Lemma 2.3.21. Let $G$ be $\mathbb{P}$-generic over $V$ with $V[G] \vDash \exists x \psi(x, y)$. $V[G]$ does not have new bounded subsets of $\kappa$ by assumption. Then $[T]$ has a branch in $V[G]$ by the property of $T$ in Lemma 2.3.21. Since wellfoundedness is absolute, $T$ has a branch $\left\langle s_{n}, t_{n}: n \in \omega\right\rangle$ in $V$. Then $\bigcup_{n \in \omega} s_{n}=f_{x}$ for some $x \in 2^{\kappa}$ by the properties of $T$. Since

$$
\psi(x, y) \Longleftrightarrow \exists g\left(f_{x}, g\right) \in[T]
$$

we have $V \models \psi(x, y)$.
$(3) \Rightarrow(5)$ : Note that the implication holds vacuously if $\kappa$ is collapsed in some $\mathbb{P}$-generic extension of $V$. In this case, both (3) and (5) fail, since the statement " $\kappa$ is not a cardinal" is $\Sigma_{1}^{1}(\kappa)$.

We next show: if $q \in \mathbb{P}$ forces that $\kappa^{+}$is preserved, then $q \Vdash H_{\kappa^{+}}^{V} \prec_{\Sigma_{1}} H_{\kappa^{+}}^{V[\dot{G}]}$ holds. To see this, let $G$ be $\mathbb{P}$-generic over $V$ with $q \in G$. Suppose $\psi=\exists x \varphi(x, y)$ is a $\Sigma_{1}$-formula with a parameter $y \in H_{\kappa^{+}}$. We follow the proof of $(3) \Rightarrow(4)$ in Corollary 2.3 .17 to construct a $\Sigma_{1}^{1}(\kappa)$-formula $\theta$ that is equivalent to $\psi$. However, we replace the first condition by:

[^9]- $\epsilon_{M}$ is extensional and wellfounded of rank $\gamma$
for a fixed $\gamma<\left(\kappa^{+}\right)^{V}=\left(\kappa^{+}\right)^{V[G]}$. If $\psi$ is true, then for sufficiently large $\gamma, \theta$ will be true. Now we only need to modify the last step of the above proof. Let $C$ be a subset of $\kappa$ such that $\left(\kappa, p^{-1}[C]\right) \cong(\gamma,<)$. Suppose $R$ is a binary relation on $\kappa$. The condition " $R$ is wellfounded of rank $\leqslant \gamma$ " is $\Sigma_{1}^{1}(\kappa)$ in $C$, since it is equivalent to the existence of a function $f: \kappa \rightarrow \gamma$ such that for all $\alpha, \beta<\kappa,(\alpha, \beta) \in R \Rightarrow f(\alpha)<f(\beta)$.

Towards a contradiction, suppose that there is no inner model with a Woodin cardinal and in some $\mathbb{P}$-generic extension $V[G]$ of $V, H_{\kappa^{+}}^{V}{ }_{\Sigma_{1}} H_{\kappa^{+}}^{V[G]}$ fails. By the previous remarks, $\kappa$ is preserved and $\kappa^{+}$is collapsed in $V[G]$. Since there is no inner model with a Woodin cardinal, the Jensen-Steel core model $K$ from [23] is generically absolute and satisfies $\left(\lambda^{+}\right)^{K}=\lambda^{+}$for all singular cardinals $\lambda$ by [23, Theorem 1.1]. Therefore any generic extension $V[G]$ of $V$ which does not collapse $\lambda$ satisfies $\left(\lambda^{+}\right)^{V}=\left(\lambda^{+}\right)^{V[G]}$. For $\lambda=\kappa$, this contradicts our assumption.

Can one remove the assumption that there is no inner model with a Woodin cardinal? A forcing $\mathbb{P}$ that witnesses the failure of $(3) \Rightarrow(5)$ must preserve $\kappa$ and collapse $\kappa^{+}$by the above proof. The existence of a forcing $\mathbb{P}$ with these two properties is consistent relative to the existence of a $\lambda^{+}$-supercompact cardinal $\lambda$ by a result of Adolf, Apter and Koepke [2, Theorem 7]. Their forcing does not add new bounded subsets of $\kappa$ as in (4) and thus also satisfies (1)-(3). However, we do not know if it satisfies (5).

Question 4. Is it consistent that there exist an uncountable cardinal $\kappa$ with $\operatorname{Cof}(\kappa)=\omega$ and a forcing $\mathbb{P}$ with the properties:
(a) $\mathbb{P}$ does not add new bounded subsets of $\kappa$ and
(b) $\Vdash_{\mathbb{P}} H_{\kappa^{+}}^{V}<\Sigma_{1} H_{\kappa^{+}}^{V[\dot{G}]}$ fails?
(Thus $\mathbb{P}$ necessarily collapses $\kappa^{+}$.)

### 2.3.4 Boolean ultrapowers

In this section, we translate the above correspondence to Boolean ultrapowers and use this to characterise forcing axioms via elementary embeddings.

The Boolean ultrapower construction generalises ultrapowers with respect to ultrafilters on the power set of a set to ultrafilters on arbitrary Boolean algebras. We recall the basic definitions from Hamkins' and Seabold's work on Boolean ultrapowers [19, Section 3]. Suppose that $\mathbb{P}$ is a forcing and $\mathbb{B}$ its Boolean completion. Fix an ultrafilter $U$ on $\mathbb{B}$, which may or may not be in the ground model. We define two relations $={ }_{U}$ and $\epsilon_{U}$ on $V^{\mathbb{B}}$ :

$$
\begin{aligned}
\sigma==_{U} \tau & : \Leftrightarrow \llbracket \sigma=\tau \rrbracket \in U \\
\sigma \epsilon_{U} \tau & : \Leftrightarrow \llbracket \sigma \in \tau \rrbracket \in U
\end{aligned}
$$

Let $[\sigma]_{U}$ denote the equivalence class of $\sigma \in V^{\mathbb{B}}$ with respect to $=_{U}$. Let $V^{\mathbb{B}} / U=\left\{[\sigma]_{U}: \sigma \in V^{\mathbb{B}}\right\}$ denote the quotient with respect to $=_{U} . \epsilon_{U}$ is well-defined on equivalence classes and ( $V^{\mathbb{B}} / U, \epsilon_{U}$ ) is a model of ZFC [19, Theorem 3]. It is easy to see from these definitions that for any $\mathbb{P}$-generic filter $G$ over $V, V^{\mathbb{B}} / G$ is isomorphic to the generic extension $V[G]$. Moreover, we can determine the truth of sentences in $V^{\mathbb{B}} / U$ via Los' theorem [19, Theorem 10]: $V^{\mathbb{B}} / U \models \varphi\left(\left[\sigma_{0}\right]_{U}, \ldots\left[\sigma_{n}\right]_{U}\right) \Longleftrightarrow \llbracket \varphi\left(\sigma_{0}, \ldots, \sigma_{n}\right) \rrbracket \in U$. In other words, the forcing theorem holds.

The Boolean ultrapower is the subclass

$$
\check{V}_{U}=\left\{[\sigma]_{U}: \llbracket \sigma \in \check{V} \rrbracket \in U\right\}
$$

of $V^{\mathbb{B}} / U$. It is isomorphic to $V$ if and only if $U$ is generic over $V$. The Boolean ultrapower embedding is the elementary embedding

$$
j_{U}: V \rightarrow \check{V}_{U}, \quad j_{U}(x)=[\check{x}]_{U} .
$$

We are interested in the case that $U$ is an ultrafilter in the ground model. In particular, $U$ is not $\mathbb{P}$-generic over $V . j_{U}$ has the following properties:

- If $U$ is generic, then $j_{U}$ is an isomorphism.
- If $U$ is not generic, then $\check{V}_{U}$ is ill-founded and $\operatorname{crit}\left(j_{U}\right)$ equals the least size of a maximal antichain in $\mathbb{B}$ not met by $U\left[19\right.$, Theorem 17]. For example, if $\mathbb{P}$ is c.c.c. then $\operatorname{crit}\left(j_{U}\right)=\omega$.
For any $x \in V^{\mathbb{B}} / U$, let $x^{\epsilon_{U}}=\left\{y \in V^{\mathbb{B}} / U: y \epsilon_{U} x\right\}$ denote the set of all $\epsilon_{U}$-elements of $x$. If $\kappa$ is a cardinal and $\sigma$ is a name for a subset of $\kappa$, then $[\sigma]_{U}^{\epsilon_{U}} \cap j[\kappa]=j\left[\sigma^{(U)}\right]$, since

$$
V^{\mathbb{B}} / U \models j_{U}(\alpha)=[\check{\alpha}]_{U} \in[\sigma]_{U} \Leftrightarrow \llbracket \check{\alpha} \in \sigma \rrbracket \in U \Leftrightarrow \alpha \in \sigma^{(U)}
$$

for all $\alpha<\kappa$.
Theorem 2.3.23. The following statements are equivalent:
(1) $\mathrm{FA}_{\mathbb{P}, \kappa}$
(2) For any transitive set $M \in H_{\kappa^{+}}$and for every $\kappa$-small $M$-name $\sigma$, there is an ultrafilter $U \in V$ on $\mathbb{P}$ such that

$$
j_{U} \upharpoonright M: M \rightarrow j_{U}(M)^{\epsilon_{U}}
$$

is an elementary embedding from $\left(M, \in, \sigma^{U}\right)$ to $\left(j_{U}(M)^{\epsilon_{U}}, \epsilon_{U},[\sigma]_{U}\right)$.
(3) For any transitive set $M \in H_{\kappa^{+}}$and for any $\kappa$-small $M$-name $\sigma$, there is an ultrafilter $U$ on $\mathbb{P}$ such that

$$
\left(M, \in, \sigma^{U}\right) \equiv\left(j_{U}(M)^{\epsilon_{U}}, \epsilon_{U},[\sigma]_{U}\right)
$$

i.e. these structures are elementarily equivalent.

Proof. (1) $\Rightarrow(2)$ : Recall from Lemma 2.3 .7 that for any finite sequence $\vec{\sigma}=\sigma_{0}, \ldots, \sigma_{k}$ of $\kappa$-small names and and every $\Sigma_{0}$-formula $\varphi\left(x_{0}, \ldots, x_{k}\right)$, there is a collection $\mathcal{D}_{\varphi(\vec{\sigma})}$ of $\leqslant \kappa$ many dense subsets of $\mathbb{P}$ with the following property: if $g$ is any filter meeting every set in $\mathcal{D}_{\varphi(\vec{\sigma})}$ and $g$ contains some $p$ such that $p \Vdash \varphi(\vec{\sigma})$, then in fact $\varphi\left(\overrightarrow{\sigma^{g}}\right)$ holds in $V$. Let $\mathcal{D}$ be the union of all collections $\mathcal{D}_{\varphi(\vec{\sigma})}$, where $k \in \omega, \varphi\left(x_{0}, \ldots, x_{k}\right)$ is a $\Sigma_{0}$-formula and each $\sigma_{i}$ is $\sigma, \check{M}$ or $\check{x}$ for some $x \in M$. By $\mathrm{FA}_{\mathbb{P}, \kappa}$, there is a filter $g$ which meets all sets in $\mathcal{D}$. We extend $g$ to an ultrafilter $U$.

Suppose that $\psi\left(x_{0}, \ldots, x_{k}\right)$ is a formula such that $\left(j_{U}(M)^{\epsilon_{U}}, \epsilon_{U},[\sigma]_{U}\right) \models \psi\left(j_{U}\left(y_{0}\right), \ldots, j_{U}\left(y_{k}\right)\right)$. We obtain $\varphi\left(x_{0}, \ldots, x_{k+2}\right)$ by replacing the unbounded quantifiers in $\psi$ by quantifiers bounded by $x_{k+1}$, and any occurence of $[\sigma]_{U}$ by $x_{k+2}$. Then

$$
\left(V^{\mathbb{B}} / U, \in_{U}\right) \models \varphi\left(j_{U}\left(y_{0}\right), \ldots, j_{U}\left(y_{k}\right), j_{U}(M),[\sigma]_{U}\right)
$$

Recall that $j_{U}(y)=[\check{y}]_{U}$ for all $u \in M$. Therefore by Łos' theorem, we have $\llbracket \varphi\left(\check{y}_{0}, \ldots, \check{y}_{k}, \check{M}, \sigma\right) \rrbracket \in U$. So there is some $p \in U$ with $p \Vdash \varphi\left(\check{y}_{0}, \ldots, \check{y}_{k}, \check{M}, \sigma\right)$. Since $U$ meets all dense sets in $\mathcal{D}_{\varphi\left(\check{y}_{0}, \ldots, \check{y}_{k}, \check{M}, \sigma\right)}$,

$$
(V, \in) \models \varphi\left(y_{0}, \ldots, y_{k}, M, \sigma^{U}\right)
$$

Hence $\left(M, \in, \sigma^{U}\right) \models \psi\left(y_{0}, \ldots, y_{k}\right)$.
$(2) \Rightarrow(3)$ : This is clear.
$(3) \Rightarrow(1)$ : Let $M=\kappa$ and suppose that $\sigma$ is a rank $1 M$-name such that $\mathbb{P} \Vdash \sigma=\check{\kappa}$. Then $\sigma^{(g)}=\kappa$ for any filter $g$ on $\mathbb{P}$. It suffices to find a filter $g$ with $\sigma^{g}=\kappa$ by Lemma 2.2.2. Let $U$ be an ultrafilter as in (3). Since $M=\kappa$ and $j_{U}(M)=j_{U}(\kappa)=[\check{\kappa}]_{U}=[\sigma]_{U}$, we have $\left(j_{U}(M)^{\epsilon_{U}}, \in_{U},[\sigma]_{U}\right) \models \forall x \in_{U}$ [ $\left.\sigma\right]_{U}$. Thus $\left(\kappa, \in, \sigma^{U}\right) \models \forall x x \in_{U} \sigma^{U}$ by elementary equivalence. Thus $\sigma^{U}=\kappa$.

A version of Theorem 2.3.23 for $\mathrm{BFA}_{\mathbb{P}, \kappa}^{\lambda}$ and $\lambda$-bounded names also holds for any cardinal $\lambda \geqslant \kappa$. The proof is essentially the same.

### 2.3.5 An application to ub-FA

Lemma 2.3.24. If $\mathbb{P}$ is a complete Boolean algebra that does not add reals, then

$$
\left(\forall q \in \mathbb{P} u b-\mathrm{FA}_{\mathbb{P}_{q}, \omega_{1}}\right) \Longrightarrow \mathrm{BFA}_{\mathbb{P}, \omega_{1}}^{\omega_{1}} .
$$

More generally, if $\kappa$ is an uncountable cardinal and $\mathbb{P}$ is a complete Boolean algebra that does not add bounded subsets of $\kappa$, then

$$
\left(\forall q \in \mathbb{P} \text { ub- } \mathrm{FA}_{\mathbb{P}_{q}, \kappa}\right) \Longrightarrow \mathrm{BFA}_{\mathbb{P}, \kappa}^{\kappa} .
$$

Proof. If $\operatorname{Cof}(\kappa)=\omega$, then adding no new bounded subsets of $\kappa$ already implies $\mathrm{BFA}_{\mathbb{P}, \kappa}^{\kappa}$ by the proof of $(4) \Rightarrow$ (3) in Theorem 2.3.22. Now suppose that $\operatorname{Cof}(\kappa)>\omega$. Towards a contradiction, suppose that $\mathrm{BFA}_{\mathbb{P}, \kappa}^{\kappa}$ fails. Then $\Sigma_{1}^{1}(\kappa)$-absoluteness fails for some $\Sigma_{1}^{1}(\kappa)$-formula $\exists x \psi(x, y)$ and some $y \in\left(2^{\kappa}\right)^{V}$ by Theorem 2.3.17. Take a subtree $T$ of $\left(2^{<\kappa} \times \kappa^{<\kappa}\right)^{<\operatorname{Cof}(\kappa)}$ for $\psi$ as in Lemma 2.3.21. Then $[T] \neq \varnothing$ in $V[G]$ in some $\mathbb{P}$-generic extension $V[G]$, but $[T]=\varnothing$ in $V$. Let $\sigma$ denote a rank $1 T$-name and let $q \in \mathbb{P}$ such that $q \Vdash_{\mathbb{P}} \sigma \in[T]$. Let

$$
\tau=\left\{(\alpha, p): p \leqslant q \wedge \exists s \in \operatorname{Lev}_{\alpha}(T) p \Vdash_{\mathbb{P}}^{+} \check{s} \in \sigma\right\}
$$

Then $\Vdash_{\mathbb{P}_{q}} \tau=\kappa$. For any filter $g \in V$ on $\mathbb{P}_{q}$ we have $\tau^{g}=\operatorname{dom}\left(\sigma^{g}\right)$. But $\operatorname{dom}\left(\sigma^{g}\right) \in \kappa$, since $[T]=\varnothing$. Therefore ub- $\mathbb{N}_{\mathbb{P}_{q}, \kappa}$ fails and hence ub-FA $\mathbb{P}_{\mathbb{P}_{q}, \kappa}$ fails by Lemma 2.2.10.

We will see in Lemma 2.4.1 that for any $<\kappa$-distributive forcing $\mathbb{P}$, ub-FA $\mathbb{P}_{\mathbb{P}, \kappa}$ implies $\mathrm{FA}_{\mathbb{P}, \kappa}$. In combination with the previous lemma, this begs the question:

Question 5. If $\lambda>\kappa$ is a cardinal and $\mathbb{P}$ is a complete Boolean algebra that does not add new elements of ${ }^{<\kappa} \lambda$, then does the implication

$$
\left(\forall q \in \mathbb{P} \text { ub- } \mathrm{FA}_{\mathbb{P}_{q}, \omega_{1}}\right) \Longrightarrow \mathrm{BFA}_{\mathbb{P}, \omega_{1}}^{\lambda}
$$

hold?

### 2.4 Specific classes of forcings

### 2.4.1 Classes of forcings

We now move on to look, over the next few sections, at what further results we can prove if we assume $\mathbb{P}$ is some specific kinds of forcing. We shall mostly return to the rank 1 cases for this and discuss the club, stat, ub and $\omega$-ub axioms in Figure 2.1.

## $\sigma$-distributive forcings

We begin with a relatively simple case, where $\mathbb{P}$ is $<\kappa$-distributive. In this case, several of our axioms turn out to be equivalent to one another. The implications for the class of $<\kappa$-distributive forcings are summarised in the next diagram.


Figure 2.5: Forcing axioms and name principles for any $<\kappa$-distributive forcing for regular $\kappa$. Lemma 2.4.3 shows that stat- $\mathbb{N}_{\mathbb{P}, \omega_{1}}$ is strictly stronger than the remaining principles for some $\sigma$-closed forcing $\mathbb{P}$.

Lemma 2.4.1. For any $<\kappa$-distributive forcing $\mathbb{P}$, ub- $\mathrm{FA}_{\mathbb{P}, \kappa} \Longrightarrow \mathrm{FA}_{\mathbb{P}, \kappa}$.

Proof. Given a sequence $\vec{D}=\left\langle D_{i}: i<\kappa\right\rangle$ of open dense subsets of $\mathbb{P}$, let $E_{j}=\bigcap_{i \leqslant j} D_{i}$ for $j<\kappa$. If for a filter $g, g \cap E_{j} \neq \varnothing$ for unboundedly many $j<\kappa$, then $g \cap D_{i} \neq \varnothing$ for all $i<\kappa$.
Lemma 2.4.2. Let $\mathbb{P}$ be $<\kappa$-distributive. stat- $\mathbb{N}_{\mathbb{P}, \kappa} \Longrightarrow \mathrm{FA}_{\mathbb{P}, \kappa}^{+}$
Proof. Suppose that $\vec{D}=\left\langle D_{i}: i<\kappa\right\rangle$ is a sequence of open dense subsets of $\mathbb{P}$ and $\sigma=\left\{(\check{\alpha}, p): p \in S_{\alpha}\right\}$ is a name with $1 \Vdash_{\mathbb{P}}$ " $\sigma$ is stationary". For each $j<\kappa$, let $E_{j}=\bigcap_{i \leqslant j} D_{i}$. For $j<\kappa$ and $p \in \mathbb{P}$, let $E_{j, p}$ denote a subset of $\left\{q \in E_{j}: q \leqslant p\right\}$ that is dense below $p$. Let

$$
\tau=\left\{(\check{\alpha}, q): \alpha<\kappa, \exists p \in S_{\alpha} q \in E_{j, p}\right\}
$$

$1 \Vdash_{\mathbb{P}}$ " $\tau$ is stationary", since $1 \Vdash_{\mathbb{P}} \sigma=\tau$. By stat- $\mathbb{N}_{\mathbb{P}, \kappa}$, there is a filter $g$ such that $\tau^{g}$ is stationary. By the definition of $\tau, \tau^{g} \subseteq \sigma^{g}$. Thus $\sigma^{g}$ is stationary. We further have $g \cap E_{j}$ for unboundedly many $j<\kappa$ and hence $g \cap D_{i} \neq \varnothing$ for all $i<\kappa$.

An equivalent argument can be made with names for unbounded sets, or for sets containing a club.

## $\sigma$-closed forcings

Note that $\mathrm{FA}_{\mathbb{P}, \omega_{1}}$ fails for some $\sigma$-distributive forcings, for instance for Suslin trees. But $\mathrm{FA}_{\sigma-\text { closed }, \omega_{1}}$ is provable: if $\left\langle D_{\alpha}: \alpha<\omega_{1}\right\rangle$ is a sequence of dense subsets of a $\sigma$-closed $\mathbb{P}$, let $\left\langle p_{\alpha}: \alpha<\omega_{1}\right\rangle$ be a decreasing sequence of conditions in $\mathbb{P}$ with $p_{\alpha} \in D_{\alpha}$ and let $g=\left\{q \in \mathbb{P}: \exists \alpha<\omega_{1} p_{\alpha} \leqslant q\right\}$. Therefore, the other principles in Figure 2.5 are provable, with the exception of stat- $\mathbb{N}_{\mathbb{P}, \omega_{1}}$ by the next lemma. The lemma follows from known results.

Lemma 2.4.3. It is consistent that there is a $\sigma$-closed forcing $\mathbb{P}$ such that stat- $\mathbb{N}_{\mathbb{P}}$ fails.
Proof. It suffices to argue that stat- $\mathrm{N}_{\mathbb{P}}$ has large cardinal strength for some $\sigma$-closed forcing $\mathbb{P}$. Note that stat- $\mathbb{N}_{\mathbb{P}}$ implies $\mathrm{FA}_{\mathbb{P}}^{+}$for any $\sigma$-closed forcing $\mathbb{P}$ by Lemma 2.4.2. There is a cardinal $\mu \geqslant \omega_{2}$ such that $\mathrm{FA}_{\operatorname{Col}\left(\omega_{1}, \mu\right)}^{+}$implies the failure of $\square(\kappa)$ for all regular $\kappa \geqslant \omega_{2}$ by [14, Page $20 \&$ Proposition 14] and [33, Theorem 2.1]. ${ }^{11}$ The proofs show that a single collapse suffices for the conclusion. The failure of $\square\left(\kappa^{+}\right)$and thus Jensen's $\square_{\kappa}$ at a singular strong limit cardinal $\kappa$ implies the existence of an inner model with a proper class of Woodin cardinals (and more) by [34, Theorem 0.1] and [39, Theorem 15.1].

Presaturation of the nonstationary ideal on $\omega_{1}$ is another interesting consequence of stat- $\mathrm{N}_{\sigma \text {-closed, } \omega_{1}}$ (equivalently, of $\mathrm{FA}_{\sigma \text {-closed }, \omega_{1}}^{+}$) [14, Theorem 25]. Even for very simple $\sigma$-closed forcings $\mathbb{P}$, stat- $\mathbb{N}_{\mathbb{P}, \omega_{1}}$ is an interesting axiom. For instance, Sakai showed in [32, Section 3] that $\mathrm{FA}_{\operatorname{Add}\left(\omega_{1}\right), \omega_{1}}^{+}$and thus stat- $\mathrm{N}_{\text {Add }\left(\omega_{1}\right), \omega_{1}}$ is not provable in ZFC. We do not know much about the weakest stationary name principle for $\sigma$-closed forcing:
Question 6. Is stat- $\mathrm{BN}_{\sigma \text {-closed }}^{1}$ provable in ZFC?

## c.c.c. forcings

The class of c.c.c. forcings is rather more interesting. It has also historically been a class where forcing axioms have been frequently used; for example $\mathrm{FA}_{\text {c.c.c. }, \omega_{1}}$ is the well-known Martin's Axiom $\mathrm{MA}_{\omega_{1}}$. Note that $\mathrm{FA}_{\mathbb{P}, \kappa}$ is equivalent to $\mathrm{BFA}_{\mathbb{P}, \kappa}^{\omega}$.


Figure 2.6: Forcing axioms and name principles at $\omega_{1}$ for the class of all c.c.c. forcings.

[^10]All principles in Figure 2.1 for $\kappa=\omega_{1}$ turn out to be equivalent to $\mathrm{FA}_{\omega_{1}}$. The implications are valid for the class of all c.c.c. forcings, but not for all single c.c.c. forcings. For instance, for the class of $\sigma$-centred forcings, the right side of Figure 2.1 is provable in ZFC by Lemma 2.4.6, but the left side is not.

We first derive the implication ub-FA ${\text { c.c.c. }, \omega_{1}}^{\Longrightarrow F_{\text {c.c.c. }, \omega_{1}} \text { from well-known results. Note that this }}$ implication does not hold for individual c.c.c. forcings, for instance it fails for Cohen forcing by Lemma 2.4.6 and Remark 2.4.14. We need the following definition:

Definition 2.4.4. Suppose that $\mathbb{P}$ is a forcing.
(1) A subset $A$ of $\mathbb{P}$ is centred if every finite subset of $A$ has a lower bound in $\mathbb{P} . A$ is $\sigma$-centred if it is a union of countably many centred sets.
(2) $\mathbb{P}$ is precaliber $\kappa$ if, whenever $A \in[\mathbb{P}]^{\kappa}$, there is some $B \in[A]^{\kappa}$ that is centred.

The hard implications in the next lemma are due to Todorčević and Veličković [42].
Lemma 2.4.5. The following conditions are equivalent:
(1) ub-FA ${\text { c.c.c. }, \omega_{1}}$ holds.
(2) Every c.c.c. forcing is precaliber $\omega_{1}$.
(3) Every c.c.c. forcing of size $\omega_{1}$ is $\sigma$-centred.
(4) $\mathrm{FA}_{\text {c.c.c. }, \omega_{1}}$ holds.

Proof. $(1) \Rightarrow(2)$ : This follows immediately from the proof of [21, Theorem 16.21]. The proof only requires meeting unboundedly many dense sets.
$(2) \Rightarrow(3)$ : See [42, Corollary 2.7].
$(3) \Rightarrow(4)$ : See [42, Theorem 3.3].
$(4) \Rightarrow(1)$ : This is immediate.
Given Lemma 2.4.5, one wonders whether the equivalence of (1) and (4) also holds for $\sigma$-centered forcings instead of c.c.c. forcings. The next lemma together with the fact that $\mathrm{FA}_{\sigma \text {-centred }}$ is equivalent to $\mathfrak{p}>\omega_{1}$ (see [42, Theorem 3.1]) shows that this is not the case.

Lemma 2.4.6. For any cardinal $\kappa$ with $\operatorname{Cof}(\kappa)>\omega$, stat- $\mathrm{N}_{\sigma \text {-centred, } \kappa}$ holds.
Proof. Suppose that $\sigma$ is name for a stationary subset of $\omega_{1}$. Let $f: \mathbb{P} \rightarrow \omega$ witness that $\mathbb{P}$ is $\sigma$-centered. Let $S$ be the stationary set of $\alpha$ such that $p \Vdash \alpha \in \sigma$ for some $p \in \mathbb{P}$. For each $\alpha \in S$, let $p_{\alpha}$ be such that $\left(\alpha, p_{\alpha}\right) \in \sigma$. There is a stationary subset $R$ of $S$ and $n \in \omega$ with $f\left(p_{\alpha}\right)=n$ for all $\alpha \in R$. Let $g$ be a filter containing $p_{\alpha}$ for all $\alpha \in S$. Then $R \subseteq \sigma^{g}$, as required.

This suggests to ask whether $\mathrm{FA}_{\sigma \text {-centred }}$ implies $\mathrm{FA}_{\sigma \text {-centred }}^{+}$as well. A further, long-standing, open question is whether one can replace precaliber $\omega_{1}$ by Knaster in the implication $(2) \Rightarrow(4)$ of Lemma 2.4.5. Recall that a subset of $\mathbb{P}$ is linked if it consists of pairwise compatible conditions. $\mathbb{P}$ is called Knaster if, whenever $A \in[\mathbb{P}]^{\omega_{1}}$, there is some $B \in[A]^{\omega_{1}}$ that is linked.
Question 7. [41, Problem 11.1] Does the statement that every c.c.c. forcing is Knaster imply FA ${\text { c.c.c. }, \omega_{1}}^{\text {? }}$ ?
We now turn to the implication $\mathrm{FA}_{\text {c.c.c. }, \omega_{1}} \Longrightarrow$ stat- $\mathrm{N}_{\text {c.c.c. }, \omega_{1}}$. To this end, we reconstruct Baumgartner's unpublished result $\mathrm{FA}_{\text {c.c.c. }, \kappa} \Longrightarrow \mathrm{FA}_{\text {c.c.c. }, \kappa}^{+n}$ that is mentioned without proof in [8, Section 8] and [9, Page 14]. Here $\mathrm{FA}_{\kappa}^{+n}$ denotes the version of $\mathrm{FA}^{+}$with $n$ many names for stationary subsets of $\kappa$.

Lemma 2.4.7 (Baumgartner). For any uncountable cardinal $\kappa$ and for any $n \in \omega$, $\mathrm{FA}_{\text {c.c.c., } \kappa}$ implies $\mathrm{FA}_{\text {c.c.c., } \kappa}^{+n}$.
Proof. Fix an uncountable $\kappa$. Suppose that for each $i<n, \sigma_{i}$ is a rank $1 \mathbb{P}$-name for a stationary subset of $\omega_{1}$. For each $\vec{\alpha}=\left\langle\alpha_{i}: i\langle n\rangle \in \kappa^{n}\right.$, let $A_{\vec{\alpha}}$ be a maximal antichain of conditions which strongly decide $\alpha \in \sigma_{i}$ for each $i<k$. Let $A=\bigcup_{\vec{\alpha} \in \kappa^{n}} A_{\vec{\alpha}}$. Since $\mathbb{P}$ satisfies the c.c.c. and $|A| \leqslant \omega_{1}$, there exists a subforcing $\mathbb{Q} \subseteq \mathbb{P}$ with $A \subseteq \mathbb{Q}$ and $|\mathbb{Q}| \leqslant \omega_{1}$ such that compatibility is absolute between $\mathbb{P}$ and $\mathbb{Q}$. In particular, $\mathbb{Q}$ is c.c.c.

Since every c.c.c. forcing of size $\omega_{1}$ is $\sigma$-centred by $\mathrm{MA}_{\omega_{1}}$ (see [44, Theorem 4.5]), there is a sequence $\vec{g}=\left\langle g_{k}: k \in \omega\right\rangle$ of filters $g_{k}$ on $\mathbb{P}$ with $\mathbb{Q} \subseteq \bigcup_{k \in \omega} g_{k}$. Morover, it follows from the proof of [44, Theorem 4.5] (by a density argument) that we can choose the filters $g_{k}$ such that $g_{k} \cap B_{\alpha} \neq \varnothing$ for all $(k, \alpha) \in \omega \times \kappa$, where $\vec{B}=\left\langle B_{\alpha}: \alpha<\kappa\right\rangle$ is any sequence of dense subsets of $\mathbb{P}$. (The conditions in the c.c.c. forcing consists of finite approximations to finitely many filters.)

It remains to find some $k \in \omega$ such that for all $i<n$, the set $\sigma_{i}^{g_{k}}$ is stationary. Let $G$ be $\mathbb{P}$-generic over $V$. We claim that

$$
\prod_{i<n} \sigma_{i}^{G} \subseteq \bigcup_{k \in \omega} \prod_{i<n} \sigma_{i}^{g_{k}}
$$

To see this, suppose that $\vec{\alpha}=\left\langle\alpha_{i}: i<n\right\rangle \in \prod_{i<n} \sigma_{i}^{G}$ and let $p \in A_{\vec{\alpha}} \cap G$. Then $p \Vdash^{+} \bigwedge_{i<n} \alpha_{i} \in \sigma_{i}$. Since $p \in \mathbb{Q}$, we have $p \in g_{k}$ for some $k \in \omega$. Hence $\vec{\alpha} \in \prod_{i<n} \sigma_{i}^{g_{k}}$. Since $\sigma_{i}^{G}$ is stationary for all $i<n$, the above inclusion easily yields that there is some $k \in \omega$ such that $\prod_{i<n} \sigma_{i}^{g_{k}}$ is stationary.

Our proof of the previous lemma does not work for $\mathrm{MA}^{+\omega}$. In fact, Baumgartner asked in [8, Section 8]:
Question 8 (Baumgartner 1984). Does $\mathrm{MA}_{\omega_{1}}$ imply $\mathrm{MA}_{\omega_{1}}^{+\omega_{1}}$ ?
We finally turn to bounded name principles for c.c.c. forcings.

## Lemma 2.4.8.

(1) club- $\mathrm{BN}_{\text {c.c.c. }}^{1}$ holds.
(2) For any c.c.c. forcing $\mathbb{P}$, ub- $\mathrm{BN}_{\mathbb{P}}^{1}$ implies ub- $\mathrm{FA}_{\mathbb{P}}$.

Proof. (1) If $\sigma$ is a $\mathbb{P}$-name for a set that contains a club, then by the c.c.c. there is a club $C$ with $1 \Vdash C \subseteq \sigma$. Since $\sigma$ is 1-bounded, $(\alpha, 1) \in \sigma$ for all $\alpha \in C$. Thus for every filter $g$, we have $C \subseteq \sigma^{g}$.
(2) Suppose that $\mathbb{P}$ satisfies the c.c.c., and $\vec{D}=\left\langle D_{\alpha}: \alpha<\omega_{1}\right\rangle$ is a sequence of dense subsets of $\mathbb{P}$. Let $A_{\alpha}$ be a maximal antichain in $D_{\alpha}$ and let $\vec{a}_{\alpha}=\left\langle a_{\alpha}^{n}: n \in \omega\right\rangle$ enumerate $A_{\alpha}$. (For ease of notation, we assume for that each $A_{\alpha}$ is infinite.) Let $\sigma=\left\{\left(\omega \cdot \alpha+n, a_{\alpha}^{n}\right): \alpha<\omega_{1}, n \in \omega\right\}$. By ub- $\mathrm{BN}_{\mathbb{P}}^{1}$, there is a filter $g$ such that $\sigma^{g}$ is unbounded. Hence $D_{\alpha} \cap g \neq \varnothing$ for unboundedly many $\alpha<\omega_{1}$.

For any c.c.c. forcing $\mathbb{P}$, the principles ub- $B N_{\mathbb{P}}^{1}$, ub- $N_{\mathbb{P}}$ and ub-FA $\mathbb{P}_{\mathbb{P}}$ are equivalent by Lemma 2.4.8 (2) and the implications in Figure 2.6. We do not know what is their relationship with stat- $\mathrm{BN}_{\text {c.c.c. }}^{1}$. However, we will show in Lemma 2.4.18 below that stat- $\mathrm{BN}_{\text {random }, \omega_{1}}^{1}$ is not provable in ZFC.

Regarding Lemma 2.4.8 (1), it is also easy to see that club- $\mathrm{BN}_{\sigma \text {-closed }}^{1}$ is provable. This suggests to ask:
Question 9. Is club- $\mathrm{BN}_{\mathbb{P}}^{1}$ is provable for any proper forcing $\mathbb{P}$ ?

### 2.4.2 Specific forcings

## Cohen forcing

We will now drop down from classes of forcings, to forcing axioms on specific forcings. This is also where we prove most of the negative results in the diagram from earlier. We start with the simplest, Cohen forcing and let $\kappa=\omega_{1}$. For Cohen forcing, all principles in the right part of the next diagram are provable in ZFC by Lemma 2.4.6 (on $\sigma$-centred forcing) and the basic implications in Figure 2.1. The left part is not provable by Remark 2.4.14 below.


Figure 2.7: Forcing axioms and name principles at $\omega_{1}$ for Cohen forcing.

Our first result is an improvement to Lemma 2.4.6. It shows that a simultaneous version of the stationary forcing axiom for countably many sequences of dense sets holds.
Lemma 2.4.9. Let $\mathbb{P}$ be Cohen forcing and $\kappa$ a cardinal with $\operatorname{Cof}(\kappa)>\omega$. For each $n \in \omega$, let $\vec{D}_{n}=\left\langle D_{\alpha}^{n}\right.$ : $\alpha\langle\kappa\rangle$ be a sequence of dense sets. Then there exists a filter $g \in V$ such that for all $n$, the trace $\operatorname{Tr}_{g, \vec{D}_{n}}$ is stationary in $\kappa .{ }^{12}$
Proof. Suppose that there is no filter $g$ as described. For $x \in 2^{\omega}$, let us write $g_{x}$ to denote the filter $\{x \mid n: n \in \omega\}$. Then for each $x \in 2^{\omega}$, the filter $g_{x}$ does not have the required property. So there is a natural number $n_{x}$ and a club $C_{x} \subseteq \kappa$ with $g_{x} \cap D_{\alpha}^{n_{x}}=\varnothing$ for all $\alpha \in C_{x}$. Then the sets $A_{n}:=\left\{x \in 2^{\omega}: n_{x}=n\right\}$ partition $2^{\omega}$. By the Baire Category Theorem, not all $A_{n}$ are nowhere dense. So there is some $n \in \omega$ and basic some open subset $N_{t}=\left\{x \in 2^{\omega}: t \subseteq x\right\}$ for some $t \in 2^{<\omega}$ such that $A_{n} \cap N_{t}$ is dense in $N_{t}$. Fix a countable set $D \subseteq A_{n} \cap U$ which is dense in $U$. Let $\alpha$ be an element of the club $\bigcap_{x \in D} C_{x}$. Let further $u \in D_{\alpha}^{n}$ with $u \leqslant t$. Since $D$ is dense in $N_{t}$, there is some $x \in D \cap N_{u}$. Then $u \in g_{x} \cap D_{\alpha}^{n}$ and hence $g_{x} \cap D_{\alpha}^{n} \neq \varnothing$. On the other hand, we have $x \in A_{n}$ and hence $n_{x}=n$. Since also $\alpha \in C_{x}$, we have $g_{x} \cap D_{\alpha}^{n}=\varnothing$.

Using a variant of the previous proof, we can also improve stat- $\mathrm{N}_{\mathbb{P}}$ to work for finitely many names.
Lemma 2.4.10. Let $\mathbb{P}$ be Cohen forcing and $\kappa$ a cardinal with $\operatorname{Cof}(\kappa)>\omega$. Suppose that $\vec{\sigma}=\left\langle\sigma_{i}: i \leqslant n\right\rangle$ is a sequence of rank $1 \mathbb{P}$-names such that for each $i \leqslant n, \mathbb{P} \Vdash \sigma_{i}$ is stationary in $\kappa$. Then there is a filter $g$ on $\mathbb{P}$ such that for all $i \leqslant n, \sigma_{i}^{g}$ is stationary in $\kappa$. In particular, stat- $\mathbb{N}_{\mathbb{P}, \kappa}$ holds.

Proof. As in the previous proof, let $g_{x}=\{x \uparrow n: n \in \omega\}$ for $x \in 2^{\omega}$. The result will follow from the next claim.
Claim 2.4.11. If $D$ is any dense subset of $2^{\omega}$, then there is some $x \in D$ such that $\sigma_{i}^{g_{x}}$ is stationary in $\kappa$ for all $i \leqslant n$.

Proof. We can assume that $D$ is countable. If the claim fails, then for each $x \in D$, there is some $i \leqslant n$ and a club $C_{x}$ such that $\sigma_{i}^{g_{x}} \cap C_{x}=\varnothing$. Then $C:=\bigcap_{x \in D} C_{x}$ is a club. Moreover, for each $x \in D$, there is some $i \leqslant n$ such that $\sigma_{i}^{g_{x}} \cap C=\varnothing$. There is some $p \in \mathbb{P}$ such that for each $i \leqslant n$, there is some $\alpha_{i} \in C$ such that $p \Vdash \check{\alpha}_{i} \in \sigma_{i}$. By Lemma 2.1.21, we can assume that $p \Vdash^{+} \check{\alpha}_{i} \in \sigma_{i}$ for all $i \leqslant n$. Now, since $D$ is dense, we can find some $x \in D$ with $p \subseteq x$. Then $p \in g_{x}$, so by 2.1 .22 we conclude $\alpha_{i} \in \sigma_{i}^{g_{x}}$ for all $i \leqslant n$. This contradicts the above property of $C$.

This completes the proof of Lemma 2.4.10.
Given the previous result about stat-FA, we might expect to be able to correctly interpret $\omega$ many names. But the above proof does not work: it breaks down where we introduce $p$. For each $i$, we can find $p_{i}$ strongly forcing $\alpha_{i} \in \sigma_{i}$; but then we would want to take some $p$ that was below every $p_{i}$ and that is only possible in $\sigma$-closed forcings.

We can, however, apply the same technique in the presence of FA to prove $\mathrm{FA}^{+}$.
Lemma 2.4.12. Let $\mathbb{P}$ be Cohen forcing and $\kappa$ a cardinal with $\operatorname{Cof}(\kappa)>\omega$. Then $\mathrm{FA}_{\mathbb{P}, \kappa}$ implies $\mathrm{FA}_{\mathbb{P}, \kappa}^{+}$.
Proof. We will in fact prove a stronger version for finitely many names. Suppose that $\vec{\sigma}=\left\langle\sigma_{i}: i \leqslant n\right\rangle$ is a sequence of rank $1 \mathbb{P}$-names such that for each $i \leqslant n, \mathbb{P} \Vdash \sigma_{i}$ is stationary in $\kappa$. Suppose that $\vec{D}=\left\langle D_{\alpha}: \alpha<\kappa\right\rangle$ is a sequence of dense open sets. Then

$$
D:=\left\{x \in 2^{\omega}: \forall \alpha<\kappa \exists p \in D_{\alpha} p \subseteq x\right\}
$$

consists of all reals $x$ such that $g_{x} \cap D_{\alpha} \neq \varnothing$ for all $\alpha<\omega_{1}$.
The next claim suffices. By Claim 2.4.11, it implies that for some $x \in D, \sigma_{i}^{g_{x}}$ is stationary for all $i \leqslant n$.
Claim 2.4.13. $D$ is dense in $2^{\omega}$.
Proof. Fix $q \in \mathbb{P}$; we will find some $x \in D$ with $q \subseteq x$. Since the forcing $\mathbb{P}_{q}:=\{p \in \mathbb{P}: p \leqslant q\}$ is isomorphic to Cohen forcing via the map $r \mapsto q^{\wedge} r, \mathrm{FA}_{\mathbb{P}_{q}}$ holds. Hence, we can find a filter $g$ on $\mathbb{P}_{q}$ which meets $D_{\alpha} \cap \mathbb{P}_{q}$ for every $\alpha<\omega_{1} . \cup g$ is an element of $2 \leqslant \omega$ with $q \subseteq \cup g$ by compatibility of elements of a filter. Then any real $x$ with $\cup g \subseteq x$ satisfies $x \in D$ and $q \subseteq x$.

[^11]Lemma 2.4.12 follows.
Remark 2.4.14. Note that $\mathrm{FA}_{\text {Cohen }, \omega_{1}}$ also has a well known characterisation via sets of reals: it is equivalent to the statement that the union of $\omega_{1}$ many meagre sets does not cover $2^{\omega}$. In particular, $\mathrm{FA}_{\text {Cohen }, \omega_{1}}$ is not provable in ZFC.

## Random forcing



Figure 2.8: Forcing axioms and name principles at $\omega_{1}$ for random forcing.
We have seen in Lemma 2.4.6 and the following remark that ub-FA $\mathbb{P}_{\mathbb{P}}$ implies $\mathrm{FA}_{\mathbb{P}}$ for $\sigma$-centred forcings. However, random forcing is not $\sigma$-centred by [12, Lemma 3.7]. The implication still holds:

Lemma 2.4.15. Let $\mathbb{P}$ denote random forcing. The following are equivalent:
(1) $\mathrm{FA}_{\mathbb{P}, \omega_{1}}$
(2) ub-FA $\mathbb{P}_{\mathbb{P}, \omega_{1}}$
(3) $2^{\omega}$ is not the union of $\omega_{1}$ many null sets

The equivalence of (1) and (3) is a well-known fact, but we really interested in the equivalence of (1) and (2). The proof of $(2) \Rightarrow(3)$ also works for certain forcings of the form $\mathbb{P}_{I} . \mathbb{P}_{I}$ consists of all Borel subsets $B \notin I$ of $2^{\omega}$, where $I$ is a $\sigma$-ideal on the Borel subsets of the Cantor space, ordered by inclusion up to sets in $I$. For $(2) \Rightarrow(3)$, it suffices that the set of closed $p \in \mathbb{P}$ is dense in $\mathbb{P}$ and $N_{t} \notin I$ for all $t \in 2^{<\omega}$. If additionally $(3) \Rightarrow(1)$ holds, then ub-FA $\mathbb{P}_{\mathbb{P}_{I}, \omega_{1}}$ implies $\mathrm{FA}_{\mathbb{P}_{I}, \omega_{1}}$.

Proof. (1) $\Rightarrow(2)$ : Immediate.
$(2) \Rightarrow(3)$ : We prove the contrapositive. Suppose $2^{\omega}=\bigcup_{\alpha<\omega_{1}} S_{\alpha}$, where $S_{\alpha} \subseteq 2^{\omega}$ has measure 0 . Without loss of generality, we may assume that $\left\langle S_{\alpha}: \alpha<\omega_{1}\right\rangle$ is an increasing sequence; i.e. $\alpha<\beta<\omega_{1} \Rightarrow S_{\alpha} \subseteq S_{\beta}$. Then

$$
D_{\alpha}=\left\{B \in \mathbb{P}: B \subseteq 2^{\omega} \backslash S_{\alpha} \text { and } B \text { is closed }\right\}
$$

is dense.
Let $g \in V$ be a filter. Without loss of generality, assume $g$ is an ultrafilter. Then for any $n \in \omega$, there is some $t \in 2^{n}$ with $N_{t} \in g$. It follows that there is a unique $x \in 2^{\omega}$ such that $N_{t} \in g$ for all $t \subseteq x$. It is easy to check that $x$ is in the closure of any element of $g$.

Towards a contradiction, suppose that for unboundedly many $\alpha$ we can find $B_{\alpha} \in D_{\alpha} \cap g$. Then $B_{\alpha}$ is closed, so $x \in B_{\alpha} \subseteq 2^{\omega} \backslash S_{\alpha}$ so $x \notin S_{\alpha}$. This contradicts the assumptions that $2^{\omega}=\bigcup S_{\alpha}$ and the $S_{\alpha}$ are increasing.
$(3) \Rightarrow(1)$ : Again we prove the contrapositive. Let $\left\langle D_{\alpha}: \alpha<\omega_{1}\right\rangle$ be a sequence of predense sets such that there is no filter in $V$ meeting all of them. $\mathbb{P}$ has the c.c.c., so without loss of generality we may assume every $D_{\alpha}$ is countable.

Fix the following notation. Recall that $x \in 2^{\omega}$ is a density point of $B$ if $\frac{\mu\left(B \cap N_{(x \mid k)}\right)}{\mu\left(N_{(x \mid k)}\right)}$ tends to 1 as $k$ tends to infinity. For $B \in \mathbb{P}$, let $D(B)$ be the set of density points of $B$. For $\alpha<\omega_{1}$, let

$$
T_{\alpha}=\bigcup_{B \in D_{\alpha}} D(B) \quad \text { and } \quad S_{\alpha}=2^{\omega} \backslash T_{\alpha}
$$

We first show that $S_{\alpha}$ is a null set. To see this, suppose that $S_{\alpha}$ has positive measure. Then we can find a closed subset $C \subseteq S_{\alpha}$ with positive measure. Since $D_{\alpha}$ is predense, we can find some $B \in D_{\alpha}$ with $\mu(B \cap C)>0$. Since $B \triangle D(B)$ is null by Lebesgue's Density Theorem, we have $\mu(D(B) \cap C)>0$. This contradicts $D(B) \cap C \subseteq T_{\alpha} \cap C=\varnothing$.

We now show $\bigcup_{\alpha<\omega_{1}} S_{\alpha}=2^{\omega}$. To see this, take any $x \in 2^{\omega}$ and let

$$
g_{x}=\{B \in \mathbb{P}: x \in D(B)\}
$$

denote the filter generated by $x$. Take $\alpha<\omega_{1}$ such that $g_{x} \cap D_{\alpha}=\varnothing$. We show that $x \in S_{\alpha}$, as required. Otherwise $x \in T_{\alpha}$, so we can find $B \in D_{\alpha}$ with $x \in D(B)$. But then $B \in g_{x} \cap D_{\alpha}$. This contradicts $g_{x} \cap D_{\alpha}=\varnothing$.

Combining the proofs of $(2) \Rightarrow(3)$ and $(3) \Rightarrow(1)$, we can obtain the following refinement:
Lemma 2.4.16. Let $\mathbb{P}$ be random forcing. Let $\left\langle D_{\alpha}: \alpha<\omega_{1}\right\rangle$ be a collection of predense sets. There exists another collection $\left\langle D_{\alpha}^{\prime}: \alpha<\omega_{1}\right\rangle$ of dense sets, such that if a filter $g$ meets unboundedly many $D_{\alpha}^{\prime}$, then it can be extended to a filter $g^{\prime}$ which meets every $D_{\alpha}$.

Proof. Define $S_{\alpha}$ as in the proof of $(3) \Rightarrow(1)$. Then for any $x \in 2^{\omega}$, we have $g_{x} \cap D_{\alpha} \neq \varnothing$ or $x \in S_{\alpha}$. Consider the null sets $S_{\alpha}^{\prime}=\bigcup_{\beta<\alpha} S_{\beta}$. Then define $D_{\alpha}^{\prime}$ from $S_{\alpha}^{\prime}$ in the same way we defined $D_{\alpha}$ from $S_{\alpha}$ in the proof of $(2) \Rightarrow(3)$. As in the proof of $(2) \Rightarrow(3)$, we obtain the following for any $x \in 2^{\omega}$ and $\alpha<\omega_{1}$ : if $g_{x} \cap D_{\alpha}^{\prime} \neq \varnothing$, then $x \notin S_{\alpha}^{\prime}$. Let $g$ be a filter which meets unboundedly many $D_{\alpha}^{\prime}$. Then $g \subseteq g_{x}$ for some $x \in 2^{\omega}$. We have seen that $x \notin S_{\alpha}^{\prime}$ for unboundedly many $\alpha$. Therefore $x$ misses all $S_{\alpha}^{\prime}$ and all $S_{\alpha}$. By the choice of the $S_{\alpha}$, we have $g_{x} \cap D_{\alpha} \neq \varnothing$ for all $\alpha<\omega_{1}$.

This then allows us to prove that stat- N alone gives us the full $\mathrm{FA}^{+}$.
Lemma 2.4.17. Let $\mathbb{P}$ be random forcing. Then stat- $\mathbb{N}_{\mathbb{P}} \Longrightarrow \mathrm{FA}_{\mathbb{P}}^{+}$.
Proof. Suppose that $\left\langle D_{\alpha}: \alpha<\omega_{1}\right\rangle$ is a sequence of dense subsets of $\mathbb{P}$. Suppose further that $\sigma$ is a rank 1 name which is forced to be stationary. Let $\left\langle D_{\alpha}^{\prime}: \alpha<\omega_{1}\right\rangle$ be a sequence as in Lemma 2.4.16 and

$$
\tau=\left\{(\check{\alpha}, p): p \in D_{\alpha}^{\prime} \wedge p \Vdash^{+} \check{\alpha} \in \sigma\right\} .
$$

Note that $\mathbb{P} \Vdash \sigma=\tau$. By stat- $\mathrm{N}_{\mathbb{P}}$, we obtain a filter $g$ such that $\tau^{g}$ is stationary. Since $\tau^{h} \subseteq \sigma^{h}$ for all filters $h$, $\sigma^{g}$ is stationary as well. Moreover, $g \cap D_{\alpha}^{\prime} \neq \varnothing$, for stationarily many $\alpha$. By the choice of $\left\langle D_{\alpha}^{\prime}: \alpha<\omega_{1}\right\rangle$, we can extend $g$ to a filter $g^{\prime}$ such that $g^{\prime} \cap D_{\alpha} \neq \varnothing$ for all $\alpha<\omega_{1}$. Moreover, $\sigma^{g} \subseteq \sigma^{g^{\prime}}$, so $\sigma^{g^{\prime}}$ is stationary.

The missing link in Figure 2.8 is:
Question 10. If $\mathbb{P}$ denotes random forcing, does $\mathrm{FA}_{\mathbb{P}, \omega_{1}}$ imply stat- $\mathrm{N}_{\mathbb{P}, \omega_{1}}$ ?
We finally show that the 1-bounded stationary name principle for random forcing is non-trivial, as we discussed at the end of Section 2.4.1.

Lemma 2.4.18. Let $\kappa=2^{\aleph_{0}}$ and assume that every set of size $<\kappa$ is null. ${ }^{13}$ Then stat- $\mathrm{BN}_{\mathbb{P}, \kappa}^{1}$ fails for random forcing $\mathbb{P}$. In particular, CH implies that stat $-\mathrm{BN}_{\mathbb{P}, \omega_{1}}^{1}$ fails.
Proof. It suffices to show that stat- $\mathrm{BN}_{\mathbb{P}, \kappa}^{\omega}$ fails. To see this, apply Corollary 2.2 .14 and use the fact that random forcing is well-met and for any $q \in \mathbb{P}$, the forcing $\mathbb{P}_{q}$ is isomorphic to $\mathbb{P}$ by [25, Theorem 17.41]. Let $\vec{x}=\left\langle x_{\alpha}: \alpha<\kappa\right\rangle$ enumerate all reals. Then $C_{\beta}:=\left\{x_{\alpha}: \alpha<\beta\right\}$ is null for all $\beta<\kappa$. For each $\alpha<\kappa$, let $A_{\alpha}$ be a countable set of approximations to the complement of $C_{\alpha}$ in the following sense:
(a) Each element of $A_{\alpha}$ is a closed set disjoint from $C_{\alpha}$, and
(b) For all $\epsilon>0, A_{\alpha}$ contains a set $C$ with $\mu(C) \geqslant 1-\epsilon$.

[^12]Let $\sigma=\left\{(\check{\alpha}, p): p \in A_{\alpha}\right\}$. Then $\Vdash_{\mathbb{P}} \sigma$ is stationary, since each $A_{\alpha}$ is predense by (b). We claim that there is no filter $g$ in $V$ such that $\sigma^{g}$ is unbounded. If $g$ were such a filter, then we could assume that for every $n \in \omega, g$ contains $N_{t_{n}}$ for some (unique) $t_{n} \in 2^{n}$ by extending $g$. (Clearly $\sigma^{g}$ will remain unbounded.) Let $x=\bigcup_{n \in \omega} t_{n}$ and suppose that $x=x_{\alpha}$. Since $\sigma^{g}$ is unbounded, there is some $\gamma>\alpha$ in $\sigma^{g}$. Find some $p \in A_{\gamma}$ with $p \in g$. By the definition of $A_{\gamma}, p$ is a closed set with $x_{\alpha} \notin p$. Hence $p \cap N_{t_{n}}=\varnothing$ for some $n \in \omega$. But this contradicts the fact that both $p$ and $N_{t_{n}}$ are in $g$.

## Hechler forcing

For $\sigma$-centred forcings $\mathbb{P}$, the principles on the right side of Figure 2.1 are provable in ZFC (see Lemma 2.4.6). A subtle difference appears when we add the requirement that the filter has to meet countably many fixed dense sets. We write $\omega$-ub-FA for this axiom (see Definition 2.1.3). For some forcings, this axiom is stronger that ub-FA. To see this, we will make use of the fact that for Hechler forcing, a filter that meets certain countably many dense sets corresponds to a real. Recall that a subset $A \subseteq \omega^{\omega}$ is unbounded if no $y \in \omega^{\omega}$ eventually strictly dominates all $x \in A$, i.e. $\exists m \forall n \geqslant m x(n)<y(n)$. The next result shows that $\omega$-ub-FA $\omega_{\omega_{1}}$ for Hechler forcing implies the negation of the continuum hypothesis.

Lemma 2.4.19. Let $\mathbb{P}$ denote Hechler forcing. If $\omega$-ub- $\mathrm{FA}_{\mathbb{P}}$ holds, then the size of any unbounded family is at least $\omega_{2}$.

Proof. Towards a contradiction, suppose $\omega$-ub- $\mathrm{FA}_{\mathbb{P}}$ holds and $A$ is an unbounded family of size $\omega_{1}$. Let us enumerate its elements as $\vec{x}=\left\langle x_{\alpha}: \alpha<\omega_{1}\right\rangle$. We define the following dense sets: For $\alpha<\omega_{1}$, we define a real $y_{\alpha}$ by taking a sort of "diagonal maximum" of $\vec{x}$. Let $\pi: \alpha \rightarrow \omega$ be a bijection and let

$$
y_{\alpha}(n)=\max \left\{x_{\gamma}(n): \pi(\gamma) \leqslant n\right\}
$$

It is easy to check that $y_{\alpha}$ is well defined, and that it eventually dominates $x_{\gamma}$ for all $\gamma<\alpha$. We now define

$$
D_{\alpha}=\left\{(s, x) \in \mathbb{P}: x \text { eventually strictly dominates } y_{\gamma}\right\}
$$

For $n<\omega$, let

$$
E_{n}=\{(s, x) \in \mathbb{P}: \text { length }(s) \geqslant n\}
$$

Now let $g \in V$ be a filter meeting unboundedly many $D_{\alpha}$ and all $E_{n}$. Since $g$ meets all $E_{n}$, the first components of its conditions are arbitrarily long. Since all its elements are compatible, this means that the union $\cup\{s:(s, x) \in g\}$ is a real $y$. And $y$ must eventually strictly dominate $x$ for every $(s, x) \in g$. But there are unboundedly many $\alpha$ such that $g$ meets $D_{\alpha}$. For any such $D_{\alpha}$, then, we have $(s, x) \in g$ where $x$ eventually strictly dominates $y_{\alpha}$. Hence, $y$ must eventually strictly dominate unboundedly many $y_{\alpha}$ and hence every $x \in A$. But $A$ was assumed to be unbounded.

## Suslin trees

A Suslin tree is a tree of height $\omega_{1}$, with no uncountable branches or antichains. The existence of Suslin trees is not provable from ZFC, but follows from the axiom $\diamond_{\omega_{1}}$ (below). We can think of a Suslin tree $T$ as a forcing, with $p \leqslant q$ if $p$ is below $q$ in the tree. It is easy to see that the forcing satisfies the c.c.c., and that a generic filter will add a cofinal branch through the tree. Suslin trees are useful tools when we want to prove that axioms about forcing can fail, because adding a Suslin tree using forcing and then and forcing over that tree to collapse it is effectively a variant of Cohen forcing, but the existence of Suslin trees are incompatible with various standard axioms which are preserved by Cohen forcing.

Here, we will also be using Suslin trees to show the failure of simple axioms; in particular, we can show that stat- $\mathrm{BN}_{T, \omega_{1}}^{1}$ fails for most Suslin trees.

Lemma 2.4.20. Suppose $T$ is a Suslin tree. Then stat- $\mathrm{BN}_{T, \omega_{1}}^{\omega}$ fails.
Proof. Let $\sigma=\left\{\langle\alpha, p\rangle: \alpha<\omega_{1}, p \in T\right.$, $\left.\operatorname{height}(p)=\alpha\right\}$. It is easy to see that $\sigma$ is $\omega$ bounded, and is forced to be equal to $\omega_{1}$. But any filter $g \in V$ is a subset of a branch in $V$, and therefore countable. So $\sigma^{g}$ is not stationary, or even unbounded.

Corollary 2.4.21. Suppose that a Suslin tree exists. Then there exists a Suslin tree $T$ such that stat $-\mathrm{BN}_{T, \omega_{1}}^{1}$ fails.

Proof. Let $T$ be any Suslin tree. By the previous lemma we know that stat- $\mathrm{BN}_{T, \omega_{1}}^{\omega}$ fails. But then by Corollary 2.2.13, $T$ contains a subtree $S$ such that stat- $\mathrm{BN}_{S, \omega_{1}}^{1}$ fails.

This also tells us that stat- $\mathrm{BN}_{\mathbb{P}, \omega_{1}}^{1}$ is not equivalent to stat- $\mathrm{BFA}_{\mathbb{P}, \omega_{1}}^{1}$, since the latter is trivially provable for any forcing in ZFC.

In fact, if we assume $\diamond \omega_{1}$ (which is somewhat stronger than the existence of a Suslin tree, see [31, Section 3]) then we can do better than this: we can show that stat- $\mathrm{BN}_{\omega_{1}}^{1}$ fails for every Suslin tree.

Definition 2.4.22. $\diamond \omega_{1}$ says: There is a sequence $\left(A_{\gamma}\right)_{\gamma<\omega_{1}}, A_{\gamma} \subset \gamma$ such that for any stationary set $S \subset \omega_{1}$, the set $\left\{\gamma<\omega_{1}: S \cap \gamma=A_{\gamma}\right\}$ is stationary.

Lemma 2.4.23. Suppose $\diamond_{\omega_{1}}$ holds. If $T$ is a Suslin tree, then stat- $\mathrm{BN}_{T, \omega_{1}}^{1}$ fails.
Proof. Let $\left(A_{\gamma}\right)$ be the sequence given by $\diamond \omega_{1}$. We build up a rank 1 name $\sigma=\left\{(\check{\alpha}, p): \alpha<\gamma, p \in B_{\alpha}\right\}$ recursively as follows.

Suppose we have defined $B_{\gamma}$ for all $\gamma<\alpha$. Consider $\bigcup_{\gamma \in A_{\alpha}} B_{\gamma}$. If this union is predense, then we let $B_{\alpha}=\varnothing$. Otherwise, choose a condition $p \in T$, sitting beyond level $\alpha$ of the Suslin tree, such that $p$ is incompatible with every element of that union. Let $B_{\alpha}=\{p\}$.

If $G$ is a generic filter, then every club $C^{\prime} \subseteq \omega_{1}$ in $V[G]$ contains a club $C \in V$. Hence, to show that $T \Vdash$ " $\sigma$ is stationary" we only need to show that for every club $C \in V$, the set $\bigcup_{\alpha \in C} B_{\alpha}$ is predense. Suppose for some club $C$ that is not the case. For stationarily many $\alpha$, we have that $C \cap \alpha=A_{\alpha}$ and hence the union we are looking at in defining $B_{\alpha}$ is $\bigcup_{\gamma \in A_{\alpha}} B_{\gamma}=\bigcup_{\gamma \in C \cap \alpha} B_{\gamma}$. Hence, the union is not predense, and $B_{\alpha}$ contains an element that is incompatible with every element of $\bigcup_{\gamma \in C \cap \alpha} B_{\gamma}$. But this is true for unboundedly many such $\alpha$, so this gives us an $\omega_{1}$ long sequence of pairwise incompatible conditions, i.e. an uncountable antichain. Since a Suslin tree is by definition c.c.c., this is a contradiction. Hence $T \Vdash$ " $\sigma$ is stationary".

But now let $g \in V$ be a filter. By extending it if necessary, without loss of generality we can assume $g$ is a maximal branch of the tree. Since $g \in V$, we know that $g$ is countable, so let the supremum of the heights of its elements be $\gamma$. Let $\alpha>\gamma$, and let $q \in g$. Since $B_{\alpha}$ is at most a singleton $\{p\}$ with $\operatorname{ht}(p) \geqslant \alpha>\gamma>\operatorname{ht}(q)$, and since $T$ is atomless, we know there is some $r \leqslant q$ with $r \Vdash \alpha \notin \sigma$. Hence $q \Vdash \nVdash \alpha \in \sigma$. Since this is true for all $q \in g$, it follows that $\alpha \notin \sigma^{(g)}$. Hence far from being stationary, $\sigma^{(g)}$ is not even unbounded!

So (assuming the existence of Suslin trees) there are certainly some Suslin trees in which stat-BN ${ }^{1}$ fails. And with strong enough assumptions, we can show that stat- $\mathrm{BN}^{1}$ fails for every tree. So it's natural to ask:

Question 11. Can we show in ZFC that stat- $\mathrm{BN}_{T, \omega_{1}}^{1}$ fails for every Suslin tree $T$ ?
Note that we can show the failure of ub- $\mathrm{BN}_{T, \omega_{1}}^{1}$ for any Suslin tree. Enumerate its level $\alpha$ elements as $\left\{p_{\alpha, n}: n \in \omega\right\}$. Now let

$$
\sigma=\left\{\left(\check{\beta}, p_{\alpha, n}\right): \alpha<\omega_{1}, n \in \omega, \beta=\omega \cdot \alpha+n\right\}
$$

Then $\sigma$ is forced to be unbounded but if $g \in V$ is such that $\sigma^{g}$ is unbounded, then $g$ defines an uncountable branch through $T$.

## Club shooting

The next lemma is a counterexample to the implication club- $\mathrm{BFA}_{\kappa}^{\lambda} \Rightarrow$ club- $\mathrm{BN}_{\kappa}^{\lambda}$ in Figure 2.3. It is open whether there is such a counterexample for complete Boolean algebras.

Suppose that $S$ is a stationary and co-stationary subset of $\omega_{1}$. Let $\mathbb{P}_{S}$ denote the forcing that shoots a club through $S$. Its conditions are closed bounded subsets of $S$, ordered by end extension.

## Lemma 2.4.24.

(1) $\mathrm{BFA}_{\mathbb{P}_{S}, \omega_{1}}^{\omega}$ holds.
(2) club- $\mathrm{BN}_{\mathbb{P}_{S}, \omega_{1}}^{1}$ fails.

In particular, for no $1 \leqslant \lambda \leqslant \omega$ does $\mathrm{BFA}_{\mathbb{P}_{S}, \omega_{1}}^{\lambda}$ imply club- $\mathrm{BN}_{\mathbb{P}_{S}, \omega_{1}}^{\lambda}$.
Proof. (1): We claim that every maximal antichain $A \neq\left\{1_{\mathbb{P}_{S}}\right\}$ is uncountable. (This shows that $\mathrm{BFA}_{\mathbb{P}_{S}, \omega_{1}}^{\omega}$ holds vacuously.) To see this, suppose that $A$ is countable. Let $\alpha=\sup \{\min (p): p \in A\}$ and find some $\beta>\alpha$ in $S$. Then $q=\{\beta\}$ is incompatible with all $p \in A$, so $A$ cannot be maximal.
(2): $\sigma=\check{S}$ is 1-bounded and $\mathbb{P}_{S} \Vdash$ " $\sigma$ contains a club". But for every filter $g, \sigma^{g}=S$ does not contain a club, since $S$ is co-stationary.

### 2.5 Summing Up

The above results show that often, name principles are equivalent to forcing axioms. This provides an understanding of basic name principles $\mathrm{N}_{\mathbb{P}, \kappa}$ and of simultaneous name principles for $\Sigma_{0}$-formulas. For bounded names, the results provide new characterisations of the bounded forcing axioms BFA ${ }^{\lambda}$ for $\lambda \geqslant \kappa$. Name principles are closely related with generic absoluteness and can be used to reprove Bagaria's equivalence between bounded forcing axioms of the form $\mathrm{BFA}^{\kappa}$ and generic absoluteness principles. Bagaria's result has been recently extended by Fuchs [16]. He introduced a notion of $\Sigma_{1}^{1}(\kappa, \lambda)$-absoluteness for cardinals $\lambda \geqslant \kappa$ and proved that it is equivalent to $\mathrm{BFA}_{\kappa}^{\lambda}$. It remains to see if this can be derived from our results.

Several problems about the unbounded name principle ub-FA ${ }_{\kappa}$ remain unclear. The results in Lemmas 2.3.24 and 2.4.1 about obtaining (bounded) forcing axioms from ub-FA for forcings that do not add reals or $<\kappa$-sequences, respectively, hint at possible generalisations (see Question 5). For forcings which add reals, we have that ub-FA $A_{\omega_{1}}$ is trivial for all $\sigma$-linked forcings and implies $\mathrm{FA}_{\omega_{1}}$ for random forcing. In all these cases, ub-FA $\omega_{\omega_{1}}$ and stat- $\mathrm{FA}_{\omega_{1}}$ are either both trivial or both equivalent to $\mathrm{FA}_{\omega_{1}}$. Can we separate ub-FA $\mathrm{A}_{\omega_{1}}$ from stat-FA $\omega_{\omega_{1}}$ (See Question 2)? Can ub-FA $\omega_{\omega_{1}}$ be nontrivial but not imply $\mathrm{FA}_{\omega_{1}}$ ? It remains to study other forcings adding reals and Baumgartner's forcing [8, Section 3] (see Question 3).

The stationary name principle stat- $\mathrm{N}_{\omega_{1}}$ follows from the forcing axiom $\mathrm{FA}_{\omega_{1}}$ for some classes of forcings. For example, for the class of c.c.c. forcings both stat- $N_{\omega_{1}}$ and $\mathrm{FA}_{\omega_{1}}^{+}$are equivalent to $\mathrm{FA}_{\omega_{1}}$ by results of Baumgartner (see Lemma 2.4.7), Todorčević and Veličković [42] (see Lemma 2.4.5). In general, FA ${ }^{+}$goes beyond FA, since being stationary is not first-order over $(\kappa, \epsilon)$. For example, for the class of proper forcings, PFA ${ }^{+}$is strictly stronger that PFA by results of Beaudoin [9, Corollary 3.2] and Magidor (see [38]). So FA ${ }^{+}$ and $\mathrm{BFA}^{+}$do not fall in the scope of generic absoluteness principles, unless one artificially adds a predicate for the nonstationary ideal. Can one formulate PFA ${ }^{+}$as a generic absoluteness or name principle for a logic beyond first order? Some questions remain about the weak variant stat- $\mathrm{BN}_{\mathbb{P}, \omega_{1}}^{1}$ of stat- $\mathrm{N}_{\omega_{1}}$. It is nontrivial for random forcing (see Lemma 2.4.18) and for Suslin trees (see Corollary 2.4.21). What is its relation with other principles? Does stat- $\mathrm{BN}_{\text {c.c.c. }, \omega_{1}}^{1}$ imply $\mathrm{MA}_{\omega_{1}}$ ?

## Chapter 3

## Some more standard concepts

For the other two threads of this thesis, we're going to need a bit more background material. This chapter is devoted to going over that material. We start with a brief look at several large cardinals we're going to be using. After that, we give a detailed exposition of the theory of mice. This takes several pages, and should be suitable even for readers with no prior experience of mice at all. However, many of the proofs are omitted, and we use a simplified definition which omits certain complex variants of mice which we won't be using in the thesis. Readers who want to see these details are advised to look at [36] and [47].

### 3.1 Large Cardinals

"Large cardinal" is a general term for classes of cardinals which may or may not exist. There are a wide variety of them that have been defined, and we will be looking at only a selection of smaller ones. In this section, we define some nonstandard notation which we will be using in the ensuing chapters, so even readers familiar with large cardinals should study the section on inaccessibles.

### 3.1.1 Inaccessibles and Hyperinaccessibles

The smallest kind of large cardinal is the inaccessible.
Definition 3.1.1. An inaccessible (or weakly inaccessible) is an uncountable regular limit cardinal.
We say an inaccessible is simple if it is not a limit of other inaccessibles.
The main focus of the next chapters of this thesis is to investigate these inaccessibles, by partitioning them into simple inaccessibles, simple limits of simple inaccessibles, etc. Motivated by this, we shall define the following (non-standard) heirarchy.

Definition 3.1.2. Reg is the class of all infinite regular cardinals. For $\epsilon \in \operatorname{On}, \operatorname{Reg}_{\epsilon}$ is the class of all regular cardinals of Cantor-Bendixson rank $\epsilon$.

More formally, for $\epsilon \in$ On, we recursively define $\operatorname{Reg}_{<\epsilon}, \operatorname{Reg}_{\geqslant \epsilon}$ and $\operatorname{Reg}_{\epsilon}$ as follows:

$$
\begin{gathered}
\operatorname{Reg}_{<\epsilon}=\bigcup_{\delta<\epsilon} \operatorname{Reg}_{\delta} \\
\operatorname{Reg}_{\geqslant \epsilon}=\operatorname{Reg} \backslash \operatorname{Reg}_{<\epsilon} \\
\operatorname{Reg}_{\epsilon}=\left\{\kappa \in \mathrm{On}: \kappa \text { is a successor of the club generated by } \operatorname{Reg}_{\geqslant \epsilon}\right\}
\end{gathered}
$$

So $\operatorname{Reg}_{<0}$ is empty, $\operatorname{Reg}_{0}=\operatorname{Reg}_{<1}$ is the class of all successor cardinals, $\operatorname{Reg}_{1}=\operatorname{Reg}_{<2}$ is the class of all simple (weak) inaccessibles, and so on. Note that the definition implies that $\omega$ is not in $\operatorname{Reg}_{\epsilon}$ for any $\epsilon$. However, any other regular cardinal will be in some $\operatorname{Reg}_{\epsilon}$ :

$$
\bigcup_{\epsilon<\mathrm{On}} \operatorname{Reg}_{\epsilon}=\operatorname{Reg} \backslash\{\omega\}
$$

Notice that $\operatorname{Reg}_{0}=\operatorname{Reg}_{<1}$ can easily both generate and be generated from Card, the class of all cardinals. So if a statement is provable about Card, then we can sensibly ask if it's also true about $\operatorname{Reg}_{\epsilon}$ or $^{\operatorname{Reg}_{<\epsilon}}$, for larger $\epsilon \in$ On. This is what we shall do, in several different situations, in the following two chapters of this thesis.

Another piece of notation which we shall need is the hyperinaccessible.
Definition 3.1.3. A cardinal $\kappa>0$ is (weakly) hyperinaccessible if it is an element of $\operatorname{Reg}_{\kappa}$.
It is easy to check that such a $\kappa$ must be the smallest element of $\operatorname{Reg}_{\kappa}$, and that no $\kappa$ is an element of $\operatorname{Reg}_{\delta}$ for any $\delta>\kappa$. It should be noted in passing that some authors use "hyperinaccessible" to refer to any element of $\operatorname{Reg} \backslash \operatorname{Reg}_{<2}$; that is, any inaccessible which is a limit of other inaccessibles. In this thesis, we shall only use the definition we gave above: a $\kappa$ which is an element of $\operatorname{Reg}_{\kappa}$.

Unfortunately, there is also another kind of cardinal which is also called inaccessible, and this is more difficult to make unambiguous.

Definition 3.1.4. A cardinal $\kappa$ is inaccessible (or strongly inaccessible) if it is weakly inaccesible, and $\alpha<\kappa$ implies $2^{\alpha}<\kappa$.

Originally, "inaccessible" exclusively meant "weakly inaccessible". However, for several decades, "inaccessible" has also been used to mean "strongly inaccessible", and this has slowly taken over as the default definition. Obviously, this sometimes leads to confusion. We will try to specify whether we mean weak or strong inaccessibility whenever an ambiguity pops up.

For the heirarchy of $\operatorname{Reg}_{\epsilon}$ defined above, dealing with weak inaccessibles is somehow more natural to work with: then $\bigcup_{\epsilon \in \text { On }} \operatorname{Reg}_{\epsilon}$ is simply the class of all regular cardinals. But strong inaccessibles are more interesting in other areas of set theory. So should we instead work with a heirarchy of strong inaccessibles?

Fortunately, most of the results here actually work whether we are talking about weak or strong inaccessibles, so long as we are consistent. We do have to make some minor tweaks to our notation, however:

Definition 3.1.5. Reg $^{s}$ is defined as the class of all cardinals $\kappa$ which are either successor cardinals or strongly inaccessible.
$\operatorname{Reg}_{<\epsilon}^{s}$ and $\operatorname{Reg}_{\epsilon}^{s}$ are defined in the same way as $\operatorname{Reg}_{<\epsilon}$ and $\operatorname{Reg}_{\epsilon}$ respectively, but with $\operatorname{Reg}^{s}$ in place of Reg.

Note that the class Reg ${ }^{s}$ omits the weak inaccessibles which are not strongly inaccessible. $\mathrm{Reg}_{0}^{s}$ is the class of successor cardinals, Reg ${ }_{1}^{s}$ the class of simple strong inaccessibles, etc. Of course, a successor cardinal will never satisfy the extra $2^{\alpha}<\kappa$ property called for in the definition of a strong inaccesible. So in a sense, it would have been more natural to make $\operatorname{Reg}_{0}^{s}$ the simple strong inaccessibles, $\mathrm{Reg}_{1}^{s}$ the simple strong limits of strong inaccessibles, and so on. But then our notation would no longer line up with Reg and it would be much more inconvenient to state our results properly. We can intuitively justify the notation if we think of $R^{s}{ }^{s}$ as coding both the class Card, and the class of all the strong inaccessibles.

Definition 3.1.6. A (strong) hyperinaccessible is a cardinal $\kappa$ which is in $\operatorname{Reg}_{\kappa}^{s}$.
Again, some authors consider a (strong) hyperinaccesible to be an element of $\operatorname{Reg}^{s} \backslash \operatorname{Reg}_{<2}^{s}$. And of course, authors who use "inaccessible" to mean "strong inaccessible" apply the same convention to hyperinaccessibles as well. So there are four different conventions for a hyperinaccessible in the literature.

Notice that if $\kappa \in \operatorname{Reg}_{\epsilon}^{s}$, then $\kappa \in \operatorname{Reg}_{\delta}$ for some $\delta \geqslant \epsilon$. It follows that a strong hyperinaccesible will always also be a weak hyperinaccessible.

### 3.1.2 Mahlo Cardinals

The next level up above these inaccessibles is the Mahlo cardinals (which, yet again, some authors refer to as hyperinaccessible!)

Definition 3.1.7. A strongly inaccessible cardinal $\kappa$ is a Mahlo cardinal if the set of strong inaccessibles below $\kappa$ is stationary.

Proposition 3.1.8. A Mahlo cardinal $\kappa$ is strongly hyperinaccessible.

Proof. Suppose that $\kappa$ is a counterexample. Clearly $\kappa$ is strongly inaccessible, so $\kappa \in \operatorname{Reg}_{\epsilon}^{s}$ for some $0<\epsilon<\kappa$. It follows immediately by definition of $\operatorname{Reg}_{\epsilon}^{s}$ that $\kappa$ is not a limit of elements of $\operatorname{Reg}_{\epsilon}^{s}$, but that it is a limit of elements of $\operatorname{Reg}_{\delta}^{s}$ for all $\delta<\epsilon$. Let $\lambda<\kappa$ be large enough that $[\lambda, \kappa) \cap \operatorname{Reg}_{\epsilon}^{s}=\varnothing$. Let $C \subset \kappa$ be the set of all cardinals in the interval $[\lambda, \kappa)$ which are limits of $\operatorname{Reg}_{\delta}^{s}$ for every $\delta<\epsilon$. It is easy to see that $C$ is closed, and contains no strong inaccessibles.
$C$ is also unbounded: if $\lambda \leqslant \mu<\kappa$ then for $\delta<\epsilon$ let $\mu_{\delta}$ be the first limit of $\operatorname{Reg}_{\delta}^{s}$ above $\mu$. We know that $\kappa$ is a limit of $\operatorname{Reg}_{\delta}$ so $\mu_{\delta} \leqslant \kappa$. In fact since $\operatorname{Cof}\left(\mu_{\delta}\right)=\omega$ we know $\mu_{\delta}<\kappa$. Now let $\nu=\sup _{\delta<\epsilon} \mu_{\delta}$. Then $\nu \geqslant \mu$ is a limit of $\operatorname{Reg}_{\delta}^{S}$ for all $\delta<\epsilon$, and by regularity of $\kappa$ we know that $\nu<\kappa$.

So $C$ is a club below $\kappa$ which doesn't contain any strong inaccessibles, so $\kappa$ is not Mahlo.

### 3.1.3 Measurable Cardinals

The next large cardinal up is one we have met before, back when we were introducing Prikry forcing.
Definition 3.1.9. A cardinal $\kappa$ is measurable if there exists $U \subset \mathcal{P}(\kappa)$ which satisfies the following:

1. $U$ is an ultrafilter on $\mathcal{P}(\kappa)$. That is, $U$ is closed under supersets and intersections, and for any $X \subset \kappa$, exactly one of $X$ and $\kappa \backslash X$ is in $U$.
2. $U$ is non-principal: There is no $\alpha \in \kappa$ such that $\{\alpha\} \in U$.
3. $U$ is $\kappa$ complete: it is closed under intersections of $<\kappa$ many of its elements.

We say a subset of $\kappa$ is measure 1 if it is in $U$, and we call $U$ a $\kappa$ complete measure on $\kappa$.
It can be shown $[21,10.20$ ] that if $\kappa$ is measurable, then we can choose $U$ to satisfy an additional property:
4. $U$ is closed under diagonal intersections of length $\kappa$ : If $\left\{X_{\alpha}: \alpha<\kappa\right\}$ is a sequence of elements of $U$, then

$$
\bigwedge_{\alpha<\kappa} X_{\alpha}:=\left\{\alpha<\kappa: \alpha \in \bigcap_{\beta<\alpha} X_{\beta}\right\} \in U
$$

If $U$ satisfies this additional condition, we call it a normal measure.
Note that in Definition 1.3 .8 we incorporated the fourth condition into the definition for simplicity.
It is possible, by a technical argument, to prove that any measurable cardinal is a Mahlo cardinal. [21, 10.21]

At this level, something rather odd starts to happen. We can use the normal measure $U$ to define an elementary map from $V$ to some new model, by taking an ultrapower.

Proposition 3.1.10. [24, Ch.1, S.5] Let $\kappa$ be a measurable cardinal, with corresponding normal measure $U$. We define the following relations on $V^{\kappa}$ :

$$
\begin{aligned}
f=U g & \Longleftrightarrow\{\alpha \in \kappa: f(\alpha)=g(\alpha)\} \in U \\
f \in_{U} g & \Longleftrightarrow\{\alpha \in \kappa: f(\alpha) \in g(\alpha)\} \in U
\end{aligned}
$$

Then $=_{U}$ is an equivalence relation, and $\epsilon_{U}$ is consistent across members of its equivalence classes. Let $\tilde{V}$ be the structure $\left(V^{\kappa} /=_{U}, \in_{U}\right)$. Then $\tilde{V}$ is elementarily equivalent to $V$, and the map

$$
j_{U}: x \mapsto\left[c_{x}\right]
$$

is an elementary embedding, where $c_{x}$ is the constant function with output $x$. Moreover, $\tilde{V}$ is well-founded.
Since $\tilde{V}$ is well-founded, we tend to also use $\tilde{V}$ to denote its Mostowski collapse, and $j_{U}$ to denote the corresponding map from $V$ into the collapse. It can be shown that the Mostowski collapse is closed under $\kappa$ sequences: $\tilde{V}^{\kappa} \subset \tilde{V}$.

Proposition 3.1.11. The map $j_{U}$ has critical point $\kappa$. (This means it is constant below $\kappa$, but $\left.j_{U}(\kappa)>\kappa\right)$. For $X \subset \kappa, \kappa \in j_{U}(X)$ if and only if $X \in U$.

This gives us a way to generate $U$ from $j_{U}$.
Proposition 3.1.12. [24, 5.6] Let $\kappa$ be a cardinal such that there is an elementary embedding $j: V \rightarrow M$ with critical point $\kappa$, for some transitive class $M$. Then $\kappa$ is measurable in $V$ with normal measure $U$, where $X \in U \Longleftrightarrow X \subset \kappa \wedge \kappa \in j(X)$.

One further simple result will be useful later.
Proposition 3.1.13. Let $j_{U}: V \rightarrow \tilde{V}$ be an ultrapower map, as described above, with critical point $\kappa$. Let $\lambda>\kappa$ be strongly inaccessible in $V$. Then $j_{U}(\lambda)=\lambda$.

Proof. $H_{\lambda}$ is a model of ZFC so we can define the ultrapower construction over it, instead of over $V$. This gives us the map $j_{U} \upharpoonleft H_{\lambda}: H_{\lambda} \rightarrow \tilde{H}_{\lambda}$. The ultrapower construction doesn't add any new ordinals to the universe, so $\tilde{H}_{\lambda} \cap \mathrm{On}=H_{\lambda} \cap \mathrm{On}=\lambda$. So for $\gamma<\lambda$ we know $j_{U}(\gamma)<\lambda$.

Suppose that $j_{U}(\lambda)>\lambda$. We know there is some function $f \in V^{\kappa}$ such that the collapse of $[f]$ is $\lambda$. Since $[f]<\left[c_{\lambda}\right]$, we know that for measure 1 many $\alpha<\kappa, f(\alpha)<c_{\lambda}(\alpha)=\lambda$. Without loss of generality, we can adjust $f$ so that this is true for all $\alpha<\kappa$. We also know that for all $\gamma<\lambda$ there exists (measure 1 many) $\alpha$ such that $f(\alpha)>\gamma$; otherwise [ $f$ ] would be less than $\left[c_{\gamma}\right]$, and we know that [ $c_{\gamma}$ ] collapses to $\gamma$ by the previous paragraph. But then $f$ is a cofinal sequence below $\lambda$ of length $\kappa<\lambda$, contradicting inaccessibility.

### 3.1.4 Supercompacts

Cardinals above measurables are generally defined in terms of the existence of elementary embeddings, rather than in terms of the structure of the cardinal itself. Our final large cardinals, the supercompacts, are a good example of this.

Definition 3.1.14. Let $\lambda \in$ On. A cardinal $\kappa$ is $\lambda$ supercompact if there is an elementary embedding $j: V \rightarrow M$ into a transitive class $M$ which is closed under $\lambda$ sequences, such that $j$ has critical point $\kappa$ and $j(\kappa)>\lambda$. A cardinal $\kappa$ is supercompact if it is $\lambda$ supercompact for all $\lambda \in$ On.

Obviously, supercompact implies $\lambda$ supercompact, which in turn implies measurable by Proposition 3.1.12. A cardinal $\kappa$ being $\lambda$ supercompact can also be expressed in terms of the existence of a certain kind of ultrafilter. This time, however, the ultrafilter is not on $\kappa$, but on the $\kappa$ size subsets of $\lambda$.

Definition 3.1.15. Let $\kappa<\lambda$ be cardinals. Let $U$ be a collection of subsets $\mathcal{P}_{\kappa}(\lambda)$. (So the elements of a member of $U$ are all subsets of $\lambda$ of cardinality $\kappa$. The members of $U$ themselves can more than $\kappa$ many elements.) We say that $U$ is a normal ultrafilter on $\mathcal{P}_{\kappa}(\lambda)$ if:

1. $U$ is an ultrafilter. That is, $U$ is closed under intersections and supersets, and for all $X \subset \mathcal{P}_{\kappa}(\lambda)$, either $X \in U$ or $\mathcal{P}_{\kappa}(\lambda) \backslash X \in U$;
2. $<\kappa$ completeness: $U$ is closed under intersections of size $<\kappa$;
3. Fineness: For any $\alpha<\lambda,\left\{x \in P_{\kappa}(\lambda): \alpha \in x\right\} \in U$; and
4. Normality: For any sequence $\left\{X_{\alpha}: \alpha<\lambda\right\}$ of elements of $U$, the diagonal intersection

$$
\bigwedge_{\alpha<\lambda} X_{\alpha}:=\left\{x \in P_{\kappa}(\lambda): x \in \bigcap_{\alpha \in x} X_{\alpha}\right\} \in U
$$

Lemma 3.1.16. [24, 22.7,22.11] Let $\kappa \leqslant \lambda$. Then $\kappa$ is $\lambda$ supercompact if and only if there is a normal ultrafilter on $\mathcal{P}_{\kappa}(\lambda)$ (the set of cardinality $\leqslant \kappa$ subsets of $\lambda$ ). More specifically, this ultrafilter generates the embedding $j: V \rightarrow M$, using the ultrapower construction given in Proposition 3.1.10 (with $\kappa$ replaced by $\left.\mathcal{P}_{\kappa}(\lambda)\right)$ followed by a Mostowski collapse.

Moreover, when we do this construction, we find for $\alpha<\lambda$ that $\alpha$ is the equivalence class of the function $f$ which takes $x \in \mathcal{P}_{\kappa}(\lambda)$ to o.t. $(x \cap \alpha)$. Similarly, $j^{\prime \prime} \alpha$ is the equivalence class of the function $g$ which takes $x \in \mathcal{P}_{\kappa}(\lambda)$ to $x \cap \alpha$.

If $\kappa$ is $\kappa^{+}$supercompact, then there is an embedding and ultrafilter which are minimal, in the sense that they destroy that supercompactness.
Lemma 3.1.17. Let $\kappa$ be a $\lambda$ supercompact cardinal for some $\lambda>\kappa$. Then there is an embedding $j: V \rightarrow M$, consistent with the definition of $\lambda$ supercompactness, such that $\kappa$ is no longer $\lambda$ supercompact in $M$.

Proof. We adapt a technique from [30,1.1]. Let $j: V \rightarrow M$ be a $\lambda$ supercompact embedding with critical point $\kappa$, such that $j(\kappa)$ is minimal. Suppose that $\kappa$ is $\lambda$ supercompact in $M$. Then we can find a $\lambda$ supercompact embedding $j^{*}: M \rightarrow M^{*}$. This corresponds to a normal ultrafilter $U^{*}$ on $P_{\kappa}(\lambda)$ in $M$. Since $M \subset V$ is closed under $\lambda$ sequences, $U^{*}$ is also normal in $V$. So it also defines a $\lambda$ supercompact embedding $\tilde{j}: V \rightarrow \tilde{M}$. Since $j(\kappa)$ is an inaccessible above $\lambda$ in $M$, a simple cofinality argument shows that $j^{*}(\kappa)<j(\kappa)$. But $j^{*}(\kappa)=\tilde{j}(\kappa)$ since $M$ and $V$ contain the same subsets of $\kappa$, and by assumption $\tilde{j}(\kappa) \geqslant j(\kappa)$. Contradiction.

There is one further standard fact about supercompacts that we will be using.
Lemma 3.1.18 (Laver). [27] There is a function $h: \kappa \rightarrow V_{\kappa}$ such that given any $x \in V$ and any $\mu \geqslant \kappa$, there is an $M$ with $M^{\mu} \subset M$ and an embedding $j: V \rightarrow M$ with critical point $\kappa$, such that $j(\kappa)>\mu$ and $j(h)(\kappa)=x$.

### 3.2 Introduction to Mice

Consider again a measurable $\kappa$ with normal measure $U$. As we have seen, there is an elementary map $j_{U}: V \rightarrow M$ (for some transitive class $M$ ) with critical point $\kappa$. By elementarity, $M$ believes that $U^{\prime}:=j_{U}(U)$ is a normal measure on $\kappa^{\prime}:=j_{U}(\kappa)$. So we can find another elementary map $j_{U^{\prime}}: M \rightarrow M^{\prime}$ with critical point $\kappa^{\prime}$. By composing $j_{U}$ and $j_{U^{\prime}}$, we can of course get an elementary map $j: V \rightarrow M^{\prime}$, which has critical point $\kappa$ and which sends $\kappa$ to $\kappa^{\prime \prime}:=j_{U^{\prime}}\left(\kappa^{\prime}\right)$.

We can continue this process indefinitely, and if we have multiple measurables, then we can jump around taking ultrapowers with respect to different measurables at each stage.

This allows us to construct some interesting models of ZFC. But the true power of this process is only realised if we can continue it transfinitely often, which requires us to have some way to handle limit stages. The theory of mice was designed to allow us to do this in a controlled environment.

A mouse $M$ is a special kind of set-sized model $J$ of ZFC ${ }^{-}$(that is, ZFC without powerset), together with a predicate giving a normal measure on its largest cardinal, and another predicate giving a list of normal measures contained within $J$ on smaller cardinals. It is defined in such a way that we can take transfinitely many ultrapowers using any normal measures in $M$, including the normal measure on the largest cardinal.

The full definition of a mouse requires a great many properties to hold, and is therefore quite long. The entirety of Section 3.2 is devoted to formally defining mice, and their class-size analogues (weasels). However, there's no need to get bogged down in the details of the definitions in the early subsections. They're just stepping stones to defining a mouse, and most of them will never be explicitly invoked in this thesis.

The section will also contain several results, but the proofs will mostly be omitted in the interests of brevity. They can be found in [36] and [47].

### 3.2.1 Rudimentary Functions and J Structures

The first step to defining a mouse is to introduce the " $J$ heirarchy", which is a slightly different variant of Gödel's $L$-hierarchy of the constructible universe. Recall that $L_{0}=\varnothing, L_{\alpha+1}$ is the set of all definable subsets of $L_{\alpha}$, and at limits we take unions.

The $J$ heirarchy is similar, but instead of throwing in every definable set at each successor stage of the hierarchy, we just include sets which can be reached by some rather simple functions. There are a few different ways to define these "rudimentary" functions and the specific definition isn't very important; here, we use the formulation of $[36,1.7]$.

Definition 3.2.1. A function $f: V^{2} \rightarrow V$ is basic rudimentary if it is of one of the following:

1. $f(x, y)=\{x, y\}$
2. $f(x, y)=x \backslash y$
3. $f(x, y)=x \times y$
4. $f(x, y)=\{\langle u, z, v\rangle: z \in x,\langle u, v\rangle \in y\}$
5. $f(x, y)=\{\langle u, v, z\rangle: z \in x,\langle u, v\rangle \in y\}$
6. $f(x, y)=\cup x$
7. $f(x, y)=\operatorname{dom}(x)$
8. $f(x, y)=\in 1(x \times x)$
9. $f(x, y)=\left\{x^{\prime \prime}(z): z \in y\right\}$
10. $f(x, y)=\langle x, y\rangle$
11. $f(x, y)=x^{\prime \prime}(y)$
12. $f(x, y)=\langle u, x, v\rangle$ if $y=\langle u, v\rangle$; or $\varnothing$ if $y$ is not an ordered pair
13. $f(x, y)=\langle v, w, x\rangle$ if $y=\langle v, w\rangle$; or $\varnothing$ if $y$ is not an ordered pair
14. $f(x, y)=\{u,\langle v, x\rangle\}$ if $y=\langle v, w\rangle$; or $\varnothing$ if $y$ is not an ordered pair
15. $f(x, y)=\{u,\langle x, v\rangle\}$ if $y=\langle v, w\rangle$; or $\varnothing$ if $y$ is not an ordered pair

If $\vec{A}=\left\langle A_{0}, \ldots, A_{n}\right\rangle$ is a finite collection of sets or classes, we say $f$ is basic rudimentary in $\vec{A}$ if it is either basic rudimentary or is

$$
f(x, y)=A_{i} \cap x \text { for some } 0 \leqslant i \leqslant n
$$

If $n=0$ then we instead say that $f$ is basic rudimentary in $A_{0}$.
We say a function $f: V^{k} \rightarrow V$ is rudimentary (or rudimentary in $\vec{A}$ or $A_{0}$ ) if it can be generated by composition of functions which are basic rudimentary (resp. basic rudimentary in $\vec{A}$ or $A_{0}$ ).

Given a set $X$, let us write $S(X)$ to denote $X \cup \bigcup\left\{f^{\prime \prime} X: f\right.$ is basic rudimentary $\}$. We write $\operatorname{rud}(X)$ to denote the closure of $X$ under rudimentary functions, which is equal to $\bigcup_{n<\omega} S^{n}(X)$.

Similarly, we will write $S^{A}$ and rud ${ }^{A}$ to define analogous concepts with rudimentary in $A$ functions (where $A$ is a finite tuple of predicates, or is a single predicate).

We use rudimentary functions to define an analogue of the $L$ hierarchy.
Definition 3.2.2. We recursively define:

$$
\begin{aligned}
J_{0} & =\varnothing \\
J_{\alpha+1} & =\operatorname{rud}\left(J_{\alpha}\right) \\
J_{\lambda} & =\bigcup_{\alpha<\lambda} J_{\alpha}
\end{aligned}
$$

We write $J_{\mathrm{On}}$ to denote the union of all the $J_{\alpha}$ for $\alpha \in \mathrm{On}$.
Occasionally, it's useful to look more carefully at these jumps from $J_{\alpha}$ to $J_{\alpha+1}$, and for that purpose we define the $S$ hierarchy:

$$
\begin{aligned}
S_{0} & =\varnothing \\
S_{\alpha+1} & =S\left(S_{\alpha}\right) \\
S_{\lambda} & =\bigcup_{\alpha<\lambda} S_{\alpha}
\end{aligned}
$$

For $A$ a predicate or a finite tuple of predicates, we define $J_{\alpha}^{A}$ and $S_{\alpha}^{A}$ similarly.
For any $\alpha$ we can easily see $J_{\alpha}=S_{\omega \alpha}$ and $J_{\alpha}^{A}=S_{\omega \alpha}^{A}$. It is also possible to show that $L[A]=\bigcup_{\text {On }} J_{\alpha}^{A}$.
Confusingly, some authors (including [36]) write $J_{\omega \alpha}^{A}$ to denote what we are calling $J_{\alpha}^{A}$, and then leave $J_{\beta}^{A}$ undefined where $\beta$ is a successor ordinal. This makes it match up more neatly with the $S$ hierarchy, but means the notation becomes more messy everywhere else. We will not be using that definition here, but it is worth keeping in mind when reading [36] and other literature about fine structure.

We're going to be defining a mouse to be a level $J_{\alpha}^{A}$ of some $J$ hierarchy, together with some predicates (including $A$ ) telling us about the measurable cardinals the mouse has.

### 3.2.2 J structures

We shall now meet, in quick succession, a series of properties that $A$ and $J_{\alpha}^{A}$ should satisfy.
Definition 3.2.3. Let $A_{0}, \ldots, A_{n}$ be predicates. A structure $\left\langle X, \epsilon, A_{0}, \ldots, A_{n}\right\rangle$ is amenable if for all $0 \leqslant$ $i \leqslant n$ and $x \in X$, we have $x \cap A_{i} \in X$.

Definition 3.2.4. $[36,1.9]$ A $J$-structure is a structure of the form $\left\langle J_{\alpha}^{A}, \epsilon, A, B\right\rangle$ which is amenable.
We often write $\left\langle J_{\alpha}^{A}, B\right\rangle$ as a shorthand for $\left\langle J_{\alpha}^{A}, \in, A, B\right\rangle$, leaving the $\in$ and $A$ implicit.
Definition 3.2.5. A $J$-structure $\left\langle J_{\alpha}^{A}, \in, A, B\right\rangle$ is acceptable if the following holds: For all ordinals $\gamma<\beta<\alpha$, with $\beta \in \operatorname{Lim}$, if $\mathcal{P}(\gamma) \cap J_{\beta+1}^{A} \notin J_{\beta}^{A}$ then we can find a surjection $f: \gamma \rightarrow \beta$ in $J_{\beta+1}^{A}$.

Unpicking this definition, it essentially says that if we add a new subset of $\gamma$ at stage $\beta+1$, then $|\gamma|=|\beta|$ and $J_{\beta+1}^{A}$ knows it.

We always have that $\alpha \in J_{\alpha}^{A}$. If the $J$-structure is acceptable, then every (infinite) cardinal $\kappa$ of $V$ is a "fixed point" of the hierarchy, in the sense that $J_{\kappa}^{A}$ is closed under subsets of cardinality less than $\kappa$ which exist anywhere in the $J$-structure.

Lemma 3.2.6. If $M=\left\langle J_{\alpha}^{A}, A, B\right\rangle$ is an acceptable $J$ structure, and $\rho \in M$ is an infinite successor cardinal of $M$, then $J_{\rho}^{A}$ is a model of ZFC ${ }^{-}$(i.e. ZFC without powerset) relativised to $A$. Moreover, if $\rho=\kappa^{+}$and $\kappa$ is strongly inaccessible in $M$, then $J_{\rho}^{A}$ is a model of Set MK** relativised to $A$.

Proof. The proof for ZFC ${ }^{-}$is [36, 1.25]. The additional axioms of Set MK ${ }^{* *}$ aren't explicitly proved there, but work in essentially in the same way: as part of $[36,1.24]$ we show that if a suitably small set is in $M$ then it's actually in $J_{\rho}^{A}$, and we can invoke rudimentary closure to show the sets required by the extra axioms of Set MK ${ }^{* *}$ are in $M$.

### 3.2.3 Soundness

This section is particularly technical, working towards defining a single "soundness" property we want our mice to have. Nonetheless, it may be worthwhile to at last skim through it, because we will be using the $n^{\prime}$ th projectum briefly a few times in Chapter 4 . Throughout this section, we fix an acceptable $J$-structure $M=\left\langle J_{\alpha}^{A}, B\right\rangle$.
Definition 3.2.7. [36, 2.1] We define the first projectum $\rho(M)$ to be the least ordinal $\rho$ such that there is a $\boldsymbol{\Sigma}_{1}^{M}$ subset of $\rho$ which is not an element of $M$.

Definition 3.2.8. For $p \in(\mathrm{On} \cap M)^{<\omega}$, the standard code $A^{p}$ of $p$ is the set of Gödel codes for formulae with parameter $p$ and parameters in $H_{\rho}$ which $M$ believes. Formally,

$$
A^{p}=\left\{\langle i, q\rangle: i \in \omega, \vec{q} \in\left(H_{\rho}^{M}\right)^{<\omega}, M \models \varphi_{i}(p, q)\right\}
$$

Note that $A^{p}$ depends on $A, B$ and $M$. From the perspective of $A^{p}$, the only interesting parts of the parameter $p$ are those which are at least $\rho(M)$, as the rest is simply contained in $H_{\rho}^{M}$ anyway. So we'll only work with $p \in[\rho(M) \text {, On } \cap M)^{<\omega}$ when defining properties about $p$.

We can define two classes of parameters $p$.
Definition 3.2.9. Let $p \in[\rho(M) \text {, On } \cap M)^{<\omega}$. We say $p$ is good if there is some set $Y$ which is $\Sigma_{1}(p)$ such that $Y \cap \rho$ is not in $M$. We write $P$ or $P_{M}$ for the set of good parameters.

We say $p$ is very good if the $\Sigma_{1}$ Skolem hull $h_{M}(\rho \cup\{p\})=J_{\alpha}^{A}$ and write $R$ or $R_{M}$ for the set of very good parameters.

So essentially, a parameter is good if we can use it to get to a set which is outside $M$, in a $\Sigma_{1}$ way. The parameter is very good if we can get to every element of $M$ in a $\Sigma_{1}$ way. It isn't directly relevant to us, but it can be shown that any very good parameter is also good, so the terminology here is sensible. It's also possible to show that $P_{M} \neq \varnothing$.

We can extend these concepts to the $n$ 'th projectum $\rho_{n}$ for $n \in \omega$.
Definition 3.2.10. [36,5.1] Let $n \in \omega$. We recursively define the ordinal $\rho_{n}$, the set of parameters $\Gamma_{n}$, and the predicate $A^{n, p}$ and model $M^{n, p}$ for each $p \in \Gamma_{n}$ as follows:

$$
\begin{aligned}
\rho_{0} & =\alpha \\
\Gamma_{0} & =\{\varnothing\} \\
A^{0, \varnothing} & =\varnothing \\
M^{0, \varnothing} & =M \\
\rho_{n+1} & =\min \left\{\rho\left(M^{n, p}\right): p \in \Gamma_{n}\right\} \\
\Gamma_{n+1} & =\left\{\left(p_{0}, \ldots, p_{n}\right):\left(p_{0}, \ldots, p_{n-1}\right) \in \Gamma_{n}, p_{n} \in\left[\rho_{n+1}, \rho_{n}\right)^{<\omega}\right\} \\
A^{n+1, p} & =A^{p_{n}} \text { in the sense of } M^{n, p \uparrow n} \\
M^{n+1, p} & =\left\langle J_{\rho_{n+1}}^{A}, A^{n+1, p}\right\rangle
\end{aligned}
$$

Note that $\left(\rho_{n}\right)_{n \in \omega}$ is a descending sequence of ordinals, so it stabilises at some minimum value $\rho_{k}$. We call that minimum $\rho_{\omega}$.

We again define two classes of parameters in $\Gamma_{n}$.
Definition 3.2.11. The class of good parameters $P_{n}=P_{n}^{M} \subset \Gamma_{n}$ is defined recursively:

$$
\begin{aligned}
P_{0} & =\varnothing \\
P_{n+1} & =\left\{p \in \Gamma_{n+1}: p \upharpoonleft n \in P_{n}, \rho\left(M^{n, p 1 n}\right)=\rho_{n+1}, p_{n} \in P_{M^{n, p 1 v}}\right\}
\end{aligned}
$$

The class of very good parameters $R_{n}=R_{n}^{M}$ is defined similarly:

$$
\begin{aligned}
R_{0} & =\varnothing \\
R_{n+1} & =\left\{p \in \Gamma_{n+1}: p \upharpoonleft n \in R_{n}, \rho\left(M^{n, p \uparrow n}\right)=\rho_{n+1}, p_{n} \in R_{M^{n, p 1 v}}\right\}
\end{aligned}
$$

We can easily see by induction that $R_{n} \subset P_{n} \neq \varnothing$ for all $n$, given the result that $R_{M^{\prime}} \subset P_{M^{\prime}} \neq \varnothing$ for any $M^{\prime}$.

We can now give the definition we've been working towards in this section:
Definition 3.2.12. [36,5.7] For $n<\omega, M$ is $n$-sound if $R_{n}=P_{n} . M$ is sound (or $\omega$-sound) if it is $n$-sound for all $n$.

More generally, $M$ is $\alpha$ sound for an ordinal $\alpha$ if for all $n \in \omega$, any element of $P_{n}$ which has no terms below $\alpha$ is in fact in $R_{n}$. (If there is some $n$ such that $\rho_{n+1} \leqslant \alpha<\rho_{n}$ then we can equivalently just ask for this statement to hold for that $n$.)

There's one extra tool which it's appropriate to mention here. Occasionally, it's useful to be able to take some arbitrary parameter, in some definable and absolute way.

Definition 3.2.13. Let $n \in \omega$. The $n$ 'th standard parameter is the first element of $P_{n}$, ordered lexicographically.

Proposition 3.2.14. [36, 6.5] Suppose $M$ is $n$ sound. Let $m<n \in \omega$. The $n$ 'th standard parameter is an end extension of the $m$ 'th standard parameter.

### 3.2.4 Ultrapowers of Rudimentary Closed Structures

We now have enough to say what the domain of a mouse should be: it will be an acceptable $J$ structure $J_{\gamma}^{A}$, whose initial segments are all sound, where $A$ also satisfies some other properties we'll introduce shortly. We're now ready to start considering the measurables of our mouse, and working our way up to iterating a sequence of ultrapowers.

The first difficulty we encounter is that $J$ structures are not models of the whole of ZFC: for a start, the ones we're interested in will have a largest cardinal. So we'll have to do some work to ensure that, given a measurable cardinal, we can take a well-defined ultrapower to get a transitive structure, and more work to ensure that it will be elementarily equivalent to the original structure. That is the goal of this chapter.

We can get pretty close to what we want by naively applying Proposition 3.1.10 to $M$ directly. Recall that any $J$ structure is rudimentary closed by definition.

Lemma 3.2.15. [36, 8.4] Let $M=\left\langle J_{\alpha}^{A}, A, B\right\rangle$ be an acceptable $J$ structure. Let $U$ be a collection of subsets of some $M$ cardinal $\kappa$ such that either:

1. $\kappa$ is not the largest cardinal of $M$ and $U \in M$; or
2. $\kappa$ is the largest cardinal of $M$, and $U=B$ for some $i$.

Suppose further that $M$ believes $U$ is a normal measure on $\kappa$.
Let $=_{U}$ and $\epsilon_{U}$ be defined as in Proposition 3.1.10. Let

$$
A_{U}(f) \Longleftrightarrow\{\alpha<\kappa: A(f(\alpha))\} \in U
$$

and define $B_{U}$ likewise.
Let $\tilde{M}=\left\langle\left(\left(J_{\alpha}^{A}\right)^{\kappa} \cap\left(J_{\alpha}^{A}\right)\right) /={ }_{U}, \epsilon_{U}, A_{U}, B_{U}\right\rangle$. Then the map $j_{U}: M \rightarrow \tilde{M}$ defined in Proposition 3.1.10 is $\Sigma_{1}$ elementary (even in the language with symbols for $P_{0}, \ldots, P_{n}$ ) and has critical point $\kappa$.

In particular, this implies that $\left(\left(J_{\alpha}^{A}\right)^{\kappa} \cap\left(J_{\alpha}^{A}\right) /=_{U}\right)=J_{\beta}^{A_{U}}$ for some $\beta$, and therefore that $\tilde{M}$ is a $J$ structure.

This is all well and good. But we really want the ultrapower embedding to preserve more than just $\Sigma_{1}$ formulas. We will get full elementarity if $M \vDash \mathrm{ZFC}^{-}$. But we can't ensure that will always be the case, and in fact it's not possible to get a fully elementary embedding in most cases. But we can get a bit closer than just $\Sigma_{1}$ elementarity, if we define the ultrapower in a slightly different way. Rather than taking the quotient of $\left(J_{\alpha}^{A}\right)^{\kappa} \cap\left(J_{\alpha}^{A}\right)$, we will take the quotient of $\left(J_{\alpha}^{A}\right)^{\kappa} \cap \operatorname{Def}_{S}\left(J_{\alpha}^{A}\right)$, where $S$ is some class of formulas, and $\operatorname{Def}_{S}\left(J_{\alpha}^{A}\right)$ is the collection of all sets which are $S$ definable over $M$.

What should $S$ be? If we want $\Sigma_{n}$ elementarity, then our first instinct might be to take $S=\boldsymbol{\Sigma}_{n}$. But this turns out not to work: we need to use a rather more complicated class. For this definition, let $\mathcal{L}$ be a first order language where each variable $v_{m}^{n}$ has two indices.

Definition 3.2.16. [47, 1.6] Let $\varphi(\vec{v})$ be a formula of $\mathcal{L}$. Recursively, we say $\varphi$ is $\Sigma_{0}^{(n)}$ if it is of the form $\psi\left(\chi_{0}(\vec{v}), \ldots, \chi_{k}(\vec{v})\right)$, where $\chi_{0}(\vec{v}), \ldots, \chi_{k}(\vec{v})$ are all $\Sigma_{0}^{(n-1)}$ formulas, and all the quantifiers of $\psi$ are of the form $\forall v_{m}^{n} \in v_{i}^{j}$ or $\exists v_{m}^{n} \in x_{i}^{j}$, with $j \geqslant n$.

We say $\varphi$ is $\Sigma_{1}^{(n)}$ if it is of the form $\exists v_{m}^{n} \psi$ where $\psi$ is a $\Sigma_{0}^{(n)}$ formula.
Recall from the last chapter that $\rho_{n}$ is the $n$ 'th projectum of $M$. If $M$ is a $J$-structure, we interpret a formula in $\mathcal{L}$ by letting $x_{i}^{j}$ range over $H_{\rho_{j}}$ :

Definition 3.2.17. Let $M$ be a $J$-structure, and let $X \subset M$. We say that $X$ is $\boldsymbol{\Sigma}_{1}^{(n)}$ definable over $M$ if we can find a $\Sigma_{1}^{(n)}$ formula $\varphi\left(v_{i_{0}}^{j_{0}}, \ldots, v_{i_{m}}^{j_{m}}, v^{*}\right)$ and parameters $p_{k} \in H_{\rho_{j_{k}}}^{M}$ for $0 \leqslant k \leqslant m$, such that

$$
X=\left\{x \in M: \tilde{\varphi}\left(p_{0}, \ldots, p_{m}, x\right)\right\}
$$

where $\tilde{\varphi}$ is obtained from $\varphi$ by replacing all instances of $\forall v_{i}^{j}$ and $\exists v_{i}^{j}$ with $\forall v_{i, j} \in H_{\rho_{j}}^{M}$ and $\exists v_{i, j} \in H_{\rho_{j}}^{M}$ respectively.

We say $X$ is $\boldsymbol{\Sigma}_{1}^{(\omega)}$ definable over $M$ if it is $\boldsymbol{\Sigma}_{1}^{(n)}$ definable for some $n \in \omega$.
We write $\boldsymbol{\Sigma}_{1}^{(n)}(M)$ for the class of all sets which are $\boldsymbol{\Sigma}_{1}^{(n)}$ definable over $M$, and likewise for $\boldsymbol{\Sigma}_{1}^{(\omega)}$.
Lemma 3.2.18. [47, 3.5] Let $M=\left\langle J_{\alpha}^{A}, A, B\right\rangle$ be an acceptable $J$ structure. Let $U$ be a collection of subsets of some $M$ cardinal $\kappa$ such that either:

1. $\kappa$ is not the largest cardinal of $M$ and $U \in M$; or
2. $\kappa$ is the largest cardinal of $M$, and $U=B$ for some $i$.

Suppose further that $M$ believes $U$ is a normal measure on $\kappa$.
Let $m \leqslant \omega$ and suppose $\kappa<\rho_{m}$. For $f, g \in\left(J_{\alpha}^{A}\right)^{\kappa} \cap \boldsymbol{\Sigma}_{1}^{(m)}\left(J_{\alpha}^{A}\right)$, let us define:

$$
\begin{aligned}
f=_{U} g & \Longleftrightarrow \exists X \in U: X \subset\{\alpha<\kappa: f(\alpha)=g(\alpha)\} \\
f \epsilon_{U} g & \Longleftrightarrow \exists X \in U: X \subset\{\alpha<\kappa: f(\alpha) \in g(\alpha)\} \\
A_{U}(f) & \Longleftrightarrow \exists X \in U: X \subset\{\alpha<\kappa: A(f(\alpha))\} \\
B_{U}(f) & \Longleftrightarrow \exists X \in U: X \subset\{\alpha<\kappa: B(f(\alpha))\}
\end{aligned}
$$

Then $=_{U}$ is an equivalence relation, and $\epsilon_{U}$ and $F_{U}$ are consistent across members of its equivalence classes. Let

$$
\tilde{M}=\left\langle\left(\left(J_{\alpha}^{A}\right)^{\kappa} \cap \Sigma^{*}\left(J_{\alpha}^{A}\right)\right) /==_{U}, \epsilon_{U}, A_{U} B_{U}\right\rangle
$$

Then $\tilde{M}$ is $\Sigma_{1}^{(m)}$ elementarily equivalent to $M$, and the map

$$
j_{U}: x \mapsto\left[c_{x}\right]
$$

is a $\Sigma_{1}^{(m)}$, cofinal elementary embedding. We call $\tilde{M}$ the $\boldsymbol{\Sigma}_{1}^{(m)}$ ultrapower of $M$ with critical point $\kappa$.
Again, $\tilde{M}$ is a $J$-structure.
Although the embedding isn't fully elementary, that's only because of the extra information added by $A$ and $B$. The embdedding of the underlying domains is fully elementary.
Lemma 3.2.19. Let $M=\left\langle J_{\alpha}^{A}, A, B\right\rangle$ be as above, and let $U, \tilde{M}$ and $j_{U}$ be as above. $S a y \tilde{M}=\left\langle J_{\beta}^{A_{U}}, A_{U}, B_{U}\right\rangle$. Then $j_{U}: J_{\alpha}^{A} \rightarrow J_{\beta}^{A_{U}}$ is a fully elementary embedding.

Taking a single ultrapower never increases the cardinality of $M$ in the sense of $V$ :
Proposition 3.2.20. Let $\tilde{M}$ be as above. Then $|\tilde{M}|=|M|$
Proof. For simplicity, assume $M$ is transitive. We know $\kappa \in M$ so $|M| \geqslant|\kappa|$. Now $\tilde{M}$ is a quotient of some subclass of $\left(M^{\kappa}\right)^{V}$, and so has cardinality at most $\left|\left(M^{\kappa}\right)^{V}\right|=|M|$. On the other hand, $j_{U}: M \rightarrow \tilde{M}$ is injective, so $|M| \leqslant|\tilde{M}|$.

Notice that in the context of a model of ZFC, this definition of an ultrapower would agree with the simpler one in Proposition 3.1.10. The only thing missing now is well-foundedness of $\tilde{M}$. We cannot prove that $\tilde{M}$ is well founded. So instead, we'll make it part of the definition of a mouse that all the ultrapowers we could want to take are well founded. We'll do this later on, since we're going to need some more technology before we can formalise "all the ultrapowers we could want to take". For now, we shall simply introduce a piece of notation for use when the ultrapower is well founded:

Definition 3.2.21. Let $M$ and $\kappa$ be as in Lemmas 3.2.15 and 3.2.18. Let $m \leqslant \omega$ be maximal such that $\kappa<\rho_{m}$. Let $\tilde{M}$ be the $\Sigma_{1}^{(m)}$ ultrapower as above, and suppose it is well founded. Then we define Ult $(M, U)$ to be the transitive collapse of $\tilde{M}$, and we define the ultrapower map $\pi_{U}: M \rightarrow \operatorname{Ult}(M, U)$ as the composition of $j_{U}$ and the transitive collapsing map.

Recall that an ultrapower of $V$ is closed under $\kappa$ sequences, so in particular it agrees with $V$ on subsets of $\kappa$. This is not automatically true here, but will be true for a $J$ structure as a consequence of amenability.

Proposition 3.2.22. [36, 8.10] Let $M=\left\langle J_{\alpha}^{A}, \in, A, B\right\rangle$ be an n-sound $\underset{\sim}{J}$ structure, with a measurable cardinal $\kappa \in\left(\rho_{n+1}, \rho_{n}\right)$. Let $\tilde{M}$ be the ultrapower defined above. Then $M$ and $\tilde{M}$ contain the same subsets of $\kappa$.

Actually, we can go slightly further than this: the ultrapower is closed under $\kappa$ sequences.
Proposition 3.2.23. Let $M, \tilde{M}$ and $\kappa$ be as in the previous proposition. If $S=\left(x_{\alpha}\right)_{\alpha<\kappa} \in M$ is a sequence of length $\kappa$, and for all $\alpha<\kappa x_{\alpha} \in \tilde{M}$, then $S \in \tilde{M}$.

Connecting with the previous section, we can also say despite not being definable within $M$, the projecta of $M$ are to some extent respected by the embedding.

Proposition 3.2.24. [47, 3.2.1, 3.2.2] Suppose that $\rho_{n+1}^{M} \leqslant \kappa<\rho_{n}^{M}$. Then $\rho_{n+1}^{\tilde{M}} \leqslant \rho_{n+1}^{M}$, and $\rho_{n}^{\tilde{M}}=\rho_{n}^{M}$.

### 3.2.5 Extender Sequences

It is now time to start introducing the collections of measurables we are going to be working with. Recall that a $J$ structure has the form $\left\langle J_{\alpha}^{A}, \epsilon, A, B\right\rangle$ where $A$ and $B$ are predicates. For a mouse, we're going to want $B$ to be a normal measure on its largest cardinal, and $A$ to be a (possibly empty) sequence of normal measures on smaller cardinals. We formalise this with the concepts of a filter sequence and of an extender sequence.

First, we need an important piece of technical notation:
Definition 3.2.25. Let $M=\left\langle J_{\gamma}^{E}, \in, E, F\right\rangle$ be a $J$ structure, where $\gamma \in \operatorname{On} \cup\{\mathrm{On}\}$. Let $\kappa$ be the largest cardinal of $M$; or let $\kappa=\mathrm{On} \cap M$ if $M$ has no largest cardinal. Suppose that $E: \kappa \rightarrow J_{\gamma}^{E}$ is some class function which maps each $\alpha<\kappa$ to some $E_{\alpha} \subset J_{\alpha}^{E}$. For $\beta<\kappa$, we define

$$
\begin{aligned}
M \upharpoonleft \beta & :=\left\langle J_{\beta}^{E}, \epsilon, E \upharpoonleft \beta, E_{\beta}\right\rangle \\
& =\left\langle J_{\beta}^{E \upharpoonleft \beta}, \in, E \upharpoonleft \beta, E_{\beta}\right\rangle
\end{aligned}
$$

For notational purposes, if $\beta=\kappa$ then we define $M 1 \beta$ to be $M$.
Now, a filter sequence is exactly what it sounds like: a sequence of (normal) measures on cardinals. For the time being, we shall simply ignore $F$ and allow it to float around being anything it likes.
Definition 3.2.26. Let $M=\left\langle J_{\gamma}^{E}, \in, E, F\right\rangle, \kappa$ and $E: \kappa \rightarrow J_{\gamma}^{E}$ be as above. $E$ is a filter sequence over $M$ if for all $\alpha<\kappa$, either

1. $E_{\alpha}=\varnothing$; or
2. $M \upharpoonleft \alpha$ has a largest cardinal $\beta<\alpha$, and

$$
M \upharpoonleft \alpha \vDash \text { " } E_{\alpha} \text { is a normal measure on } \beta \text { " }
$$

and for all $\beta<\gamma<\alpha$,

$$
M \upharpoonleft \alpha \not \vDash " E_{\gamma} \text { is a normal measure on } \beta "
$$

We have cheated slightly with this notation. A filter sequence, as we have defined it above, can only have at most a single measure on any given ordinal $\alpha$. But usually, especially complex mice are allowed to have multiple measures on the same measurable: the mouse called O-Sword is a famous example of this. So really,
we should define a filter sequence in a way which allows the same cardinal to get multiple measures, and later specify under what circumstances this is allowed to happen in a mouse.

But in this thesis, we will not actually be working with any mice which have multiple measures on the same cardinal. And without this limit, we would have to give various tedious statements about when certain results in this chapter hold.

Definition 3.2.27. Let $M=\left\langle J_{\gamma}^{E}, \in, E, F\right\rangle$ be a $J$ structure, and let $E$ is a filter sequence over $M$. Let $\kappa$ again be the largest cardinal of $M$, or let $\kappa=\mathrm{On} \cap M$ if $M$ has no largest cardinal. Then $E$ is an extender sequence over $M$ if $M$ and $E$ satisfy the following properties whenever $\alpha<\kappa$ and $E_{\alpha} \neq \varnothing$ :

1. Acceptability: $J_{\alpha}^{E}$ is acceptable and $\alpha$ is its largest cardinal;
2. Amenability: $M \upharpoonleft \alpha$ is amenable;
3. Coherency: If the ultrapower of $M \upharpoonleft \alpha$ with respect to $E_{\alpha}$ is well-founded, and so $\pi_{E_{\alpha}}: M \upharpoonleft \alpha \rightarrow$ $\operatorname{Ult}\left(M \upharpoonleft \alpha, E_{\alpha}\right)$ is defined, then for all $\beta<\alpha$,

$$
E_{\beta}:=E(\beta)=\pi_{E_{\alpha}}(\beta)
$$

and

$$
\pi_{E_{\alpha}}(E)(\alpha)=\varnothing
$$

We say that $(E, F)$ is an extender sequence over $M$ if:

1. $E$ is an extender sequence over $M$;
2. Either $F=\varnothing$ or $M$ believes $F$ is a normal measure on $\kappa<\mathrm{On} \cap M$; and
3. If $F \neq \varnothing$ then the coherency property holds with $\kappa$ in place of $\alpha$ and $F$ in place of $E_{\alpha}$

If so, we often informally write $E_{\kappa}$ for $F$.
We have now almost reached the point of defining a mouse!
Definition 3.2.28. [47, 4.1] A premouse is an acceptable $J$ structure $M=\left\langle J_{\gamma}^{E}, \in, E, F\right\rangle$, such that ( $E, F$ ) is an extender sequence over $M, \gamma<$ On, and for all $\alpha<\kappa$, the $J$ structure $M 1 \alpha$ is sound. If $F \neq \varnothing$ (and so $M$ has a largest cardinal and $F$ is a normal measure on it) we say that $M$ is active.

There is no requirement that $M$ itself should be sound. In fact, a (pre)mouse itself being sound implies several other special properties, as we will see later.

Notice that if $M$ is a premouse and $\alpha<\kappa$ then $M \upharpoonleft \alpha$ will also be a premouse. Also, if $E_{\alpha}$ is a normal measure on $\alpha$, and the ultrapower of $M$ with respect to it is well founded, then its transitive collapse will be another premouse, whose extender sequence agrees with $E$ up to $\kappa$.

### 3.2.6 Iterations and Mice

We're nearly there! A premouse defines all the structure we need for a mouse: we have both an underlying domain, and a list of measurables. We just need one last step: we need to make sure that we can take repeated ultrapowers using those measurables. Recall that the ultrapower of a premouse with respect to a given measure always exists, but is not necessarily well-founded.

We carefully give a formal definition of what it means to perform an iteration of ultrapowers. The definition is complicated, but the idea is fairly simple. At each stage $i$ of the iteration we just pick some measurable $\kappa_{i}$ in the extender sequence of the current mouse $M_{i}$, perhaps cut the mouse down to some level $\alpha_{i} \geqslant \kappa_{i}$, and then take the ultrapower to get the next mouse $M_{i+1}$. But to make limit stages work properly, we have to insist that we only do finitely many of these cut-downs.

Definition 3.2.29. [47, 4.2] Let $M=\left\langle J_{\gamma}^{E}, E, F\right\rangle$ be a premouse. An iteration $\mathcal{I}$ of $M$ of length $\delta \in$ On $\cup\{$ On $\}$ is a tuple

$$
\left\langle M_{i}, \pi_{i, j}\right\rangle_{i \leqslant j<\delta}
$$

together with a tuple of pairs of ordinals

$$
\left\langle\kappa_{i}, \alpha_{i}\right\rangle_{i+1<\delta}
$$

such that $\mathcal{I}$ satisfies the following:

1. For $i \leqslant j<\delta, M_{i}=\left\langle J_{\gamma_{i}}^{E^{i}}, E^{i}, F^{i}\right\rangle$ is a premouse, and $\pi_{i, j}$ is a map (not necessarily total) from $M_{i} \upharpoonleft \alpha_{i}$ into $M_{j}$.
2. $M_{0}=M$
3. For $i<\delta, \pi_{i, i}: M_{i} \rightarrow M_{i}$ is the (full) identity map.
4. For $i+1<\delta, \kappa_{i} \leqslant \alpha_{i} \leqslant \lambda_{i}$ where $\lambda_{i}$ is the supremum of the cardinals of $M_{i}$. (In particular, if $M_{i}$ is active then $\lambda_{i}$ is the largest cardinal of $M_{i}$.)
5 . The sequence $\left(\kappa_{i}\right)_{i<\delta}$ is strictly increasing. ${ }^{1}$
5. For all but finitely many $i+1<\delta, \alpha_{i}=\lambda_{i}$.
6. If $\kappa_{i}<\lambda_{i}$ and $E^{i}\left(\kappa_{i}\right)=\varnothing$ then $\alpha_{i}=\lambda_{i}$ and $M_{i+1}=M_{i}$ and $\pi_{i, i+1}: M_{i} \rightarrow M_{i+1}$ is the identity. The same is true if $\kappa_{i}=\lambda_{i}$ and $F^{i}$ is trivial.
7. If $\kappa_{i}<\lambda_{i}$ and $E^{i}\left(\kappa_{i}\right) \neq \varnothing$ then $M_{i} \upharpoonleft \alpha_{i}$ believes that $E^{i}\left(\kappa_{i}\right)$ is a normal measure; and when we take the ultrapower of $M_{i} \upharpoonleft \alpha_{i}$ with respect to that measure, we get a well-founded set. $M_{i+1}$ is the transitive collapse $\operatorname{Ult}\left(M_{i} \upharpoonleft \alpha_{i}, E^{i}\left(\kappa_{i}\right)\right)$ of that ultrapower, and $\pi_{i, i+1}: M_{i} \uparrow \alpha_{i} \rightarrow M_{i+1}$ is the corresponding elementary embedding.
8. If $\kappa_{i}=\lambda_{i}$ and $F^{i}$ is nontrivial, then when we take the ultrapower of $M_{i}$ (which equals $M_{i} \upharpoonleft \alpha_{i}$ by 4) with respect to $F^{i}$ we get a well-founded set. $M_{i+1}=\operatorname{Ult}\left(M_{i}, F^{i}\right)$ is the transitive collapse of this ultrapower, and $\pi_{i, i+1}: M_{i} \rightarrow M_{i+1}$ is the corresponding ultrapower map.
9. For $i \leqslant j<j+1<\delta$, the map $\pi_{i, j+1}$ is obtained by composing $\pi_{j, j+1}$ with $\pi_{i, j}$, the latter restricted to the preimage of the domain of the former:

$$
\pi_{i, j+1}=\pi_{j, j+1} \circ\left(\pi_{i, j} \upharpoonleft\left(\left(\pi_{i, j}^{-1}\right) "\left(M_{j} \upharpoonleft \alpha_{j}\right)\right)\right)
$$

11. If $j<\delta$ is a limit ordinal, then letting $i_{0}$ be large enough that for all $i_{0} \leqslant i<j$ we have $\alpha_{i}=\lambda_{i}$ (recall that there are only finitely many $i$ for which this fails, so this $i_{0}$ must exist) the direct limit of the system $\left\langle M_{i}, \pi_{i, k}\right\rangle_{i_{0} \leqslant i \leqslant k<j}$ is well founded, and $M_{j}$ is its transitive collapse. (This makes sense as a definition, because $\pi_{i, k}$ is always at least $\Sigma_{1}$ elementary in this system, so the direct limit must at least exist.) For $i_{0} \leqslant i<j$, the map $\pi_{i, j}$ is the corresponding elementary embedding.
12. For $j$ and $i_{0}$ as above, and $i<i_{0}<j, \pi_{i, j}=\pi_{i_{0}, j} \circ \pi_{i, i_{0}}$.

If $\alpha_{i}$ is always equal to $\lambda_{i}$ and $\delta<$ On then we say $\mathcal{I}$ is "simple". If for some $i, \alpha_{i}<\lambda_{i}$ we say that the mouse $M_{i}$ is "cut down", or that there is a "truncation", at stage $i$. (We also say that $\mathcal{I}$ is simple if it contains no truncations, $\delta=$ On, and a tail of its critical points are trivial - so it's "really" only set long.)

We say that a (simple) iteration $\mathcal{I}$ has no "drops in degree" if for all $i<\delta$, the least $n<\omega$ such that $\rho_{n}^{M_{i}} \leqslant \kappa_{i}$ exists and does not depend on $i$.

We call the $\kappa_{i}$ the critical points of the iteration $\mathcal{I}$, and say that $\delta$ is the length of the iteration. Notice that an iteration is allowed to have "do nothing" stages where $E^{i}\left(\kappa_{i}\right)=\varnothing$. This is important when we are doing coiterations (see section 3.3.2), because we will want to do two different iterations simultaneously and give them the same critical points. But most of the time, it's irrelevant clutter, and without loss of generality we can usually assume that there is no stage in $\mathcal{I}$ where $E^{i}\left(\kappa_{i}\right)=\varnothing$.

[^13]Definition 3.2.30. Let $M$ be a premouse, and $\mathcal{I}$ be an iteration of length $\delta<0$. We say that $\mathcal{I}$ can be continued if either:

1. $\delta$ is a limit, and if $i_{0}$ is large enough that for all $i_{0}<i<\delta, \alpha_{i}=\lambda_{i}$ then the direct limit of the system $\left\langle M_{i}, \pi_{i, j}\right\rangle_{i_{0}<i \leqslant j<\delta}$ is well-founded; or
2. $\delta=i+1$ for some $i$, and $M_{i}=\left\langle J_{\gamma_{i}}^{E^{i}}, E^{i}, F^{i}\right\rangle$ is such that:
(a) If $\kappa$ is below the largest cardinal of $M_{i}$ and $E^{i}(\kappa) \neq \varnothing$ then we can find $\alpha \geqslant \kappa$ such that $M_{i} \upharpoonleft \alpha$ believes $E^{i}(\kappa)$ is a normal measure on $\kappa$; and
(b) For every such $\alpha \geqslant \kappa$, if we take the ultrapower of $M_{i} \mid \alpha$ with respect to the measure $E^{i}(\kappa)$, we get a well-founded structure; and
(c) If $F^{i}$ is nontrivial then the ultrapower of $M_{i}$ with respect to $F^{i}$ is well founded.

The idea is that we can take the iteration $\mathcal{I}$ of length $\delta$ and extend it to an iteration of length $\delta+1$ using any measure the premice know about.

We can now, finally, define a mouse.
Definition 3.2.31. [47, 4.2] A mouse is a premouse $M$ such that every iteration $\mathcal{I}$ of $M$ of length less than On can be continued.

Lemma 3.2.32. Let $M=\left\langle J_{\gamma}^{E}, E, F\right\rangle$ be a mouse. Let $\alpha$ be a successor cardinal of $M$. Then $M \upharpoonleft \alpha$ believes ZFC ${ }^{-}$relativised to $E$. Similarly, if $M$ is active then $\left\langle J_{\gamma}^{E}, E\right\rangle$ believes ZFC $^{-}$relativised to $E$. Moreover, both also believe the additional axioms of Set MK** (relativised to E).

Proof (Sketch). The successor cardinal case follows immediately from Lemma 3.2.6. Let $M^{\prime}$ be the result of iterating the top measure $F$ of $M$ once. Then $M^{\prime}$ is also a mouse, and thinks that On $\cap M$ is a successor cardinal, so $M^{\prime} \upharpoonleft$ On $\cap M$ believes the required axioms. But $M^{\prime} \upharpoonleft$ On $\cap M$ is just $\left\langle J_{\gamma}^{E}, E\right\rangle$, because the extender sequence $E^{\prime}$ of $M^{\prime}$ agrees with $E$ below On $\cap M$, and is trivial on On $\cap M$.

The technique in this proof involved creating a larger mouse $M^{\prime}$ in which On $\cap M$ was $\kappa^{+}$, and which agreed with $M$ below that $\kappa^{+}$. So in a sense, we can informally think of On $\cap M$ as being a successor cardinal. Motivated by this, if $M$ is an active mouse then we informally write $H_{\kappa^{+}}^{M}$ to denote $H_{\kappa^{+}}^{M^{\prime}}=\left\langle J_{\gamma}^{E}, E\right\rangle$.

### 3.2.7 Weasels

What about if we do an iteration of length On? We can still get a direct limit structure. The structure will be class-sized, and therefore we can't do a transitive collapse, even if it is well-founded. (If we tried, we would end up generating "sets" which contained all the ordinals of $V$, for example.) But we can take the collapse of those parts of the structure which are simple enough to be sets. This gives us a weasel (so named because it's like a very long mouse).

Definition 3.2.33. A weasel is a structure $W=\left\langle J_{\infty}^{E}, E\right\rangle$, where $E$ is an extender sequence over $W$, and for all $\beta<\mathrm{On}, W 1 \beta$ is a mouse.

A major aim of this subsection is to prove that whenever we take a class-length iteration, we can generate a weasel from it. The first step - after taking a direct limit - is to find an "ordinal" of the resulting structure which is analogous to On. To do this, we use Fodor's Lemma.

Lemma 3.2.34 (Fodor's Lemma). [47, 6.1.4] Let $\omega<\lambda \in \operatorname{Reg}^{V}$, or $\lambda=$ On. Let $T \subset \lambda$ be stationary. Let $f: T \rightarrow \lambda$ be a (class) function which is regressive (i.e. $f(\alpha)<\alpha$ for all $\alpha \in \lambda$ ). Then there is some stationary set $S \subset T$ and some $\beta \in \lambda$ such that for all $\alpha \in S, f(\alpha)=\beta$.

To show that we generate a weasel, we actually only need to prove the following two lemmas for the case $\lambda=$ On. But the proof for $\lambda \in \operatorname{Reg}^{V}$ is exactly the same, and we'll need it later.

Lemma 3.2.35. [47, 6.1.5] Let $\lambda \in \operatorname{Reg}^{V}$, or $\lambda=$ On. Let $M_{0}$ be a mouse, such that $\lambda>\left|M_{0}\right|$. Let $\mathcal{I}=\left\langle M_{i}, \pi_{i, j}\right\rangle$ be a simple iteration of length $\lambda$ of $M_{0}$ with no nontrivial critical points, whose set/class of critical points is cofinal below $\lambda$, but does not go above $\lambda$. Then there exists a cofinal increasing sequence $\left(i_{\gamma}\right)_{\gamma<\lambda}$ such that for all $\gamma<\delta<\lambda$,

$$
\pi_{i_{\gamma}, i_{\delta}}\left(\kappa_{i_{\gamma}}\right)=\kappa_{i_{\delta}}
$$

Proof. We define a regressive function $f:(\operatorname{Lim} \cap \lambda) \rightarrow \lambda$. Let $i \in \operatorname{Lim} \cap \lambda$. Then $\kappa_{i} \in M_{i}$, so (since $M_{i}$ is a direct limit model) $\kappa_{i}$ is the image of something in an earlier model. Let $f(i)$ be the least $j<i$ such that there exists $\bar{\kappa}_{i} \in M_{j}$ with $\pi_{j, i}\left(\bar{\kappa}_{i}\right)=\kappa_{i}$. Applying Fodor's Lemma, take a stationary $S \subset \operatorname{Lim} \cap \lambda$ and fixed $j<\lambda$ such that for all $i \in S, f(i)=j$.

So for all $i \in S, \bar{\kappa}_{i} \in M_{j}$. By Proposition 3.2.20 and a cofinality argument, we can see that $\left|M_{j}\right|<\lambda$. So by the pigeonhole principle, there is an unbounded $U \subset S$ and $\kappa \in M_{j}$ such that for all $i \in U, \bar{\kappa}_{i}=\kappa$. But then for $i \in U, \pi_{j, i}(\kappa)=\kappa_{i}$; it follows immediately that if $i, i^{\prime} \in U$ and $i<i^{\prime}$ then $\pi_{i, i^{\prime}}\left(\kappa_{i}\right)=\kappa_{i^{\prime}}$. So we can get our sequence $\left(i_{\gamma}\right)$ by enumerating the elements of $U$ in increasing order.

Lemma 3.2.36. Let $\lambda, M_{0}$ and $\mathcal{I}$ be as in the previous lemma. If $\lambda \neq \mathrm{On}$, let $M_{\lambda}$ be the new mouse in the continuation of $\mathcal{I}$ (i.e. the transitive collapse of the direct limit of $\left\langle M_{i}, \pi_{i, j}\right\rangle_{i \leqslant j<\lambda}$ ). Let $\pi_{i, \lambda}: M_{i} \rightarrow M_{\lambda}$ be the corresponding maps. Then $\lambda$ is measurable in $M_{\lambda}$; specifically,

$$
\pi_{i_{0}, \lambda}\left(\kappa_{i_{0}}\right)=\lambda
$$

Similarly, if $\lambda=$ On (and so we don't have a well defined transitive collapse) then let $\bar{M}_{\lambda}$ be the structure which is the direct limit of the system $\left\langle M_{i}, \pi_{i, j}\right\rangle$ and let $\bar{\pi}_{i, \lambda}: M_{i} \rightarrow \bar{M}_{\lambda}$ be the corresponding maps. Then $\bar{\pi}_{i_{0}, \lambda}\left(\kappa_{i_{0}}\right)$ is an ordinal in the sense of $\bar{M}_{\lambda}$, and has order type On.
Proof. First we prove the case $\lambda \neq$ On. Clearly $\pi_{i_{0}, \lambda}\left(\kappa_{i_{0}}\right) \geqslant \pi_{i_{0}, i_{\gamma}}\left(\kappa_{i_{0}}\right)=\kappa_{i_{\gamma}}$ for all $\gamma<\lambda$, and hence $\pi_{i_{0}, \lambda}\left(\kappa_{i_{0}}\right) \geqslant \lambda$. On the other hand, if $\pi_{i_{0}, \lambda}\left(\kappa_{i_{0}}\right)>\lambda$, then for large enough $i_{\gamma}<\lambda$, we can find a preimage $\bar{\lambda}=\pi_{i_{\gamma}, \lambda}^{-1}(\lambda) \in M_{i_{\gamma}}$ of $\lambda$ in $M_{i_{\gamma}}$. Note that the cardinality of $M_{i_{\gamma}}$ is less than $\lambda$, and hence $\bar{\lambda}<\lambda$. By elementarity $\bar{\lambda}<\pi_{i_{0}, i_{\gamma}}\left(\kappa_{i_{0}}\right)=\kappa_{i_{\gamma}}$. But then $\pi_{i_{\gamma}, \lambda}(\bar{\lambda})=\bar{\lambda}<\lambda$. Contradiction.

The case $\lambda=$ On is proved in exactly the same way; the notation is just a bit messier.
We can use this to define a part of $\bar{M}_{\mathrm{On}}$ which can be collapsed. Let $\mathcal{I}$ be a class length iteration. Without loss of generality, we may assume all its critical points are nontrivial. Let $\bar{M}=\bar{M}_{\mathrm{On}}$ be the direct limit model as above, and let $X=\bar{\pi}_{i_{0}, \text { On }}\left(\kappa_{i_{0}}\right)$. Now, $\bar{M}$ is a model of most of ZFC and $X$ is an ordinal of $\bar{M}$, so we can define $\left(H_{X}\right)^{\bar{M}}$ in the usual way.
Lemma 3.2.37. $\left(H_{X}\right)^{\bar{M}}$ is a wellfounded class size structure, and for all $x \epsilon_{\bar{M}}\left(H_{X}\right)^{\bar{M}}$, the collection $\left\{y \epsilon_{\bar{M}} x\right\}$ of $\epsilon_{\bar{M}}$ elements of $x$ is set size.

Thus, we can define the transitive collapse $\tilde{V}$ of $\left(H_{X}\right)^{\bar{M}}$, and the collapsing map $\iota:\left(H_{X}\right)^{\bar{M}} \rightarrow \tilde{V}$. We can also define a predicate $E_{\tilde{V}}$, as the image under $\iota$ of $E_{\bar{M}}$.
Lemma 3.2.38. [47, 6.1.6] The structure $W=\left\langle\tilde{V}, \epsilon, E_{\tilde{V}}\right\rangle$ is a weasel. Moreover, for any $\alpha \leqslant \kappa_{j}<$ On such that all the truncations of $\mathcal{I}$ happen before stage $j$,

$$
W \upharpoonleft \alpha=M_{j} \upharpoonleft \alpha
$$

As a shorthand, we will refer to an iteration of length On +1 to mean an iteration of length On together with the weasel it generates.

The weasel we get is not dependent on what iterations we did above $X$.
Lemma 3.2.39. Let $\mathcal{I}=\left\langle M_{i}, \pi_{i, j}\right\rangle$ be a class length iteration, and let $W$ be as above. Let $\mathcal{J}=\left\langle N_{i}, \tau_{i, j}\right\rangle$ be another iteration of $M_{0}$, which does the same as $\mathcal{I}$ below $X$, but does nothing at all above $X .{ }^{2}$

Then $\mathcal{I}$ and $\mathcal{J}$ generate the same weasel.

[^14]Proof. If $M$ is any mouse or weasel, then we know by 3.3.4 that taking an ultrapower (or a direct limit of ultrapowers), with critical points above $X$ will not change $M \upharpoonleft X$. So by induction, it follows that $M_{i} \upharpoonleft X=$ $N_{i} \upharpoonleft X$. But the previous lemma implies that if $j$ is large enough that $\mathcal{I}$ (and therefore $\mathcal{J}$ ) involves no cut-downs after stage $j$

$$
\begin{aligned}
\tilde{V} & =\bigcup_{\alpha \in W} W \upharpoonleft \alpha \\
& =\bigcup_{j<\alpha \in \mathrm{On}} M_{\alpha} \\
& =\bigcup_{j<\alpha \in \mathrm{On}} N_{\alpha} \\
& =\bigcup_{j<\alpha \in \mathrm{On}} W^{\prime} \upharpoonleft \alpha
\end{aligned}
$$

where $W^{\prime}$ is the weasel generated by $\mathcal{J}$.
It follows easily that $W=W^{\prime}$.
Not every weasel can be produced by iterating mice, but if a weasel can be, then we can actually inherit a little more structure from $\bar{M}$ and get a model of MK.
Lemma 3.2.40. Let $\bar{M}, X, \iota, \tilde{V}$ and $E_{\tilde{V}}$ be as above. For $Y \in \bar{M}$, let

$$
C_{Y}=\left\{x \in \tilde{V}: \iota^{-1}(x) \in_{\bar{M}} Y\right\}
$$

be the class of $\tilde{V}$ defined by $Y . \operatorname{Let} \mathcal{C}$ be the collection of all $C_{Y}$ for $Y \in \bar{M} .{ }^{3}$ Then the structure $\langle\tilde{V}, \in, \mathcal{C}\rangle$ is a model of $\mathrm{MK}^{* *}$, relativised to $E$.

Proof. Mostly follows from Lemma 3.2.32. The only "trick" is the powerset axiom, which holds because $X$ is a strong limit cardinal of $\bar{M}$.

If we abuse notation slightly, this effectively says that $H_{X^{+}}^{\bar{M}}$ is a model of MK, and its transitive collapse is $\langle\tilde{V}, \in, \mathcal{C}\rangle$. This means that if we know a weasel was generated by iterating mice, then we can informally treat it as if it were a model of MK. In particular, this lets us use definable class forcings over the weasel, which is something we're going to need in the next chapter.

It is possible to iterate weasels in exactly the same way as mice, and it can be shown that any set-length iteration of a weasel can be continued. Essentially, because each iteration only really affects a set size portion of $W$, so if the iteration is set-length then it's only "really" happening on some set size portion $W 1 \beta$ of $W$. By definition, $W \upharpoonleft \beta$ is a mouse, so any iteration on it can be continued. We can use this to construct a weasel which continues the original iteration of $W$.

This also means that we can define a class long iteration of weasels. There's one definition that we do need to adjust here: that of a simple iteration. With mice, we only allowed simple iterations that were set long, but with weasels, we allow certain class-long iterations to be simple too.

Definition 3.2.41. Let $W_{0}$ be a weasel, and let $\mathcal{I}=\left\langle W_{i}, \pi_{i, j}\right\rangle_{i \leqslant j<\theta}$ be an iteration of $W_{0}$ of length $\theta \leqslant$ On. We say that $\mathcal{I}$ is simple if it has no cut-downs, and either $\theta<$ On or $\theta=$ On and for all $\alpha \in$ On, $\sup _{j \in \mathrm{On}} \pi_{0, j}(\alpha)<\mathrm{On}$.

Essentially, this means that in order for a class-length iteration of a weasel to be simple, we can't iterate any measurable up onto On in the manner we described in Lemma 3.2.35. As we saw above, this always happens for (genuinely) class-length iterations of mice, justifying the way we excluded class long simple iterations of mice altogether.

If a class-length iteration of weasels is simple, then we can take a direct limit and take its transitive collapse in the usual way for mice, and it can be shown that this will be another well-defined weasel. This is a consequence of the following simple lemma, which is itself a complement to Lemma 3.2.36:

[^15]Lemma 3.2.42. Let $\lambda \in \operatorname{Reg}^{V}$, let $\mathcal{I}=\left\langle M_{i}\right\rangle_{i<\theta}$ be a set-length simple iteration of mice or weasels which all contain $\lambda$, whose critical points are all below $\lambda$. Suppose that there is no sequence of the sort described in Lemma 3.2.35, i.e. no sequence $\left(i_{\gamma}\right)_{\gamma<\theta}$ of ordinals below $\theta$ such that for all $\gamma<\delta<\theta, \pi_{i_{\gamma}, i_{\delta}}\left(\kappa_{i_{\gamma}}\right)=\kappa_{i_{\delta}}$, and such that $\left(\kappa_{i_{\gamma}}\right)_{\gamma<\theta}$ is cofinal below $\lambda$. Then $\lambda$ is a fixed point of $\mathcal{I}$.

If an iteration is not simple, then we can still construct a direct limit, but it's more complicated: we work in exactly the manner described in this chapter for mice. This implicitly involves a cut-down to On at the final stage, justifying our refusal to call the iteration simple.

### 3.3 More about Mice

There are a plethora of standard results about mice and how they interact. In this section, we present those which we're going to be using later.

### 3.3.1 Extending results about a single ultrapower

Several results about ultrapowers extend upward automatically to iterations. For example, the following result extends the elementarity we obtained in Lemma 3.2.18.

Lemma 3.3.1. Let $M_{0}$ be a mouse (or weasel) and let $\mathcal{I}=\left\langle M_{i}, \pi_{i, j}\right\rangle$ be a simple iteration of length $\theta \leqslant$ On. Suppose that $n \in \omega$ is such that for all $i<\theta, \rho_{n}^{M_{i}}>\kappa_{i}$. Then for all $i \leqslant j<\theta$, the map $\pi_{i, j}: M_{i} \rightarrow M_{j}$ is $\Sigma_{1}^{(n)}$ elementary and cofinal.

Proof. An easy induction on $j$. Successor stages follow from Lemma 3.2.18; limit stages from basic properties of a direct limit.

Proposition 3.3.2. [47, 3.2.2] Let $M_{0}$ be a mouse (or weasel), let $\mathcal{I}$ be a simple iteration of length $\theta \leqslant$ On, and let $n \in \omega$ be such that for $i<\theta, \rho_{n+1}^{M_{i}} \leqslant \kappa_{i}<\rho_{n}^{M_{i}}$. Then for all $i<\theta, \rho_{n+1}^{M_{i}}=\rho_{n+1}^{M_{0}}$, and $\pi_{0, i}\left(\rho_{n}^{M_{0}}\right)=\rho_{n}^{M_{i}}$.

Consider a single (transitive collapse of an) ultrapower $\bar{V}$ of $V$ with respect to some measurable $\kappa$. Let $\pi: V \rightarrow \bar{V}$ be the corresponding map. Recall that every element of the ultrapower is (the transitive collapse of) an equivalence class [f] for some function $f \in V^{\kappa}$. If $\pi: V \rightarrow \bar{V}$ is the ultrapower map, then we know that $\pi(f) \in \bar{V}^{\pi(\kappa)}$ agrees with $f$ up to $\kappa$. Also, recall that $\kappa$ is the transitive collapse of [id $1 \kappa$ ]. Consider $\pi(f)(\kappa) \in \bar{V}$. What are its elements? Let $x \in \bar{V}$ be the transitive collapse of (say) [g]. Then:

$$
\begin{aligned}
x \in \pi(f)(\kappa) & \Longleftrightarrow x \in(\pi(f))(\pi(\mathrm{id} \upharpoonleft \kappa)) \\
& \Longleftrightarrow x \in \pi(f \circ \mathrm{id} \upharpoonleft \kappa) \\
& \Longleftrightarrow\{\gamma<\kappa: g(\gamma) \in f(\gamma)\} \\
& \Longleftrightarrow[g] \in[f]
\end{aligned}
$$

So $\pi(f)(\kappa)$ is the transitive collapse of $f$, and so we can express every element of $\bar{V}$ in this way for some choice of $f$.

It turns out that this result extends to iterations.
Theorem 3.3.3. [47, 4.2.4] Let $M_{0}$ be a mouse or weasel, and let $\mathcal{I}$ be a simple iteration of length $\theta \leqslant \mathrm{On}+1$. Let $i<\theta$ and let $x \in M_{i}$. Then there exists a function $f \in M_{0}, f: \mathrm{On} \cap M_{0} \rightarrow M_{0}$, and $n \in \omega$ and $i_{0}<\ldots<i_{n}<i$ such that

$$
x=\pi_{0, i}(f)\left(\kappa_{i_{0}}, \ldots, \kappa_{i_{n}}\right)
$$

Recall now Proposition 3.2.22, which says that we never change the subsets of $\kappa$ by taking an ultrapower of a measurable cardinal $\kappa$ in a $J$ structure. This extends to iterations.

Proposition 3.3.4. [47, 4.2.1(d)] Let $M_{0}$ be an $n$ sound mouse, let $\mathcal{I}$ be an iteration of length $\theta+1 \leqslant \mathrm{On}+1$, and let $\rho_{n+1}^{M_{0}} \leqslant \kappa_{0}<\rho_{n}^{M_{0}}$ be the first critical point. Then $M_{0}$ and $M_{\theta}$ agree about sequences of length $\leqslant \kappa_{0}$.

### 3.3.2 The Mouse-Weasel Ordering

Our next trick is to construct a prewellordering of mice (and weasels). First we define a coiteration, a way of iterating two mice to make them look similar. For this definition, we really do need to allow iterations to have non-measurable critical points $\kappa_{i}$, in which case we do nothing and $M_{i}=M_{i+1}$.

Definition 3.3.5. Let $M$ and $N$ be mice or weasels. A coiteration of $M$ and $N$ is a pair of iterations $\mathcal{I}:=\left\langle M_{i}, \pi_{i}, j\right\rangle$ and $\mathcal{J}:=\left\langle N_{i}, \tau_{i, j}\right\rangle$ such that:

1. $M_{0}=M$ and $N_{0}=N$. (Recall that it's possible to iterate weasels as well as mice, so this makes sense.)
2. $\mathcal{I}$ and $\mathcal{J}$ are the same length $\theta+1 \leqslant$ On +1 .
3. At every $i<\theta$, the critical point $\kappa_{i}$ is the same in both $\mathcal{I}$ and $\mathcal{J}$, and is an element of both $M_{i}$ and $N_{i}$.
4. For $i<\theta$, either $M_{i} \neq M_{i+1}$ or $N_{i} \neq N_{i+1}$ (or both).
5. For $i<\theta$, if both $M_{i} \neq M_{i+1}$ and $N_{i} \neq N_{i+1}$ then the measures on $\kappa_{i}$ in $M_{i}$ and $N_{i}$ must be different.
6. At least one of $\mathcal{I}$ and $\mathcal{J}$ is simple. The only cut-downs that happen on the other are those which are necessary to make the next critical points into a measurable, and there we cut down only to some maximal $\alpha$ which does this.
7. Either $M_{\theta}=N_{\theta}$; or there is some $\alpha$ such that $M_{\theta}=N_{\theta} \upharpoonleft \alpha$ and the iteration $\mathcal{I}$ is simple; or vice versa there is some $\alpha$ such that $N_{\theta}=M_{\theta} \upharpoonleft \alpha$ and $\mathcal{J}$ is simple.

So a coiteration involves iterating $M$ and $N$ until they end up looking the same as one another (except that perhaps one of them has had its top cut off), and then stopping as soon as we achieve that.

Lemma 3.3.6 (The Comparison Lemma). [47, 4.4.1] Let $M$ and $N$ be any mice or weasels. Then there exists a unique coiteration of $M$ and $N$. This coiteration is set-length if both $M$ and $N$ are mice.

Proof (sketch). We basically do the only thing the definition allows. At each stage $i$, we compare $M_{i}$ and $N_{i}$, and find the least $\kappa$ which is either measurable in just one of $M_{i}$ and $N_{i}$, or is measurable in both but with different measures. We choose that as our next critical point. If there is no such $\kappa$, then without loss of generality the extender sequence of $M_{i}$ is either the same as, or an initial segment of, the extender sequence of $N_{i}$. But then the domains of $M_{i}$ and $N_{i}$ are levels of the same $J$ hierarchy, so are either equal or one is a cut down of the other.

We can use this to define a pre-well-ordering:
Definition 3.3.7. Let $M$ and $N$ be two mice or weasels. Let $\mathcal{I}=\left\langle M_{i}\right\rangle_{i<\theta+1}$ and $\mathcal{J}=\left\langle N_{i}\right\rangle_{j<\theta+1}$ be the coiteration of $M$ and $N$. We say $M=^{*} N$ if $M_{\theta}=N_{\theta}$ and both $\mathcal{I}$ and $\mathcal{J}$ are simple. We say $M<^{*} N$ if either $\mathcal{J}$ is not simple, or $M_{\theta}$ is a proper initial segment of $N_{\theta}$.
Lemma 3.3.8. [47, 5.4.4] $=^{*}$ is an equivalence relation. $<^{*}$ and $=^{*}$ constitute a prewellordering of mice and weasels.

If we want to show that one mouse is $<^{*}$ another, we can often use the following result.
Proposition 3.3.9. Let $M$ be a mouse or weasel, and let $N \in M$ be another mouse. Then $N<^{*} M$.
There are a couple of other easy results which are useful to know.
Proposition 3.3.10. Let $M$ be a mouse, and $W$ be a weasel. Then $M \not \neq *^{*} W$.
Proof. If the coiteration of $M$ and $W$ involves any cut-downs of $W$, then we immediately know that $M<{ }^{*} W$. If not, then the iterate of $W$ will still be class-sized. If the iterate of $M$ is set-size, then it has to be a proper initial segment of $W$ (so again $M<* W$ ). If the iterate of $M$ is class-size, then the $M$ side of the coiteration must be genuinely class-length, and so $M>^{*} W$.

Proposition 3.3.11. Let $M$ be an active mouse, and $N=* M$. Then $N$ is also active.

Proof. $N$ is also a mouse, and $M$ and $N$ can both be iterated to produce the same mouse $\bar{M}=\bar{N}$. By elementarity, $\bar{M}$ is active; so again by elementarity so is $N$.

It's natural to want to improve $<^{*}$ to an actual well-ordering so that we can talk about the minimal mouse with some property. Unfortunately, in general an equivalence class of $=$ * mice (or weasels) can't really be well ordered in a sensible way. But we can still find a mouse in the class which is somehow minimal. To do this, we take any arbitrary mouse $M$ in the equivalence class, and then find a new mouse in the class which is known as the "core" of $M$. This core is the same whichever $M$ we chose, and it can be iterated to produce any mouse in the class.

How we find this core is the subject of the next section.

### 3.3.3 Universal Weasels and the Core Model

We will start by defining the core $K$ of the universe $V$. To define $K$, we have to introduce a new concept, a universal weasel. Recall that a "weasel" is a class-sized mouse, or more formally, a structure that can be obtained by iterating some mouse with an iteration of length On.

Definition 3.3.12. Let $W$ be a weasel. We say $W$ is universal if every coiteration of $W$ with a mouse $M$ has length less than On. In other words, given any mouse $M$ we can embed some set-size iterate of $M$ in some simple iterate of $W$ as a proper initial segment.

Lemma 3.3.13. [47, 6.3.2] If $V$ is a weasel, then it is universal.
Actually, [47] uses the term "weakly universal" for this, and reserves "universal" for weasels for which this property also holds with $M$ in a certain class of premice. But for the small mice we are dealing with in this thesis, the two concepts agree.

Note that this implies that $W>^{*} M$ for every mouse $M$, so a universal weasel is in some sense in the class of the most complex sort of weasels that exist in $V$. In sufficiently nice situations, we can always find a universal weasel. [47, 6.4.4] [35]
Definition 3.3.14. Let $M=\left\langle J_{\alpha}^{E}, E\right\rangle$ be a mouse. We say $M$ is strong if we can extend $M$ to a universal weasel. That is, $M$ is strong if we can find a universal weasel $W$ such that $M=W \upharpoonleft \alpha$.

We're now ready to define the core $K$ of $V$.
Definition 3.3.15. The core model $K$ is the weasel of the form $\bigcup_{\text {On }} J_{\alpha}^{E}$, where $E$ is a (class long) extender sequence defined recursively as follows. Suppose we have defined $E \upharpoonleft \alpha$. If $V$ contains a normal measure $F$ such that $\left\langle J_{\alpha}^{E \upharpoonleft \alpha}, E \upharpoonleft \alpha, F\right\rangle$ is strong, then we define $E_{\alpha}=F$. (It can be shown that if $F$ exists then it must be unique.) If $V$ does not contain any such $F$, then we instead define $E_{\alpha}=\varnothing$.

This construction throws every normal measure $F$ it can find into $E$, except those which would immediately mean that there was something which was $>^{*}$ above $K$.
Lemma 3.3.16. [47, 7.1] $K$ is well defined, and can be defined within ZFC. Thus $K$ is a subclass of $V$. Also, it is a model of ZFC (since it's a weasel) and the core model of $K$ is $K$ itself.

The construction of the core involves throwing in every measure which won't cause problems, so it contains a lot of the measures on $K$ that $V$ knows about.

Lemma 3.3.17. [47, 7.3.7] If $U \in V$ is a normal ultrafilter on some ordinal of $K$, and $U$ is $\omega$ complete, then $U$ is in the extender sequence of $K$.

We can also say something further about the structure of $K$.
Theorem 3.3.18. [47, 7.3.7] The core $K$ of $V$ is a universal weasel.
Theorem 3.3.19. [47, 7.4.9] Any universal weasel is a simple iterate of K. In particular, by Lemma 3.3.13 if $V$ is a weasel, then it will be a simple iterate of $K$.

Note that any weasel $W$ is a model of ZFC, so we can relativise these results to define the core of $W$. Then $W$ will be a simple iterate of its core. Inspired by this, we define an analogous "core" of a mouse.

Definition 3.3.20. The core of a mouse $M$ is the unique sound mouse which can be simply iterated to produce $M$, if it exists.
Proposition 3.3.21. [45, 9.6] The core exists for any mouse with no measurables below $\rho_{\omega}$.
We can also generalise this.
Definition 3.3.22. Let $n \in \omega$. The $n$ 'th core of a mouse $M$ is the unique $n$ sound $N$ mouse which can be simply iterated to produce $M$, and which agrees with $M$ below $\rho_{n+1}$.

Proposition 3.3.23. The $n$ 'th core of a mouse always exists.
Going back to the full core, we can show that it agrees between mice which are $=^{*}$.
Lemma 3.3.24. Let $M=* N$ be two mice with no measurables below their respective $\rho_{\omega} s$, and the same ordinal height. Then they have the same core.
Proof. Let $K^{M}$ be the core of $M$ and $K^{N}$ be the core of $N$. Let $\tilde{M}=\tilde{N}$ be the mouse produced by coiterating $M$ and $N$. Then $\tilde{M}$ is a simple iterate of both $M$ and $N$, and hence of both $K^{M}$ and $K^{N}$. So both $K^{M}$ and $K^{N}$ are cores of $\tilde{M}$. By uniqueness, $K^{M}=K^{N}$.

In particular, this means that every mouse in $[M]$ will be a simple iterate of $K^{M}$. So it makes sense to talk about $K^{M}$ as the "least mouse".

Definition 3.3.25. We say that a mouse $M$ is the least mouse of $[M]$ if $M$ has no measurables below $\rho_{\omega}$ and $M$ is its own core.

Proposition 3.3.26. If $M$ is the least mouse of $[M]$, then every $N \in[M]$ is a simple iterate of $M$.
Definition 3.3.27. Let $P$ be a property such that if $P$ holds for a mouse or weasel $M$, it holds for all mice (resp. weasels) $N$ such that $M==^{*} N$. Then the least mouse (resp. least weasel) such that $P$ holds is defined to be the least mouse/weasel $\tilde{M}$ of $[M]$, where $[M]$ is the $<^{*}$ smallest equivalence class such that $P$ holds on the elements of $[M]$.

### 3.3.4 Universal Iterations and Fixed Points

In this section, we shall look at a test which lets us identify some fixed points of an iteration. This is less well-known material, and can be found in [40].

First, we shall define a "universal" iteration. In some ways, this is related to a universal weasel. A universal weasel "almost" contains an iterate of every mouse; a universal iteration of a mouse $M$ actually does contain an iterate of every mouse which is itself an iterate of $M$.

Definition 3.3.28. Let $M_{0}=\left\langle J_{\gamma}^{E}, E, F\right\rangle$ be a mouse, and $n \in \omega$. An iteration $\mathcal{I}=\left\langle M_{i}, \pi_{i, j}\right\rangle$ of length $\delta+1$ with no trivial steps is n-universal if for all $i$, for all $\alpha<\rho_{n}^{M_{i}}$ such that $\pi_{0, i}(E)(\alpha) \neq \varnothing$, there are unboundedly many $i<j<\delta$ such that $\kappa_{j}=\pi_{i, j}\left(\pi_{0, i}(E)(\alpha)\right)$.

Note that this definition is absolute between different class-size models of ZFC. This definition can also be extended to work for weasels, but we won't need universal iterations of weasels in this thesis, and there are a few extra technicalities we would have to deal with. Notice that if $m>n$ and $\mathcal{I}$ is $n$-universal, then it is also $m$ universal.

In the right circumstances, it can be shown that universal iterations not only exist, but are definable in $L[M]$. Recall the definition of an admissible ordinal.

Definition 3.3.29. We say an ordinal $\alpha$ is admissible if $L_{\alpha}$ is a model of Kripke-Platek set theory KP. If $X$ is a predicate, then we say $\alpha$ is $X$-admissible if $L_{\alpha}[X]$ is a model of KP.

KP is a weaker version of ZFC, which has only $\Sigma_{0}$ separation and $\Sigma_{0}$ collection (together with induction, and extensionality, emptyset, pairs and unions). It can be shown that if $X$ is a set then any sufficiently large regular cardinal is $X$-admissible, so there are plenty of them around.

Lemma 3.3.30. [40, 2.8] Let $M$ be a mouse, and let $\mathrm{On}^{M}<\lambda<\kappa$ be two $M$-admissible ordinals. Then for some $n \in \omega$, there is an n-universal iteration $\mathcal{I}$ of $M$ of length $\lambda+1$ which is an element of $L_{\kappa}[M]$.

The universal iteration is not unique (there are many trivial ways we can tweak it). But any two iterations of the same length produce the same results.

Lemma 3.3.31. [40, 2.9] Let $M$ be a mouse, and let $\lambda$ be an $M$-admissible ordinal. Let $\mathcal{I}=\left\langle M_{i}\right\rangle$ and $\mathcal{J}=\left\langle N_{i}\right\rangle$ be iterations of length $\lambda+1$, which are both $n$ universal for large enough $n$. Then $M_{\lambda}=N_{\lambda}$.

The key feature of a universal iteration is that any other iteration can be embedded into it, even if that iteration isn't in $L[M]$.

Theorem 3.3.32. [40, 2.10] Let $M=M_{0}=N_{0}$ be a mouse, let $\lambda$ be $M$-admissible, let $n \in \omega$, and let $\mathcal{I}=\left\langle M_{i}\right\rangle_{i \leqslant \lambda}$ be an n-universal iteration of $M$ of length $\lambda+1$. Let $\mathcal{J}=\left\langle N_{i}\right\rangle_{i<\delta}$ be a simple iteration of $M$ of length $\delta+1<\lambda$ with no drops in degree. Then for some $\delta \leqslant \epsilon<\lambda$, there is some extension $\mathcal{J}^{\prime}=\left\langle N_{i}\right\rangle_{i \leqslant \epsilon}$ of $\mathcal{J}$ (still with no truncations or drops in degree) such that $N_{\epsilon}=M_{j}$ for some $j<\lambda$, and the embeddings $M \rightarrow M_{j}=N_{\epsilon}$ defined by $\mathcal{I}$ and $\mathcal{J}^{\prime}$ are the same.

This allows us to prove that certain ordinals are fixed points of a given iteration.
Lemma 3.3.33. [46, 2.10] Let $M$ be a mouse, and let $\lambda \in M$ be $M$ admissible. Let $\mu \in M$ be strongly inaccessible in the sense of $W$, and larger than $\lambda$.

Let $\mathcal{I}=\left\langle M_{i}, \pi_{i, j}\right\rangle$ be a simple iteration of $M$ of length $\delta+1<\lambda$. Then $\pi_{0, \delta}(\mu)=\mu$.
Proof. Let $\kappa=$ On $\cap M$. Note that $\kappa$ is $M$-admissible, since $L_{\kappa}[M]=M \vDash$ ZFC $^{-}$. Let us define (for some $n$ ) an $n$-universal iteration $\mathcal{J}=\left\langle N_{i}, \tau_{i, j}\right\rangle$ of $M$ of length $\lambda+1$, which is contained in $L_{\kappa}[M]=M$.

Claim 3.3.34. For $j \leqslant \lambda, \tau_{0, j}(\mu)=\mu$
Proof. We use induction on $j$. The case $j=0$ is of course trivial, and successors follow from Proposition 3.1.13. If $j$ is a limit ordinal, and the result is proved for $i<j$, then the only way that $\tau_{0, j}(\mu)>\mu$ can happen is if there is some $\beta<\mu$ and some $i<j$ such that $\tau_{i, j}(\beta)=\mu$. This implies that the sequence $\left(\tau_{i, k}(\beta)\right)_{i<k<j}$ (which we know by inductive hypothesis does not reach $\mu$ ) is cofinal below $\mu$. This contradicts inaccessibility of $\mu$ in $M$, because $\mu>\lambda>j$.

Let $\mathcal{I}^{\prime}$ be an extension of $\mathcal{I}$, of total length $\epsilon+1$, such that $M_{\epsilon}=N_{j}$ for some $j$, and such that $\pi_{0, \epsilon}=\tau_{0, j}$. We know by the claim that $\pi_{\delta, \epsilon}\left(\pi_{0, \delta}(\mu)\right)=\pi_{0, \epsilon}(\mu)=\tau_{0, j}(\mu)=\mu$. But $\pi_{0, \delta}$ and $\pi_{\delta, \epsilon}$ are increasing functions, so this implies that $\pi_{0, \delta}(\mu)=\mu$.

## Chapter 4

## Of Mice and Machetes

Recall that we defined the classes $\operatorname{Reg}_{\epsilon}$ and $\operatorname{Reg}_{\epsilon}^{s}$ near the start of the previous chapter. We suggested that results about the predicate Card could be extended to $\mathrm{Reg}_{<\epsilon}$. In this chapter, we present two such extensions.

In [46], Welch proves the following result:
Suppose there exists a mouse with an unbounded sequence of measurables in it. Then $L$ [Card] is a generic extension of an iterate of the smallest such mouse.

The analogous result, which we shall prove here, is the following:
Let $\epsilon>0$. Suppose there exists a mouse containing "enough" measurables (in a sense to be specified shortly). Then $L\left[\operatorname{Reg}_{<\epsilon}\right]$ and $L\left[\operatorname{Reg}_{<\epsilon}^{s}\right]$ are both generic extensions of iterates of the smallest mouse with "enough" measurables.

Welch calls the mouse he uses O-Kukri, since it fits somewhere between two mice called O-Dagger and O-Sword. The mice we need to use will contain more measurables, so they will be somewhat larger than O-Kukri; but they are still strictly smaller than O-Sword. Continuing the established pattern, we shall name them after a weapon which is somewhere between a kukri and a sword in size - a machete.

Definition 4.0.1. Let $\epsilon \in$ On. Suppose there is an active sound mouse $M$ whose extender sequence $E$ contains unboundedly many measurables of all Cantor-Bendixson ranks below $\epsilon$. That is, $M$ contains (definable) sequences $C_{\delta}, \delta<\epsilon$ of ordinals, cofinal below the largest cardinal $\kappa$ of $M$, such that:

1. For all $\delta<\epsilon$, and for all $\alpha \in C_{\delta}, E(\alpha) \neq \varnothing$ is a normal measure on $\alpha$ from the perspective of $M$ (and so can be iterated without cutdowns)
2. For all $\delta<\epsilon, C_{\delta}$ does not contain any of its own limits.
3. For all $\gamma<\delta<\epsilon$, if $\alpha \in C_{\delta}$ then $\alpha$ is a limit of $C_{\gamma}$.

If such a mouse exists, then we define the mouse $O^{\text {Machete- } \epsilon}$, or $\mathrm{O}^{\mathrm{M} \epsilon}$ for short, to be the least such sound mouse. (If no such mice exist, then we simply say that $\mathrm{O}^{\mathrm{M} \mathrm{\epsilon}}$ does not exist.)
Proposition 4.0.2. Suppose that an active (but not necessarily sound) mouse $M$ exists with measurables of all ranks below $\epsilon$. Then $\mathrm{O}^{\mathrm{M} \epsilon}$ exists, and is $\leqslant^{*} M$. Moreover, if $\mathrm{O}^{\mathrm{M} \epsilon}$ exists, then every mouse which is $=^{*}$ to $\mathrm{O}^{\mathrm{M} \epsilon}$ has unboundedly many measurables of all ranks below $\epsilon$, but none has any (full) measure below $\epsilon$, or any measurables of Cantor Bendixson rank $\epsilon$ (other than the top measure). Also, $\left|\mathrm{O}^{\mathrm{M} \epsilon}\right|=\max \left(\aleph_{0},|\epsilon|\right)$.
Proof. Without loss of generality, suppose that $M$ is in the smallest $=^{*}$ equivalence class containing a mouse with unboundedly many measurables of all ranks $<\epsilon$. (Clearly $M \leqslant{ }^{*} \mathrm{O}^{\mathrm{M} \mathrm{\epsilon} \epsilon}$ if the latter exists; once we prove the first sentence of the proposition we will know that $M={ }^{*} \mathrm{O}^{\mathrm{M} \mathrm{\epsilon}}$ as well.) We start by proving that $M$ has no measurables of rank $\epsilon$ or measurables $\leqslant \epsilon$.

Suppose that the extender sequence of $M$ contains a measurable $\kappa$ of rank $\epsilon$. Then the extender sequence of $M 1 \kappa<{ }^{*} M$ would also have unboundedly many measurables of all ranks below $\epsilon$, and be active, so [ $M$ ] would not be the least $=*$ class containing such a mouse after all.

Next, suppose that the least measurable $\lambda \in M$ in the extender sequence of $M$ is $\leqslant \epsilon$. Let us construct a simple set long iteration of $M$ where we repeatedly iterate its bottom measure until we reach a mouse $\bar{M} \in[M]$ whose least measurable is above $\epsilon$. This gives us an elementary map $\pi: M \rightarrow \bar{M}$ such that $\pi(\epsilon)>\pi(\lambda)>\epsilon$. By elementarity, $\bar{M}$ contains unboundedly many measurables of all ranks below $\pi(\epsilon)$, and hence its extender sequence contains a measurable $\mu$ (below its top measure) which is rank $\epsilon$. But also, $\bar{M}$ is another active mouse in [ $M$ ] whose extender sequence contains unboundedly many measurables of all ranks below $\epsilon$, and we just saw this implies that no such $\mu$ can exist in $\bar{M}$. So the least measurable $\lambda$ of $M$ must be above $\epsilon$.

Let $\tilde{M}$ be the transitive collapse of the $\Sigma_{1}$ Skolem hull of $\epsilon$ in $M$. Then $\tilde{M}$ is elementarily equivalent to $M$, so it is a premouse. Moreover, any iteration of $\tilde{M}$ corresponds to a unique iteration of $M$, and hence can be continued. So $\tilde{M}$ is a mouse. By elementarity, $\tilde{M}$ contains unboundedly many measurables of all ranks $<\epsilon$, and no measurables below $\epsilon$, and hence $\tilde{M} \geqslant * M$ by minimality of $M$. But $\tilde{M}$ is (elementarily equivalent to) a subset of $M$. Hence $\tilde{M}={ }^{*} M$. But a simple cardinality argument shows $\rho_{\omega}^{\tilde{M}} \leqslant \rho_{1}^{\tilde{M}} \leqslant|\tilde{M}|=\max \left\{\aleph_{0},|\epsilon|\right\} \leqslant \epsilon$. In particular, $\tilde{M}$ contains no measurables below $\rho_{\omega}$, and hence there exists a (unique) sound mouse which is $=*$ to $\tilde{M}$ and hence also to $M .{ }^{1}$

Now let $\bar{M}$ be the unique sound mouse which is $={ }^{*}$ to $M$. Note that $|\bar{M}| \leqslant|\tilde{M}|=\max \left\{\aleph_{0}, \epsilon\right\}$. We know by elementarity that there is some $\bar{\epsilon} \in \bar{M}$ such that $\tilde{M}$ contains unboundedly many measurables of all ranks $<\bar{\epsilon}$, but none of rank $\bar{\epsilon}$ other than its top measure. Moreover, there is some simple iteration $\bar{M} \rightarrow M$, which (by elementarity) sends $\bar{\epsilon}$ to $\epsilon$. But there are no measurables below $\epsilon$ in $M$, and hence none below $\bar{\epsilon}$ in $\bar{M}$. So in fact $\bar{\epsilon}=\epsilon$. Hence, $\bar{M}$ satisfies the requirement for $\mathrm{O}^{\mathrm{M} \epsilon}$; by minimality of $M$ we therefore know that $\bar{M}=\mathrm{O}^{\mathrm{M} \epsilon}$. This also tells us that $\epsilon \in \bar{M}$, so $|\bar{M}| \geqslant \epsilon$. Hence,

$$
\left|\mathrm{O}^{\mathrm{M} \epsilon}\right|=\max \left\{\aleph_{0}, \epsilon\right\}
$$

as required.
Now, if $N$ is a mouse such that $N=* \mathrm{O}^{\mathrm{M} \epsilon}$, then $N$ is a simple iterate of $\mathrm{O}^{\mathrm{M} \epsilon}$ and $\epsilon$ is not moved by the iteration, so elementarity means $N$ contains unboundedly many measurables of all ranks $<\epsilon$, none of rank $\epsilon$ other than its top measure, and none below $\epsilon$.

Saying "the full measurables contained in the extender sequence of $M$ " is a bit of a mouthful, and will quickly get tiresome. To save time, we will informally talk about the measurables of a mouse $M$, meaning that collection of measurables which have normal measures defined by the extender sequence.

To generate $L\left[\operatorname{Reg}_{<\epsilon}\right]$ or $L\left[\operatorname{Reg}_{<\epsilon}^{s}\right]$, the mouse we need is precisely $\mathrm{O}^{\mathrm{M} \mathrm{\epsilon} \epsilon}$. Our main aim in this section is to prove the following theorem:
Theorem 4.0.3. Let $\epsilon \in \aleph_{\omega}$, and suppose $\mathrm{O}^{\mathrm{M} \epsilon}$ exists. Then there exists an iteration $\mathcal{I}=\left\langle M_{i}, \pi_{i, j}\right\rangle$ of $M_{0}=\mathrm{O}^{\mathrm{M} \mathrm{\epsilon}}$, of length $\mathrm{On}+1$, such that $L\left[\mathrm{Reg}_{<\epsilon}\right]$ is a generic extension of $M_{\mathrm{On}}$ by a hyperclass forcing, and another similar iteration such that $L\left[\mathrm{Reg}_{<\epsilon}{ }^{s}\right]$ is a generic extension of $M_{\mathrm{On}}$.

There's nothing inherently special about the statement "has unboundedly many measurables of a certain Cantor-Bendixson rank", from the perspective of the definition. We could reasonably define Machete mice for other first order statements.

Definition 4.0.4. Let $\varphi\left(v_{0}, \ldots, v_{n}, v_{n+1}\right)$ be any first order formula. Let $\alpha_{0}, \ldots, \alpha_{n}$ be ordinals of $V$. Suppose there exists an active mouse $M$, whose least measurable is above $\alpha:=\max \left\{\alpha_{0}, \ldots, \alpha_{n}\right\}$, and whose top measurable $\kappa$ is such that

$$
H_{\kappa^{+}}^{M} \vDash \varphi\left(\alpha_{0}, \ldots, \alpha_{n}, \kappa\right)
$$

Suppose further that a least such mouse $M$ is sound. Then we say that $M$ is $O^{M \varphi, \alpha_{0}, \ldots, \alpha_{n}}$. If no such mouse exists, then we say that $O^{M \varphi, \alpha_{0}, \ldots, \alpha_{n}}$ does not exist.

If the parameters are obvious, we will suppress them and just write $\mathrm{O}^{\mathrm{M} \varphi}$.

[^16]Note that unlike when we defined $\mathrm{O}^{\mathrm{M} \epsilon}$, we are explicitly requiring that $\mathrm{O}^{\mathrm{M} \varphi}$ be not just the least sound mouse which satisfies the criteria, but also the least mouse in general (in the sense that there is no mice that is $<^{*}$ it and satisfies the criteria too). This is necessary, as we can see from the following trivial example.

Example 1. Let $\varphi\left(v_{0}, v_{1}\right):=v_{0}=v_{0}$. Suppose that there exists an active sound mouse of uncountable cardinality, which has no measurables below $\aleph_{1}$. (This is a fairly weak assumption.) Let $M$ be the least such mouse. Then $M=O^{M \varphi, \aleph_{1}}$, because $\aleph_{1}$ must be in the mouse $O^{M \varphi, \aleph_{1}}$.

On the other hand, let $N$ be the smallest nontrivial sound mouse possible: a mouse with a single measure. ${ }^{2}$ A Löwenheim-Skolem argument shows that $N<{ }^{*} M$. But there exists a simple iterate $N^{\prime}$ of $N$ which contains $\aleph_{1}$; and then

$$
H_{\kappa^{+}}^{N^{\prime}} \vDash \varphi\left(\aleph_{1}, \kappa\right)
$$

However, we can get close to $\mathrm{O}^{\mathrm{M} \varphi}$. We can at least find a mouse which is $\alpha$ sound.
Proposition 4.0.5. Suppose that there exists a (not necessarily sound) mouse $M$ of the kind described in the definition above. Then there is an $\alpha$ sound mouse $N \leqslant * M$, also of the kind described above. Moreover, we can choose $N$ so that there is a sound mouse which is $=^{*}$ to $N$.

Proof. Take the $\Sigma_{1}$ Skolem hull of $\alpha$ in $M$, and let $N$ be its transitive collapse. By construction, $\rho_{1}^{N} \leqslant \alpha$, and hence $\rho_{\omega}$ is below the least measurable of $N$. So the core of $N$ exists.

In particular, if $\varphi$ has no parameters other than $\kappa$, then the existence of any such mouse is sufficient to conclude that $\mathrm{O}^{\mathrm{M} \varphi}$ exists.

A similar argument, applied to $\mathrm{O}^{\mathrm{M} \varphi}$ itself, shows:
Proposition 4.0.6. For any $\varphi$ and $\alpha_{0}, \ldots, \alpha_{n}$, if $\mathrm{O}^{\mathrm{M} \varphi}$ exists then

$$
\rho_{1}^{\mathrm{O}^{\mathrm{M} \varphi}} \leqslant \max \left\{\alpha_{0}, \ldots, \alpha_{n}\right\}
$$

Proposition 4.0.7. Suppose that $\mathrm{O}^{\mathrm{M} \varphi}$ exists. Then for all mice $M={ }^{*} \mathrm{O}^{\mathrm{M} \varphi}$, the same statement about $\varphi$ holds true for $M$ as for $\mathrm{O}^{\mathrm{M} \varphi}$.

Proof. Follows from soundness, elementarity and the fact that there are no measurables below $\alpha$.
For any $\epsilon, \mathrm{O}^{\mathrm{M} \epsilon}$ is an $\mathrm{O}^{\mathrm{M} \varphi}$ mouse. The $\varphi$ we use has two variables, and says "Below $v_{1} \mathrm{I}$ have unboundedly many measurables of all Cantor-Bendixson ranks less than $v_{0}$ ". We saw above that the least measurable of $\mathrm{O}^{\mathrm{M} \epsilon}=O^{M \varphi, \epsilon}$ is greater than $\epsilon$.

In the later sections of this chapter, we will see that any sufficiently nice Machete mouse, including $\mathrm{O}^{\mathrm{M} \epsilon}$ for any reasonable $\epsilon$, is contained in a different regularity related inner model:

Suppose that On is Mahlo. Let $\alpha$ be the least measurable in the core of $L\left[\mathrm{Reg}^{s}\right]$. Let $O^{M, \alpha_{0}, \ldots, \alpha_{n}}$ be a hereditarily $\alpha$ friendly, hereditarily tidy machete mouse which is friendly with respect to $\varphi$. Then $\mathrm{O}^{\mathrm{M} \varphi} \in L\left[\operatorname{Reg}^{s}\right]$.

We will define what friendliness and tidiness mean in this context later in the chapter.

### 4.1 Magidor Forcings, Iterations and the Mathias Criterion

The hyperclass forcing we shall be using in proving Theorem 4.0 .3 will be a Magidor iteration of Prikry forcings. Before we start proving the theorem, we shall lay some groundwork about these iterations for us to use later. First, let's define what the forcing actually is. Recall that a Prikry forcing singularises a measurable cardinal, and has two relations $\leqslant$ and $\leqslant^{*}$. The Magidor iteration singularises an infinite collection of measurables, by doing a Prikry forcing on each one. If we are working in ZFC, then we can do an iteration for any set of measurables; if we are in a model of MK then we can go further and define the iteration for a proper class.

[^17]Definition 4.1.1. [28] Let $C$ be a set of measurables (if we are working in ZFC) or a class of measurables (if we are working in MK), and for $\kappa \in C$ let $U_{\kappa}$ be a normal measure on $\kappa$. We recursively define forcings $\mathbb{P}_{\kappa}$ for $\kappa \in C \cup\{\sup C\}$ and $\mathbb{P}_{\kappa}$ names $\dot{\mathbb{Q}}_{\kappa}$ of Prikry forcings for $\kappa \in C$. The two definitions depend on one another, and so are part of the same recursion. The Magidor iteration of $C$ is the forcing $\mathbb{P}_{\text {sup }} C$.
$\mathbb{P}_{\kappa}$ : For $\kappa \in C \cup\{\sup C\}, \mathbb{P}_{\kappa}$ consists of sequences $p=\left(p_{\lambda}\right)_{\lambda \in C \cap \kappa}$, such that:

1. For all $\lambda \in C \cap \kappa, p 1 \lambda:=\left(p_{\delta}\right)_{\delta \in C \cap \lambda} \in \mathbb{P}_{\lambda}$
2. For all $\lambda \in C \cap \kappa, p \upharpoonleft \lambda \Vdash p_{\lambda} \in \dot{\mathbb{Q}}_{\lambda}$
3. For all but finitely many $\lambda \in C \cap \kappa, p 1 \lambda \Vdash p_{\lambda} \leqslant_{\mathbb{Q}_{\lambda}}^{*} \mathbb{1}_{\dot{\mathbb{Q}}_{\lambda}}$

The order $\leqslant$ on $\mathbb{P}_{\kappa}$ is defined in the natural way: if $p=\left(p_{\lambda}\right)_{\lambda \in C \cap \kappa}$ and $q=\left(q_{\lambda}\right)_{\lambda \in C \cap \kappa}$ then $q \leqslant p$ if for all $\lambda \in C \cap \kappa, q \upharpoonleft \lambda \Vdash q_{\lambda} \leqslant_{\dot{\mathbb{Q}}_{\lambda}} p_{\lambda}$. We also define a second partial order $\leqslant^{*}$ on $\mathbb{P}_{\kappa}: q \leqslant{ }^{*} p$ if for all $\lambda \in C \cap \kappa$, $q 1 \lambda \Vdash q_{\lambda} \leqslant_{\mathbb{Q}_{\lambda}}^{*} p_{\lambda}$.
$\dot{\mathbb{Q}}_{\kappa}$ : For $\kappa \in C$, let $j_{U_{\kappa}}$ be the ultrapower map defined by $U_{\kappa}$. Let $\tilde{\mathbb{P}}_{\kappa}=j_{U_{\kappa}}\left(\mathbb{P}_{\kappa}\right)$. Let $\dot{U}_{\kappa}^{*}$ be the following $\mathbb{P}_{\kappa}$ name:

$$
\dot{U}_{\kappa}^{*}=\left\{(\dot{A}, p): \exists q \leqslant_{\tilde{\mathbb{P}}_{\kappa}}^{*}\left(j_{U_{\kappa}}(p) \backslash \kappa\right), p^{\wedge} q \Vdash_{\tilde{\mathbb{P}}_{\kappa}} \check{\kappa} \in j_{U_{\kappa}}(\dot{A})\right\}
$$

It can be shown $[28,2.5]$ that $\dot{U}_{\kappa}^{*}$ is a name for a normal measure on $\kappa$ in the $\mathbb{P}_{\kappa}$ generic extension, and that $\mathbb{1}_{\mathbb{P}_{\kappa}} \Vdash \dot{U}_{\kappa}^{*} \cap V=\check{U}_{\kappa}$.

We define $\dot{\mathbb{Q}}_{\kappa}$ to be a name for the Prikry forcing on $\kappa$ in the generic extension defined by $\dot{U}_{\kappa}^{*}$.
The Magidor iteration adds a cofinal $\omega$ sequence below each measurable in $C$. Like a Prikry forcing, a generic filter is determined by the generic sequence (in $\prod_{\kappa \in C} \kappa^{\omega}$ ) it adds. Notice that in MK, if $C$ is a proper class then the iteration is a hyperclass forcing and a generic filter is technically a hyperclass, but its equivalent sequence is just a class.

We can also say a little more about the nature of the measure $\dot{U}_{\kappa}^{*}$.
Proposition 4.1.2. [10] In the forcing defined above, let $\kappa \in C$. Let $G$ be $\mathbb{P}_{\kappa}$ generic, and for $\lambda \in C \cap \kappa$ let $G_{\lambda}$ be the $\omega$ sequence added by $G$ below $\lambda$. Then $\left(\dot{U}_{\kappa}^{*}\right)^{G}$ is generated by the measure 1 sets of $U_{\kappa}$, together with the set

$$
\Sigma_{\kappa}^{G}:=\left\{\nu<\kappa: \forall \lambda \in C \cap \kappa \backslash \nu,(\nu+1) \cap G_{\lambda}=\varnothing\right\}
$$

We can show an analogue of Lemma 1.3.11. (In fact, this lemma is essentially always true in Prikry style forcings, and is one of their defining features.)

Lemma 4.1.3. [28, 2.1] Let $\varphi$ be some sentence (perhaps with parameters) in the language of $\mathbb{P}_{\kappa}$, and let $p \in \mathbb{P}_{\kappa}$. There is some $q \leqslant^{*} p$ such that either $q \Vdash \varphi$ or $q \Vdash \neg \varphi$.

Recall that the Prikry forcing on $\kappa$ consists of two components: a finite stem which is an element of $\kappa^{<\omega}$, and a measure 1 set $X$. So an element $p=\left(p_{\lambda}\right)$ of the Magidor forcing can be rearranged into a sequence of names $\dot{s}_{\lambda}$ for stems $s_{\lambda} \in \lambda^{<\omega}$, and a sequence of names $\dot{X}_{\lambda}$ for measure 1 subsets $X_{\lambda} \subset \lambda$.
Lemma 4.1.4. Let $p, q \in \mathbb{P}_{\kappa}$. Let $p=\left(\dot{s}_{\lambda}, \dot{X}_{\lambda}\right)_{\lambda \in C \cap \kappa}$, and let $q=\left(\dot{t}_{\lambda}, \dot{Y}_{\lambda}\right)_{\lambda \in C \cap \kappa}$.
Suppose that for all $\lambda, p \upharpoonleft \lambda \Vdash \dot{s}_{\lambda}=\dot{t}_{\lambda}$, or $q 1 \lambda \Vdash \dot{s}_{\lambda}=\dot{t}_{\lambda}$.
Then $r:=\left(\dot{s}_{\lambda}, \dot{X}_{\lambda} \cap \dot{Y}_{\lambda}\right)_{\lambda \in C_{n} \kappa}$ is a condition of $\mathbb{P}_{\kappa}$, and $r \leqslant^{*} p$ and $r \leqslant^{*} q$.
Proof. Induction on the order type of $C \cap \kappa$. The case o.t. $=0$ is trivial: it boils down to saying that the intersection of two measure 1 sets is measure 1.

Suppose $C \cap \kappa$ has a largest element $\lambda$. By inductive assumption $r 1 \lambda \in \mathbb{P}_{\lambda}$, and $r 1 \lambda \leqslant * p 1 \lambda$ and $r \upharpoonleft \lambda \leqslant{ }^{*} q \upharpoonleft \lambda$. Then we know that $r \upharpoonleft \lambda \Vdash \dot{s}_{\lambda}=\dot{t}_{\lambda} \in \check{\lambda}<\omega$, and that $r \upharpoonleft \lambda \Vdash \dot{X}_{\lambda}, \dot{Y}_{\lambda} \in \dot{U}_{\lambda}^{*}$. So $r \upharpoonleft \lambda \Vdash \dot{X}_{\lambda} \cap \dot{Y}_{\lambda} \in \dot{U}_{\lambda}^{*}$. Hence $r \upharpoonleft \lambda \Vdash\left\langle\dot{s}_{\lambda}, \dot{X}_{\lambda}\right\rangle \cap \dot{Y}_{\lambda} \in \dot{\mathbb{Q}}_{\lambda}$.

Hence, $r=r \upharpoonleft \lambda^{\wedge}\left(\dot{s}, \dot{X}_{\lambda} \cap \dot{Y}_{\lambda}\right) \in \mathbb{P}_{\kappa}$. It is now trivial to see that $r \leqslant{ }^{*} p, q$.
If $C \cap \kappa$ has a limit order type, then it is immediate that $r \in \mathbb{P}_{\kappa}$ and $r \leqslant^{*} p, q$, just by the definition of $\mathbb{P}_{\kappa}$ and $\leqslant^{*}$ and the inductive hypothesis.

Corollary 4.1.5. [28, 4.4] $\mathbb{P}_{\kappa}$ satisfies the $\kappa^{+}$chain condition.
Proof. Induction on the order type of $C \cap \kappa$. If the order type is a successor (say it has a largest element $\lambda$ ), then $\mathbb{P}_{\kappa}=\mathbb{P}_{\lambda} * \dot{\mathbb{Q}}_{\lambda}$, and the result follows by inductive hypothesis and the fact that the Prikry forcing on $\lambda$ has the $\lambda^{+}$chain condition. So we just need to consider the case where the order type is a limit.

Suppose that this is the case, and that $A$ is an antichain of $\mathbb{P}_{\kappa}$ of size $\kappa^{+}$. For each $p \in \mathbb{P}_{\kappa}$, let $\operatorname{supp}(p)$ be the support of the stem of $p$, i.e. the finite set of $\lambda \in C \cap \kappa$ such that $p 1 \lambda \neg \Vdash p_{\lambda} \leqslant_{\mathbb{Q}_{\lambda}}^{*} \mathbb{1}_{\dot{\mathbb{Q}}_{\lambda}}$. Note that this is defined in the ground model $V$. By the pigeonhole principle, we can find some finite subset $S \subset C \cap \kappa$, and some $A^{\prime} \subset A$ of cardinality $\kappa^{+}$, such that for all $p \in A^{\prime}, \operatorname{supp}(p)=S$. Since $C \cap \kappa$ has limit order type, we know that $S$ is bounded below $\kappa$, by some $\lambda$ say.

Let $A^{\prime \prime}=\left\{p 1 \lambda: p \in A^{\prime}\right\} \subset \mathbb{P}_{\lambda}$. We know that $\mathbb{P}_{\lambda}$ has the $\lambda^{+}$chain condition by assumption, so either $A^{\prime \prime}$ has cardinality less than $\kappa^{+}$, or it contains two compatible elements. Either way, we can find $p, q \in A^{\prime}$ such that $p 1 \lambda$ and $q 1 \lambda$ are compatible. Let $r \in \mathbb{P}_{\lambda}$ be below both of them. Let $p^{\prime}, q^{\prime} \in \mathbb{P}_{\kappa}$ be obtained by sticking the parts of $p$ and $q$ (respectively) above $\lambda$ onto the end of $r$. Since $r \leqslant p 1 \lambda$ and $r \leqslant q 1 \lambda$, we know $p^{\prime}$ and $q^{\prime}$ are conditions, and that $p^{\prime} \leqslant p$ and $q^{\prime} \leqslant q$.

But the parts of $p$ and $q$ which are above $\lambda$ have empty support. So the stems of $p^{\prime}$ and $q^{\prime}$ are both exactly the same as the stem of $p^{\prime} \upharpoonleft \lambda=r=q^{\prime} \upharpoonleft \lambda$. By the previous lemma, then, $p^{\prime}$ and $q^{\prime}$ are compatible. But then $p \geqslant p^{\prime}$ and $q \geqslant q^{\prime}$ are also compatible. Contradiction.

In [11], Ben Neria gives a generalisation of the Mathias criterion to test whether a given sequence is generic.

Theorem 4.1.6. [11] [ZFC] Let $\mathbb{P}$ be the Magidor iteration on a set of measurables $C$, with corresponding normal measures $U_{\kappa}$ as above. Let $S=\left(S_{\kappa}\right)_{\kappa \in C} \in \prod_{\kappa \in C} \kappa^{\omega}$ be a sequence, not necessarily in the ground model $V$. Then the filter corresponding to $S$ is $\mathbb{P}$ generic if and only if it satisfies the following two conditions:

1. The Mathias Criterion: For every $X \in \prod_{\kappa \in C} U_{\kappa}$ in $V$, the set $\bigsqcup_{\kappa \in C} S_{\kappa} \backslash X_{\kappa}$ is finite;
2. The Separation Property: There are only finitely many tuples $\nu \leqslant \nu^{\prime}<\kappa<\kappa^{\prime}$ such that $\kappa, \kappa^{\prime} \in C$ and $\nu \in S_{\kappa}$ and $\nu^{\prime} \in S_{\kappa^{\prime}}$.

We will be using this to show that the sequence we want to add to create $L\left[\operatorname{Reg}_{<\epsilon}\right]$ or $L\left[\operatorname{Reg}_{<\epsilon}^{s}\right]$ is actually generic.

More precisely, we shall use this to prove that a new criterion is sufficient for a sequence to be generic; and will then verify that the sequence we're interested in satisfies this criterion.

Lemma 4.1.7. Let $M=M_{0}$ be a mouse whose extender sequence contains a bounded set of measurables $C$. Let $\mathcal{I}=\left\langle M_{i}, \pi_{i, j}\right\rangle_{i \leqslant j \leqslant \theta}$ be a (set or class) length iteration of $M$. For $\lambda \in \pi_{0, \theta}(C)$, let $S_{\lambda} \in \lambda^{\omega}$ be an increasing and cofinal sequence (necessarily outside $M_{\alpha}$ ) of critical points of $\mathcal{I}$, such that if $\kappa_{i} \in S_{\lambda}$ then $\pi_{i, \theta}\left(\kappa_{i}\right)=\lambda$.

Then $S=\left(S_{\lambda}\right)_{\lambda \in \pi_{0, \theta}(C)}$ is a generic sequence for the Magidor iteration of $\pi_{0} \theta(C)$.
Proof. We shall show that $S$ satisfies the criteria in 4.1.6.
First, the Separation Property. Suppose that $\nu \in S_{\lambda}$ and $\nu^{\prime} \in S_{\lambda}^{\prime}$, and $\nu \leqslant \nu^{\prime}<\lambda<\lambda^{\prime}$. We know, by definition of $S_{\lambda}$, that there exists $i \in \theta$ such that $\nu=\kappa_{i}$ and $\pi_{i, \theta}\left(\kappa_{i}\right)=\lambda$. Likewise, there exists $j \in \theta$ such that $\nu^{\prime}=\kappa_{j}$ and $\pi_{j, \theta}\left(\kappa_{j}\right)=\lambda^{\prime}$. We know that $\kappa_{i} \leqslant \kappa_{j}$, so $i \leqslant j$. Since

$$
\pi_{i, \theta}\left(\kappa_{i}\right)=\lambda<\lambda^{\prime}=\pi_{j, \theta}\left(\kappa_{j}\right)
$$

it follows that $i<j$. By elementarity, then,

$$
\pi_{i, j}\left(\kappa_{i}\right)<\pi_{j, j}\left(\kappa_{j}\right)=\kappa_{j}
$$

But $\pi_{j, \theta} \upharpoonleft \kappa_{j}=$ id, so $\pi_{i, j}\left(\kappa_{i}\right)=\pi_{i, \theta}\left(\kappa_{i}\right)=\lambda$. This implies that $\lambda<\kappa_{j}$, contradicting our assumption that $\nu \leqslant \nu^{\prime}<\kappa<\kappa^{\prime}$. So there are no such interleaved pairs at all, and $S$ very much satisfies the Separation Property.

We now turn to the more difficult part of this proof, showing that the Mathias Criterion is satisfied.

To simplify notation, let $C^{*}=\pi_{0, \theta}(C)$, let $U=\left\{\left(\lambda, U_{\lambda}\right): \lambda \in C\right\}$ be the sequence of ultrapowers on $C$ in $M_{0}$, and let $U^{*}=\pi_{0, \theta}(U)$ be the corresponding sequence of ultrapowers on $C^{*}$. For $\kappa \in C^{*}$, let $U_{\lambda}^{*}=U^{*}(\lambda)$. Fix $X \in \prod_{\lambda \in C^{*}} U_{\lambda}^{*}$ in $M_{\theta}$, and for $\kappa \in C^{*}$ let $X_{\lambda}=X(\lambda)$. Let

$$
\tau_{X}=\sup \left\{\lambda \in C^{*}: X_{\lambda} \neq \lambda\right\} \leqslant \sup C^{*}
$$

We want to show that $\bigsqcup S_{\lambda} \backslash X_{\lambda}$ is finite. Suppose we can find a counterexample: an $X$ for which the is infinite. Then let $X$ be a counterexample which minimises $\tau_{X} \cdot{ }^{3}$ Let $\bar{X}=X \upharpoonleft \tau_{X}$.

Recall that by Theorem 3.3.3, we can write

$$
\bar{X}=\pi_{0, \theta}(f)\left(\kappa_{i_{0}}, \ldots, \kappa_{i_{n}}\right) 1 \tau_{X}
$$

for some function $f \in M_{0}$, some $n \in \omega$ and some $i_{0}<\ldots<i_{n}<\theta$, where $\kappa_{i}$ denotes the $i$ 'th critical point of the iteration $\mathcal{I}$ as usual. In fact, we know it's possible to do this even without the $1 \tau_{X}$. Fix a way of writing this (including the $1 \tau_{X}$ ) which minimises $i_{n}$ for our chosen $\bar{X}$, and let $i=i_{n}$. (Note that we do not require that $\pi_{0, \theta}(f)\left(\kappa_{i, 0}, \ldots, \kappa_{i_{n}}\right)$ be equal to either $X$ or $\bar{X}$, only that it agree with $\bar{X}=X \upharpoonleft \tau_{X}$ up to $\tau_{X}$.)

This is an opportune point to prove a short technical result we're going to need later.
Claim 4.1.8. $\kappa_{i}<\tau_{X}$
Proof. This is trivial if $\tau_{X}>\kappa_{j}$ for all $j<\theta$. Suppose otherwise, and let $j<\theta$ be least such that $\kappa_{j} \geqslant \tau_{X}$. Now $\bar{X} \in M_{\theta}$ can be coded easily as a subset of $\tau_{X}$ and hence as a subset of $\kappa_{j}$. By Proposition 3.3.4, we know that this coding already exists in $M_{j}$, and hence that $\bar{X} \in M_{j}$. Since $\pi_{j, \theta}$ acts as the identity on ordinals below $\kappa$, it is also easy to see that $\pi_{j, \theta}(\bar{X}) \upharpoonleft \tau_{X}=\bar{X}$.

We can write $\bar{X}=\pi_{0, j}(g)\left(\kappa_{j_{0}}, \ldots, \kappa_{j_{m}}\right)$ for some $g \in M_{0}$, some $m \in \omega$ and some $j_{0}<\ldots<j_{m}<j$. But then

$$
\bar{X} \upharpoonleft \tau_{X}=\pi_{0, \theta}(g)\left(\kappa_{j_{0}}, \ldots, \kappa_{j_{m}}\right)
$$

Hence $i \leqslant j_{m}<\tau_{X}$.
Returning from this diversion, we now prove the central claim of this lemma.
Claim 4.1.9. Let $\lambda \in C^{*}$, let $i<j<\theta$, and suppose that $\pi_{j, \theta}\left(\kappa_{j}\right)=\lambda$. Then $\kappa_{j} \in X_{\lambda}$.
Proof. If $\lambda \geqslant \tau_{X}$, then $X_{\lambda}=\lambda$ and so the claim is trivial. So suppose $\lambda \in C^{*} \cap \tau_{X}$.
Now, $j+1>i$, so $\pi_{0, j+1}(f)\left(\kappa_{i_{0}}, \ldots, \kappa_{i_{n}}\right)$ is well defined, and is a function in $M_{j}$ which chooses a measure 1 subset of each measurable in $\pi_{0, j+1}(C)$. We know that $\pi_{j, \theta}\left(\kappa_{j}\right)=\lambda \in C^{*}=\pi_{0, \theta}(C)$, so by elementarity $\kappa_{j} \in \pi_{0, j}(C)$ and $\pi_{j, j+1}\left(\kappa_{j}\right) \in \pi_{0, j+1}(C)$. So it makes sense to talk about the set

$$
Y:=\pi_{0, j+1}(f)\left(\kappa_{i_{0}}, \ldots, \kappa_{i_{n}}\right)\left(\pi_{j, j+1}\left(\kappa_{j}\right)\right) \in M_{j+1}
$$

$M_{j+1}$ believes that this $Y$ is a measure 1 subset of $\pi_{j, j+1}\left(\kappa_{j}\right)$.
But (since $\kappa_{j}$ is the critical point the iteration $\mathcal{I}$ at stage $j$ ) we know that we generated $M_{j+1}$ by taking the ultrapower of $\kappa_{j}$ in $M_{j}$, and $\pi_{j, j+1}$ is the corresponding ultrapower map. So Proposition 3.1.11 (or rather, the equivalent proposition for rudimentary closed structures) tells us that $\kappa_{j} \in Y$, since $Y$ is measure 1 in $M_{j+1}$.

The critical point of $\pi_{j+1, \alpha}$ is $\kappa_{j+1}>\kappa_{j}$. So

$$
\kappa_{j}=\pi_{j+1, \theta}\left(\kappa_{j}\right) \in \pi_{j+1, \theta}(Y)
$$

But

$$
\begin{aligned}
\pi_{j+1, \theta}(Y) & =\pi_{j+1, \theta}\left(\pi_{0, j+1}(f)\left(\kappa_{i_{0}}, \ldots, \kappa_{i_{n}}\right)\left(\pi_{j, j+1}\left(\kappa_{j}\right)\right)\right) \\
& =\pi_{0, \theta}(f)\left(\pi_{j+1, \theta}\left(\kappa_{i_{0}}\right), \ldots, \pi_{j+1, \theta}\left(\kappa_{i_{n}}\right)\right)\left(\pi_{j, \theta}\left(\kappa_{j}\right)\right) \\
& =\pi_{0, \theta}(f)\left(\kappa_{i_{0}}, \ldots, \kappa_{i_{n}}\right)(\lambda) \\
& =X_{\lambda}
\end{aligned}
$$

[^18]So $\kappa_{j} \in X_{\lambda}$.
Let $\lambda \in C^{*} \cap \tau_{X}$. It follows immediately from the second Claim that $S_{\lambda} \backslash X_{\lambda} \subset \kappa_{i}$. Of course, this tells us nothing if $\lambda \leqslant \kappa_{i}$, but if $\lambda>\kappa_{i}$ then since $S_{\lambda}$ is cofinal in $\lambda$, we know that $S_{\lambda} \backslash X_{\lambda}$ is finite. Moreover, if the least $j<\theta$ such that $\kappa_{j} \in S_{\lambda}$ is greater than $i$, then $S_{\lambda} \backslash X_{\lambda}=\varnothing$.

We shall show that this second, stronger statement holds for all but finitely many $\lambda \in C^{*} \backslash \kappa_{i}$. That is, there are only finitely many $\lambda \geqslant \kappa_{i}$ in $C^{*}$ such that $\kappa_{j} \in S_{\lambda}$ for some $j<i$. Suppose there were infinitely many such elements of $C^{*}$. Call them $\lambda_{0}<\lambda_{1}<\ldots$ say, and call the corresponding indices $j_{0}, j_{1}, \ldots$... Without loss of generality, say that $\lambda_{0} \neq \kappa_{i}$. For $m<n$, we know that $\kappa_{j_{m}} \in S_{\lambda_{m}}$, that $\kappa_{j_{n}} \in S_{\lambda_{n}}$, and that $\kappa_{j_{n}} \leqslant \kappa_{i}<\lambda_{m}<\lambda_{n}$. If $\kappa_{j_{m}} \leqslant \kappa_{j_{n}}$, then we are in exactly the situation we were dealing with back when we were proving the Separation Property. We have already seen that this leads to a contradiction, so instead it must be that $\kappa_{j_{n}}<\kappa_{j_{m}}$. But now $\kappa_{j_{0}}>\kappa_{j_{1}}>\ldots$ is an infinite decreasing sequence of ordinals. So our assumption about the existence of the $\lambda_{n}$ 's and $j_{n}$ 's is contradictory - there are only finitely many $\lambda>\kappa_{i}$ such that we can find $j<i$ with $\kappa_{j} \in S_{\lambda}$.

Hence, the set $\bigsqcup_{\lambda \in C * \backslash \kappa_{i}} S_{\lambda} \backslash X_{\lambda}$ is finite. We are assuming that $\bigsqcup_{C *} S_{\lambda} \backslash X_{\lambda}$ is infinite, so $\bigsqcup_{\lambda \in C * \cap \kappa_{i}} S_{\lambda} \backslash X_{\lambda}$ must be infinite.

Let us define a new sequence $\tilde{X} \in \prod_{C^{*}} U_{\lambda}^{*}$ :

$$
\tilde{X}(\lambda)= \begin{cases}X(\lambda) & \text { if } \lambda \in C^{*} \cap \kappa_{i} \\ \lambda & \text { if } \lambda \in C^{*} \backslash \kappa_{i}\end{cases}
$$

Let $\tilde{X}_{\lambda}=\tilde{X}(\lambda)$. We have just seen that $\bigsqcup_{\lambda \in C^{*}} S_{\lambda} \backslash \tilde{X}_{\lambda}=\bigsqcup_{\lambda \in C^{*} \cap \kappa_{i}} S_{\lambda} \backslash X_{\lambda}$ is infinite. Also, $\tau_{\tilde{X}}:=\sup \{\lambda$ : $\left.\tilde{X}_{\lambda} \neq \lambda\right\} \leqslant \kappa_{i}$.

We saw earlier that $\kappa_{i}<\tau_{X}$, so $\tau_{\tilde{X}}<\tau_{X}$. But then the existence of $\tilde{X}$ contradicts minimality of $\tau_{X}$. So we have a contradiction, and there is no $X$ such that $\bigsqcup_{C *} S_{\lambda} \backslash X_{\lambda}$ is infinite.

So the Mathias Criterion and Separation Property both hold, and hence $S$ is generic by Theorem 4.1.6.

### 4.2 Generating an inner model

With those preliminaries done, we shall now return to O-Machete and Theorem 4.0.3. We shall actually prove a slightly more general statement, of which both the Reg and $\mathrm{Reg}^{s}$ statements are special cases:
Theorem 4.2.1. Suppose $\mathrm{O}^{\mathrm{M} \epsilon}$ exists. Let $R$ be any (proper) class of regular uncountable cardinals of $V$, none of whose elements have Cantor-Bendixson rank $\epsilon$ in $R$, and which has at most finitely many elements below $\epsilon$. Then there is a class long iterate $M_{\mathrm{On}}$ of $\mathrm{O}^{\mathrm{M} \epsilon}$ such that $L[R]$ is a hyperclass generic extension of $M_{\mathrm{On}}$. The forcing we use is the Magidor forcing on all the measurables of the extender sequence of $M_{\mathrm{On}}$.

We get Theorem 4.0 .3 by taking $R=\operatorname{Reg}_{\epsilon}$ or $R=\operatorname{Reg}_{\epsilon}{ }^{s}$. (The assumption in that theorem that $\epsilon<\aleph_{\omega}$ gives us the "at most finitely many elements below $\epsilon$ statement here.)

### 4.2.1 Iterating O-Machete

Proof. For $\gamma<\epsilon$, let $R_{\gamma}$ be the class of elements of $R$ of rank $\gamma$. (So if $R=\operatorname{Reg}_{\epsilon}$ then $R_{\gamma}=\operatorname{Reg}_{\gamma}$, and likewise for $\operatorname{Reg}_{\epsilon}^{s}$.) Let $W_{\gamma}$ be the class of $\omega$ limits of $R_{\gamma}$. For $\lambda \in W_{\gamma}$, let $S_{\gamma}^{\lambda}$ be the $\omega$ sequence of cardinals in $R_{\gamma}$ immediately preceding $\lambda$ : i.e. $S_{\gamma}^{\lambda}$ is the $\omega$ many elements of $\left(R_{\gamma} \cap \lambda\right) \backslash \sup \left(W_{\gamma} \cap \lambda\right)$, ordered as an increasing sequence. Notice that $\bigcup_{\lambda \in W_{\gamma}} S_{\gamma}^{\lambda} \subset R_{\gamma}$, and $R_{\gamma} \backslash \bigcup_{\lambda \in W_{\gamma}} S_{\gamma}^{\lambda}$ is finite.

Let $M_{0}=\mathrm{O}^{\mathrm{M} \epsilon}$. We saw in 4.0.2 that the extender sequence of $M_{0}$ contains unboundedly many measurables of every rank below $\epsilon$, but none of rank $\epsilon$ or above. As in that proposition, let $C$ denote the collection of all the measurables in the extender sequence of $M_{0}$.

Consider the On +1 -long iteration $\mathcal{I}=\left\langle M_{i}, \pi_{i, j}\right\rangle$ which is defined by the following rule:
If $i \in$ On and $M_{i}$ is defined, then the critical point used to generate $M_{i+1}$ is the first measurable $\kappa$ in $\pi_{0, i}(C)$ which is not an element of $W_{\gamma}$ for any $\gamma<\epsilon$. If no such measurables exist, then instead the critical point used is the top measure of $M_{i}$.

It is immediately clear that we can follow this rule in any iterate $M_{i}$, so this gives us a well-defined iteration with no trivial stages.
Lemma 4.2.2. 1. $\pi_{0, \mathrm{On}}(C)=\bigcup_{\gamma<\epsilon} W_{\gamma}$
2. For all $\gamma<\epsilon$, if $\lambda \in R_{\gamma}$, and $\lambda>\epsilon$, then $\lambda$ is a critical point $\kappa_{i}$ of the iteration $\mathcal{I}$. Moreover, if $\lambda<\sup W_{\gamma}$ then $\pi_{i, \mathrm{On}}(\lambda)$ is the least element of $W_{\gamma}$ above $\lambda$. On the other hand, if $\lambda \geqslant \sup W_{\gamma}$ then $\pi_{i, j}(\lambda)$ becomes arbitrarily high as $j$ increases, and therefore $\pi_{i, \mathrm{On}}(\lambda)$ is undefined. (This means that the image of $\lambda$ is not in the set size part of the direct limit of the $M_{j}$, so it is lost when we do a "cut down" to produce the weasel $M_{\mathrm{On}}$.)

Proof. We shall first prove that every element of $R_{\gamma}$ above $\epsilon$ is a critical point of $\mathcal{I}$. Let $\lambda \in R_{\gamma}$ be greater than $\epsilon$. Let $i<$ On be least such that $\kappa_{i} \geqslant \lambda$.
Claim 4.2.3. $i=\lambda$
Proof. By 4.0 .2 we know $\mathrm{O}^{\mathrm{M} \epsilon}$ has cardinality equal to $\max (\omega,|\epsilon|)<\lambda$. On the other hand, obviously $M_{i}$ has $V$ cardinality at least $\lambda$. Let $j \leqslant i$ be least such that $\left|M_{j}\right| \geqslant \lambda$. It is easy to see that for $\alpha$ an infinite ordinal, $\left|J_{\alpha}^{E}\right|=|\alpha|$, so $j$ is also least such that $\lambda \in M_{j}$.

By Proposition 3.2.20, we know that $j$ is not a successor cardinal. So $M_{j}$ is a direct limit model, and for some $k<j$, there exists $\bar{\lambda} \in M_{k}$ such that $\pi_{k, j}(\bar{\lambda})=\lambda$.

The sequence $\left\{\pi_{k, j^{\prime}}(\bar{\lambda}): k<j^{\prime}<j\right\}$ is cofinal below $\lambda$. Since $\lambda$ is regular, it follows that $\lambda \leqslant j \leqslant i$. On the other hand, by definition of $i$, there is an $i$ long sequence of critical points below $\lambda$, so $i \leqslant \lambda$.

So the sequence of critical points $\left(\kappa_{j}\right)_{j<i}$ of the iteration $M_{0} \rightarrow M_{i}$ is a $\lambda$ long increasing sequence below $\lambda$, and is therefore unbounded. We've also already seen that $\mathrm{On} \cap \mathrm{O}^{\mathrm{M} \epsilon}<\lambda$. By Lemma 3.2.36, in $M_{i}=M_{\lambda}$ there is a measurable on $\lambda$. Moreover, this measurable is an image of an earlier critical point and therefore (by how we constructed the iteration) it is an element of $\pi_{0, i}(C)$.

Suppose that there is some $\mu \in \pi_{0, i}(C) \cap \lambda$ which is not an element of any $W_{\gamma}$. Then for large enough $j<i, \kappa_{j}>\mu$ and $\mu \in \pi_{0, j}(C)$. But then we would have chosen $\mu$ to be the critical point of the iteration at stage $j$, which we didn't since $\kappa_{j}>\mu$. So no such $\mu$ exists: in $M_{i}$, all the measurables of $\pi_{0, i}(C)$ below $\lambda$ are on elements of $W_{\gamma}$ for some $\gamma$ as they're supposed to be.

On the other hand, $\lambda$ itself is regular in $V$, so is certainly not an element of any $W_{\gamma}$ (or it would have $V$ cofinality $\omega$ ). So $\lambda$ is the least element of $\pi_{0, i}(C)$ which is not on an element of $W_{\gamma}$ for any $\gamma$, and hence $\lambda=\kappa_{i}$.

We must still show that $\pi_{i, \mathrm{On}}(\lambda)$ is the smallest element of $W_{\gamma}$ above $\lambda$ (or does not exist if $\lambda>\sup W_{\gamma}$ ). We do this by proving the following two statements together, using induction on $\gamma$ :

Claim 4.2.4. 1. For all $\lambda \in R_{\gamma}$, if $\lambda^{\prime}$ is the immediate successor of $\lambda$ in $R_{\gamma}$, and $\lambda=\kappa_{i}$ and $\lambda^{\prime}=\kappa_{j}$, then $\pi_{i, j}(\lambda)=\lambda^{\prime} .{ }^{4}$
2. For all $\lambda \in R_{\gamma}$, if $\mu=\min W_{\gamma} \backslash \lambda$ and $\lambda=\kappa_{i}$, then $\pi_{i, \mathrm{On}}(\lambda)=\mu$. If $W_{\gamma} \backslash \lambda=\varnothing$ then $\pi_{i, j}(\lambda)$ is unbounded as $j$ increases.

Proof. Induction on $\gamma$. Suppose that both 1 and 2 hold for all $\delta<\gamma$.
1: We know that $\lambda=\kappa_{i}$ is the critical point at stage $i$. The first step is to show that its Cantor-Bendixson rank in the class $\pi_{0, i}(C)$ of measurables of $M_{i}$ is at least (in fact, exactly) $\gamma$. Since we are assuming 2 holds for $\delta<\gamma$, for any $\delta<\gamma$ and for any $\mu \in W_{\delta} \cap \lambda$, we know that $\mu=\pi_{\alpha, \mathrm{On}}(\alpha)$ for some $\alpha \in R_{\delta}$. But we also know that $\alpha=\kappa_{\alpha} \in \pi_{0, \kappa}(C)$. Hence $\mu \in \pi_{0, \mathrm{On}}(C)$.

Since $\kappa_{i}=\lambda$, the map $\pi_{i, \text { On }}$ acts as the identity below $\lambda$, and hence $\mu=\pi_{i, \text { On }}^{-1}(\mu) \in \pi_{0, i}(C)$. This gives us an unbounded collection of measurables below $\lambda$ of every rank below $\gamma$. So the rank of $\lambda$ in $\pi_{0, i}(C)$ is at least $\gamma$.

On the other hand, since $\lambda$ is the critical point at stage $i$ of the iteration, we know that every element of $\pi_{0, i}(C)$ below $\lambda$ is in $W_{\delta}$ for some $\delta<\epsilon$. And $\lambda$ has Cantor-Bendixon rank $\gamma$ in the class $\bigcup W_{\delta}$. So its rank in $\pi_{0, i}(C)$ is at most $\gamma$.

[^19]Let $\nu=\sup \left(W_{\gamma} \cap \lambda\right)<\lambda$. We have shown that $\lambda$ is the least measurable in $\pi_{0, i}(C)$ which is strictly above $\nu$ and has Cantor Bendixson rank $\gamma$ in $\pi_{0, i}(C)$. The same argument shows that $\lambda^{\prime}$ is the least measurable in $\pi_{0, j}(C)$ which is strictly above $\sup \left(W_{\gamma} \cap \lambda^{\prime}\right)$ and has rank $\gamma$.

But $W_{\gamma}$ is a collection of $\omega$ limits of $R_{\gamma}$, and $\lambda^{\prime}$ is the immediate successor of $\lambda$ in $R_{\gamma}$, so $\sup \left(W_{\gamma} \cap \lambda^{\prime}\right)=\nu$. And since $\nu, \gamma<\lambda=\kappa_{i}$, we know that $\pi_{i, j}(\nu)=\nu$ and $\pi_{i, j}(\gamma)=\gamma$. Since $\pi_{i, j}$ is elementary, $\pi_{i, j}(\lambda)=\lambda^{\prime}$ as required.

2: Suppose that $\lambda<\sup R_{\gamma}$. Let $\lambda=\kappa_{i}=\kappa_{i_{0}}<\kappa_{i_{1}}<\ldots$ be the first $\omega$ many elements of $R_{\gamma}$ above $\lambda$, whose supremum $\mu$ is the least element of $W_{\gamma}$ above $\lambda$. Let $j=\sup \left\{i_{n}: n \in \omega\right\}$. By induction on 1 we know $\pi_{i_{0}, i_{n}}(\lambda)=\kappa_{i_{n}}$ for all $n \in \omega$. So since $M_{j}$ is a direct limit model, $\pi_{i, j}(\lambda)=\sup \left\{\kappa_{n}: n \in \omega\right\}=\mu$.

By stage $j$ of the iteration $\mathcal{I}$, we have seen unboundedly large critical points below $\mu$ (namely, $\kappa_{i_{0}}, \kappa_{i_{1}}, \ldots$ ). So from $j$ onwards, the remainder of the iteration has no critical points below $\mu$. Moreover, $\mu \in W_{\gamma}$ and we defined $\mathcal{I}$ such that no element of $W_{\gamma}$ would ever be a critical point. So all the critical points of the iteration from $M_{j}$ to $M_{\text {On }}$ are strictly greater than $\mu$. Hence $\pi_{j, \text { On }}(\mu)=\mu$ and so $\pi_{i, \infty}(\lambda)=\mu$.

Now suppose $\lambda \geqslant \sup W_{\gamma}$. As we saw earlier, $\lambda$ is of rank at least $\gamma$ in $\pi_{0, i}(C)$. This will also be true (by elementarity) about $\pi_{i, j}(\kappa)$ in $\pi_{0, j}(C)$, for $i<j<$ On. So for all such $j$, we know $M_{j}$ will contain at least one measurable $\lambda \leqslant \pi_{i, j}(\kappa)$ of $\pi_{0, j}(C)$ which is not an element of $\bigcup_{\delta} W_{\delta}$. Hence for all $i<j<$ On we know $\kappa_{j} \leqslant \pi i, j(\lambda)$. Since the critical points of a class long iteration always become arbitrarily high, it follows that $\pi_{i, j}(\lambda)$ increases unboundedly as $j$ increases and $\pi_{i, \mathrm{On}}(\lambda)$ is undefined.

This immediately shows the remainder of the second part of the lemma, and also that $\pi_{0, \mathrm{On}}(C) \supset \bigcup_{\gamma} W_{\gamma}$. The last step is to show that this is an equality. Suppose that $\lambda \in \pi_{0, \mathrm{On}}(C) \backslash \bigcup W_{\gamma}$. Then for large enough $i$, we know there is some $\bar{\lambda} \leqslant \lambda$ such that $\pi_{i, \mathrm{On}}(\bar{\lambda})=\lambda$. Choosing $i$ such that $\kappa_{i}>\lambda \geqslant \bar{\lambda}$ it is immediately clear that $\bar{\lambda}=\lambda \notin \bigcup W_{\gamma}$. It is then clear by elementarity that $\lambda \in \pi_{0, i}(C)$. But then we would have chosen $\lambda$ as our critical point at stage $i$, and we said that $\kappa_{i}>\lambda$. Contradiction. So $\pi_{0, \mathrm{On}}(C)=\bigcup_{\gamma} W_{\gamma}$.

Corollary 4.2.5. Let $\kappa \in \pi_{0, \mathrm{On}}(C)$. In $M_{\mathrm{On}}$, let $\mathbb{P}_{\kappa}$ be the Magidor iteration of $\pi_{0, \mathrm{On}}(C) \upharpoonleft \kappa$. Then

$$
T^{\kappa}:=\left\{\left(\lambda, S_{\gamma}^{\lambda}\right): \gamma<\epsilon, \lambda \in W_{\gamma} \cap \kappa\right\}
$$

is generic for $\mathbb{P}_{\kappa}$.
Proof. By Lemma 4.2.2, we know that $T^{\kappa}$ satisfies the condition for genericity proved in Lemma 4.1.7.
So we now have our generic extension. There is a subtle issue we still need to deal with, however. The corollary only talks about $\mathbb{P}_{\kappa}$, a set size initial segment of the Magidor iteration. We want to show that it holds for the Magidor iteration of the whole of $\pi_{0} \mathrm{On}_{\mathrm{n}}(C)$. But this is a proper class of $M_{\infty}$, and so the Magidor iteration is a hyperclass forcing. Lemma 4.1.7 only tells us about set size forcings. We must find a way to get around this difficulty if we want to force with the full iteration $\mathbb{P}$. First, we must check that $\mathbb{P}$ actually satisfies the conditions from Chapter 1 to make forcing work properly.
Lemma 4.2.6. Let $\mathbb{P}$ be the Magidor iteration of $\pi_{0, \mathrm{On}}(C)$, defined in the $\mathrm{MK}^{* *}$ model generated by $\mathcal{I}$ whose (set) domain is $M_{\mathrm{On}}$. Then $\mathbb{P}$ is pretame.

Proof. Follows immediately from the fact that it has the "On ${ }^{+}$chain condition" (see Corollary 4.1.5; the proof goes through for classes in exactly the same way), so every dense subhyperclass of $\mathbb{P}$ contains a predense subclass.

Lemma 4.2.7. Let $\mathbb{P}$ be as above. Then

$$
T:=\left\{\left(\lambda, S_{\gamma}^{\lambda}\right): \gamma<\epsilon, \lambda \in R_{\gamma}\right\}
$$

is generic for $\mathbb{P}$.
Proof. Let $\left(i_{\gamma}\right)_{\gamma \in \mathrm{On}}$ be the sequence given in Lemma 3.2.35 for $\lambda:=$ On. For $\gamma \in$ ZFC, let

$$
\mathcal{C}_{\gamma}=\left(\mathcal{P}\left(H_{\kappa_{i_{\gamma}}}\right)\right)^{M_{i_{\gamma}}}
$$

and let

$$
\mathcal{H}_{\gamma}=\left\langle H_{\kappa_{i_{\gamma}}}^{M_{i_{\gamma}}}, \mathcal{C}_{\gamma}\right\rangle
$$

We know that the analogous structure in the direct limit of $\left\langle M_{i}\right\rangle_{i<0 n}$ is a model of MK** (see the remarks after Lemma 3.2.40) so by elementarity $\mathcal{H}_{\gamma}$ is a model of $\mathrm{MK}^{* *}$, with set-part $H_{\kappa_{i_{\gamma}}}^{M_{i_{\gamma}}}$. So we can also view $\mathbb{P} 1 \kappa_{i_{\gamma}}=\mathbb{P}_{\kappa_{i_{\gamma}}}$ as a hyperclass forcing over $\mathcal{H}_{\gamma}$.
Claim 4.2.8. Let $\gamma \in$ On. A filter $G$ is generic over $\mathbb{P}_{\kappa_{i_{\gamma}}}$ for $\mathcal{H}_{\gamma}$ (as a hyperclass forcing) if and only if it is generic over $\mathbb{P}_{\kappa_{i \gamma}}$ for $M_{\mathrm{On}}$ (as a set forcing).

Proof. To keep notation tidier, let us write $\overline{\mathbb{P}}$ for $\mathbb{P}_{\kappa_{i \gamma}}$ just for this claim.
The critical points of the iteration from $M_{i_{\gamma}}$ to $M_{\mathrm{On}}$ are all $\geqslant \kappa_{i_{\gamma}}$. So by Proposition 3.3.4 $M_{\mathrm{On}}$ and $M_{i_{\gamma}}$ agree on $H_{\kappa_{i_{\gamma}}}$ and its subsets. So $\mathcal{H}_{\gamma}$ is a set in $M_{\text {On }}$ :

$$
\mathcal{H}_{\gamma}=\left\langle H_{\kappa_{i_{\gamma}}}^{M},\left(\mathcal{P}\left(H_{\kappa_{i_{\gamma}}}\right)\right)^{M_{\mathrm{On}}}\right\rangle \in M_{\mathrm{On}}
$$

Hence any definable dense subhyperclass of $\overline{\mathbb{P}}$ in $\mathcal{H}_{\gamma}$ is a set in $M_{\mathrm{On}}$. So a filter which is generic in the sense of $M_{\mathrm{On}}$ is generic in the sense of $\mathcal{H}_{\gamma}$ as well.

On the other hand, we saw in Corollary 4.1.5 that $\overline{\mathbb{P}}$ has the $\kappa_{i_{\gamma}}^{+}$chain condition. So letting $D \in M_{\text {On }}$ be any dense subset of $\overline{\mathbb{P}}$, we can find some predense subset $D^{\prime} \subset D$ of cardinality $\leqslant \kappa_{i_{\gamma}}$. Since $\overline{\mathbb{P}} \subset H_{\kappa_{i_{\gamma}}^{+}}^{M_{0 n}}$, we know $D^{\prime} \in H_{\kappa_{\gamma_{\gamma}}^{+}}^{M \text { on }}$ and hence that $D^{\prime}$ can be coded using a canonical Skolem function as a set $S \subset \kappa_{i_{\gamma}}$ in $M_{\text {On }}$. Since $S \subset \kappa_{i_{\gamma}}$, it follows immediately that $S$ is a class in the MK** model $\mathcal{H}_{\gamma}$. Hence, $D^{\prime}$ is a definable hyperclass over $\mathcal{H}_{\gamma}$. So any filter $G$ which is generic in the sense of $\mathcal{H}_{\gamma}$ will meet $D^{\prime}$ and hence meet $D$. Since $D$ was arbitrary, any such filter $G$ is generic in the sense of $M_{\infty}$.

In particular, this claim means that $T^{\kappa_{i \gamma}}$ is generic over $\mathcal{H}_{\gamma}$ in the sense of hyperclass forcing, because Corollary 4.2.5 tells us it is generic over $M_{\mathrm{On}}$.

We now need a way to transfer this up to genericity of $T$ over $\mathbb{P}$ in $M_{\text {On }}$. To help us here, we need the following technical result.
Claim 4.2.9. Let $i<$ On and let $\kappa_{i}<\kappa \leqslant$ On. Let $j \leqslant$ On be minimal such that $\kappa_{j} \geqslant \kappa$ (interpreting $\kappa_{\mathrm{On}}$ as On). If $p \in\left(\mathbb{P}_{\kappa_{i}}\right)^{M_{i}}$ agrees with $T 1 \kappa_{i}$, then $\pi_{i, j}(p) 1 \kappa$ agrees with $T 1 \kappa$.

By " $p$ agrees with $T 1 \kappa_{i}$ " we mean that $p$ is in the generic filter generated by $T 1 \kappa_{i}$, i.e. that all the Prikry conditions named by $p$ are interpreted by the appropriate initial segments of $T$ as Prikry conditions that are compatible with the relevant part of the sequence. Note that the claim makes sense: if $\kappa_{i}$ is the $i$ 'th critical point, then $M_{i}$ and $M_{\mathrm{On}}$ agree on $\pi(C) 1 \kappa_{i}$, so $T 1 \kappa_{i}$ is a collection of ordinals with the correct suprema for $\mathbb{P}_{\kappa_{i}}$. Similarly for $M_{j}$ and $T 1 \kappa$.
Proof. We prove this by induction on $\kappa$, but simultaneously for all $i$ below the $j$ defined by $\kappa$. The notation in this proof gets a bit fiddly, so to simplify things, we shall introduce some shorthand. Let $\pi=\pi_{i, j}$. Let $\tilde{C}=\pi_{0, i}(C) \cap \kappa_{i}$, and $\tilde{T}=T 1 \kappa_{i}$. Let $\bar{C}=\pi_{0, j}(C) \cap \kappa$ and $\bar{T}=T 1 \kappa$. By a small abuse of notation, let the Magidor forcing on $\bar{C}$ be $\mathbb{P}_{\kappa}$. Let $\bar{p}=\pi(p) 1 \kappa$. Note that $\mathbb{P}_{\kappa_{i}}$ is the Magidor forcing on $\tilde{C}$.

As usual, we can think of $p$ as a combination of two parts: a name $\dot{s}=\left(\dot{s}_{\lambda}\right)_{\lambda \in \tilde{C}}$ for some stem of the generic sequence we are adding, and another name $\dot{X}=\left(\dot{X}_{\lambda}\right)_{\lambda \in \tilde{C}}$ for a sequence of measure 1 sets. We know that $\bar{p}$ is an end extension of $p$, so let's extend this notation by writing $\dot{s}_{\lambda}=\pi(\dot{s})(\lambda)$ and $\dot{X}_{\lambda}$ for all $\lambda \in \bar{C}$. Since $\pi(p)$ is an end extension, the two definitions of $\dot{s}_{\lambda}$ and $\dot{X}_{\lambda}$ agree where they are both defined. For clarity, we will avoid writing $\dot{s}$ to denote the overall sequence, instead writing $\bar{s}$ to denote the sequence $\dot{s}=\left(\dot{s}_{\lambda}\right)_{\lambda \in \bar{C}}$ and $\tilde{s}$ for the sequence $\left(\dot{s}_{\lambda}\right)_{\lambda \in \tilde{C}}$. We will define $\bar{X}$ and $\tilde{X}$ likewise. Note that $\pi(p)$ consists of $\bar{s}$ together with $\bar{X}$. We know that $\tilde{S}^{\tilde{T}}$ and $\tilde{X}^{T}$ both agree with $\tilde{T}$; we want to show that $\bar{s}^{T}$ and $\bar{X}^{T}$ agree with $\bar{T}$.

Now $\tilde{s}$ has finite support which is bounded below $\kappa_{i}$, so $\pi(\tilde{s})=\bar{s}$ differs from $\tilde{s}$ only by a trivial end extension. So $\bar{s}$ is forced by $\mathbb{1}_{\mathbb{P}_{k}}$ to be equal to $\tilde{s}$, and thus, $\bar{s}$ agrees with $\tilde{T}$ and hence also $\bar{T}$ in any generic
extension whose filter contains (an end extension of) $p$. In particular, we know that $\bar{T}$ itself contains many end extensions of $p$, so $\bar{s}^{\bar{T}}$ agrees with $\bar{T}$.

The more difficult task is showing that $\bar{X}$ agrees with $T$. More precisely, we're aiming to show that if $\lambda \in \bar{C}$ (and so $\lambda \in W_{\beta}$ say), then the measure 1 set $\left(\dot{X}_{\lambda}\right)^{T 1 \lambda}$ contains all of the $\omega$ sequence $S_{\beta}^{\lambda}$ except for any elements of $\left(\dot{s}_{\lambda}\right)^{T 1 \lambda}$. This is given to us automatically if $\lambda \in \tilde{C}$ by our assumption that $p \in \tilde{T}$. Also, notice that $\kappa_{i} \notin \tilde{C}$ (since it's not measurable in $M_{j}$ ). So the only case we actually need to check is where $\kappa_{i}<\lambda \in \bar{C}$. Fix such a $\lambda$.

Recall that by Proposition 4.1.2, any measure 1 set $\dot{Y}^{G}$ in a $\mathbb{P}_{\lambda}$ generic extension (by any generic $G$ ) will contain the intersection of some measure 1 set $Y$ in the ground model with the set

$$
\Sigma_{\lambda}^{G}:=\left\{\nu<\lambda: \forall \kappa \in \bar{C} \cap \lambda \backslash \nu,(\nu+1) \cap G_{\kappa}=\varnothing\right\}
$$

Decoding this for $G=T 1 \lambda$, we find that $\nu \in \Sigma_{\lambda}^{T 1 \lambda}$ if and only if $\nu<\lambda$ and $\nu$ is not in the interval $\left(\min S_{\kappa}^{\delta}, \kappa\right)$ for any $\kappa<\lambda$ in $\bar{C}$. So in particular, every term of $S_{\lambda}^{\beta}$ is in $\Sigma_{\lambda}^{T 1 \lambda}$. We now need to show that $S_{\lambda}^{\beta}$ is also contained in the relevant ground model measure 1 set in $M_{j}$.

Let $\kappa_{k}$ be the first element of $S_{\lambda}^{\beta}$; and let $l$ be least such that $\pi_{k, l}\left(\kappa_{k}\right)=\lambda$. We know that the $\kappa_{k}$ portion of the condition $\pi_{i, k}(p)$ contains a $\mathbb{P} \upharpoonleft \kappa_{k}$ name $\dot{X}^{*}$ for a measure 1 subset of $\kappa_{k}$. As we said above, in $M_{k}\left[T \upharpoonleft \kappa_{j}\right]$ there is some measure 1 subset $Y$ of $\kappa_{k}$ such that $Y \cap \Sigma_{\kappa_{k}}^{T 1 \kappa_{k}} \subset\left(\dot{X}_{\lambda}^{\prime}\right)^{T 1 \kappa_{k}}$. Let $q \in T 1 \kappa_{k}$ be a condition which decides which $Y \in M_{k}$ this is.

Now, since $\lambda \in \bar{C} \cap \kappa$, we know that $\lambda<\kappa$. And $l$ is least such that $\pi_{k, l}\left(\kappa_{j}\right)=\lambda$, and hence is least such that $\kappa_{l} \geqslant \lambda$. (We've already seen that $\kappa_{j}$ will be sent through limit many critical points before reaching $\lambda$, so certainly $\kappa_{l} \geqslant \lambda$.) So by inductive hypothesis, $\pi_{k, l}(q)=\pi_{k, l}(q) \upharpoonleft \lambda \in T \upharpoonleft \lambda$. By elementarity, $\pi_{k, l}(q)$ forces that $\pi_{k, l}(Y) \cap \Sigma_{\lambda}$ is a subset of $\pi_{k, l}\left(\dot{X}^{*}\right)$. And we saw during the proof of Lemma 4.1.7 that $\pi_{k, l}(Y)$ contains all the critical points of the iteration taking $\kappa_{k}$ to $\lambda$, including all the elements of $S_{\lambda}^{\beta}$. Finally, note that $\kappa_{l}>\lambda$ (since $\lambda \in \bar{C}$ it can't be a critical point) so $\pi_{l, j}$ acts as the identity below $\lambda$. So by elementarity, $\pi_{k, j}(q)=\pi_{k, l}(q)$ forces that $\pi_{k, j}(Y) \cap \Sigma_{\lambda}=\pi_{k, l}(Y) \cap \Sigma_{\lambda}$ is a subset of $\pi_{k, j}\left(\dot{X}^{*}\right)=\dot{X}_{\lambda}$. Since both $\pi_{k, l}(Y)$ and $\Sigma_{\lambda}^{T 1 \lambda}$ contain all of $S_{\lambda}^{\beta}$ it follows that $S_{\lambda}^{\beta}$ is contained in $\dot{X}_{\lambda}^{T 1 \lambda}$, as required.

This is what we were aiming to show, and implies that $\overline{( } p)$ agrees with $T 1 \kappa$.
Now, let $D \subset \mathbb{P}$ be a dense definable hyperclass over $M_{\text {On }}$. Let $\varphi(\vec{x})$ be the formula defining it. By an elementarity argument, we know that for large enough $\gamma, \varphi(\vec{x})$ defines a hyperclass $D_{\gamma}$ over $\mathcal{H}_{\gamma}$, which is dense in the $M_{i_{\gamma}}$ analogue of $\mathbb{P}$, i.e. $\mathbb{P}_{\kappa_{i_{\gamma}}}$. Since $T^{\kappa_{i}}$ is generic for $\mathbb{P}_{\kappa_{i_{\gamma}}}$ over $\mathcal{H}_{\gamma}$, it must meet $D_{i_{\gamma}}$. So let $p \in T^{\kappa} \in D_{i_{\gamma}}$. Then $\mathcal{H}_{\gamma} \vDash \varphi(\vec{x})(p)$. By elementarity, $\varphi(\vec{x})\left(\pi_{i, \mathrm{On}}(p)\right)$ holds in the MK ${ }^{* *}$ model with set-domain $M_{\mathrm{On}}$, so $\pi_{i, \mathrm{On}}(p) \in D$. But $T^{\kappa_{i \gamma}}=T 1 \kappa_{i_{\gamma}}$, so by the claim we just proved, $\pi(p) \in T 1$ On $=T$. So $T$ meets every dense subhyperclass of $\mathbb{P}$ over the MK model extending $M_{\mathrm{On}}$, and is therefore generic over that model.

### 4.2.2 Proving the theorem

It should now be clear how we're going to prove theorem 4.2.1. Starting with $M_{0}=\mathrm{O}^{\mathrm{M} \epsilon}$, we perform the iteration described in the previous section to get $M_{\text {On }}$, together with a collection of classes $\mathcal{C}$ over $M_{\text {On }}$ which make it a model of MK. We then take the hyperclass generic extension of this MK model with respect to the generic sequence $T$, to get a model ( $M_{\mathrm{On}}[T], \mathcal{C}[T]$ ). The last step, which we do in this section, is to show that $M_{\mathrm{On}}[T]=L[R]$.

First, let us make some easy observations. Abusing notation slightly, we shall write $\cup T$ to denote the collection of all terms in sequences in $T$, i.e. all $\mu$ such that $(n, \mu) \in S_{\gamma}^{\lambda}$ for some $n, \lambda$ and $\gamma$.
Lemma 4.2.10. $R \backslash \cup T$ is finite.
Proof. Clearly, for any $\gamma<\epsilon, R_{\gamma} \backslash \cup T$ will just consist of the top few elements of $\operatorname{Reg}_{\gamma}$, above which there is no element of $W_{\gamma}$. Since $W_{\gamma}$ is defined as the class of $\omega$ limits of $R_{\gamma}$, there can only be finitely many of these.

Let $X$ be the set of all $\gamma<\epsilon$ such that $R_{\gamma} \backslash \cup T \neq \varnothing$. As we've seen, if this holds for some $\gamma$ then $R_{\gamma}$ has a finite (but nonempty) final segment, and so contains a maximum element $\lambda_{\gamma}$.

If $\delta, \gamma \in X$ and $\delta<\gamma$, then $\lambda_{\gamma}$ is a limit of elements of $R_{\delta}$, and so $\lambda_{\delta}>\lambda_{\gamma}$. Hence, $X$ must be finite, or $\left(\lambda_{\gamma}\right)_{\gamma \in X}$ would be an infinite decreasing sequence of ordinals.

So $R \backslash \cup T=\bigcup_{\gamma \in S}\left(R_{\gamma} \backslash \cup T\right)$ is a finite union of finite sets, and hence finite.
Lemma 4.2.11. If $M$ is any model of ZFC, the $T$ definable sets over $M$ are precisely the $R$ definable sets over $M$. Hence, if $M$ is closed under $T$ definability then it contains $L[R]$ as a subclass, definable in any language which includes $T$. Likewise, if $M$ is closed under $R$ definability and contains $M_{\mathrm{On}}$ as an $R$-definable subclass, then it also contains $M_{\mathrm{On}}[T]$ as an $R$-definable subclass.

Proof. $R$ is simply $\cup T$ together with a finite collection of extra ordinals, so any formula in terms of $R$ can easily be turned into one in terms of $T$. Conversely, in a model of ZFC there is a class function taking any set $S$ to the Cantor-Bendixson rank of its largest element. We can apply this class function to the $R$ definable set $R \cap(\kappa+1)$ to determine the Cantor-Bendixson rank of any $\kappa \in R$, and thus calculate the classes $R_{\gamma}$ for $\gamma<\epsilon$ within $M$. Once we have done that, it is easy to make a formula to calculate where, if anywhere, a given $\kappa$ will appear in $T$, and thus turn any formula in terms of $T$ into one in terms of $R$.

So we automatically know that $L[R]$ is a $T$-definable subclass of $M_{\mathrm{On}}[T]$, and to complete the theorem, it suffices to show that $M_{\mathrm{On}}$ is an $R$-definable subclass of $L[R]$.
Lemma 4.2.12. $M_{\mathrm{On}}$ is an $R$-definable subclass of $L[R]$.
Proof. Recall that by definition, $M_{\mathrm{On}}=\left(J_{\mathrm{On}}^{E}, E\right)=(L[E], E)$, where $E$ is the extender sequence of $M_{\mathrm{On}}$. So it suffices to show that $L[R]$ can calculate $E$, i.e. that it can determine what the measurables of $E$ are, and which of the sets in $M_{\text {On }}$ are measure 1 . We can then carry out the usual recursive construction to calculate $L_{\gamma}[E]$ for all $\gamma$, and hence find $M_{\text {On }}$. Note that we do not need to devise an explicit test for whether a given set is an element of $M_{\mathrm{On}}$ : it suffices to determine the properties of sets we already know to be in $M_{\mathrm{On}}$.

The class of measurables of $E$ is just $\pi_{0, \mathrm{On}}(C)=\bigcup_{\gamma} W_{\gamma}=: W$. This is clearly definable over $L[R]$.
Now we must find a way to express in $L[R]$ the statement, for $\lambda \in W$ and $X \in L[R]$,
"If $X \in L[E]$ then $X$ is a measure 1 subset of $\lambda$."
Of course, it is easy to express " $X$ is a subset of $\lambda$ "; the challenge is in the "measure 1 " part. Suppose that $X \in L[E]$, and that (say) $\lambda \in W_{\gamma}$. If $X$ is measure 1 , then by Theorem 4.1.6 ${ }^{5}$ and the fact that $T$ is generic in $L[E]$, we know $X$ will contain all but finitely many terms of $S_{\lambda}^{\gamma}$.

On the other hand, if $X \subset \lambda$ is not measure 1 but is an element of $L[E]$, then $\lambda \backslash X$ will be measure 1 instead. So as we just saw, $\lambda \backslash X$ will contain all but finitely many terms of $S_{\lambda}^{\gamma}$, and therefore $X$ itself will contain at most finitely many terms of $S_{\lambda}^{\gamma}$.

So if $X \in L[E]$ is a subset of $\lambda$, then $X$ is measure 1 if and only if it contains all but finitely many terms of $S_{\lambda}^{\gamma}$, which is a statement which can be expressed in $L[R]$. So $L[R]$ knows what $E$ looks like, and can therefore calculate the whole of $(L[E], E)=M_{\text {On }}$.

Corollary 4.2.13. The domain of $M_{\mathrm{On}}[T]$ is precisely $L[R]$, and its extender sequence is $R$-definable over $L[R]$.

This ends the proof of Theorem 4.0.3.

### 4.3 Friendly Machetes

We end this chapter with an existence result about O-Machetes. So far, we've been assuming we've already found an O-Machete, and then used it to generate some structure $L[R]$. But maybe we could do the opposite: we could start with some interesting $L[S]$ type structure for some predicate $S$, and show that every nice enough O-Machete must be an element of that model.

The predicate $S$ we are going to look at is $\mathrm{Reg}^{s}$, a natural choice when we want to find a mouse which contains mice that we've just seen can generate $L\left[\operatorname{Reg}_{<\alpha}^{s}\right]$. Of course, it's possible that $\mathrm{Reg}^{s}$ is almost empty and doesn't give us any useful information: for example, if there are no inaccessibles then Reg ${ }^{s}$ is effectively

[^20]just Card. So we need to assume some largeness criterion for $\mathrm{Reg}^{s}$. The condition we'll be using is " $\mathrm{Reg}^{s}$ is a stationary class", which can also be expressed as "On is Mahlo".

Assuming this condition, it turns out that every nice enough machete mouse will be in $L\left[\mathrm{Reg}^{s}\right]$.

### 4.3.1 Definitions

To state this properly, we must first introduce some new definitions so we can express "nice enough". Importantly, these definitions are not restricted to machete mice.

Definition 4.3.1. Let $\varphi\left(v_{0}, \ldots, v_{n+1}\right)$ be a formula, and let $\alpha_{0}, \ldots, \alpha_{n}$ be ordinals. We say the tuple $\left\langle\varphi, \alpha_{0}, \ldots, \alpha_{n}\right\rangle$ is upwards reflecting if for all weasels $W$, and for all sequences of ordinals $\left(\beta_{\gamma}\right)_{\gamma<\delta}$ above $\max \left\{\alpha_{0}, \ldots, \alpha_{n}\right\}$ such that for all $\gamma<\delta$,

$$
H_{\beta_{\gamma}^{+}}^{W} \vDash \varphi\left(\alpha_{0}, \ldots, \alpha_{n}, \beta_{\gamma}\right)
$$

we have

$$
H_{\beta^{+}}^{W} \vDash \varphi\left(\alpha_{0}, \ldots, \alpha_{n}, \beta\right)
$$

where $\beta=\sup _{\gamma} \beta_{\gamma}$.
Essentially, this rather complicated looking statement is just asking that the statement associated with $\mathrm{O}^{\mathrm{M} \varphi}$ be preserved by limits.

As a specific example, the formula associated with $\mathrm{O}^{\mathrm{M} \epsilon}$ ("There are unboundedly many measurables of all ranks $<\epsilon$ below $v_{n+1}$ ") is upwards-reflecting: if for all $\gamma$, there are unboundedly many measurables of ranks $<\epsilon$ below $\beta_{\gamma}$ then there will also be unboundedly many measurables of ranks $<\epsilon$ below $\beta=\sup \beta_{\gamma}$.

Definition 4.3.2. Let $\alpha$ be an ordinal, and let $M$ be a mouse. We say that $M$ is $\alpha$ friendly if there exists an upwards-reflecting formula $\varphi\left(v_{0}, \ldots, v_{n+1}\right)$ and parameters $\alpha_{0}, \ldots, \alpha_{n} \in M$ such that:

1. $M$ is active, and has largest cardinal $\kappa$;
2. For all $0 \leqslant i \leqslant n, \alpha_{i}$ is below the least measurable of $M$ and $\alpha_{i}<\alpha$;
3. $H_{\kappa^{+}}^{M} \vDash \varphi\left(\alpha_{0}, \ldots, \alpha_{n}, \kappa\right)$ (so $M$ is a candidate for $\mathrm{O}^{\mathrm{M} \varphi}$ );
4. There are at most boundedly many measurables $\lambda$ of $M$ below $\kappa$ such that $H_{\lambda^{+}}^{M} \vDash \varphi\left(\alpha_{0}, \ldots, \alpha_{n}, \lambda\right)$
$M$ is hereditarily $\alpha$ friendly if it is $\alpha$ friendly, and for all active $N<^{*} M, N$ is $\alpha$ friendly.
We say that $M$ is $\alpha$ friendly with respect to $\varphi$ and $\alpha_{0}, \ldots, \alpha_{n}$ if this formula and parameters satisfy the criteria above.

Of course, any machete mouse $\mathrm{O}^{\mathrm{M} \varphi}$ will be $\alpha$ friendly, provided that $\varphi$ is $\Sigma_{1}$ and its parameters are below $\alpha$. Condition 4 holds because there are no such measurables $\lambda$ at all: if there were, then $\mathrm{O}^{\mathrm{M} \varphi} 1 \lambda<* M$ would be a candidate for $\mathrm{O}^{\mathrm{M} \varphi}$.

We saw earlier that for all $\epsilon, \mathrm{O}^{\mathrm{M} \epsilon}=\mathrm{O}^{\mathrm{M} \varphi}$ for a certain formula $\varphi$ with parameter $\epsilon$. It follows that if $\epsilon<\alpha$ then $\mathrm{O}^{\mathrm{M} \epsilon}$ is $\alpha$ friendly.

Proposition 4.3.3. In fact, $\mathrm{O}^{\mathrm{M} \epsilon}$ is hereditarily $\alpha$ friendly for $\epsilon<\alpha$.
Proof. Let $N<{ }^{*} \mathrm{O}^{\mathrm{M} \epsilon}$ be active. Let $\delta$ be the Cantor Bendixson rank of the largest measurable $\kappa$ of $N$. We know that $\delta<\epsilon<\alpha$.
$N$ believes that there are only boundedly many measurables below $\kappa$ which are rank $\delta$. So it is friendly, with $\varphi(\delta, v)$ saying "There are unboundedly many measurables of all ranks $<\delta$ below $v$ ".

There is another property which we will need to state the theorem, and it looks a bit technical.
Definition 4.3.4. Let $M_{0}$ be some mouse, and let $\mathcal{I}$ be an iteration of $M_{0}$ of length On +1 . As usual, let $\kappa_{i}$ denote the $i$ 'th critical point of $\mathcal{I}$, and let $M_{i}$ be the $i$ 'th mouse of the iteration.

We say that $\mathcal{I}$ is tidy if there exist cardinals $\lambda \in \operatorname{Card}$ and $\mu \in \operatorname{Reg}$ such that if $\kappa>\lambda$ is strongly inaccessible in $M_{\text {On }}$, one of the following holds:

1. $\kappa$ is measurable in $M_{\mathrm{On}}$
2. $\kappa=\kappa_{i}$ for some $i<$ On (meaning $\kappa$ used to be measurable earlier in the iteration, but was used as a critical point)
3. $\operatorname{Cof}^{V}(\kappa)=\mu$

We say that a mouse $M$ is tidy if every class length iteration $\mathcal{I}$ of $M$ is tidy. We say that $M$ is hereditarily tidy if it is tidy and every active mouse $N \leqslant{ }^{*} M$ is tidy.

Although this seems like a very specific condition, it's actually extremely common. It turns out that with the definition of a mouse we gave in Chapter 3, almost every mouse is tidy automatically, provided that $\rho_{\omega}$ drops below the first measurable. (This is also true for any mouse whose consistency strength is below that of a strong cardinal, even if it has multiple measures on the same cardinal.) The proof is an expansion of $[46,6.3]$.

Lemma 4.3.5. Let $M_{0}$ be a mouse, and let $\mathcal{I}=\left\langle M_{i}, \pi_{i, j}\right\rangle_{i<\theta+1}$ be a set-length simple iteration with no drops in degree, whose iteration takes place above $\rho_{\omega}$.

Let $n<\omega$ be the unique natural number such that $\rho_{n+1}^{M_{i}} \leqslant \kappa_{i}<\rho_{n}^{M_{i}}$ for all $i<\theta$. If $\kappa$ is strongly inaccessible in $M_{\theta}$, and $\rho_{n+1}^{M_{\theta}}<\kappa<\rho_{n}^{M_{\theta}}$, then one of the following holds:

1. $\kappa$ is measurable in $M_{\theta}$
2. $\kappa=\kappa_{i}$ for some $i<\theta$
3. $\operatorname{Cof}^{V}(\kappa)=\operatorname{Cof}^{V}\left(\rho_{n}^{M_{0}}\right)$
4. $\kappa$ is below the supremum of the critical points of the iteration from the core of $M_{0}$ to $M_{0}$ itself.

Proof. First, some easy results from [47]. Let $0 \leqslant i \leqslant j \leqslant \theta$. By Lemma 3.3.1, the map $\pi_{i, j}$ is $\Sigma_{1}^{(n)}$ preserving and cofinal, and by Proposition 3.3.2, $\rho_{n+1}^{M_{i}}=\rho_{n+1}^{M_{j}}$.

We can't say the same about $\rho_{n}$, which can increase as we do the iteration. But its cofinality in $V$ remains constant.
Claim 4.3.6. [46, 6.3] There is a $\Sigma_{1}^{(n)}$ formula with a single parameter, which defines, over any $M_{i}$ for $i<\theta$, a cofinal sequence below $\rho_{n}^{M_{i}}$ in $M_{i}$ whose order type does not depend on $i$ and is $\leqslant \rho_{n+1}^{M_{0}}$.

Hence, $\operatorname{Cof}^{V}\left(\rho_{n}^{M_{i}}\right)$ does not depend on $i$.
Proof. Let $\bar{M}$ be the $n$ 'th core of $M$. Let $\mathcal{K}: \bar{M} \rightarrow M_{0}$ be the corresponding iteration, with iteration map $\tau: \bar{M} \rightarrow M_{0}$.

Since $\bar{M}$ is $n$ sound, we can map $\rho_{n+1}$ onto $\bar{M}^{n, p 1 n}$ in a $\Sigma_{1}(p)$ way (where $p$ is the $n+1$ 'th standard parameter of $\bar{M})$. In particular, this gives us a $\Sigma_{1}^{(n)}$ definable partial map $f_{0}: \rho_{n+1}^{\bar{M}} \rightarrow \rho_{n}^{\bar{M}}$, with parameter $p$, whose range is cofinal below $\rho_{n}^{M_{0}}$. By preservation of $\Sigma_{1}^{(n)}$ formulae (or just by the fact that $\pi_{0, i}$ is cofinal) we know that for all $i \leqslant \theta$, the same formula (with parameter $\pi_{0, i}(\tau(p))$ ) defines a partial map $f_{i}: \rho_{n+1}^{M_{i}} \rightarrow \rho_{n}^{M_{i}}$ whose range is cofinal below $\rho_{n}^{M_{i}}$. But $\pi_{0, i}\left(\tau\left(\rho_{n+1}^{\bar{M}}\right)\right)=\rho_{n+1}^{\bar{M}}=\rho_{n+1}^{M_{i}}$. So the domains of $f_{0}$ and $f_{i}$ are the same, and hence the order types of their ranges are the same.

Let $\mu$ denote the fixed order type of this sequence. Note that $\operatorname{Cof}^{V}(\mu)=\operatorname{Cof}^{V}\left(\rho_{n}^{M_{0}}\right)$.
Now, we prove the result by induction on the length $\theta+1$ of $\mathcal{I}$. Assume it holds for all shorter iterations, and in particular the lemma holds with $M_{i}$ in place of $M_{\theta}$ for any $i<\theta$.

Fix some $\kappa \in\left(\rho_{n+1}^{M_{0}}, \rho_{n}^{M_{\theta}}\right)$ strongly inaccessible in $M_{\theta}$, and large enough that the fourth condition fails. Let $\lambda_{\theta}=\sup \left\{\kappa_{i}: i<\theta\right\}$. We will divide into three cases. Case $1: \kappa<\lambda_{\theta}$. Then for some $i<\theta$, we have $\kappa<\kappa_{i}$. In particular (since $\kappa_{i}<\rho_{n}^{M_{i}}$ for all $i$ ) this means that $\kappa<\rho_{n}^{M_{i}}$. Since $\pi_{i, \theta}(\kappa)=\kappa$ we know that $\kappa$ is strongly inaccessible in $M_{i}$. So by inductive hypothesis, we know that either $\kappa$ is measurable in $M_{i}$ (and hence also in $M_{\theta}$ ) or it was a critical point $\kappa_{j}$ for some $j<i$, or its $V$ cofinality is $\mu$. So we're done.

Case 2: $\kappa>\lambda_{\theta}$. We'll start by introducing some more notation. Let $p$ be the $n$ 'th standard parameter of $\tilde{M}$, let $p_{n}=p \backslash \rho_{n}^{\tilde{M}}$, and let $q_{n}=p \cap \rho_{n}^{\tilde{M}} \backslash \rho_{n+1}^{M_{0}}$. Let $\tilde{\pi}: \tilde{M} \rightarrow M_{0}$ be the ultrapower map given by $\mathcal{J}$. Let
$p_{n}^{0}=\tilde{\pi}\left(p_{n}\right)$ and $q_{n}^{0}=\tilde{\pi}\left(q_{n}\right)$, and let $p_{n}^{\theta}=\pi_{0, \theta}\left(p_{n}^{0}\right)$ and $q_{n}^{\theta}=\pi_{0, \theta}\left(q_{n}^{0}\right)$. Finally, let $K$ be the set of critical points of $\mathcal{J}$.

We prove the following claim, which will also be useful in proving Case 3 .
Claim 4.3.7. Suppose that $\kappa^{\prime}>\lambda_{\theta}$ is any regular cardinal of $M_{\theta}$ below $\rho_{n}^{M_{\theta}}$, but above the supremum of the critical points of $\mathcal{J}$. Then there is a (fixed) $\Sigma_{1}$ formula, with parameters $\theta, \kappa^{\prime},\left\{\kappa_{j}: j<\theta\right\}, \rho_{n+1}^{M_{\theta}}=\rho_{n+1}^{M_{0}}, K$, $q_{n}^{\theta}$, and the parameter from the previous claim, which defines in $M_{\theta}^{n, p_{n}^{\theta}}$ a sequence $\left(\beta_{k}^{\theta}\right)_{k<\mu}$ which is cofinal below $\kappa^{\prime}$.

Proof. Let $\left(\alpha_{i}\right)_{i<\mu}$ be the cofinal $\mu$ sequence below $\rho_{n}^{M_{\theta}}$ given by the previous claim. Let us write $\bar{M}$ to denote $M_{\theta}^{n, p_{n}^{\theta}}$.

We know $\tilde{M}$ is sound (since it's the core of $M_{0}$ ) so $p \in P_{n}^{\tilde{M}}=R_{n}^{\tilde{M}}$. Hence, $(\tilde{M})^{n, p_{n}}$ is the $\Sigma_{1}$ Skolem hull of $\rho_{n+1}^{\tilde{M}} \cup\left\{q_{n}\right\}$. By Theorem 3.3.3, applied to the iteration that combines $\mathcal{I}$ and $\mathcal{J}$, any element of $\bar{M}$ can be expressed in terms of a function $f \in M^{n, p_{n}}$ and a tuple of elements of $\left\{\kappa_{j}: j<\theta\right\} \cup K$. So $\bar{M}$ is the $\Sigma_{1}$ Skolem hull of

$$
S:=\left\{\kappa_{j}: j<\theta\right\} \cup K \cup \tilde{\pi}\left(\rho_{n+1}^{\tilde{M}}\right) \cup\left\{q_{n}\right\}
$$

in $\bar{M}$ itself.
For $i<\mu$, let $\beta_{i}:=\sup \kappa^{\prime} \cap h^{\bar{M} 1 \alpha_{i}}(S)$. Now, $\bar{M} 1 \alpha_{i}$ and $h^{\bar{M} 1 \alpha_{i}}(S)$ are elements of $\bar{M}$, and $S$ has cardinality less than $\kappa^{\prime}$. And $\kappa^{\prime}$ is regular from the perspective of $M_{\theta}$ and hence the perspective of $\bar{M}$. So we know that $\beta_{i}<\kappa^{\prime}$. On the other hand,

$$
\begin{aligned}
\sup \beta_{i} & =\sup _{i<\mu} \sup \kappa^{\prime} \cap h^{\bar{M} 1 \alpha_{i}}(S) \\
& =\sup \kappa \cap \bigcup_{i<\mu} h^{\bar{M} 1 \alpha_{i}}(S) \\
& =\sup \kappa \cap h^{\cup \bar{M} 1 \alpha_{i}}(S) \\
& =\sup \kappa \cap h^{\bar{M}}(S) \\
& =\sup \kappa \cap \bar{M} \\
& =\kappa
\end{aligned}
$$

The claim immediately finishes case 2 by showing $\operatorname{Cof}^{V}(\kappa)=\operatorname{Cof}^{V}(\mu)$.
Case 3: $\kappa=\lambda_{\theta}$. If $\theta$ is a successor ordinal $i+1$, then $\lambda_{\theta}=\kappa_{i}$, and so we're done immediately. Suppose that $\theta$ is a limit ordinal. For all large enough $j<\theta$, we can find $\sigma_{j} \in M_{j}$ such that $\pi_{j, \theta}\left(\sigma_{j}\right)=\kappa$. By elementarity, we know that $M_{j} \vDash$ " $\sigma_{j}$ is inaccessible". Also, since $\pi_{j, \theta}\left(\sigma_{j}\right)=\kappa=\lambda_{\theta}>\kappa_{j}$ we know that $\sigma_{j} \geqslant \kappa_{j}$. If $\sigma_{j}=\kappa_{j}$ then $\pi_{j, \theta}\left(\kappa_{j}\right)$ is measurable in $M_{\theta}$ and we're done. So suppose that $\sigma_{j}>\kappa_{j}$ for all $j$. In particular, this means that $\sigma_{j}>\sup \left\{\kappa_{i}: i<j\right\}$. By Proposition 3.3.2, we know that $\sigma_{j}<\rho_{n}^{M_{j}}$ as well. And a tail of the $\sigma_{j}$ are above all the critical points of $\mathcal{J}$. So we can apply the claim from Case 2 with $j$ in place of $\theta$, and $\sigma_{j}$ in place of $\kappa$. This gives us a (canonically defined) sequence $\left(\beta_{i}^{j}\right)_{i<\mu}$ of ordinals, cofinal below $\sigma_{j}$.

Since every sequence $\left(\beta_{i}^{j}\right)(j<\theta)$ is defined using the same $\Sigma_{1}^{(n)}$ formula, elementarity tells us that for $j_{1}<j_{2}<\theta$ and $i<\mu$,

$$
\pi_{j_{1}, j_{2}}\left(\beta_{i}^{j_{1}}\right)=\beta_{i}^{j_{2}}
$$

All the sequences are also the same length $\mu$. So if for $i<\mu$ we define, for some suitably large $j<\theta$,

$$
\beta_{i}:=\pi_{j, \theta}\left(\beta_{i}^{j}\right)
$$

then the value of $\beta_{i}$ does not depend on our choice of $j$, and by elementarity the sequence $\left(\beta_{i}\right)_{i<\mu}$ is cofinal below $\pi_{j, \theta}\left(\sigma_{j}\right)=\kappa$.

Corollary 4.3.8. If $M_{0}$ is a mouse, and $\rho_{\omega}^{M_{0}}$ is below the smallest measurable of $M_{0}$, then $M_{0}$ is tidy.

Proof. Let $\tilde{M}$ be the core of $M_{0}$, and let $\mathcal{J}$ be the iteration from $\tilde{M}$ to $M_{0}$. Let $\lambda_{0}$ be the supremum of its critical points.

Let $\mathcal{I}$ be an iteration of $M_{0}$ of length $\mathrm{On}+1$. We know that $\mathcal{I}$ contains only finitely many cut-downs (by definition of an iteration). After the final cut-down, there can only be finitely many drops in degree (since each one indicates a move from $\rho_{n}$ to $\rho_{m}$ for some $m<n$, and there's no way to reverse this move).

Let $i$ be beyond all the cut-downs and drops in degree, and let $n<\omega$ be such that $\rho_{n+1}^{M_{i}} \leqslant \kappa_{i}<\rho_{n}^{M_{i}}$. Let $\lambda=\max \left(\kappa_{i}, \lambda_{0}\right)+1$ and let $\mu=\operatorname{Cof}^{V}\left(\rho_{n}^{M_{i}}\right)$.

Applying the definition of tidiness, let $\kappa>\lambda$ be inaccessible in $M_{\text {On }}$. Let On $>\theta>i$ be large enough that $\kappa_{\theta}>\kappa$, and therefore $\pi_{\theta, \mathrm{On}}(\kappa)=\kappa$. By elementarity, $\kappa$ is inaccessible in $M_{\theta}$. Apply the previous lemma to the iteration $M_{i} \rightarrow M_{\theta}$. This iteration is simple and has no drops. We know that

$$
\rho_{n+1}^{M_{\theta}}=\rho_{n+1}^{M_{i}}<\lambda<\kappa<\kappa_{\theta}<\rho_{n}^{M_{\theta}}
$$

So the lemma tells us that one of the following holds:

1. $\kappa$ is measurable in $M_{\theta}$
2. $\kappa=\kappa_{j}$ for some $j<\theta$
3. $\operatorname{Cof}^{V}(\kappa)=\operatorname{Cof}^{V}\left(\rho_{n}^{M_{i}}\right)=: \mu$

This is exactly what we need to show that $\mathcal{I}$ is tidy. Since $\mathcal{I}$ was arbitrary, $M_{0}$ is tidy.

### 4.3.2 Finding Friendly Machetes

We are now ready to state and prove the final result of this chapter.
Theorem 4.3.9. Suppose that On is Mahlo; i.e., that $\mathrm{Reg}^{s}$ is stationary. Let $\alpha$ be the least measurable in the core model of $L\left[\mathrm{Reg}^{s}\right] .{ }^{6}$ Let $O^{M \varphi, \alpha_{0}, \ldots, \alpha_{n}}$ be a hereditarily $\alpha$ friendly, hereditarily tidy machete mouse, which is friendly with respect to $\varphi$ and $\alpha_{0}, \ldots, \alpha_{n}$. Then $\mathrm{O}^{\mathrm{M} \varphi} \in L\left[\mathrm{Reg}^{s}\right]$.

Proof. Suppose this is false for a given $\mathrm{O}^{\mathrm{M} \varphi}$. Let $K$ be the core of $V$, and let $K^{\prime}$ be the core of $L\left[\mathrm{Reg}^{s}\right]$. By Theorem 3.3.18 we know that $K$ is a universal weasel, so $\mathrm{O}^{\mathrm{M} \varphi}<{ }^{*} K$.
Claim 4.3.10. $K^{\prime}<{ }^{*} \mathrm{O}^{\mathrm{M} \varphi}$
Proof. Of course $\mathrm{O}^{\mathrm{M} \varphi} \not \neq *_{*} K^{\prime}$ as $\mathrm{O}^{\mathrm{M} \varphi}$ is a mouse and $K^{\prime}$ is a weasel. Suppose $\mathrm{O}^{\mathrm{M} \varphi}<^{*} K^{\prime}$. We will show that this implies $\mathrm{O}^{\mathrm{M} \varphi} \in K^{\prime}$, a contradiction. We immediately know that some (not necessarily simple) iterate $N_{\theta}$ of $K^{\prime}$ contains some simple iterate $M_{\theta}$ of $\mathrm{O}^{\mathrm{M} \varphi}$ as an initial segment (not necessarily proper).

Let $\pi: \mathrm{O}^{\mathrm{M} \varphi} \rightarrow M_{\theta}$ and $\tau: K^{\prime} \rightarrow N_{\theta}$ be the corresponding embeddings. Recall from Proposition 4.0.6 that $\rho_{1}^{\mathrm{O}^{\mathrm{M} \varphi}}$ is at most the largest parameter of $\varphi$, and so by definition of a machete, it is below the least measurable of $\mathrm{O}^{\mathrm{M} \varphi}$.

Let $p$ be the first standard parameter of $\mathrm{O}^{\mathrm{M} \varphi}$. Then $A^{p} \subset \omega \times \rho_{1}^{\mathrm{O}^{\mathrm{M} \varphi}}$, so $\pi$ sends $A^{p}$ to itself. In particular then, $\left(A^{p}\right)^{\mathrm{O}^{\mathrm{M} \varphi}} \in M_{\theta} \subset N_{\theta}$. Since, again, $\rho_{1}^{\mathrm{O}^{\mathrm{M} \varphi}}$ is at most the largest parameter of the $\varphi$, by $\alpha$ friendliness it follows that $\rho_{1}^{\mathrm{O}^{\mathrm{M} \varphi}}<\alpha$, the least measurable of $K^{\prime}$. So $\tau$ acts as the identity on subsets of $\omega \times \rho_{1}^{\mathrm{O}^{\mathrm{M} \varphi}}$, and hence $A^{p} \in K^{\prime}$.

But we can now use $A^{p}$ to define, in $K^{\prime}$, a structure that is isomorphic to $M_{0}$. It will be well-founded, and its transitive collapse, which is $M_{0}$ itself, will also be an element of $K^{\prime}$. So $\mathrm{O}^{\mathrm{M} \varphi} \in K^{\prime}$, which contradicts our assumption that $\mathrm{O}^{\mathrm{M} \varphi} \notin L\left[\mathrm{Reg}^{s}\right]$.

So $K^{\prime}<^{*} \mathrm{O}^{\mathrm{M} \varphi}<^{*} K$. Now (abandoning the notation in the preceding claim), for the rest of this proof, let $M_{0}=K^{\prime}$, let $N_{0}=K$, and let $(\mathcal{I}, \mathcal{J})$ be the coiteration of $M_{0}$ and $N_{0}$. As usual, let us say that $\mathcal{I}=\left\langle M_{i}, \pi_{i, j}\right\rangle_{i \leqslant j \leqslant \text { On }}$ and $\mathcal{J}=\left\langle N_{i}, \tau_{i, j}\right\rangle_{i \leqslant j \leqslant \text { On }}$. Since $M_{0}<^{*} N_{0}$ and $N_{0}$ is a weasel, we know that $\mathcal{J}$ is class-length. $\mathcal{I}$ is also class length (and simple) but doesn't iterate any measurables up to On: every measurable gets left behind after a set-long iteration. (Recall that by definition this will always be true for the $<^{*}$ side of the coiteration.) We also know that $M_{\mathrm{On}}=N_{\mathrm{On}}$ : since $M_{\mathrm{On}}$ is class sized, it can't be a proper

[^21]initial segment of $N_{\text {On }}$. (Recall that $N_{\text {On }}$ isn't the direct limit model of the system $\left\langle N_{i}\right\rangle$, it's the cut-down of that direct limit model of On.)

It's possible that the $N$ side of the coiteration pushes some ordinals all the way up above On, where they disappear in the jump to $N_{\text {On }}$. (In fact, we know that it pushes at least one measurable up onto On: see below.) These ordinals play no actual role in the proof, and make certain concepts we need rather fiddly to express. So to tidy things up, we'll start by rearranging the $N$ side of the coiteration to get rid of them by doing a suitable cut-down earlier on.

Claim 4.3.11. There is some $i$ and some ordinal $\nu \in N_{i}$ such that the class-length iteration $\overline{\mathcal{J}}=\left\langle\bar{N}_{j}, \bar{\tau}_{j_{0}, j_{1}}\right\rangle$ defined by

$$
\bar{N}_{j}= \begin{cases}N_{j} & j \leqslant i \\ N_{j} \upharpoonleft \bar{\tau}_{i, j}(\nu) & j>i\end{cases}
$$

is well defined and satisfies $\bar{N}_{\mathrm{On}}=N_{\mathrm{On}}$. Moreover, we can choose $i$ and $\nu$ such that for all $j>i$, all the cardinals of $\bar{N}_{j}$ other than its top one are in the domain of $\bar{\tau}_{j, \mathrm{On}}$.

Proof. We know the direct limit model $N=\lim \left\langle N_{j}\right\rangle_{j<\text { On }}$ contains an "ordinal" of order type On. For any large enough $i$, we can find some $\nu \in N_{i}$ which is sent to that "ordinal" of $N$. It is easy to see that $i$ and $\nu$ then have the required properties.

Fix $i, \nu$ and $\overline{\mathcal{J}}$ as given in the previous claim. Note that $\overline{\mathcal{J}}$ has no cut-downs after stage $i$ (because all the cardinals of $\bar{N}_{j}, j>i$ are in the domain of $\left.\pi_{j, \text { On }}\right)$.
Claim 4.3.12. $\bar{N}_{i+1} \leqslant{ }^{*} \mathrm{O}^{\mathrm{M} \varphi}$
Proof. Recall that $\bar{N}_{\text {On }}<^{*} \bar{N}_{i+1}$, and that $\bar{N}_{i+1}$ is an active mouse.
Suppose first $\mathrm{O}^{\mathrm{M} \varphi}<^{*} \bar{N}_{\text {On }}$. Then since $\bar{N}_{\text {On }}=N_{\text {On }}=M_{\text {On }}$ is a set-long simple iterate of $K^{\prime}$, we know $\mathrm{O}^{\mathrm{M} \varphi}<{ }^{*} K^{\prime}$. But we saw earlier that $K^{\prime}<{ }^{*} \mathrm{O}^{\mathrm{M} \varphi}$.

Next, suppose that $\bar{N}_{\mathrm{On}}<^{*} \mathrm{O}^{\mathrm{M} \varphi}<^{*} \bar{N}_{i+1}$. (Obviously $\bar{N}_{\mathrm{On}} \not \neq^{*} \mathrm{O}^{\mathrm{M} \varphi}$ as the former is a weasel and the latter a mouse.) Coiterate $\mathrm{O}^{\mathrm{M} \varphi}$ and $\bar{N}_{i+1}$. The $\mathrm{O}^{\mathrm{M} \varphi}$ side of the coiteration is simple, so its result $\tilde{M}$ satisfies $\tilde{M}>^{*} \bar{N}_{\text {On }}$. Coiterating $\tilde{M}$ with $\bar{N}_{\text {On }}$, we get a class-long iteration of $\tilde{M}$ whose final model is some simple set-long iterate of $\bar{N}_{\text {On }}$. But $\tilde{M}$ itself is an initial segment of some iterate of $\bar{N}_{i+1}$, and either the initial segment is proper or the iteration wasn't simple. So this gives us a iteration of $\bar{N}_{i}$ which involves at least one cut-down, and produces a simple set-long iterate of $\bar{N}_{\text {On }}$. This is, by definition, one half of a coiteration of $\bar{N}_{i+1}$ with $\bar{N}_{\text {On }}$. But the iteration $\bar{N}_{i+1} \rightarrow \bar{N}_{\text {On }}$ is also (the nontrivial half of) a coiteration of these two objects, and it is simple. So there are two different coiterations of $\bar{N}_{i+1}$ with $\bar{N}_{\text {On }}$, contradicting uniqueness of coiterations.

The only remaining possibility is that $\bar{N}_{i+1} \leqslant{ }^{*} \mathrm{O}^{\mathrm{M} \varphi}$ as claimed.
If $\bar{N}_{i+1}<^{*} \mathrm{O}^{\mathrm{M} \varphi}$ then (since $\bar{N}_{i+1}$ has a top measure $\nu^{*}:=\tau_{i, i+1}(\nu)$ and $\mathrm{O}^{\mathrm{M} \varphi}$ is hereditarily $\alpha$ friendly) we know that $\bar{N}_{i+1}$ is $\alpha$ friendly. So there exists an upwards-preserving formula $\psi\left(v_{0}, \ldots, v_{m+1}\right)$ and ordinals $\alpha_{0}^{\prime}, \ldots, \alpha_{m}^{\prime}$ below $\alpha$ and below all the measurables of $\bar{N}_{i+1}$ such that

$$
H_{\left(\nu^{*}\right)^{+}}^{\bar{N}_{i+1}} \vDash \psi\left(\alpha_{0}^{\prime}, \ldots, \alpha_{m}^{\prime}, \nu^{*}\right)
$$

but such that $\bar{N}_{i+1}$ contains only boundedly many smaller measurables $\lambda<\nu^{*}$ such that

$$
H_{\lambda^{+}}^{\bar{N}_{i+1}} \models \psi\left(\alpha_{0}^{\prime}, \ldots, \alpha_{m}^{\prime}, \lambda\right)
$$

Trivially, this is also true if $\bar{N}_{i+1}=^{*} \mathrm{O}^{\mathrm{M} \varphi}$ : we can take $\psi:=\varphi$ and $\alpha_{k}^{\prime}:=\alpha_{k}$ for all $k \leqslant n$. Indeed, in this case there are no $\lambda<\nu^{*}$ at all satisfying the above equation.

In either case, fix some $\psi$ and $\alpha_{0}^{\prime}, \ldots, \alpha_{m}^{\prime}$ as described. By elementarity, $\bar{N}_{\text {On }}=N_{\text {On }}$ contains only boundedly many measurables $\lambda$ such that

$$
H_{\lambda+}^{\bar{N}_{\mathrm{O}_{\mathrm{n}}}} \vDash \psi\left(\alpha_{0}^{\prime}, \ldots, \alpha_{m}^{\prime}, \lambda\right)
$$

This equation is a bit of a mouthful to keep writing all the time, so we'll informally say that " $\lambda$ believes $\psi$ in $M "$ to mean that

$$
H_{\lambda^{+}}^{M} \vDash \psi\left(\alpha_{0}^{\prime}, \ldots, \alpha_{m}^{\prime}, \lambda\right)
$$

Let us now fix some $j \in$ On which is large enough that $\mathcal{I}$ and $\mathcal{J}$ have already done everything interesting by stage $j$ :

1. $j>i$;
2. $j \in \mathrm{On} \backslash \bar{N}_{i+1}$;
3. $j$ is larger than the $\lambda$ given in Definition 4.3.4 for the iteration $\bar{N}_{i+1} \rightarrow \bar{N}_{\text {On }}=N_{\text {On }}$ (which holds for $\bar{N}_{i+1}$ since $\mathrm{O}^{\mathrm{M} \varphi}$ is hereditarily tidy);
4. There is a $K^{\prime}$ admissible ordinal in the interval $(i, j)$, and $j$ is more than the cardinality of its powerset; and
5. $\bar{N}_{\text {On }}=N_{\text {On }}=M_{\text {On }}$ contains no measurables $\lambda>j$ such that $\lambda$ believes $\psi$ in $M_{\text {On }}$

Let us also choose $j$ to be strongly inaccessible in $V$.
The final item means (by elementarity and the fact that $\pi(j) \geqslant j$ ) that $M_{0}=K^{\prime}$ also contains no measurables $\lambda>j$ such that

$$
H_{\lambda^{+}}^{\bar{N}_{\mathrm{On}^{\prime}}} \vDash \psi\left(\alpha_{0}^{\prime}, \ldots, \alpha_{m}^{\prime}, \lambda\right)
$$

Our goal for this proof is to derive a contradiction, by finding such a measurable of $K^{\prime}$.
From Definition 4.3.4 and condition 3 above, we know that there is some $\mu \in \operatorname{Reg}{ }^{V}$ such that if $\kappa>j$ is strongly inaccessible in $N_{\text {On }}$, then one of the following holds.

1. $\kappa$ is measurable in $N_{\text {On }}$
2. $\kappa=\kappa_{h}$ for some $i<h<$ On
3. $\operatorname{Cof}^{V}(\kappa)=\mu$

We shall now use this to show that $K^{\prime}$ can almost calculate the critical points of the iteration $\bar{N}_{j} \rightarrow N_{\text {On }}$ at which we iterate the top measure. But only "almost": the test we use misses some of the critical points, and in fact, it may miss quite a lot of them.
Definition 4.3.13. Let us call an ordinal $\beta$ useful if:

1. $\beta>\kappa_{j}($ and hence $\beta>j)$;
2. $\psi\left(\alpha_{0}^{\prime}, \ldots, \alpha_{m}^{\prime}, \beta\right)$ holds in $H_{\beta^{+}}^{K^{\prime}}$;
3. $\beta \in \operatorname{Reg}^{s}$

Claim 4.3.14. Let $\beta \in \mathrm{On}$ be useful. Then $\beta$ is a fixed point of the iteration $M_{0} \rightarrow M_{\mathrm{On}}$, and a nontrivial critical point $\kappa_{h}$ of the iteration $\bar{N}_{j} \rightarrow N_{\text {On }}$, for some $j<h<$ On. Moreover, $\beta$ is the top measurable of $\bar{N}_{h}$.

Proof. Let $k$ be least such that $\kappa_{k} \geqslant \beta$. Since $\beta$ really is strongly inaccessible in $V$, we know it's a fixed point of $\pi_{0, k}$ by Lemma 3.2.42. Also, since $\beta>j$ and believes $\psi$ in $M_{0}=K^{\prime}$, we know that $\beta$ can't be measurable in $M_{0}$, and hence can't be measurable in $M_{k}$ either. So either $\kappa_{k}>\beta$, or $\kappa_{k}=\beta$ is trivial on the $M$ side of the coiteration; and hence $\beta$ is a fixed point of $\pi_{0, \mathrm{On}}$.

So $\beta$ believes $\psi$ in $M_{\mathrm{On}}=N_{\mathrm{On}}$. Now $\beta$ is also strongly inaccessible in $N_{\mathrm{On}}($ and indeed in $V)$ so one of the following holds:

1. $\beta$ is measurable in $N_{\mathrm{On}}$;
2. $\beta=\kappa_{h}$ for some nontrivial $i<h<$ On; or
3. $\operatorname{Cof}^{V}(\beta)=\mu$

We know that $\beta$ isn't measurable in $N_{\text {On }}$ since it wasn't measurable in $M_{0}$. We also know that $\beta \in \operatorname{Reg}^{s}$, so $\operatorname{Cof}^{V}(\beta)=\beta>\mu$. So by process of elimination, $\beta$ is a critical point of the iteration $\bar{N}_{i+1} \rightarrow N_{\text {On }}$, say $\beta=\kappa_{h}$. It remains to show that $\beta=\kappa_{h}$ is the top measure of $\bar{N}_{h}$. But since $\beta$ is the critical point $\kappa_{h}$, we know

$$
H_{\beta^{+}}^{\bar{N}_{h}}=H_{\beta^{+}}^{N_{\mathrm{On}}} \models \psi\left(\alpha_{0}^{\prime}, \ldots, \alpha_{m}^{\prime}, \beta\right)
$$

By elementarity between $N_{\text {On }}$ and $\bar{N}_{h}$, we know that $\bar{N}_{h}$ doesn't contain any measurables $\lambda>j$ which believe $\psi$ in $\bar{N}_{h}$, other than its top measurable (which is sent up to On). So $\beta$ must be the top measurable of $\bar{N}_{h}$.

Of course, not every such critical point will be useful: there will be lots of times we iterated the top measure but weren't on a strong inaccessible of $V$, for example. But we can still say that a lot of them will be useful.
Claim 4.3.15. There are unboundedly many useful ordinals.
Proof. In the iteration $\bar{N}_{i+1} \rightarrow N_{\text {On }}$ we iterate the top measure unboundedly often. Hence, in $N_{\text {On }}=M_{\text {On }}$ there are unboundedly many ordinals $\lambda$ which believe $\psi$. (Recall that friendliness merely tells us that there are only boundedly many measurables which believe $\psi$.)

By elementarity, there are also unboundedly many ordinals $\lambda$ which believe $\psi$ in $M_{0}=K^{\prime}$. Since $\psi$ is upwards reflecting there are actually a club of these ordinals. But we're assuming Reg ${ }^{s}$ is stationary in $V$, so there are unboundedly many ordinals $\lambda \in \operatorname{Reg}^{s}$ which believe $\psi$ in $K^{\prime}$. Hence there are unboundedly many useful ordinals.
" $\beta$ is a useful ordinal" can be expressed as a statement about $K^{\prime}, \beta$ and $\operatorname{Reg}^{s}$, so $L\left[\mathrm{Reg}^{s}\right]$ can identify the class of all useful ordinals. For $\gamma \in$ On let $\beta_{\gamma}$ be the $\gamma$ 'th useful ordinal. Let $C$ be the club of all limits of useful ordinals, and (again exploiting the fact that $\operatorname{Reg}^{s}$ is a club) let $\beta \in C \cap \operatorname{Reg}^{s}$. Then (since $\beta$ is inaccessible) we know $\beta=\sup _{\gamma<\beta} \beta_{\gamma}$. Since $\psi$ is upwards reflecting and for all $\gamma<\beta$, usefulness tells us that $\beta_{\gamma}$ believes $\psi$ in $K^{\prime}$, it follows immediately that $\beta$ also believes $\psi$ in $K^{\prime}$. Hence, $\beta$ is itself useful. (We could, rather repetitively, write $\beta=\beta_{\beta}$.) Also, we'll have our desired contradiction if we can show that $\beta$ is measurable in $K^{\prime}$.

Now, by the claim before last, we know that $\beta$ and each $\beta_{\gamma}$ is a fixed point of $\pi_{0, \mathrm{On}}$ :

$$
\begin{gathered}
\pi_{0, \mathrm{On}}\left(\beta_{\gamma}\right)=\beta_{\gamma} \\
\pi_{0, \mathrm{On}}(\beta)=\beta
\end{gathered}
$$

The same claim also tells us that for all $\gamma, \beta_{\gamma}$ is a nontrivial critical point $\kappa_{h_{\gamma}}$ of the iteration $\bar{N}_{j} \rightarrow N_{\text {On }}$, and moreover that $\kappa_{h_{\gamma}}$ is the top measure of $\bar{N}_{h_{\gamma}}$. So in particular, since the iteration $\bar{N}_{i+1} \rightarrow N_{\text {On }}$ is simple, we know that for $\delta<\gamma<\beta$,

$$
\bar{\tau}_{h_{\delta}, h_{\gamma}}\left(\beta_{\delta}\right)=\beta_{\gamma}
$$

So if $h:=\sup \left\{h_{\gamma}: \gamma<\omega_{1}\right\}$ then $\bar{\tau}_{h_{\gamma}, h}\left(\beta_{\gamma}\right)=\beta$ is the top measure of $\bar{N}_{h}$. Since stage $h$ is a limit of stages where we iterated the top measure, the critical point $\kappa_{h}$ must be the top measure of $\bar{N}_{h}$; ie. $\beta=\kappa_{h} .{ }^{7}$

Claim 4.3.16. A subset $X \in \bar{N}_{h}$ of $\beta$ is measure 1 if and only if contains a tail of the sequence $\left(\beta_{\gamma}\right)_{\gamma<\beta}$. Hence, $K^{\prime}$ can identify whether a subset of $\beta$ in $\bar{N}_{h} \cap K^{\prime}$ is measure 1 or not.

8

[^22]Proof. Suppose that $X$ is measure 1. Let $f \in \bar{N}_{j}$ and $i_{0}<\ldots<i_{n}<h$ be such that $X=\bar{\tau}_{j, h}(f)\left(\kappa_{i_{0}}, \ldots, \kappa_{i_{n}}\right)$. Let $\gamma<\beta$ be large enough that $h_{\gamma}>i_{n}$, so $\beta_{\gamma}>\kappa_{i_{n}}$. Now we know $\bar{\tau}_{h_{\gamma}, h}\left(\beta_{\gamma}\right)=\beta$ so by elementarity, $X_{\gamma}:=\bar{\tau}_{j, h_{\gamma}+1}(f)\left(\kappa_{i_{0}}, \ldots, \kappa_{i_{n}}\right)$ is a measure 1 subset of $\bar{\tau}_{h_{\gamma}, h_{\gamma}+1}\left(\beta_{\gamma}\right)$. So by Proposition 3.1.11, $\beta_{\gamma} \in X_{\gamma}$. But then

$$
\beta_{\gamma}=\tau_{h_{\gamma}+1, h}\left(\beta_{\gamma}\right) \in \tau_{h_{\gamma}+1, h}\left(X_{\gamma}\right)=X
$$

So $X$ contains $\beta_{\gamma}$ for all sufficiently large $\gamma<\beta$.
On the other hand, if $X \subset \beta$ is not measure 1 , then $\beta \backslash X$ is measure 1 instead. So $\beta \backslash X$ contains a tail of the $\beta_{\gamma}$, and therefore $X$ does not contain such a tail.

Since $\beta$ has the same subsets in $\bar{N}_{h}$ and $N_{\text {On }}=M_{\text {On }}$ we know that the normal measure $F_{h}$ on $\beta$ in $\bar{N}_{h}$ is also a normal measure on $\beta$ in $M_{\text {On }}$. (Of course, $\beta$ isn't actually measurable in $M_{\text {On }}$, but that's because it doesn't know about $F_{h}$, not because something has changed about the subsets of $\beta$.)

So the pull-back $U$ of $F_{h}$ to $M_{0}$, defined by

$$
X \in U \Longleftrightarrow X \in M_{0} \wedge X \subset \beta \wedge \pi_{0, \mathrm{On}}(X) \in F_{h}
$$

is a normal measure on $\beta=\pi_{0, \text { On }}^{-1}(\beta)$ over $M_{0}=K^{\prime}$. So we've found our normal measure. But we need to show that $U$ is in the extender sequence of $K^{\prime}$.

By the preceding claims, we know for $X \in K^{\prime}$ that

$$
\begin{aligned}
X \in U & \Longleftrightarrow X \subset \beta \wedge \pi_{0, \mathrm{On}}(X) \in F_{h} \\
& \Longleftrightarrow X \subset \beta \wedge \pi_{0, \mathrm{On}}(X) \text { contains a tail of }\left(\beta_{\gamma}\right)_{\gamma<\beta} \\
& \Longleftrightarrow X \subset \beta \wedge \exists \alpha<\beta \forall \gamma \in(\alpha, \beta) \beta_{\gamma} \in \pi_{0, \mathrm{On}}(X) \\
& \Longleftrightarrow X \subset \beta \wedge \exists \alpha<\beta \forall \gamma \in(\alpha, \beta) \pi_{0, \mathrm{On}}\left(\beta_{\gamma}\right) \in \pi_{0, \mathrm{On}}(X) \\
& \Longleftrightarrow X \subset \beta \wedge \exists \alpha<\beta \forall \gamma \in(\alpha, \beta) \beta_{\gamma} \in X
\end{aligned}
$$

This final line can be expressed within $L\left[\mathrm{Reg}^{s}\right]$, so $U \in L\left[\mathrm{Reg}^{s}\right]$.
Now, note that within $L\left[\operatorname{Reg}^{s}\right]$ (or indeed $\left.V\right) \beta$ is regular, so $U$ is $\omega$ complete. Hence, by Lemma 3.3.17 we know $U$ will appear on the extender sequence of $K^{\prime}$.

So $\beta>j$ is now a measurable of $K^{\prime}$ such that

$$
H_{\beta^{+}}^{K^{\prime}} \vDash \psi\left(\alpha_{0}^{\prime}, \ldots, \alpha_{m}^{\prime}\right)
$$

But there were supposed to be no such $\beta>j$. Contradiction!

### 4.4 Open Questions

In this final section, we suggest some extensions of these results, which are beyond the scope of this thesis but would be interesting to prove.

Perhaps the most interesting of these relates to Theorem 4.0.3.
Question 12. Assuming the existence of complex enough mice, can we prove that $L[\mathrm{Reg}]$ is a generic extension of an iterate of a certain mouse?

The natural mouse to start from would probably be O-Sword. This is the smallest mouse which has two measures on its largest cardinal. (This means that it doesn't fit into the scheme of mice that we've defined in this thesis.) It is also the smallest mouse which has measure 1 many measurables below that largest cardinal. This means that any iterate of O-Sword will have lots of measurables of all Cantox-Bendixson ranks, so it seems likely that we can line them up onto the $\omega$ limits of subclasses of Reg in the same way we did for $\operatorname{Reg}_{\epsilon}$ in Lemma 4.2.2.

However, there's one important hurdle to overcome with this approach. We need to partition Reg into $\omega$ sequences, in such a way that knowing all the $\omega$ sequences is equivalent to knowing Reg itself. A simple Cantor-Bendixson partition will no longer work: Lemma 4.2 .2 will fail at hyperinaccessibles. (This is one
reason why we have to assume there are only finitely many elements of $R$ below $\epsilon$ in Theorem 4.2.1: it makes sure there are only finitely many " $R$-hyperinaccessibles".) And other natural ways to divide up Reg by, say, quarantining the hyperinaccessibles in new $\omega$ sequences, tend to use information that can't be calculated from Reg alone.

There are also interesting questions to be answered about the more general class of $\mathrm{O}^{\mathrm{M} \varphi}$ mice. Firstly, recall that we only defined $\mathrm{O}^{\mathrm{M} \varphi}$ if the "least $\alpha$ sound" mouse which believes $\varphi$ is actually sound. (We put this in quotes because the concept of a least $\alpha$ sound mouse hasn't really been defined.) This was necessary for the definition to make sense, since otherwise either we define $\mathrm{O}^{\mathrm{M} \varphi}$ to be a mouse which isn't sound (and then $\varphi$ might not be preserved across $=^{*}$ mice) or we define $\mathrm{O}^{\mathrm{M} \varphi}$ to be the least sound mouse believing $\varphi$ and lose minimality.

Question 13. For which $\varphi$ and parameters $\alpha_{0}, \ldots, \alpha_{n}$ is it consistent that $\mathrm{O}^{\mathrm{M} \varphi}$ exists? Of those, for which will $\mathrm{O}^{\mathrm{M} \varphi}$ always exist if there exists any mouse that believes $\varphi$ ?

We saw in Proposition 4.0.2 that (assuming large mice can exist) both of these are true for $\mathrm{O}^{\mathrm{M} \epsilon}$, for any $\epsilon$. So there are definitely nontrivial choices of $\varphi$ and $\alpha_{0}, \ldots, \alpha_{n}$ for which the question is answered positively. On the other hand, the answer is not always yes, even when mice exist that believe $\varphi$ : see Example 1.

Other questions can be asked about the theorem in the previous section, and the axioms leading up to it.
Question 14. Which mice are $\alpha$ friendly? Which mice are tidy?
Again, this is nontrivial. We know that $\mathrm{O}^{\mathrm{M} \epsilon}$ is $\alpha$ friendly for $\epsilon<\alpha$. We know that a mouse is tidy if $\rho_{\omega}$ is below the least measurable. And we know this is true for any actual $\mathrm{O}^{\mathrm{M} \varphi}$ (by Proposition 4.0.6). But can we show that it's true for all mice below $\mathrm{O}^{\mathrm{M} \varphi}$ too?

Finally, it would be interesting to examine the role of the upwards-reflecting property in the proof of Theorem 4.3.9. It's used in only two places, and in both of them it seems plausible that it could be removed.

Question 15. Can we prove Theorem 4.3 .9 if we drop the requirement (in the definition of $\alpha$-friendliness) for the formula to be upwards-reflecting?

The first place the upwards-reflecting property came up when we were showing that there are unboundedly many useful ordinals. We showed that in $M_{\text {On }}$ there are unboundedly many ordinals which believe $\psi$; so there are also unboundedly many in $M_{0}$. Since $\psi$ is upwards-reflecting this means that there's a club of such points, and so there are examples of them which are in $\mathrm{Reg}^{s}$.

But actually, even if $\psi$ were not upwards-reflecting, we know that the class of all ordinals in $M_{\text {On }}$ which believe $\psi$ contains a club: namely, all the critical points of $\mathcal{J}$ where we iterated the top measure. If we could somehow transfer this property down to $M_{0}$ then we'd be home and dry: we could find elements of that club which are in $\mathrm{Reg}^{s}$ and we'd be done. This is nontrivial, however, because the club is not necessarily definable in $M_{\text {On }}$ so we can't just invoke elementarity.

If this can be solved, it is likely that the only other point we use upwards-reflection - showing that a limit of useful ordinals which is in $\mathrm{Reg}^{s}$ is useful - could also be handled in a similar way, since every useful ordinal is a top measure critical point of $\mathcal{J}$.

## Chapter 5

## Lowenheim-Skolem-Tarski Numbers

In this final chapter, we shall examine a question about logics that sit between first and second order. The Löwenheim-Skolem theorem famously says that for any first-order language $\mathcal{L}$, any first order $\mathcal{L}$ structure contains an elementary substructure of size less than $\max \left(|\mathcal{L}|, \omega_{1}\right)$. The concept of the Löwenheim-SkolemTarski number generalises this to simple second order logics. The LST number of a second-order logic is the smallest cardinal $\kappa$ such that every structure contains a substructure of size less than $\kappa$.

In [29], Magidor and Väänänen investigate LST numbers for two syntactical quantifiers: the Härtig quantifier $I$ and the equal cofinality quantifier $Q^{\text {e.c. }}$. Roughly speaking, we can think of $I$ as telling us about the class Card of all cardinals (or, equivalently, about the class $\operatorname{Reg}_{0}$ of all successor cardinals), while $Q^{\text {e.c. }}$ tells us about the class Reg of all (infinite) regular cardinals. They prove an exact lower bound for both $\operatorname{LST}(I)$ and $\operatorname{LST}\left(I, Q^{\text {e.c. }}\right)$.

We shall introduce two schemes of analogous quantifiers $Q^{\epsilon}$ and $R^{\epsilon}$, which both tell us about $\operatorname{Reg}_{<\epsilon}$. We will then investigate what results about their LST numbers can be proved in ZFC, in the process justifying our choice of those particular quantifiers. Finally, we will derive lower bounds for $\operatorname{LST}\left(I, Q^{\epsilon}\right)$ and $\operatorname{LST}\left(I, R^{\epsilon}\right)$, and - as the main result of this chapter - prove that these lower bounds are exact (assuming the consistency of supercompacts).

### 5.1 Preliminaries

We open with some definitions. The logics we will be looking at are first order, but expanded with some extra quantifier symbols. These "quantifiers" are so-named because they hold the same positions in a formula that the standard quantifiers $\forall$ and $\exists$ do, and both $\forall$ and $\exists$ can be understood as particular syntactic quantifiers.

Definition 5.1.1. A quantifier is a formal symbol $Q$, which is equipped with an arity $n \in \omega$ and a number $m \in \omega$, called the number of variables it quantifies over.

Definition 5.1.2. Let $\mathcal{L}$ be a first order language, and let $Q$ be an $n$-ary quantifier over $m$ variables. The language $\mathcal{L} \cup\{Q\}$ consists of the following formulas:

- Atomic formulas of $\mathcal{L}$
- $\neg \varphi$ and $\varphi \Longrightarrow \psi$, whenever $\varphi$ and $\psi$ are formulas
- $\forall v(\varphi)$ where $v$ is any variable and $\varphi$ is a formula
- $Q v_{1}, \ldots, v_{m}\left(\varphi_{1}, \ldots, \varphi_{n}\right)$ where $v_{1}, \ldots, v_{m}$ are any variables (not necessarily distinct) and $\varphi_{1}, \ldots, \varphi_{n}$ are formulas

If $Q_{1}, \ldots, Q_{k}$ are quantifiers then the language $\mathcal{L} \cup\left\{Q_{1}, \ldots, Q_{n}\right\}$ is defined similarly.
We are specifically interested in quantifiers which (like $\forall$ and $\exists$ ) have a single canonical interpretation.

Definition 5.1.3. A predefined quantifier is an $n$-ary quantifier symbol $Q$ in $m$ variables (for some $n$ and $m$ ), together with a fixed rule of how to interpret $Q v_{1}, \ldots, v_{m}\left(\varphi_{1}, \ldots, \varphi_{n}\right)$, for any second-order formulas $\varphi_{1}, \ldots, \varphi_{n}$ in any language $\mathcal{L}$ including $Q$, over any $\mathcal{L}$ structure $\mathcal{A}$ and assignment of variable symbols to elements of $\mathcal{A}$. Usually, this guideline will be expressed in a recursive manner, with the interpretation of $Q v_{1}, \ldots, v_{m}\left(\varphi_{1}, \ldots, \varphi_{n}\right)$ dependent on the interpretations of $\varphi_{1}, \ldots, \varphi_{n}$.

The details of how we formalise this concept are left to the reader: there are several different options, all of which are fiddly to write down formally. See [7] for one method.

A trivial example might make the idea here clearer.
Example 2. $\forall$ is a predefined 1-ary quantifier, which quantifies over one variable. $\forall v_{1} \varphi$ is true (for a given $\varphi=\varphi\left(v_{1}\right), \mathcal{A}$ and assignment) if and only if for all $x \in \mathcal{A}, \varphi(x)$ is interpreted as true. Similarly, $\exists$ is also a predefined 1-ary quantifier, which again quantifies over one variable.

There are two well-known quantifiers we are interested in, each with this predefined property.
Definition 5.1.4. The Härtig quantifier $I$ is 2 -ary, and quantifies over two variables. $I v_{1}, v_{2}(\varphi, \psi)$ is interpreted as true (in a structure $\mathcal{A}$ over which we can interpret $\varphi$ and $\psi$, and have defined an assignment for variable symbols) if and only if the two sets

$$
X:=\left\{x \in A: \mathcal{A} \vDash \varphi\left[x / v_{1}\right]\right\}
$$

and

$$
Y:=\left\{x \in A: \mathcal{A} \vDash \psi\left[x / v_{2}\right]\right\}
$$

have the same cardinality in $V$.
Definition 5.1.5. The Equal Cofinality quantifier $Q^{\text {e.c. }}$ is 2 -ary, and quantifies over 4 variables. $Q^{\text {e.c. }} v_{1}, \ldots, v_{4}(\varphi, \psi)$ is interpreted as true (over $\mathcal{A}$, etc.) if the two sets

$$
X:=\left\{(x, y): \mathcal{A} \vDash \varphi\left[x / v_{1}, y / v_{2}\right]\right\}
$$

and

$$
Y:=\left\{(x, y): \mathcal{A} \vDash \psi\left[x / v_{3}, y / v_{4}\right]\right\}
$$

are both linear orders, and have the same cofinality.
Intuitively, we can think of $I$ as telling us about Card and $Q^{\text {e.c. telling us about Reg. }}$
Throughout this chapter, we will simplify notation by committing a slight abuse: if the free variables of a formula $Q v_{1}, \ldots, v_{k} \varphi_{1}, \ldots, \varphi_{n}$ are, say, some subset of $v_{m+1}, \ldots, v_{m+k}$ and $y_{1}, \ldots, y_{k}$ are a tuple of elements of some structure $\mathcal{A}$, we will write

$$
\mathcal{A} \vDash Q\left(\varphi_{1}, \ldots, \varphi_{n}, y_{1}, \ldots, y_{k}\right)
$$

to denote the statement
$Q v_{1}, \ldots, v_{m}\left(\varphi_{1}, \ldots, \varphi_{n}\right)$ is interpreted as true over $\mathcal{A}$ with the assignment of $y_{i}$ to $v_{m+i}$ for $0<i \leqslant k$.

We will also use the same notation if instead of $v_{1}, \ldots, v_{m}$ and $v_{m+1}, \ldots, v_{m+k}$, we are quantifying over some (otherwise unused) variables $v_{i_{1}}, \ldots, v_{i_{m}}$ and assign the variables $v_{j_{1}}, \ldots, v_{j_{k}}$.

The final standard concept to define is the LST number.
Definition 5.1.6. Let $Q_{0}, \ldots, Q_{n}$ be predefined quantifier symbols of second order logic. The Löwenheim-Skolem-Tarski number $\operatorname{LST}\left(Q_{0}, \ldots, Q_{n}\right)$ of $Q_{0}, \ldots, Q_{n}$ is the least infinite cardinal $\kappa$ such that the following holds: For every first order language $\mathcal{L}$ whose vocabulary has cardinality less than $\kappa$, for every $\mathcal{L}$ structure $\mathcal{A}$, there is an $\mathcal{L} \cup\left\{Q_{0}, \ldots, Q_{n}\right\}$ elementary substructure $\mathcal{B} \leqslant \mathcal{A}$ of size less than $\kappa$.

If no such $\kappa$ exists, then we say that $\operatorname{LST}\left(Q_{0}, \ldots, Q_{n}\right)=\infty$ or that it does not exist.
[29] appears to define the LST number in a slightly different way: they start by (implicitly) fixing a particular first order language $\mathcal{L}$ and then only discuss $\mathcal{L}$ structures. However, it is easy to check that the results we will be looking at still apply in this general setting. (In fact, the formulation in [29] has a minor error, in that they do not specify a limit on the size of the language $\mathcal{L}$ in their theorems about its LST number. The formulation given here bypasses this issue.)

It was shown in [29] that $\operatorname{LST}(I)$ can be the least inaccessible (but not less than that) and that $\operatorname{LST}\left(I, Q^{\text {e.c. })}\right.$ can be the least Mahlo cardinal (but again, no less):

Theorem 5.1.7 (Magidor,Väänänen). [29, Theorems 7, 20 \& 21] If it exists, then $\operatorname{LST}(I)$ is at least the first inaccessible cardinal. Similarly, $\operatorname{LST}\left(I, Q^{e . c .}\right)$ is at least the first Mahlo cardinal, if it exists. Moreover, if it is consistent that a supercompact cardinal exists then it is also consistent that either one of these LST numbers is exactly equal to the bound given.

### 5.2 The New Quantifiers

As we discussed in the introduction, in this chapter we are looking at LST numbers for quantifiers analogous to $I$ and $Q^{\text {e.c. }}$, which tell us about $\mathrm{Reg}_{<\epsilon}$. It is now time to define the quantifiers we're going to be looking at. The first is similar to $Q^{\text {e.c. }}$. However, in order to restrict it to $\operatorname{Reg}_{<\epsilon}$ in a natural way, we add an extra requirement: as well as requiring that the two linear orders have the same cofinality, we ask for an auxiliary set deciding the Cantor-Bendixson rank of that cofinality.

Definition 5.2.1. Let $\epsilon \in$ On or $\epsilon=$ On. The quantifier $Q^{\epsilon}$ is 3 -ary, and quantifies over six variables. $Q^{\epsilon} v_{1}, \ldots, v_{6}(\varphi, \psi, \chi)$ is interpreted as true (over $\mathcal{A}$, etc.) if the three sets

$$
\begin{aligned}
X & :=\left\{(x, y): \mathcal{A} \vDash \varphi\left[x / v_{1}, y / v_{2}\right]\right\} \\
Y & :=\left\{(x, y): \mathcal{A} \vDash \psi\left[x / v_{3}, y / v_{4}\right]\right\} \\
Z & :=\left\{(x, y): \mathcal{A} \vDash \chi\left[x / v_{5}, y / v_{6}\right]\right\}
\end{aligned}
$$

satisfy the following four conditions.

1. $X$ and $Y$ are both linear orders with the same $V$ cofinality
2. $Z$ is a well order
3. The order type of $Z$ is less than $\epsilon$
4. The equal cofinality of $X$ and $Y$ is an element of $\operatorname{Reg}_{\text {o.t. }}(Z)$

These extra requirements on $Z$ mean that for $\delta \leqslant \epsilon \leqslant$ On, we can naturally define $Q^{\delta}$ from $Q^{\epsilon}$ and $\delta$. We simply have to restrict $Q^{\epsilon}$ to those cases where the third argument of the quantifier defines a set with order type less than $\delta$. This is something which can be defined universally over any $\mathcal{L}$ structure $\mathcal{A}$ which knows even basic set theory and contains all the ordinals below $\delta$. By converse, if we had attempted to define a quantifier which simply restricted the domain of $Q^{\text {e.c. without using the auxiliary set, then we would have to }}$ produce a complicated definition - inside $\mathcal{A}$ - of the Cantor-Bendixson rank of a definable class.

Note that $Q^{\mathrm{On}}$ is effectively $Q^{\text {e.c. }}$, but with this auxiliary set requirement. ${ }^{1}$ Fortunately, it turns out that the extra requirement is unimportant when studying LST numbers:
Theorem 5.2.2. $\operatorname{LST}\left(I, Q^{\mathrm{On}}\right)=\operatorname{LST}\left(I, Q^{\text {e.c. }}\right)$

[^23]Thus, we can reasonably say that (for the purposes of LST numbers) $Q^{\text {On }}$ is just $Q^{\text {e.c. }}$ expressed in a more convenient form. And therefore, that for $\epsilon<\mathrm{On}$, the quantifier $Q^{\epsilon}$ that we have introduced is a fragment of $Q^{\text {e.c. }}$.

Before we get into proving the theorem, we finish off our definitions by introducing the second class of quantifiers we're going to be looking at. These tell us only about cardinalities (not cofinalities), and specifically whether a given cardinality is regular. It uses the same auxiliary set technique as $Q^{\alpha}$.

Definition 5.2.3. Let $\epsilon \leqslant$ On. The quantifier $R^{\epsilon}$ is 2 -ary, and quantifies over three variables. $R^{\epsilon} v_{1}, \ldots, v_{3}(\varphi, \psi, \chi)$ is interpreted as true (over $\mathcal{A}$, etc.) if the two sets

$$
\begin{gathered}
X:=\left\{x: \mathcal{A} \vDash \varphi\left[x / v_{1}\right]\right\} \\
Z:=\left\{(x, y): \mathcal{A} \vDash \chi\left[x / v_{2}, y / v_{3}\right]\right\}
\end{gathered}
$$

satisfy the following three conditions.

1. $Z$ is a well order
2. The order type of $Z$ is less than $\epsilon$
3. $|X| \in \operatorname{Reg}_{\text {o.t.(Z) }}$

Morally, $R^{\alpha}$ is true of $X$ if $|X| \in \operatorname{Reg}_{<\alpha}$. We are interested in looking at the LST numbers of these predicates together with $I$. (Examining LST numbers of this kind of symbol in contexts where $I$ is not available is notoriously difficult: little is known even about $\operatorname{LST}\left(Q^{\text {e.c. }}\right)$.)

### 5.3 Inequalities

There is an implicit hierarchy of complexity in the new predicates: if $\delta<\epsilon$ then $Q^{\epsilon}$ is intuitively giving all the information that $Q^{\delta}$ is, and more. So we would expect the LST number to be higher for $Q^{\epsilon}$ than $Q^{\delta}$. This turns out to be the case, with one exception: in the right structures, we can use the fact that $Q^{\delta}$ stops working at $\delta$ to define an order type of exactly $\delta$ using $Q^{\delta}$. So if $\delta$ is particularly large, $Q^{\delta}$ might give a small piece of information which is not given by $Q^{\epsilon}$ and so the relationship breaks down. A similar hierarchy exists among the $R^{\epsilon}$ as well, and the LST numbers of the two schemes of predicates are also related to one another.

Theorem 5.3.1. In the following statements, if an LST number does not exist we consider it to be equal to $\infty$. For $\epsilon, \delta \in \mathrm{On} \cup\{\mathrm{On}\}$ and $\delta<\epsilon$ :

1. $\operatorname{LST}\left(I, Q^{\text {e.c. }}\right)=\operatorname{LST}\left(I, Q^{\mathrm{On}}\right)$
2. $\operatorname{LST}\left(I, Q^{\epsilon}\right) \geqslant \operatorname{LST}\left(I, R^{\epsilon}\right)$
3. Either $\operatorname{LST}\left(I, Q^{\epsilon}\right) \geqslant \operatorname{LST}\left(I, Q^{\delta}\right)$, or both $\delta \geqslant \operatorname{LST}\left(I, Q^{\epsilon}\right)$ and $\operatorname{LST}\left(I, Q^{\delta}\right)>\min \left(\operatorname{Reg}_{\delta}\right)$.
4. Either $\operatorname{LST}\left(I, R^{\epsilon}\right) \geqslant \operatorname{LST}\left(I, R^{\delta}\right)$, or both $\delta \geqslant \operatorname{LST}\left(I, R^{\epsilon}\right)$ and $\operatorname{LST}\left(I, R^{\delta}\right)>\min \left(\operatorname{Reg}_{\delta}\right)$.
5. $\operatorname{LST}\left(I, R^{1}\right)=\operatorname{LST}\left(I, R^{0}\right)=\operatorname{LST}(I)$.

The second possibilities in 3 and 4 always fail unless $\delta$ is above certain large cardinals. To see this, we will need certain other results, so that result is deferred until 5.3.5.

Proof. The idea of all these proofs is the same. Although the predicates are not definable from one another in arbitrary structures, we show that they can be defined from each other in a suitable extension of any $\mathcal{L}$ structure.

Throughout these proofs, we shall use the following two formulas (in any language containing the symbols $\in$ and $=)$, for use with $I, Q$ and $R$ :

$$
\begin{gathered}
\varphi_{\in}\left(v_{0}, v_{1}\right):=v_{0} \in v_{1} \\
\varphi_{\in}^{\prime}\left(v_{0}, v_{1}, v_{2}\right):=v_{0} \in v_{2} \wedge v_{1} \in v_{2} \wedge v_{0} \in v_{1}
\end{gathered}
$$

$$
\begin{gathered}
\varphi_{\forall}\left(v_{0}\right):=v_{0}=v_{0} \\
\varphi_{\mathrm{On}}\left(v_{0}\right):=" v_{0} \text { is transitive and well-ordered by } \in "
\end{gathered}
$$

These are very simple formulas, of course, but having names for them will be useful when we use them as parameters for second order predicates. We will always be quantifying over $v_{0}$, or $v_{0}$ and $v_{1}$ in the case of $\varphi_{\epsilon}^{\prime}$.

Note that $\varphi_{\text {On }}$ is saying that $v_{0}$ is an ordinal of the structure it is interpreted over, not an ordinal of $V$. We will only use it in situations where we already know these two notions to agree before using it.

Proof (1). First, we shall show that $\operatorname{LST}\left(I, Q^{\text {e.c. }}\right) \geqslant \operatorname{LST}\left(I, Q^{\mathrm{On}}\right)$. Obviously, this is trivial if the former is $\infty$. So suppose that $\operatorname{LST}\left(I, Q^{\text {e.c. }}\right)=\kappa<\infty$. Let $\mathcal{L}$ be a first order language of cardinality less than $\kappa$, and let $\mathcal{A}$ be an $\mathcal{L}$ structure of cardinality $\lambda \geqslant \kappa$. Without loss of generality, assume that $\mathcal{A}$ does not contain any ordinals, but is an element of $H_{\lambda^{+}}$, and that $\mathcal{L}$ does not include the symbol $\in$.

Let $P$ be a new 1-ary first order predicate symbol, and let $\mathcal{L}^{\prime}=\mathcal{L} \sqcup\{\epsilon, P\}$. Let $\mathcal{A}^{\prime}$ be the $\mathcal{L}^{\prime}$ structure with domain $H_{\lambda^{+}}$, with $\in$ interpreted in the usual way, the predicate $P$ interpreted as true of elements of $\mathcal{A}$, and all the symbols of $\mathcal{L}$ interpreted in line with $\mathcal{A}$ on the elements of $\mathcal{A}$, and in some trivial way everywhere else.

Let $\mathcal{B}^{\prime}$ be an $\mathcal{L}^{\prime} \cup\left\{I, Q^{\text {e.c. }}\right\}$ elementary substructure of $\mathcal{A}^{\prime}$ of cardinality less than $\kappa$ (which we are assuming exists), and let $\mathcal{C}^{\prime}$ be its transitive collapse. Let $j: \mathcal{C}^{\prime} \rightarrow \mathcal{A}^{\prime}$ be the $\mathcal{L}^{\prime} \cup\left\{I, Q^{\text {e.c. }}\right\}$ elementary embedding. Let

$$
\mathcal{B}=\left\{x \in \mathcal{B}^{\prime}: P(x)\right\}=\mathcal{B}^{\prime} \cap \mathcal{A}
$$

viewed as an $\mathcal{L}$ structure, and let $\mathcal{C}$ be the analogous substructure of $\mathcal{C}^{\prime}$.
We shall show that $j$ is an $\mathcal{L}^{\prime} \cup\left\{I, Q^{\mathrm{On}}\right\}$ elementary embedding of $\mathcal{C}^{\prime}$ into $\mathcal{A}^{\prime}$. From this it immediately follows that $\mathcal{B}^{\prime}=\operatorname{ran}(j)$ is an $\mathcal{L}^{\prime} \cup\left\{I, Q^{\mathrm{On}}\right\}$ elementary substructure of $\mathcal{A}^{\prime}$, and it hence that $\mathcal{B}=\mathcal{B}^{\prime} \cap P$ is an $\mathcal{L} \cup\left\{I, Q^{\mathrm{On}}\right\}$ elementary substructure of $\mathcal{A}=\mathcal{A}^{\prime} \cap P$.

We show $\mathcal{L}^{\prime} \cup\left\{I, Q^{\mathrm{On}}\right\}$ elementarity of $j$ in several stages.
Claim 5.3.2. 1. Both $\mathcal{A}^{\prime}$ and $\mathcal{C}^{\prime}$ believe a set is an ordinal if and only if it is an ordinal of $V$.
2. Both $\mathcal{A}^{\prime}$ and $\mathcal{C}^{\prime}$ believe an ordinal is a cardinal if and only if it is a cardinal of $V$.
3. Both $\mathcal{A}^{\prime}$ and $\mathcal{C}^{\prime}$ believe a cardinal is regular if and only if it is a regular cardinal of $V$.
4. There is an $\mathcal{L}^{\prime} \cup\left\{I, Q^{\text {e.c. }\}}\right.$ formula $\Phi_{R}(x, y)$ which both $\mathcal{A}^{\prime}$ and $\mathcal{C}^{\prime}$ interpret as true of $(x, y)$ if and only if $y$ is an ordinal and $x \in \operatorname{Reg}_{y}$.
5. Suppose that $\varphi, \psi, \chi$ are $\mathcal{L} \cup\left\{I, Q^{\text {e.c. }}\right\}$ formulas, which are elementary between $\mathcal{A}^{\prime}$ and $\mathcal{C}^{\prime}$. Suppose further that $\varphi\left(v_{1}, \ldots, v_{k}\right) \Longrightarrow P\left(v_{1}\right) \wedge P\left(v_{2}\right)$, and similarly for $\psi$ and $\chi$. So $\varphi, \psi$ and $\chi$, together with any assignment $\vec{y}$, define subsets of $\mathcal{A}^{2}$ and $\mathcal{C}^{2}$ in $\mathcal{A}^{\prime}$ and $\mathcal{C}^{\prime}$ respectively.
Then there is a formula $\Psi(\varphi, \psi, \chi, \vec{v})$ such that for any assignment $\vec{y}$ of variables to elements of $\mathcal{A}$ (resp. of $\mathcal{C}$ ), $\mathcal{A}^{\prime}$ (resp. $\mathcal{C}^{\prime}$ ) interprets $\Psi(\varphi, \psi, \chi, \vec{y})$ as true if and only if $\mathcal{A}$ (resp. $\left.\mathcal{C}\right)$ believes $Q^{\mathrm{On}}(\varphi, \psi, \chi, \vec{y})$.

Proof (Claim, 1). Trivial, since $\mathcal{A}^{\prime}$ and $\mathcal{C}^{\prime}$ are transitive.
Proof (Claim, 2). First, notice that since $\mathcal{A}^{\prime}$ has domain $H_{\lambda^{+}}$, it is true that the cardinals of $\mathcal{A}^{\prime}$ are precisely the cardinals of $V$ which are in $\mathcal{A}^{\prime}$.

In both $\mathcal{A}^{\prime}$ and $\mathcal{C}^{\prime}$, the $\mathcal{L}^{\prime} \cup I$ formula

$$
\Phi_{C}(x):=x \in \text { On } \wedge \forall y \in x \neg I\left(\varphi_{\epsilon}, \varphi_{\epsilon}, x, y\right)
$$

is true of a set $x$ if and only if $x$ is a cardinal of $V$. So we have a way to "test" whether an ordinal is a cardinal. But then as we saw a moment ago,

$$
\mathcal{A}^{\prime} \vDash \forall x x \in \operatorname{Card} \Longleftrightarrow \Phi_{C}(x)
$$

As usual, " $x \in$ Card" is shorthand for the first order formula in the language of set theory which says that $x$ is a cardinal.

By elementarity, then,

$$
\mathcal{C}^{\prime} \vDash \forall x x \in \operatorname{Card} \Longleftrightarrow \Phi_{C}(x)
$$

Hence, the cardinals of $\mathcal{C}^{\prime}$ are precisely the cardinals of $V$ that are in $\mathcal{C}^{\prime}$.
Proof (Claim, 3). Similar to the previous case, but instead of $\Phi_{C}$ we use the formula:

$$
\Phi_{R}^{\prime}(x):=\Phi_{C}(x) \wedge \forall y \in x \neg Q^{\text {e.c. }}\left(\varphi_{\epsilon}^{\prime}, \varphi_{\in}^{\prime}, x, y\right)
$$

which is true in both $\mathcal{A}^{\prime}$ and $\mathcal{C}^{\prime}$ if and only if $x$ is a $V$-regular cardinal. We know that $\mathcal{A}^{\prime}$ is correct about the regular cardinals because its domain is $H_{\lambda^{+}}$, so by elementarity $\mathcal{C}^{\prime}$ is as well.

Proof (Claim, 4). Note that $\mathcal{A}^{\prime}$ believes in recursion. Since Reg $\mathcal{A}^{\prime}$ is a definable subclass of $\mathcal{A}^{\prime}$, we know that $\mathcal{A}^{\prime}$ can recursively calculate the Cantor-Bendixson ranks of elements of $\operatorname{Reg} \mathcal{A}^{\prime}$ and therefore can calculate $\operatorname{Reg}_{\epsilon} \mathcal{A}^{\prime}$ for all $\epsilon \in \mathcal{A}^{\prime}$. Let $\Phi_{R}(x, y)$ be the (first order) formula which does this. That is, for $x, y \in \mathcal{A}^{\prime}, \Phi_{R}(x, y)$ is true if $y \in \operatorname{On}$ and $x \in \operatorname{Reg}_{y}^{\mathcal{A}^{\prime}}$.

By elementarity, this property of $\Phi_{R}$ is also true in $\mathcal{C}^{\prime}$. That is, $\mathcal{C}^{\prime}$ believes that $\Phi_{R}(x, y)$ is true if and only if $y \in \mathrm{On}^{\mathcal{C}^{\prime}}$ and $x \in \operatorname{Reg}_{y}^{\mathcal{C}^{\prime}}$.

But we know that both Reg $\mathcal{A}^{\prime}$ and $\operatorname{Reg}^{\mathcal{C}^{\prime}}$ are simply $\operatorname{Reg}^{V}$ intersected with their respective models. So in fact $\operatorname{Reg}_{y}^{\mathcal{A}^{\prime}}=\operatorname{Reg}_{y}^{\mathcal{C}^{\prime}}=\operatorname{Reg}_{y}^{V}$.

Proof (Claim, 5). We begin by defining a formula $\Psi_{0}\left(\chi, \vec{v}, v_{k}\right)$ (where $v_{k}$ is an otherwise unused variable).

$$
\Psi_{0}(\chi, \vec{y}, z):=z \in \operatorname{On} \wedge \exists s\left(\forall x_{0}, x_{1}\left(x_{0}, x_{1}\right) \in s \Longleftrightarrow \chi\left(x_{0}, x_{1}, \vec{y}\right)\right) \wedge s \text { is a well order } \wedge \text { o.t. }(s)=z
$$

So $\Psi_{0}$ holds if and only if $\chi$ and $\vec{y}$ define a well order, and that well order is an element of the structure, and it has order type $z \in$ On.

Since $\mathcal{A}^{\prime}$ has domain $H_{\lambda^{+}}$, we know that if $\chi$ and $\vec{y}$ define a well order of size less than $\lambda^{+}$, then that well order will be an element of $\mathcal{A}^{\prime}$. So in particular, if $\chi$ and $\vec{y}$ define a linear order over $\mathcal{A}$ (which, remember, has cardinality $\lambda$ ) then that linear order will be an element of $\mathcal{A}^{\prime}$, and if it is a well order then its order type will then also be an ordinal in $\mathcal{A}^{\prime}$.

We claim the same is true of $\mathcal{C}$ and $\mathcal{C}^{\prime}$. If $\chi$ and $\vec{y}$ define a linear order of elements of $\mathcal{C}$ over $\mathcal{C}^{\prime}$, then by elementarity they define a linear order of elements of $\mathcal{A}$ over $\mathcal{A}^{\prime}$. That linear order is in $\mathcal{A}^{\prime}$. By elementarity then the original linear order over $\mathcal{C}$ is in $\mathcal{C}^{\prime}$. For the order type part, note that $\mathcal{A}^{\prime}$ and hence $\mathcal{C}^{\prime}$ believe that for all $\alpha \in \mathrm{On}, \neg I\left(\varphi_{\mathrm{On}}, \varphi_{\epsilon}, \alpha\right)$, meaning that $\mathrm{On} \cap \mathcal{C}^{\prime}$ is a cardinal of $V$. And letting $\varphi_{P}(x):=P(x)$, we also know (by elementarity) that $\mathcal{C}^{\prime}$ believes there is some cardinal $\alpha$ such that $I\left(\varphi_{P}, \varphi_{\in}, \alpha\right)$. So the cardinality of $\mathcal{C}$ is below On $\cap \mathcal{C}^{\prime}$. Hence all the ordinals of cardinality $\leqslant|\mathcal{C}|$ are in $\mathcal{C}^{\prime}$, and so any well order of elements of $\mathcal{C}$ will have order type in $\mathcal{C}^{\prime}$.

So if $\chi$ and $\vec{y}$ define a subset of $\mathcal{C}^{2}$ or $\mathcal{A}^{2}$ over $\mathcal{C}^{\prime}$ or $\mathcal{A}^{\prime}$ then they define a well order if and only if there is some ordinal $z$ of $\mathcal{C}^{\prime}$ or $\mathcal{A}^{\prime}$ such that $\mathcal{C}^{\prime}$ or $\mathcal{A}^{\prime}$, respectively, believes $\Psi_{0}(\chi, \vec{y}, z)$; and if so, then the well order has order type $z$.

Now let

$$
\begin{aligned}
\Psi(\varphi, \psi, \chi, \vec{y}):= & Q^{\text {e.c. }}(\varphi, \psi, \vec{y}) \wedge \\
& \exists \alpha \in \operatorname{On} \Psi_{0}(\chi, \vec{y}, \alpha) \wedge \\
& \exists \beta \in \operatorname{On} \Phi_{R}(\beta, \alpha) \wedge Q^{\text {e.c. }}\left(\varphi, \varphi_{\epsilon}^{\prime}, \vec{y}, \beta\right)
\end{aligned}
$$

An easy definition chase shows that if $\varphi, \psi$ and $\chi$, together with $\vec{y}$, define subsets of $\mathcal{A}^{2}$ or $\mathcal{C}^{2}$ over $\mathcal{A}^{\prime}$ or $\mathcal{C}^{\prime}$ (respectively) then $\mathcal{A}^{\prime}$ or $\mathcal{C}^{\prime}$ believe $\Psi(\varphi, \psi, \chi, \vec{y})$ if and only if $\mathcal{A}$ believes $Q^{\mathrm{On}}(\varphi, \psi, \chi, \vec{y})$. Since we already know how $\Psi_{0}$ is interpreted, the only "trick" is showing that the equal cofinality of the orders defined by $\varphi$ and $\psi$ must be an element of $\mathcal{A}^{\prime}$ or $\mathcal{C}^{\prime}$. This is trivial for $\mathcal{A}^{\prime}$, and for $\mathcal{C}^{\prime}$ it follows from the fact that (as we saw above) $\mathcal{C}^{\prime}$ contains all the cardinals $\leqslant|\mathcal{C}|$.

The existence of $\Psi$ shows that $\mathcal{C}$ is $\mathcal{L} \cup\left\{I, Q^{\mathrm{On}}\right\}$ elementarily equivalent to $\mathcal{A}$ by standard arguments, and therefore that $\mathcal{B}$ is an $\mathcal{L} \cup\left\{I, Q^{\mathrm{On}}\right\}$ elementary substructure of $\mathcal{A}$. $\operatorname{So} \operatorname{LST}\left(I, Q^{\mathrm{On}}\right) \leqslant \operatorname{LST}\left(I, Q^{\text {e.c. }}\right)$.

To show the converse, we use a similar trick, but the technique is much simpler: given an $\mathcal{L}$ structure $\mathcal{A}$ of cardinality $\lambda>\kappa=\operatorname{LST}\left(I, Q^{\mathrm{On}}\right)$, let $\mathcal{A}^{\prime}=(\lambda+1, \epsilon)$ and let $\mathcal{A}^{\prime \prime}=\mathcal{A} \sqcup \mathcal{A}^{\prime}$. Let $\mathcal{B}^{\prime \prime}$ be an $\mathcal{L} \cup\left\{I, Q^{\mathrm{On}}\right\}$ elementary substructure of size less than $\kappa$, and let $\mathcal{B}$ be the part corresponding to $\mathcal{A}$.

Now, the cardinality of $\mathcal{A}$ is equal to the cardinality of $\mathcal{A}^{\prime}$, and $\mathcal{A}^{\prime \prime}$ can easily express this using $I$. Hence, by elementarity $\mathcal{B}$ and $\mathcal{B}^{\prime}$ also have the same cardinality as each other. Since $\mathcal{B}^{\prime}$ has a largest element, if $X$ is any linear order of elements of $\mathcal{B}^{\prime}$ then its $V$ cofinality, and the Cantor-Bendixson rank of its cofinality in $\operatorname{Reg}^{V}$, will be in the transitive collapse of $\mathcal{B}^{\prime}$.

Hence, in both $\mathcal{A}^{\prime \prime}$ and $\mathcal{B}^{\prime \prime}, Q^{\text {e.c. }}(\varphi, \psi, \vec{y})$ is equivalent to " $\exists \epsilon \in \operatorname{On} Q^{\mathrm{On}}\left(\varphi, \psi, \varphi_{\epsilon}^{\prime}, \vec{y}, \epsilon\right)$ ". So $\mathcal{B}$ is an $\mathcal{L} \cup\left\{I, Q^{\text {e.c. }}\right\}$ elementary substructure of $\mathcal{A}$. Hence $\operatorname{LST}\left(I, Q^{\text {e.c. }}\right) \leqslant \operatorname{LST}\left(I, Q^{\mathrm{On}}\right)$.

Proof (2). Similar to the previous part. Let $\mathcal{A}$ be an $\mathcal{L}$ structure with cardinality $\lambda \geqslant \kappa=\operatorname{LST}\left(I, Q^{\epsilon}\right)$. Without loss of generality assume $\mathcal{A} \in H_{\lambda^{+}}$. Let $\mathcal{L}^{\prime}=\mathcal{L} \sqcup\{\epsilon, P\}$ and let $\mathcal{A}^{\prime}$ be a $\mathcal{L}^{\prime}$ structure which extends $\mathcal{A}$, with $P^{\mathcal{A}^{\prime}}=\mathcal{A}$ and all the symbols of $\mathcal{L}$ interpreted trivially outside $\mathcal{A}$.

Let $\mathcal{B}^{\prime}$ be an $\mathcal{L} \sqcup\left\{I, Q^{\epsilon}\right\}$ substructure of cardinality $<\kappa$, and let $\mathcal{B}=\mathcal{B}^{\prime} \cap P \subset \mathcal{A}$. Now, $R^{\epsilon}$ can be defined in terms of $Q^{\epsilon}$ in both $\mathcal{A}^{\prime}$ and $\mathcal{B}^{\prime}$. Let

$$
\begin{aligned}
\Phi\left(\varphi, \chi, \vec{y}_{0}, \vec{y}_{1}\right):=\exists \alpha \in \operatorname{On} & \left(I\left(\varphi, \varphi_{\in}, \vec{y}_{0}, \alpha\right) \wedge\right. \\
& \forall \beta<\alpha \neg I\left(\phi_{\epsilon}, \phi_{\epsilon}, \alpha, \beta\right) \wedge \\
& \exists \gamma \in \operatorname{On} Q^{\epsilon}\left(\varphi_{\epsilon}^{\prime}, \varphi_{\epsilon}^{\prime}, \varphi_{\epsilon}^{\prime}, \alpha, \alpha, \gamma\right) \wedge \\
& \left.\forall \beta<\alpha \forall \gamma \in \operatorname{On} \neg Q^{\epsilon}\left(\varphi_{\epsilon}^{\prime}, \varphi_{\epsilon}^{\prime}, \varphi_{\epsilon}^{\prime}, \alpha, \beta, \gamma\right)\right)
\end{aligned}
$$

Then in both $\mathcal{A}^{\prime}$ and $\mathcal{B}^{\prime}, \varphi\left(x, \vec{y}_{0}\right) \Longrightarrow P(x)$ implies that $\Phi\left(\varphi, \chi, \vec{y}_{0}, \vec{y}_{1}\right)$ holds if and only if $R^{\epsilon}\left(\varphi, \chi, \vec{y}_{0}, \vec{y}_{1}\right)$ is true.

To see that this works in $\mathcal{B}^{\prime}$, it suffices to know that the transitive collapse of $\mathcal{B}^{\prime}$ contains ordinals of every order type below $|\mathcal{B}|$. This follows as before: in $\mathcal{B}^{\prime}, I\left(\varphi_{\mathrm{On}}, \varphi_{\epsilon}, \alpha\right)$ fails for all $\alpha<\mathrm{On} \cap \mathcal{B}^{\prime}$, so On $\cap \mathcal{B}^{\prime}$ is a cardinal of $V$. But there is some $\alpha \in$ On $\cap \mathcal{B}^{\prime}$ such that $I\left(\varphi_{P}, \varphi_{\epsilon}, \alpha\right)$ holds, where $\varphi_{P}(x)=P(x)$. So On $\cap \mathcal{B}^{\prime}$ is a cardinal of $V$ which is larger than $|\mathcal{B}|$.

Now $\mathcal{B}^{\prime}$ is an $\mathcal{L}^{\prime} \cup\{I, \Phi\}$ elementary substructure of $\mathcal{A}^{\prime}$. But $\Phi$ agrees with $R^{\epsilon}$ on $\mathcal{A}$ and $\mathcal{B}$, so $\mathcal{B}$ is an $\mathcal{L} \cup\{I, \Phi\}$ elementary substructure of $\mathcal{A}$ of cardinality less than $\kappa$.
Proof (3). First suppose that $\delta<\operatorname{LST}\left(I, Q^{\epsilon}\right)$. We shall show that $\operatorname{LST}\left(I, Q^{\epsilon}\right) \geqslant \operatorname{LST}\left(I, Q^{\delta}\right)$. The technique is (once again) similar. However, we need some way to name $\delta$.

Let $\mathcal{A}$ be an $\mathcal{L}$ structure, of cardinality $\lambda \geqslant \operatorname{LST}\left(I, Q^{\epsilon}\right)$. Assume that $\mathcal{L}$ has cardinality less than $\operatorname{LST}\left(I, Q^{\epsilon}\right)$. Expand $\mathcal{L}$ to a language $\mathcal{L}^{\prime}$ which contains $\in$, a predicate $P$ and $\delta$ many constant symbols $c_{i}: i<\delta+1$. Note that $\mathcal{L}^{\prime}$ has cardinality less than $\operatorname{LST}\left(I, Q^{\epsilon}\right)$ still. Let $\mathcal{A}^{\prime}=\mathcal{A} \sqcup(\lambda+1)$ be an expansion of $\mathcal{A}$ which adds the first $\lambda+1$ ordinals, and assigns constant symbols to the first $\delta+1$ many of those ordinals, and interprets $P$ as true on $\mathcal{A}$. Let $\mathcal{B}^{\prime} \leqslant \mathcal{A}^{\prime}$ be an $\mathcal{L}^{\prime} \cup\left\{I, Q^{\epsilon}\right\}$ elementary substructure of cardinality $<\operatorname{LST}\left(I, Q^{\epsilon}\right)$. We know that $\mathcal{B}^{\prime}$ contains all the ordinals up to and including $\delta=c_{\delta}$, so the set defined over $\mathcal{B}^{\prime}$ by $\varphi_{\in}$ and $c_{\delta}$ is just $\delta$ itself.

It follows that we can define $Q^{\delta}$, in both $\mathcal{A}^{\prime}$ and $\mathcal{B}^{\prime}$, via:

$$
Q^{\delta}(\varphi, \psi, \chi, \vec{y}) \Longleftrightarrow Q^{\epsilon}(\varphi, \psi, \chi, \vec{y}) \wedge \exists \alpha \in c_{\delta} Q^{\epsilon}\left(\varphi, \psi, \varphi_{\epsilon}, \vec{y}, \alpha\right)
$$

So $\mathcal{B}=\mathcal{B}^{\prime} \cap \mathcal{A}$ is a $\mathcal{L} \cup\left\{I, Q^{\delta}\right\}$ elementary substructure of $\mathcal{A}$, and $\operatorname{LST}\left(I, Q^{\delta}\right) \leqslant \operatorname{LST}\left(I, Q^{\epsilon}\right)$.
Now suppose instead that $\operatorname{LST}\left(I, Q^{\delta}\right) \leqslant \min \left(\operatorname{Reg}_{\delta}\right)$, and suppose (seeking a contradiction) that $\operatorname{LST}\left(I, Q^{\epsilon}\right)<$ $\operatorname{LST}\left(I, Q^{\delta}\right)$. Let $\mathcal{A}$ be an $\mathcal{L}$ structure (where the cardinality of $\mathcal{L}$ is less than $\operatorname{LST}\left(I, Q^{\epsilon}\right)$ ). Then $\mathcal{A}$ contains an $\mathcal{L} \cup\left\{I, Q^{\delta}\right\}$ elementary substructure $\mathcal{B}$ of cardinality less than $\operatorname{LST}\left(I, Q^{\delta}\right)$. And $\mathcal{B}$ has an $\mathcal{L} \cup\left\{I, Q^{\epsilon}\right\}$ substructure $\mathcal{C}$ of cardinality less than $\operatorname{LST}\left(I, Q^{\epsilon}\right)$. But $\mathcal{B}$ is small enough that every linear order we can construct from its elements has cofinality in $\operatorname{Reg}_{<\delta}$. Hence, in $\mathcal{B}$ and all its substructures, $Q^{\epsilon}$ and $Q^{\delta}$ agree with each other. So $\mathcal{C}$ is an $\mathcal{L} \cup\left\{I, Q^{\delta}\right\}$ elementary substructure of $\mathcal{B}$ and hence of $\mathcal{A}$. So any $\mathcal{L}$ structure contains an $\mathcal{L} \cup\left\{I, Q^{\delta}\right\}$ elementary substructure of cardinality $<\operatorname{LST}\left(I, Q^{\epsilon}\right)<\operatorname{LST}\left(I, Q^{\delta}\right)$. Contradiction.

Proof (4). Just like the previous case.

Proof (5). First note that $R^{0}$ is simply always false, since there are no ordinals below 0 . So the claim $\operatorname{LST}\left(I, R^{0}\right)=\operatorname{LST}(I)$ is trivial.
$\operatorname{LST}\left(I, R^{1}\right) \geqslant \operatorname{LST}(I)$ is also trivial. The only thing to show is $\operatorname{LST}\left(I, R^{1}\right) \leqslant \operatorname{LST}(I)$. We prove this in the same way as the previous cases. Let $\mathcal{A}$ be an $\mathcal{L}$ structure of cardinality $\lambda \geqslant \operatorname{LST}(I)$. Let $\mathcal{L}^{\prime}=\mathcal{L} \cup\{\in, P\}$ and let $\mathcal{A}^{\prime}=\mathcal{L} \sqcup \lambda+1$ with $P$ interpreted as true on $\mathcal{A}^{\prime}$. Now, $R^{1}(\varphi, \chi, \vec{y})$ holds in $\mathcal{A}^{\prime}$ if and only if $\chi$ (and the relevant part of $\vec{y}$ ) defines the empty set, and $\varphi$ (and $\vec{y}$ ) defines a set whose cardinality is an (infinite) successor cardinal. Both of these properties can be easily defined in $\mathcal{A}^{\prime}$ :

$$
\begin{aligned}
\mathcal{A}^{\prime} \vDash R^{1}(\varphi, \chi, \vec{y}) \Longleftrightarrow \mathcal{A}^{\prime} \vDash & \forall x \neg \chi(x, \vec{y}) \wedge \\
& \exists \kappa \in \operatorname{On} I\left(\varphi, \varphi_{\in}, \vec{y}, \kappa\right) \wedge \\
& \forall \alpha \in \kappa \neg I\left(\varphi_{\in}, \varphi_{\in}, \kappa, \alpha\right) \wedge \\
& \exists \mu \in \kappa \forall \alpha \in(\mu, \kappa) I\left(\varphi_{\in}, \varphi_{\in} \mu, \alpha\right)
\end{aligned}
$$

So $\mathcal{B}:=P\left(\mathcal{B}^{\prime}\right)$ is an $\mathcal{L} \cup\left\{I, R^{1}\right\}$ elementary substructure of $\mathcal{A}=P\left(\mathcal{A}^{\prime}\right)$.

This theorem shows that if we exclude very large values of $\epsilon, \operatorname{LST}\left(I, Q^{\epsilon}\right)$ and $\operatorname{LST}\left(I, R^{\epsilon}\right)$ move downward as we decrease $\epsilon$ or move from $Q$ to $R$. At the bottom of this hierarchy is $\operatorname{LST}(I)$, and we saw earlier (from [29]) that the minimum possible value of this is precisely the first inaccessible. At the top is $\operatorname{LST}\left(I, Q^{\mathrm{On}}\right)=$ $\operatorname{LST}\left(I, Q^{\text {e.c. }}\right)$, and we saw that the minimum possible value of this is precisely the first Mahlo cardinal. The remainder of the section is devoted to finding a similar result for the other members of the hierarchy. The tagline for this section is:

For $\epsilon>0$, the minimum consistent values for $\operatorname{LST}\left(I, Q^{\epsilon}\right)$ and $\operatorname{LST}\left(I, R^{\epsilon}\right)$ are both precisely the first element of $\operatorname{Reg}_{\epsilon}$, provided $\epsilon$ is not too large.

So for example, this says the minimum value of $\operatorname{LST}\left(I, R^{1}\right)=\operatorname{LST}(I)$ is the first element of $\operatorname{Reg}_{1}$, which is the least inaccessible as we had expected. We will, of course, explain what we mean by "too large" shortly; it depends on whether we're dealing with $Q$ or $R$.

First, we shall prove that these cardinals are lower bounds for the LST numbers.
Theorem 5.3.3. Let $\epsilon>0$ be such that there are no hyperinaccessibles below $\epsilon$. Then if it exists, $\operatorname{LST}\left(I, R^{\epsilon}\right)$ is at least the first element of $\operatorname{Reg}_{\epsilon}$.

Proof. Suppose $\kappa=\operatorname{LST}\left(I, R^{\epsilon}\right)$ is less than the first element of $\operatorname{Reg}_{\epsilon}$. Let $\mathcal{A}=\left(H_{\kappa^{+}}, \epsilon\right)$. The cardinals of $\mathcal{A}$ are precisely the cardinals of $V$ up to (and including) $\kappa$.

By definition of the LST number, we can find an elementary substructure $\mathcal{B} \subset \mathcal{A}$ of cardinality less than $\kappa$. Since $\mathcal{A}$ is well founded, so is $\mathcal{B}$, so we can take its transitive collapse $\mathcal{C}=(C, \epsilon)$ and the elementary embedding

$$
j: \mathcal{C} \rightarrow \mathcal{A}
$$

Now, letting $\varphi_{\in}$ be as in the previous theorem,

$$
\begin{aligned}
\gamma \in \operatorname{Card}^{\mathcal{C}} & \Longleftrightarrow j(\gamma) \in \operatorname{Card}^{\mathcal{A}} \\
& \Longleftrightarrow j(\gamma) \in \operatorname{Card}^{V} \\
& \Longleftrightarrow \mathcal{A} \vDash \forall \alpha \in j(\gamma) \neg I\left(\varphi_{\epsilon}, \varphi_{\epsilon}, \alpha, j(\gamma)\right) \\
& \Longleftrightarrow \mathcal{C} \vDash \forall \alpha \in \gamma \neg I\left(\varphi_{\epsilon}, \varphi_{\epsilon}, \alpha, \gamma\right) \\
& \Longleftrightarrow \gamma \in \operatorname{Card}^{V}
\end{aligned}
$$

The last $\Longleftrightarrow$ follows because $\mathcal{C}$ is transitive. Similarly,

$$
\begin{aligned}
\gamma \in \operatorname{Reg}^{\mathcal{C}} & \Longleftrightarrow j(\gamma) \in \operatorname{Reg}^{\mathcal{A}} \\
& \Longleftrightarrow j(\gamma) \in\left(\operatorname{Reg}_{<\epsilon}\right)^{V} \\
& \Longleftrightarrow \mathcal{A} \vDash \exists \beta \in \operatorname{On} R^{\epsilon}\left(\varphi_{\epsilon}, \varphi_{\epsilon}^{\prime}, j(\gamma), \beta\right) \\
& \Longleftrightarrow \mathcal{C} \vDash \exists \beta \in \operatorname{On} R^{\epsilon}\left(\varphi_{\epsilon}, \varphi_{\epsilon}^{\prime}, \gamma, \beta\right) \\
& \Longleftrightarrow \gamma \in\left(\operatorname{Reg}_{<\epsilon}\right)^{V} \\
& \Longleftrightarrow \gamma \in \operatorname{Reg}^{V}
\end{aligned}
$$

$\mathcal{A}$ and $\mathcal{C}$ can then (using the same recursion) calculate which cardinals are in $\operatorname{Reg}_{\delta}^{V}$ for each $\delta<\epsilon$.
Now, let $\gamma \in \mathcal{C}$ be the least ordinal such that $j(\gamma)>\gamma$. (Such a $\gamma$ must exist: $\mathcal{C}$ has a largest cardinal $\beta$ which must be smaller than $\kappa$, and $j(\beta)=\kappa>\beta$.)

Clearly $\gamma$ is a cardinal of $\mathcal{C}$. Otherwise there would be some bijection $f: \alpha \rightarrow \gamma$ for some $\alpha<\gamma$; and then $j(f)=f$ would be a bijection from $j(\alpha)=\alpha$ to $j(\gamma)>\gamma$, which is clearly nonsense. So $\gamma$ is a cardinal of $\mathcal{C}$, and hence of $V$. Moreover, $\gamma$ cannot be a successor cardinal: if $\gamma=\left(\beta^{+}\right)^{\mathcal{C}}$ then

$$
j(\gamma)=\left(j(\beta)^{+}\right)^{\mathcal{A}}=\left(j(\beta)^{+}\right)^{V}=\left(\beta^{+}\right)^{V}=\left(\beta^{+}\right)^{\mathcal{C}}=\gamma
$$

Suppose that $\gamma$ is singular (in $\mathcal{C}$ or equivalently in $V$ ). Then we can take some sequence $\mu=\left(\mu_{\delta}\right)_{\delta<\beta} \in \mathcal{C}$ which is cofinal in $\gamma$, with $\beta<\gamma$. Since $j 1 \gamma=\mathrm{id}, j\left(\mu_{\delta}\right)=\mu_{\delta}$ for all $\delta$, and the length of $j(\mu)$ is $j(\beta)=\beta$. Hence $j(\mu)=\mu$. But $j(\mu)$ is cofinal in $j(\gamma)$, so then $j(\gamma)=\gamma$. Contradiction.

So $\gamma$ is a regular limit cardinal, and hence is in $\operatorname{Reg}_{\alpha}^{V}$ for some $0<\alpha$. We know that $\alpha<\epsilon$ since $|\mathcal{C}|<\min \left(\operatorname{Reg}_{\epsilon}^{V}\right)$. As usual, since $\gamma \in \operatorname{Reg}_{\alpha}$ we know $\alpha \leqslant \gamma$. Moreover, if $\alpha=\gamma$ then it is a hyperinaccessible, and we are assuming none exist below $\epsilon$. So $\alpha<\gamma$ and $j(\alpha)=\alpha$.

Say $\gamma$ is the $\beta$ 'th element of $\operatorname{Reg}_{\alpha}$. Since $\operatorname{Reg}_{\alpha} \cap \gamma$ must be bounded below $\gamma$, we know $\beta<\gamma$ and hence $j(\beta)=\beta$. But then

$$
\mathcal{C} \vDash " \gamma \text { is the } \beta \text { 'th element of } \operatorname{Reg}_{\alpha}^{\mathcal{C}} "
$$

and hence by elementarity

$$
\mathcal{A} \vDash " j(\gamma) \text { is the } \beta \text { 'th element of } \operatorname{Reg}_{\alpha}^{\mathcal{A}} "
$$

But this is saying that $\gamma$ and $j(\gamma)$ are both the same element of $\operatorname{Reg}_{\alpha}^{V}$. Contradiction.
By the above and Theorem 5.3.1 it also follows, under the above conditions, that $\operatorname{LST}\left(I, Q^{\epsilon}\right) \geqslant \min \left(\operatorname{Reg}_{\epsilon}\right)$. In fact, if $\min \left(\operatorname{Reg}_{\epsilon}\right)$ is strongly inaccessible (e.g. because we are assuming GCH) then this will hold under weaker conditions:

Theorem 5.3.4. Let $\epsilon>0$ and suppose that the first element of $\operatorname{Reg}_{\epsilon}$ is strongly inaccessible. Suppose there are no Mahlo cardinals below $\epsilon$. Then if it exists, $\operatorname{LST}\left(I, Q^{\epsilon}\right)$ is at least the first element of $\operatorname{Reg}_{\epsilon}$.

Proof. Suppose that $\operatorname{LST}\left(I, Q^{\epsilon}\right)<\min \left(\operatorname{Reg}_{\epsilon}\right)=: \kappa$. Since $\kappa$ is strongly inaccessible, $M:=H_{\kappa}$ is a model of ZFC. It is easy to see that $\operatorname{LST}^{M}\left(I, Q^{\epsilon}\right) \leqslant \operatorname{LST}^{V}\left(I, Q^{\epsilon}\right) \in M$ : if $\mathcal{A} \in M$ is an $\mathcal{L}$ structure, then it contains some $\mathcal{L} \cup\left\{I, Q^{\epsilon}\right\}$ elementary substructure $\mathcal{B} \in V$ of cardinality $<\operatorname{LST}\left(I, Q^{\epsilon}\right)^{V}$. Then $\mathcal{B} \in H_{\kappa}=M$.

Moreover, $M$ contains no Mahlo cardinals. This is because (by assumption) it has none below $\epsilon$, and since any Mahlo cardinal is hyperinaccessible, there cannot be any in the interval $\left[\epsilon, \min \left(\operatorname{Reg}_{\epsilon}\right)\right)=[\epsilon, \kappa)$. So we know that $\operatorname{LST}^{M}\left(I, Q^{\text {e.c. }}\right)=\infty$.

But $M$ doesn't contain any cardinals which are in $\operatorname{Reg}_{\epsilon}, \operatorname{Reg}_{\epsilon+1}, \ldots$ So as far as $M$ is concerned $Q^{\epsilon}$ is evaluated in exactly the same way as $Q^{\mathrm{On}}$. Hence, $\operatorname{LST}^{M}\left(I, Q^{\epsilon}\right)=\operatorname{LST}^{M}\left(I, Q^{\mathrm{On}}\right)=\infty$. Contradiction.

A similar argument shows that if there are no Mahlo cardinals below $\epsilon$ and $\operatorname{Reg}_{\epsilon}=\varnothing$ then $\operatorname{LST}\left(I, Q^{\epsilon}\right)=$ $\infty$.

This gives us the promised condition for 3 and 4 of Theorem 5.3.1:

Corollary 5.3.5. If $\delta<\epsilon$ and there are no hyperinaccessibles below $\epsilon$ then $\operatorname{LST}\left(I, R^{\delta}\right) \leqslant \operatorname{LST}\left(I, R^{\epsilon}\right)$ and $\operatorname{LST}\left(I, Q^{\delta}\right) \leqslant \operatorname{LST}\left(I, Q^{\epsilon}\right)$. If instead there are no Mahlo cardinals below $\epsilon$ and $\min \left(\operatorname{Reg}_{\epsilon}\right)$ is strongly inaccessible, then $\operatorname{LST}\left(I, Q^{\delta}\right) \leqslant \operatorname{LST}\left(I, Q^{\epsilon}\right)$.

Proof. If $\delta=0$ then this is trivial, as both $Q^{0}$ and $R^{0}$ are always false. So assume $\delta>0$.
We have just seen that this implies $\operatorname{LST}\left(I, R^{\epsilon}\right) \geqslant$. But $\min \left(\operatorname{Reg}_{\epsilon}\right) \geqslant \epsilon>\delta$, so $\delta<\operatorname{LST}\left(I, R^{\epsilon}\right)$. We saw in Theorem 5.3.1 that this implies $\operatorname{LST}\left(I, R^{\delta}\right) \leqslant \operatorname{LST}\left(I, R^{\epsilon}\right)$.

So we have seen that if $\epsilon$ is not too large, the first element of $\mathrm{Reg}_{\epsilon}$ is a lower bound. We now want to show these lower bounds are optimal. This is the focus of most of the rest of this chapter.

Just before we begin, we shall note in passing that there is also an upper bound for $\operatorname{LST}\left(I, Q^{\epsilon}\right)$ and $\operatorname{LST}\left(I, R^{\epsilon}\right)$ : they can never be above a supercompact. The proof is fairly simple, but introduces several of the techniques we will use in proving the main theorem.

Theorem 5.3.6. Let $\kappa$ be a supercompact, and let $\epsilon<\kappa$. Then $\operatorname{LST}\left(I, Q^{\epsilon}\right)$ exists, and is no larger than $\kappa$. Hence, the same is true of $\operatorname{LST}\left(I, R^{\epsilon}\right)$.

Proof. Let $\mathcal{L}$ be a first order language of cardinality less than $\kappa$, and let $\mathcal{A} \in V$ be an $\mathcal{L}$ structure, of cardinality $\lambda \geqslant \kappa$. Without loss of generality, let us say that the domain of $\mathcal{A}$ is $\lambda$. Without loss of generality, let us further say that $\mathcal{L} \in H_{\kappa}$. We want to show that there is a substructure of cardinality less than $\kappa$.

Let $j: V \rightarrow M$ be an elementary embedding with critical point $\kappa$, such that $j(\kappa)>\lambda$ and $M^{\lambda} \subset M$. (Such an embedding exists by definition of a supercompact). Note that since $\mathcal{L} \in H_{\kappa}, j$ acts as the identity on $\mathcal{L}$. Moreover, $\mathcal{A}$ can be coded as a $\lambda$ sequence. Since $M$ contains all its $\lambda$ sequences, we know that $\mathcal{A} \in M$. Also notice that since $\epsilon<\kappa \leqslant \lambda$ we know that $j\left(\left(Q^{\epsilon}\right)^{V}\right)=\left(Q^{\epsilon}\right)^{M}$. And of course, $j\left(I^{V}\right)=I^{M}$.

Another consequence of $M$ containing all its $\lambda$ sequences is that it correctly calculates the cardinalities and cofinalities of all subsets of $\lambda$ and $\lambda \times \lambda$. In particular, both $I$ and $Q^{\epsilon}$ are evaluated the same way over $\mathcal{A}$ in both $M$ and $V$. It follows that if $\Phi$ is an $\mathcal{L} \cup\left\{I, Q^{\epsilon}\right\}$ formula (with parameters from $\mathcal{A}$ and $\mathcal{L}$ ) then

$$
(\mathcal{A} \vDash \Phi)^{M} \Longleftrightarrow(\mathcal{A} \vDash \Phi)^{V}
$$

By elementarity, we know that

$$
(\mathcal{A} \vDash \Phi)^{V} \Longleftrightarrow(j(\mathcal{A}) \vDash \Phi)^{M}
$$

There is a small technicity which is disguised by this notation. $\Phi$ can have parameters from $\mathcal{A}$ and $\mathcal{L}$, which we have suppressed. For elementarity to hold, the $\Phi$ on the right must take the images of those parameters under $j$. But in fact, since $j$ acts as the identity on $\mathcal{L}$ and on the domain $\lambda$ of $\mathcal{A}$, doing $j$ to those parameters just gives us back the original parameters.

This argument implies that in $M$,

$$
\mathcal{A} \vDash \Phi \Longleftrightarrow j(\mathcal{A}) \vDash \Phi
$$

Hence, from the perspective of $M, \mathcal{A}$ is an $\mathcal{L} \cup\left\{I, Q^{\epsilon}\right\}$ elementary substructure of $j(\mathcal{A})$ of cardinality $\lambda<j(\kappa)$. So $M$ believes: " $j(\mathcal{A})$ contains an $\mathcal{L} \cup\left\{I, Q^{\epsilon}\right\}$ elementary substructure of cardinality less than $j(\kappa)$ ". So by elementarity, $V$ believes: " $\mathcal{A}$ contains an $\mathcal{L} \cup\left\{I, Q^{\epsilon}\right\}$ elementary substructure of cardinality less than $\kappa$." Which is what we wanted to show.

Without further ado, we now state and prove the main result of this section.

### 5.4 The main theorem

Theorem 5.4.1. Let $0<\epsilon \in$ On. If the universe believes $G C H$ and contains a supercompact cardinal larger than $\epsilon$, then there is a generic extension in which $\epsilon$ is countable (and so has no hyperinaccessibles below it) and $\operatorname{LST}\left(I, Q^{\epsilon}\right)$ and $\operatorname{LST}\left(I, R^{\epsilon}\right)$ are both no larger than the first element of $\operatorname{Reg}_{\epsilon}$.

Proof. We adapt a technique from [29]. The first step is to show that without loss of generality, we may assume that $\epsilon$ is countable. Take a universe $V_{0}$ which believes GCH and has a supercompact $\kappa>\epsilon$. Let $V$ be a generic extension of $V_{0}$ by the collapsing forcing $\operatorname{Col}(\omega,|\epsilon|)$. We show that $V$ inherits the properties of $V_{0}$ : $\kappa$ is supercompact, and GCH holds. GCH follows from Lemma 1.2 .7 but the supercompactness of $\kappa$ requires a little work. We need the following lemmas, which is known as the Silver Lifting Criterion. We will prove it for an arbitrary universe $\tilde{V}$ and forcing $\mathbb{P}$, because we'll need it again in a different context later on in the proof.
Lemma 5.4.2. Let $j: \tilde{V} \rightarrow M$ be an elementary embedding from some universe $\tilde{V}$ to a model $M$. Let $\mathbb{P} \in \tilde{V}$ be a forcing. Let $G$ be $\mathbb{P}$ generic, and let $H$ be $j(\mathbb{P})$ generic. Suppose that for all $p \in G, j(p) \in H$. Then there is an elementary embedding $j^{*}: \tilde{V}[G] \rightarrow M[H]$, which extends $j$ and is such that the following diagram commutes.


Proof. We define $j^{*}$ in the only way we can, and verify that the definition works. Let $S \in \tilde{V}[G]$ be a set. Let $\sigma$ be a $\mathbb{P}$ name with $\sigma^{G}=S$. We fix

$$
j^{*}(S)=(j(\sigma))^{H}
$$

We must verify that $j^{*}$ is well defined. Suppose that $\sigma$ and $\tau$ are two $\mathbb{P}$ names, and $\sigma^{G}=\tau^{G}$. Then there is some $p \in G$ such that $p \Vdash \sigma=\tau$. By elementarity $j(p) \Vdash j(\sigma)=j(\tau)$, and by assumption $j(p) \in H$. Hence $(j(\sigma))^{H}=(j(\tau))^{H}$.

It is trivial to verify that $j^{*}$ extends $j$, and that the diagram commutes. The last thing to check is that $j^{*}$ is elementary, and this is done in a similar way to the "well-defined" proof. Let $\varphi$ be a formula with parameters $\vec{S}=\vec{\sigma}^{G}$ and suppose that $\tilde{V}[G] \vDash \varphi(\vec{S})$. Then let $p \in G$ force $\varphi(\vec{\sigma})$; by elementarity $j(p) \Vdash \varphi(j(\vec{\sigma})$ and by assumption $j(p) \in H$. So $M[H] \vDash \varphi\left(j^{*}(\vec{S})\right)$.

If we add some extra assumptions, we can also prove that $j^{*}$ preserves $\lambda$ sequences like a supercompact embedding.
Lemma 5.4.3. Suppose the conditions of the above lemma hold, that $j(\mathbb{P})=\mathbb{P}, H=G$ and that $\mathbb{P}$ satisfies the $\lambda^{+}$chain condition. Then $\tilde{V}[G]$ believes that $M[G]^{\lambda} \subset M[G]$.

In fact, this can be proved in much more general circumstances: rather than assuming that $j(\mathbb{P})=\mathbb{P}$ and $H=G$ we only need that $G \in M[H]$ and $H \in \tilde{V}[G]$. But that's overkill for this proof.

Proof. First, note that since $M$ is definable in $\tilde{V}, M[G]$ is definable in $V[G]$. Let $S=\left(S_{\gamma}\right)_{\gamma<\lambda} \in \tilde{V}[G]$ be a $\lambda$ sequence of elements of $M[G]$. Since $M$ and $G$ are definable in $\tilde{V}[G]$, we can also define a sequence $\dot{S}=\left(\dot{S}_{\gamma}\right)_{\gamma<\lambda} \in \tilde{V}[G]$ of names for the terms of $S$. (Note that $S$ itself is not a name, it's just a sequence of names.) Let $\dot{\Sigma} \in \tilde{V}$ be a $\mathbb{P}$ name for $\dot{S}$. Let $p \in G$ force $\dot{\Sigma}$ to be a $\lambda$ sequence of $\mathbb{P}$ names, each of which is an element of $M$.

For $\gamma<\lambda$, let $A_{\gamma}$ be a maximal antichain of conditions below $p$ which decide which name $\dot{\Sigma}(\gamma)$ is going to be interpreted as. So for any $q \in A_{\gamma}$ there is some name $\sigma_{q, \gamma} \in M$ such that $q \Vdash \dot{\Sigma}(\gamma)=\sigma_{q, \gamma}$. Then $\dot{\Sigma}$ is forced by $p$ to be equal to

$$
\tau:=\left\{\left\langle\left(\sigma_{q, \gamma}, \check{\gamma}\right), q\right\rangle: \gamma<\lambda, q \in A_{\gamma}\right\}
$$

By the chain condition, $\left|A_{\gamma}\right| \leqslant \lambda$ for all $\gamma<\lambda$, and hence $\tau$ has cardinality $\leqslant \lambda$. All the elements of $\tau$ are in $M, \tau \in \tilde{V}$ and $M$ is closed under $\lambda$ sequences, so $\tau \in M$. So $\dot{S}=\tau^{G} \in M[G]$. The conclusion that $S \in M[G]$ is now immediate.

Corollary 5.4.4. $\kappa$ is supercompact in $V$.

Proof. Let $\lambda>\kappa$. Let $j: V_{0} \rightarrow M$ be a $\lambda$ embedding: i.e. elementary, with critical point $\kappa, j(\kappa)>\lambda$ and $M^{\lambda} \subset M$. Let $G$ be the $\mathbb{P}:=\operatorname{Col}(\omega,|\epsilon|)$ generic filter used in constructing $V$ (so $V=V_{0}[G]$ ). Note that $\mathbb{P}$ is small compared to $\kappa$, so $j(\mathbb{P})=\mathbb{P}$ and $G$ is also generic over $M$. Clearly for all $p \in G, j(p)=p \in G$. So $j$ extends to an elementary embedding $j^{*}: V_{0}[G] \rightarrow M[G]$ by the first lemma. Since $j^{*}$ extends $j$ it has critical point $\kappa$ and sends $\kappa$ up above $\lambda$. Since $\mathbb{P}$ is small compared to $\kappa$ it certainly satisfies the $\lambda^{+}$chain condition, and therefore by the second lemma $V=V_{0}[G]$ believes that $M[G]^{\lambda} \subset M[G]$.

From now on, we will forget about $V_{0}$ and the collapsing forcing, and just work in $V$, a universe where $\epsilon$ is countable, GCH holds and $\kappa$ is supercompact, and where $\operatorname{Reg}_{<\epsilon}$ is unbounded. Note that since $\kappa$ is supercompact it is hyperinaccessible, and therefore $\operatorname{Reg}_{<\epsilon}$ is also unbounded below $\kappa$. We will also assume (by cutting off the top of the model if necessary) that $\operatorname{Reg}_{\epsilon} \backslash \kappa^{+}=\varnothing$. Note that $\kappa$ will be an inaccessible limit of elements of $\operatorname{Reg}_{<\epsilon}$ (so it is in $\operatorname{Reg}_{\delta}$ for some $\delta>\epsilon$ ), and by the assumption we just made it is the largest element of this $\operatorname{Reg}_{\delta}$.

### 5.4.1 The Structure of the Proof

We will construct a forcing which will make $\kappa$ the first element of $\operatorname{Reg}_{\epsilon}$ and ensure $\operatorname{LST}\left(I, Q^{\epsilon}\right), \operatorname{LST}\left(I, R^{\epsilon}\right) \leqslant \kappa$. The actual forcing we want to use is complex to define - it's a delicate combination of several other forcings, some of which are rather complicated in their own right. We'll start by giving an informal sketch of how the proof is going to work, before we dive into the formalities.

First, we will singularise all the cardinals below $\kappa$ which are dangerously close to being supercompact. To be precise, we singularise any $\lambda<\kappa$ which is $\lambda^{+}$supercompact. Any such cardinal will be measurable, so we could just use Prikry forcing to do this; but for reasons which will shortly be explained we actually need a more complicated forcing we call $\mathbb{Q}_{\lambda}$, which combines elements of both Prikry forcing and some other forcings.

After we have done this, we next force $\kappa$ to be non-Mahlo. We knew that we don't want $\kappa$ to be Mahlo in the final model, since the set of all limits of $\operatorname{Reg}_{<\epsilon}$ below $\kappa$ should end up being a club of singular cardinals. The non-Mahlo forcing $\mathrm{NM}_{\kappa}$ we use gives us a club $C$ of $\kappa$ whose limits are all singular, and whose successors are all elements of $\mathrm{Reg}_{\epsilon}$. We will also somehow contrive $C$ to be such that for "a short distance" above any of its elements, we can find many elements of $\operatorname{Reg}_{<\epsilon}$ but no elements of $\operatorname{Reg}_{\epsilon}$. Of course, once we begin the actual proof we'll formalise what we mean by "a short distance", and explain what forcing we use to make these things happen. For now, it's enough to know that given any cardinal $\mu>\kappa$, there is some embedding $j: V \rightarrow M$ (with critical point $\kappa$ ) such that $\mu$ is considered "a short distance" above $\kappa$ in $M$.

Finally, we collapse every cardinal below $\kappa$ that is not a "short distance" above some element of $C$. We do this using an Easton product $\operatorname{Col}(C)$ of collapsing forcings.

The overall forcing $\mathbb{P}$ that we use is the usual $*$ iteration of these forcings: first we do $\mathbb{Q}_{\lambda}$ for every $\lambda^{+}$ supercompact $\lambda<\kappa$, then we do the non-Mahlo forcing $\mathrm{NM}_{\kappa}$, and finally the collapsing forcing $\mathrm{Col}(C)$.

This gives us a universe $V^{\mathbb{P}}$ in which, below $\kappa$, there are unboundedly many elements of Reg $\operatorname{Re}$, but no elements of $\operatorname{Reg}_{\epsilon}$. Meanwhile, $\kappa$ is still inaccessible. This immediately tells us it is the first element of $\operatorname{Reg}_{\epsilon}$. And of course $\epsilon$ is countable in $V^{\mathbb{P}}$.

How do we show that $\operatorname{LST}\left(I, Q^{\epsilon}\right), \operatorname{LST}\left(I, R^{\epsilon}\right) \leqslant \kappa$ ? We already know that $\operatorname{LST}\left(I, R^{\epsilon}\right) \leqslant \operatorname{LST}\left(I, Q^{\epsilon}\right)$ from the previous lemmas; we need to show that given any language $\mathcal{L} \in V^{\mathbb{P}}$ (of cardinality $<\kappa$ ) and any structure $\mathcal{A} \in V^{\mathbb{P}}$, we can find a substructure of cardinality less than $\kappa$ which is $\mathcal{L} \cup\left(I, Q^{\epsilon}\right)$-elementarily equivalent to $\mathcal{A}$. We use a similar approach to in Theorem 5.3.6.

Say that $\mathcal{A}$ has cardinality $\mu>\kappa$. Because $\kappa$ is supercompact in $V$, we can find an embedding $j: V \rightarrow M$ such that $j(\kappa)>\mu$ and such that $\mu$ is only a "short distance" above $\kappa$ (in the sense we were discussing a moment ago). With a good deal of effort, we can show that $\mathbb{P}$ embeds nicely into $j(\mathbb{P})$, and that the conditions of Lemma 5.4 .2 and Lemma ?? hold for some suitable $j(\mathbb{P})$ generic extension of $M$. For this to be happen, we must be able to embed the non-Mahlo forcing $\mathrm{NM}_{\kappa}$ on $\kappa$ into $\mathbb{Q}_{\kappa}$, and we must also be able to make the collapsing forcing components of $\mathbb{P}$ and $j(\mathbb{P})$ interact nicely with each other. This is why we can't just use a Prikry forcing for $\mathbb{Q}_{\lambda}$ : it must also somehow contain $\mathrm{NM}_{\lambda}$ and decide some things about the collapsing forcing. The Lemmas tell us that $j: V \rightarrow M$ extends naturally to an embedding $j^{*}: V^{\mathbb{P}} \rightarrow M^{j(\mathbb{P})}$, and that $M^{j(\mathbb{P})}$ is closed under $\lambda$ sequences from the perspective of $V^{\mathbb{P}}$.

Now, in $M^{j(\mathbb{P})}$, we know that no cardinals between $\kappa$ and $\mu$ are collapsed or singularised, because $\mu$ is only a "short distance" above $\kappa$. Because we made the collapsing forcings somehow interact nicely with
each other, we can also contrive for the club $C$ in $V^{\mathbb{P}}$ added by $\mathrm{NM}_{\kappa}$ to be contained in the equivalent club in $M^{j(\mathbb{P})}$ added by $\mathrm{NM}_{j(\kappa)}$. So $\left(\operatorname{Card} \cap \mu^{+}\right)$, $\left(\operatorname{Reg}_{0} \cap \mu^{+}\right), \ldots,\left(\operatorname{Reg}_{\delta} \cap \mu^{+}\right), \ldots(\delta<\epsilon)$ are each unchanged between $M^{j(\mathbb{P})}$ and $V^{\mathbb{P}}$. Hence, $\mathcal{L} \cup\left\{I, Q^{\epsilon}\right\}$ interprets all statements about $\mathcal{A}$ in the same way in $V^{\mathbb{P}}$ and $M^{j(\mathbb{P})}$.

The proof now follows Theorem 5.3.6. Without loss of generality, we can easily arrange that $\mathcal{L}, \mathcal{A} \in M^{j(\mathbb{P})}$ and that $\mathcal{A}$ is a substructure of $j^{*}(\mathcal{A})$. And since $j^{*}: V^{\mathbb{P}} \rightarrow M^{j(\mathbb{P})}$ is elementary, an $\mathcal{L} \cup\left\{I, Q^{\epsilon}\right\}$ sentence will be true about $j^{*}(\mathcal{A})$ in $M^{j(\mathbb{P})}$ if and only if it's true about $\mathcal{A}$ in $V^{\mathbb{P}}$, and thus if and only if it's true about $\mathcal{A}$ in $M^{j(\mathbb{P})}$ too. So in $M^{j(\mathbb{P})}, \mathcal{A}$ is an $\mathcal{L} \cup\left\{I, Q^{\epsilon}\right\}$-elementary substructure of $j^{*}(\mathcal{A})$.

So $M^{j(\mathbb{P})}$ believes the statement " $j^{*}(\mathcal{A})$ contains an $\mathcal{L}\left\{I, Q^{\epsilon}\right\}$-elementary substructure of cardinality less than $j^{*}(\kappa)$." Pulling back to $V^{\mathbb{P}}$ using elementarity of $j^{*}$, we see that $\mathcal{A}$ contains an $\mathcal{L}\left\{I, Q^{\epsilon}\right\}$-elementary substructure of cardinality less than $\kappa$, as required.

So in summary, the proof will consist of the following steps:

1. Define a function $f$ that formalises the concept of "a short distance"
2. Define the non-Mahlo and collapsing forcings $\mathrm{NM}_{\lambda}$ and Col we discussed above
3. Define the forcing $\mathbb{Q}_{\lambda}$, which singularises $\lambda$ and somehow contains $\mathrm{NM}_{\lambda}$ and some information about Col
4. Show that $\mathrm{NM}_{\lambda} * \mathrm{Col}$ embeds nicely in $\mathbb{Q}_{\lambda} * \mathrm{Col}$
5. Put these forcings together in some way to get the forcing $\mathbb{P}$ we will be using
6. Show that in a $\mathbb{P}$ generic extension, the LST is number at most $\kappa$

### 5.4.2 Defining $f$

Recall Lemma 3.1.18, for any supercompact cardinal $\kappa$ :
Lemma. There is a function $h: \kappa \rightarrow V_{\kappa}$ such that given any $x \in V$ and any $\mu \geqslant \kappa$, there is an $M$ with $M^{\mu} \subset M$ and an embedding $j: V \rightarrow M$ with critical point $\kappa$, such that $j(\kappa)>\mu$ and $j(h)(\kappa)=x$.

We use this $h$ to define a related function $f$ specifically for the ordinals.
Lemma 5.4.5. There is a strictly increasing function $f: \kappa \rightarrow \kappa$ such that:

1. For all $\alpha<\kappa$, the interval $(\alpha, f(\alpha))$ contains unboundedly many elements of $\operatorname{Reg}_{\delta}$ for every $\delta<\epsilon$, but no elements of $\operatorname{Reg}_{\epsilon}$
2. For any $\mu>\kappa$, there is a model $M$ with $M^{\mu} \subset M$ and an elementary embedding $j: V \rightarrow M$ with critical point $\kappa$ such that $j(\kappa)>\mu$ and $j(f)(\kappa)>\mu$.

Proof. For $\alpha<\kappa$, let $g(\alpha):=h(\alpha)$ if the latter is an ordinal greater than $\alpha$ but smaller than $\kappa$, and $(\alpha, g(\alpha)) \cap \operatorname{Reg}_{\epsilon}=\varnothing$, and otherwise let $g(\alpha):=\alpha$. Let $f(\alpha)$ be the least element of $\operatorname{Reg}_{\epsilon}$ which is (strictly) above $g(\alpha)$. It is trivial to see that $f$ is strictly increasing and satisfies the first requirement.

Fix $\mu>\kappa$. Using the previous result, let $M$ and $J: V \rightarrow M$ be such that $j(h)(\kappa)=\mu$. We know that $M$ is closed under $\mu$ sequences, so it correctly calculates the cardinalities and cofinalities of all cardinals below $\mu$. In particular, $\operatorname{Reg}_{\epsilon}^{M}$ agrees with $\operatorname{Reg}_{\epsilon}^{V}$ up to $\mu$, and hence $(\kappa, \mu) \cap \operatorname{Reg}_{\epsilon}=\varnothing$ from the perspective of $M$. Hence $j(g)(\kappa)=j(h)(\kappa)=\mu$. So $j(f)(\kappa)>\mu$.

In the language we used in the preamble to this proof, $\beta$ is "a short distance" above $\alpha$ if $\beta \in(\alpha, f(\alpha))$.

### 5.4.3 NM and Col

Now we have $f$, we can define the simpler forcings used in this construction, $\mathrm{NM}_{\lambda}$ and Col. Throughout this section, let $\tilde{V}$ be any universe which contains the $f$ we found in the previous section. We will also assume that $f$ has the same relation to $\operatorname{Reg}$ in $\tilde{V}$ as it does in $V$ (although we do not assume that $\operatorname{Reg}_{\delta}^{\tilde{V}}=\operatorname{Reg}_{\delta}^{V}$ for any particular $\delta \leqslant \epsilon$ ), and (in preparation for the future) we assume that GCH holds in $\tilde{V}$ except that there are perhaps a few singular cardinals $\lambda$ such that $2^{\lambda}=\lambda^{++}$. Let us fix $\lambda \leqslant \kappa$ to be a $\lambda^{+}$supercompact cardinal in $\tilde{V}$.

A Mahlo cardinal is one for which the class of regulars below it is stationary, so to stop it being Mahlo we need to add a club which consists only of singular cardinals. We use a variant of the usual club shooting forcing which also chooses some elements of $\mathrm{Reg}_{\epsilon}$.

Definition 5.4.6. We define the non-Mahlo forcing $\mathrm{NM}_{\lambda}$ as follows. Its elements are closed bounded subsets of $\lambda$, whose minimum element is $\omega$, whose successors are all taken from $\operatorname{Reg}_{\epsilon}$, and whose limits are all singular. We order $\mathrm{NM}_{\lambda}$ by end inclusion.

Note that this is not quite the usual non-Mahlo forcing: it gives us a club whose limits are singular, but whose successors are elements of $\operatorname{Reg}_{\epsilon}$. (Since $\operatorname{Reg}_{\epsilon}$ is unbounded below $\lambda$, we do get an unbounded subset of $\lambda$ despite the odd requirement for successors.) By contrast, a standard non-Mahlo forcing would just add a club of singular cardinals. Of course, we can obtain a fully singular club here simply by deleting all the successors, so $\mathrm{NM}_{\lambda}$ does indeed force that $\lambda$ is no longer Mahlo.

Although $\mathrm{NM}_{\lambda}$ is not strictly $<\lambda$-closed (since the limit of a sequence of conditions could be inaccessible) it is almost $<\lambda$-closed, and we can prove the usual results of $<\lambda$-closed-ness.

Lemma 5.4.7. Let $\mu<\lambda$. Then there are densely many conditions $p$ such that $\mathrm{NM}_{\lambda} \uparrow p$ is $\mu$-closed.
Proof. Let $p \in \mathrm{NM}_{\lambda}$ be any condition whose final element is larger than $\mu$. Let $p \geqslant p_{0} \geqslant p_{1} \geqslant \ldots$ be a descending sequence of conditions of length $\mu$. Then $p_{0} \subset p_{1} \subset \ldots$ are all end extensions of one another, so $\bigcup_{i<\mu} p_{i}$ is a set whose successors are all in $\operatorname{Reg}_{\epsilon}$ and whose limit elements are all singular. It is closed, except that it might not contain its supremum $\alpha$. But $\alpha \geqslant \sup p>\mu$ has cofinality $\operatorname{Cof}(\mu)$, so it is singular. This implies that $\alpha<\lambda$, and that $\bigcup_{i<\mu} p_{i} \cup\{\alpha\} \in \mathrm{NM}_{\lambda}$.

Corollary 5.4.8. $\mathrm{NM}_{\lambda}$ is $<\lambda$ distributive, does not collapse or singularise any cardinals, and preserves GCH, except perhaps that $2^{\lambda}$ becomes $\lambda^{++}$.

Proof. Let $\mu<\lambda$. Let $D_{i}: i<\mu$ be a collection of dense sets. Fix some $p$ such that $\mathrm{NM}_{\lambda} \upharpoonleft p$ is $\mu$-closed. For $i<\mu$ let $D_{i}^{\prime}=D_{i} \cap\left(\mathrm{NM}_{\lambda} \upharpoonleft p\right)$. Then $D_{i}^{\prime}$ is a dense subset of $\mathrm{NM}_{\lambda} \upharpoonleft p$. We know that $\mathrm{NM}_{\lambda} \upharpoonleft p$ is $\mu$ distributive since it is $\mu$-closed, and hence $\varnothing \neq \bigcap D_{i}^{\prime} \subset \bigcap D_{i}$.

It follows immediately from Theorem 1.2.6 that $\mathrm{NM}_{\lambda}$ preserves all cardinals $<\lambda$ and does not singularise any of them, and from Lemma 1.2.7 that it preserves GCH below $\lambda$.

Showing that $\lambda$ is not collapsed or singularised requires a slightly more technical argument, where we essentially mimic the proof of Theorem 1.2.6. Suppose that $\tilde{V}[G]$ collapses $\lambda$. Let $g: \mu \rightarrow \lambda$ be a bijection for some $\mu<\lambda$, and let $\dot{g}$ be a name for $g$. Let $p \in G$ force " $g$ is a bijection from $\mu$ to $\lambda$ ". Let $q \leqslant p$ be such that $\mathrm{NM}_{\lambda} 1 q$ is $\mu$-closed. We construct a descending sequence of conditions $q=q_{0} \geqslant q_{1} \geqslant \ldots$ (not necessarily elements of $G$ ) of length $\mu+1$ as follows. If $i=j+1$ then we choose $q_{i} \leqslant q_{j}$ which decides the value of $\dot{g}(j)$. If $i$ is a limit, then by $\mu$-closed-ness, we can choose some $q_{i}$ below every earlier $q_{j}$. At the end, $q_{\mu}$ has decided $\dot{g}(i)$ for every $i$, and hence has defined a bijection from $\mu$ to $\lambda$ in $\tilde{V}$. Contradiction. A similar proof shows that $\lambda$ is not singularised.

Finally, note that $\mathrm{NM}_{\lambda} \subset \lambda^{<\lambda}$, so

$$
\left|\mathrm{NM}_{\lambda}\right| \leqslant \sum_{\alpha<\lambda} \lambda^{\alpha}=\sum_{\alpha<\lambda} \lambda=\lambda
$$

Hence $\mathrm{NM}_{\lambda}$ has the $\lambda^{+}$chain condition, and so does not collapse or singularise any cardinals $\geqslant \lambda^{+}$. By Lemma 1.2.7 it also preserves GCH for cardinals above $\lambda$.

Finally, then, in the generic extension $2^{\lambda} \leqslant 2^{\lambda^{+}}=\lambda^{++}$.

Throughout this proof, we shall write $C$, and variants thereof, to refer to the generic club added by $\mathrm{NM}_{\lambda}$. It should (hopefully) always be clear from context to which $\lambda$ we are referring. Notice that for any $\alpha \in C$, we know $\operatorname{Succ}(\alpha) \geqslant f(\alpha)$, because there are no $\operatorname{Reg}_{\epsilon}$ 's between $\alpha$ and $f(\alpha)$.

Next, we define the collapsing forcing. Recall Definition 1.3.15, where we defined the forcing $\operatorname{Col}(\alpha,<\beta)$ collapsing all the cardinals below $\beta$ down to $\alpha$.

For the rest of this section, let us fix a club $C$ which is generic for ${\underset{\tilde{V}}{\lambda}}^{\mathrm{NM}_{\lambda}}$. We will work in some generic extension $\tilde{V}[G]$ of $\tilde{V}$ containing $C$, but we do not assume that $\tilde{V}[G]=\tilde{V}[C]$. We will, however, assume that $[G]$ does not collapse or singularise any cardinals below $\lambda$, and preserves GCH where it holds, except perhaps at $\lambda$.

The forcing we want to work with is a combination of these collapsing forcings, and is defined in terms of $C$. We want a forcing which will collapse, for each $\alpha \in C$, every cardinal between $f(\alpha)$ and $\operatorname{Succ}_{C}(\alpha)$. Once we have done this, it will be easy to see that it makes $\lambda$ the first element of $\operatorname{Reg}_{\epsilon}$. The forcing we want to use is the Easton product of all the collapsing forcings:
Definition 5.4.9. Let $\tilde{V}[G]$ be some generic extension of $\tilde{V}$, which contains an $\mathrm{NM}_{\lambda}^{\tilde{V}}$ generic club $C$ and is such that all the successors of $C$ are strongly inaccessible in $\tilde{V}[G]$. We define the forcing $\operatorname{Col}(C)$ to be the set of all the elements $h$ of

$$
\prod_{\alpha \in C} \operatorname{Col}\left(f(\alpha),<\operatorname{Succ}_{C}(\alpha)\right)
$$

such that for all regular cardinals $\mu \leqslant \lambda$, the set

$$
\{\alpha \in C \cap \mu: h(\alpha) \neq \varnothing\}
$$

is bounded in $\mu$.
We order $\operatorname{Col}(C)$ in the obvious way.
More generally, we define $\operatorname{Col}(c)$ in the same way for any closed set $c$ of cardinals whose successors are all strong inaccessibles below $\kappa$. In particular, we can define $\operatorname{Col}(c)$ for any $c \in \mathrm{NM}_{\lambda}$. Notice that in this case $c \in \tilde{V}$, so $\operatorname{Col}(c)$ is actually defined in $\tilde{V}$. Unless otherwise specified, when we write $\operatorname{Col}(c)$ for $c \in \tilde{V}$ we shall always mean $\mathrm{Col}^{\tilde{V}}(c)$. Also, if $\tilde{V}[G]$ is some generic extension of $\tilde{V}$ (with respect to any forcing) and if $C \in \tilde{V}[G]$ is an $\mathrm{NM}_{\lambda}$ generic club containing $c$, then with a minor abuse of notation we can say $\operatorname{Col}^{\tilde{V}}(c) \subset \operatorname{Col}^{\tilde{V}[G]}(C)$. (The abuse is that an element of $\operatorname{Col}(c)$ is technically a function with domain $c \backslash\{\max (c)\}$, and we are identifying it with a function whose domain is $C$ but which is trivial above $c$ ).

We can think of $\operatorname{Col}(c)$ as the set of all conditions that $\tilde{V}$ knows will be in $\operatorname{Col}(C)$ if $c$ is an initial segment of $C$.

Note that $\operatorname{Col}(C)$ depends not just on $C$ but also on which cardinals are regular. Thus, two different universes may have different opinions on what $\operatorname{Col}(C)$ should look like, even if both universes contain the same club $C$.
Proposition 5.4.10. Let $c$ be a closed set of cardinals (in $\tilde{V}$ or $\tilde{V}[G]$ ) whose successors are all inaccessibles below $\kappa$. If $\alpha=\min (c)$ then $\operatorname{Col}(c)$ is $<f(\alpha)$-closed. In particular, this means it is $\alpha^{+}$-closed.

Proof. An easy definition chase we leave to the reader.
Lemma 5.4.11. Let $C \in \tilde{V}[G]$. Then $\operatorname{Col}(C)$, defined over $\tilde{V}[G]$, does not collapse any cardinals except those which it is supposed to (i.e. those in the interval $\left(f(\alpha), \operatorname{Succ}_{C}(\alpha)\right)$ for some $\left.\alpha \in C\right)$. Nor does it singularise any other cardinals.

Proof. Let $\mu$ be a cardinal which is not in any of the intervals that are supposed to be collapsed by $\operatorname{Col}(C)$. Let $\mathbb{P}=\operatorname{Col}(C \upharpoonleft \mu)$ (in the sense of $\tilde{V}[C]$, not $\tilde{V})$ and let $\mathbb{R}=\operatorname{Col}(C \backslash \mu)$. Then $\operatorname{Col}(C)=\mathbb{P} \times \mathbb{R}$. Now $\mathbb{P}$ has cardinality less than $\mu$ (by GCH) and therefore does not collapse or singularise $\mu$. On the other hand, $\mathbb{R}$ is $\mu$-closed, and so again it does not collapse or singularise $\mu$.
Corollary 5.4.12. Let $C \in \tilde{V}[G]$ as above. If $0<\delta<\epsilon$ then $\operatorname{Col}(C)$ does not modify $\operatorname{Reg}_{\delta}$, other than removing those elements which are in an interval $\left(f(\alpha), \operatorname{Succ}_{C}(\alpha)\right)$.

Proof. Let $\mu \in \operatorname{Card}$ not be in $\left(f(\alpha), \operatorname{Succ}_{C}(\alpha)\right)$ for any $\alpha \in C$. We show that $\mu \in \operatorname{Reg}_{\delta}$ in the $\operatorname{Col}(C)$ generic extension if and only if $\mu \in \operatorname{Reg}_{\delta}$ in $\tilde{V}[G]$.

Case 1: $\mu<\min (C), \mu \in(\alpha, f(\alpha)]$ or $\mu>\lambda$. All the cardinals in these intervals are preserved, and not singularised, by $\operatorname{Col}(C) . \operatorname{So~} \operatorname{Reg}_{\delta}$ is preserved over these intervals too.

Case 2: $\mu \in C$ is a successor of $C$. By definition of $\mathrm{NM}_{\lambda}$, we know $\mu$ is an element of $\operatorname{Reg}_{\epsilon}$ in $\tilde{V}[G]$, and hence not in $\operatorname{Reg}_{\delta}$. In the generic extension $\mu$ becomes a successor cardinal, and hence again not in $\operatorname{Reg}_{\delta}$.

Case 3: $\mu \in C$ is a limit of $C$. By definition of $\mathrm{NM}_{\lambda}$, we know $\mu$ is a successor cardinal in $\tilde{V}[G]$ and hence also in the $\operatorname{Col}(C)$ generic extension.

Case 4: $\mu=\lambda$. In $\tilde{V}$, we know that $\lambda>\delta$ is hyperinaccessible, and therefore a limit of elements of $\operatorname{Reg}_{\delta}$. By assumption, $\tilde{V}[G]$ agrees with $\tilde{V}$ on $\operatorname{Reg} 1 \lambda$, so in particular in $\tilde{V}[G]$ we know that $\lambda$ is still a limit of $\operatorname{Reg}_{\delta}$. So it cannot be an element of $\operatorname{Reg}_{\delta}$. Recall that every interval ( $\left.\alpha, f(\alpha)\right]$ contains an element of $\operatorname{Reg}_{\delta}$. So by Case 1, we know that in the generic extension $\lambda$ is still a limit of elements of $\mathrm{Reg}_{\delta}$, and hence cannot be in $\operatorname{Reg}_{\delta}$.

Notice also that in the $\operatorname{Col}(C)$ generic extension, there are no elements of $\operatorname{Reg}_{\epsilon}$ below $\lambda$, and that $\lambda$ is a limit of elements of $\operatorname{Reg}_{\delta}$ for all $\delta<\epsilon$. So if $\lambda$ is regular in $\tilde{V}[G]$ (and hence in the $\operatorname{Col}(C)$ generic extension) then it will be the first element of $\operatorname{Reg}_{\epsilon}$.

### 5.4.4 The Prikry-style forcing $\mathbb{Q}$

Again, we fix a $\lambda^{+}$supercompact $\lambda \leqslant \kappa$ in a universe $\tilde{V}$ which knows about $f$. Assume also that $\operatorname{Col}(C)$ is well defined for every $\mathrm{NM}_{\lambda}^{\tilde{V}}$ generic club $C$.

The forcing $\mathbb{Q}_{\lambda}$ is similar to a Prikry forcing, but with some extra components. We want any $\mathbb{Q}_{\lambda}$ generic filter to not only singularise $\lambda$, but also to define a $\mathrm{NM}_{\lambda}$ generic club $C$. For reasons that will become clearer later, we also want it to define some kind of "generic element" of $\operatorname{Col}(C)$.

Where a standard Prikry forcing would add an $\omega$ sequence of single ordinals, we will arrange for $\mathbb{Q}_{\lambda}$ to add an $\omega$ sequence whose terms are taken from the following set:
Definition 5.4.13. $K_{\lambda}$ is the set of all triples $\langle c, x, \gamma\rangle$ where:

1. $c \in \mathrm{NM}_{\lambda}$;
2. $h \in \operatorname{Col}(c)$;
3. $\gamma \in$ On and $\max (c)<\gamma<\lambda$.

For $\delta<\lambda, K_{\lambda}^{\delta}$ is the subset of $K_{\lambda}$ consisting of all the triples $\langle c, h, \gamma\rangle$ where the least element of $c$ is greater than $\delta$ (and thus also $\gamma>\delta$ ).

The Prikry-style generic sequence we're aiming for will consist of an element $\left\langle c_{0}, h_{0}, \gamma_{0}\right\rangle$ of $K_{\lambda}$, then an element $\left\langle c_{1}, h_{1}, \gamma_{1}\right\rangle$ of $K_{\lambda}^{\gamma_{0}}$, and so on up through the all $n \in \omega$.

From such a sequence $G$, we will be able to extract a club $C=\bigcup_{n \in \omega} c_{n}$ in $\lambda$, an element $H=\bigcup_{n \in \omega} h_{n}$ of $\prod_{\alpha \in C} \operatorname{Col}\left(\alpha,<\operatorname{Succ}_{C}(\alpha)\right)$ and an $\omega$-sequence $\left(\gamma_{n}\right)$. If we set up the forcing correctly, we will later discover that $C$ is $\mathrm{NM}_{\lambda}$ generic, that $H \in \mathrm{Col}{ }^{\tilde{V}[G]}(C)$, and that $\left(\gamma_{n}\right)$ is cofinal in $\lambda$.

Recall that a condition in a Prikry forcing consists of two components: a finite stem of the sequence we're constructing, and a "large" (i.e. measure 1) set of places that are allowed to be in later parts of the sequence.

It's fairly easy to see what the finite stem should look like in this context. But how can we find an analogue of a measure 1 set of ordinals? We must define a measure on $K_{\lambda}$, and in fact on $K_{\lambda}^{\delta}$ for every $\delta<\lambda$.

To do this, we first define an ordering on $K_{\lambda}$ :
Definition 5.4.14. Let $\langle c, h, \gamma\rangle$ and $\langle\tilde{c}, \tilde{h}, \tilde{\gamma}\rangle$ be elements of $K_{\lambda}$. We say that $\langle\tilde{c}, \tilde{h}, \tilde{\gamma}\rangle \leqslant *\langle c, h, \gamma\rangle$ if:

1. $\tilde{c} \leqslant c$, that is, $\tilde{c}$ is an end extension of $c$;
2. $\tilde{h} \upharpoonleft \max (c)=h$;
3. $\tilde{\gamma} \geqslant \gamma$.

Notice the unusual nature of the second clause. It's not enough that $\tilde{h} \leqslant h$ in the $\operatorname{Col}(\tilde{c})$ ordering. It must actually agree completely with $h$, although it is allowed to add more things once we're above the area where $h$ is defined. This is because $H=\bigcup h_{n}$ is supposed to be a condition in $\operatorname{Col}(C)$, not a $\operatorname{Col}(C)$ generic filter.

Of course, this also defines an ordering on $K_{\lambda}^{\delta} \subset K_{\lambda}$ for every $\delta<\lambda$.
Lemma 5.4.15. Let $\delta<\lambda$. Let $F$ be the family of all subsets of $K_{\lambda}^{\delta}$ which contain $a \leqslant *$ dense open subset of $K_{\lambda}^{\delta}$. Then $F$ is a $\lambda$ complete filter in $K_{\lambda}^{\delta}$ in the usual $\subset$ ordering of $\mathcal{P}\left(K_{\lambda}^{\delta}\right)$.

Proof. Clearly, $F$ is upwards closed and nonempty. The intersection of fewer than $\lambda$ many $\leqslant *$ dense open sets can be easily seen to be dense open, so $F$ is a $\lambda$ complete filter.

The following is a standard consequence of $\lambda^{+}$supercompactness:
Lemma 5.4.16. [24, 22.17] Let $S$ be a set of size $\lambda$, and let $F$ be a $\lambda$ complete filter on $\mathcal{P}(S)$. Then $F$ can be extended to a $\lambda$ complete ultrafilter $U$.

Proof. Without loss of generality, let us assume $S=\lambda$. Note that $|F| \leqslant 2^{\lambda}=\lambda^{+}$. Let $j: \tilde{V} \rightarrow M$ be an embedding with critical point $\lambda$ such that $j(\lambda)>\lambda^{+}$and $M^{\lambda^{+}} \subset M$. By elementarity $j(F) \in M$ is $j(\lambda)$ complete. Since $\left|j^{\prime \prime}(F)\right| \leqslant \lambda^{+}$(in $\tilde{V}$ ) and $M$ is closed under $\lambda^{+}$sequences, we know that $j^{\prime \prime}(F) \in M$. Since also $\lambda^{+} \leqslant j(\lambda)$ and $j^{\prime \prime}(F) \subset j(F)$, we know by $j(\lambda)$ completeness of $j(F)$ that $\cap j^{\prime \prime}(F) \in j(F)$. In particular, there is some $\alpha \in \cap j^{\prime \prime}(F)$. Now define an ultrafilter $U \in \tilde{V}$ by:

$$
X \in U \Longleftrightarrow X \subset \lambda \wedge \alpha \in j(X)
$$

It is easy to check that $F \subset U$, that $U$ is an ultrafilter, and that it is $\lambda$ complete.
Corollary 5.4.17. Let $\delta<\lambda$. There is an ultrafilter $U_{\delta}$ on $K_{\lambda}^{\delta}$ which contains all the $\leqslant^{*}$ dense open subsets of $K_{\lambda}^{\delta}$.

In fact, of course, there will be many such ultrafilters, but we will fix a single one for the rest of this section to call $U_{\delta}$.

With $U_{\delta}$ in hand, we can define the analogue of the measure 1 set of ordinals in a Prikry forcing.
Definition 5.4.18. Let $T$ be a tree of height $\omega$, whose nodes are all elements of $K_{\lambda}$. We abuse notation by allowing the same element of $K_{\lambda}$ to appear multiple times, provided no element appears twice as direct successors of the same node. We say $T$ is nice if the following hold:

1. If $\langle\tilde{c}, \tilde{h}, \tilde{\gamma}\rangle{ }_{T}\langle c, h, \gamma\rangle \in T$, then $\langle\tilde{c}, \tilde{h}, \tilde{\gamma}\rangle \in K_{\lambda}^{\gamma}$;
2. If $\langle c, h, \gamma\rangle \in T$ then the set of its direct successors (which is a subset of $K_{\lambda}^{\gamma}$ by the previous condition) is in $U_{\gamma}$.

Recall that a condition in a Prikry forcing contains two components: a finite sequence of ordinals, and a measure 1 set. The finite sequence fixes an initial segment of the $\omega$ sequence we are going to add, and the rest of the sequence is chosen from elements of the measure 1 set. Analogously, in the forcing $\mathbb{Q}_{\lambda}$, our conditions will have two components: a finite sequence $s$ of terms from $K_{\lambda}$ and a nice tree $T$ which gives us a map of where the sequence is allowed to go from there. The root of $T$ will the final term of $s$. Then for $n>0$, the $n$ 'th level contains all the elements of $K_{\lambda}$ which we are allowing to appear as the $n$ 'th term in the undetermined part of the sequence. A branch through the tree corresponds to a (not necessarily generic) way to complete the sequence.
Definition 5.4.19. The forcing $\mathbb{Q}_{\lambda}$ has conditions of the form

$$
\left(\left(\left\langle c_{0}, h_{0}, \gamma_{0}\right\rangle,\left\langle c_{1}, h_{1}, \gamma_{1}\right\rangle, \ldots,\left\langle c_{n-1}, h_{n-1}, \gamma_{n-1}\right\rangle\right), T\right)
$$

for some $n \in \omega$, where

1. $\left\langle c_{0}, h_{0}, \gamma_{0}\right\rangle \in K_{\lambda}$ if $n>0$;
2. For $0<i<n,\left\langle c_{i}, h_{i}, \gamma_{i}\right\rangle \in K_{\lambda}^{\gamma_{i-1}}$;
3. $T$ is a nice tree
4. The root of $T$ is $\left\langle c_{n-1}, h_{n-1}, \gamma_{n-1}\right\rangle$.

We call $\gamma_{n-1}$ the height of the condition (writing ht in symbols).
The conditions are ordered in the usual way for a Prikry style forcing: $\left(s^{\prime}, T^{\prime}\right) \leqslant(s, T)$ if $s^{\prime}$ is an end extension of $s$ and there is a path $B=b_{0}, b_{1}, \ldots, b_{k}$ through $T$ of length $k:=\left|s^{\prime} \backslash s\right|+1$ such that:

1. $b_{0}$ is the root of $T$ (i.e. the last element of $s$ )
2. For all $0<i \leqslant k, b_{i}$ is the $i$ 'th element of $s^{\prime} \backslash s$
3. $T^{\prime}$ is a subtree of $T$ whose root is $b_{k}$

As usual for Prikry forcings, if $s^{\prime}=s$ we say $\left(s^{\prime}, T^{\prime}\right)$ is a direct extension of $(s, T)$ and write $\left(s^{\prime}, T^{\prime}\right) \leqslant *$ $(s, T)$.

Note: We now have two different definitions of $\leqslant^{*}$. One talks about elements of $K_{\lambda}$ and the other about conditions in $\mathbb{Q}_{\lambda}$, so it should be easy to understand which one we are talking about.
Proposition 5.4.20. The forcing $\left(\mathbb{Q}_{\lambda}, \leqslant^{*}\right)$ is $<\lambda$-closed. That is, any descending sequence $T_{0} \supset T_{1} \supset T_{2} \ldots$ of nice trees with the same root will have a nice tree as their intersection.

Proof. Follows from the fact that $U_{\gamma}$ is closed under $<\lambda$ intersections.
The following lemma is very standard for Prikry style forcings.
Lemma 5.4.21. Let $p \in \mathbb{Q}_{\lambda}$. Let $\varphi$ be first order (perhaps with parameters). Then there is some $q \leqslant^{*} p$ deciding $\varphi$.

Proof. We'll essentially follow the standard proof of this result for Prikry forcings. However, the argument gets rather technical to state, because we need to construct a tree analogue of the diagonal intersection of measure 1 sets. The way we do this is really quite simple and natural, but unfortunately it's also rather messy to write out.

Let $p=(s, T)$. We shall recursively construct a condition $r=\left(s, T^{\prime}\right) \leqslant{ }^{*} p$, together with a collection of nice subtrees $\left\{T_{t} \subset T: t \in T^{\prime}\right\}$, where $T_{t}$ has root $t$. For $t \in T^{\prime}$, we define $s_{t}$ to be the sequence of predecessors of $t$ in $T^{\prime}$, starting at the root of $T^{\prime}$ and ending at $t$ itself. For each $t \in T^{\prime}$, the intention is that $\left(s \cup s_{t}, T^{\prime} \upharpoonleft t\right)$ should be a condition of $\mathbb{Q}_{\lambda}$ below $\left(s \cup s_{t}, T_{t}\right)$; and that if at all possible both will decide $\varphi$.

First, suppose $t$ is the root of $T$ (which will also have to be the root of $T^{\prime}$, since $\left(s, T^{\prime}\right)$ is supposed to be a condition of $\mathbb{Q}_{\lambda}$ ). If there is some $r_{t} \leqslant^{*} p$ which decides $\varphi$, then let $T_{t}$ be its associated tree. (Of course, if such an $r_{t}$ exists, then we're already done!). Otherwise, let $T_{t}=T$.

Now, let $n>0$ and assume that we have already defined levels $1 \ldots, n$ of $T^{\prime}$, as well as $T_{t}$ for all $t$ in these levels of $T^{\prime}$. Let $t \in T^{\prime}$ be at level $n$. Then we define the direct successors of $t$ in $T^{\prime}$ to be the level 1 elements of $T_{t}$ (all of which are, by inductive hypothesis, direct successors of $t$ in $T$ ). In doing this, we have define the whole of level $n+1$ of $T^{\prime}$. It remains to define the trees $T_{u}$ of elements $u$ of this level.

If $u \in T^{\prime}$ is a direct successor of $t$, then let $T_{u}^{0}$ be the restriction $T_{t} \upharpoonleft u$ of $T_{t}$ to $u$. (So $T_{u}^{0}$ has root $u$ and consists of all elements of $T_{t}$ which are below $u$.) Consider the condition $p_{u}:=\left(s \cup s_{u}, T_{u}^{0}\right)$. Since $u \in T$ and $T_{t}$ is a subtree of $T$, we know that $p_{u} \leqslant p$. If there is some direct extension $r_{u} \leqslant{ }^{*} p_{u}$ which decides $\varphi$, then let $T_{u}$ be the tree in $r_{u}$. If no such $r_{t}$ exists, then we define $T_{u}:=T_{u}^{0}$. This completes the recursive definition.

It is easy to verify that $r:=\left(s, T^{\prime}\right) \leqslant{ }^{*} p$, and that it has the following property: if $\tilde{r}=(\tilde{s}, \tilde{T}) \leqslant r$ decides $\varphi$, then so does $\left(\tilde{s}, T^{\prime} \upharpoonleft \max \tilde{s}\right)$.

The next step is to recursively construct another condition $q=\left(s, T^{\prime \prime}\right) \leqslant^{*} r$ which actually decides $\varphi$ itself. This time, the construction is a little simpler: we don't need the auxiliary $T_{t}$ trees. The root of $T^{\prime \prime}$ is the same as that of $T^{\prime}$ and $T$, of course.

Say that we have built the first $n$ levels of $T^{\prime \prime}$. Let $t$ be in the $n$ 'th level of $T^{\prime \prime}$, and let $A \subset K_{\lambda}$ be the set of all its direct successors in $T^{\prime}$. Recall that $A$ is measure 1 in the sense of $U_{\delta}$, where $\delta$ is the height of $t$.

We partition $A$ into three parts: we say that $u \in A$ is in $A_{1}$ if $q_{u}:=\left(s \cup s_{u}, T^{\prime} \upharpoonleft u\right)$ decides that $\varphi$ is true, $u$ is in $A_{2}$ if $q_{u}$ decides $\varphi$ is false, and $u$ is in $A_{3}$ if $q_{u}$ doesn't decide $\varphi$ at all.

One of $A_{1}, A_{2}$ and $A_{3}$ will be measure 1 in the sense of $U_{\delta}$. The direct successors of $t$ in $T^{\prime \prime}$ are defined to be the elements of that measure 1 set. It is again easy to verify that this defines a nice tree $T^{\prime \prime}$ and that $q:=\left(s, T^{\prime \prime}\right) \in \mathbb{Q}_{\lambda}$. It is also easy to check that $q \leqslant^{*} r \leqslant^{*} p$ and that $q$ has the following two properties:

1. If $\tilde{q}=(\tilde{s}, \tilde{T}) \leqslant q$ decides $\varphi$, then so does $q^{\prime}:=\left(\tilde{s}, T^{\prime \prime} \upharpoonleft \tilde{s}\right)$ (this follows from $q \leqslant r$ ); and
2. For any extension $q^{\prime}=\left(\tilde{s}, T^{\prime \prime} \upharpoonleft \tilde{s}\right)$ of $q$, either all the one step extensions of $q^{\prime}$ fail to decide $\varphi$, or they all decide it and make the same decision.

Let $\tilde{q}=(\tilde{s}, \tilde{T}) \leqslant q$ be some condition which decides $\varphi$, and let it be such that $\tilde{s}$ has minimal length among conditions which decide $\varphi$. Then by the first property, $\varphi$ is decided by $q^{\prime}:=\left(\tilde{s}, T^{\prime \prime} \upharpoonleft \tilde{s}\right)$. Suppose, seeking a contradiction, that $\tilde{s}$ is a proper extension of $s$, and let $\tilde{s}^{\prime}$ be $\tilde{s}$ with its final term $t$ omitted. (So by assumption, $\tilde{s}^{\prime}$ still extends $s$.)

Then $q^{\prime}$ is a one step extension of $q^{\prime \prime}:=\left(\tilde{s}^{\prime}, T^{\prime \prime} \upharpoonleft \tilde{s}^{\prime}\right)$, and decides $\varphi$. So by the second property, all the one-step extensions of $q^{\prime \prime}$ decide $\varphi$, and agree on that decision. But then $q^{\prime \prime}$ decides $\varphi$ as well, and this is a contradiction since $\tilde{s}^{\prime}$ is shorter than $\tilde{s}$.

So in fact $\tilde{s}=s$ and $\tilde{q} \leqslant^{*} q$ decides $\varphi$.
Corollary 5.4.22. Let $\varphi(x)$ be a formula with one free variable, let $\mu \leqslant \lambda$, and let $p \Vdash \exists x<\check{\mu} \varphi(x)$. Then there is some $q \leqslant^{*} p$ and some $\alpha<\mu$ such that $q \Vdash \varphi(\check{\alpha})$.

Proof. If no such $q$ exists, then let $p=q_{0} \geqslant{ }^{*} q_{1} \geqslant{ }^{*} q_{2} \ldots$ be a descending sequence of conditions of length $\mu+1$ defined as follows. For $i=j+1$, we choose $q_{i} \leqslant q_{j}$ deciding $\varphi(\check{j})$; by assumption $q_{i} \Vdash \neg \varphi(\check{j})$. At limit $i$ we take some $q_{i}$ which is $\leqslant *$ every $q_{j}, j<i$. This exists by $<\lambda$ closure of $\left(\mathbb{Q}_{\lambda}, \leqslant^{*}\right)$. Then $q_{\mu} \leqslant^{*} p \Vdash \forall x<\check{\nu} \neg \varphi(x)$. Contradiction.

This tells us that $\mathbb{Q}_{\lambda}$ does not do any unexpected collapsing or singularising of cardinals.
Lemma 5.4.23. $\mathbb{Q}_{\lambda}$ does not add any new bounded subsets of $\lambda$, collapse any cardinals, or singularise any cardinals apart from $\lambda$. It preserves $G C H$ where it holds in $\tilde{V}$, except at $\lambda$.

Proof. First, note that $\mathbb{Q}_{\lambda}$ has the $\lambda^{+}$chain condition, because any two conditions of $\mathbb{Q}_{\lambda}$ with the same stem are compatible. Hence, it does not collapse or singularise any cardinals $\geqslant \lambda^{+}$, or change the cardinalities of their power sets.

Let $\mu<\lambda$ be a cardinal of $\tilde{V}$, and suppose that $\mathbb{Q}_{\lambda}$ collapses it to some cardinal $\nu<\mu$. Let $\dot{g}$ be a name for a bijection $g: \nu \rightarrow \mu$. Let $p \Vdash \dot{g}: " \check{\nu} \rightarrow \check{\mu}$ is a bijection". We construct a descending chain of direct extensions $p=p_{0} \geqslant{ }^{*} p_{1} \geqslant * \ldots$ of length $\nu+1$. If $i$ is a successor, then we use the previous corollary to take $p_{i}$ deciding the value of $\dot{g}(i)$; at limit $i$ we take some $p_{i}$ which is $\leqslant^{*} p_{j}$ for all $j<i$. Then $p_{\nu}$ decides what $g$ is, and hence $g \in \tilde{V}$. Contradiction. A similar proof shows that $\mathbb{Q}_{\lambda}$ does not singularise $\mu$ either, and that it adds no bounded subsets of $\lambda$. (For the latter, we start with some $p \in \mathbb{Q}_{\lambda}$ which decides what the bound will be, and then take $a \leqslant^{*}$ descending chain whose length is that bound, deciding which elements of $\lambda$ will be in the new subset.)

This also means that $\lambda$ is a limit of cardinals in $\tilde{V}[G]$, and hence is still a cardinal, and that the power sets of cardinals below $\lambda$ are preserved.

All our definitions of $\mathrm{NM}_{\lambda}, \operatorname{Col}(C)$ and $\mathbb{Q}_{\lambda}$ have been given only for $\lambda \leqslant \kappa$. The only reason for this is because we use $f$ in their definition, and $f$ is only defined up to $\kappa$. We will later want to deal with elementary embeddings $j: \tilde{V} \rightarrow M$ with critical point $\kappa$. From the perspective of such a model $M$, we can extend the definition by introducing analogous forcings in terms of $j(f)$, for any $\lambda \leqslant j(\kappa)$. We will extend the notation by referring to these forcings also as $\mathrm{NM}_{\lambda}, \operatorname{Col}(C)$ and $\mathbb{Q}_{\lambda}$. Since $j(f)$ will be an end-extension of $f$, for $\lambda \leqslant \kappa$ the forcings $\mathrm{NM}_{\lambda}, \operatorname{Col}(C)$ and $\mathbb{Q}_{\lambda}$ are defined the same way in $M$ whether we use $f$ or $j(f)$ in their definitions, so there is no ambiguity in doing this.

### 5.4.5 Embedding NM* ${ }^{*}$ Col into $\mathrm{QQ}^{*} \mathrm{Col}$

Once again, fix a $\lambda^{+}$supercompact $\lambda \leqslant \kappa$ in a universe $\tilde{V}$ which knows about $f$. As we discussed earlier, we can extract elements of $\mathrm{NM}_{\lambda}$ and Col from a $\mathbb{Q}_{\lambda}$ condition.
Definition 5.4.24. We define two groups of abbreviations.

1. Let

$$
p=\left(\left(\left\langle c_{0}, h_{0}, \gamma_{0}\right\rangle,\left\langle c_{1}, h_{1}, \gamma_{1}\right\rangle, \ldots,\left\langle c_{n-1}, h_{n-1}, \gamma_{n-1}\right\rangle\right), T\right)
$$

be a condition in $\mathbb{Q}_{\lambda}$. We define $c_{p}=\bigcup_{i<n} c_{i}$ and $h_{p}=\bigcup_{i<n} h_{i}$.
2. Let $G$ be $\mathbb{Q}_{\lambda}$ generic. We define $C_{G}=\bigcup_{p \in G} c_{p}$ and $H_{G}=\bigcup_{p \in G} h_{p}$.

An equivalent, but less friendly, definition is that

$$
C_{G}=\bigcup\{c: \exists h, \gamma\langle c, h, \gamma\rangle \text { is a term in the first part of some } p \in G\}
$$

and that

$$
H_{G}=\bigcup\{h: \exists c, \gamma\langle c, h, \gamma\rangle \text { is a term in the first part of some } p \in G\}
$$

We shall now establish what these four objects actually are, in terms of $\mathrm{NM}_{\lambda}$ and Col. The first two objects, $c_{p}$ and $h_{p}$, are simply conditions of the relevant forcings in $\tilde{V}$ :
Proposition 5.4.25. Let $p \in \mathbb{Q}_{\lambda}$. Then $c_{p} \in \mathrm{NM}_{\lambda}$ and $h_{p} \in \operatorname{Col}\left(c_{p}\right)$ in $\tilde{V}$.
Proof. $c_{p}$ is a union of finitely many conditions in $\mathrm{NM}_{\lambda}$, so it is certainly closed, bounded, and its successors and limits have the right properties. So $c_{p} \in \mathrm{NM}_{\lambda}$.

In the notation used in the definition, $h_{p}$ is a union of one condition from each $\operatorname{Col}\left(c_{i}\right), i<n$. Thus it's certainly an element of

$$
\prod_{\alpha \in c\{\{\max (c)\}} \operatorname{Col}\left(f(\alpha),<\operatorname{Succ}_{C}(\alpha)\right)
$$

Moreover, $\max \left(c_{i-1}\right)<\gamma_{i-1}<\min \left(c_{i}\right)$ for all $0<i<n$, so the domain of $h_{p}$ is trivially bounded below $\min \left(c_{i}\right)$ for $i<n$. It is also bounded below all other regular ordinals, by definition of $\operatorname{Col}\left(c_{i}\right)$. Hence, $h_{p} \in \operatorname{Col}\left(c_{p}\right)$.
$C_{G}$ will be generic for $\mathrm{NM}_{\lambda}$, and there is a useful correspondence between $\mathrm{NM}_{\lambda}$ and $\mathbb{Q}_{\lambda}$ :
Lemma 5.4.26. If $G$ is $\mathbb{Q}_{\lambda}$ generic, then $C_{G}$ is a club which is $\mathrm{NM}_{\lambda}$ generic (over $\tilde{V}$ ). Moreover, given any $\mathrm{NM}_{\lambda}$ name $\sigma$, there is a $\mathbb{Q}_{\lambda}$ name $\varphi(\sigma)$ such that for any $\mathbb{Q}_{\lambda}$ filter $G$ (not necessarily generic), $\sigma^{C_{G}}=\varphi(\sigma)^{G}$.
(As usual, in the statement of the lemma we are muddling the definition of a generic filter over $\mathrm{NM}_{\lambda}$, and the club corresponding to that generic filter.)

Proof. It is easy to see that $C_{G}$ is a club in $\lambda$, and that all its closed initial segments are elements of $\mathrm{NM}_{\lambda}$. We must show that it is generic.

Let $D \subset \mathrm{NM}_{\lambda}$ be open dense. We will show that $X_{D}:=\left\{p \in \mathbb{Q}_{\lambda}: c_{p} \in D\right\}$ is dense in $\mathbb{Q}_{\lambda}$. This implies $G \cap X_{D} \neq \varnothing$ and so $C_{G}$ contains an element of $D$ as required.

Fix a condition $p=(s, T) \in \mathbb{Q}_{\lambda}$.
$D$ is dense in $\mathrm{NM}_{\lambda}$, so the set

$$
S_{D, p}:=\left\{\langle c, h, \gamma\rangle \in K_{\lambda}^{\mathrm{htt}(p)}: c_{p} \cup c \in D\right\}
$$

is $\leqslant^{*}$ dense in $K_{\lambda}^{\mathrm{ht}(p)}$. Clearly, it is also open.
So $S_{D, p} \in U_{\mathrm{ht}(p)}$ because $U_{\mathrm{ht}(p)}$ contains all the open dense subsets of $K_{\lambda}^{\mathrm{ht}(p)}$. Since the set of all successors of the root of $T$ is also in $U_{\mathrm{ht}(p)}$, there must be a level 1 element of the tree $T$ which is in $S_{D, p}$. But then that gives us a 1 step extension $q \leqslant p$ in $\mathbb{Q}_{\lambda}$, such that the extra term $\langle c, h, \gamma\rangle$ in $q$ is an element of $S_{D, p}$. It follows that $c_{q}=c_{p} \cup c \in D$, and hence $q \in X_{D}$ as required.

The second part of the lemma is an easy recursive definition: we take

$$
\varphi(\sigma):=\left\{\langle\varphi(\tau), p\rangle: p \in \mathbb{Q}_{\lambda} \wedge\left\langle\tau, c_{p}\right\rangle \in \sigma\right\}
$$

and verify that the required equality holds for any $\mathbb{Q}_{\lambda}$ filter $G$.
So $\mathbb{Q}_{\lambda}$ behaves well regarding $\mathrm{NM}_{\lambda}$. What about $\operatorname{Col}(C)$ ? We want $\mathbb{Q}_{\lambda} * \operatorname{Col}^{\tilde{V}[G]}\left(C_{G}\right)$ to play nicely with $\mathrm{NM}_{\lambda} * \mathrm{Col}^{\tilde{V}\left[C_{G}\right]}\left(C_{G}\right)$.

This is not as simple as it looks, because (fixing some $\mathbb{Q}_{\lambda}$ generic filter $G$ ) we know $\operatorname{Col}^{\tilde{V}[G]}\left(C_{G}\right)$ and $\operatorname{Col}^{\tilde{V}\left[C_{G}\right]}\left(C_{G}\right)$ are not the same. Remember that the Easton product we used to define Col only asks for its conditions to be bounded below regular ordinals. $\lambda$ is regular in $\tilde{V}\left[C_{G}\right]$ but is singular in $\tilde{V}[G]$. So there are sets in $\tilde{V}\left[C_{G}\right]$ which are in $\mathrm{Col}^{\tilde{V}[G]}\left(C_{G}\right)$ but not in $\mathrm{Col}^{\tilde{V}\left[C_{G}\right]}\left(C_{G}\right)$.

On the other hand, we do at least know that $\operatorname{Col}^{\tilde{V}\left[C_{G}\right]}\left(C_{G}\right) \subset \operatorname{Col}^{\tilde{V}[G]}\left(C_{G}\right) \cap \tilde{V}\left[C_{G}\right]$.
This is where the condition $H_{G}$ named by $G$ comes in. It ise a sort of "generic element" of $\operatorname{Col}{ }^{\tilde{V}[G]}\left(C_{G}\right)$ which forces any generic extension containing it to cooperate in the way we want despite this difficulty.

First, we must verify that $H_{G}$ really is a condition.
Lemma 5.4.27. Let $G$ be $\mathbb{Q}_{\lambda}$ generic. Then $H_{G} \in \operatorname{Col}^{\tilde{V}[G]}\left(C_{G}\right)$.
Proof. It is easy to see that $H_{G} \in \prod_{\alpha \in C_{G}} \operatorname{Col}\left(\alpha,<\operatorname{Succ}_{C_{G}}(\alpha)\right)$. We must verify that its support is bounded below every $\tilde{V}[G]$ regular cardinal $\mu \in[0, \lambda]$. If $\mu<\lambda$, this follows from Proposition 5.4.25: take $p \in G$ such that $\sup c_{p}>\mu$, and then $H_{p} \mid \mu=h_{p} \in \operatorname{Col}\left(c_{p}\right)$ and hence (since $\mu$ is regular in $\tilde{V}$ ) the support of $H_{p}$ is bounded below $\mu$.

On the other hand, we know that $\mathbb{Q}_{\lambda}$ singularises $\lambda$, so the case $\mu=\lambda$ is vacuous.
The value of $H_{G}$ is shown in the following rather technical lemma.
Lemma 5.4.28. Let $G$ be $\mathbb{Q}_{\lambda}$ generic. Let $G^{*}$ be $\operatorname{Col}^{\tilde{V}[G]}\left(C_{G}\right)$ generic over $\tilde{V}[G]$, and contain $H_{G}$. Then the filter $G^{* *}:=G^{*} \cap \mathrm{Col}^{\tilde{V}\left[C_{G}\right]}\left(C_{G}\right)$ is $\mathrm{Col}^{\tilde{V}\left[C_{G}\right]}\left(C_{G}\right)$ generic over $\tilde{V}\left[C_{G}\right]$.
Proof. It is easy to check that $G^{* *}$ is indeed a filter; the challenge is showing that it's generic. So let $\dot{D}$ be an $\mathrm{NM}_{\lambda}$ name for a dense open subset of $\mathrm{Col}^{\tilde{V}\left[C_{G}\right]}\left(C_{G}\right)$. (Formally, this means $\mathbb{1}_{\mathrm{NM}_{\lambda}}$ should force that $\dot{D}$ is a dense subset of $\operatorname{Col}(C)$, where $C$ is the generic club added by $\mathrm{NM}_{\lambda}$.)

For any $\mathbb{Q}_{\lambda}$ generic filter $G$, and for any $\operatorname{Col}^{\tilde{V}[G]}\left(C_{G}\right)$ generic filter $G^{*}$ containing $H_{G}$, the set $G^{*} \cap D \neq \varnothing$, where $D=\dot{D}^{C_{G}}$.

To begin with, we shall work in $\tilde{V}\left[C_{G}\right]$ for a fixed filter $G$. Let $D=\dot{D}^{C_{G}}$ as above. For $\beta \in \operatorname{Succ}\left(C_{G}\right)$, consider the two forcings $\mathbb{P}^{\beta}:=\operatorname{Col}^{\tilde{V}\left[C_{G}\right]}\left(C_{G} \backslash \beta+1\right)$ and $\mathbb{P}_{\beta}:=\operatorname{Col}^{\tilde{V}\left[C_{G}\right]}\left(C_{G} \cap(\beta+1)\right)$.

For $h \in \mathbb{P}^{\beta}$ we define the set

$$
D_{h}:=\left\{h^{\prime} \in \mathbb{P}_{\beta}: h^{\prime} \cup h \in D\right\}
$$

Then we define

$$
D^{\beta}:=\left\{h \in \mathbb{P}^{\beta}: D_{h} \text { is open dense }\right\}
$$

Claim 5.4.29. $D^{\beta}$ is an open dense subset of $\mathbb{P}^{\beta}$.
Proof. $D$ is open, so if $\tilde{h} \leqslant h \in \mathbb{P}^{\beta}$ then $D_{\tilde{h}} \supset D_{h}$. Hence $D_{\tilde{h}}$ is dense. Moreover, $D_{\tilde{h}}$ is also open, since $D$ is open. Hence $D^{\beta}$ is open.

To show $D^{\beta}$ is also dense, let us fix $h \in \mathbb{P}^{\beta}$.
Now $\mathbb{P}_{\beta}$ has cardinality $\beta$, so we can enumerate its elements $\left\{h_{\alpha}: \alpha<\beta\right\}$. We construct a decreasing sequence $\left(h_{\alpha}^{\prime}\right)_{\alpha<\beta+1}$ of length $\beta$ of conditions in $\mathbb{P}^{\beta}$. Let $h_{0}^{\prime}=h$. For $\gamma<\beta$, we choose some $h_{\gamma+1}^{\prime} \leqslant h_{\gamma}^{\prime}$ such that for some $h^{*} \leqslant h_{\gamma}$, the condition $h^{*} \cup h_{\gamma+1}^{\prime} \in D$. We can do this easily, since $D$ is dense: just take some element of $D$ below $h_{\gamma} \cup h_{\gamma}^{\prime}$ and let $h_{\gamma+1}^{\prime}$ be the part of it which is above $\beta$.

For limit $\alpha \leqslant \beta$, we take $h_{\alpha}$ to be below every earlier term of the sequence, which we can do since $\mathbb{P}^{\beta}$ is $\beta^{+}$-closed by Proposition 5.4.10.

Now, $h_{\beta}^{\prime} \leqslant h$ is such that for all $h_{\alpha} \in \mathbb{P}_{\beta}$, there exists $h^{*} \leqslant h_{\alpha}$ such that $h^{*} \cup h_{\beta}^{\prime} \in D$, since $h_{\beta}^{\prime} \leqslant h_{\alpha+1}^{\prime}$ and $D$ is open. It follows immediately that $D_{h_{\beta}^{\prime}}$ is dense. Since $D$ is open, it's also immediate that $D_{h_{\beta}^{\prime}}$ is open. Hence $h_{\beta}^{\prime} \in D^{\beta}$. But $h_{\beta}^{\prime} \leqslant h$ so $D^{\beta}$ is dense.

We now work in $\tilde{V}$. We shall show that $\mathbb{1}_{\mathbb{Q}_{\lambda}}$ forces the following statement:
"There are two ordinals $\beta<\delta$ in $C_{G}$, with $\beta$ a successor element of $C_{G}$, such that $H_{G} 1[\beta, \delta] \in$ $D^{\beta} . "$

This statement makes sense, since $\tilde{V}\left[C_{G}\right]$ is a definable subclass of $\tilde{V}[G]$ and so $\tilde{V}[G]$ knows what $D^{\beta}$ looks like for any $\beta$. Note that it implicitly assumes that $H_{G} 1[\beta, \delta] \in \tilde{V}\left[C_{G}\right]$, but that's automatically true since it's an element of $\operatorname{Col}\left(C_{G} \cap \delta+1\right) \in \tilde{V}$.

So fix $p \in \mathbb{Q}_{\lambda}$. Let $\operatorname{ht}(p)=\gamma$. We shall show there is a one step extension of $p$ which forces the above statement.

Let us say an element $\langle c, h, \alpha\rangle$ of $K_{\lambda}^{\gamma}$ is cooperative if there is some element $\beta \in \operatorname{Succ}(c)$ such that $c_{p} \cup c \Vdash_{\mathrm{NM}_{\lambda}} h \upharpoonleft(c \backslash \beta) \in D^{\beta}$. (Recall that by definition of $K_{\lambda}$, we know $h \in \operatorname{Col}^{\tilde{V}}(c)$, so $h \in \operatorname{Col}^{\tilde{V}\left[C_{G}\right]}\left(C_{G}\right)$. So $h \upharpoonleft(c \backslash \beta) \in \mathbb{P}^{\beta}$ in $\tilde{V}\left[C_{G}\right]$, and thus it makes sense ask whether it is in $D^{\beta}$.)
Claim 5.4.30. The set of all cooperative elements of $K_{\lambda}^{\gamma}$ is $\leqslant^{*}$ dense.
Proof. Let $\langle c, h, \alpha\rangle \in K_{\lambda}^{\gamma}$. Without loss of generality, we can assume $c$ has a largest element. (If it doesn't, then we can simply extend $c$ arbitrarily by one step to get a $\left(c^{\prime}, h, \alpha^{\prime}\right)<^{*}(c, h, \alpha)$ and work with that instead.) Let $\beta$ be that largest element.
$c_{p} \cup c$ forces that $\beta$ is a successor element of the club that $\mathrm{NM}_{\lambda}$ adds, so it forces that $D^{\beta}$ is an open dense subset of $\mathbb{P}^{\beta}$. Hence, there is some end extension $c^{\prime} \leqslant c$ and some name $\dot{h}^{\prime}$ for an element of $\mathbb{P}^{\beta}$ such that $c_{p} \cup c^{\prime} \Vdash_{\mathrm{NM}_{\lambda}} \dot{h}^{\prime} \in D^{\beta}$.

Now, in $\tilde{V}\left[C_{G}\right]$ an element of $\mathbb{P}^{\beta}$ is a sequence of conditions in collapsing forcings whose support is bounded below $\lambda$. Each of these collapsing forcings also has cardinality less than $\lambda$. So any element of $\mathbb{P}^{\beta}$ in $\tilde{V}\left[C_{G}\right]$ can be coded as a subset of $\lambda . \mathrm{NM}_{\lambda}$ is $\lambda$ distributive, so by Theorem 1.2 .6 all the elements of $\mathbb{P}^{\beta}$ in $\tilde{V}\left[C_{G}\right]$ actually already existed in $\tilde{V}$. (Of course, $\mathbb{P}^{\beta}$ itself does not exist in $\tilde{V}$, though!)

So without loss of generality, we can choose $\dot{h}^{\prime}$ to be a check name for some $h^{\prime} \in \tilde{V}$. Now $h^{\prime}$ is certainly bounded below $\lambda$, so without loss of generality we may assume that $c^{\prime}$ is longer than the support of $h^{\prime}$, that is, that $h^{\prime} \in \operatorname{Col}^{\tilde{V}}\left(c^{\prime}\right)=\operatorname{Col}^{\tilde{V}\left[C_{G}\right]}\left(c^{\prime}\right)$. In fact, since $h^{\prime} \in P^{\beta}$, we know $h^{\prime} \in \operatorname{Col}\left(c^{\prime} \backslash \beta\right)$. In particular, since $h \in \operatorname{Col}(c)$ and $\sup (c)=\beta$, we know $h \cup h^{\prime}$ is a well defined element of $\operatorname{Col}\left(c^{\prime}\right)$. Take $\alpha^{\prime} \geqslant \alpha$ to be some ordinal which is larger than $\sup \left(c^{\prime}\right)$.

Then $\left\langle c^{\prime}, h \cup h^{\prime}, \alpha^{\prime}\right\rangle \in K_{\lambda}^{\gamma}$ and $\left\langle c^{\prime}, h \cup h^{\prime}, \alpha^{\prime}\right\rangle \leqslant *\langle c, h, \alpha\rangle$. Using the same $\beta$ as above, we can see by construction that $c_{p} \cup c^{\prime} \Vdash_{\mathrm{NM}_{\lambda}}\left(h \cup h^{\prime}\right) \upharpoonleft\left(c^{\prime} \backslash \beta\right)=h^{\prime} \in D^{\beta}$. Hence $\left(c^{\prime}, h \cup h^{\prime}, \alpha^{\prime}\right)$ is cooperative.

So the set $\tilde{K}$ of all cooperative elements of $K_{\lambda}^{\gamma}$ is in $U_{\gamma}$. Since the set of all valid one step extensions of $p$ is also in $U_{\gamma}$, there is a cooperative $\langle c, h, \alpha\rangle$ which is a valid way to extend $p$ by one step. Let $q$ be this one step extension of $p$. Since $\langle c, h, \alpha\rangle$ is cooperative, we can find $\beta \in \operatorname{Succ}(c)$ such that $c_{p} \cup c \Vdash_{\mathrm{NM}_{\lambda}} h \upharpoonleft(c \backslash \beta) \in D^{\beta}$. Let $\delta=\sup (c)$.

Now, let $G$ be a $\mathbb{Q}_{\lambda}$ generic filter with $q \in G$. Then $H_{G} 1[\beta, \delta]=h \uparrow(c \backslash \beta)$, and $c_{p} \cup c \in C_{G}$. So $H_{G} 1[\beta, \delta] \in D^{\beta}$. Hence $q$ forces "There are two ordinals $\beta<\delta$ in $C_{G}$, with $\beta$ a successor element of $C_{G}$, such that $H_{G} 1[\beta, \delta] \in D^{\beta "}$. The condition $p$ was arbitrary, so $\mathbb{1}_{Q_{\lambda}}$ forces the statement, which is what we wanted to show.

Now let $G$ be an arbitrary $\mathbb{Q}_{\lambda}$ generic filter $G$, and work over $\tilde{V}[G]$. The statement is true of $\tilde{V}[G]$, so find $\beta<\delta$ that fit it. Let $G^{*}$ be a $\operatorname{Col}^{V[G]}\left(C_{G}\right)$ generic filter containing $H_{G}$. Recall that we are aiming to show that $G^{*} \cap D=G^{*} \cap \dot{D}^{C_{G}} \neq \varnothing$. Let $h=H_{G} 1[\beta, \delta]$. So $h \in D^{\beta}$. Hence, $D_{h}$ is open and dense in $\mathbb{P}_{\beta}=\operatorname{Col}^{\tilde{V}\left[C_{G}\right]}\left(C_{G} \cap(\beta+1)\right)$. Also, $h \in G^{*}$ since $H_{G} \leqslant h$.

Now, $\mathbb{Q}_{\lambda}$ adds no new bounded subsets of $\lambda$ by Lemma 5.4.23, so in fact $\operatorname{Col}^{\tilde{V}}\left(C_{G} \cap(\beta+1)\right)=$ $\mathrm{Col}^{\tilde{[ }\left[C_{G}\right]}\left(C_{G} \cap(\beta+1)\right)=\operatorname{Col}^{\tilde{V}[G]}\left(C_{G} \cap(\beta+1)\right)$. So $D_{h^{2}}$ is open dense over the latter forcing. Since $G^{*}$ is generic over $\operatorname{Col}^{\tilde{V}[G]}\left(C_{G}\right)$, the restriction of $G^{*}$ to $\operatorname{Col}^{\tilde{V}[G]}\left(C_{G} \cap(\beta+1)\right)$ is generic over that forcing. Hence $G^{*} \cap D_{h} \neq \varnothing$, so it contains some $h^{\prime}$. But then $h^{\prime} \cup h \in D$ by definition of $D_{h}$ (and hence also
$\left.h^{\prime} \cup h \in \operatorname{Col}{ }^{\tilde{V}\left[C_{G}\right]}\left(C_{G}\right)\right)$. Since $h^{\prime}, h \in G^{*}$ also $h^{\prime} \cup h \in G^{*}$. So $G^{* *} \cap D=G^{*} \cap \operatorname{Col}{ }^{\tilde{V}\left[C_{G}\right]}\left(C_{G}\right) \cap D$ contains $h^{\prime} \cup h$ and is therefore nonempty.

So fixing an arbitrary $\mathbb{Q}_{\lambda}$ filter $G$, we know that if $G^{*}$ is any $\operatorname{Col}{ }^{\tilde{V}[G]}\left(C_{G}\right)$ generic filter containing $H_{G}$ and $D$ is any $\mathrm{Col}^{\tilde{V}\left[C_{G}\right]}\left(C_{G}\right)$ dense set in $\tilde{V}\left[C_{G}\right]$, then $G^{* *}:=G^{*} \cap \operatorname{Col}^{\tilde{V}\left[C_{G}\right]}\left(C_{G}\right)$ meets $D$. Hence this $G^{* *}$ is generic over $\tilde{V}\left[C_{G}\right]$, as required.

As a consequence of our earlier results about NM, $\mathbb{Q}$ and Col, we can also conclude that the generic extensions agree about Card and $\operatorname{Reg}_{\delta}$ for all $\delta<\epsilon$

Lemma 5.4.31. Let $G$ be $\mathbb{Q}_{\lambda}$ generic, let $G^{*}$ be $\mathrm{Col}^{\tilde{V}[G]}\left(C_{G}\right)$ generic and contain $H_{G}$, and let $G^{* *}$ be obtained from $G^{*}$ as in lemma 5.4.28. The cardinals of $\tilde{V}[G]\left[G^{*}\right]$ and $\tilde{V}\left[C_{G}\right]\left[G^{* *}\right]$ are both precisely the cardinals of $\tilde{V}$ which are not in the interval $\left(f(\alpha), \operatorname{Succ}_{C}(\alpha)\right)$ for any $\alpha \in C$. Moreover, in both these generic extensions, for all $0<\delta<\epsilon$, the class $\operatorname{Reg}_{\delta}$ in the generic extension is precisely $\operatorname{Reg}_{\delta}^{\tilde{V}}$ but with the intervals $\left(f(\alpha), \operatorname{Succ}_{C}(\alpha)\right)$ omitted. In particular, $\operatorname{Reg}_{\delta}^{\tilde{V}}$ and Card agree up in $\tilde{V}[G]\left[G^{*}\right]$ and $\tilde{V}\left[C_{G}\right]\left[G^{* *}\right]$, for all $\delta<\epsilon$.

Proof. Corollary 5.4.8 tells us that $\tilde{V}\left[C_{G}\right]$ has the same cardinals, and regular cardinals, as $\tilde{V}$. Likewise, Lemma 5.4.11 tells us that $\tilde{V}[G]$ has the same cardinals and regular cardinals as $\tilde{V}$, except that $\lambda$ has been singularised (which does not modify $\operatorname{Reg}_{\delta}$ for $\delta<\epsilon$ ). Then Lemma 5.4.23 tells us that the generic extensions of $\tilde{V}[G]$ and $\tilde{V}\left[C_{G}\right]$ by $G^{*}$ and $G^{* *}$ respectively both remove all the cardinals in the intervals $\left(f(\alpha), \operatorname{Succ}_{C}(\alpha)\right)$ but do not otherwise change Card or $\operatorname{Reg}_{\delta}$ for $0<\delta<\epsilon$. For the final sentence, we've covered the case $\delta>0$; and showing that the two models agree on $\operatorname{Reg}_{0}$ just requires us to recall that $\operatorname{Reg}_{0}$ can be trivially calculated from Card.

Corollary 5.4.32. Let $G, G^{*}, G^{* *}$ be as above. Then $\tilde{V}\left[C_{G}\right]\left[G^{* *}\right]$ and $\tilde{V}[G]\left[G^{*}\right]$ have the same cardinals, and their $\operatorname{Reg}_{\delta}$ 's are the same for all $\delta<\epsilon$.

### 5.4.6 Putting the forcings together

We are finally ready to put together these forcings, and define the overall forcing we're going to be using. We use a Prikry style iteration of $\mathbb{Q}_{\lambda}$ forcings, followed by an $\mathrm{NM}_{\kappa}$ forcing and the corresponding Col forcing. This will give us a universe in which $\kappa$ is the first element of $\operatorname{Reg}^{\epsilon}$, and is also $\operatorname{LST}\left(I, Q^{\epsilon}\right)$. We now drop our discussions of $\tilde{V}$, and just work in the universe $V$ we fixed near the start of the proof. Recall that $V$ believes GCH.

Definition 5.4.33. Recursively, we define forcings $P_{\alpha}(\alpha \leqslant \kappa)$ with two orders $\leqslant$ and $\leqslant *$, and (for $\alpha<\kappa$ ) $P_{\alpha}$ names $\dot{Q}_{\alpha}$ for forcings, also with two orders $\leqslant$ and $\leqslant^{*}$, as follows.

- For $\alpha \leqslant \kappa$, the elements of $P_{\alpha}$ are sequences $\left\langle\tau_{\gamma}\right\rangle_{\gamma<\alpha}$ of length $\alpha$ such that:

1. For $\beta<\alpha$, the sequence $\left\langle\tau_{\gamma}\right\rangle_{\gamma<\beta} \in P_{\beta}$ and forces $\tau_{\beta} \in \dot{Q}_{\beta}$.
2. The sequence has Easton support. That is, for every $V$ regular $\lambda \leqslant \alpha$, the set

$$
\left\{\beta<\lambda:\left\langle\tau_{\gamma}\right\rangle_{\gamma<\beta} \Vdash \tau_{\beta}=\mathbb{1}_{\dot{Q}_{\beta}}\right\}
$$

is bounded below $\lambda$.

- The $\leqslant$ order of $P_{\alpha}$ is defined as follows: $\left\langle\tau_{\gamma}^{\prime}\right\rangle_{\gamma<\alpha} \leqslant\left\langle\tau_{\gamma}\right\rangle_{\gamma<\alpha}$ if:

1. For all $\beta<\alpha,\left\langle\tau_{\gamma}^{\prime}\right\rangle_{\gamma<\beta} \Vdash \tau_{\beta}^{\prime} \leqslant \tau_{\beta}$.
2. For all but finitely many $\beta$, either $\tau_{\beta}$ is forced to be $\mathbb{1}_{\dot{Q}_{\beta}}$ by $\left\langle\tau_{\gamma}\right\rangle_{\gamma<\beta}$, or we can replace $\leqslant$ with $\leqslant *$ (in the sense of $\dot{Q}_{\beta}$ ) on the previous line.

- $\left\langle\tau_{\gamma}^{\prime}\right\rangle_{\gamma<\alpha} \leqslant{ }^{*}\left\langle\tau_{\gamma}\right\rangle_{\gamma<\alpha}$ if in the above, 2 holds for every $\beta$.
- For any $\alpha<\kappa, \dot{Q}_{\alpha}$ is a $P_{\alpha}$ name for a forcing:

1. If $\alpha$ is $\alpha^{+}$supercompact in $V$ and $P_{\alpha}$ forces: " $\alpha$ is a cardinal which is $\alpha^{+}$supercompact", then $\dot{Q}_{\alpha}$ is a name for the forcing $\mathbb{Q}_{\alpha}$ we defined earlier.
2. Otherwise, $\dot{Q}_{\alpha}$ is the canonical name for the trivial forcing.

This gives us a well defined forcing $P_{\kappa}$. But we don't immediately know very much about it. For the rest of the proof to work, we need to verify that it does in fact singularise every $\lambda^{+}$supercompact $\lambda<\kappa$ with a $\mathbb{Q}_{\lambda}$ style forcing, but that $\kappa$ remains Mahlo. There is also a further complication: we need to make sure that when we do our $\mathrm{NM}_{\kappa} * \operatorname{Col}(C)$ portion of the forcing, we will end up with $\kappa=\min \left(\operatorname{Reg}_{\epsilon}\right)$. Showing that there are no elements of $\operatorname{Reg}_{\epsilon}$ below $\kappa$ is pretty much immediate. But making sure that unboundedly many elements of $\operatorname{Reg}_{\delta}$ survive below $\kappa$, for every $\delta<\epsilon$, is harder.

Of course, once we know that $\kappa$ is a Mahlo cardinal in the $P_{\kappa}$ generic extension, it follows that $\mathrm{Reg}_{\delta}$ will certainly be unbounded in $\kappa$ at that point. But remember that $\operatorname{Col}(C)$ collapses nearly all the cardinals below $\kappa$. So if we're not careful, we might get unlucky and find that we've collapsed all the elements of $\operatorname{Reg}_{\delta}$ in the final stage of hte forcing. To avoid this, we need to make sure that there are elements of $\operatorname{Reg}_{\delta}$ in the intervals $(\alpha, f(\alpha)), \alpha \in C$.

Now because of how we defined $f$, we know that there are such regulars in that interval in $V$. But when we choose $\alpha \in C$, we need to be sure that these regulars have survived in the $P_{\kappa}$ generic extension. This is non-trivial, since we cannot (easily) prove $P_{\kappa}$ preserves all the cardinals, let alone that it doesn't singularise anything we didn't ask it to.

We start with some technical lemmas. The first is a special case of $[18,1.3]$ and we do not re-prove it here.
Lemma 5.4.34. Let $\alpha \leqslant \kappa$ be a Mahlo cardinal (of $V$ ). Then $P_{\alpha}$ has the $\alpha$ chain condition, and has cardinality $\leqslant \alpha$.
Corollary 5.4.35. Let $\alpha \leqslant \kappa$. Then $\left|P_{\alpha}\right| \leqslant \alpha^{++}$.
Proof. Case 1: $\alpha$ is a Mahlo cardinal. Then $\left|P_{\alpha}\right| \leqslant \alpha$ by the previous lemma. Case 2: $\alpha$ is a limit of Mahlo cardinals. Let $\left(\gamma_{i}\right)_{i<\operatorname{cf}(\alpha)} \subset \alpha$ be a sequence of Mahlo cardinals which is cofinal below $\mu$. Again by Lemma 5.4.34, for all $i,\left|P_{\gamma_{i}}\right|=\gamma_{i}$. A condition of $P_{\alpha}$ can be expressed as a collection of conditions, one from each $P_{\gamma_{i}}$, which all agree with each other. So

$$
\begin{aligned}
\left|P_{\alpha}\right| & \leqslant \prod_{i<\operatorname{cf}(\alpha)}\left|P_{\gamma_{i}}\right| \\
& =\prod_{i<\operatorname{cf}(\alpha)} \gamma_{i} \\
& \leqslant \prod_{i<\operatorname{cf}(\alpha)} \alpha \\
& \leqslant \alpha^{\alpha} \\
& =\alpha^{+}
\end{aligned}
$$

Case 3: $\alpha$ is neither a Mahlo cardinal nor a limit of Mahlo cardinals. Let $\beta<\alpha=\sup \{$ Mahlo cardinals $\} \cap \alpha$. Any $\lambda^{+}$supercompact cardinal $\lambda$ is Mahlo, so we know that $\dot{Q}_{\gamma}$ is trivial for all $\gamma \in(\beta, \alpha)$. So

$$
\begin{aligned}
\left|P_{\alpha}\right| & \leqslant\left|P_{\beta} * \dot{Q}_{\beta}\right| \\
& \leqslant \beta^{++} \\
& <\alpha^{++}
\end{aligned}
$$

Similarly, this next lemma is a special case of $[18,1.4]$, and again we don't prove it here:
Lemma 5.4.36. Let $\varphi$ be a statement (with parameters) and $p \in P_{\alpha}$. There is some $q \leqslant^{*} p$ which decides $\varphi$. The same is true of the forcing $P_{\alpha} / P_{\beta}$ if $\beta<\alpha$ is a Mahlo cardinal.

Here $P_{\alpha} / P_{\beta}$ is the ( $P_{\beta}$ name for the) forcing defined in the usual way in a $P_{\beta}$ generic extension $V[G]$ : we simply take $P_{\alpha}$ and delete all those conditions which are incompatible with an element of $G$.
Lemma 5.4.37. For any $\beta<\alpha \leqslant \kappa$, the forcing $P_{\alpha} / P_{\beta}$ is closed under $\leqslant{ }^{*}$ sequences of length less than $\beta$.
Proof. Let $p_{0} \geqslant * p_{1} \geqslant \ldots$ be $a \geqslant *$ decreasing sequence of $P_{\alpha} / P_{\beta}$ length $\lambda<\beta$. We want to find a $p=p_{\lambda}=\tau_{\gamma \beta \gamma<\alpha}$ which is $\leqslant^{*}$ below every $p_{i}$. We define it recursively as follows. Let $\beta \leqslant \gamma<\alpha$.

If for all $i$, the $\gamma$ component of $p_{i}$ is trivial (either because $\dot{Q}_{\gamma}$ is trivial, or because $\gamma$ is not in the support of $p_{i}$ ) then we define $\tau_{\gamma}$ to be trivial as well.

Otherwise, if say $p_{j}(\gamma)$ is nontrivial, then we know that $p_{j}(\gamma) \geqslant{ }^{*} p_{j+1}(\gamma) \geqslant * \ldots$ are names for an $\leqslant^{*}$ decreasing sequence of conditions in $\mathbb{Q}_{\gamma}$ of length $\lambda<\gamma$. We know by Proposition 5.4.20 that there is a condition in $\mathbb{Q}_{\gamma}$ below them; let $\tau_{\gamma}$ be a name which is forced to be that condition by $p 1 \gamma$.

If this construction works, it will obviously give a condition which is $\leqslant^{*} p_{i}$ for all $i<\lambda$. We must check that $p 1 \gamma$ is a condition of $P_{\alpha} / P_{\beta}$, for all $\gamma$. It suffices to check that $p$ has Easton support, i.e. its support is bounded below every regular cardinal of $V[G]$. But we can easily see that

$$
\operatorname{supp}(p)=\bigcup_{i<\lambda} \operatorname{supp}\left(p_{i}\right)
$$

For all $i$, we know that $\operatorname{supp}\left(p_{i}\right)$ is bounded below every regular cardinal of $V[G]$, and $\operatorname{supp}\left(p_{i}\right) \cap \beta=\varnothing$. So if $\mu$ is regular in $V[G]$, then either $\mu \leqslant \beta(\operatorname{and} \operatorname{supp}(p) \cap \mu=\varnothing)$ or $\operatorname{supp}(p) \cap \mu$ is a union of $\lambda<\mu$ many bounded sets, and therefore is bounded below $\mu$.

Lemma 5.4.38. If $\beta<\alpha$ is a Mahlo cardinal, then the forcing $P_{\alpha} / P_{\beta}$ does not collapse or singularise any cardinals below $\beta$, or add any new bounded subsets of $\beta$.

Proof. This is similar to Lemma 5.4.23. Let us write $P$ for $P_{\alpha} / P_{\beta}$. First, let $p \in P$, and let $\varphi(x)$ be a statement such that

$$
p \Vdash \exists x \in \check{\lambda} \varphi(x)
$$

We claim there is some condition $q \leqslant^{*} p$ which forces, for some $\gamma \in \lambda$, that $\varphi(\check{\gamma})$ holds. By Lemmas 5.4.36 and 5.4.37 we can construct a descending chain of conditions $p=p_{0} \geqslant{ }^{*} p_{1} \geqslant * \ldots$ in $P$ of length $\lambda+1$ such that $p_{i}$ decides $\varphi(\check{i})$. But then $q:=p_{\lambda}$ decides $\varphi(\check{\gamma})$ for every $\alpha \in \lambda$, so it must decide at least one of them positively.

Now let $\lambda<\beta$ be a cardinal of $V[G]$, a generic extension of $V$ by $P_{\beta}$. Suppose that $\lambda$ is collapsed by $P$. Let $\dot{h}$ be a name for a bijection $h: \nu \rightarrow \lambda$ for some $\nu<\lambda$. Let $p \in P$ be a condition forcing this. Using the above and Lemma 5.4.37, we construct a descending sequence $p=p_{0} \geqslant * p_{1} \geqslant * \ldots$ of length $\nu+1$, such that for all $i, p_{i}$ decides the value of $\dot{h}(\check{i})$. Then $p_{\nu}$ defines bijection between $\nu$ and $\lambda$ in $V[G]$. Contradiction.

Showing that $\lambda$ is not singularised by $P$, and that $P$ adds no bounded subsets of $\beta$, are both similar.
Corollary 5.4.39. If $\alpha$ is a Mahlo cardinal, and $G$ is a $P_{\alpha}$ generic filter, then $G C H$ holds in $V[G]$ except perhaps at the cardinals $\lambda$ for which $\dot{Q}_{\lambda}$ is nontrivial. (In other words, if $\lambda$ is not $\lambda^{+}$supercompact in $V$, or $\lambda \geqslant \alpha$, then $2^{\lambda}=\lambda^{+}$in $\left.V[G].\right)$

Proof. Let $\lambda$ be a cardinal of $V$ which is not $\lambda^{+}$supercompact. We first examine the case $\lambda \geqslant \alpha$. By Lemma 1.2.7 and Lemma 5.4.34 we know that if $\lambda>\alpha^{+}$then $2^{\lambda}=\lambda^{+}$in any $P_{\alpha}$ generic extension. Similarly, if $\lambda=\alpha$ or $\lambda=\alpha^{+}$, then by regularity we know that for all $\beta<\lambda, \lambda^{\beta}=\lambda$. So again by the same two lemmas, we know $P_{\alpha}$ preserves $2^{\lambda}$.

Now suppose that $\lambda<\alpha$. Let $\mu \leqslant \lambda=\sup \left\{\beta \leqslant \lambda: \beta\right.$ is $\beta^{+}$supercompact $\}$and let $\nu$ be the least $\beta>\lambda$ such that $\beta$ is $\beta^{+}$supercompact. Then up to some trivial notation changes,

$$
P_{\alpha}=P_{\mu} * \dot{Q}_{\mu} * P_{\alpha} / P_{\nu}
$$

We have just seen that $P_{\mu}$ preserves $2^{\lambda}$. If $\dot{Q}_{\mu}$ is nontrivial, then $\mu$ is $\mu^{+}$supercompact in $V$, and hence $\mu<\lambda$. If so, then $\dot{Q}_{\mu}$ is $\mathbb{Q}_{\mu}$, which we saw in Lemma 5.4.23 preserves $2^{\lambda}$. And finally, $P_{\alpha} / P_{\nu}$ adds no new bounded subsets of $\nu>\lambda$, so it doesn't change $\mathcal{P}(\lambda)$ at all. Hence $P_{\alpha}$ preserves $2^{\lambda}$.

Lemma 5.4.40. If $\alpha \leqslant \kappa$ is $\alpha^{+}$supercompact in $V$ then it is still $\alpha^{+}$supercompact in the $P_{\alpha}$ generic extension.

Proof. Let $V[G]$ be a $P_{\alpha}$ generic extension of $V$. Let $j: V \rightarrow M$ be an $\alpha^{+}$supercompact embedding with critical point $\alpha \geqslant \kappa$, such that $\alpha$ is no longer $\alpha^{+}$supercompact in $M$, which exists by Lemma 3.1.17. Let $U$ be the ultrafilter on $\mathcal{P}_{\alpha}\left(\alpha^{+}\right)$generated by $j$ in Lemma 3.1.16. Let

$$
\left(P_{\beta}^{*}: \beta \leqslant j(\alpha)\right):=j\left(\left(P_{\beta}: \beta \leqslant \alpha\right)\right)
$$

and

$$
\left(\dot{Q}_{\beta}^{*}: \beta<j(\alpha)\right):=j\left(\left(\dot{Q}_{\beta}: \beta<\alpha\right)\right)
$$

Now $\alpha$ is strongly inaccessible in $V$, so by Corollary 5.4 .35 we know that for $\beta<\alpha,\left|P_{\beta}\right| \leqslant \beta^{++}<\alpha$. Also, $\left|\mathbb{Q}_{\beta}\right|<\lambda$ for $\beta<\alpha$. Hence, for all $\beta<\alpha, P_{\beta}^{*}=P_{\beta}$ and $\dot{Q}_{\beta}^{*}=\dot{Q}_{\beta}$. It follows that also $P_{\alpha}^{*}=P_{\alpha}$. So $j\left(P_{\alpha}\right)=P_{j(\alpha)}^{*}$ is an end extension of $P_{\alpha}$. Moreover, $\dot{Q}_{\alpha}^{*}$ is trivial, as $\alpha$ is not $\alpha^{+}$supercompact in $M$. Note that $G$ is also generic for $P_{\alpha}$ over $M$.

By Lemma 5.4.34, $P_{\alpha}$ satisfies the $\alpha$ chain condition and has cardinality $\alpha$ from the perspective of $V$, and hence also from the perspective of $M$. Also, $M$ is closed under $\alpha^{+}$sequences. So $\alpha^{+}$and $\alpha^{++}$are both absolute between $V, M, V[G]$ and $M[G]$. In particular, since $2^{\alpha^{+}}=\alpha^{++}$from the perspective of $V$ (and $V[G]$, see Corollary 5.4.39) we can find in $V$ a collection of $\alpha^{++}$many names such that in any $P_{\alpha}$ generic extension, every element of $\mathcal{P}_{\alpha}\left(\alpha^{+}\right)$will be equal to one of these names. To ward off a technical issue later in the proof, we will insist that we include in this collection all the check names $\check{X}$ for elements $X$ of $\left(\mathcal{P}_{\alpha}\left(\alpha^{+}\right)\right)^{V}$. Let us enumerate these names $\sigma_{i}: i<\alpha^{++}$. Since $M^{\alpha^{+}} \subset M$ and $P_{\alpha} \in M$, we know for $k<\alpha^{++}$that $\left(j\left(\sigma_{i}\right)\right)_{i<k} \in M$.

Now $\alpha$ is a Mahlo cardinal in $M$, because $M$ and $V$ agree on the singular cardinals below $\kappa$ and any club of such singulars in $M$ would also be definable in $V$. We can localise the proof of Lemmas 5.4.37 and 5.4.36 to $M$ and $P_{j(\alpha)}^{*}$. In particular, since $\alpha$ is still a Mahlo cardinal in $M$, we know that $P_{j(\alpha)}^{*} / P_{\alpha}$ is closed under $\leqslant^{*}$ sequences of length $\alpha^{+}$, and any statement can be decided by $\leqslant^{*}$ densely many conditions of $P_{j(\alpha)}^{*} / P_{\alpha}$. Note also that for $i<\alpha^{++}, j\left(\sigma_{i}\right)$ is a $P_{j(\alpha)}^{*}$ name for a subset of $j\left(\alpha^{++}\right)$; in $V[G]$ we can think of it as a $P_{j(\alpha)}^{*} / P_{\alpha}$ name.

We shall use this to recursively define a descending $\alpha^{++}$sequence of conditions $p_{0} \geqslant{ }^{*} p_{1} \geqslant * \ldots$ of $P_{j(\alpha)}^{*} / P_{\alpha}$. (Technically, this means $p_{i}$ will be a $P_{\alpha}$ name in $M$ for a condition of $P_{j(\alpha)}^{*}$ which is compatible with the $P_{\alpha}$ generic filter.)

To get a canonical choice function for the definition, let us fix some well ordering $>$ in $M$ on the elements of $P_{j(\alpha)}^{*} / P_{\alpha}$. (Note that $>$ need not have any particular relation to $\geqslant *$.) As part of the recursion, we will also inductively show that all the initial segments $\left(p_{l}\right)_{l<i}$ of the sequence are definable in terms of $G, j^{\prime \prime} \alpha^{+}$, and $\left(j\left(\sigma_{k}\right)\right)_{k_{i}}$. The sequence itself, however, will not be in $M$ but only in $V$.

Now, we take $p_{0}=\mathbb{1}$. For $i=k+1<\alpha^{++}$, we define $p_{i}$ to be the $>$ least element of $M$ which is $\leqslant^{*} p_{k}$ and which decides

$$
j^{\prime \prime} \check{\alpha}^{+} \in j\left(\sigma_{k}\right)
$$

$p_{i}$ is definable in terms of $>, j^{\prime \prime} \alpha^{+}, j\left(\sigma_{k}\right)$ and $p_{k}$, which by inductive hypothesis is itself definable in terms of $>, j^{\prime \prime} \alpha^{+}$and $\left(j\left(\sigma_{l}\right)\right)_{l<k}$. So $\left(p_{l}\right)_{l<i+1}$ is definable in terms of $>, j^{\prime \prime} \alpha^{+}$and $\left(j\left(\sigma_{l}\right)\right)_{l<i+1}$.

Now suppose that $i<\alpha^{++}$is a limit. Then $\left(p_{l}\right)_{l<i}$ is definable in terms of $>, j^{\prime \prime} \alpha^{+}$and $\left(j\left(\sigma_{k}\right)\right)_{k<i}$. These are in $M$, so the sequence $\left(p_{l}\right)_{l<i} \in M$. Hence there exist $p \in M$ such that $p \leqslant^{*} p_{l}$ for all $l<i$. Let $p_{i}$ be the $>$ least such $p . p$ is definable in terms of $\left(p_{l}\right)_{l<i}$ and $>$, so it is also definable in terms of $>, j^{\prime \prime} \alpha^{+}$and $\left(j\left(\sigma_{k}\right)\right)_{k<i+1}$.

Now, the overall sequence $\left(p_{i}\right)_{i<\alpha^{++}}$does not exist in $M$, but it does exist in $V$ and hence in $V[G]$. Working in $V[G]$, let $U^{*}$ be defined by:

$$
\sigma_{i}^{G} \in U^{*} \Longleftrightarrow p_{i} \Vdash j^{\prime \prime} \alpha^{+} \in j\left(\check{\sigma}_{i}\right)
$$

Here we are viewing $j\left(\sigma_{i}\right)$ as the $P_{j(\alpha)}^{*} / P_{\alpha}$ name for a subset of $j\left(\alpha^{+}\right)$.
Claim 5.4.41. $U^{*}$ is a well-defined normal ultrafilter on $\mathcal{P}_{\alpha}\left(\alpha^{+}\right)$.

Proof. We go through the properties of a normal ultrafilter on $\mathcal{P}_{\alpha}\left(\alpha^{+}\right)$.
Well-defined: Suppose that $\sigma_{i}^{G}=\sigma_{k}^{G}, i<k$. Then $p_{i}$ decides whether $j^{\prime \prime} \alpha^{+} \in j\left(\sigma_{i}\right)$ or not. $p_{k} \leqslant * p_{i}$ so it agrees with $p_{i}$ about this.

Filter: If $\sigma_{i}^{G}, \sigma_{k}^{G} \in U^{*}$ then let $l$ be such that $\sigma_{l}^{G}=\sigma_{i}^{G} \cap \sigma_{k}^{G}$. Let $m=\max \{i, l, m\}$. Then $p_{m}$ forces that $j^{\prime \prime} \alpha^{+} \in j\left(\sigma_{i}\right) \cap j\left(\sigma_{k}\right)=j\left(\sigma_{l}\right)$. Also $p_{m} \leqslant{ }^{*} p_{l}$ so $p_{l}$ cannot decide $j^{\prime \prime} \alpha^{+} \notin j\left(\sigma_{l}\right)$. So $p_{l}$ decides $j^{\prime \prime} \alpha^{+} \in j\left(\sigma_{l}\right)$ and hence $\sigma_{l}^{G} \in U^{*}$. Similarly, if $\sigma_{i}^{G} \subset \sigma_{k}^{G}$ and $\sigma_{i}^{G} \in U^{*}$ then let $l=\max \{i, k\}$; then $p_{l}$ cannot decide that $j^{\prime \prime} \alpha^{+} \notin j\left(\sigma_{k}\right)$, so $p_{k}$ decides that $j^{\prime \prime} \alpha^{+} \in j\left(\sigma_{k}\right)$.

Ultrafilter: Suppose that $\sigma_{i}^{G} \notin U^{*}$; let $\alpha^{+} \backslash \sigma_{i}^{G}=\sigma_{k}^{G}$, and let $l=\max \{i, k\}$. Then $p_{l}$ cannot decide that $j^{\prime \prime} \alpha^{+} \notin j\left(\sigma_{k}\right)$ so $p_{k}$ decides that $j^{\prime \prime} \alpha^{+} \in j\left(\sigma_{k}\right)$.

Complete: Let $\sigma_{i_{\gamma}}^{G}: \gamma<\beta$ be a collection of elements of $U^{*}$, with $\beta<\alpha$. Let $k$ be such that $\sigma_{k}^{G}=\bigcap \sigma_{i_{\gamma}}$. Let $l=\max \left\{i_{\gamma}: \gamma<\beta\right\} \cup\{k\}<\alpha$. Then $p_{l}$ cannot decide that $j^{\prime \prime} \alpha^{+} \notin j\left(\sigma_{k}\right)$ so $p_{k}$ decides that $j^{\prime \prime} \alpha^{+} \in j\left(\sigma_{k}\right)$.

Fine: Fix $\beta<\alpha^{+}$. Let $X=\left\{x \in\left(P_{\alpha}\left(\alpha^{+}\right)\right)^{V}: \beta \in x\right\}$. By definition of $U$, we know that $X \in U$ and hence that $j^{\prime \prime} \alpha^{+} \in j(X)$. Since $X \in V$, we know that $\check{X}=\sigma_{i}$ for some $i$. (This is the reason we insisted on including all the check names in our collection of $\sigma_{i}$ 's near the start of the lemma.) And of course, $j(\check{X})=(j(\check{X}))$. So clearly $\mathbb{1} \Vdash j^{\prime \prime} \alpha^{+} \in j(X)$, and hence $X \in U^{*}$. But $X \subset\left\{x \in P_{\alpha}\left(\alpha^{+}\right)^{V[G]}: \beta \in x\right\}$, so the latter is also in $U^{*}$.

Normal: Let $\sigma_{i_{\gamma}}^{G}: \gamma<\alpha^{+}$be a collection of elements of $U^{*}$. Let $k$ be such that $\sigma_{k}^{G}=\Delta_{\gamma<\alpha^{+}} \sigma_{\gamma}^{G}$. Let $l=\max \left\{i_{\gamma}: \gamma<\alpha^{+}\right\} \cup\{k\}<\alpha^{++}$. Then $p_{l}$ cannot decide that $j^{\prime \prime} \alpha^{+} \notin j\left(\sigma_{k}\right)$, so $p_{k}$ decides that $j^{\prime \prime} \alpha^{+} \in j\left(\sigma_{k}\right)$.

Hence $\alpha$ is $\alpha^{+}$supercompact in $V[G]$ by Lemma 3.1.16.
Corollary 5.4.42. For $\alpha<\kappa, \dot{Q}_{\alpha}$ is nontrivial if any only if $\alpha$ is $\alpha^{+}$supercompact in $V$.
Lemma 5.4.43. Let $\alpha \leqslant \kappa$ and let $G$ be a $P_{\alpha}$ generic filter. Suppose that $\beta<\alpha$ is in $\mathrm{Reg}^{\epsilon}$ from the perspective of $V[G]$. Then $V[G]$ believes that $(\beta, f(\beta))$ contains unboundedly many elements of $\operatorname{Reg}_{\delta}$ for all $\delta<\epsilon$, but no elements of $\operatorname{Reg}_{\epsilon}$. Also, $V[G]$ does not believe that $f(\beta)$ is in $\operatorname{Reg}_{\epsilon}$.

Proof. Since the interval contains no elements of $\operatorname{Reg}_{\epsilon}$ even in $V$ by definition of $f$, it is immediate that it contains none in $V[G]$ either. Similarly, $f(\beta) \notin \operatorname{Reg}_{\epsilon} V[G]$.

Let $\mu \leqslant \beta$ be be the supremum of the class of $\lambda^{+}$supercompacts $\lambda$ which are $\leqslant \beta$. Let $\nu>\beta$ be the least $\lambda>\beta$ which is $\lambda^{+}$supercompact. Note that $P_{\alpha}=P_{\mu} * \dot{Q}_{\mu} *\left(P_{\alpha} / P_{\nu}\right)$ (or, if we want to be very formal, these two forcings are equivalent via a trivial relabelling of conditions).

It follows trivially from Corollary 5.4 .35 that $P_{\mu}$ satisfies the $\mu^{++}$chain condition, and therefore does not collapse or singularise any cardinals $\geqslant \mu^{++}$. But we know that $f(\beta)$ is a limit of elements of $\operatorname{Reg}_{0}=\operatorname{Succ}$, so $f(\beta)$ is a limit cardinal. Hence, $\mu^{++} \leqslant \beta^{++}<f(\beta)$.
$\dot{Q}_{\mu}$ is either trivial, or is of the form $\mathbb{Q}_{\mu}$, which we know by Lemma 5.4.23 does not collapse or singularise any cardinals other than $\mu$.
$\nu$ is a successor element of $X$, and hence is $\nu^{+}$supercompact in $V$. In particular, then, it is Mahlo. So by Lemma 5.4.38, $P_{\alpha} / P_{\nu}$ does not collapse or singularise any cardinals below $\nu$. But since $\nu>\beta>\epsilon$, and (again) $\nu$ is Mahlo in $V$, there are certainly elements of $\operatorname{Reg}_{\epsilon}$ in the interval $(\beta, \nu)$. It follows that $\nu>f(\beta)$.

Putting these facts together, we find that $P_{\alpha}$ does not collapse or singularise any cardinals in the interval $\left(\left(\beta^{++}\right)^{V}, f(\beta)\right)$, and hence that in $V[G]$ there are unboundedly many elements of $\operatorname{Reg}_{\delta}$ for all $\delta<\epsilon$.

So now we know that $P_{\kappa}$ is a well-defined forcing which combines a $\mathbb{Q}_{\lambda}$ on every $\lambda^{+}$supercompact $\lambda<\kappa$. Also, $\kappa$ is still $\kappa^{+}$supercompact and hence Mahlo in the generic extension, so the following forcing is well-defined.

Definition 5.4.44. The forcing $\mathbb{P}$ is defined as

$$
P_{\kappa} * \mathrm{NM}_{\kappa} * \operatorname{Col} \dot{C}
$$

Here, $\dot{C}$ is a $P_{\kappa} * \mathrm{NM}_{\kappa}$ name for the generic club added by $\mathrm{NM}_{\kappa}$. (We are abusing notation slightly here: strictly speaking, the forcing is $P_{\kappa} * \mathrm{NM} * \dot{\mathrm{Col}}$ where NM is a name for $\mathrm{NM}_{\kappa}$ and $\dot{\mathrm{Col}}$ is a name for $\operatorname{Col}(\dot{C})$. This kind of abuse is standard shorthand.)

### 5.4.7 The LST number

Now, let us fix a $\mathbb{P}$ generic extension $V^{* *}$ of $V$. Let the corresponding generic filter be $\tilde{G} * C * G^{* *}$. So $\tilde{G}$ is a $P_{\kappa}$ generic filter, and $C$ is an $\mathrm{NM}_{\kappa}$ generic club over $V[\tilde{G}]$, and $G^{* *}$ is a $\operatorname{Col}(C)$ generic extension over $V[\tilde{G}][C]$.

To tidy up our notation, let us write $\tilde{V}$ for $V[\tilde{G}]$. By Corollary 5.4 .8 and Lemma 5.4.11 we know that the cardinals of $V^{* *}$ below $\kappa$ are precisely the cardinals of $\tilde{V}$ which are either below $\min (C)=\omega$ or are in some interval $[\alpha, f(\alpha)]$ for some $\alpha \in C$. By Lemma 5.4.43 it follows that in $V^{* *}$ there are unboundedly many elements of $\operatorname{Reg}_{\delta}$ below $\kappa=\sup (C)$ for all $\delta<\epsilon$, but no elements of $\operatorname{Reg}_{\epsilon}$.

Also, $\kappa$ is regular (since it was Mahlo in $\tilde{V}$ and neither $\mathrm{NM}_{\kappa}$ nor $\operatorname{Col}(C)$ singularise it). So $\kappa=$ $\min \left(\operatorname{Reg}_{\epsilon}^{V^{* *}}\right)$.

Our goal now is to show that in $V^{* *}$,

$$
\operatorname{LST}\left(I, Q^{\epsilon}\right)=\operatorname{LST}\left(I, R^{\epsilon}\right)=\kappa
$$

By Theorem 5.3.1 and 5.3.3, we know that $\operatorname{LST}\left(I, Q^{\epsilon}\right) \geqslant \operatorname{LST}\left(I, R^{\epsilon}\right) \geqslant \kappa$. (Recall that $\epsilon$ is countable, so there are certainly no hyperinaccessibles below it.) So it suffices to show that $\operatorname{LST}\left(I, Q^{\epsilon}\right) \leqslant \kappa$.

Let $\mathcal{A} \in V^{* *}$ be a structure in some first order language $\mathcal{L}$ for cardinality less than $\kappa$. Without loss of generality, let us assume $|\mathcal{A}|=\mu$ is much larger than $\kappa$. (Any smaller structure can be easily padded out by gluing on a extra structure of large cardinality over which all the symbols of $\mathcal{L}$ are interpreted trivially, and adding one extra predicate to let us identify the original structure.) In fact, without loss of generality, we can just assume that the domain of $\mathcal{A}$ is $\mu$ itself. We want to find an $\mathcal{L} \cup\left\{I, Q^{\epsilon}\right\}$ elementary substructure of $\mathcal{A}$ of cardinality less than $\kappa$.

By Lemma 5.4.5, we can find a model $M$ with $M^{\mu} \subset M$ (from the perspective of $V$ ) and an elementary embedding $j: V \rightarrow M$ with critical point $\kappa$, such that $j(\kappa)>\mu$ and $j(f)(\kappa)>\mu$.

We want to use Lemma 5.4.2 to extend $j$ to an elementary embedding $j^{*}$ of $V^{* *}$ into some $j(\mathbb{P})$ generic extension of $M$. The first step is to work out what $j(\mathbb{P})$ generic filter we're going to use.

Let's start by working out what $j(\mathbb{P})$ looks like. Since the critical point of $j$ is $\kappa$, we know that $j\left(P_{\alpha}\right)=P_{\alpha}$ for $\alpha<\kappa$. (We saw the proof of this in Lemma 5.4.40.) What about $j\left(P_{\kappa}\right)$ ? It will start off by just constructing $P_{\kappa}$, but will then have some extra steps of the iteration to get it up to $j(\kappa)$ stages. To be precise, it will add in $\mathbb{Q}_{\lambda}$ for all $\lambda^{+}$supercompacts $\lambda$ in the interval $[\kappa, j(\kappa))$ in $M$.

Now, since $M^{\mu} \subset M$ we know that below $\mu, M$ and $V$ agree on the cardinals and their cofinalities. Also, we know that $\kappa$ is $\kappa^{+}$supercompact in $M$ : by Lemma 3.1.16 this is equivalent to the existence of a normal ultrafilter on $P_{\kappa}\left(\kappa^{+}\right)$, and since $M$ is closed under $\mu$ sequences and $\mu>\kappa^{++}$this is absolute between $V$ and $M$. On the other hand, there are no other $\lambda^{+}$supercompact cardinals $\lambda$ between $\kappa$ and $\mu$ in $M$ : there are no even any elements of $\operatorname{Reg}_{\epsilon}$ in this interval, since $\operatorname{Reg}_{\epsilon}^{V} \subset \kappa$. So $j\left(P_{\kappa}\right)$ is the iterated forcing:

$$
P_{\kappa} * \mathbb{Q}_{\kappa} * P_{j(\kappa)} / P_{\nu}
$$

where $j(\kappa) \geqslant \nu>\mu$ is the least cardinal above $\mu$ such that $M$ believes $\nu$ is $\nu^{+}$supercompact. (We are slightly abusing notation: strictly speaking, we only defined $P$ up to $P_{\kappa}$, within $V$. By $P_{\nu}$ for $\nu \leqslant j(\kappa)$ we mean the forcing in $M$ which is defined analogously to $P_{\alpha}$ for $\alpha \leqslant \kappa$ in $V$.)

Then $j(\mathbb{P})$ is

$$
P_{\kappa} * \mathbb{Q}_{\kappa} * P_{j(\kappa)} / P_{\nu} * \mathrm{NM}_{j(\kappa)} * \operatorname{Col}(\dot{D})
$$

where $\dot{D}$ is the name for the club added by $\mathrm{NM}_{j(\kappa)}$.
In order to make $j^{*}$ exist, we need to find a $j(\mathbb{P})$ generic filter which extends $\tilde{G} * C * G^{* *}$. We will need the following well-known lemma.
Lemma 5.4.45. Let $\mathbb{Q}$ be a complete Boolean algebra, and let $\mathbb{P} \subset \mathbb{Q}$ be a complete Boolean subforcing. Suppose that $H$ is a $\mathbb{Q}$ generic filter, and that $G:=H \cap \mathbb{P}$ is $\mathbb{P}$ generic. Then there is a forcing $\mathbb{Q}_{G} \subset \mathbb{Q}$ (defined in $V[G]$ ) such that $H$ is a $\mathbb{Q}_{G}$ generic filter.
Proof. We define a function $\pi: \mathbb{Q} \rightarrow \mathbb{P}$ by

$$
\pi(q)=\inf \{p \in \mathbb{P}: p \geqslant q\}
$$

Notice that $\pi(q) \geqslant q$ for all $q \in \mathbb{Q}$. We then define

$$
\mathbb{Q}_{G}:=\{q \in \mathbb{Q}: \pi(q) \in G\}
$$

Although we will avoid using this terminology, our goal is essentially to show that $\pi$ is a projection map from $\mathbb{Q}=\mathbb{P} * \mathbb{Q}_{G}$ onto $\mathbb{P}$.

First, we shall establish that $H \subset \mathbb{Q}_{G}$. Let $q \in H$. Let

$$
D_{q}:=\{\pi(q)\} \cup\{\neg p \in \mathbb{P}: p \geqslant q\} \subset \mathbb{P}
$$

Now, $D_{q}$ is clearly predense: if $r \in \mathbb{P}$ is incompatible with $\neg p$ for all $p \geqslant q$, then $r \leqslant p$ for all $p \geqslant q$, and hence $r \leqslant \pi(q)$ Since $G$ is generic, $G \cap D_{q} \neq \varnothing$. But for all $p \geqslant q$, we know $p \in H$ and hence $p \in G$, so $\neg p \notin G$. By process of elimination, $\pi(q) \in G$, and so $q \in \mathbb{Q}_{G}$. Hence $H \subset \mathbb{Q}_{G}$.

We know $H$ is a filter; it remains to show that it is generic over $\mathbb{Q}_{G}$.
Let $D \in V[G]$ be an open dense subset of $\mathbb{Q}_{G}$. Let $\dot{D}$ be a $\mathbb{P}$ name for $D$, which is forced by $\mathbb{1}$ to be dense and open in $\mathbb{Q}_{\dot{G}}$. Let

$$
D^{\prime}:=\{q \in \mathbb{Q}: \pi(q) \Vdash \check{q} \in \dot{D}\}
$$

Note that $D^{\prime} \in V$. We shall show that it is dense in $\mathbb{Q}$. Fix $q \in \mathbb{Q}$. Let $\bar{G}$ be any $\mathbb{P}$ generic filter containing $\pi(q)$. Then $q \in \mathbb{Q}_{\bar{G}}$.

Now, $\dot{D}^{\bar{G}}$ is dense in $\mathbb{Q}_{\bar{G}}$, so let $r \in \dot{D}^{\bar{G}}$ be an element of $\mathbb{Q}_{\bar{G}}$ which is below $q$. Find $p \in \bar{G}$ which forces $\check{r} \in \dot{D}$.

Since $r \in \mathbb{Q}_{\bar{G}}$ we know $\pi(r) \in \bar{G}$. So $p \| \pi(r)$. Without loss of generality, say $p \leqslant \pi(r)$.
Claim 5.4.46. There is a condition $s \leqslant r$ with $\pi(s) \leqslant p$.
In fact, the existence of such an $s$ for all $p \leqslant \pi(r)$ is one way to define a projection map $\pi$.
Proof. We claim that $r \| p$. Note that $\neg p=\sup \left\{p^{\prime} \in \mathbb{P}: p^{\prime} \perp p\right\}$. So if $r \perp p$ then $r \leqslant \neg p$. But then $\pi(r) \leqslant \neg p$, by definition of $\pi$ and the fact that $\neg p \in \mathbb{P}$. Hence $p \leqslant \pi(r) \leqslant \neg p$, a contradiction. So $r \| p$.

So let $s \leqslant p, r$. Since $p \in \mathbb{P}$, we know by definition of $\pi$ that $\pi(s) \leqslant p$, so $s$ is as required.
Now, $p \Vdash \check{r} \in \dot{D}$, so since $\dot{D}$ is forced to be open and $s \leqslant r, p \Vdash \check{s} \in \dot{D}$ as well. And $\pi(s) \leqslant p$, so $\pi(s) \Vdash \check{s} \in \dot{D}$. So $s \in D^{\prime}$. Since $s \leqslant r \leqslant q$ and $q$ was arbitrary, $D^{\prime}$ is dense.

Now, since $H$ is generic over $\mathbb{Q}$, we know there is some $q \in H \cap D^{\prime}$; that is, a $q \in H$ such that $\pi(q) \Vdash q \in \dot{D}$. But then $\pi(q) \geqslant q \in H$ so $\pi(q) \in H \cap \mathbb{P}=G$. Hence, $q \in \dot{D}^{G}=D$. We also know $q \in H$, so $H \cap D \neq \varnothing$. Hence, $H$ is $\mathbb{Q}_{G}$ generic.

We use this lemma to extend $\tilde{G} * C * G^{* *}$ to a $P_{\kappa} * \mathbb{Q}_{\kappa} * \operatorname{Col}(C)$ generic filter.
Lemma 5.4.47. There exists a $\left(\mathbb{Q}_{\kappa}\right)^{\tilde{V}}$ generic filter $G$ and $a \operatorname{Col}{ }^{\tilde{V}[G]}\left(C_{G}\right)$ generic filter $G^{*}$ such that:

1. $C_{G}=C$
2. $H_{G} \in G^{*}$
3. $G^{* *}=G^{*} \cap \operatorname{Col}^{\tilde{V}[C]}(C)$

Proof. Just for this proof, let us write $B(\mathbb{Q})$ for the Boolean completion of a forcing $\mathbb{Q}$.
First, we show that the claim will hold if and only if $V^{* *}$ believes the following statement. (The technical details of this formalisation aren't very important; we just need to establish that it's something we can express within $V^{* *}$.)
"There is a forcing $\mathbb{R} \subset B\left(\mathbb{Q}_{\kappa} * \operatorname{Col}\left(\dot{C}_{G}\right)\right)$ such that some condition in $\mathbb{R}$ forces that the $\mathbb{R}$ generic filter is also generic over $B\left(\mathbb{Q}_{\kappa} * \operatorname{Col}\left(\dot{C}_{G}\right)\right)$, and that the equivalent generic filter $G * G^{*}$ on $\mathbb{Q}_{\kappa} * \operatorname{Col}\left(\dot{C}_{G}\right)$ is such that $C_{G}=C, \dot{H}_{G} \in G^{*}$ and $G^{* *}=G^{*} \cap \operatorname{Col}^{\tilde{V}[C]}(C)$."

It is clear that if $V^{* *}$ believes this, then we can find our filters $G$ and $G^{*}$ as required by the claim. On the other hand, suppose we can find $G$ and $G^{*}$ as the claim describes. Then apply Lemma 5.4.45 in $\tilde{V}$, using $B\left(\mathrm{NM}_{\kappa} * \operatorname{Col}(\dot{C})\right)$ as $\mathbb{P}$ and $B\left(\mathbb{Q}_{\kappa} * \operatorname{Col}\left(\dot{C}_{G}\right)\right)$ as $\mathbb{Q}$. We use the Boolean analogue of $C * G^{* *}$ in place of $G$ and the analogue of $G * G^{*}$ in place of $H$. This gives us a subforcing $\mathbb{R}$ (denoted $Q_{G}$ in the lemma) of $B\left(\mathbb{Q}_{\kappa} * \operatorname{Col}\left(\dot{C}_{G}\right)\right)$, which exists in $\tilde{V}\left[C * G^{* *}\right]=V^{* *}$, and which is such that the Boolean analogue of $G * G^{*}$ is generic over this subforcing. This witnesses that the statement is then true. So $V^{* *}$ knows whether such a $G$ and $G^{*}$ exist (although of course it won't know what they look like).

Now, suppose that for some choice of $C$ and $G^{* *}$, no such $G$ and $G^{*}$ exist. Then we can find some condition $(c, \dot{h}) \in \mathrm{NM}_{\kappa} * \operatorname{Col}(C)$ forcing that $G$ and $G^{*}$ do not exist. Since $\operatorname{Col}^{V^{* *}[C]}(C) \subset \tilde{V}$, we know $h:=\dot{h}^{C} \in \tilde{V}$, and so without loss of generality we can choose $c$ such that $c \Vdash \dot{h}=\check{h}$. Further, without loss of generality, we can then assume that $\dot{h}=\check{h}$.

Working in $\tilde{V}$, let $G$ be a $\mathbb{Q}_{\kappa}$ generic filter such that $c$ is an initial segment of $C_{G}$ and $h \subset H_{G}$. (Note that we aren't insisting that $C_{G}=C$. It is easy to see that such a $G$ exists: $h$ is bounded below $\kappa$, so by extending $c$ if necessary we may assume $h \in \operatorname{Col}(c)$, and then we can just take a condition $p$ with $c_{p}=c, h_{p}=h$ and take a generic containing $p$.) Let $G^{*}$ be a $\mathrm{Col}^{\tilde{V}[G]}\left(C_{G}\right)$ generic filter containing $H_{G}$, and hence also containing $h$. Let $G_{\text {new }}^{* *}:=G^{*} \cap \operatorname{Col} \tilde{V}^{\left[C_{G}\right]}\left(C_{G}\right)$. As we saw in Lemma 5.4.28, $G_{\text {new }}^{* *}$ is $\operatorname{Col}{ }^{\tilde{V}\left[C_{G}\right]}$ generic. Let $C_{\text {new }}=C_{G}$.

Consider the model $\tilde{V}\left[C_{\text {new }}\right]\left[G_{\text {new }}^{* *}\right]$. Clearly, there exist $G$ and $G^{*}$ corresponding to $C_{\text {new }}$ and $G_{\text {new }}^{* *}$ in the way described in the claim: the $G$ and $G^{*}$ we have already defined are the ones we need. And since $c \in C_{\text {new }}=C_{G}$ and $h \in G^{*} \cap \operatorname{Col}^{\tilde{V}\left[C_{G}\right]}\left(C_{G}\right)=G_{\text {new }}^{* *}$ we know that $(c, \check{h}) \in C_{\text {new }} * G_{\text {new }}^{* *}$. But ( $c, \check{h}$ ) was supposed to force that there were no such $G$ and $G^{*}$. Contradiction.

Now, let us define the $j(\mathbb{P})$ filter $G^{M}$. Recall that

$$
j(\mathbb{P})=P_{\kappa} * \mathbb{Q}_{\kappa} * P_{j(\kappa)} / P_{\nu} * \operatorname{NM}_{j(\kappa)} * \operatorname{Col}\left(\dot{C}^{\prime}\right)
$$

We shall define generic filters $G_{1}, \ldots, G_{5}$ on each of these forcings in turn, which together will make up $G^{M}$. Recall that we already have fixed generic sets $\tilde{G}$ (over $P_{\kappa}$ ), $C$ (over $\mathrm{NM}_{\kappa}$ in $V[\tilde{G}]$ ) and $G^{* *}$ (over $\operatorname{Col}(C)$ in $V[\tilde{G}][C]), G$ (over $\mathbb{Q}_{\kappa}$ in $\left.V[\tilde{G}]\right)$ and $G^{*}\left(\right.$ over $\operatorname{Col}\left(C_{G}\right)$ in $\left.V[\tilde{G}][G]\right)$. The generic filters $G_{1}, \ldots, G_{5}$ are defined as follows:

- $G_{1}:=\tilde{G} ;$
- $G_{2}:=G$;
- $G_{3}$ is some arbitrary $P_{j(\kappa)} / P_{\nu}$ generic filter over $V\left[G_{1} * G_{2}\right]$;
- $G_{4}$ is an $\mathrm{NM}_{\kappa}$ generic club extending $C=C_{G}$. This is possible since $\kappa$ is singular in $V\left[G_{1} * G_{2} * G_{3}\right]$, and $C \in \tilde{V}[C] \subset V\left[G_{1} * G_{2}\right]$, and neither $\mathbb{Q}_{\kappa}$ nor $P_{j(\kappa)} / P_{\nu}$ collapse or singularise any cardinals below $\kappa$, so $C$ is a condition of $\mathrm{NM}_{j(\kappa)}$ in $V\left[G_{1} * G_{2} * G_{3}\right]$.
Defining $G_{5}$ is a little more complicated. We know that $C$ is an initial segment of the generic club $G_{4}$, so we can break down

$$
\mathrm{Col}^{V\left[G_{1} * \ldots * G_{4}\right]}\left(G_{4}\right)=\mathrm{Col}^{V\left[G_{1} * \ldots * G_{4}\right]}(C) * \mathrm{Col}^{V\left[G_{1} * \ldots * G_{4}\right]}\left(G_{4} \backslash \kappa\right)
$$

We know that $P_{j(\kappa)} / P_{\nu}$ and $\mathrm{NM}_{j(\kappa)}$ do not collapse or singularise any cardinals below $\kappa$, or otherwise add any new elements or subsets of $\operatorname{Col}(C)$. So

$$
\mathrm{Col}^{V\left[G_{1} * \ldots * G_{4}\right]}(C)=\mathrm{Col}^{V\left[G_{1} * G_{2}\right]}(C)=\mathrm{Col}^{\tilde{V}[G]}\left(C_{G}\right)
$$

Moreover, this forcing has the same dense subsets in $\tilde{V}[G]$ and $V\left[G_{1} * \ldots * G_{4}\right]$. So the filter $G^{*}$, defined in Lemma 5.4.47, is $\mathrm{Col}^{V\left[G_{1} * \ldots * G_{4}\right]}(C)$ generic over $V\left[G_{1} * \ldots * G_{4}\right]$.

Now, let $G^{\prime}$ be some arbitrary $\mathrm{Col}^{V\left[G_{1} * \ldots * G_{4}\right]}\left(G_{4} \backslash \kappa\right)$ generic filter over $V\left[G_{1} * \ldots * G_{4}\right]\left[G^{*}\right]$.
Finally, we define $G_{5}:=G^{*} * G^{\prime}$, and

$$
G^{M}:=G_{1} * G_{2} * G_{3} * G_{4} * G_{5}
$$

We now verify that $G^{M}$ satisfies the requirements for Lemma 5.4.2.

Claim 5.4.48. Let $p \in \tilde{G} * C * G^{* *}$. Then $j(p) \in G^{M}$.
Proof. Let us write $p=(q, c, h)$. (Formally, $c$ is a name for an element of $\mathrm{NM}_{\kappa}$, etc. In order to reduce notation, we will write $c$ for both this name and its interpretation in $V[\tilde{G}]$, and similarly for $h$.) To verify that $j(p) \in G^{M}$, it suffices to check that $j(q) \in \tilde{G}$, that $j(c) \subset C^{M}$ and that $j(h) \in G^{*} * G^{\prime}$. (Again, strictly speaking we mean that $j(c)$ is a name whose interpretation in $V\left[G_{1} * G_{2} * G_{3}\right]$ is an element of $C^{M}$, and give a similar statement for $h$.)
$q$ is easy: $j\left(P_{\kappa}\right)=P_{\kappa}$, so $j(q)=q \in \tilde{G}$ by assumption.
Similarly, $j(c)=c \in \mathrm{NM}_{j(\kappa)}$. By assumption $c \subset C$, and by definition of $C^{M}$, we know $C \subset C^{M}$.
Once again, $j(t)=t$. By assumption, $t \in G^{* *}$, and so since $G^{* *}=G^{*} \cap \tilde{V}[C]$, we know $t \in G^{*}$.
Hence, by Lemma 5.4.2 we can extend $j$ to an elementary embedding $j^{*}: V^{* *} \rightarrow M\left[G^{M}\right]$.
Recall that the domain of $\mathcal{A}$ is $\mu$. Consider $j \upharpoonleft \mu=j^{*} 1 \mu$. Since $\mu \subset V$ and $M$ is closed under $\mu$ sequences, we know that range $(j \upharpoonleft \mu) \in M$ and hence range $\left(j^{*} \upharpoonleft \mu\right) \in M\left[G^{M}\right]$. Also, since the critical point of $j^{*}$ is $\kappa$ and $|\mathcal{L}|<\kappa$, we also know that $j^{*}(\mathcal{L})=\mathcal{L}$. Hence, $j^{*}(\mathcal{A})$ is an $\mathcal{L}$ structure in $M\left[G^{M}\right]$, which is elementarily equivalent to $\mathcal{A}$ in $V^{* *}$ (even in the language $\mathcal{L} \cup\left\{I, Q^{\epsilon}, R^{\epsilon}\right\}$ since $j^{*}$ is fully elementary). Let $\mathcal{B}$ be the substructure of $j^{*}(\mathcal{A})$ whose domain is range $\left(j^{*} \upharpoonleft \mu\right)$. Note that $\mathcal{B} \in M\left[G^{M}\right]$, and since $j^{*}$ is elementary and $\mathcal{A}$ has domain $\mu$, we can easily see that $\mathcal{B}$ is isomorphic to $\mathcal{A}$. Also, $M\left[G^{M}\right]$ believes that $\mathcal{B}$ has cardinality less than $j^{*}(\kappa)$.

By construction, $M\left[G^{M}\right]$ and $V^{* *}$ have the same cardinals and regulars below $\kappa$. Also, since $\mathbb{P}$ and $j(\mathbb{P})$ do not collapse or singularise any cardinals in the interval $(\kappa, \mu]$, and the cardinals and cofinalities of $V$ and $M$ agree up to $\nu, M\left[G^{M}\right]$ and $V^{* *}$ have the same cardinals and regulars in that interval. So in fact, the two models agree completely on the cardinals and regulars $\leqslant \mu$, except that $M\left[G^{M}\right]$ thinks that $\kappa$ has cofinality $\omega$ and $V^{* *}$ thinks it is in $\mathrm{Reg}^{\alpha}$.

In particular, this means that $I$ and $Q^{\alpha}$ are interpreted the same way about subsets of $\mathcal{A}$ in $V^{* *}$ and the $\mathcal{L}$ isomorphic structure $\mathcal{B}$ in $M\left[G^{M}\right]$. The only case which is nontrivial is when we are using $Q^{\alpha}$ to compare two linear orders, one or both of which have cofinality $\kappa$ in $V^{* *}$ and therefore $\omega$ in $M\left[G^{M}\right]$. In that case, since $Q^{\alpha}$ only tells us about cofinalities in $\operatorname{Reg}_{<\alpha}$, we know that in $V^{* *}$ it will always be false regardless of whether the two linear orders actually have the same cofinality, and regardless of the third (auxiliary) formula we choose. Likewise, we know that $\omega \notin \operatorname{Reg}_{<\alpha}$, so $Q^{\alpha}$ will also be false in $M\left[G^{M}\right]$.

So the map

$$
j^{*} \upharpoonleft \mu: \mathcal{A}^{V^{* *}} \rightarrow \mathcal{B}^{M\left[G^{M}\right]}
$$

is an $\mathcal{L} \cup\left\{I, Q^{\epsilon}\right\}$ isomorphism.
But we also know that

$$
j^{*} \upharpoonleft \mathcal{A}: \mathcal{A}^{V^{* *}} \rightarrow j^{*}(\mathcal{A})^{M\left[G^{M}\right]}
$$

is an $\mathcal{L} \cup\left\{I, Q^{\epsilon}\right.$ ) elementary embedding, since $j^{*}: V^{* *} \rightarrow M\left[G^{M}\right]$ is fully elementary. So $M\left[G^{M}\right]$ believes that $\mathcal{B}$ is an $L\left(I, Q^{\epsilon}\right)$ elementary substructure of $j^{*}(\mathcal{A})$.

But now we know $M\left[G^{M}\right]$ believes
" $j^{*}(\mathcal{A})$ contains an $\mathcal{L} \cup\left\{I, Q^{\epsilon}\right\}$ elementary substructure of cardinality less than $j^{*}(\kappa)$ ".
Since $j^{*}: V^{* *} \rightarrow M\left[G^{M}\right]$ is elementary, $j^{*}(\mathcal{L})=\mathcal{L}$ and $j^{*}(\epsilon)=\epsilon$, it follows that $V^{* *}$ believes
" $\mathcal{A}$ contains an $\mathcal{L} \cup\left\{I, Q^{\epsilon}\right\}$ elementary substructure of cardinality less than $\kappa$ ".
But $\mathcal{A}$ was an arbitrary structure in $V^{* *}$, so

$$
\operatorname{LST}\left(I, Q^{\epsilon}\right) \leqslant \kappa
$$

in $V^{* *}$. As we've already seen, $\operatorname{LST}\left(I, Q^{\epsilon}\right) \geqslant \operatorname{LST}\left(I, R^{\epsilon}\right) \geqslant \kappa$, so

$$
\operatorname{LST}\left(I, Q^{\epsilon}\right)=\operatorname{LST}\left(I, R^{\epsilon}\right)=\kappa
$$

### 5.5 Open Questions

This proof suggests several possible extensions, and improvements to the result. These are beyond the scope of this thesis, but would be interesting avenues for future research.

Firstly, the results in this section have shown that $\operatorname{LST}\left(I, Q^{\epsilon}\right)$ and $\operatorname{LST}\left(I, R^{\epsilon}\right)$ can be the least element of $\mathrm{Reg}_{\epsilon}$. Could they be a larger element?

Question 16. Let $\alpha>0$ and $\epsilon>0$ be ordinals. Granting the consistency of a supercompact, is it consistent that $\operatorname{LST}\left(I, Q^{\epsilon}\right)$ and $\operatorname{LST}\left(I, R^{\epsilon}\right)$ are the $\alpha^{\prime}$ th element of $\operatorname{Reg}_{\epsilon}$ ?

One could also ask for an element of $\operatorname{Reg}_{\delta}$ for some $\delta>\epsilon$. I hypothesise that the answer to this question is "yes", and that we can adjust the proof as above to preserve an initial segment of the regular cardinals. The main difficulty would be in making sure that the LST number doesn't drop below $\kappa$ : in the proof above, we knew this automatically by Theorem 5.3.3. If the question were answered positively, then we would have shown that $\operatorname{LST}\left(I, R^{\epsilon}\right)$ and $\operatorname{LST}\left(I, Q^{\epsilon}\right)$ can be any element of $\operatorname{Reg}_{\geqslant \epsilon}$.

Question 17. Let $\epsilon>0$ be an ordinal. Is it possible that $\operatorname{LST}\left(I, Q^{\epsilon}\right)$ or $\operatorname{LST}\left(I, R^{\epsilon}\right)$ exist, and are not elements of $\mathrm{Reg}_{\geqslant \epsilon}$ ?

I hypothesise the answer to this question to be "no". An element of $\mathrm{Reg}_{\geqslant \epsilon}$ is somehow the simplest object which $Q^{\epsilon}$ and $R^{\epsilon}$ don't know about, so it seems natural that this should be the dividing line. A natural approach to proving it would be to imitate the proof of Theorem 5.3.3, but let $\mathcal{A}$ contain constant symbols for all the elements $H_{\lambda}$ for all $\lambda \in \operatorname{Reg}_{\epsilon} \cap \kappa$. But this would not work for limits of $\operatorname{Reg}_{\epsilon}$, or for cardinals which were only slightly above an element of $\mathrm{Reg}_{\epsilon}$.

These two questions, together, could give us an exact characterisation of the possible cardinals which can be LST numbers of these predicates. But there's still some information missing here. The technique we used for proving this rather critically assumed that there were no elements of $\operatorname{Reg}_{\epsilon}$ above $\kappa$. So $\kappa$ is not just the smallest element of $\operatorname{Reg}_{\epsilon}$, but the only one. The adaptations I propose above address the idea of adding inaccessibles below $\kappa$, but would still leave $\kappa$ as the largest element of $\operatorname{Reg}_{\epsilon}$.

Question 18. Let $\epsilon>0$ be an ordinal. Assuming sufficient large cardinal hypotheses, is it consistent that $\operatorname{LST}\left(I, Q^{\epsilon}\right)$ or $\operatorname{LST}\left(I, R^{\epsilon}\right)$ are elements of $\operatorname{Reg}_{\epsilon}$, but not its largest element? How about that $\operatorname{Reg}_{\epsilon}$ is unbounded?

Another natural question to ask is whether we can separate $\operatorname{LST}\left(I, Q^{\epsilon}\right)$ from $\operatorname{LST}\left(I, R^{\epsilon}\right)$.
Question 19. For $\epsilon>0$, is it consistent that $\operatorname{LST}\left(I, Q^{\epsilon}\right)>\operatorname{LST}\left(I, R^{\epsilon}\right)=\min \left(\operatorname{Reg}_{\epsilon}\right)$ ?
To avoid strange answers related to Theorem 5.3.1, it might be necessary to add the assumption that $\operatorname{LST}\left(I, R^{\delta}\right) \geqslant \delta$ for all $\delta \leqslant \epsilon$.

Finally, thoughout this thesis we have been looking at hierarchies of inaccessibles. Can we prove similar results with hierarchies of other large cardinals?

Question 20. Let $H$ be some large cardinal property (e.g. hyperinaccessibility, being Mahlo, etc.) For $\epsilon>0$, let $H_{\epsilon}$ be the class of elements of $H$ of Cantor-Bendixson rank $\epsilon$. Let $S^{\epsilon}$ be some second order predicate expressing $H_{<\epsilon}$, in the same way that $R^{\epsilon}$ and $Q^{\epsilon}$ express $\mathrm{Reg}_{<\epsilon}$.

For what choices of $H$ and $\epsilon$ can we show that $\operatorname{LST}\left(I, S^{\epsilon}\right) \geqslant \min H_{\epsilon}$, and that this inequality is optimal?

## Bibliography

[1] Uri Abraham and Saharon Shelah. Forcing closed unbounded sets. The Journal of Symbolic Logic, 48(3):643-657, 1983.
[2] Dominik Adolf, Arthur Apter, and Peter Koepke. Singularizing successor cardinals by forcing. Proceedings of the American Mathematical Society, 146(2):773-783, 2018.
[3] Carolin Antos. Class forcing in class theory. In The hyperuniverse project and maximality, pages 1-16. Springer, 2018.
[4] Carolin Antos and Sy-David Friedman. Hyperclass forcing in Morse-Kelley class theory. The Journal of Symbolic Logic, 82(2):549-575, 2017.
[5] Joan Bagaria. A characterization of Martin's axiom in terms of absoluteness. The Journal of Symbolic Logic, 62(2):366-372, 1997.
[6] Joan Bagaria. Bounded forcing axioms as principles of generic absoluteness. Archive for Mathematical Logic, 39(6):393-401, 2000.
[7] Jon Barwise, Solomon Feferman, and Solomon Feferman. Model-theoretic logics, volume 8. Cambridge University Press, 2017.
[8] James E. Baumgartner. Applications of the proper forcing axiom. In Handbook of set-theoretic topology, pages 913-959. Elsevier, 1984.
[9] Robert Beaudoin. The proper forcing axiom and stationary set reflection. Pacific journal of mathematics, 149(1):13-24, 1991.
[10] Omer Ben-Neria. Forcing Magidor iteration over a core model below O pistol. Archive for Mathematical Logic, 53(3):367-384, 2014.
[11] Omer Ben-Neria. A Mathias criterion for the Magidor iteration of Prikry forcings. 2021.
[12] Jörg Brendle. Forcing and the structure of the real line: The Bogotá lectures. Available at http: //www. logic. univie. ac. at/~ykhomski/ST2013/bogotalecture. pdf, 2009.
[13] Vincenzo Dimonte and Luca Motto Ros. Generalized descriptive set theory at singular cardinals of countable cofinality. In preparation.
[14] Matthew Foreman, Menachem Magidor, and Saharon Shelah. Martin's maximum, saturated ideals, and non-regular ultrafilters. Part I. Annals of Mathematics, pages 1-47, 1988.
[15] Rodrigo A Freire and Peter Holy. An axiomatic approach to forcing in a general setting. Bulletin of Symbolic Logic, pages 1-21.
[16] Gunter Fuchs. Aronszajn tree preservation and bounded forcing axioms. The Journal of Symbolic Logic, pages 1-16, 2021.
[17] Gunter Fuchs and Kaethe Minden. Subcomplete forcing, trees, and generic absoluteness. The Journal of Symbolic Logic, 83(3):1282-1305, 2018.
[18] Moti Gitik. Changing cofinalities and the nonstationary ideal. Israel Journal of Mathematics, 56(3):280314, 1986.
[19] Joel David Hamkins and Daniel Evan Seabold. Well-founded Boolean ultrapowers as large cardinal embeddings. arXiv preprint arXiv:1206.6075, 2012.
[20] Peter Holy, Regula Krapf, and Philipp Schlicht. Sufficient conditions for the forcing theorem, and turning proper classes into sets. Fundamenta Mathematicae, 246:27-44, 2019.
[21] Thomas Jech. Set theory. Springer Science \& Business Media, 2013.
[22] Ronald Jensen. Subcomplete forcing and $\mathcal{L}$-forcing. In $E$-recursion, forcing and $C^{*}$-algebras, pages 83-182. World Scientific, 2014.
[23] Ronald Jensen and John Steel. K without the measurable. The Journal of Symbolic Logic, 78(3):708-734, 2013.
[24] Akihiro Kanamori. The higher infinite: Large cardinals in set theory from their beginnings. Springer Science \& Business Media, 2008.
[25] Alexander Kechris. Classical descriptive set theory, volume 156. Springer Science \& Business Media, 2012.
[26] Kenneth Kunen. Set theory: An introduction to independence proofs. Elsevier, 2014.
[27] Richard Laver. Making the supercompactness of $\kappa$ indestructible under $\kappa$-directed closed forcing. Israel Journal of Mathematics, 29(4):385-388, 1978.
[28] Menachem Magidor. How large is the first strongly compact cardinal? or a study on identity crises. Annals of Mathematical Logic, 10(1):33-57, 1976.
[29] Menachem Magidor and Jouko Väänänen. On Löwenheim-Skolem-Tarski numbers for extensions of first order logic. Journal of Mathematical Logic, 11(01):87-113, 2011.
[30] William J Mitchell. Sets constructible from sequences of ultrafilters. The Journal of Symbolic Logic, 39(1):57-66, 1974.
[31] Assaf Rinot. Jensen's diamond principle and its relatives. Set theory and its applications, 533:125-156, 2011.
[32] Hiroshi Sakai. Separation of $M A^{+}(\sigma$-closed) from stationary reflection principles. Preprint, 2014.
[33] Hiroshi Sakai and Boban Veličković. Stationary reflection principles and two cardinal tree properties. Journal of the Institute of Mathematics of Jussieu, 14(1):69-85, 2015.
[34] Grigor Sargsyan. Nontame mouse from the failure of square at a singular strong limit cardinal. Journal of Mathematical Logic, 14(01):1450003, 2014.
[35] Ralf Schindler. A universal weasel without large cardinals in V. arXiv preprint math/0011077, 2000.
[36] Ralf Schindler and Martin Zeman. Fine structure. In Handbook of set theory, pages 605-656. Springer, 2010.
[37] Philipp Schlicht and Christopher Turner. Forcing axioms via ground model interpretations. Annals of Pure and Applied Logic.
[38] Saharon Shelah. Semiproper forcing axiom implies Martin's maximum but not PFA+. The Journal of Symbolic Logic, 52(2):360-367, 1987.
[39] John R Steel. Derived models associated to mice. In Computational Prospects Of Infinity: Part I: Tutorials, pages 105-193. World Scientific, 2008.
[40] John R Steel and Philip D Welch. $\Sigma_{3}^{1}$ absoluteness and the second uniform indiscernible. Israel Journal of Mathematics, 104(1):157-190, 1998.
[41] Stevo Todorčević. Forcing with a coherent Suslin tree. Preprint, 2011.
[42] Stevo Todorčević and Boban Veličković. Martin's axiom and partitions. Compositio Mathematica, 63(3):391-408, 1987.
[43] Boban Veličković. Forcing axioms and stationary sets. Advances in Mathematics, 94(2):256-284, 1992.
[44] William Weiss. Versions of Martin's axiom. In Handbook of set-theoretic topology, pages 827-886. Elsevier, 1984.
[45] P Welch. An introduction to inner model theory. lecture notes, 2019.
[46] PD Welch. Closed and unbounded classes and the härtig quantifier model. The Journal of Symbolic Logic, pages 1-22, 2019.
[47] Martin Zeman. Inner models and large cardinals. In Inner Models and Large Cardinals. de Gruyter, 2011.


[^0]:    - Your contact details
    -Bibliographic details for the item, including a URL
    -An outline nature of the complaint

[^1]:    ${ }^{1}$ It should be noted that this definition is from [21]. This diverges from [26], where $\kappa$-closed denotes the property here defined as $<\kappa$-closed.

[^2]:    ${ }^{2}$ Two forcings are equivalent if they both embed densely into some third forcing. Equivalent forcings produce the same generic extensions.

[^3]:    ${ }^{1}$ This follows from Theorem 2.3.1 (2) for $X=\kappa$ and $\alpha=1$.

[^4]:    ${ }^{2}$ Recall that $\mathrm{N}_{\mathbb{P}, X, \kappa}(\alpha)$ is only defined if $X$ has size at most $\kappa$.

[^5]:    ${ }^{3}$ Note the somewhat delicate nature of this statement: we cannot first take an arbitrary $\gamma$ such that $\sigma_{\gamma}^{G}=B$ then try to find $q$ such that $q \Vdash^{+} \sigma_{\gamma} \in \sigma$.

[^6]:    ${ }^{4}$ The equivalence $(1) \Leftrightarrow(4)$ is equivalent to Bagaria's version, since his definition of BFA refers to Boolean completions.
    ${ }^{5}$ The version $\Sigma_{0}-\mathrm{BN}_{\mathbb{P}, \kappa}^{1}(1)$ for single $\Sigma_{0}$-formulas is also equivalent by the proof below.

[^7]:    ${ }^{6} \operatorname{Cof}(\kappa)>\omega$ is in fact necessary to ensure that the set of codes on $\kappa$ for elements of $H_{\kappa^{+}}$is $\Sigma_{1}^{1}(\kappa)$-definable with parameters in $\mathcal{P}(\kappa)$. If $\operatorname{Cof}(\kappa)=\omega$ and $\kappa$ is a strong limit, then this set is $\Pi_{1}^{1}(\kappa)$-complete and hence not $\Sigma_{1}^{1}(\kappa)$ by a result of Dimonte and Motto Ros [13].

[^8]:    ${ }^{7}$ If $\kappa_{i}$ is multiplicatively closed, i.e. $\forall \alpha<\kappa \alpha \cdot \alpha<\kappa_{i}$, then this holds for Gödel's pairing function.
    ${ }^{8}$ This includes the case $V[G]=V$.

[^9]:    ${ }^{9}$ The assumption $2^{<\kappa}=\kappa$ is not needed for $(4) \Rightarrow(3)$.
    ${ }^{10}$ The assumption that there is no inner model with a Woodin cardinal is not used for $(5) \Rightarrow(3)$.

[^10]:    ${ }^{11}$ A more direct argument using [14, Page 20] and [43, Theorem 3.8] should be possible, but the required results are not explicitly mentioned there.

[^11]:    ${ }^{12}$ See Definition 2.1.1.

[^12]:    ${ }^{13}$ This assumption is equivalent to non(null) $=2^{\aleph_{0}}$. It follows from MA, but not from FA ${ }_{\text {random }}$ by known facts about Cichon's diagram.

[^13]:    ${ }^{1}$ If we were allowing multiple measures on the same cardinal in our premice, then we would have to drop the "strictly" here.

[^14]:    ${ }^{2}$ Formally, this means that if $i<$ On, then stage $i$ of $\mathcal{J}$ is trivial unless the critical point $\kappa_{i}$ of $\mathcal{I}$ is such that $\pi_{i, \text { On }}\left(\kappa_{i}\right) \leqslant \bar{M} X$. If this is true, then letting $\alpha_{j}$ be the cut-down at stage $j$ of $\mathcal{I}$, we have $\mathcal{J}$ do a cut-down to $\min _{j<i} \pi_{j, i}\left(\alpha_{j}\right)$, and then take the ultrapower with respect to $\kappa_{i}$. Verifying that this gives a well-defined iteration is an exercise for the reader.

[^15]:    ${ }^{3}$ Notice that this is not something that is definable in $V$, since it's a collection of classes. This is a definition in the metatheory. If this bothers you, then you can avoid going outside $V$ by working with $\bar{M}$ itself instead of $\tilde{V}$, but the notation becomes nasty.

[^16]:    ${ }^{1}$ We can actually show that $\bar{M}=\tilde{M}$, but we don't need that here.

[^17]:    ${ }^{2}$ This mouse is called $O^{\#}$, "O Sharp".

[^18]:    ${ }^{3}$ At this point, it's entirely plausible at this stage that $\tau_{X}$ can't ever be smaller than $\sup C^{*}$.

[^19]:    ${ }^{4}$ Of course, we've just shown that in fact $i=\lambda$ and $j=\lambda^{\prime}$, but we use $i$ and $j$ to emphasise that we're thinking of the ordinals as stages of the iteration.

[^20]:    ${ }^{5}$ Invoking genericity of $T$ and Ben Neria's result is actually very much overkill for showing this simple fact. It can be proved directly with an adjusted version of the second Claim in the proof of 4.1.7. But since we've proved genericity already, we might as well save ourselves some time by making use of it.

[^21]:    ${ }^{6}$ If the core contains no measurables then we say $\alpha=\infty$.

[^22]:    ${ }^{7}$ It might seem like we're making a mountain out of a molehill with this proof. Can't we just use the fact that $\beta$ is useful to conclude immediately that it's a top-measure critical point? But doing it this way tells us that it's specifically the critical point at stage $h=\sup \left\{h_{\gamma}: \gamma<\beta\right\}$.
    ${ }^{8}$ Note the similarities between this claim and the proof of Lemma 4.2.12. The only change is that we're now dealing with a $\beta$ sequence of critical points, whereas back there it was just an $\omega$ sequence. This makes no difference to the underlying proof, but it means we can't use the same shortcut we did in that lemma.

[^23]:    ${ }^{1}$ There's one small difference: $Q^{\text {On }}$ never tells us that two linear orders have the same cofinality if they happen to both have cofinality $\omega$. But in a moderately rich $\mathcal{L}$ structure, that's not a problem: we can discover whether the linear order has cofinality $\omega$ by comparing it to itself using $Q^{\mathrm{On}}$ with different auxiliary sets $Z$. If $Q^{\mathrm{On}}$ is never true, whatever auxiliary set we use, then we know that the order has cofinality $\omega$. And of course any two orders of cofinality $\omega$ have the same cofinality, so we can define the "missing" bit of $Q^{\text {e.c. }}$. The details are left to the reader.

