First Steps in Twisted Rabinowitz–Floer Homology

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To Neil and Jil. In memory of Will J. Merry. A brilliant teacher and a guiding light.

Abstract

Rabinowitz–Floer homology is the Morse–Bott homology in the sense of Floer associated with the Rabinowitz action functional introduced by Kai Cieliebak and Urs Frauenfelder in 2009. In our work, we consider a generalisation of this theory to a Rabinowitz–Floer homology of a Liouville automorphism. As an application, we show the existence of noncontractible periodic Reeb orbits on quotients of symmetric star-shaped hypersurfaces. In particular, our theory applies to lens spaces. Moreover, we show a forcing theorem, which guarantees the existence of a contractible twisted closed characteristic on a displaceable twisted stable hypersurface in a symplectically aspherical geometrically bounded symplectic manifold if there exists a contractible twisted closed characteristic belonging to a Morse–Bott component, with energy difference smaller or equal to the displacement energy of the displaceable hypersurface.

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Contents

Chapter 1
Introduction

The existence of closed Reeb orbits on lens spaces is important in the study of celestial mechanics. Indeed, by [23, Corollary 5.7.5], the Moser regularised energy hypersurface near the earth or the moon of the planar circular restricted three-body problem for energy values below the first critical value is diffeomorphic to the real projective space $\mathbb{RP}^3$. See also [35, Introduction] for more details. An explicit noncontractible periodic orbit can be found via Birkhoff's shooting method [23, Theorem 8.3.2]. We present the main result of this thesis.

Theorem 1.1 ([14, Theorem 1.2]). *Let $\Sigma \subseteq \mathbb{C}^n$, $n \geq 2$, be a compact and connected star-shaped hypersurface invariant under the rotation*

$$\varphi\colon \mathbb{C}^n \to \mathbb{C}^n, \quad \varphi(z_1, \ldots, z_n) := \left(e^{2\pi i k_1/m} z_1, \ldots, e^{2\pi i k_n/m} z_n\right)$$

for some even $m \geq 2$ and $k_1, \ldots, k_n \in \mathbb{Z}$ coprime to m. Then $\Sigma/\mathbb{Z}_m$ admits a noncontractible periodic Reeb orbit generating $\pi_1(\Sigma/\mathbb{Z}_m) \cong \mathbb{Z}_m$.

Theorem 1.1 has similarities with the following two recent results.

Theorem 1.2 ([29, Corollary 1.6 (iv)]). *Any contact form on $\mathbb{S}^{2n-1}/\mathbb{Z}_m$ defining the standard contact structure admits a closed Reeb orbit.*

Using the fact that there is a natural bijection between contact forms on the odd-dimensional sphere equipped with the standard contact structure and star-shaped hypersurfaces, Theorem 1.2 is actually stronger than Theorem 1.1 in that it does not restrict the parity of the lens space. However, Theorem 1.2 does not say anything about the topological nature of the Reeb orbit. The proof of this theorem uses a generalisation of Givental's nonlinear Maslov index to lens spaces.

Theorem 1.3 ([41, Theorem 1.2]). *Every dynamically convex star-shaped C^3-hypersurface $\Sigma \subseteq \mathbb{C}^n$, $n \geq 2$, satisfying $\Sigma = -\Sigma$ admits at least two symmetric geometrically distinct closed characteristics.*

Theorem 1.3 has the advantage of being a *multiplicity result*, but in disadvantage requires the assumption that the hypersurface is dynamically convex and does only work for $\mathbb{Z}_2$-symmetry. To the authors knowledge, the first named author of [41] is working on extending Theorem 1.3 to lens spaces. As many multiplicity results, the proof of this theorem makes use of index theory and in particular Ekeland–Hofer theory. The proof of Theorem 1.1 relies on a generalisation of *Rabinowitz–Floer homology*. Rabinowitz–Floer homology is the Morse–Bott homology in the sense of Floer associated with the Rabinowitz action functional introduced by Kai Cieliebak and Urs Frauenfelder in 2009. See the excellent survey article [8] for a brief introduction to Rabinowitz–Floer homology and [2] for an overview of common Floer theories. One important feature of this homology in our work is that it provides an affirmative answer to the *Weinstein conjecture* in some instances. Specifically, we introduce an analogue of the twisted Floer homology [54] in the Rabinowitz–Floer setting. Following [17] and [5], we construct a Morse–Bott homology for a suitable twisted version of the standard Rabinowitz action functional, that is, the Lagrange multiplier functional of the symplectic area functional.

Theorem 1.4 ([14, Theorem 1.1]). *Let (M, λ) be the completion of a Liouville domain (W, λ) and let $\varphi \in \operatorname{Diff}(W)$ be of finite order and such that $\varphi^*\lambda - \lambda = df_\varphi$ for some smooth compactly supported function $f_\varphi \in C_c^\infty(\operatorname{Int} W)$.*

(a) *The semi-infinite dimensional Morse–Bott homology $\operatorname{RFH}^\varphi(\partial W, M)$ in the sense of Floer of the twisted Rabinowitz action functional exists and is well-defined. Moreover, twisted Rabinowitz–Floer homology is invariant under twisted homotopies of Liouville domains.*
(b) *If ∂W is simply connected and does not admit any nonconstant twisted Reeb orbit, then $\operatorname{RFH}^\varphi_*(\partial W, M) \cong \operatorname{H}_*(\operatorname{Fix}(\varphi|_{\partial W}); \mathbb{Z}_2)$.*
(c) *If ∂W is displaceable by a compactly supported Hamiltonian symplectomorphism in (M, λ), then $\operatorname{RFH}^\varphi(\partial W, M) \cong 0$.*

Twisted Rabinowitz–Floer homology does indeed generalise standard Rabinowitz–Floer homology as

$$\operatorname{RFH}^{\operatorname{id}_W}(\partial W, M) \cong \operatorname{RFH}(\partial W, M).$$

The proof of Theorem 1.1 is straightforward, once we have computed the $\mathbb{Z}_m$-equivariant twisted Rabinowitz–Floer homology of the spheres $\mathbb{S}^{2n-1} \subseteq \mathbb{C}^n$. Indeed, by invariance we may assume that $\Sigma = \mathbb{S}^{2n-1}$, as Σ is star-shaped. Then we use the following elementary topological fact (see Lemma 1.5 below). Let Σ be a simply connected topological manifold and let $\varphi \colon \Sigma \to \Sigma$ be a homeomorphism of finite order m that is not equal to the identity. If the induced discrete action

$$\mathbb{Z}_m \times \Sigma \to \Sigma, \qquad [k] \cdot x := \varphi^k(x)$$

is free, then $\pi \colon \Sigma \to \Sigma/\mathbb{Z}_m$ is a normal covering map [38, Theorem 12.26]. For a point $x \in \Sigma$ define the ***based twisted loop space of Σ and φ*** by

$$\mathscr{L}_\varphi(\Sigma, x) := \{\gamma \in C(I, \Sigma) : \gamma(0) = x \text{ and } \gamma(1) = \varphi(x)\},$$

where $I := [0, 1]$. Then we have the following result. See Figure 1.1.

Lemma 1.5. *If $\gamma \in \mathscr{L}_\varphi(\Sigma, x)$, then $\pi \circ \gamma \in \mathscr{L}(\Sigma/\mathbb{Z}_m, \pi(x))$ is not contractible. Conversely, if $\gamma \in \mathscr{L}(\Sigma/\mathbb{Z}_m, \pi(x))$ is not contractible, then there exists $1 \leq k < m$ such that $\widetilde{\gamma}_x \in \mathscr{L}_{\varphi^k}(\Sigma, x)$ for the unique lift $\widetilde{\gamma}_x$ of γ with $\widetilde{\gamma}_x(0) = x$.*

For a more detailed study of twisted loop spaces of universal covering manifolds as well as a proof of Lemma 1.5 see Appendix A.

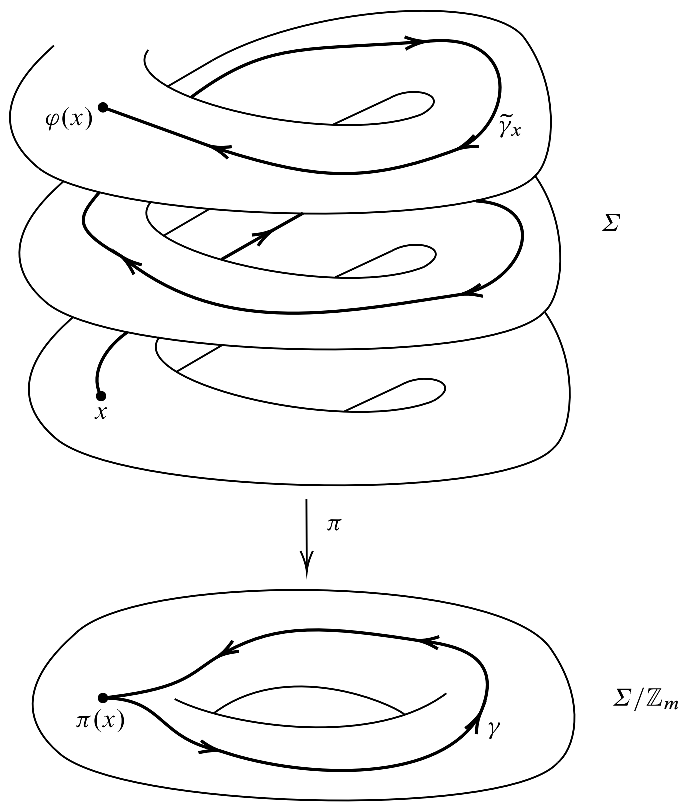

Fig. 1.1: The projection $\pi \circ \gamma \in \mathscr{L}(\Sigma/\mathbb{Z}_m, \pi(x))$ of $\gamma \in \mathscr{L}_\varphi(\Sigma, x)$ is not contractible for the deck transformation $\varphi \neq \mathrm{id}_\Sigma$.

Another interesting application of Theorem 1.4 is the following *forcing result*. Suppose that ∂W is Hamiltonianly displaceable in the completion (M, λ) and simply connected. If $\mathrm{Fix}(\varphi|_{\partial W}) \neq \varnothing$, then ∂W does admit a twisted periodic Reeb orbit. Indeed, if there does not exist any twisted periodic Reeb orbit on ∂W, we compute

$$\mathrm{RFH}^\varphi(\partial W, M) \cong \mathrm{H}(\mathrm{Fix}(\varphi|_{\partial W}); \mathbb{Z}_2) = \bigoplus_{k \geq 0} \mathrm{H}_k(\mathrm{Fix}(\varphi|_{\partial W}); \mathbb{Z}_2) \neq 0$$

by part (b) of Theorem 1.4, contradicting part (c) of Theorem 1.4.

Theorem 1.6 (Forcing). *Let Σ be a twisted stable displaceable hypersurface in a symplectically aspherical, geometrically bounded, symplectic manifold (M, ω) for some $\varphi \in \operatorname{Symp}(M, \omega)$ and suppose that v_0 is a contractible twisted closed characteristic on Σ belonging to a Morse–Bott component C. Then there exists a contractible twisted closed characteristic $v \notin C$ such that*

$$\int_{\mathbb{D}} \overline{v}^* \omega - \int_{\mathbb{D}} \overline{v}_0^* \omega \leq \operatorname{ord}(\varphi) e(\Sigma),$$

where $e(\Sigma)$ denotes the displacement energy of Σ.

The proof of Theorem 1.6 is an adaptation of [19, Theorem 4.9]. This result was initially shown by Felix Schlenk using quite different methods.

Finally, we put Theorem 1.1 into context. If $\Sigma^{2n-1}/\mathbb{Z}_m$ satisfies the index condition

$$\mu_{\mathrm{CZ}}(\gamma) > 4 - n \tag{1.1}$$

for all contractible Reeb orbits γ, the $\bigvee$-shaped symplectic homology $\check{\mathrm{SH}}(\Sigma)$ can be defined in the positive cylindrical end $[0, +\infty) \times \Sigma$ by [53, Corollary 3.7]. If $\Sigma/\mathbb{Z}_m$ admits a Liouville filling W, then we have

$$\check{\mathrm{SH}}_*(\Sigma/\mathbb{Z}_m, M) \cong \mathrm{RFH}_*(\Sigma/\mathbb{Z}_m, M),$$

where M denotes the completion of W. Note that even in the case of lens spaces this need not be the case, as for example $\mathbb{RP}^{2n-1}$ is not Liouville fillable for any odd $n \geq 2$ by [28, Theorem 1.1]. As the index condition (1.1) is only required for contractible Reeb orbits and they come from the universal covering manifold Σ, we can say something in the case where Σ is strictly convex. Indeed, the Hofer–Wysocki–Zehnder Theorem [23, Theorem 12.2.1] then implies that Σ is dynamically convex, that is,

$$\mu_{\mathrm{CZ}}(\gamma) \geq n + 1$$

holds for all periodic Reeb orbits γ. Thus for $n \geq 2$, the index condition is satisfied and we can compute $\check{\mathrm{SH}}_*(\mathbb{S}^{2n-1}/\mathbb{Z}_m)$ via the $\mathbb{Z}_m$-equivariant version of the symplectic homology $\check{\mathrm{SH}}_*(\mathbb{S}^{2n-1})$.

In the case of hypertight contact manifolds, there is a similar construction without the index condition (1.1). See for example [44, Theorem 1.1]. By [44, Theorem 1.7], there do exist noncontractible periodic Reeb orbits on hypertight contact manifolds under suitable technical conditions. Moreover, one can show the existence of invariant Reeb orbits in this setting. See [44, Corollary 1.6] as well as [45, Theorem 1.6] in the Liouville-fillable case.

The thesis is organised as follows. In Chapter 2, we review the basics of Hamiltonian Floer homology defined as the Morse–Bott homology associated with the symplectic action functional. We follow the cascade approach introduced by Urs Frauenfelder and define Hamiltonian Floer homology in the simplest case, that is, in the symplectically aspherical case. This is sufficient for our purposes. A detailed proof of the compactness of the relevant moduli spaces is given in Appendix C and in order to deal with transversality, we use the polyfold approach which is sketched in Appendix D. We also review stable Hamiltonian manifolds, a generalisation of contact manifolds. Appendix C is based on lecture notes written for a course on Hamiltonian Floer homology in the winter semester 2021/2022 taught by Urs Frauenfelder at the university of Augsburg.

In Chapter 3, we introduce the main machinery for defining our new homology theory and prove Theorem 1.4. This material is an extended version of [14] and additional details may also be found in [21].

In Chapter 4, we prove Theorem 1.1 and the Forcing theorem 1.6. Theorem 1.1 and its proof is also taken from [14].

In the final Chapter 5, we indicate two further results that may be obtained using the theory developed in this thesis.

Chapter 2
Hamiltonian Floer Homology

Floer homology was introduced by Andreas Floer around 1988 to tackle the *homological Arnold conjecture*. Roughly speaking, the conjecture says in its simplest form that the number of nondegenerate solutions of a 1-periodic Hamiltonian equation is bounded below by the dimension of the singular homology of the symplectic manifold with coefficients in $\mathbb{Z}_2$. That is, the number of such solutions is always bounded below by a topological invariant. This resembles the famous *Morse inequalities*, and thus it is not suprising that the construction of Floer homology was largely influenced by Morse homology. See the excellent article [2]. However, a key technical ingredient for this semi-infinite dimensional version of Morse homology was Gromov's analysis of pseudoholomorphic curves introduced in 1985. Today there are many flavours of Floer theories, and we shall focus on *Rabinowitz–Floer homology*. This homology was introduced in 2009 by Kai Cieliebak and Urs Frauenfelder. Crucial is the observation, that Floer homology can also be constructed in a more general way, namely in the *Morse–Bott* case. In contrast to standard Hamiltonian Floer homology, Rabinowitz–Floer homology considers a fixed energy but arbitrary period problem. This leads to particular instances of the *Weinstein conjecture* formulated in 1979, including the result of Rabinowitz in 1978. Weinstein conjectured that on every compact manifold admitting a contact form, there must exist a closed Reeb orbit. For an extensive historical treatment see [43].

The aim of this introductory chapter is to explain the fundamental concepts required later on. In the first section we discuss the finite-dimensional version of Morse–Bott homology, a generalisation of Morse homology.

The second section discusses some basic facts coming from Hamiltonian dynamics, focusing on theory we need in subsequent chapters.

The third section introduces the archetypical version of Floer homology, *Hamiltonian Floer homology*, on compact symplectic manifolds. We discuss the Morse–Bott approach, as this one will be useful in the discussion of Rabinowitz–Floer homology.

In the last section we discuss suitable structures on regular hypersurfaces in symplectic manifolds, including hypersurfaces of restricted contact type.

2.1 Morse–Bott Homology

Morse–Bott homology is a generalisation of Morse homology to functions with degenerate critical points. See [17, Appendix A] for a short introduction via the cascade approach and [22, Appendix A] for a more extensive treatment. Morse–Bott functions often occur in the presence of symmetries. Indeed, let G be a Lie group acting on a manifold M. If $f \in C^\infty(M)$ is G-invariant, that is, $f(gx) = f(x)$ holds for all $g \in G$ and $x \in M$, then $\operatorname{Crit} f$ is also G-invariant. In particular, f is not a Morse function in general. However, the presence of symmetry usually simplifies the explicit computation of the Morse homology.

Definition 2.1 (Morse–Bott Function, [43, p. 232]). Let M be a smooth manifold. A ***Morse–Bott function on M*** is defined to be a function $f \in C^\infty(M)$ such that

(i) $\operatorname{Crit} f \subseteq M$ is an embedded submanifold.
(ii) $T_x \operatorname{Crit} f = \ker \operatorname{Hess} f|_x$ for all $x \in \operatorname{Crit} f$.

Remark 2.2. Assumption (ii) is crucial for proving the Morse–Bott Lemma [11, Lemma 3.51], an analogue of the Morse Lemma. The Morse–Bott Lemma is a key technical ingredient for proving exponential decay of gradient flow lines.

Example 2.3 ($\mathbb{S}^{2n-1}$). Let $f : \mathbb{S}^{2n-1} \to \mathbb{R}$ on the odd-dimensional sphere

$$\mathbb{S}^{2n-1} := \left\{ (z_1, \dots, z_n) \in \mathbb{C}^n : \sum_{j=1}^n |z_j|^2 = 1 \right\}$$

be defined by

$$f(z_1, \dots, z_n) := \sum_{j=1}^n j\, |z_j|^2 .$$

Then f is a Morse–Bott function with critical manifold being a disjoint union of n copies of $\mathbb{S}^1$.

Let (M, g) be a compact Riemannian manifold and $f \in C^\infty(M)$ a Morse–Bott function. Choose an additional Morse function $h \in C^\infty(\operatorname{Crit} f)$ and a Riemannian metric g_0 on $\operatorname{Crit} f$ such that (h, g_0) is a Morse–Smale pair, that is, the stable and unstable manifolds intersect transversally. Using Theorem D.12, one can define a nonnegative $\mathbb{Z}$-graded chain complex $(\mathrm{CM}_*(f), \partial_*)$ of $\mathbb{Z}_2$-vector spaces by

$$\mathrm{CM}_k(f) := \operatorname{Crit}_k h \otimes \mathbb{Z}_2, \quad \operatorname{Crit}_k h := \{x \in \operatorname{Crit} h : \operatorname{ind}_f(x) + \operatorname{ind}_h(x) = k\},$$

for all $k \in \mathbb{Z}$, where ind denotes the ordinary Morse index, that is, the number of negative eigenvalues of the Hessian at that point, with boundary operator

$$\partial_k \colon \mathrm{CM}_k(f) \to \mathrm{CM}_{k-1}(f), \qquad \partial_k x^- := \sum_{x^+ \in \operatorname{Crit}_{k-1} h} n(x^-, x^+) x^+,$$

where

$$n(x^-, x^+) := \#_2\mathcal{M}(x^-, x^+) \in \mathbb{Z}_2$$

denotes the $\mathbb{Z}_2$-count of the abstractly perturbed unparametrised negative gradient flow lines with cascades from x^- to x^+. Then ∂_* is indeed a boundary operator by

$$\begin{aligned}(\partial_k \circ \partial_{k+1})x^- &= \sum_{x^+ \in \mathrm{Crit}_{k-1} h} \sum_{x \in \mathrm{Crit}_k h} \#_2\mathcal{M}(x^-, x)\#_2\mathcal{M}(x, x^+)x^+ \\ &= \sum_{x^+ \in \mathrm{Crit}_{k-1} h} \#_2\partial\mathcal{M}(x^-, x^+)x^+ \\ &= 0\end{aligned}$$

for all $x^- \in \mathrm{Crit}_{k+1} h$ as

$$\partial\mathcal{M}(x^-, x^+) \cong \coprod_{x \in \mathrm{Crit}\, h} \mathcal{M}(x^-, x) \times \mathcal{M}(x, x^+).$$

Thus we can define the ***Morse–Bott homology of f*** by

$$\mathrm{HM}_k(f) := \frac{\ker \partial_k}{\mathrm{im}\, \partial_{k+1}}, \qquad \forall k \in \mathbb{Z}.$$

As our notation suggests, $\mathrm{HM}(f)$ is independent of any auxiliary data up to natural isomorphisms. In particular, as every Morse function is a Morse–Bott function, Morse–Bott homology is canonically isomorphic to the ordinary Morse homology of M, and thus to the singular homology of M with coefficients in $\mathbb{Z}_2$, that is,

$$\mathrm{HM}_*(f) \cong \mathrm{H}_*(M; \mathbb{Z}_2).$$

Example 2.4 ($\mathbb{S}^{2n-1}$). Consider the odd-dimensional sphere $\mathbb{S}^{2n-1}$ with Morse–Bott function $f \in C^\infty(\mathbb{S}^{2n-1})$ defined in Example 2.3. Choose the standard height function $h \in C^\infty(\mathrm{Crit}\, f)$ on each critical component $\mathbb{S}^1$. Then the associated chain complex is given by

$$\mathrm{CM}_k(f) \cong \begin{cases} \mathbb{Z}_2 & 0 \le k \le 2n-1, \\ 0 & \text{else.} \end{cases}$$

with boundary operator

$$\partial_k = \begin{cases} 1 & k \text{ even and } 1 \le k \le 2n-1, \\ 0 & \text{else.} \end{cases}$$

See Figure 2.1. Thus the resulting homology is

$$\mathrm{HM}_k(f) \cong \begin{cases} \mathbb{Z}_2 & k = 0, 2n-1, \\ 0 & \text{else.} \end{cases}$$

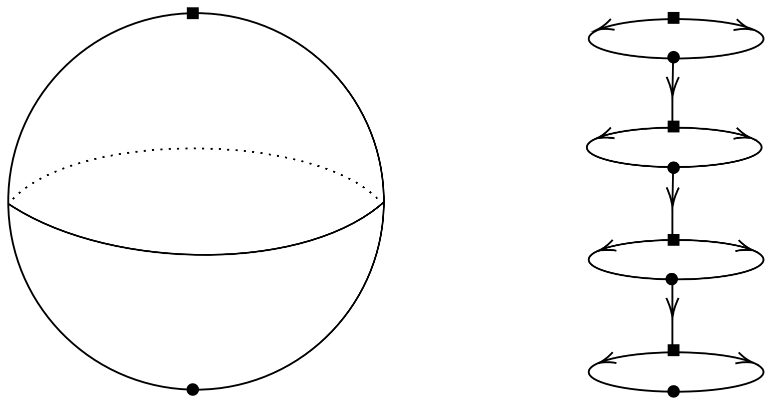

Fig. 2.1: The sphere $\mathbb{S}^{2n-1}$ with standard height function is depicted on the left and on the right we see the critical submanifold Crit f with standard height function on each critical component.

Example 2.5 (The Teapot). Consider the deformed sphere $\mathbb{S}^2$ as in Figure 2.2. Then the standard height function is a Morse–Bott function on the teapot with critical manifold being the disjoint union of a circle $\mathbb{S}^1$ and four nondegenerate critical points. Choose also the standard height function on $\mathbb{S}^1$. Then the resulting chain complex is given by

$$0 \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \xrightarrow{\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \xrightarrow{\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \longrightarrow 0$$

Thus the homology coincides again with $\mathrm{H}_*(\mathbb{S}^2; \mathbb{Z}_2)$.

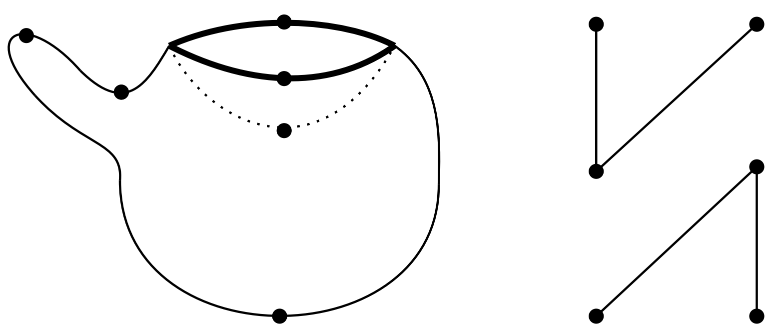

Fig. 2.2: The deformed sphere $\mathbb{S}^2$ resembling a teapot with the standard height function.

2.2 Hamiltonian Dynamics

The modern language of classical mechanics is provided by symplectic geometry. For an introduction to symplectic geometry see [50] and for a more sophisticated treatment [43]. For an introduction to Hamiltonian dynamics see [4] as well as [34] for a view towards symplectic invariants. Here we just briefly review the basics needed later on and to fix our conventions.

Definition 2.6 (Hamiltonian System). A ***Hamiltonian system*** is a symplectic manifold (M, ω), called the ***phase space*** together with a smooth function $H \in C^\infty(M)$, called a ***Hamiltonian function***. We write (M, ω, H) for a Hamiltonian system.

Definition 2.7 (Hamiltonian Vector Field). Let (M, ω, H) be a Hamiltonian system. The ***Hamiltonian vector field*** is defined to be the vector field $X_H \in \mathfrak{X}(M)$ given implicitly by

$$i_{X_H}\omega = -dH.$$

Lemma 2.8 (Jacobi, [4, Theorem 3.3.19]). *Let (M, ω, H) be a Hamiltonian system and let $\varphi \in \operatorname{Symp}(M, \omega)$ be a symplectomorphism. Then*

$$\varphi^* X_H = X_{\varphi^* H}.$$

Proof. We compute

$$i_{X_{\varphi^* H}}\omega = -d\varphi^* H = -\varphi^* dH = \varphi^*(i_{X_H}\omega) = i_{\varphi^* X_H}(\varphi^*\omega) = i_{\varphi^* X_H}\omega.$$

Thus we conclude by the uniqueness of the Hamiltonian vector field. □

Lemma 2.9. *Let (M, ω, H) be a Hamiltonian system and let $\varphi \in \operatorname{Symp}(M, \omega)$ be a symplectomorphism. Then*

$$\phi_t^{X_{\varphi^* H}} = \varphi^{-1} \circ \phi_t^{X_H} \circ \varphi,$$

whenever either side is defined, where ϕ denotes the smooth flow of a vector field.

Proof. Using Lemma 2.8 we compute

$$\begin{aligned}
\frac{d}{dt}\varphi^{-1} \circ \phi_t^{X_H} \circ \varphi &= D\varphi^{-1} \circ \frac{d}{dt}\phi_t^{X_H} \circ \varphi \\
&= D\varphi^{-1} \circ X_H \circ \phi_t^{X_H} \circ \varphi \\
&= D\varphi^{-1} \circ X_H \circ \varphi \circ \varphi^{-1} \circ \phi_t^{X_H} \circ \varphi \\
&= \varphi^* X_H \circ \varphi^{-1} \circ \phi_t^{X_H} \circ \varphi \\
&= X_{\varphi^* H} \circ \varphi^{-1} \circ \phi_t^{X_H} \circ \varphi,
\end{aligned}$$

and the result follows by the uniqueness of integral curves. □

Definition 2.10 (Algebra of Classical Observables, [51, p. 46]). Let (M, ω) be a symplectic manifold. The commutative real algebra $C^\infty(M)$ of smooth functions on M is called the ***algebra of classical observables***.

Definition 2.11 (Poisson Bracket, [39, p. 578]). Let (M, ω) be a symplectic manifold. Define a mapping, called the ***Poisson bracket on the algebra of classical observables***,

$$\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

by

$$\{f, g\} := \omega(X_f, X_g).$$

Remark 2.12 ([39, Corollary 22.20]). Let (M, ω) be a symplectic manifold. Then

$$(C^\infty(M), \{\cdot, \cdot\}) \to (\mathfrak{X}(M), [\cdot, \cdot]), \qquad f \mapsto X_f,$$

is a Lie algebra homomorphism, where $[\cdot, \cdot]$ denotes the Lie bracket given by

$$[X, Y] = L_X Y = \left.\frac{d}{dt}\right|_{t=0} \left(\phi_t^X\right)^* Y$$

for all $X, Y \in \mathfrak{X}(M)$.

Lemma 2.13 (Evolution Equation, [4, Corollary 3.3.15]). *Let (M, ω, H) be a Hamiltonian system. Then*

$$\frac{d}{dt} f \circ \phi_t^{X_H} = \{H, f\} \circ \phi_t^{X_H} \qquad \forall f \in C^\infty(M),$$

whenever either side is defined.

Proof. Using Fisherman's formula [39, Proposition 22.14] we compute

$$\begin{aligned} \frac{d}{dt} f \circ \phi_t^{X_H} &= \frac{d}{dt} \left(\phi_t^{X_H}\right)^* f \\ &= \left(\phi_t^{X_H}\right)^* L_{X_H} f \\ &= \left(\phi_t^{X_H}\right)^* \{H, f\} \\ &= \{H, f\} \circ \phi_t^{X_H} \end{aligned}$$

for all $f \in C^\infty(M)$. □

Corollary 2.14 (Preservation of Energy, [23, Theorem 2.2.2]). *Let (M, ω, H) be a Hamiltonian system. Then*

$$H\left(\phi_t^{X_H}(x)\right) = H(x) \qquad \forall x \in M,$$

whenever the left side is defined.

Proof. Using Lemma 2.13 we compute

$$\frac{d}{dt} H \circ \phi_t^{X_H} = \{H, H\} \circ \phi_t^{X_H} = 0$$

by antisymmetry of the Poisson bracket. □

We describe a particularly interesting class of Hamiltonian systems. Let (M^n, g) be a compact Riemannian manifold and denote by $\pi \colon T^*M \to M$ its cotangent bundle. For a smooth potential function $V \in C^\infty(M)$ define $H \in C^\infty(T^*M)$ by

$$H(q, p) := \frac{1}{2} \|p\|_{g^*}^2 + V(q). \tag{2.1}$$

For $\sigma \in \Omega^2(M)$ closed, the form $\omega_\sigma := dp \wedge dq + \pi^*\sigma$ is a symplectic form on T^*M where (q, p) denote the standard coordinates on the cotangent bundle. The symplectic manifold (T^*M, ω_σ) is called a ***magnetic cotangent bundle*** and the Hamiltonian system (T^*M, ω_σ, H) is called a ***magnetic Hamiltonian system***. If $\sigma = 0$, the system is called a ***mechanical Hamiltonian system***. The dynamics of a magnetic Hamiltonian system are given by the flow of the associated Hamiltonian vector field

$$X_H(q, p) = \sum_{i=1}^n \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} + \sum_{i=1}^n \left(\sum_{j=1}^n \sigma_{ij}(q) \frac{\partial H}{\partial p_j} - \frac{\partial H}{\partial q_i} \right) \frac{\partial}{\partial p_i}, \tag{2.2}$$

where σ is locally given by

$$\sigma = \frac{1}{2} \sum_{i,j=1}^n \sigma_{ij}(q) dq_i \wedge dq_j, \qquad \sigma_{ji} = -\sigma_{ij}.$$

Assume that σ is exact, that is, there exists $\lambda \in \Omega^1(M)$ with $\sigma = d\lambda$. We claim that

$$\varphi_\sigma \colon (T^*M, \lambda_{T^*M} + \pi^*\lambda) \to (T^*M, \lambda_{T^*M}), \quad \varphi_\sigma(q, p) := (q, p + \lambda_q)$$

is an exact symplectomorphism, where λ_{T^*M} denotes the canonical Liouville form on T^*M. For $(q, p) \in T^*M$ and $v \in TT^*_{(q,p)}M$ we compute

$$\begin{aligned}
(\varphi_\sigma^* \lambda_{T^*M})_{(q,p)}(v) &= \varphi_\sigma(q, p)(D\pi \circ D\varphi_\sigma(v)) \\
&= (p + \lambda_q)(D(\pi \circ \varphi_\sigma)(v)) \\
&= p(D\pi(v)) + \lambda_q(D\pi(v)) \\
&= \lambda_{T^*M}|_{(q,p)}(v) + (\pi^*\lambda)_{(q,p)}(v).
\end{aligned}$$

The mechanical Hamiltonian (2.1) is transformed to the ***magnetic Hamiltonian***

$$H \circ \varphi_\sigma^{-1}(q, p) = \frac{1}{2} \|p - \lambda_q\|_{g^*}^2 + V(q).$$

Definition 2.15 (Cotangent Lift). Let $\varphi \in \mathrm{Diff}(M)$ be a diffeomorphism of a smooth manifold M. Define a map $D\varphi^\dagger \colon T^*M \to T^*M$, called the ***cotangent lift of the diffeomorphism*** $\boldsymbol{\varphi}$, by

$$D\varphi^\dagger(q,p)(v) := p\left(D\varphi^{-1}(v)\right), \qquad \forall v \in T_{\varphi(q)}M.$$

Proposition 2.16 (Physical Transformation, [23, p. 10]). *Let* $\varphi \in \mathrm{Diff}(M)$ *be a diffeomorphism and denote by* λ_{T^*M} *the Liouville form on* T^*M*. Then*

$$(D\varphi^\dagger)^*\lambda_{T^*M} = \lambda_{T^*M}.$$

Proof. Let $(q,p) \in T^*M$. Then we compute

$$\begin{aligned}
(D\varphi^\dagger)^*\lambda_{T^*M}|_{(q,p)}(v) &= \lambda_{T^*M}|_{D\varphi^\dagger(q,p)}\left(D(D\varphi^\dagger)(v)\right)\\
&= \lambda_{T^*M}|_{(\varphi(q),p\circ D\varphi^{-1})}\left(D(D\varphi^\dagger)(v)\right)\\
&= (p\circ D\varphi^{-1})\left(D\pi_{(\varphi(q),p\circ D\varphi^{-1})}\left(D(D\varphi^\dagger)(v)\right)\right)\\
&= p\left(D\varphi^{-1}\circ D(\pi\circ D\varphi^\dagger)(v)\right)\\
&= p\left(D\varphi^{-1}\circ D(\varphi\circ\pi)(v)\right)\\
&= \lambda_{T^*M}|_{(q,p)}(v)
\end{aligned}$$

for all $v \in T_{(q,p)}T^*M$. □

Example 2.17 (Holomorphic Function). Let $U \subseteq \mathbb{C}$ be an open subset and suppose that $\varphi \in C^\infty(U,\mathbb{C})$ is holomorphic with $\varphi' \neq 0$ on U for the complex derivative φ' of φ. Then the cotangent lift $D\varphi^\dagger$ of φ is given by

$$D\varphi^\dagger \colon T^*U \to T^*\mathbb{C}, \qquad D\varphi^\dagger(z,w) = \left(\varphi(z), \frac{w}{\overline{\varphi'(z)}}\right).$$

2.3 Morse–Bott Homology for the Symplectic Action Functional

In this section we briefly describe how to construct a Morse–Bott homology in a semi-infinite dimensional setting following [24].

Definition 2.18 (The Symplectic Action Functional, [43, p. 446]). Let (M,ω) be a connected symplectic manifold such that $[\omega]|_{\pi_2(M)} = 0$ and denote by ΛM the connected component of contractible loops in $C^\infty(\mathbb{T},M)$. For $H \in C^\infty(M\times\mathbb{T})$, define the ***symplectic action functional***

$$\mathcal{A}_H \colon \Lambda M \to \mathbb{R}, \qquad \mathcal{A}_H(\gamma) := \int_{\mathbb{D}} \overline{\gamma}^*\omega - \int_0^1 H_t(\gamma(t))dt, \tag{2.3}$$

where $\overline{\gamma} \in C^\infty(\mathbb{D},M)$ is a filling of γ, that is, $\overline{\gamma}(e^{2\pi i t}) = \gamma(t)$ for all $t \in \mathbb{T}$.

The gradient $\operatorname{grad}_J \mathcal{A}_H$ of the symplectic action functional is given by

$$\operatorname{grad}_J \mathcal{A}_H|_\gamma(t) = J(\dot\gamma(t) - X_{H_t}(\gamma(t))) \qquad \forall t \in \mathbb{T},$$

for all $\gamma \in \Lambda M$ and for some ω-compatible almost complex structure J on (M, ω) with respect to the L^2-metric

$$\langle X, Y\rangle_J := \int_0^1 \omega(JX(t), Y(t))dt$$

for all $X, Y \in \Gamma(\gamma^* TM)$. Thus a negative gradient flow line of the symplectic action functional $\mathcal{A}_H$ is a map $u \in C^\infty(\mathbb{R} \times \mathbb{T}, M)$ that is a solution of the ***Floer equation***

$$\partial_s u(s,t) + J(\partial_t u(s,t) - X_{H_t}(u(s,t))) = 0 \qquad \forall (s,t) \in \mathbb{R} \times \mathbb{T}. \tag{2.4}$$

Assume that (M, ω) is compact and we are given a sequence (u_k) of negative gradient flow lines of the symplectic action functional such that the derivatives Du_k are uniformly bounded. Then by [42, Theorem 4.1.1] there exists a negative gradient flow line u of the symplectic action functional such that

$$u_k \xrightarrow{C^\infty_{\text{loc}}} u, \qquad k \to \infty,$$

up to a subsequence. Thus if the derivatives of the sequence of negative gradient flow lines are uniformly bounded, (u_k) converges to a broken negative gradient flow line in the Floer–Gromov sense. In contrast to finite-dimensional Morse–Bott homology, a new phenomenon occurs if the derivatives explode. Indeed, if the derivatives explode, by Theorem C.3 there exists a nonconstant J-holomorphic sphere $v \in C^\infty(\mathbb{S}^2, M)$, see Figure 2.3. This cannot happen in our setting as we assumed $[\omega]|_{\pi_2(M)} = 0$ and thus

$$\int_{\mathbb{S}^2} v^*\omega = 0,$$

contradicting [42, Lemma 2.2.1].

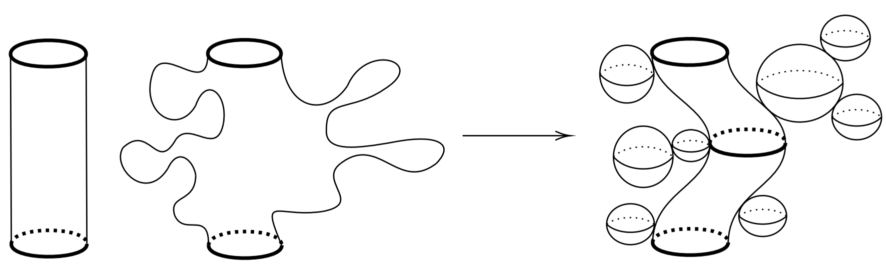

Fig. 2.3: The bubbling phenomenon of a sequence of solutions of the Floer equation.

By adapting Theorem D.12 to the semi-infinite dimensional case as sketched in [15, Corollary 8.13], we can define the ***Floer Homology of H*** by

$$\mathrm{HF}(H) := \mathrm{HM}(\mathcal{A}_H)$$

if the symplectic action functional $\mathcal{A}_H$ is a Morse–Bott function. More precisely, choose an additonal Morse function $h \in C^\infty(\operatorname{Crit}\mathcal{A}_H)$ on $\operatorname{Crit}\mathcal{A}_H \subseteq M$ via the obvious identification $\gamma \mapsto \gamma(0)$. Define a $\mathbb{Z}_2$-vector space $\mathrm{CF}(H) := \operatorname{Crit} h \otimes \mathbb{Z}_2$ and a boundary operator

$$\partial\colon \mathrm{CF}(\mathcal{A}_H) \to \mathrm{CF}(\mathcal{A}_H), \qquad \partial\gamma^- := \sum_{\gamma^+ \in \operatorname{Crit} h} n(\gamma^-, \gamma^+)\gamma^+,$$

where

$$n(\gamma^-, \gamma^+) := \#_2 \mathcal{M}^0(\gamma^-, \gamma^+) \in \mathbb{Z}_2$$

denotes the $\mathbb{Z}_2$-count of the zero dimensional component of the moduli space of all abstractly perturbed unpartametrised negative gradient flow lines with cascades. Then the ungraded Floer homology of $H \in C^\infty(M \times \mathbb{T})$ is given by

$$\mathrm{HF}(H) = \frac{\ker \partial}{\operatorname{im} \partial}.$$

Again, one can show that the definition of Hamiltonian Floer homology does not depend on any auxiliary choices. In fact, Hamiltonian Floer homology is also independent of the choice of time-dependent Hamiltonian function H. Consequently, we have a chain of natural isomorphisms

$$\mathrm{HF}(H) = \mathrm{HM}(\mathcal{A}_H) \cong \mathrm{HM}(\mathcal{A}_0) = \bigoplus_{k \geq 0} \mathrm{HM}_k(h) \cong \bigoplus_{k \geq 0} \mathrm{H}_k(M; \mathbb{Z}_2). \tag{2.5}$$

Consequently, we have that

$$\#\operatorname{Crit}\mathcal{A}_H = \dim_{\mathbb{Z}_2} \mathrm{CF}(H) \geq \dim_{\mathbb{Z}_2} \mathrm{HF}(H) = \sum_{k=0}^{\dim M} \dim_{\mathbb{Z}_2} \mathrm{H}_k(M; \mathbb{Z}_2)$$

This resembles the famous Morse inequalities and yields a proof of a special case for the following conjecture.

Conjecture 2.19 (Homological Arnold Conjecture). Let (M, ω) be a compact symplectic manifold and $H \in C^\infty(M \times \mathbb{T})$ such that $\mathcal{A}_H$ is a Morse function. Then the number of contractible periodic orbits $\mathcal{P}(H)$ of H satisfies the inequality

$$\#\mathcal{P}(H) \geq \sum_{k=0}^{\dim M} \dim_{\mathbb{Z}_2} \mathrm{H}_k(M; \mathbb{Z}_2).$$

For a discussion of the general homological Arnold conjecture, see [2, p. 29].

We finally discuss a $\mathbb{Z}$-grading for Hamiltonian Floer homology. There is an obvious $\mathbb{Z}_2$-grading, but the $\mathbb{Z}$-grading requires an additional assumption. First, we observe that the ordinary Morse index and coindex are both infinite for the symplectic action functional. Indeed, as $\mathcal{A}_H$ is a zero-order perturbation of the symplectic area functional $\mathcal{A}_0$, it is enough to consider that case. Using a Darboux chart, it is actually enough to consider a loop $z \in C^\infty(\mathbb{T}, \mathbb{C}^n)$. This loop can be represented by its Fourier series

$$z = \sum_{k=-\infty}^{+\infty} z_k e^{2\pi i k t}, \qquad z_k \in \mathbb{C}^n.$$

Then

$$\mathcal{A}_0(z) = -\pi \sum_{k=-\infty}^{+\infty} k \, \|z_k\|^2 =: \mathcal{A}(z). \tag{2.6}$$

Indeed, we have that $\mathcal{A}_0(0) = 0 = \mathcal{A}(0)$ and as $C^\infty(\mathbb{T}, \mathbb{C}^n)$ is connected, it suffices to show that the differential on both sides of (2.6) coincide. Represent the tangent vector $v \in T_z C^\infty(\mathbb{T}, \mathbb{C}^n) \cong C^\infty(\mathbb{T}, \mathbb{C}^n)$ by its Fourier series as

$$v = \sum_{k=-\infty}^{+\infty} v_k e^{2\pi i k t}, \qquad v_k \in \mathbb{C}^n.$$

Then we compute

$$\begin{aligned} d\mathcal{A}_0|_z(v) &= \int_0^1 \omega(v(t), \dot{z}(t)) dt \\ &= \int_0^1 \operatorname{Re}(i \overline{v}(t) \dot{z}(t)) dt \\ &= -2\pi \operatorname{Re}\left(\sum_{k=-\infty}^{+\infty} k \overline{v}_k z_k \right) \\ &= -\pi \sum_{k=-\infty}^{+\infty} k(\overline{v}_k z_k + v_k \overline{z}_k) \\ &= d\mathcal{A}|_z(v). \end{aligned}$$

Thus the gradient $\operatorname{grad} \mathcal{A}_0$ with respect to the standard real inner product on $\mathbb{C}^n$ is given by

$$\operatorname{grad} \mathcal{A}_0|_z^k = -2\pi k z_k \qquad \forall k \in \mathbb{Z}.$$

Consequently, the spectrum of the Hessian of $\mathcal{A}_0$ is $2\pi\mathbb{Z}$ and we see that $\mathcal{A}_0$ has infinite Morse index and coindex. Let $\gamma \in \operatorname{Crit} \mathcal{A}_H$ be a critical point of the symplectic action functional and choose a filling disk $\overline{\gamma} \in C^\infty(\mathbb{D}, M^{2n})$ of γ. Fix a symplectic trivialisation of the pullback tangent bundle $\Phi \colon \mathbb{D} \times \mathbb{R}^{2n} \to \overline{\gamma}^* TM$. This is possible by [43, Proposition 2.6.7]. Then we can associate to the pair $(\gamma, \overline{\gamma})$

an integer by

$$\mu(\gamma, \overline{\gamma}) := \mu_{\mathrm{CZ}}(\Psi) \in \mathbb{Z},$$

where μ_{CZ} denotes the Conley–Zehnder index [23, Definition 10.4.1] of the path of symplectic matrices Ψ defined by

$$\Psi\colon [0,1] \to \mathrm{Sp}(n), \qquad \Psi(t) := \Phi^{-1}_{e^{2\pi i t}} \circ D\phi_t^{X_{H_t}} \circ \Phi_1.$$

If we choose another filling disk $\overline{\gamma}' \in C^\infty(\mathbb{T}, M)$ of γ, then we have the formula

$$\mu(\gamma, \overline{\gamma}) - \mu(\gamma, \overline{\gamma}') = 2c_1([\overline{\gamma}' \# \overline{\gamma}^-]),$$

where $[\overline{\gamma}' \# \overline{\gamma}^-] \in \pi_2(M)$ and c_1 denotes the first Chern class [43, p. 85]. So there is always a well-defined $\mathbb{Z}_2$-grading of $\mathrm{CF}(H)$. If $c_1(TM)|_{\pi_2(M)} = 0$, there is also a well-defined $\mathbb{Z}$-grading. More precisely, if $c_1(TM)|_{\pi_2(M)} = 0$, then the Floer chain group $\mathrm{CF}(H)$ is graded by the signature index

$$\mu_{\mathrm{CZ}}(\gamma) - \frac{1}{2}\operatorname{sgn}\operatorname{Hess} h(\gamma) \qquad \forall \gamma \in \operatorname{Crit} h.$$

See [17, p. 297]. Note that the chain of canonical isomorphisms (2.5) gives rise to a natural isomorphism $\mathrm{HF}_*(M) \cong \mathrm{H}_{*+n}(M; \mathbb{Z}_2)$.

2.4 Regular Energy Surfaces

In contrast to Floer homology, in Rabinowitz–Floer homology, we study an arbitrary period but fixed energy problem. Thus we need to consider hypersurfaces in Hamiltonian systems.

Definition 2.20 (Regular Energy Surface). Let (M, ω, H) be a Hamiltonian system. The level set $\Sigma := H^{-1}(0)$ is a ***regular energy surface***, if $\operatorname{Crit} H \cap \Sigma = \varnothing$.

Definition 2.21 (Hamiltonian Manifold, [23, Definition 2.4.1]). A ***Hamiltonian manifold*** is defined to be a pair (Σ, ω), where Σ is an odd-dimensional smooth manifold and $\omega \in \Omega^2(\Sigma)$ is closed such that $\ker \omega$ is a line distribution. The foliation inducing the line distribution $\ker \omega$ is called the ***characteristic foliation***.

Example 2.22 (Regular Energy Surface). Let Σ be a regular energy surface in a Hamiltonian system (M, ω, H). Then $(\Sigma, \omega|_\Sigma)$ is a Hamiltonian manifold. Moreover, the line distribution $\ker \omega|_\Sigma$ is spanned by the Hamiltonian vector field $X_H|_\Sigma$.

Definition 2.23 (Stable Hamiltonian Manifold, [19, p. 1773]). A Hamiltonian manifold (Σ, ω) is called ***stable***, if there exists $\lambda \in \Omega^1(\Sigma)$ which is nowhere-vanishing on $\ker \omega$ and such that $\ker \omega \subseteq \ker d\lambda$. We write $(\Sigma, \omega, \lambda)$ for a stable Hamiltonian manifold.

Remark 2.24. Equivalently, a Hamiltonian manifold (Σ^{2n-1}, ω) is stable, if and only if there exists $\lambda \in \Omega^1(\Sigma)$ with $\ker \omega \subseteq \ker d\lambda$ and such that $\lambda \wedge \omega^{n-1}$ is a volume form on Σ.

Example 2.25 (Regular Energy Surface). Let Σ be a regular energy surface in a Hamiltonian system (M, ω, H). Suppose that there exists a vector field X in a neighbourhood of Σ with X nowhere-tangent to Σ and $\ker \omega|_\Sigma \subseteq \ker L_X \omega|_\Sigma$. Then $(\Sigma, \omega|_\Sigma, i_X \omega|_\Sigma)$ is a stable Hamiltonian manifold.

Example 2.26 ([19, Section 6.1]). Let $\mathbb{T}^n$ be the standard flat torus for $n \geq 2$ and let $J : \mathbb{R}^n \to \mathbb{R}^n$ be an antisymmetric nonzero linear map. Define $\sigma \in \Omega^2(\mathbb{T}^n)$ by setting $\sigma(\cdot, \cdot) := \langle \cdot, J \cdot \rangle$ and denote by $\omega_\sigma = dp \wedge dq + \pi^* \sigma$ the magnetic symplectic form on $T^*\mathbb{T}^n \cong \mathbb{T}^n \times \mathbb{R}^n$. For an energy value $c \in \mathbb{R}$ set $\Sigma_c := H^{-1}(c)$ for the mechanical Hamiltonian function

$$H(q, p) := \frac{1}{2} \|p\|^2 \qquad \forall (q, p) \in \mathbb{T}^n \times \mathbb{R}^n.$$

Define $A := (J|_{\operatorname{im} J})^{-1}$ and $\alpha \in \Omega^1(\operatorname{im} J)$ by

$$\alpha_x(v) := \frac{1}{2} \langle x, Av \rangle.$$

Then Σ_c is a stable Hamiltonian manifold for every $c > 0$ by [19, Proposition 6.3]. The stabilising form λ on Σ_c is given by

$$\lambda := f^*(p\,dq) + (\mathrm{pr}_\parallel \circ \mathrm{pr})^* \alpha, \tag{2.7}$$

where

$$\mathrm{pr}_\perp : \mathbb{R}^n \to \ker J, \qquad \mathrm{pr}_\parallel : \mathbb{R}^n \to \operatorname{im} J, \qquad \mathrm{pr} : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}^n$$

denote the projections with respect to the orthogonal splitting

$$\mathbb{R}^n = \ker J \oplus \operatorname{im} J,$$

and

$$f : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{T}^n \times \mathbb{R}^n, \qquad f(q, p) := \left(q, \mathrm{pr}_\perp(p)\right).$$

Definition 2.27 (Reeb Vector Field, [19, p. 1773]). Let $(\Sigma, \omega, \lambda)$ be a stable Hamiltonian manifold. The unique vector field $R \in \mathfrak{X}(\Sigma)$ implicitly defined by

$$i_R \omega = 0 \qquad \text{and} \qquad i_R \lambda = 1$$

is called the ***Reeb vector field***.

Example 2.28. The flow ϕ_t of the magnetic Hamiltonian system in Example 2.26 is given by

$$\phi_t(q, p) = \left(\int_0^t e^{sJ} p\,ds + q, e^{tJ} p \right),$$

as one can explicitly compute this flow using (2.2), and $(q, p) \in \Sigma_c$ gives rise to a contractible closed orbit of period τ if and only if

$$\mathrm{pr}_\perp(p) = 0, \qquad e^{\tau J}\,\mathrm{pr}_\parallel(p) = \mathrm{pr}_\parallel(p), \qquad \text{and} \qquad \left\|\mathrm{pr}_\parallel(p)\right\|^2 = 2c. \tag{2.8}$$

It is illustrative to consider the special case $n = 2$ and $\sigma = dq_1 \wedge dq_2$. Then

$$\lambda = -\frac{1}{2}(p_1 dp_2 - p_2 dp_1),$$

and the projection of a contractible periodic orbit to $\mathbb{T}^2$ is depicted in Figure 2.4.

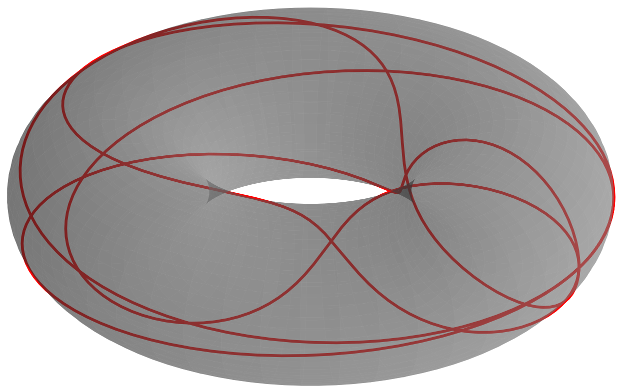

Fig. 2.4: A contractible periodic Reeb orbit on $\mathbb{T}^2$.

The following provides a large class of examples of stable Hamiltonian manifolds.

Definition 2.29 (Contact Manifold, [23, Definition 2.5.1]). A ***contact manifold*** is defined to be a stable Hamiltonian manifold $(\Sigma, d\lambda, \lambda)$. We simply write (Σ, λ) for a contact manifold.

Example 2.30 (Regular Energy Surface, [1, Theorem 1.2.2]). Let Σ be a compact regular energy surface in a mechanical Hamiltonian system (T^*M, ω_0, H). Then there exists $\lambda \in \Omega^1(\Sigma)$ such that $d\lambda = \omega|_\Sigma$ and (Σ, λ) is a contact manifold.

Definition 2.31 (Liouville Domain, [23, Definition 2.6.2]). A ***Liouville domain*** is a compact connected exact symplectic manifold (W, λ) with connected boundary such that the Liouville vector field $X \in \mathfrak{X}(W)$, implicitly defined by $i_X d\lambda = \lambda$, is outward-pointing along the boundary ∂W.

Remark 2.32 ([23, p. 24]). For a Liouville domain (W^{2n}, λ) we have $\partial W \neq \varnothing$. Indeed, if $\partial W = \varnothing$, then using Stokes Theorem we compute

$$0 < \int_W d\lambda^n = \int_W d(\lambda \wedge d\lambda^{n-1}) = \int_{\partial W} \lambda \wedge d\lambda^{n-1} = 0.$$

Remark 2.33 ([23, Lemma 2.63]). Let (W, λ) be a Liouville domain. Then the boundary $(\partial W, \lambda|_{\partial W})$ is a contact manifold.

Remark 2.34. In our definition of a Liouville domain (W, λ) it would actually suffice to assume that the Liouville vector field $X \in \mathfrak{X}(W)$ is nowhere-tangent to the boundary $\Sigma := \partial W$. Indeed, if the Liouville vector field is inward-pointing at the boundary, we get an exact symplectic embedding

$$\psi \colon \left([0, +\infty) \times \Sigma, e^r \lambda|_\Sigma\right) \hookrightarrow (W, \lambda)$$

defined by

$$\psi(r, x) := \phi_t^X(x),$$

where ϕ^X denotes the smooth flow of X. But ψ expands volume as $\psi_r^* \lambda = e^r \lambda|_\Sigma$.

Example 2.35 (Star-Shaped Domain, [23, Example 2.6.6]). Denote by $(\mathbb{C}^n, \lambda)$ the standard exact linear symplectic manifold where

$$\lambda := \frac{1}{2} \sum_{j=1}^{n} \left(y_j dx_j - x_j dy_j\right) = \frac{i}{4} \sum_{j=1}^{n} \left(\overline{z}_j dz_j - z_j d\overline{z}_j\right), \tag{2.9}$$

with coordinates $z_j := x_j + i y_j$. Then the Liouville vector field $X \in \mathfrak{X}(\mathbb{C}^n)$ is given by

$$X = \frac{1}{2} \sum_{j=1}^{n} \left(x_j \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j}\right) = \frac{1}{2} \sum_{j=1}^{n} \left(z_j \frac{\partial}{\partial z_j} + \overline{z}_j \frac{\partial}{\partial \overline{z}_j}\right). \tag{2.10}$$

If $U \subseteq \mathbb{C}^n$ is a bounded connected open subset which is star-shaped with respect to the origin and with a smooth connected boundary ∂U, then $(\overline{U}, \lambda|_{\overline{U}})$ is a Liouville domain. All star-shaped Liouvile domains are diffeomorphic to $(B^n, \lambda|_{B^n})$ via the radial projection, where $B^n := \{z \in \mathbb{C}^n : \|z\| \leq 1\}$ denotes the closed unit disc. However, the Reeb flow of these star-shaped contact type boundaries can be very different as in [2, Example 5.1].

Example 2.36 (Cotangent Bundle, [2, Example 5.2]). Let (M, g) be a compact connected Riemannian manifold and consider the cotangent bundle (T^*M, pdq). Then the Liouville vector field is locally given by $p\frac{\partial}{\partial p}$. Suppose $U \subseteq T^*M$ is a bounded connected open set with smooth boundary containing the zero section and such that the fibrewise intersection $U \cap T_q^*M$ is star-shaped with respect to the origin for all $q \in M$. Then $(\overline{U}, pdq|_{\overline{U}})$ is a Liouville domain. Any such star-shaped Liouville domain in the cotangent bundle T^*M is diffeomorphic to the ***unit cotangent bundle***

$$D^*M := \{(q, p) \in T^*M : \|p\|_{g^*} \leq 1\},$$

with contact type boundary the spherisation $(S^*M, pdq|_{S^*M})$. Once more, the Reeb flow of such star-shaped hypersurfaces can be very different.

Example 2.37 (Magnetic Hamiltonian System, [2, Example 5.2]). Consider an exact magnetic Hamiltonian system $(T^*M, pdq + \pi^*\lambda, H)$ for $\lambda \in \Omega^1(M)$. For

$$c > \max_{q \in M} \left(\frac{1}{2} \|\lambda_q\|^2_{m^*} + V(q) \right)$$

the level set $H^{-1}(c)$ is fibrewise star-shaped.

Definition 2.38 (Liouville Automorphism, [16, p. 237]). Let (W, λ) be a Liouville domain with boundary Σ. A diffeomorphism $\varphi \in \operatorname{Diff}(W)$ is said to be a ***Liouville automorphism***, if $\varphi^*\lambda - \lambda$ is exact and compactly supported in the interior Int W, and ord $\varphi < \infty$. We denote by $\operatorname{Aut}(W, \lambda)$ the set of all Liouville automorphisms on a given Liouville domain (W, λ).

Remark 2.39. Let $\varphi \in \operatorname{Aut}(W, \lambda)$ be a Liouville automorphism. Then there exists a unique function $f_\varphi \in C^\infty_c(\operatorname{Int} W)$ such that $\varphi^*\lambda - \lambda = df_\varphi$.

Remark 2.40. For a Liouville domain (W, λ), the set $\operatorname{Aut}(W, \lambda)$ of Liouville automorphisms is in general not a group. Indeed, for $\varphi, \psi \in \operatorname{Aut}(W, \lambda)$ it is not necessarily true that $\varphi \circ \psi$ is of finite order unless φ and ψ commute.

Remark 2.41. Any $\varphi \in \operatorname{Aut}(W, \lambda)$ induces a strict contactomorphism $\varphi|_{\partial W}$.

Example 2.42 (Rotation). For $m \geq 1$ consider the rotation

$$\varphi \colon \mathbb{C}^n \to \mathbb{C}^n, \qquad \varphi(z_1, \dots, z_n) := \left(e^{2\pi i k_1/m} z_1, \dots, e^{2\pi i k_n/m} z_n\right),$$

where $k_1, \dots, k_n \in \mathbb{Z}$ are coprime to m. Let (W, λ) be a star-shaped Liouville domain in $\mathbb{C}^n$ as in Example 2.35 invariant under the rotation φ, that is, $\varphi(\partial W) = \partial W$. Then φ is a Liouville automorphism as $\varphi^*\lambda = \lambda$ by (2.9) and ord $\varphi = m$.

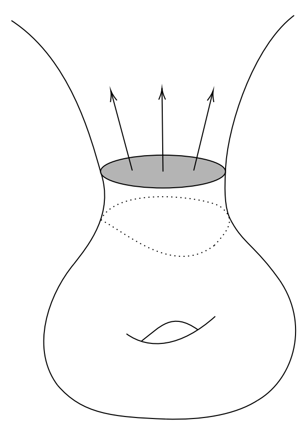

Fig. 2.5: The completion of a Liouville domain.

One can complete a Liouville domain (W, λ) to a noncompact exact symplectic manifold without boundary by attaching the positive cylindrical end $[0, +\infty) \times \partial W$ to its boundary ∂W. See Figure 2.5.

Definition 2.43 (Completion of a Liouville Domain, [43, p. 148]). Let (W, λ) be a Liouville domain with boundary Σ. The ***completion of* (W, λ)** is defined to be the exact symplectic manifold (M, λ), where

$$M := W \cup_\Sigma [0, +\infty) \times \Sigma \qquad \text{and} \qquad \lambda|_{[0,+\infty)\times\Sigma} := e^r \lambda|_\Sigma.$$

Example 2.44 (Star-Shaped Domain). Let (W, λ) be a star-shaped Liouville domain as in Example 2.35. Then the completion (M, λ) of (W, λ) is symplectomorphic to the exact linear symplectic manifold $(\mathbb{C}^n, \lambda)$ via the flow of the Liouville vector field. See Figure 2.6. The completion of a star-shaped Liouville domain in a cotangent bundle T^*M as in Example 2.36 is (T^*M, pdq).

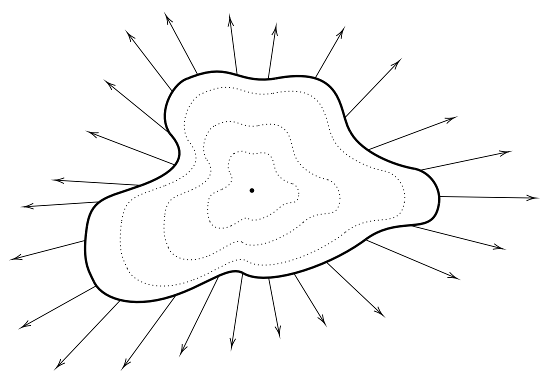

Fig. 2.6: The completion $(\mathbb{C}^n, \lambda)$ of a star-shaped Liouville domain.

Finally, we consider more general hypersurfaces in symplectic manifolds.

Definition 2.45 (Stable Hypersurface, [19, p. 1774]). Let (M, ω) be a connected symplectic manifold. A ***stable hypersurface in* (M, ω)** is a compact connected hypersurface $\Sigma \subseteq M$ such that the following conditions are satisfied.

(i) Σ is separating, that is, $M \setminus \Sigma$ consists of two connected components $M^\pm$, where M^- is bounded and M^+ is unbounded.
(ii) There exists a vector field X in a neighbourhood of Σ such that X is outward-pointing to $\Sigma \cup M^-$ and such that $\ker \omega|_\Sigma \subseteq \ker L_X \omega|_\Sigma$.

Definition 2.46 (Displaceability, [43, p. 411]). A subset $A \subseteq M$ of a symplectic manifold (M, ω) is said to be ***Hamiltonianly displaceable***, if there exists a compactly supported Hamiltonian symplectomorphism $\varphi_F \in \mathrm{Ham}_c(M, \omega)$, such that

$$\varphi_F(A) \cap A = \varnothing.$$

Example 2.47 ([25, p. 4]). Every compact subset of $(M \times \mathbb{C}, \omega \oplus \omega_0)$ is Hamiltonianly displaceable, where (M, ω) is any symplectic manifold.

Definition 2.48 (Hofer Norm, [43, p. 466]). Let (M, ω) be a symplectic manifold and $F \in C_c^\infty(M \times [0,1])$. Define the ***Hofer norm of F*** by

$$\|F\| := \|F\|_+ + \|F\|_- ,$$

where

$$\|F\|_+ := \int_0^1 \max_{x \in M} F_t(x) dt \qquad \text{and} \qquad \|F\|_- := -\int_0^1 \min_{x \in M} F_t(x) dt.$$

Definition 2.49 (Displacement Energy, [43, p. 469]). Let (M, ω) be a symplectic manifold and $A \subseteq M$ a compact subset. The ***displacement energy of A*** is

$$e(A) := \inf_{\substack{F \in C_c^\infty(M \times [0,1]) \\ \varphi_F(A) \cap A = \varnothing}} \|F\| .$$

Example 2.50. Consider the displaceable hypersurface $\Sigma_c \subseteq (T^*\mathbb{T}^n, \omega_\sigma, H)$ as in Example 2.26. Then by [43, Theorem 12.3.4], the displacement energy of Σ_c is given by

$$e(\Sigma_c) = e\left(\overline{B}^n_{\sqrt{2c}}(0)\right) = 2\pi c \qquad \forall c > 0,$$

where $\overline{B}^n_{\sqrt{2c}}(0) \subseteq \mathbb{R}^n$ denotes the closed ball around the origin with radius $\sqrt{2c}$.

Chapter 3
Twisted Rabinowitz–Floer Homology

In this chapter we construct the generalisation of Rabinowitz–Floer homology and prove Theorem 1.4.

To begin, we define the twisted Rabinowitz action functional for an exact symplectic manifold and compute its first and second variation.

In the second section we prove a compactness result for the moduli space of twisted negative gradient flow lines in a restricted geometrical setting. This follows a standard procedure, but one has to carefully adapt the constructions and proofs to this more general case.

In the third section we define ungraded and graded twisted Rabinowitz–Floer homology and prove part (b) of Theorem 1.4 in Proposition 3.39.

In the fourth section we briefly illustrate how to prove part (a) of Theorem 1.4 (see Theorem 3.41) and define the notion of twisted homotopies of Liouville domains.

In the last section we prove an important vanishing result for twisted Rabinowitz–Floer homology, that is, part (c) of Theorem 1.4 (see Theorem 3.48).

3.1 The Twisted Rabinowitz Action Functional

Definition 3.1 (Free Twisted Loop Space). Let $\varphi \in \mathrm{Diff}(M)$ be a diffeomorphism of a smooth manifold M. Define the ***free twisted loop space of M and φ*** by

$$\mathscr{L}_\varphi M := \{\gamma \in C^\infty(\mathbb{R}, M) : \gamma(t+1) = \varphi(\gamma(t)) \ \forall t \in \mathbb{R}\}.$$

Let (M, ω) be a symplectic manifold and let $\varphi \in \mathrm{Symp}(M, \omega)$. Given a twisted loop $\gamma \in \mathscr{L}_\varphi M$ and $\varepsilon_0 > 0$, we say that a curve

$$(-\varepsilon_0, \varepsilon_0) \to \mathscr{L}_\varphi M, \qquad \varepsilon \mapsto \gamma_\varepsilon$$

starting at γ is ***smooth***, if the induced variation

$$\mathbb{R} \times (-\varepsilon_0, \varepsilon_0) \to M, \qquad (t, \varepsilon) \mapsto \gamma_\varepsilon(t)$$

is smooth. Since $\gamma_\varepsilon(t+1) = \varphi(\gamma_\varepsilon(t))$ holds for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ and $t \in \mathbb{R}$, we call such a variation a ***twisted variation***. Then the infinitesimal variation

$$\delta\gamma := \left.\frac{\partial \gamma_\varepsilon}{\partial \varepsilon}\right|_{\varepsilon=0} \in \Gamma(\gamma^* TM),$$

satisfies

$$\delta\gamma(t+1) = D\varphi(\delta\gamma(t)) \qquad \forall t \in \mathbb{R}.$$

Lemma 3.2. *Let (M, ω) be a symplectic manifold and let $\varphi \in \operatorname{Symp}(M, \omega)$ be of finite order. Let $\gamma \in \mathscr{L}_\varphi M$ and let $X \in \Gamma(\gamma^* TM)$ be such that*

$$X(t+1) = D\varphi(X(t)) \qquad \forall t \in \mathbb{R}.$$

Then there exists a twisted variation of γ such that $\delta\gamma = X$.

Proof. As φ is assumed to be of finite order, there exists a φ-invariant ω-compatible almost complex structure J on M by [43, Lemma 5.5.6]. With respect to the induced Riemannian metric

$$m_J := \omega(J\cdot, \cdot),$$

the symplectomorphism φ is an isometry. Define the exponential variation

$$\mathbb{R} \times (-\varepsilon_0, \varepsilon_0) \to M, \qquad \gamma_\varepsilon(t) := \exp^{\nabla_J}_{\gamma(t)}(\varepsilon X(t)),$$

for $\varepsilon_0 > 0$ sufficiently small and ∇_J denoting the Levi–Civita connection associated with m_J. Such an $\varepsilon_0 > 0$ exists by the naturality of geodesics [40, Corollary 5.14]. Then we compute

$$\begin{aligned}
\gamma_\varepsilon(t+1) &= \exp^{\nabla_J}_{\gamma(t+1)}(\varepsilon X(t+1)) \\
&= \exp^{\nabla_J}_{\varphi(\gamma(t))}(D\varphi(\varepsilon X(t))) \\
&= \varphi\left(\exp^{\nabla_J}_{\gamma(t)}(\varepsilon X(t))\right) \\
&= \varphi(\gamma_\varepsilon(t))
\end{aligned}$$

by naturality of the exponential map [40, Proposition 5.20]. □

Remark 3.3. The statement of Lemma 3.2 remains true if $\operatorname{ord}\varphi = \infty$.

This discussion together with Lemma A.3 motivates the following definition of the tangent space to the free twisted loop space.

Definition 3.4 (Tangent Space to the Free Twisted Loop Space). Let (M, ω) be a symplectic manifold and $\varphi \in \operatorname{Symp}(M, \omega)$. For $\gamma \in \mathscr{L}_\varphi M$ define the ***tangent space to the free twisted loop space at γ*** by

$$T_\gamma \mathscr{L}_\varphi M := \{X \in \Gamma(\gamma^* TM) : X(t+1) = D\varphi(X(t))\ \forall t \in \mathbb{R}\}.$$

Definition 3.5 (Twisted Hamiltonian Function). Let (M, ω) be a symplectic manifold and $\varphi \in \mathrm{Symp}(M, \omega)$. A function $H \in C^\infty(M \times \mathbb{R})$ is said to be a ***twisted Hamiltonian function***, if

$$\varphi^* H_{t+1} = H_t \qquad \forall t \in \mathbb{R}.$$

We denote the space of all twisted Hamiltonian functions by $C^\infty_\varphi(M \times \mathbb{R})$ and the subspace of all autonomous twisted Hamiltonian functions by $C^\infty_\varphi(M)$.

Recall, that an exact symplectic manifold is a pair (M, λ) such that $(M, d\lambda)$ is a symplectic manifold. Moreover, an exact symplectomorphism of an exact symplectic manifold (M, λ) is a diffeomorphism $\varphi \in \mathrm{Diff}(M)$ such that $\varphi^*\lambda - \lambda$ is exact.

Definition 3.6 (Perturbed Twisted Rabinowitz Action Functional). Let (M, λ) be an exact symplectic manifold and $\varphi \in \mathrm{Diff}(M)$ an exact symplectomorphism such that $\varphi^*\lambda - \lambda = df$. For $H, F \in C^\infty_\varphi(M \times \mathbb{R})$ define the ***perturbed twisted Rabinowitz action functional***

$$\mathcal{A}^{(H,F)}_\varphi \colon \mathscr{L}_\varphi M \times \mathbb{R} \to \mathbb{R}$$

by

$$\mathcal{A}^{(H,F)}_\varphi(\gamma, \tau) := \int_0^1 \gamma^*\lambda - \tau \int_0^1 H_t(\gamma(t))dt - \int_0^1 F_t(\gamma(t))dt - f(\gamma(0)).$$

If $F = 0$ and $H \in C^\infty_\varphi(M)$, we write $\mathcal{A}^H_\varphi$ for $\mathcal{A}^{(H,F)}_\varphi$ and call $\mathcal{A}^H_\varphi$ the ***twisted Rabinowitz action functional***.

Remark 3.7. Assume that $m := \mathrm{ord}\,\varphi < \infty$. Then

$$\mathcal{A}^{(H,F)}_\varphi(\gamma, \tau) = \frac{1}{m}\mathcal{A}^{(H,F)}(\overline{\gamma}, \tau) - \frac{1}{m}\sum_{k=0}^{m-1} f(\gamma(k)),$$

for all $(\gamma, \tau) \in \mathscr{L}_\varphi M$, where $\overline{\gamma} \in \mathscr{L} M$ is defined by $\overline{\gamma}(t) := \gamma(mt)$.

Definition 3.8 (Differential of the Perturbed Twisted Rabinowitz Action Functional). Let $\varphi \in \mathrm{Diff}(M)$ be an exact symplectomorphism of an exact symplectic manifold (M, λ). For $H, F \in C^\infty_\varphi(M \times \mathbb{R})$, define the ***differential of the perturbed twisted Rabinowitz action functional***

$$d\mathcal{A}^{(H,F)}_\varphi|_{(\gamma,\tau)} \colon T_\gamma \mathscr{L}_\varphi M \times \mathbb{R} \to \mathbb{R}$$

for all $(\gamma, \tau) \in \mathscr{L}_\varphi M \times \mathbb{R}$ by

$$d\mathcal{A}^{(H,F)}_\varphi|_{(\gamma,\tau)}(X, \eta) := \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathcal{A}^{(H,F)}_\varphi(\gamma_\varepsilon, \tau + \varepsilon\eta),$$

where γ_ε is a twisted variation of γ such that $\delta\gamma = X$.

Proposition 3.9 (Differential of the Perturbed Twisted Rabinowitz Action Functional). *Let $\varphi \in \mathrm{Diff}(M)$ be an exact symplectomorphism of an exact symplectic manifold (M, λ) and $H, F \in C^\infty_\varphi(M \times \mathbb{R})$. Then*

$$d\mathcal{A}_\varphi^{(H,F)}|_{(\gamma,\tau)}(X,\eta) = \int_0^1 d\lambda(X(t), \dot{\gamma}(t) - \tau X_{H_t}(\gamma(t)) - X_{F_t}(\gamma(t)))dt - \eta \int_0^1 H_t(\gamma(t))dt \tag{3.1}$$

for all $(\gamma, \tau) \in \mathscr{L}_\varphi M \times \mathbb{R}$ and $(X, \eta) \in T_\gamma \mathscr{L}_\varphi M \times \mathbb{R}$. Moreover, $(\gamma, \tau) \in \mathrm{Crit}\, \mathcal{A}_\varphi^{(H,F)}$ if and only if

$$\dot{\gamma}(t) = \tau X_{H_t}(\gamma(t)) + X_{F_t}(\gamma(t)) \qquad \textit{and} \qquad \int_0^1 H_t(\gamma(t))dt = 0 \tag{3.2}$$

for all $t \in \mathbb{R}$.

Proof. In order to show (3.1), we compute

$$\begin{aligned} d\mathcal{A}_\varphi^{(H,F)}|_{(\gamma,\tau)}(X,\eta) &= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \mathcal{A}_\varphi^{(H,F)}(\gamma_\varepsilon, \tau + \varepsilon\eta) \\ &= \int_0^1 \gamma^* i_X d\lambda + \int_0^1 d i_X \lambda - \tau \int_0^1 dH_t(X(t))dt \\ &\quad - \eta \int_0^1 H_t(\gamma(t))dt - \int_0^1 dF_t(X(t))dt - df_\varphi(X(0)) \\ &= \int_0^1 d\lambda(X(t), \dot{\gamma}(t) - \tau X_{H_t}(\gamma(t)) - X_{F_t}(\gamma(t)))dt \\ &\quad - \eta \int_0^1 H_t(\gamma(t))dt + \lambda(X)|_0^1 - df_\varphi(X(0)) \\ &= \int_0^1 d\lambda(X(t), \dot{\gamma}(t) - \tau X_{H_t}(\gamma(t)) - X_{F_t}(\gamma(t)))dt \\ &\quad - \eta \int_0^1 H_t(\gamma(t))dt + (\varphi^*\lambda - \lambda)(X(0)) - df_\varphi(X(0)) \\ &= \int_0^1 d\lambda(X(t), \dot{\gamma}(t) - \tau X_{H_t}(\gamma(t)) - X_{F_t}(\gamma(t)))dt \\ &\quad - \eta \int_0^1 H_t(\gamma(t))dt. \end{aligned}$$

Let $(\gamma, \tau) \in \mathrm{Crit}\, \mathcal{A}_\varphi^{(H,F)}$. It follows immediately from (3.1) that

$$\int_0^1 H_t(\gamma(t))dt = 0$$

and
$$\int_0^1 d\lambda(X(t), \dot{\gamma}(t) - \tau X_{H_t}(\gamma(t)) - X_{F_t}(\gamma(t)))dt = 0$$
for all $X \in T_\gamma \mathscr{L}_\varphi M$. Suppose there exists $t_0 \in \operatorname{Int} I$ such that
$$\dot{\gamma}(t_0) - \tau X_{H_{t_0}}(\gamma(t_0)) - X_{F_{t_0}}(\gamma(t_0)) \neq 0.$$
By nondegeneracy of the symplectic form $d\lambda$ there exists $v \in T_{\gamma(t_0)}M$ with
$$d\lambda(v, \dot{\gamma}(t_0) - \tau X_{H_{t_0}}(\gamma(t_0)) - X_{F_{t_0}}(\gamma(t_0))) \neq 0.$$
Fix a Riemannian metric on M and let X_v denote the unique parallel vector field along $\gamma|_I$ such that $X_v(t_0) = v$. As $\operatorname{Int} I$ is open, there exists $\delta > 0$ such that $\overline{B}_\delta(t_0) \subseteq \operatorname{Int} I$. Fix a smooth bump function $\beta \in C^\infty(I)$ for t_0 supported in $B_\delta(t_0)$. By shrinking δ if necessary, we may assume that
$$\int_{t_0-\delta}^{t_0+\delta} d\lambda(\beta(t)X_v(t), \dot{\gamma}(t) - \tau X_{H_t}(\gamma(t)) - X_{F_t}(\gamma(t)))dt \neq 0.$$
Extending
$$(\beta X_v)(t+k) := D\varphi^k(\beta(t)X_v(t)) \qquad \forall t \in I, k \in \mathbb{Z},$$
we have that $\beta X_v \in T_\gamma \mathscr{L}_\varphi M$ and thus we compute
$$\begin{aligned} 0 &= d\mathscr{A}_\varphi^{(H,F)}|_{(\gamma,\tau)}(\beta X_v, 0) \\ &= \int_{t_0-\delta}^{t_0+\delta} d\lambda(\beta(t)X_v(t), \dot{\gamma}(t) - \tau X_{H_t}(\gamma(t)) - X_{F_t}(\gamma(t)))dt \\ &\neq 0. \end{aligned}$$
Hence
$$\dot{\gamma}(t) = \tau X_{H_t}(\gamma(t)) + X_{F_t}(\gamma(t)) \qquad \forall t \in I,$$
implying
$$\begin{aligned} \dot{\gamma}(t+k) &= D\varphi^k(\dot{\gamma}(t)) \\ &= \tau(D\varphi^k \circ X_{H_t})(\gamma(t)) + (D\varphi^k \circ X_{F_t})(\gamma(t)) \\ &= \tau(D\varphi^k \circ X_{H_t} \circ \varphi^{-k} \circ \varphi^k)(\gamma(t)) + (D\varphi^k \circ X_{F_t} \circ \varphi^{-k} \circ \varphi^k)(\gamma(t)) \\ &= \tau\varphi_*^k X_{H_t}(\gamma(t+k)) + \varphi_*^k X_{F_t}(\gamma(t+k)) \\ &= \tau X_{\varphi_*^k H_t}(\gamma(t+k)) + X_{\varphi_*^k F_t}(\gamma(t+k)) \\ &= \tau X_{H_{t+k}}(\gamma(t+k)) + X_{F_{t+k}}(\gamma(t+k)) \end{aligned}$$
for all $t \in I$ and $k \in \mathbb{Z}$. The other direction is immediate. □

Corollary 3.10. *The differential of the perturbed twisted Rabinowitz action functional is well-defined, that is, independent of the choice of twisted variation, and linear.*

Preservation of energy 2.14 implies the following corollary.

Corollary 3.11. *Let $\varphi \in \mathrm{Diff}(M)$ be an exact symplectomorphism of an exact symplectic manifold (M,λ) and $H \in C^\infty_\varphi(M)$. Then $\mathrm{Crit}\,\mathcal{A}^H_\varphi$ consists precisely of all $(\gamma,\tau) \in \mathcal{L}_\varphi M \times \mathbb{R}$ such that $\gamma(\mathbb{R}) \subseteq H^{-1}(0)$ and γ is an integral curve of τX_H.*

There is a natural $\mathbb{R}$-action on the twisted loop space $\mathcal{L}_\varphi M$ given by

$$(s\cdot\gamma)(t) := \gamma(t+s) \qquad \forall t \in \mathbb{R}.$$

If (M,λ) is an exact symplectic manifold and $H \in C^\infty_\varphi(M)$ for an exact symplectomorphism $\varphi \in \mathrm{Diff}(M)$ of finite order such that $\operatorname{supp} f \cap H^{-1}(0) = \varnothing$, then the twisted Rabinowitz action functional $\mathcal{A}^H_\varphi$ is invariant under the induced $\mathbb{S}^1$-action on $\mathrm{Crit}\,\mathcal{A}^H_\varphi$. In particular, the unperturbed twisted Rabinowitz action functional is never a Morse function.

Definition 3.12 (Hessian of the Twisted Rabinowitz Action Functional). Let φ be an exact symplectomorphism of an exact symplectic manifold (M,λ) and suppose that $H \in C^\infty_\varphi(M)$. For $(\gamma,\tau) \in \mathrm{Crit}\,\mathcal{A}^H_\varphi$, define the ***Hessian of the twisted Rabinowitz action functional***

$$\mathrm{Hess}\,\mathcal{A}^H_\varphi|_{(\gamma,\tau)} \colon (T_\gamma\mathcal{L}_\varphi M \times \mathbb{R}) \times (T_\gamma\mathcal{L}_\varphi M \times \mathbb{R}) \to \mathbb{R}$$

by

$$\mathrm{Hess}\,\mathcal{A}^H_\varphi|_{(\gamma,\tau)}((X,\eta),(Y,\sigma)) := \left.\frac{\partial^2}{\partial\varepsilon_1\partial\varepsilon_2}\right|_{\varepsilon_1=\varepsilon_2=0} \mathcal{A}^H_\varphi(\gamma_{\varepsilon_1,\varepsilon_2}, \tau+\varepsilon_1\eta+\varepsilon_2\sigma),$$

for a smooth two-parameter family $\gamma_{\varepsilon_1,\varepsilon_2}$ of twisted loops with

$$\left.\frac{\partial}{\partial\varepsilon_1}\right|_{\varepsilon_1=0}\gamma_{\varepsilon_1,0} = X \qquad \text{and} \qquad \left.\frac{\partial}{\partial\varepsilon_2}\right|_{\varepsilon_2=0}\gamma_{0,\varepsilon_2} = Y.$$

Remark 3.13. Traditionally, the differential and the Hessian of the twisted Rabinowitz action functional are called the first and second variation of the twisted Rabinowitz action functional.

Definition 3.14 (Symplectic Connection). Let (M,ω) be a symplectic manifold. A ***symplectic connection on*** $\boldsymbol{(M,\omega)}$ is defined to be a torsion-free connection ∇ in the tangent bundle TM such that $\nabla\omega = 0$.

Remark 3.15. Every symplectic manifold admits a symplectic connection by [31, p. 308], but in sharp contrast to the Riemannian case, a symplectic connection on a given symplectic manifold is in general not unique.

Lemma 3.16. *Let $\varphi \in \mathrm{Diff}(M)$ be an exact symplectomorphism of an exact symplectic manifold (M, λ). Fix a symplectic connection ∇ on $(M, d\lambda)$ and a twisted Hamiltonian function $H \in C^\infty_\varphi(M)$. If $(\gamma, \tau) \in \mathrm{Crit}\,\mathscr{A}^H_\varphi$, then*

$$\begin{aligned}\mathrm{Hess}\,\mathscr{A}^H_\varphi|_{(\gamma,\tau)}((X,\eta),(Y,\sigma)) = &\int_0^1 d\lambda(Y, \nabla_t X)\\ &- \tau\int_0^1 \mathrm{Hess}^\nabla H(X,Y) - \eta\int_0^1 dH(Y) - \sigma\int_0^1 dH(X) \end{aligned}\tag{3.3}$$

for all $(X,\eta),(Y,\sigma) \in T_\gamma\mathscr{L}_\varphi M \times \mathbb{R}$.

Proof. We compute

$$\begin{aligned}&\frac{\partial^2}{\partial\varepsilon_1\partial\varepsilon_2}\bigg|_{\varepsilon_1=\varepsilon_2=0}(\tau+\varepsilon_1\eta+\varepsilon_2\sigma)\int_0^1 H\circ\gamma_{\varepsilon_1,\varepsilon_2}\\ &\quad = \eta\int_0^1 dH(Y) + \sigma\int_0^1 dH(X) + \tau\int_0^1 \frac{\partial^2}{\partial\varepsilon_1\partial\varepsilon_2}\bigg|_{\varepsilon_1=\varepsilon_2=0} H(\gamma_{\varepsilon_1,\varepsilon_2}),\end{aligned}\tag{3.4}$$

and

$$\begin{aligned}\frac{\partial^2}{\partial\varepsilon_1\partial\varepsilon_2}\bigg|_{\varepsilon_1=\varepsilon_2=0} H(\gamma_{\varepsilon_1,\varepsilon_2}) &= \frac{\partial}{\partial\varepsilon_1}\bigg|_{\varepsilon_1=0} dH\left(\partial_{\varepsilon_2}|_{\varepsilon_2=0}\gamma_{\varepsilon_1,\varepsilon_2}\right)\\ &= -\frac{\partial}{\partial\varepsilon_1}\bigg|_{\varepsilon_1=0} d\lambda\left(X_H(\gamma_{\varepsilon_1,0}), \partial_{\varepsilon_2}|_{\varepsilon_2=0}\gamma_{\varepsilon_1,\varepsilon_2}\right)\\ &= -\,d\lambda\left(\nabla_{\varepsilon_1}|_{\varepsilon_1=0}X_H(\gamma_{\varepsilon_1,0}), Y\right)\\ &\quad -\,d\lambda\left(X_H(\gamma), \nabla_{\varepsilon_1}|_{\varepsilon_1=0}\partial_{\varepsilon_2}|_{\varepsilon_2=0}\gamma_{\varepsilon_1,\varepsilon_2}\right).\end{aligned}$$

The $d\lambda$-compatibility of ∇ implies

$$\begin{aligned}d\lambda\left(\nabla_{\varepsilon_1}|_{\varepsilon_1=0}X_H(\gamma_{\varepsilon_1,0}), Y\right) &= d\lambda\left(\nabla_{\partial_{\varepsilon_1}|_{\varepsilon_1=0}\gamma_{\varepsilon_1,0}}X_H, Y\right)\\ &= d\lambda(\nabla_X X_H, Y)\\ &= \nabla_X d\lambda(X_H, Y) - d\lambda(X_H, \nabla_X Y)\\ &= -\nabla_X dH(Y) + dH(\nabla_X Y)\\ &= -X(Y(H)) + (\nabla_X Y)H\\ &= -\,\mathrm{Hess}^\nabla H|_\gamma(X,Y).\end{aligned}\tag{3.5}$$

Next we compute

$$\begin{aligned}\frac{\partial^2}{\partial\varepsilon_1\partial\varepsilon_2}\bigg|_{\varepsilon_1=\varepsilon_2=0}\int_0^1 \gamma^*_{\varepsilon_1,\varepsilon_2}\lambda &= \int_0^1 \frac{\partial}{\partial\varepsilon_1}\bigg|_{\varepsilon_1=0}\gamma^*_{\varepsilon_1,0} i_{\partial_{\varepsilon_2}|_{\varepsilon_2=0}\gamma_{\varepsilon_1,\varepsilon_2}}(d\lambda\circ\gamma_{\varepsilon_1,0})\\ &\quad + \int_0^1 \frac{\partial}{\partial\varepsilon_1}\bigg|_{\varepsilon_1=0} d\gamma^*_{\varepsilon_1,0} i_{\partial_{\varepsilon_2}|_{\varepsilon_2=0}\gamma_{\varepsilon_1,\varepsilon_2}}(\lambda\circ\gamma_{\varepsilon_1,0})\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 \frac{\partial}{\partial \varepsilon_1}\bigg|_{\varepsilon_1=0} d\lambda \left(\partial_{\varepsilon_2}|_{\varepsilon_2=0}\gamma_{\varepsilon_1,\varepsilon_2}, \dot{\gamma}_{\varepsilon_1,0}\right) \\
&\quad + \frac{\partial}{\partial \varepsilon_1}\bigg|_{\varepsilon_1=0} \lambda(\partial_{\varepsilon_2}|_{\varepsilon_2=0}\gamma_{\varepsilon_1,\varepsilon_2})|_0^1 \\
&= \int_0^1 d\lambda \left(\nabla_{\varepsilon_1}|_{\varepsilon_1=0}\partial_{\varepsilon_2}|_{\varepsilon_2=0}\gamma_{\varepsilon_1,\varepsilon_2}, \dot{\gamma}\right) \\
&\quad + \int_0^1 d\lambda \left(Y, \nabla_{\varepsilon_1}|_{\varepsilon_1=0}\partial_t \gamma_{\varepsilon_1,0}\right) \\
&\quad + \frac{\partial}{\partial \varepsilon_1}\bigg|_{\varepsilon_1=0} \lambda(\partial_{\varepsilon_2}|_{\varepsilon_2=0}\gamma_{\varepsilon_1,\varepsilon_2})|_0^1 \\
&= \tau \int_0^1 d\lambda \left(\nabla_{\varepsilon_1}|_{\varepsilon_1=0}\partial_{\varepsilon_2}|_{\varepsilon_2=0}\gamma_{\varepsilon_1,\varepsilon_2}, X_H(\gamma)\right) \\
&\quad + \int_0^1 d\lambda \left(Y, \nabla_t \partial_{\varepsilon_1}|_{\varepsilon_1=0}\gamma_{\varepsilon_1,0}\right) \\
&\quad + \frac{\partial}{\partial \varepsilon_1}\bigg|_{\varepsilon_1=0} \lambda(\partial_{\varepsilon_2}|_{\varepsilon_2=0}\gamma_{\varepsilon_1,\varepsilon_2})|_0^1 \\
&= \tau \int_0^1 d\lambda \left(\nabla_{\varepsilon_1}|_{\varepsilon_1=0}\partial_{\varepsilon_2}|_{\varepsilon_2=0}\gamma_{\varepsilon_1,\varepsilon_2}, X_H(\gamma)\right) \\
&\quad + \int_0^1 d\lambda \left(Y, \nabla_t X\right) \\
&\quad + \frac{\partial}{\partial \varepsilon_1}\bigg|_{\varepsilon_1=0} \lambda(\partial_{\varepsilon_2}|_{\varepsilon_2=0}\gamma_{\varepsilon_1,\varepsilon_2})|_0^1.
\end{aligned} \tag{3.6}
$$

Moreover

$$
\begin{aligned}
\int_0^1 d\lambda \left(Y, \nabla_t X\right) &= \int_0^1 \frac{d}{dt} d\lambda(Y, X) - \int_0^1 d\lambda(\nabla_t Y, X) \\
&= d\lambda(Y, X)|_0^1 + \int_0^1 d\lambda(X, \nabla_t Y) \\
&= d\lambda \left(D\varphi(Y(0)), D\varphi(X(0))\right) - d\lambda(Y(0), X(0)) \\
&\quad + \int_0^1 d\lambda(X, \nabla_t Y) \\
&= \int_0^1 d\lambda(X, \nabla_t Y).
\end{aligned} \tag{3.7}
$$

Finally

$$
\begin{aligned}
\lambda(\partial_{\varepsilon_2}|_{\varepsilon_2=0}\gamma_{\varepsilon_1,\varepsilon_2})|_0^1 &= \lambda\left(\partial_{\varepsilon_2}|_{\varepsilon_2=0}\gamma_{\varepsilon_1,\varepsilon_2}(1)\right) - \lambda\left(\partial_{\varepsilon_2}|_{\varepsilon_2=0}\gamma_{\varepsilon_1,\varepsilon_2}(0)\right) \\
&= \lambda\left(\partial_{\varepsilon_2}|_{\varepsilon_2=0}\varphi(\gamma_{\varepsilon_1,\varepsilon_2}(0))\right) - \lambda\left(\partial_{\varepsilon_2}|_{\varepsilon_2=0}\gamma_{\varepsilon_1,\varepsilon_2}(0)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \lambda\left(D\varphi\left(\partial_{\varepsilon_2}|_{\varepsilon_2=0}\gamma_{\varepsilon_1,\varepsilon_2}(0)\right)\right) - \lambda\left(\partial_{\varepsilon_2}|_{\varepsilon_2=0}\gamma_{\varepsilon_1,\varepsilon_2}(0)\right)\\
&= (\varphi^*\lambda - \lambda)(\partial_{\varepsilon_2}|_{\varepsilon_2=0}\gamma_{\varepsilon_1,\varepsilon_2}(0))\\
&= df_\varphi(\partial_{\varepsilon_2}|_{\varepsilon_2=0}\gamma_{\varepsilon_1,\varepsilon_2}(0)). \qquad (3.8)
\end{aligned}
$$

Combining (3.6), (3.4), (3.7), and (3.8) yields (3.3). □

Corollary 3.17. *The Hessian of the twisted Rabinowitz action functional is a well-defined, that is, independent of the choice of twisted two-parameter family, symmetric bilinear form.*

Lemma 3.18. *Let $\varphi \in \mathrm{Diff}(M)$ be an exact symplectomorphism of an exact symplectic manifold (M, λ) and $H \in C^\infty_\varphi(M)$. If $(\gamma, \tau) \in \mathrm{Crit}\,\mathcal{A}^H_\varphi$, then*

$$
\mathrm{Hess}\,\mathcal{A}^H_\varphi|_{(\gamma,\tau)}((X,\eta),(Y,\sigma)) = \int_0^1 d\lambda(Y, L_{\tau X_H}X - \eta X_H(\gamma)) - \sigma\int_0^1 dH(X) \qquad (3.9)
$$

for all $(X,\eta),(Y,\sigma) \in T_\gamma\mathcal{L}_\varphi M \times \mathbb{R}$, where

$$
L_{\tau X_H}X(t) = \frac{d}{ds}\bigg|_{s=0} D\phi^{X_H}_{-s\tau}X(s+t) \qquad \forall t \in \mathbb{R}.
$$

Proof. Inserting $\mathrm{Hess}^\nabla(X,Y) = d\lambda(Y, \nabla_X X_H)$ into (3.3) yields

$$
\mathrm{Hess}\,\mathcal{A}^H_\varphi|_{(\gamma,\tau)}((X,\eta),(Y,\sigma)) = \int_0^1 d\lambda(Y, \nabla_t X - \tau\nabla_X X_H) - \eta\int_0^1 dH(Y) - \sigma\int_0^1 dH(X).
$$

But as ∇ has no torsion by assumption, we compute

$$
\nabla_t X - \tau\nabla_X X_H = \nabla_{\dot\gamma}X - \tau\nabla_X X_H = \nabla_{\tau X_H}X - \tau\nabla_X X_H = [\tau X_H, X],
$$

and

$$
\begin{aligned}
[\tau X_H, X](t) &= L_{\tau X_H}X(t)\\
&= \frac{d}{ds}\bigg|_{s=0} D\phi^{X_H}_{-s\tau}(X(\phi^{X_H}_{s\tau}(\gamma(t)))\\
&= \frac{d}{ds}\bigg|_{s=0} D\phi^{X_H}_{-s\tau}(X(\phi^{X_H}_{s\tau}(\phi^{X_H}_{t\tau}(\gamma(0)))))\\
&= \frac{d}{ds}\bigg|_{s=0} D\phi^{X_H}_{-s\tau}X(s+t)
\end{aligned}
$$

for all $t \in \mathbb{R}$. □

Corollary 3.19. *Let $\varphi \in \mathrm{Diff}(M)$ be an exact symplectomorphism of an exact symplectic manifold (M, λ) and let $H \in C^\infty_\varphi(M)$. The kernel of the Hessian of the twisted Rabinowitz action functional $\mathcal{A}^H_\varphi$ at $(\gamma, \tau) \in \mathrm{Crit}\,\mathcal{A}^H_\varphi$ consists precisely of all $(X, \eta) \in T_\gamma \mathcal{L}_\varphi M \times \mathbb{R}$ satisfying*

$$L_{\tau X_H} X = \eta X_H(\gamma) \qquad \textit{and} \qquad \int_0^1 dH(X) = 0.$$

Lemma 3.20. *Let $\varphi \in \mathrm{Diff}(M)$ be an exact symplectomorphism of an exact symplectic manifold (M, λ) and $H \in C^\infty_\varphi(M)$. For every $(\gamma, \tau) \in \mathrm{Crit}\,\mathcal{A}^H_\varphi$, there is a canonical isomorphism*

$$\ker \mathrm{Hess}\,\mathcal{A}^H_\varphi|_{(\gamma,\tau)} \cong \mathfrak{K}(\gamma, \tau), \tag{3.10}$$

where

$$\mathfrak{K}(\gamma, \tau) := \{(v_0, \eta) \in T_{\gamma(0)} M \times \mathbb{R} : \text{solution of (3.11)}\}$$

with

$$D(\phi^{X_H}_{-\tau} \circ \varphi) v_0 = v_0 + \eta X_H(\gamma(0)) \qquad \textit{and} \qquad dH(v_0) = 0. \tag{3.11}$$

Proof. We follow [23, p. 99–100]. Let $(X, \eta) \in \ker \mathrm{Hess}\,\mathcal{A}^H_\varphi|_{(\gamma,\tau)}$ and define

$$v \colon I \to T_{\gamma(0)} M, \qquad v(t) := D\phi^{X_H}_{-\tau t} X(t).$$

We claim that

$$\ker \mathrm{Hess}\,\mathcal{A}^H_\varphi|_{(\gamma,\tau)} \to \mathfrak{K}(\gamma, \tau), \qquad (X, \eta) \mapsto (v(0), \eta) \tag{3.12}$$

is an isomorphism. First, we show that the above homomorphism is indeed well-defined. The assumption that (X, η) lies in the kernel of the Hessian of the twisted Rabinowitz action functional at the critical point (γ, τ) is by Corollary 3.19 equivalent to the system

$$\dot{v} = \eta X_H(\gamma(0)) \qquad \text{and} \qquad \int_0^1 dH(v) = 0. \tag{3.13}$$

Integrating the first equation yields

$$v(t) = v_0 + t\eta X_H(\gamma(0)) \qquad \forall t \in I,$$

with $v_0 := v(0)$. Thus $(v_0, \eta) \in \mathfrak{K}(\gamma, \tau)$ follows from

$$\begin{aligned} v(1) &= D\phi^{X_H}_{-\tau} X(1) \\ &= D\phi^{X_H}_{-\tau} D\varphi(X(0)) \\ &= D(\phi^{X_H}_{-\tau} \circ \varphi) X(0) \\ &= D(\phi^{X_H}_{-\tau} \circ \varphi) v_0. \end{aligned} \tag{3.14}$$

That (3.12) is an isomorphism follows by considering the inverse

$$\mathfrak{K}(\gamma,\tau) \to \ker \operatorname{Hess} \mathcal{A}_{\varphi}^{H}|_{(\gamma,\tau)}, \qquad (v_0,\eta) \mapsto (X,\eta),$$

where $X \in T_{\gamma}\mathcal{L}_{\varphi}M$ is defined by

$$X(t) := D\phi_{\tau t}^{X_H}(v_0 + t\eta X_H(\gamma(0))) \qquad \forall t \in \mathbb{R}.$$

This establishes the canonical isomorphism (3.12). ☐

Recall, that a strict contactomorphism of a contact manifold (Σ,α) is defined to be a diffeomorphism $\varphi \in \operatorname{Diff}(\Sigma)$ such that $\varphi^*\alpha = \alpha$.

Definition 3.21 (Parametrised Twisted Reeb Orbit). For a contact manifold (Σ,α) and a strict contactomorphism $\varphi\colon (\Sigma,\alpha) \to (\Sigma,\alpha)$ define the set of ***parametrised twisted Reeb orbits on*** $\boldsymbol{(\Sigma,\alpha)}$ by

$$\mathcal{P}_{\varphi}(\Sigma,\alpha) := \{(\gamma,\tau) \in \mathcal{L}_{\varphi}\Sigma \times \mathbb{R} : \dot{\gamma}(t) = \tau R(\gamma(t))\ \forall t \in \mathbb{R}\}.$$

Definition 3.22 (Twisted Spectrum). For a contact manifold (Σ,α) and a strict contactomorphism $\varphi\colon (\Sigma,\alpha) \to (\Sigma,\alpha)$ define the ***twisted spectrum*** by

$$\operatorname{Spec}_{\varphi}(\Sigma,\alpha) := \{\tau \in \mathbb{R} : \exists \gamma \in \mathcal{L}_{\varphi}\Sigma \text{ such that } (\gamma,\tau) \in \mathcal{P}_{\varphi}(\Sigma,\alpha)\}.$$

Lemma 3.23. *Let $\varphi\colon (\Sigma,\alpha) \to (\Sigma,\alpha)$ be a strict contactomorphism of a compact contact manifold (Σ,α). Then*

$$\varphi \circ \phi_t^R = \phi_t^R \circ \varphi \qquad \forall t \in \mathbb{R}.$$

Proof. If $\varphi_* R = R$, then we compute

$$\frac{d}{dt}\varphi \circ \phi_t^R = D\varphi \circ R \circ \phi_t^R = \varphi_* R \circ \varphi \circ \phi_t^R = R \circ \varphi \circ \phi_t^R,$$

for all $t \in \mathbb{R}$. To prove $\varphi_* R = R$, just observe that

$$\begin{aligned} i_{\varphi_* R} d\alpha &= \varphi^* d\alpha(R \circ \varphi^{-1}, D\varphi^{-1}\cdot) \\ &= d\varphi^*\alpha(R \circ \varphi^{-1}, D\varphi^{-1}\cdot) \\ &= d\alpha(R \circ \varphi^{-1}, D\varphi^{-1}\cdot) \\ &= \varphi_*(i_R d\alpha) \\ &= 0, \end{aligned}$$

and

$$\alpha(\varphi_* R) = \alpha(D\varphi \circ R \circ \varphi^{-1}) = \varphi^*\alpha(R \circ \varphi^{-1}) = \alpha(R \circ \varphi^{-1}) = 1.$$

Hence the statement follows from the uniqueness of integral curves. ☐

Proposition 3.24 (Kernel of the Hessian of the Twisted Rabinowitz Action Functional). *Let $(\Sigma, \lambda|_\Sigma)$ be a regular energy surface of restricted contact type in an exact Hamiltonian system (M, λ, H) with $X_H|_\Sigma = R$. Suppose $\varphi \in \mathrm{Diff}(M)$ is an exact symplectomorphism such that $H \in C^\infty_\varphi(M)$ and $\varphi^*\lambda|_\Sigma = \lambda|_\Sigma$. Then*

$$\mathrm{Crit}\,\mathscr{A}^H_\varphi = \mathscr{P}_\varphi(\Sigma, \lambda|_\Sigma)$$

and

$$\ker \mathrm{Hess}\,\mathscr{A}^H_\varphi|_{(\gamma,\tau)} \cong \ker\left(D(\phi^R_{-\tau} \circ \varphi)|_{\gamma(0)} - \mathrm{id}_{T_{\gamma(0)}\Sigma}\right)$$

for all $(\gamma, \tau) \in \mathscr{P}_\varphi(\Sigma, \lambda|_\Sigma)$. Moreover, we have $R(\gamma(0)) \in \ker \mathrm{Hess}\,\mathscr{A}^H_\varphi|_{(\gamma,\tau)}$ and if $\mathscr{P}_\varphi(\Sigma, \lambda|_\Sigma) \subseteq \Sigma \times \mathbb{R}$ is an embedded submanifold, then $\mathrm{Spec}_\varphi(\Sigma, \lambda|_\Sigma)$ is discrete.

Remark 3.25. If $(\gamma, \tau) \in \mathscr{P}_\varphi(\Sigma, \lambda|_\Sigma)$, we have the period-action equality

$$\mathscr{A}^H_\varphi(\gamma, \tau) = \int_0^1 \gamma^*\lambda = \int_0^1 \lambda(\dot{\gamma}) = \tau \int_0^1 \lambda(R(\gamma)) = \tau.$$

Proof. The identity $\mathrm{Crit}\,\mathscr{A}^H_\varphi = \mathscr{P}_\varphi(\Sigma, \lambda|_\Sigma)$ immediately follows from Corollary 3.11 together with [39, Corollary 5.30]. The proof of the formula for the kernel of the Hessian of $\mathscr{A}^H_\varphi$ is inspired by [23, p. 102]. By Lemma 3.20 we have that

$$\ker \mathrm{Hess}\,\mathscr{A}^H_\varphi|_{(\gamma,\tau)} \cong \mathfrak{K}(\gamma, \tau),$$

where $(v_0, \eta) \in T_{\gamma(0)}M \times \mathbb{R}$ belongs to $\mathfrak{K}(\gamma, \tau)$ if and only if

$$D(\phi^{X_H}_{-\tau} \circ \varphi)v_0 = v_0 + \eta X_H(\gamma(0)) \qquad \text{and} \qquad dH(v_0) = 0.$$

Thus in our setting, the second condition implies $v_0 \in T_{\gamma(0)}\Sigma$. Decompose

$$v_0 = v_0^\xi + aR(\gamma(0)) \qquad v_0^\xi \in \xi_{\gamma(0)}, a \in \mathbb{R},$$

where $\xi := \ker \lambda|_\Sigma$ denotes the contact distribution. Then we compute

$$\begin{aligned} D(\phi^R_{-\tau} \circ \varphi)R(\gamma(0)) &= D(\phi^R_{-\tau} \circ \varphi)\left(\left.\frac{d}{dt}\right|_{t=0} \phi^R_t(\gamma(0))\right) \\ &= \left.\frac{d}{dt}\right|_{t=0} (\phi^R_{-\tau} \circ \varphi \circ \phi^R_t)(\gamma(0)) \\ &= \left.\frac{d}{dt}\right|_{t=0} (\phi^R_t \circ \varphi \circ \phi^R_{-\tau})(\gamma(0)) \\ &= \left.\frac{d}{dt}\right|_{t=0} \phi^R_t(\gamma(0)) \\ &= R(\gamma(0)), \end{aligned}$$

using Lemma 3.23. Hence

$$v_0 + \eta R(\gamma(0)) = D(\phi^R_{-\tau} \circ \varphi)v_0 = D^\xi(\phi^R_{-\tau} \circ \varphi)v_0^\xi + aR(\gamma(0)),$$

where

$$D^\xi(\phi^R_{-\tau} \circ \varphi) := D(\phi^R_{-\tau} \circ \varphi)|_\xi \colon \xi \to \xi,$$

implies

$$\eta = 0 \qquad \text{and} \qquad D^\xi(\phi^R_{-\tau} \circ \varphi)v_0^\xi = v_0^\xi$$

by considering the splitting $T\Sigma = \xi \oplus \langle R\rangle$. Consequently

$$\mathfrak{K}(\gamma, \tau) = \ker\left(D(\phi^R_{-\tau} \circ \varphi)|_{\gamma(0)} - \mathrm{id}_{T_{\gamma(0)}\Sigma}\right) \times \{0\}\,.$$

Finally, assume that $\mathcal{P}_\varphi(\Sigma, \lambda|_\Sigma) \subseteq \Sigma \times \mathbb{R}$ is an embedded submanifold via the obvious identification of $(\gamma, \tau) \in \mathcal{P}_\varphi(\Sigma, \lambda|_\Sigma)$ with $(\gamma(0), \tau) \in \Sigma \times \mathbb{R}$. Fix a path (γ_s, τ_s) in $\mathcal{P}_\varphi(\Sigma, \lambda|_\Sigma) = \mathrm{Crit}\,\mathcal{A}^H_\varphi$ from (γ_0, τ_0) to (γ_1, τ_1). Using Remark 3.25 we compute

$$\partial_s \tau_s = \partial_s \mathcal{A}^H_\varphi(\gamma_s, \tau_s) = d\mathcal{A}^H_\varphi|_{(\gamma_s,\tau_s)}(\partial_s\gamma_s, \partial_s\tau_s) = 0,$$

implying that τ_s is constant, and in particular $\tau_0 = \tau_1$. Consequently, $\mathcal{A}^H_\varphi$ is constant on each path-connected component of $\mathcal{P}_\varphi(\Sigma, \lambda|_\Sigma)$. As $\mathcal{P}_\varphi(\Sigma, \lambda|_\Sigma)$ is a submanifold of $\Sigma \times \mathbb{R}$, there are only countably many connected components by definition, implying that $\mathrm{Spec}_\varphi(\Sigma, \lambda|_\Sigma)$ is discrete. □

3.2 Compactness of the Moduli Space of Twisted Negative Gradient Flow Lines

Definition 3.26 (Twisted Defining Hamiltonian Function). Let (W, λ) be a Liouville domain with boundary Σ and $\varphi \in \mathrm{Aut}(W, \lambda)$. A ***twisted defining Hamiltonian function for Σ*** is a Hamiltonian function $H \in C^\infty(M)$ on the completion (M, λ) of (W, λ), satisfying the following conditions:

(i) $H^{-1}(0) = \Sigma$ and $\Sigma \cap \mathrm{Crit}\, H = \varnothing$.
(ii) $H \in C^\infty_\varphi(M)$.
(iii) dH is compactly supported.
(iv) $X_H|_\Sigma = R$ is the Reeb vector field of the contact form $\lambda|_\Sigma$.

Denote by $\mathcal{F}_\varphi(\Sigma)$ the set of twisted defining Hamiltonian functions for Σ.

Remark 3.27. A necessary condition for $\mathcal{F}_\varphi(\Sigma) \neq \varnothing$ is that $\varphi^* R = R$. This is not true in general if φ does not induce a strict contactomorphism on Σ.

Definition 3.28 (Adapted Almost Complex Structure). Let (W, λ) be a Liouville domain with boundary Σ. An ***adapted almost complex structure on (W, λ)*** is a $d\lambda$-compatible almost complex structure J on (W, λ) such that J restricts to define a compatible almost complex structure on the contact distribution $\ker \lambda|_\Sigma$ and $JR = \partial_r$ holds near the boundary.

Definition 3.29 (Rabinowitz–Floer Data). Let (M, λ) be the completion of a Liouville domain (W, λ) with boundary Σ and $\varphi \in \operatorname{Aut}(W, \lambda)$. ***Rabinowitz–Floer data for*** $\boldsymbol{\varphi}$ is defined to be a pair (H, J) consisting of a twisted defining Hamiltonian function $H \in \mathscr{F}_\varphi(\Sigma)$ for Σ and an adapted almost complex structure J on (W, λ) such that $\varphi^* J = J$.

Lemma 3.30. *Let* (W, λ) *be a Liouville domain and* $\varphi \in \operatorname{Aut}(W, \lambda)$. *Then there exists Rabinowitz–Floer data for* φ.

Proof. The construction of the twisted defining Hamiltonian H for Σ is inspired by the proof of [19, Proposition 4.1]. Fix $\delta > 0$ such that the exact symplectic embedding

$$\psi \colon \left((-\delta, 0] \times \Sigma, e^r \lambda|_\Sigma\right) \hookrightarrow (W, \lambda)$$

defined by

$$\psi(r, x) := \phi_r^X(x)$$

satisfies

$$U_\delta := \psi((-\delta, 0] \times \Sigma) \cap \operatorname{supp} f_\varphi = \varnothing. \tag{3.15}$$

Indeed, that ψ is an exact symplectic embedding follows from the computation

$$\begin{aligned}
\frac{d}{dr}\psi_r^* \lambda &= \frac{d}{dr}\left(\phi_r^X\right)^* \lambda \\
&= (\phi_r^X)^* L_X \lambda \\
&= \left(\phi_r^X\right)^* (d i_X \lambda + i_X d\lambda) \\
&= \left(\phi_r^X\right)^* (d i_X i_X d\lambda + \lambda) \\
&= \left(\phi_r^X\right)^* \lambda \\
&= \psi_r^* \lambda
\end{aligned}$$

implying

$$\psi_r^* \lambda = e^r \lambda|_\Sigma \qquad \forall r \in (-\delta, 0],$$

by $\psi_0 = \iota_\Sigma$, where $\iota_\Sigma \colon \Sigma \hookrightarrow W$ denotes the inclusion. Note that $\psi_r^* X = \partial_r$. We claim

$$\varphi(\psi(r, x)) = \psi(r, \varphi(x)) \qquad \forall (r, x) \in (-\delta, 0] \times \Sigma, \tag{3.16}$$

that is, φ and ψ commute. Note that (3.16) is well-defined since $\varphi(\Sigma) = \Sigma$ by assumption. Indeed, (3.16) follows from the uniqueness of integral curves and the computation

$$\begin{aligned}
\frac{d}{dr}\varphi(\psi(r, x)) &= \frac{d}{dr}\varphi(\phi_r^X(x)) \\
&= D\varphi(X(\phi_r^X(x))) \\
&= (D\varphi \circ X|_{U_\delta} \circ \varphi^{-1} \circ \varphi)(\phi_r^X(x)) \\
&= (\varphi_* X|_{\varphi(U_\delta)} \circ \varphi)(\phi_r^X(x))
\end{aligned}$$

$$
\begin{aligned}
&= (X|_{\varphi(U_\delta)} \circ \varphi)(\phi_r^X(x)) \\
&= X(\varphi(\psi(r, x)))
\end{aligned}
$$

where we used the φ-invariance of the Liouville vector field on U_δ, that is

$$
\varphi_* X|_{\varphi(U_\delta)} = X|_{\varphi(U_\delta)},
$$

which in turn follows from

$$
\begin{aligned}
i_{\varphi_* X} d\lambda &= d\lambda(\varphi_* X, \cdot) \\
&= d\lambda(D\varphi \circ X \circ \varphi^{-1}, \cdot) \\
&= d\lambda \left(D\varphi \circ X \circ \varphi^{-1}, D\varphi \circ D\varphi^{-1} \cdot\right) \\
&= \varphi^* d\lambda(X \circ \varphi^{-1}, D\varphi^{-1} \cdot) \\
&= d\lambda(X \circ \varphi^{-1}, D\varphi^{-1} \cdot) \\
&= \varphi_*(i_X d\lambda) \\
&= \varphi_* \lambda \\
&= \lambda - d(f_\varphi \circ \varphi^{-1})
\end{aligned}
$$

and assumption (3.15).

Next we construct the defining Hamiltonian $H \in C^\infty(M)$. Let $h \in C^\infty(\mathbb{R})$ be a sufficiently small mollification of the piecewise linear function

$$
h(r) := \begin{cases} r & r \in \left[-\frac{\delta}{2}, \frac{\delta}{2}\right], \\ \frac{\delta}{2} & r \in \left[\frac{\delta}{2}, +\infty\right), \\ -\frac{\delta}{2} & r \in \left(-\infty, -\frac{\delta}{2}\right], \end{cases}
$$

as in Figure 3.1.

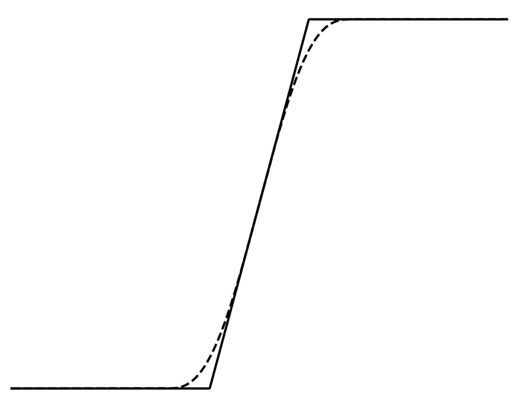

Fig. 3.1: Mollification of the piecewise linear function h.

Define $H \in C^\infty(M)$ by

$$H(p) := \begin{cases} h(r) & p = \psi(r,x) \in U_\delta, \\ h(r) & p = (r,x) \in [0,+\infty) \times \Sigma, \\ -\frac{\delta}{2} & p \in W \setminus U_\delta. \end{cases} \tag{3.17}$$

Then H is a defining Hamiltonian for Σ and dH is compactly supported by construction. Moreover, H is φ-invariant by (3.16). Finally, $X_H|_\Sigma = R$ follows from the observation $X_H = h'(r)e^{-r}R$. Indeed, on U_δ we compute

$$\begin{aligned} i_{h'(t)e^{-r}R}\psi^* d\lambda &= i_{h'(r)e^{-r}R} d(e^r \lambda|_\Sigma) \\ &= i_{h'(r)e^{-r}R}(e^r dr \wedge \lambda|_\Sigma + e^r d\lambda|_\Sigma) \\ &= -h'(r)dr \\ &= -dH. \end{aligned}$$

Next we construct the adapted almost complex structure J. Fix a φ-invariant compatible almost complex structure J^Σ on the contact distribution $\ker \lambda|_\Sigma$. Extend this family to $((-\delta,+\infty) \times \Sigma, d(e^r\lambda|_\Sigma))$ by setting

$$J^\Sigma|_{(a,x)}(b,v) := \left(\lambda_x(v), J^\Sigma|_x(\pi(v)) - bR(x)\right), \tag{3.18}$$

where $\pi\colon \ker\lambda|_\Sigma \oplus \langle R\rangle \to \ker\lambda|_\Sigma$ denotes the projection. Choose a φ-invariant $d\lambda$-compatible almost complex structure $J^{W\setminus\Sigma}$ on $W \setminus \Sigma$ and let $\{\beta^\Sigma, \beta^{W\setminus\Sigma}\}$ be a partition of unity subordinate to the open cover $\{U_\delta, W \setminus \Sigma\}$ of W. The compatible almost complex structure J associated with the Riemannian metric

$$m_J(\cdot,\cdot) := \beta^\Sigma d\lambda(J^\Sigma \cdot,\cdot) + \beta^{W\setminus\Sigma} d\lambda(J^{W\setminus\Sigma}\cdot,\cdot)$$

on W is adapted. □

Definition 3.31 (L^2-Metric). Let (H,J) be Rabinowitz–Floer data for a Liouville automorphism $\varphi \in \operatorname{Aut}(W,\lambda)$. Define an ***$L^2$-metric*** on $\mathscr{L}_\varphi M \times \mathbb{R}$

$$\langle (X,\eta),(Y,\sigma)\rangle_J := \int_0^1 d\lambda(JX(t),Y(t))dt + \eta\sigma \tag{3.19}$$

for all $(X,\eta),(Y,\sigma) \in T_\gamma \mathscr{L}_\varphi M \times \mathbb{R}$ and $\gamma \in \mathscr{L}_\varphi M$.

With respect to the L^2-metric (B.2), the gradient of the twisted Rabinowitz action functional $\operatorname{grad}_J \mathcal{A}_\varphi^H \in \mathfrak{X}(\mathscr{L}_\varphi M \times \mathbb{R})$ is given by

$$\operatorname{grad}_J \mathcal{A}_\varphi^H|_{(\gamma,\tau)}(t) = \begin{pmatrix} J(\dot\gamma(t) - \tau X_H(\gamma(t))) \\ -\int_0^1 H \circ \gamma \end{pmatrix} \qquad \forall t \in \mathbb{R}.$$

Lemma 3.32 (Fundamental Lemma). *Let (H, J) be Rabinowitz–Floer data for a Liouville automorphism $\varphi \in \operatorname{Aut}(W, \lambda)$ of a Liouville domain (W, λ). Then there exists a constant $C = C(\lambda, H, J, f_\varphi) > 0$ such that*

$$\|\operatorname{grad}_J \mathcal{A}^H_\varphi|_{(\gamma,\tau)}\|_J < \frac{1}{C} \quad \Rightarrow \quad |\tau| \leq C(|\mathcal{A}^H_\varphi(\gamma,\tau)| + 1)$$

for all $(\gamma, \tau) \in \mathcal{L}_\varphi M \times \mathbb{R}$.

Proof. We proceed in three steps.

Step 1: There exist constants $\delta > 0$ and $0 < C_\delta < +\infty$ such that if $(\gamma, \tau) \in \mathcal{L}_\varphi M$ with $\gamma(I) \subseteq H^{-1}\big((-\delta, \delta)\big) =: U_\delta$, then

$$|\tau| \leq 2\,|\mathcal{A}^H_\varphi(\gamma,\tau)| + C_\delta\,\|\operatorname{grad}_J \mathcal{A}^H_\varphi|_{(\gamma,\tau)}\|_J + 2\,\|f_\varphi\|_\infty .$$

Choose $\delta > 0$ such that $U_\delta \subseteq \operatorname{supp} X_H$ and

$$\lambda_x(X_H(x)) \geq \frac{1}{2} + \delta \qquad \forall x \in U_\delta .$$

This is possible as $X_H|_\Sigma = R$. Moreover, set

$$C_\delta := 2\,\|\lambda|_{U_\delta}\|_\infty .$$

Then $C_\delta < +\infty$ as dH is compactly supported. We estimate

$$\begin{aligned}
|\mathcal{A}^H_\varphi(\gamma,\tau)| &= \left|\int_0^1 \gamma^*\lambda - \tau\int_0^1 H(\gamma) - f_\varphi(\gamma(0))\right| \\
&= \left|\tau\int_0^1 \lambda(X_H(\gamma)) + \int_0^1 \lambda(\dot\gamma - \tau X_H(\gamma)) - \tau\int_0^1 H(\gamma) - f_\varphi(\gamma(0))\right| \\
&\geq |\tau|\left(\frac{1}{2} + \delta\right) - \left|\int_0^1 \lambda(\dot\gamma - \tau X_H(\gamma))\right| - |\tau|\,\delta - \|f_\varphi\|_\infty \\
&\geq \frac{|\tau|}{2} - \frac{C_\delta}{2}\int_0^1 \|\dot\gamma(t) - \tau X_H(\gamma(t))\|_J\,dt - \|f_\varphi\|_\infty \\
&\geq \frac{|\tau|}{2} - \frac{C_\delta}{2}\sqrt{\int_0^1 \|\dot\gamma(t) - \tau X_H(\gamma(t))\|_J^2\,dt} - \|f_\varphi\|_\infty \\
&\geq \frac{|\tau|}{2} - \frac{C_\delta}{2}\,\|\operatorname{grad}_J \mathcal{A}^H_\varphi|_{(\gamma,\tau)}\|_J - \|f_\varphi\|_\infty
\end{aligned}$$

by Jensen's inequality.

Step 2: For all $\delta > 0$, there exists $\varepsilon = \varepsilon(\delta) > 0$ with the property that if there exists $t_0 \in I$ with $|H(\gamma(t_0))| \geq \delta$ for $(\gamma, \tau) \in \mathcal{L}_\varphi M$, then $\|\operatorname{grad}_J \mathcal{A}^H_\varphi|_{(\gamma,\tau)}\|_J \geq \varepsilon$. Assume first that $\gamma(t) \in M \setminus U_{\frac{\delta}{2}}$ for all $t \in I$. In this case we estimate

$$\|\text{grad}_J \mathcal{A}_\varphi^H|_{(\gamma,\tau)}\|_J \geq \left|\int_0^1 H(\gamma)\right| \geq \frac{\delta}{2}.$$

Otherwise, we may assume without loss of generality that there exists $t_1 \in I$ such that $|H(\gamma(t_1))| \leq \frac{\delta}{2}$, $t_0 < t_1$ and $\frac{\delta}{2} \leq |H(\gamma(t))| \leq \delta$ for all $t \in [t_0, t_1]$. Set

$$\kappa := \max_{x \in \bar{U}_\delta} \|\text{grad}_J H\|_J > 0,$$

as $dH \neq 0$ in a neighbourhood of Σ. We estimate

$$\begin{aligned}
\|\text{grad}_J \mathcal{A}_\varphi^H|_{(\gamma,\tau)}\|_J &\geq \sqrt{\int_0^1 \|\dot{\gamma}(t) - \tau X_H(\gamma(t))\|_J^2 \, dt} \\
&\geq \int_0^1 \|\dot{\gamma}(t) - \tau X_H(\gamma(t))\|_J \, dt \\
&\geq \int_{t_0}^{t_1} \|\dot{\gamma}(t) - \tau X_H(\gamma(t))\|_J \, dt \\
&\geq \frac{1}{\kappa} \int_{t_0}^{t_1} \|\text{grad}_J H(\gamma(t))\|_J \, \|\dot{\gamma}(t) - \tau X_H(\gamma(t))\|_J \, dt \\
&\geq \frac{1}{\kappa} \int_{t_0}^{t_1} |m_J(\text{grad}_J H(\gamma(t)), \dot{\gamma}(t) - \tau X_H(\gamma(t)))| \, dt \\
&= \frac{1}{\kappa} \int_{t_0}^{t_1} |dH(\dot{\gamma}(t) - \tau X_H(\gamma(t)))| \, dt \\
&= \frac{1}{\kappa} \int_{t_0}^{t_1} |dH(\dot{\gamma}(t))| \, dt \\
&= \frac{1}{\kappa} \int_{t_0}^{t_1} \left|\frac{d}{dt}(H \circ \gamma)\right| \\
&\geq \frac{1}{\kappa} \left|\int_{t_0}^{t_1} \frac{d}{dt}(H \circ \gamma)\right| \\
&= \frac{1}{\kappa} |H(\gamma(t_1)) - H(\gamma(t_0))| \\
&\geq \frac{1}{\kappa} \big(|H(\gamma(t_0))| - |H(\gamma(t_1))|\big) \\
&\geq \frac{\delta}{2\kappa}
\end{aligned}$$

by Cauchy–Schwarz. Hence $\|\text{grad}_J \mathcal{A}_\varphi^H|_{(\gamma,\tau)}\|_J \geq \varepsilon$ for

$$\varepsilon = \varepsilon(\delta) := \frac{\delta}{2 \max\{1, \kappa\}}.$$

Step 3: We prove the Fundamental Lemma. Choose $\delta > 0$ and $0 < C_\delta < +\infty$ as in Step 1 and $\varepsilon = \varepsilon(\delta) > 0$ as in Step 2. Set

$$C_0 := \max\left\{2, C_\delta\varepsilon + 2\left\|f_\varphi\right\|_\infty\right\}.$$

Assume that $\left\|\operatorname{grad}_J \mathcal{A}_\varphi^H|_{(\gamma,\tau)}\right\|_J < \varepsilon$ for $(\gamma, \tau) \in \mathcal{L}_\varphi M$. Then $\gamma(I) \subseteq U_\delta$, as otherwise there exists $t_0 \in I$ with $|H(\gamma(t_0))| \geq \delta$ implying $\left\|\operatorname{grad}_J \mathcal{A}_\varphi^H|_{(\gamma,\tau)}\right\|_J \geq \varepsilon$ by Step 2. Thus with Step 1 we estimate

$$|\tau| \leq C_0(|\mathcal{A}_\varphi^H(\gamma, \tau)| + 1). \tag{3.20}$$

Finally, set

$$C := \max\left\{C_0, \frac{1}{\varepsilon}\right\}.$$

This proves the Fundamental Lemma. □

Definition 3.33 (Twisted Negative Gradient Flow Line). Let (H, J) be Rabinowitz–Floer data for a Liouville automorphism $\varphi \in \operatorname{Aut}(W, \lambda)$. A ***twisted negative gradient flow line*** is a tuple $(u, \tau) \in C^\infty(\mathbb{R}, \mathcal{L}_\varphi M \times \mathbb{R})$ such that

$$\partial_s(u, \tau) = -\operatorname{grad}_J \mathcal{A}_\varphi^H|_{(u(s),\tau(s))} \qquad \forall s \in \mathbb{R}.$$

Definition 3.34 (Energy). Let (H, J) be Rabinowitz–Floer data for a Liouville automorphism $\varphi \in \operatorname{Aut}(W, \lambda)$. The ***energy of a twisted negative gradient flow line*** $\boldsymbol{(u, \tau)}$ is defined by

$$E_J(u, \tau) := \int_{-\infty}^{+\infty} \|\partial_s(u, \tau)\|_J^2 \, ds = \int_{-\infty}^{+\infty} \left\|\operatorname{grad}_J \mathcal{A}_\varphi^H|_{(u(s),\tau(s))}\right\|_J^2 ds.$$

Theorem 3.35 (Compactness). *Let (H, J) be Rabinowitz–Floer data for a Liouville automorphism $\varphi \in \operatorname{Aut}(W, \lambda)$. Suppose (u_k, τ_k) is a sequence of negative gradient flow lines of the twisted Rabinowitz action functional $\mathcal{A}_\varphi^H$ such that there exist constants $a, b \in \mathbb{R}$ with*

$$a \leq \mathcal{A}_\varphi^H\left(u_k(s), \tau_k(s)\right) \leq b \qquad \forall k \in \mathbb{N}, s \in \mathbb{R}.$$

For every reparametrisation sequence $(s_k) \subseteq \mathbb{R}$ there exists a subsequence (s_{k_l}) and a negative gradient flow line (u_∞, τ_∞) of $\mathcal{A}_\varphi^H$ such that

$$\left(u_{k_l}(\cdot + s_{k_l}), \tau_{k_l}(\cdot + s_{k_l})\right) \xrightarrow{C^\infty_{\text{loc}}} (u_\infty, \tau_\infty) \qquad \textit{as } l \to \infty.$$

Proof. In order to show C^∞_{loc}-convergence, we need to establish

- a uniform L^∞-bound on u_k,
- a uniform L^∞-bound on τ_k,
- a uniform L^∞-bound on the derivatives of u_k.

Indeed, through elliptic bootstrapping [42, Theorem B.4.1] the negative gradient flow equation, we will obtain C^∞_{loc}-convergence by [42, Theorem B.4.2]. To obtain a uniform L^∞-bound on the sequence of twisted negative gradient flow lines u_k, observe that by definition of Rabinowitz–Floer data for φ, there exists $r \in (0, +\infty)$ such that

$$\operatorname{supp} X_H \cap [r, +\infty) \times \Sigma = \varnothing$$

and J is adapted to the boundary of $W \cup_\Sigma [0, r] \times \Sigma$. Consequently, the Maximum Principle [42, Corollary 9.2.11] implies that every u_k remains inside the compact set $W \cup_\Sigma [0, r] \times \Sigma$ as the asymptotics belong to $W \cup_\Sigma [0, r) \times \Sigma$ for all $k \in \mathbb{N}$. Indeed, this follows from

$$\begin{aligned} E_J(u_k, \tau_k) &= \int_{-\infty}^{+\infty} \|\partial_s(u_k, \tau_k)\|_J^2 \, ds \\ &= \int_{-\infty}^{+\infty} \langle \partial_s(u_k, \tau_k), \partial_s(u_k, \tau_k)\rangle_J \, ds \\ &= -\int_{-\infty}^{+\infty} \langle \operatorname{grad}_J \mathcal{A}_\varphi^H|_{(u_k(s), \tau_k(s))}, \partial_s(u_k, \tau_k)\rangle_J \, ds \\ &= -\int_{-\infty}^{+\infty} d\mathcal{A}_\varphi^H(\partial_s(u_k, \tau_k)) ds \\ &= -\int_{-\infty}^{+\infty} \partial_s \mathcal{A}_\varphi^H(u_k, \tau_k) ds \\ &= \lim_{s\to-\infty} \mathcal{A}_\varphi^H(u_k(s), \tau_k(s)) - \lim_{s\to+\infty} \mathcal{A}_\varphi^H(u_k(s), \tau_k(s)) \\ &\leq b - a, \end{aligned}$$

as this implies

$$\lim_{s\to\pm\infty} \|\partial_s(u_k, \tau_k)\|_J = \lim_{s\to\pm\infty} \left\|\operatorname{grad}_J \mathcal{A}_\varphi^H|_{(u_k(s), \tau_k(s))}\right\|_J = 0$$

by the negative gradient flow equation.

The uniform L^∞-bound on the Lagrange multipliers τ_k follows from the Fundamental Lemma 3.32. Fix a twisted negative gradient flow line (u, τ) and let $C > 0$ as in the Fundamental Lemma 3.32. For every $\sigma \in \mathbb{R}$ we can define $\zeta(\sigma) \geq 0$ by

$$\zeta(\sigma) := \inf \left\{ \zeta \geq 0 : \left\|\operatorname{grad}_J \mathcal{A}_\varphi^H|_{(u(\sigma+\zeta), \tau(\sigma+\zeta))}\right\|_J < \frac{1}{C} \right\}.$$

We estimate

$$b - a \geq E_J(u, \tau) \geq \int_\sigma^{\sigma+\zeta(\sigma)} \left\|\operatorname{grad}_J \mathcal{A}_\varphi^H|_{(u_s, \tau(s))}\right\|_J^2 ds \geq \frac{\zeta(\sigma)}{C^2}.$$

By the Fundamental Lemma 3.32 we have that

$$|\tau(\sigma + \zeta(\sigma))| < C(\max\{|a|, |b|\} + 1) \qquad \forall \sigma \in \mathbb{R},$$

and thus using the negative gradient flow equation again we estimate

$$\begin{aligned}|\tau(\sigma)| &\leq |\tau(\sigma + \zeta(\sigma))| + \int_\sigma^{\sigma+\zeta(\sigma)} |\partial_s \tau(s)|\, ds \\ &= |\tau(\sigma + \zeta(\sigma))| + \int_\sigma^{\sigma+\zeta(\sigma)} \left| \int_0^1 H(u(s,t))dt \right| ds \\ &\leq C(\max\{|a|, |b|\} + 1) + \zeta(\sigma)\, \|H\|_\infty \\ &\leq C(\max\{|a|, |b|\} + 1) + C^2(b-a)\, \|H\|_\infty .\end{aligned}$$

for all $\sigma \in \mathbb{R}$. Hence

$$\|\tau\|_\infty \leq C(\max\{|a|, |b|\} + 1) + C^2(b-a)\, \|H\|_\infty$$

is independent of the twisted negative gradient flow line (u, τ).

The uniform L^∞-bound on the derivatives of u_k follows from Corollary C.9 as an exact symplectic manifold is symplectically aspherical. □

3.3 Definition of Twisted Rabinowitz–Floer Homology

In this section we make implicit use of the requirement that a Liouville automorphism has finite order. This is crucial because then the arguments proceed as in the case of loops by Remark 3.7.

Definition 3.36 (Transverse Conley–Zehnder Index). Let (W^{2n}, λ) be a Liouville domain with boundary Σ. Let $(\gamma_0, \tau_0), (\gamma_1, \tau_1) \in \mathscr{P}_\varphi(\Sigma, \lambda|_\Sigma)$ for some Liouville automorphism $\varphi \in \operatorname{Aut}(W, \lambda)$ such that there exists a path γ_σ in $\mathscr{L}_\varphi \Sigma$ from γ_0 to γ_1. Define the ***transverse Conley–Zehnder index*** by

$$\mu((\gamma_0, \tau_0), (\gamma_1, \tau_1)) := \mu_{\mathrm{CZ}}(\Psi^1) - \mu_{\mathrm{CZ}}(\Psi^0) \in \mathbb{Z},$$

with

$$\begin{aligned} \Psi^0 \colon I &\to \mathrm{Sp}(n-1), \qquad \Psi^0_t := \Phi^{-1}_{t,0} \circ D^\xi \phi^R_{\tau_0 t} \circ \Phi_{0,0}, \\ \Psi^1 \colon I &\to \mathrm{Sp}(n-1), \qquad \Psi^1_t := \Phi^{-1}_{t,1} \circ D^\xi \phi^R_{\tau_1 t} \circ \Phi_{0,1}, \end{aligned}$$

where $\Phi_{t,\sigma} \colon \mathbb{R}^{2n-2} \to \xi_{\gamma_\sigma(t)}$ is a symplectic trivialisation of $F^*\xi$, $\xi := \ker \lambda|_\Sigma$ with $F \in C^\infty(\mathbb{R} \times I, M)$ being defined by $F(t, \sigma) := \gamma_\sigma(t)$, satisfying

$$\Phi_{t+1,\sigma} = D\varphi \circ \Phi_{t,\sigma} \qquad \forall (t, \sigma) \in \mathbb{R} \times I. \tag{3.21}$$

Lemma 3.37. *In the setup of Definition 3.36, the transverse Conley–Zehnder index is well-defined, that is, independent of the choice of symplectic trivialisation.*

Proof. First we need to show that one can always construct a symplectic trivialisation

$$\Phi_{t,\sigma}\colon \mathbb{R}^{2n-2} \to \xi_{F(t,\sigma)} \qquad \forall (t,\sigma) \in \mathbb{R} \times I,$$

of $F^*\xi$ satisfying the twist condition (3.21). By [43, Theorem 2.1.3], there exists a linear symplectomorphism $\Phi_{0,0}\colon \mathbb{R}^{2n-2} \to \xi_{F(0,0)}$. By [43, Lemma 2.6.6], we get a symplectic trivialisation $\Phi_{t,0}\colon \mathbb{R}^{2n-2} \to \xi_{F(t,0)}$ for all $t \in I$ with $\Phi_{1,0} = D\varphi \circ \Phi_{0,0}$. Extend this trivialisation to $\mathbb{R}$ by setting

$$\Phi_{t+k,0} := D\varphi^k \circ \Phi_{t,0} \qquad \forall k \in \mathbb{Z}.$$

Next, trivialise along each ray $\sigma \mapsto F(t,\sigma)$ for fixed $t \in \mathbb{R}$. Hence we get a symplectic trivialisation $\Phi_{t,\sigma}\colon \mathbb{R}^{2n-2} \to \xi_{F(t,\sigma)}$ of $F^*\xi$ satisfying (3.21).

Now we show that the transverse Conley–Zehnder index μ is independent of the choice of trivialisation. Suppose that $\widetilde{\Phi}_{t,\sigma}\colon \mathbb{R}^{2n-2} \to \xi_{F(t,\sigma)}$ is another symplectic trivialisation of $F^*\xi$ satisfying

$$\widetilde{\Phi}_{t+1,\sigma} = D\varphi \circ \widetilde{\Phi}_{t,\sigma} \qquad \forall (t,\sigma) \in \mathbb{R} \times I.$$

Then we have

$$\widetilde{\Psi}_{t,\sigma} = \widetilde{\Phi}^{-1}_{t,\sigma} \circ \Phi_{t,\sigma} \circ \Psi_{t,\sigma} \circ \Phi^{-1}_{0,\sigma} \circ \widetilde{\Phi}_{0,\sigma} \qquad \forall (t,\sigma) \in \mathbb{R} \times \partial I,$$

where

$$\Psi_{t,\sigma} = \Phi^{-1}_{t,\sigma} \circ D^{\xi}\theta_t^{\tau_\sigma R} \circ \Phi_{0,\sigma}.$$

Define

$$\eta\colon \mathbb{T} \times I \to \mathrm{Sp}(n-1), \qquad \eta(t,\sigma) := \Phi^{-1}_{0,\sigma} \circ \widetilde{\Phi}_{0,\sigma} \circ \widetilde{\Phi}^{-1}_{t,\sigma} \circ \Phi_{t,\sigma}.$$

Indeed, we compute

$$\begin{aligned}
\eta(t+1,\sigma) &= \Phi^{-1}_{0,\sigma} \circ \widetilde{\Phi}_{0,\sigma} \circ \widetilde{\Phi}^{-1}_{t+1,\sigma} \circ \Phi_{t+1,\sigma} \\
&= \Phi^{-1}_{0,\sigma} \circ \widetilde{\Phi}_{0,\sigma} \circ \widetilde{\Phi}^{-1}_{t,\sigma} \circ D\varphi^{-1} \circ D\varphi \circ \Phi_{t,\sigma} \\
&= \eta(t,\sigma),
\end{aligned}$$

for all $(t,\sigma) \in \mathbb{R} \times I$. Using the naturality as well as the loop property [48, p. 20] of the Conley–Zehnder index we compute

$$\mu_{\mathrm{CZ}}(\widetilde{\Psi}^s) = \mu_{\mathrm{CZ}}(\eta_s \cdot \Psi^s) = \mu_{\mathrm{CZ}}(\Psi^s) + 2\mu_{\mathrm{M}}(\eta_s) \qquad \forall s \in \partial I,$$

where μ_{M} denotes the Maslov index. In particular,

$$\begin{aligned}
\mu_{\mathrm{CZ}}(\widetilde{\Psi}^1) - \mu_{\mathrm{CZ}}(\widetilde{\Psi}^0) &= \mu_{\mathrm{CZ}}(\Psi^1) - \mu_{\mathrm{CZ}}(\Psi^0) + 2\left(\mu_{\mathrm{M}}(\eta_1) - \mu_{\mathrm{M}}(\eta_0)\right) \\
&= \mu_{\mathrm{CZ}}(\Psi^1) - \mu_{\mathrm{CZ}}(\Psi^0)
\end{aligned}$$

by the invariance property of the Maslov index [23, p. 195]. □

Remark 3.38. Denote by

$$\Sigma_\varphi := \frac{\Sigma \times \mathbb{R}}{(\varphi(x), t+1) \sim (x, t)}$$

the mapping torus of φ giving rise to the fibration

$$\pi_\varphi \colon \Sigma_\varphi \to \mathbb{T}, \qquad \pi_\varphi([x,t]) := [t].$$

The vertical bundle $\ker D^\xi \pi_\varphi \to \Sigma_\varphi$ is a symplectic vector bundle. If $\widetilde{F}$ is another homotopy in $\mathscr{L}_\varphi \Sigma$ from γ_0 to γ_1, the concatenation with the reversed path F^- can be identified with the map

$$\widetilde{F} \# F^- \colon \mathbb{T}^2 \to \Sigma_\varphi, \qquad (t, \sigma) \mapsto [(\widetilde{F} \# F^-(t,\sigma), t)].$$

Hence using the concatenation property of the Conley–Zehnder index [23, p. 195] as well as the functoriality of the Chern number [43, p. 85], for $s \in \partial I$ we compute

$$\begin{aligned}
\mu_{\mathrm{CZ}}(\widetilde{\Psi}^s) - \mu_{\mathrm{CZ}}(\Psi^s) &= \mu_{\mathrm{CZ}}\left(\widetilde{\Psi}^s \# (\Psi^s)^-\right) \\
&= 2\mu_{\mathrm{M}}\left(\widetilde{\Psi}^s \# (\Psi^s)^-\right) \\
&= 2c_1\left((\widetilde{F} \# F^-)^* \ker D^\xi \pi_\varphi\right) \\
&= 2c_1(\ker D^\xi \pi_\varphi).
\end{aligned}$$

Thus if the transverse Conley–Zehnder index is viewed in $\mathbb{Z}_2$ or $c_1(\ker D^\xi \pi_\varphi) = 0$, then it additionally does not depend on the choice of path in $\mathscr{L}_\varphi \Sigma$.

Let (H, J) be Rabinowitz–Floer data for $\varphi \in \operatorname{Aut}(W, \lambda)$. Set

$$\Sigma := \partial W \qquad \text{and} \qquad M := W \cup_\Sigma [0, +\infty) \times \Sigma.$$

Fix $(\eta, \tau_\eta) \in \mathscr{P}_\varphi(\Sigma, \lambda|_\Sigma)$ and denote by $[\eta]$ the corresponding class in $\pi_0 \mathscr{L}_\varphi \Sigma$. Assume that the twisted Rabinowitz action functional $\mathscr{A}^H_\varphi$ is Morse–Bott, that is, $\operatorname{Crit} \mathscr{A}^H_\varphi \subseteq \Sigma \times \mathbb{R}$ is a properly embedded submanifold by Proposition 3.24, and fix a Morse function $h \in C^\infty(\operatorname{Crit} \mathscr{A}^H_\varphi)$. Define the twisted Rabinowitz–Floer chain group $\mathrm{RFC}^\varphi(\Sigma, M)$ to be the $\mathbb{Z}_2$-vector space consisting of all formal linear combinations

$$\zeta = \sum_{\substack{(\gamma,\tau) \in \operatorname{Crit}(h) \\ [\gamma] = [\eta]}} \zeta_{(\gamma,\tau)} (\gamma, \tau)$$

satisfying the Novikov finiteness condition

$$\#\{(\gamma, \tau) \in \operatorname{Crit}(h) : \zeta_{(\gamma,\tau)} \neq 0, \mathscr{A}^H_\varphi(\gamma, \tau) \geq \kappa\} < \infty \qquad \forall \kappa \in \mathbb{R}.$$

Define a boundary operator

$$\partial \colon \mathrm{RFC}^\varphi(\Sigma, M) \to \mathrm{RFC}^\varphi(\Sigma, M)$$

by

$$\partial(\gamma^-,\tau^-) := \sum_{\substack{(\gamma^+,\tau^+)\in\mathrm{Crit}(h)\\ [\gamma^+]=[\gamma^-]}} n_\varphi(\gamma^\pm,\tau^\pm)(\gamma^+,\tau^+),$$

where

$$n_\varphi(\gamma^\pm,\tau^\pm) := \#_2\mathcal{M}^0_\varphi(\gamma^\pm,\tau^\pm) \in \mathbb{Z}_2,$$

with $\mathcal{M}^0_\varphi(\gamma^\pm,\tau^\pm)$ denoting the zero-dimensional component of the moduli space of all unparametrised twisted negative gradient flow lines with cascades from (γ^-,τ^-) to (γ^+,τ^+). This is well-defined by Theorem 3.35. Define the ***twisted Rabinowitz–Floer homology of Σ and φ*** by

$$\mathrm{RFH}^\varphi(\Sigma, M) := \frac{\ker\partial}{\mathrm{im}\,\partial}.$$

Proposition 3.39. *Let (W,λ) be a Liouville domain with simply connected boundary Σ and $\varphi \in \mathrm{Aut}(W,\lambda)$. If there do not exist any nonconstant twisted periodic Reeb orbits on Σ, then*

$$\mathrm{RFH}^\varphi_*(\Sigma, M) \cong \mathrm{H}_*(\mathrm{Fix}(\varphi|_\Sigma);\mathbb{Z}_2).$$

Proof. If there do not exist any nonconstant twisted periodic Reeb orbits,

$$\mathrm{Crit}\,\mathcal{A}^H_\varphi = \{(c_x,0) : x\in\mathrm{Fix}(\varphi|_\Sigma)\} \cong \mathrm{Fix}(\varphi|_\Sigma)$$

for any $H \in \mathcal{F}_\varphi(\Sigma)$. Since $\mathrm{Fix}(\varphi|_\Sigma)$ is a properly embedded submanifold of Σ by [40, Problem 8-32] or [43, Lemma 5.5.7], $\mathcal{A}^H_\varphi$ is a Morse–Bott function. Let $x, y \in \mathrm{Fix}(\varphi|_\Sigma)$. As Σ is simply connected by assumption, there exists some path γ from x to y in Σ and a homotopy from γ to $\varphi\circ\gamma$ with fixed endpoints. Extend this homotopy to a path in $\mathcal{L}_\varphi\Sigma$ from c_x to c_y. Choose a Morse function h on $\mathrm{Fix}(\varphi|_\Sigma)$ and any critical point $c_x \in \mathrm{Fix}(\varphi|_\Sigma)$. Then we can define a $\mathbb{Z}$-grading of $\mathrm{RFC}^\varphi(\Sigma, M)$ by

$$\mu((c_y,0),(c_x,0)) + \mathrm{ind}_h(c_y) = \mathrm{ind}_h(c_y) \qquad \forall c_y \in \mathrm{Crit}(h),$$

and consequently,

$$\mathrm{RFH}^\varphi_*(\Sigma, M) = \mathrm{HM}_*(\mathrm{Fix}(\varphi|_\Sigma);\mathbb{Z}_2) \cong \mathrm{H}_*(\mathrm{Fix}(\varphi|_\Sigma);\mathbb{Z}_2)$$

as there are only twisted negative gradient flow lines with zero cascades, that is, ordinary Morse gradient flow lines of h. Indeed, suppose that there is a nonconstant twisted negative gradient flow line (u,τ) of $\mathcal{A}^H_\varphi$ with asymptotics $(\gamma^\pm,\tau^\pm)$. Using the twisted negative gradient flow equation we estimate

$$\tau^- - \tau^+ = \int_{-\infty}^{+\infty} \left\|\mathrm{grad}_J\,\mathcal{A}^H_\varphi|_{(u(s),\tau(s))}\right\|^2_J ds > 0.$$

Hence $\tau^+ < \tau^-$, contradicting $\tau^\pm = 0$. □

3.4 Invariance of Twisted Rabinowitz–Floer Homology Under Twisted Homotopies of Liouville Domains

Definition 3.40 (Twisted Homotopy of Liouville Domains). Given the completion (M, λ) of a Liouville domain (W_0, λ) and $\varphi \in \operatorname{Aut}(W_0, \lambda)$, a ***twisted homotopy of Liouville domains in M*** is a time-dependent Hamiltonian function $H \in C^\infty(M \times I)$ such that

(i) $W_\sigma := H_\sigma^{-1}((-\infty, 0])$ is a Liouville domain with symplectic form $d\lambda|_{W_\sigma}$ and boundary $\Sigma_\sigma := H_\sigma^{-1}(0)$ for all $\sigma \in I$,
(ii) $H_\sigma \in \mathcal{F}_\varphi(\Sigma_\sigma)$ for all $\sigma \in I$,
(iii) $\Sigma_\sigma \cap \operatorname{supp} f_\varphi = \varnothing$ for all $\sigma \in I$.

We write $(H_\sigma)_{\sigma \in I}$ for a twisted homotopy of Liouville domains.

Theorem 3.41 (Invariance of Twisted Rabinowitz–Floer Homology). *If $(H_\sigma)_{\sigma \in I}$ is a twisted homotopy of Liouville domains such that both $\mathcal{A}_\varphi^{H_0}$ and $\mathcal{A}_\varphi^{H_1}$ are Morse–Bott, then there is a canonical isomorphism*

$$\mathrm{RFH}^\varphi(\Sigma_0, M) \cong \mathrm{RFH}^\varphi(\Sigma_1, M).$$

Proof. The proof follows from the same adiabatic argument as in [17, p. 275–277]. Crucial is that [17, Theorem 3.6] remains true in our setting, as well as the genericness of the Morse–Bott condition. Indeed, if (M, λ) is an exact symplectic manifold and $\varphi \in \operatorname{Diff}(M)$ is of finite order such that $\varphi^*\lambda = \lambda$, then we have the following generalisation of [17, Theorem B.1]. Adapting the proof accordingly, one can show that there exists a subset

$$\mathcal{U} \subseteq \{H \in C_\varphi^\infty(M) : \operatorname{supp} dH \text{ compact}\},$$

of the second category such that for every $H \in \mathcal{U}$, $\mathcal{A}_\varphi^H$ is Morse–Bott with critical manifold being $\operatorname{Fix}(\varphi|_{H^{-1}(0)})$ together with a disjoint union of circles. Again, this works only since the contact condition is an open condition. □

Remark 3.42. Invariance of twisted Rabinowitz–Floer homology allows us to define twisted Rabinowitz–Floer homology also in the case where $\mathcal{A}_\varphi^H$ is not necessarily Morse–Bott. Indeed, as the proof of Theorem 3.41 shows, we can perturb the hypersurface Σ slightly to make it Morse–Bott. Thus we can define the twisted Rabinowitz–Floer homology of such a hypersurface to be the twisted Rabinowitz–Floer homology of any Morse–Bott perturbation. By Theorem 3.41, this is indeed well-defined.

Corollary 3.43 (Independence). *Let $\varphi \in \operatorname{Aut}(W, \lambda)$ and $H_0, H_1 \in \mathcal{F}_\varphi(\Sigma)$ be such that either $\mathcal{A}_\varphi^{H_0}$ or $\mathcal{A}_\varphi^{H_1}$ is Morse–Bott. Then the definition of twisted Rabinowitz–Floer homology $\mathrm{RFH}^\varphi(\Sigma, M)$ is independent of the choice of a twisted defining Hamiltonian function for Σ.*

Proof. Note that $\mathscr{F}_\varphi(\Sigma)$ is a convex space. Indeed, set

$$H_\sigma := (1-\sigma)H_0 + \sigma H_1 \qquad \sigma \in I.$$

Then $\varphi^* H_\sigma = H_\sigma$, dH_σ has compact support and $X_{H_\sigma}|_\Sigma = R$ for all $\sigma \in I$. Moreover, for the Liouville vector field $X \in \mathfrak{X}(M)$ we compute

$$\frac{d}{dt}\bigg|_{t=0} H \circ \phi_t^X|_\Sigma = dH(X)|_\Sigma = d\lambda(X, X_H)|_\Sigma = \lambda(X_H)|_\Sigma = \lambda(R) = 1,$$

for any $H \in \mathscr{F}_\varphi(\Sigma)$, and thus $H < 0$ on Int W and $H > 0$ on $M \setminus W$. Consequently, $H_\sigma^{-1}(0) = \Sigma$ and so $H_\sigma \in \mathscr{F}_\varphi(\Sigma)$ for all $\sigma \in I$. Hence $(H_\sigma)_{\sigma \in I}$ is a twisted homotopy of Liouville domains in M and Theorem 3.41 implies the claim. □

3.5 Twisted Leaf-Wise Intersection Points

Definition 3.44 (Twisted Leaf-Wise Intersection Point). Let (M, λ) be the completion of a Liouville domain (W, λ) and let $\varphi \in \operatorname{Aut}(W, \lambda)$ be a Liouville automorphism. A point $x \in \Sigma$ is a ***twisted leaf-wise intersection point*** for a Hamiltonian symplectomorphism $\varphi_F \in \operatorname{Ham}(M, d\lambda)$, if

$$\varphi_F(x) \in L_{\varphi(x)} := \{\phi_t^R(\varphi(x)) : t \in \mathbb{R}\}.$$

Definition 3.45 (Twisted Moser Pair). Let $\varphi \in \operatorname{Aut}(W, \lambda)$. A ***twisted Moser pair*** is defined to be a tuple $\mathfrak{M} := (\chi H, F)$, where

(i) $H \in C_\varphi^\infty(M)$, $F \in C_\varphi^\infty(M \times \mathbb{R})$ and $\chi \in C^\infty(\mathbb{S}^1, I)$ such that $\int_0^1 \chi = 1$. Any time-dependent Hamiltonian function χH is said to be ***weakly time-dependent***.
(ii) supp $\chi \subseteq \left(0, \frac{1}{2}\right)$ and $F_t = 0$ for all $t \in \left[0, \frac{1}{2}\right]$.

Lemma 3.46. *Let $\varphi \in \operatorname{Aut}(W, \lambda)$. For all $H \in \mathscr{F}_\varphi(\Sigma)$ and $\varphi_F \in \operatorname{Ham}(M, d\lambda)$ there exists a corresponding twisted Moser pair $\mathfrak{M}$ such that the flow of χX_H is a time-reparametrisation of the flow of X_H.*

Proof. For constructing the Hamiltonian perturbation $\widetilde{F}$, fix $\rho \in C^\infty(I, I)$ such that

$$\rho(t) = \begin{cases} 0 & t \in \left[0, \frac{1}{2}\right], \\ 1 & t \in \left[\frac{2}{3}, 1\right]. \end{cases}$$

See Figure 3.2a. Then define $\widetilde{F} \in C_\varphi^\infty(M \times \mathbb{R})$ by

$$\widetilde{F}(x,t) := \dot{\rho}(t-k) F\left(\varphi^{-k}(x), \rho(t-k)\right) \qquad \forall t \in [k, k+1],$$

for $k \in \mathbb{Z}$. See Figure 3.2b. Then $\widetilde{F}_t = 0$ for all $t \in \left[0, \frac{1}{2}\right]$, and

$$\phi_t^{X_{\widetilde{F}}} = \phi_{\rho(t)}^{X_F} \qquad \forall t \in I.$$

Indeed, we compute

$$\frac{d}{dt}\phi^{X_F}_{\rho(t)} = \dot\rho(t)\frac{d}{d\rho}\phi^{X_F}_{\rho(t)} = \dot\rho(t)\left(X_{F_{\rho(t)}}\circ\phi^{X_F}_{\rho(t)}\right) = X_{\widetilde F_t}\circ\phi^{X_F}_{\rho(t)}.$$

In particular

$$\varphi_{\widetilde F} = \phi_1^{X_{\widetilde F}} = \phi^{X_F}_{\rho(1)} = \phi_1^{X_F} = \varphi_F.$$

Finally, we have that

$$\phi_t^{\chi X_H} = \phi^{X_H}_{\tau(t)} \qquad \text{with} \qquad \tau(t) := \int_0^t \chi,$$

as we compute

$$\frac{d}{dt}\phi^{X_H}_{\tau(t)} = \chi(t)\frac{d}{d\tau}\phi^{X_H}_{\tau(t)} = \chi(t)X_H\circ\phi^{X_H}_{\tau(t)},$$

and thus we conclude by the uniqueness of integral curves. □

(a) The smooth function ρ. (b) The derivative $\dot\rho$ of ρ.

Fig. 3.2

Lemma 3.47. *Let* $\varphi\in\mathrm{Aut}(W,\lambda)$ *and* $\varphi_F\in\mathrm{Ham}(M,d\lambda)$ *a Hamiltonian symplectomorphism. If* $(\gamma,\tau)\in\mathrm{Crit}\,\mathcal{A}^{\mathfrak{M}}_\varphi$*, then* $x:=\gamma\left(\frac{1}{2}\right)$ *is a twisted leaf-wise intersection point for* φ_F*.*

Proof. Let $\mathfrak{M}=(\chi H, F)$ denote the twisted Moser pair from Lemma 3.46. Using Proposition 3.9 we compute

$$\begin{aligned}\partial_t H(\gamma(t)) &= dH(\dot\gamma(t))\\ &= dH(\tau X_{\chi(t)H}(\gamma(t)) + X_{F_t}(\gamma(t)))\\ &= dH(\tau\chi(t)X_H(\gamma(t)))\\ &= \tau\chi(t)dH(X_H(\gamma(t)))\\ &= 0\end{aligned}$$

for all $t \in \left[0, \frac{1}{2}\right]$. Thus $H \circ \gamma = c \in \mathbb{R}$ on $\left[0, \frac{1}{2}\right]$ with

$$0 = \int_0^1 \chi H(\gamma) = \int_0^{\frac{1}{2}} \chi H(\gamma) = c \int_0^{\frac{1}{2}} \chi = c \int_0^1 \chi = c.$$

Consequently, $\gamma(0) \in L_x$ and $x \in \Sigma$. Moreover, also $\gamma(1) = \varphi(\gamma(0)) \in \Sigma$ by the φ-invariance of H. For $t \in \left[\frac{1}{2}, 1\right]$, $\dot{\gamma} = X_{F_t}(\gamma)$ and so $\gamma(1) = \varphi_F(x) \in \Sigma$. We conclude

$$L_{\varphi(x)} = \{\phi_t^R(\varphi(x)) : t \in \mathbb{R}\} = \{\varphi(\phi_t^R(x)) : t \in \mathbb{R}\} = \varphi(L_x),$$

and so $\varphi_F(x) = \gamma(1) = \varphi(\gamma(0)) \in L_{\varphi(x)}$. □

Theorem 3.48. *Let (W, λ) be a Liouville domain with displaceable boundary in the completion (M, λ) and $\varphi \in \mathrm{Aut}(W, \lambda)$. Then $\mathrm{RFH}^\varphi(\Sigma, M) \cong 0$.*

Proof. Suppose that $\Sigma = \partial W$ is displaceable in M via $\varphi_F \in \mathrm{Ham}_c(M, d\lambda)$ and choose Rabinowitz–Floer data (H, J) for φ. Denote by $\mathfrak{M} = (\chi H, F)$ the associated twisted Moser pair from Lemma 3.46. Then $\mathrm{Crit}\, \mathcal{A}_\varphi^{\mathfrak{M}} = \varnothing$. Indeed, if there exists $(\gamma, \tau) \in \mathrm{Crit}\, \mathcal{A}_\varphi^{\mathfrak{M}}$, then $\gamma\left(\frac{1}{2}\right)$ is a twisted leaf-wise intersection point for φ_F by Lemma 3.47. However, this is impossible as by displaceability we have that $\varphi_F(\Sigma) \cap \Sigma = \varnothing$. Consequently, the perturbed twisted Rabinowitz action functional $\mathcal{A}_\varphi^{\mathfrak{M}}$ is a Morse function. By adapting the Fundamental Lemma to the current setting as in [5, Theorem 2.9], the Floer homology $\mathrm{HF}(\mathcal{A}_\varphi^{\mathfrak{M}})$ is well-defined. By making use of continuation homomorphisms we have that

$$0 = \mathrm{HF}(\mathcal{A}_\varphi^{\mathfrak{M}}) \cong \mathrm{HF}(\mathcal{A}_\varphi^{(\chi H, 0)}) \cong \mathrm{RFH}^\varphi(\Sigma, M),$$

where the last equation is the observation that twisted Rabinowitz–Floer homology in the autonomous case extends to the weakly time-dependent case without any issues. Crucial is, that the period–action equality (see Remark 3.25) is still valid. Indeed, we compute

$$\mathcal{A}_\varphi^{(\chi H, 0)}(\gamma, \tau) = \int_0^1 \gamma^* \lambda = \int_0^1 \lambda(\dot{\gamma}) = \tau \int_0^1 \chi \lambda(R(\gamma)) = \tau \int_0^1 \chi = \tau$$

for all $(\gamma, \tau) \in \mathrm{Crit}\, \mathcal{A}_\varphi^{(\chi H, 0)}$. □

Chapter 4
Applications

In this chapter we give two applications of the abstract machinery developed in the previous chapter and prove Theorem 1.1 as well as Theorem 1.6 (see Theorem 4.11).

4.1 Existence of Noncontractible Periodic Reeb Orbits

We define an equivariant version of twisted Rabinowitz–Floer homology for the discrete group $\mathbb{Z}_m$ following [7, p. 487]. In general, suppose that a topological manifold M admits a group action by a topological group G. Then there exists a principal G-bundle $EG \to BG$, where $BG = EG/G$ denotes the classifying space of G and EG is weakly contractible. Then G acts freely on $EG \times M$ via the diagonal action. Thus we can define the ***G*-equivariant homology of *M*** by

$$\mathrm{H}_*^G(M; R) := \mathrm{H}_*(EG \times M/G; R),$$

for any coefficient ring R and where $EG \times M/G$ is the homotopy quotient of M by G. See [52, p. 30–31]. For example, if $G = \mathbb{Z}_m$, then $E\mathbb{Z}_m = \mathbb{S}^\infty$ and $B\mathbb{Z}_m = \mathbb{S}^\infty/\mathbb{Z}_m$ is a lens space. Since G acts freely on M, there is a fibre bundle

$$EG \to EG \times M/G \to M/G,$$

inducing an isomorphism

$$\mathrm{H}_*^G(M) \cong \mathrm{H}_*(M/G)$$

by [52, Corollary 9.6] and [52, Theorem 3.3]. This observation will be crucial in what follows. Explicitly, let $\Sigma \subseteq \mathbb{C}^n$, $n \geq 2$, be a star-shaped hypersurface invariant under the rotation φ from Example 2.42. As Σ is star-shaped with respect to the origin, there exists a φ-invariant function $f \in C^\infty(\mathbb{S}^{2n-1})$ such that

$$\Sigma = \left\{ e^{f(z)} z : z \in \mathbb{S}^{2n-1} \right\}.$$

Define a twisted defining Hamiltonian function $H \in \mathscr{F}_\varphi(\Sigma)$ by

$$H(z) := \begin{cases} \beta\left(\log(\|z\|) - f(z/\|z\|)\right) & z \neq 0, \\ -\frac{1}{2} & z = 0. \end{cases}$$

for some sufficiently small mollification of the piecewise linear function

$$\beta(r) := \begin{cases} -\frac{1}{2} & r \leq -\frac{1}{2}, \\ r & -\frac{1}{2} \leq r \leq \frac{1}{2}, \\ \frac{1}{2} & \frac{1}{2} \leq r. \end{cases}$$

Fix a φ-invariant ω-compatible almost complex structure on $(\mathbb{C}^n, \lambda)$, where λ is given by (2.9). Then φ induces a free $\mathbb{Z}_m$-action on $\operatorname{Crit} \mathscr{A}_\varphi^H$ and on the moduli space of twisted negative gradient flow lines with cascades of $\mathscr{A}_\varphi^H$. Therefore, we can define the ***$\mathbb{Z}_m$-equivariant twisted Rabinowitz-Floer homology***

$$\overline{\mathrm{RFH}}_k^\varphi(\Sigma/\mathbb{Z}_m) := \frac{\ker \overline{\partial}_k}{\operatorname{im} \overline{\partial}_{k+1}} \qquad \forall k \in \mathbb{Z},$$

as the homology of the $\mathbb{Z}$-graded chain complex (see Remark 3.38)

$$\overline{\partial}_k \colon \mathrm{RFC}_k^\varphi(\Sigma, \mathbb{C}^n)/\mathbb{Z}_m \to \mathrm{RFC}_{k-1}^\varphi(\Sigma, \mathbb{C}^n)/\mathbb{Z}_m$$

given by

$$\overline{\partial}_k[(\gamma, \tau)] := [\partial_k(\gamma, \tau)] \qquad (\gamma, \tau) \in \operatorname{Crit} h,$$

for some φ-invariant Morse function h on $\operatorname{Crit} \mathscr{A}_\varphi^H$. More generally, if G is a finite symmetry of a Hamiltonian system which acts freely on the displaceable regular energy hypersurface, one can define the G-equivariant twisted Rabinowitz–Floer homology as above if the twisted Rabinowitz–Floer homology is defined. Under some mild index assumption on the Conley–Zehnder index, the resulting G-equivariant twisted Rabinowitz–Floer homology is isomorphic to the Tate homology of G with coefficients in $\mathbb{Z}_2$. See [47, Theorem 5.6] for a proof and [55, Definition 6.2.4] as well as [13, p. 135] for a definition of Tate homology.

Theorem 4.1. *Let $n \geq 2$. For $m \geq 1$ consider the rotation*

$$\varphi \colon \mathbb{C}^n \to \mathbb{C}^n, \quad \varphi(z_1, \ldots, z_n) := \left(e^{2\pi i k_1/m} z_1, \ldots, e^{2\pi i k_n/m} z_n\right)$$

for $k_1, \ldots, k_n \in \mathbb{Z}$ coprime to m. Then

$$\overline{\mathrm{RFH}}_k^\varphi(\mathbb{S}^{2n-1}/\mathbb{Z}_m) \cong \begin{cases} \mathbb{Z}_2 & m \text{ even}, \\ 0 & m \text{ odd}, \end{cases} \qquad \forall k \in \mathbb{Z},$$

If m is even, then $\overline{\mathrm{RFH}}_k^\varphi(\mathbb{S}^{2n-1}/\mathbb{Z}_m)$ is generated by a noncontractible periodic Reeb orbit in the lens space $\mathbb{S}^{2n-1}/\mathbb{Z}_m$ for all $k \in \mathbb{Z}$.

Proof. First we consider the special case

$$\varphi\colon \mathbb{C}^n \to \mathbb{C}^n, \qquad \varphi(z) = e^{2\pi i/m} z.$$

The hypersurface $\mathbb{S}^{2n-1} \subseteq (\mathbb{C}^n, \lambda)$ is of restricted contact type with contact form $\lambda|_{\mathbb{S}^{2n-1}}$ and associated Reeb vector field

$$R = 2\sum_{j=1}^{n} \left(y_j \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial y_j} \right)\bigg|_{\mathbb{S}^{2n-1}} = 2i \sum_{j=1}^{n} \left(\overline{z}_j \frac{\partial}{\partial \overline{z}_j} - z_j \frac{\partial}{\partial z_j} \right)\bigg|_{\mathbb{S}^{2n-1}}.$$

Suppose $(\gamma, \tau) \in \operatorname{Crit} \mathscr{A}^H_\varphi$. If $\tau = 0$, then γ is constant. This cannot happen as $\operatorname{Fix}(\varphi|_{\mathbb{S}^{2n-1}}) = \varnothing$. So we assume $\tau \neq 0$. Define a reparametrisation

$$\gamma_\tau\colon \mathbb{R} \to \mathbb{S}^{2n-1}, \qquad \gamma_\tau(t) := \gamma(t/\tau).$$

Then γ_τ is the unique integral curve of R starting at $z := \gamma(0)$ and thus

$$\gamma_\tau(t) = e^{-2it} z \qquad \forall t \in \mathbb{R}.$$

From $\gamma(t) = \gamma_\tau(\tau t)$ and the requirement

$$e^{-2i\tau} z = \gamma(1) = \varphi(\gamma(0)) = \varphi(z) = e^{2\pi i/m} z,$$

we conclude $\tau \in \frac{\pi}{m}(m\mathbb{Z} - 1)$. Hence

$$\operatorname{Crit} \mathscr{A}^H_\varphi = \left\{ \left(\phi^{\tau_k R}(z), \tau_k\right) : k \in \mathbb{Z}, z \in \mathbb{S}^{2n-1} \right\} \cong \mathbb{S}^{2n-1} \times \mathbb{Z},$$

for any $H \in \mathscr{F}_\varphi(\mathbb{S}^{2n-1})$, where we define $\tau_k := \frac{\pi}{m}(mk - 1)$. By Proposition 3.24, $(z_0, \eta) \in T_z\mathbb{S}^{2n-1} \times \mathbb{R}$ belongs to the kernel of the Hessian at $(z, k) \in \operatorname{Crit} \mathscr{A}^H_\varphi$ if and only if $\eta = 0$ and

$$z_0 \in \ker \left(D(\phi^R_{-\tau_k} \circ \varphi)|_z - \mathrm{id}_{T_z\mathbb{S}^{2n-1}} \right).$$

A direct computation yields $D(\phi^R_{-\tau_k} \circ \varphi)|_z = \mathrm{id}_{T_z\mathbb{S}^{2n-1}}$ and thus the twisted Rabinowitz action functional $\mathscr{A}^H_\varphi$ is Morse–Bott with spheres.

The full Conley–Zehnder index [23, Definition 10.4.1] gives rise to a locally constant function

$$\widehat{\mu}_{\mathrm{CZ}}\colon \operatorname{Crit} \mathscr{A}^H_\varphi \to \mathbb{Z}, \qquad \widehat{\mu}_{\mathrm{CZ}}(z, k) = (2k - 1)n.$$

Indeed, this follows from the product property and the formula

$$\widehat{\mu}_{\mathrm{CZ}} \left((e^{it})_{t \in [0,T]} \right) = \left\lfloor \frac{T}{2\pi} \right\rfloor + \left\lceil \frac{T}{2\pi} \right\rceil.$$

Note that the definition of the Conley–Zehnder index also applies in this degenerate case, compare [23, Remark 10.4.2]. By the adapted proof of the Hofer–Wysocki–

Zehnder Theorem [23, Theorem 12.2.1] to the n-dimensional setting, the full Conley–Zehnder index coincides with the transverse Conley–Zehnder index μ_{CZ}. Indeed, for a critical point $(\gamma, \tau) \in \operatorname{Crit} \mathcal{A}_\varphi^H$ define a smooth path

$$\Psi \colon I \to \mathrm{Sp}(n), \qquad \Psi_t := D\phi_{\tau t}^H|_{\gamma(0)} \colon \mathbb{C}^n \to \mathbb{C}^n.$$

Adapting the proof of [23, Lemma 12.2.3 (iii)], we get that

$$\Psi_1(R(\gamma(0))) = R(\gamma(1)) \qquad \text{and} \qquad \Psi_1(\gamma(0)) = \gamma(1).$$

Arguing as in [23, p. 235–236] we conclude

$$\mu_{\mathrm{CZ}}(\gamma, \tau) = \widehat{\mu}_{\mathrm{CZ}}(\gamma, \tau).$$

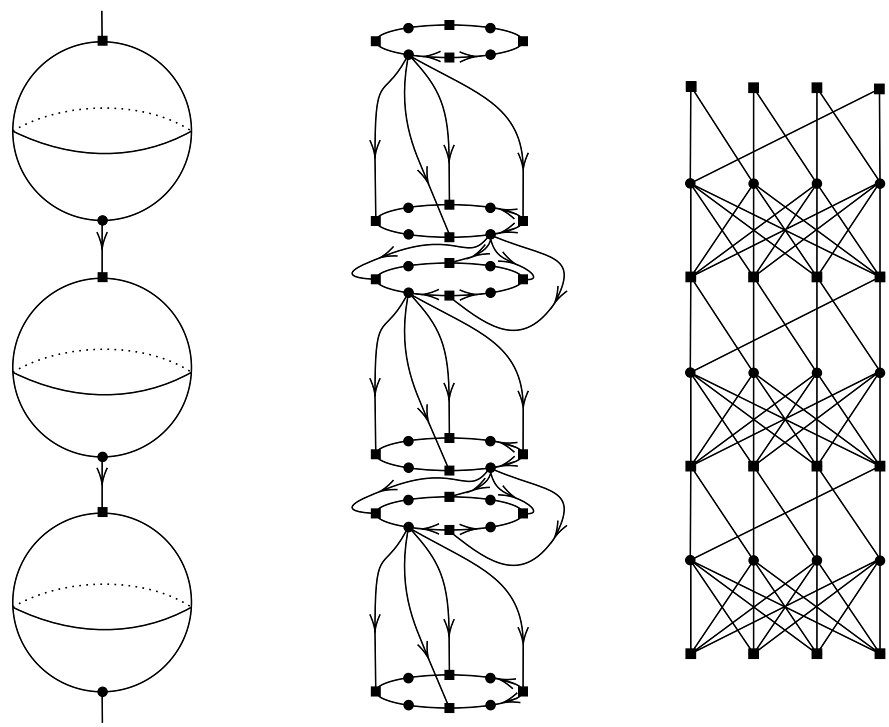

Fig. 4.1: The critical manifold $\mathbb{S}^{2n-1} \times \mathbb{Z}$ with the standard height function, the Morse–Bott function f and the resulting chain complex.

Fix $z_0 \in \mathbb{S}^{2n-1}$ and define $\eta := \phi^{\tau_0 R}(z_0)$. Note that $\phi^{\tau_k R}(z)$ belongs to the same equivalence class in $\pi_0 \mathscr{L}_\varphi \mathbb{S}^{2n-1}$ as η for all $z \in \mathbb{S}^{2n-1}$ and $k \in \mathbb{Z}$ because $\mathbb{S}^{2n-1}$

is simply connected for $n \geq 2$. Let $h \in C^\infty(\mathbb{S}^{2n-1})$ be the standard height function. By Remark 3.38, $\mathrm{RFH}^\varphi(\mathbb{S}^{2n-1}, \mathbb{C}^n)$ carries the $\mathbb{Z}$-grading

$$\mu((z,k),(z_0,0)) + \mathrm{ind}_h(z) = 2kn + \mathrm{ind}_h(z) \qquad \forall (z,k) \in \mathbb{S}^{2n-1} \times \mathbb{Z}.$$

We claim that the number of twisted negative gradient flow lines between the minimum of $\mathbb{S}^{2n-1} \times \{k+1\}$ and the maximum of $\mathbb{S}^{2n-1} \times \{k\}$ must be odd, so that the critical manifold $\mathrm{Crit}\,\mathcal{A}_\varphi^H$ looks like a string of pearls, see Figure 4.1. Indeed, if there is an even number of such negative gradient flow lines, then $\mathrm{RFH}_*^\varphi(\mathbb{S}^{2n-1}, \mathbb{C}^n) \neq 0$, contradicting Theorem 3.48 as $\mathbb{S}^{2n-1}$ is displaceable in the completion $\mathbb{C}^n$. To compute the $\mathbb{Z}_m$-equivariant twisted Rabinowitz–Floer homology, choose the additional $\mathbb{Z}_m$-invariant Morse–Bott function f from Example 2.3. Additionaly, choose a $\mathbb{Z}_m$-invariant Morse function on $\mathrm{Crit}\, f$. For example, one can take

$$h\colon \mathbb{T} \to \mathbb{R}, \qquad h(t) := \cos(2\pi m t).$$

The resulting chain complex is given by

$$\dots \longrightarrow \mathbb{Z}_2^m \xrightarrow{\mathbb{1}} \mathbb{Z}_2^m \xrightarrow{A} \mathbb{Z}_2^m \xrightarrow{\mathbb{1}} \mathbb{Z}_2^m \xrightarrow{A} \mathbb{Z}_2^m \xrightarrow{\mathbb{1}} \mathbb{Z}_2^m \longrightarrow \dots$$

where $\mathbb{1} \in M_{m\times m}(\mathbb{Z}_2)$ has every entry equal to 1 and $A \in M_{m\times m}(\mathbb{Z}_2)$ is defined by

$$A := I_{m\times m} + \sum_{j=1}^{m-1} e_{(j+1)j} + e_{1m},$$

where $e_{ij} \in M_{m\times m}(\mathbb{Z}_2)$ satisfies $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$. Thus the resulting chain complex looks like a rope ladder. Compare Figure 4.1. Passing to the quotient via the free $\mathbb{Z}_m$-action, we get the acyclic chain complex

$$\dots \longrightarrow \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \longrightarrow \dots$$

if m is even and the alternating chain complex

$$\dots \longrightarrow \mathbb{Z}_2 \xrightarrow{1} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{1} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{1} \mathbb{Z}_2 \longrightarrow \dots$$

if m is odd.

For the general case, we note that

$$\mathbb{C}^n \times [0,1] \to \mathbb{C}^n, \qquad \varphi_s(z^1, \dots, z^n) := \left(e^{2\pi i s k_1/m} z_1, \dots, e^{2\pi i s k_n/m} z_n\right)$$

is a smooth path from $\varphi_0 = \mathrm{id}_{\mathbb{C}^n}$ to $\varphi_1 = \varphi$. By adapting the proof of [54, Lemma 2.27], we get an isomorphism of chain complexes

$$\mathrm{RFC}(\mathbb{S}^{2n-1}, \mathbb{C}^n) \cong \mathrm{RFC}^\varphi(\mathbb{S}^{2n-1}, \mathbb{C}^n).$$

This isomorphism does not necessarily preserve the grading, but the relative Conley–Zehnder index is preserved. Note that also f is invariant under φ_s for all $s \in [0, 1]$. It is no problem to allow twists φ_s of infinite order as the standard Reeb flow on $\mathbb{S}^{2n-1}$ is periodic. Consider the torus action

$$\mathbb{T}^n \times \mathbb{C}^n \to \mathbb{C}^n, \quad (\theta_1, \dots, \theta_n) \cdot (z_1, \dots, z_n) := \left(e^{2\pi i\theta_1} z_1, \dots, e^{2\pi i\theta_n} z_n\right).$$

Since the torus $\mathbb{T}^n$ is abelian, we have that the $\mathbb{Z}_m$-action induced by φ and the different twists along the path $(\varphi_s)_{s\in[0,1]}$ commute. Thus we get an isomorphism of the $\mathbb{Z}_m$-equivariant chain complexes and consequently

$$\overline{\mathrm{RFH}}^{\varphi}_*(\mathbb{S}^{2n-1}/\mathbb{Z}_m) \cong \mathrm{RFH}^{\mathbb{Z}_m}_*(\mathbb{S}^{2n-1}, \mathbb{C}^n),$$

where $\mathrm{RFH}^{\mathbb{Z}_m}_*$ denotes the $\mathbb{Z}_m$-equivariant Rabinowitz–Floer homology constructed in [7, p. 487]. Performing the same computation of the latter homology as before in the special case yields

$$\mathrm{RFH}^{\mathbb{Z}_m}_k(\mathbb{S}^{2n-1}, \mathbb{C}^n) \cong \begin{cases} \mathbb{Z}_2 & m \text{ even}, \\ 0 & m \text{ odd}, \end{cases} \qquad \forall k \in \mathbb{Z}.$$

Lastly, $\overline{\mathrm{RFH}}^{\varphi}_k(\mathbb{S}^{2n-1}/\mathbb{Z}_m)$ is generated by a noncontractible periodic Reeb orbit in $\mathbb{S}^{2n-1}/\mathbb{Z}_m$ for all $k \in \mathbb{Z}$ by Lemma 1.5. □

Remark 4.2 (Coefficients). As $\overline{\mathrm{RFH}}^{\varphi}_*(\mathbb{S}^{2n-1}/\mathbb{Z}_m)$ vanishes for odd m, one should rather consider twisted Rabinowitz–Floer homology with coefficients in $\mathbb{Z}$ in this case. Using the polyfold approach, it might be possible to invoke [33, Chapter 6] to define coherent orientations on the moduli spaces. However, Lagrangian Floer homology admits an abstract polyfold description, but it is not always possible to define coherent orientations. But heuristically, the very same $\mathbb{Z}_m$-invariant chain complex from 4.1 modulo the $\mathbb{Z}_m$-action should be given by

$$\dots \longrightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{m} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{m} \mathbb{Z} \longrightarrow \dots$$

in the oriented case. Thus the resulting homology is

$$\overline{\mathrm{RFH}}^{\varphi}_k(\mathbb{S}^{2n-1}/\mathbb{Z}_m) \cong \begin{cases} \mathbb{Z}_m & k \text{ even}, \\ 0 & k \text{ odd}, \end{cases}$$

coinciding with the Tate homology $\widehat{\mathrm{H}}_*(C_m; \mathbb{Z})$.

Using Theorem 4.1 we can finally prove Theorem 1.1.

Proof (of Theorem 1.1). By assumption, Σ bounds a star-shaped domain D with respect to the origin. Thus $(D \cup \Sigma, \lambda)$ is a Liouville domain with λ given by (2.9). By rescaling we may assume that $\mathbb{S}^{2n-1} \subseteq D$. Define a smooth function

$$\delta\colon \Sigma \to (-\infty, 0)$$

by requiring $\delta(x)$ to be the unique number such that $\phi^X_{\delta(x)}(x) \in \mathbb{S}^{2n-1}$, $x \in \Sigma$, where $X \in \mathfrak{X}(\mathbb{C}^n)$ denotes the Liouville vector field (2.10). We claim that $\delta \circ \varphi = \delta$. Indeed, $\delta(\varphi(x))$ is the unique number such that $\phi^X_{\delta(\varphi(x))}(\varphi(x)) \in \mathbb{S}^{2n-1}$. As the flow of X and φ commute by the proof of Lemma 3.30, we conclude that $\phi^X_{\delta(\varphi(x))}(x) \in \mathbb{S}^{2n-1}$. Define a smooth family of star-shaped hypersurfaces $(\Sigma_\sigma)_{\sigma \in I}$

$$\Sigma_\sigma := \left\{\phi^X_{\sigma\delta(x)}(x) : x \in \Sigma\right\} \subseteq \mathbb{C}^n.$$

Then we compute

$$\begin{aligned}
\varphi(\Sigma_\sigma) &= \left\{\varphi\left(\phi^X_{\sigma\delta(x)}(x)\right) : x \in \Sigma\right\} \\
&= \left\{\phi^X_{\sigma\delta(x)}(\varphi(x)) : x \in \Sigma\right\} \\
&= \left\{\phi^X_{\sigma\delta(\varphi(x))}(\varphi(x)) : x \in \Sigma\right\} \\
&= \left\{\phi^X_{\sigma\delta(y)}(y) : y \in \varphi(\Sigma)\right\} \\
&= \left\{\phi^X_{\sigma\delta(y)}(y) : y \in \Sigma\right\} \\
&= \Sigma_\sigma
\end{aligned}$$

for all $\sigma \in I$ and therefore we can find a twisted homotopy $(H_\sigma)_{\sigma \in I}$ of Liouville domains in $\mathbb{C}^n$. By Theorem 3.41 we have that

$$\mathrm{RFH}^\varphi_*(\Sigma, \mathbb{C}^n) \cong \mathrm{RFH}^\varphi_*(\mathbb{S}^{2n-1}, \mathbb{C}^n),$$

giving rise to a canonical isomorphism of the associated $\mathbb{Z}_m$-equivariant twisted Rabinowitz–Floer homology

$$\overline{\mathrm{RFH}}^\varphi_*(\Sigma/\mathbb{Z}_m) \cong \overline{\mathrm{RFH}}^\varphi_*(\mathbb{S}^{2n-1}/\mathbb{Z}_m).$$

Indeed, this follows from observing that the continuation homomorphism [17, p. 276] is $\mathbb{Z}_m$-invariant and thus descends to the quotient. However, by Theorem 4.1 the latter does not vanish as $m \geq 2$ is even. Thus there exists a noncontractible periodic Reeb orbit on $\Sigma/\mathbb{Z}_m$ if the twisted Rabinowitz action functional is Morse–Bott. Otherwise, consider arbitrarily small symmetric perturbations of Σ such that the twisted Rabinowitz action functional is Morse–Bott. See Remark 3.42. The quotient of every such perturbation then admits a noncontractible periodic Reeb orbit and we conclude by Arzelà–Ascoli. ☐

Using Theorem 4.1 it is also possible to generalise [7, Theorem 1.2]. Define the set of ***φ-invariant Hamiltonian symplectomorphisms*** by

$$\mathrm{Ham}^\varphi(\mathbb{C}^n, d\lambda) := \{\varphi_F \in \mathrm{Ham}(\mathbb{C}^n, d\lambda) : F_t(x) = F_t(\varphi(x))\ \forall (x,t) \in \mathbb{C}^n \times I\}.$$

If $\varphi_F \in \mathrm{Ham}^{\varphi}(\mathbb{C}^n, d\lambda)$, then $\varphi \circ \varphi_F = \varphi_F \circ \varphi$. In particular $0 \in \mathrm{Fix}(\varphi_F)$, and thus no element in $\mathrm{Ham}^{\varphi}(\mathbb{C}^n, d\lambda)$ can displace a star-shaped hypersurface with respect to the origin in $\mathbb{C}^n$. We have the following result.

Theorem 4.3. *Let $\Sigma \subseteq \mathbb{C}^n$ be a compact connected star-shaped hypersurface invariant under the rotation φ. Every element in $\mathrm{Ham}^{\varphi}(\mathbb{C}^n, d\lambda)$ admits infinitely many leaf-wise intersection points on Σ or there does exist a leaf-wise intersection point on a closed leaf.*

Proof. We reproduce the proof in [7] for completeness with minor modifications. Let $\varphi_F \in \mathrm{Ham}_c^{\varphi}(\mathbb{C}^n, d\lambda)$ and for $r \in [0, 1]$ consider the smooth family of perturbed Rabinowitz action functionals

$$\mathcal{A}_r : \mathcal{L}\mathbb{C}^n \times \mathbb{R} \to \mathbb{R}$$

defined by

$$\mathcal{A}_r(\gamma, \tau) := \int_0^1 \gamma^*\lambda - \tau \int_0^1 H_r(\gamma(t))dt - r \int_0^1 F_t(\gamma(t))dt,$$

where $(H_r)_{r \in I}$ is a twisted homotopy of Liouville domains from $\mathbb{S}^{2n-1}$ to Σ. Clearly, every $\mathcal{A}_r$ is φ-invariant. As in [6, Definition 5.1], we define the spectral value $\sigma([\xi])$ of a homology class $[\xi] \in \mathrm{RFH}_*^{\mathbb{Z}_m}(\mathbb{S}^{2n-1}, \mathbb{C}^n)$ by

$$\sigma([\xi]) := \inf_{\eta \in [\xi]} \max_{\eta(\gamma,\tau) \neq 0} \mathcal{A}_0(\gamma, \tau) \in \mathbb{R} \cup \{-\infty\}.$$

Moreover, we define the set

$$\mathfrak{S} := \{\sigma([\xi]) : [\xi] \in \mathrm{RFH}_*^{\mathbb{Z}_m}(\mathbb{S}^{2n-1}, \mathbb{C}^n)\}.$$

By Theorem 4.1, we conclude that $\mathfrak{S} = 2\pi\mathbb{Z}$. Hence $\mathcal{A}_0$ has critical values of arbitrarily large critical value and so does $\mathcal{A}_1$ by [6, Corollary 5.14]. Thus $\mathcal{A}_1$ has infinitely many critical points which give rise to leaf-wise intersection points by Lemma 3.47. The map

$$\mathrm{Crit}\, \mathcal{A}_1 \to \{\text{leaf-wise intersection points}\}$$

is injective unless there exists a leaf-wise intersection point on a closed leaf. For the general case, use cut-off functions. □

4.2 A Forcing Theorem for Twisted Periodic Reeb Orbits

Definition 4.4 (Twisted Stable Hypersurface). Let $(\Sigma, \omega|_{\Sigma}, \lambda)$ be a stable hypersurface in a connected symplectic manifold (M, ω) and $\varphi \in \mathrm{Symp}(M, \omega)$. We say that Σ is ***twisted by φ***, if $\varphi(\Sigma) = \Sigma$, φ is of finite order and $\varphi^*\lambda = \lambda$.

Example 4.5. Consider the stable hypersurface $\Sigma_c \subseteq (T^*\mathbb{T}^n, \omega_\sigma, H)$ for $c > 0$ as in Example 2.26. Let $\varphi \in \mathrm{Diff}(\mathbb{T}^n)$ be an isometry of finite order such that

$$D\varphi \circ J = J \circ D\varphi \tag{4.1}$$

holds and consider the cotangent lift (2.15)

$$D\varphi^\dagger \colon \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{T}^n \times \mathbb{R}^n, \qquad D\varphi^\dagger(q, p) = \left(\varphi(q), \left(D\varphi^{-1}(q)\right)^t p\right).$$

Then clearly $\varphi(\Sigma_c) = \Sigma_c$ as φ is an isometry and $D\varphi^\dagger$ is of finite order as φ is. Moreover, $D\varphi^\dagger \in \mathrm{Symp}(T^*\mathbb{T}^n, \omega_\sigma)$, as $D\varphi^\dagger \in \mathrm{Symp}(T^*\mathbb{T}^n, \omega_0)$ by Proposition 2.16 and $D\varphi^\dagger$ preserves σ by assumption (4.1). Lastly, we have that $\varphi^*\lambda = \lambda$ as one sees by considering the formula (2.7) together with assumption (4.1).

Let $(\Sigma, \omega|_\Sigma, \lambda)$ be a twisted stable hypersurface for $\varphi \in \mathrm{Symp}(M, \omega)$ in a connected symplectically aspherical symplectic manifold (M, ω), that is, $[\omega]|_{\pi_2(M)} = 0$. As φ is of finite order by assumption, we can define the ***set of twisted contractible loops***, written $\Lambda_\varphi M \subseteq \Lambda M$, as follows. We say that a contractible free loop $v \in \Lambda M$ is in $\Lambda_\varphi M$, if there exists $\gamma \in \mathscr{L}_\varphi M$ such that

$$v(t) = \gamma(mt) \qquad \forall t \in \mathbb{T},$$

where $m := \mathrm{ord}\,\varphi$. Then we can define a generalisation of the twisted Rabinowitz action functional

$$\mathcal{A}_\varphi^H \colon \Lambda_\varphi M \times \mathbb{R} \to \mathbb{R}, \quad \mathcal{A}_\varphi^H(v, \tau) := \frac{1}{m}\int_{\mathbb{D}} \bar{v}^*\omega - \tau \int_0^1 H(v(t))dt, \tag{4.2}$$

where $\bar{v} \in C^\infty(\mathbb{D}, M)$ is a filling of v and H is any twisted defining Hamiltonian function for Σ. Then $(v, \tau) \in \mathrm{Crit}\,\mathcal{A}_\varphi^H$ if and only if $(\gamma, \tau) \in \mathscr{L}_\varphi \Sigma$ solves

$$\dot{\gamma}(t) = \tau R(\gamma(t)) \qquad \forall t \in \mathbb{R},$$

where $R \in \mathfrak{X}(\Sigma)$ denotes the stable Reeb vector field 2.27. We call the projection of the set of critical points of $\mathcal{A}_\varphi^H$ to $\Lambda_\varphi M$ ***contractible twisted closed characteristics*** and denote it by $\mathscr{C}_\varphi(\Sigma)$. Define a function, called the ***ω-energy***, by

$$\Omega \colon \mathscr{C}_\varphi(\Sigma) \to \mathbb{R}, \qquad \Omega(v) := \frac{1}{m}\int_{\mathbb{D}} \bar{v}^*\omega.$$

It follows that $\Omega(v) = \mathcal{A}_\varphi^H(v, \tau)$.

Example 4.6. Consider the twisted stable hypersurface $\Sigma_c \subseteq (T^*\mathbb{T}^n, \omega_\sigma, H)$ as in Example 4.5. By adapting Example 2.28, we have that $(q, p) \in \Sigma_k$ gives rise to a contractible twisted closed characteristic if and only if

$$\int_0^\tau e^{sJ} p\,ds + q = \varphi(q), \quad e^{\tau J} p = \left(D\varphi^{-1}(q)\right)^t p, \quad \text{and} \quad \|p\|^2 = 2c.$$

A computation similar to [19, p. 1843] shows

$$\Omega : \mathcal{C}_\varphi(\Sigma_c) \to \mathbb{R}, \qquad \Omega(v) = c\tau.$$

In order to state the main result of this section, we need two additional preliminary definitions.

Definition 4.7 (Morse–Bott Component, [5, p. 86]). Let M be a smooth manifold and $f \in C^\infty(M)$. A subset $C \subseteq \operatorname{Crit} f$ is called a ***Morse–Bott component***, if

(i) C is a connected embedded submanifold of M.
(ii) $T_x C = \ker \operatorname{Hess} f(x)$ for all $x \in C$.

Example 4.8 ([5, Lemma 2.12]). In the setting of Proposition 3.24, any connected component of $\operatorname{Fix}(\varphi|_\Sigma) \subseteq \operatorname{Crit} \mathcal{A}_\varphi^H$ is a Morse–Bott component. Indeed, we have that

$$\ker \operatorname{Hess} \mathcal{A}_\varphi^H|_{(x,0)} \cong \ker(D\varphi_x - \operatorname{id}_{T_x\Sigma}) = T_x \operatorname{Fix}(\varphi|_\Sigma)$$

for all $x \in \operatorname{Fix}(\varphi|_\Sigma)$.

Definition 4.9 ([19, p. 1768]). A symplectic manifold (M, ω) is called ***geometrically bounded***, if there exists an ω-compatible almost complex structure J and a complete Riemannian metric such that the following conditions hold.

(i) There are constants $C_0, C_1 > 0$ with

$$\omega(Jv, v) \geq C_0 \|v\|^2 \qquad \text{and} \qquad |\omega(u, v)| \leq C_1 \|u\| \|v\|$$

for all $u, v \in T_x M$ and $x \in M$.
(ii) The sectional curvature of the metric is bounded above, and its injectivity radius is bounded away from zero.

Example 4.10 ([19, p. 1768]). Twisted cotangent bundles are geometrically bounded.

Theorem 4.11 (Forcing). *Let Σ be a twisted stable displaceable hypersurface in a symplectically aspherical, geometrically bounded, symplectic manifold (M, ω) for some $\varphi \in \operatorname{Symp}(M, \omega)$ and suppose that $v^- \in \mathcal{C}_\varphi(\Sigma)$ belongs to a Morse–Bott component C of the twisted Rabinowitz action functional (4.2). Then there exists a contractible twisted closed characteristic $v \in \mathcal{C}_\varphi(\Sigma) \setminus C$ such that*

$$\Omega(v) - \Omega(v^-) \leq e(\Sigma).$$

Corollary 4.12. *Let Σ be a twisted stable displaceable hypersurface in a symplectically aspherical, geometrically bounded, symplectic manifold (M, ω) for some symplectomorphism $\varphi \in \operatorname{Symp}(M, \omega)$. If $\operatorname{Fix}(\varphi|_\Sigma) \neq \emptyset$, then there exists a contractible twisted closed characteristic $v \in \mathcal{C}_\varphi(\Sigma) \setminus \operatorname{Fix}(\varphi|_\Sigma)$ such that*

$$\Omega(v) \leq e(\Sigma).$$

Corollary 4.13 ([43, Theorem 12.3.4], [34, p. 171]). *We have that*

$$e(\overline{B}_r^{2n}(0)) = \pi r^2 \qquad \forall r > 0,$$

where $\overline{B}_r^{2n}(0) \subseteq \mathbb{R}^{2n}$ *denotes the closed ball around the origin of radius* r.

Proof. By monotonicity and [43, Exercise 12.3.7] we have that

$$e(\partial \overline{B}_r^{2n}(0)) \leq e(\overline{B}_r^{2n}(0)) \leq \pi r^2 \qquad \forall r > 0.$$

The Reeb flow on $\partial \overline{B}_r^{2n}(0)$ is given by

$$\phi_t^{R_r}(z) = e^{-2it/r^2} z \qquad \forall z \in \partial \overline{B}_r^{2n}(0).$$

Hence the parametrised periodic Reeb orbits are $(\phi^{R_r}(z), \tau)$ with $\tau \in \pi r^2 \mathbb{Z}$. But Corollary 4.12 implies the existence of a nonconstant closed characteristic v on the hypersurface $\partial \overline{B}_r^{2n}(0)$ such that

$$0 < \tau = \Omega(v) \leq e(\partial \overline{B}_r^{2n}(0)) \leq \pi r^2.$$

This is only possible for $\tau = \pi r^2$ and the statement follows. □

Proof (of Theorem 4.11). This proof uses a method called a "homotopy of homotopies argument". Fix $\varepsilon > 0$ and choose a Hamiltonian function $F \in C_c^\infty(M \times I)$ satisfying

$$\|F\| < e(\Sigma) + \varepsilon \qquad \text{and} \qquad \varphi_F(\Sigma) \cap \Sigma = \varnothing.$$

For an appropriate twisted defining Hamiltonian function H for Σ we denote by $\mathfrak{M}$ the associated twisted Moser pair. The actual construction of H is very cumbersome and is carried out in [19]. The crucial observation here is that [19, Proposition 2.6] gives a φ-invariant stable tubular neighbourhood of Σ as $\varphi^*\lambda = \lambda$ by invoking the equivariant Darboux–Weinstein Theorem [30, Theorem 22.1]. Moreover, we choose a smooth family $(\beta_r)_{r \in [0,+\infty)}$ of cutoff functions $\beta_r \in C^\infty(\mathbb{R}, I)$ such that

$$\begin{cases} \beta_r(s) = 0 & |s| \geq r, \\ \beta_r(s) = 1 & |s| \leq r - 1, \\ s\beta_r'(s) \leq 0 & \forall s \in \mathbb{R}, \end{cases}$$

for all $r \in [0, +\infty)$. Define a family of twisted Rabinowitz action functionals

$$\mathscr{A}_r \colon \Lambda_\varphi M \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

by

$$\mathscr{A}_r(v, \tau, s) := \mathscr{A}_\varphi^H(v, \tau) - \beta_r(s) \int_0^1 F_t(v(t)) dt$$

for all $r \in [0, +\infty)$. Note that $\mathscr{A}_0 = \mathscr{A}_\varphi^H$. For a suitable φ-invariant ω-compatible almost complex structure we consider the moduli space

$$\mathcal{M} := \{(u, \tau, r) \in C^{\infty}(\mathbb{R}, \mathcal{L}_{\varphi} M \times \mathbb{R}) \times [0, +\infty) : (u, \tau, r) \text{ solution of (4.3)}\},$$

where

$$\begin{cases} \partial_s(u, \tau) = \operatorname{grad} \mathcal{A}_r|_{(u(s),\tau(s),s)} & \forall s \in \mathbb{R}, \\ \lim_{s\to-\infty}(u(s), \tau(s)) = (v^-, \tau^-), \\ \lim_{s\to+\infty}(u(s), \tau(s)) \in C. \end{cases} \tag{4.3}$$

Note that always $(v^-, \tau^-, 0) \in \mathcal{M}$. The proof is now based on the following observation. If

$$\Omega(v) > \|F\| + \Omega(v^-) \qquad \forall v \in \mathcal{C}_{\varphi}(\Sigma) \setminus C \tag{4.4}$$

holds, then $\mathcal{M}$ is compact. This is absurd. Indeed, the moduli space $\mathcal{M}$ is the zero level set of a Fredholm section of a bundle over a Banach manifold. As v^- belongs to a Morse–Bott component, the Fredholm section is regular at the point v^-, that is, the linearisation of the gradient flow equation is surjective there. By compactness, we can therefore perturb the Fredholm section to make it transverse. Hence $\mathcal{M}$ is a compact smooth manifold with boundary consisting precisely of the point v^-. There do not exist such manifolds. Thus we conclude that there exists $v \in \mathcal{C}_{\varphi}(\Sigma) \setminus C$ such that

$$\Omega(v) - \Omega(v^-) \le \|F\| < e(\Sigma) + \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, the statement follows. We prove the compactness of $\mathcal{M}$ under assumption (4.4) in four steps.

Step 1: If $(u, \tau, r) \in \mathcal{M}$, then $E(u, \tau) \le \|F\|$. We estimate

$$\begin{aligned} E(u, \tau) &= \int_{-\infty}^{+\infty} \|\partial_s(u, \tau)\|^2 \, ds \\ &= \int_{-\infty}^{+\infty} d\mathcal{A}_r(\partial_s(u, \tau), s) ds \\ &= \int_{-\infty}^{+\infty} \frac{d}{ds} \mathcal{A}_r(u, \tau, s) ds - \int_{-\infty}^{+\infty} (\partial_s \mathcal{A}_r)(u, \tau, s) ds \\ &= \lim_{s\to+\infty} \mathcal{A}_r(u, \tau, s) - \lim_{s\to-\infty} \mathcal{A}_r(u, \tau, s) - \int_{-\infty}^{+\infty} (\partial_s \mathcal{A}_r)(u, \tau, s) ds \\ &= \mathcal{A}_0(v^+, \tau^+) - \mathcal{A}_0(v^-, \tau^-) - \int_{-\infty}^{+\infty} (\partial_s \mathcal{A}_r)(u, \tau, s) ds \\ &= -\int_{-\infty}^{+\infty} (\partial_s \mathcal{A}_r)(u, \tau, s) ds \\ &= \int_{-\infty}^{+\infty} \dot{\beta}_r(s) \int_0^1 F_t(u(s, t)) dt ds \\ &\le \|F\|_+ \int_{-\infty}^{0} \dot{\beta}_r(s) ds - \|F\|_- \int_0^{+\infty} \dot{\beta}_r(s) ds \\ &= \beta_r(0)(\|F\|_- + \|F\|_+) \end{aligned}$$

$$= \beta_r(0)\,\|F\|$$
$$\leq \|F\|\,,$$

as $\mathcal{A}_0(v^+, \tau^+) = \mathcal{A}_0(v^-, \tau^-)$ since C is connected.

Step 2: There exists $r_0 \in \mathbb{R}$ such that $r \leq r_0$ for all $(u, \tau, r) \in \mathcal{M}$. Crucial is the existence of a constant $\delta > 0$ such that

$$\|\mathrm{grad}\,\mathcal{A}_r|_{(v,\tau,s)}\| \geq \delta \qquad \forall (v, \tau, s) \in \Lambda_\varphi M \times \mathbb{R} \times \mathbb{R}.$$

This is proven along the lines of [17, Lemma 3.9]. With the above inequality and Step 1 we estimate

$$\|F\| \geq E(u, \tau) \geq \int_{-r}^{r} \left\|\mathrm{grad}\,\mathcal{A}_r|_{(u(s),\tau(s),s)}\right\|^2 ds \geq 2r\delta^2,$$

and thus we can set

$$r_0 := \frac{\|F\|}{2\delta^2}.$$

Step 3: There exists $C > 0$ such that $\|\tau\|_\infty \leq C$ for all $(u, \tau, r) \in \mathcal{M}$. This is a delicate estimate based on the construction of the defining Hamiltonian H for Σ as well as an extension of the stablising form and proceeds as in [19]. Particularly crucial is [19, Proposition 4.1].

Step 4: If (4.4) holds, then $\mathcal{M}$ is compact. Let (u_k, τ_k, r_k) be a sequence in the moduli space $\mathcal{M}$. By Step 2 and Step 3, the sequences (r_k) and (τ_k) are uniformly bounded. Thus (u_k, τ_k, r_k) admits a C^∞_{loc}-convergent subsequence by standard arguments. Indeed, the uniform L^∞-bound on the sequence (u_k) follows from the assumption that (M, ω) is geometrically bounded and the uniform L^∞-bound on the derivatives (Du_k) follows from Corollary C.9 by the assumption that (M, ω) is symplectically aspherical. Denote the limit of this subsequence by (u, τ, r). This limit clearly satisfies the first equation in (4.3), thus one only needs to check the asymptotic conditions in (4.3). Again by compactness, (u, τ) converges to critical points $(w^\pm, \tau^\pm)$ of $\mathcal{A}_0$ at its asymptotic ends. We claim that

$$\mathcal{A}_r(u(s), \tau(s), s) \in [-\|F\| + \Omega(v^-), \|F\| + \Omega(v^-)] \qquad \forall s \in \mathbb{R}. \tag{4.5}$$

In particular, $\Omega(w^\pm) \in [-\|F\| + \Omega(v^-), \|F\| + \Omega(v^-)]$. So if (4.5) holds, then by assumption (4.4) we conclude $(w^\pm, \tau^\pm) \in C$ and $\mathcal{M}$ is indeed compact. It remains to prove (4.5). It is enough to prove

$$\mathcal{A}_r(u_k(s), \tau_k(s), s) \in [-\|F\| + \Omega(v^-), \|F\| + \Omega(v^-)] \qquad \forall s \in \mathbb{R}$$

for every $k \in \mathbb{N}$. As in the proof of [5, Lemma 2.8] we estimate

$$0 \leq \int_{s_0}^{+\infty} d\,\mathcal{A}_r(\partial_s(u_k, \tau_k), s)ds$$

$$
\begin{aligned}
&= \int_{s_0}^{+\infty} \frac{d}{ds}\mathcal{A}_r(u_k,\tau_k,s)ds - \int_{s_0}^{+\infty} (\partial_s \mathcal{A}_r)(u_k,\tau_k,s)ds \\
&= \lim_{s\to+\infty} \mathcal{A}_r(u_k,\tau_k,s) - \mathcal{A}_r(u_k(s_0),\tau_k(s_0),s_0) - \int_{s_0}^{+\infty} (\partial_s \mathcal{A}_r)(u_k,\tau_k,s)ds \\
&= \mathcal{A}_0(v^+,\tau^+) - \mathcal{A}_r(u_k(s_0),\tau_k(s_0),s_0) + \int_{s_0}^{+\infty} \dot{\beta}_r(s)\int_0^1 F_t(u_k(s,t))dtds \\
&\le \mathcal{A}_0(v^+,\tau^+) - \mathcal{A}_r(u_k(s_0),\tau_k(s_0),s_0) + \int_{-\infty}^{+\infty} \|\dot{\beta}_r(s)F\|_+ \, ds \\
&\le \mathcal{A}_0(v^+,\tau^+) - \mathcal{A}_r(u_k(s_0),\tau_k(s_0),s_0) + \|F\| \\
&= \Omega(v^-) - \mathcal{A}_r(u_k(s_0),\tau_k(s_0),s_0) + \|F\|
\end{aligned}
$$

for all $s_0 \in \mathbb{R}$. Similarly, we compute

$$
\begin{aligned}
0 &\le \int_{-\infty}^{s_0} d\mathcal{A}_r(\partial_s(u_k,\tau_k),s)ds \\
&= \int_{-\infty}^{s_0} \frac{d}{ds}\mathcal{A}_r(u_k,\tau_k,s)ds - \int_{-\infty}^{s_0} (\partial_s \mathcal{A}_r)(u_k,\tau_k,s)ds \\
&= \mathcal{A}_r(u_k(s_0),\tau_k(s_0),s_0) - \lim_{s\to-\infty} \mathcal{A}_r(u_k,\tau_k,s) - \int_{-\infty}^{s_0} (\partial_s \mathcal{A}_r)(u_k,\tau_k,s)ds \\
&= \mathcal{A}_r(u_k(s_0),\tau_k(s_0),s_0) - \mathcal{A}_0(v^-,\tau^-) + \int_{-\infty}^{s_0} \dot{\beta}_r(s)\int_0^1 F_t(u_k(s,t))dtds
\end{aligned}
$$

and thus we estimate

$$
\begin{aligned}
\mathcal{A}_r(u_k(s_0),\tau_k(s_0),s_0) &\ge \mathcal{A}_0(v^-,\tau^-) - \int_{-\infty}^{s_0} \dot{\beta}_r(s)\int_0^1 F_t(u_k(s,t))dtds \\
&\ge \mathcal{A}_0(v^-,\tau^-) - \int_{-\infty}^{+\infty} \|\dot{\beta}(s)F\|_+ \, ds \\
&\ge \Omega(v^-) - \|F\| \, .
\end{aligned}
$$

This completes the proof of the Forcing Theorem 4.11. □

We conlcude this section by applying the Forcing Theorem 4.11 to a displaceable twisted stable hypersurface.

Example 4.14. Consider the isometry

$$\varphi\colon \mathbb{T}^2 \times \mathbb{R}^2 \to \mathbb{T}^2 \times \mathbb{R}^2, \qquad \varphi(q_1,q_2) := (q_2,-q_1)$$

and its cotangent lift

$$D\varphi^\dagger\colon \mathbb{T}^2 \times \mathbb{R}^2 \to \mathbb{T}^2 \times \mathbb{R}^2, \quad D\varphi^\dagger(q_1,q_2,p_1,p_2) = (q_2,-q_1,p_2,-p_1).$$

Then $\Sigma_c \subseteq (T^*\mathbb{T}^2, \omega_\sigma, H)$ is a displaceable twisted stable hypersurface for the area form $\sigma = dq_1 \wedge dq_2$ by Example 4.5. By Example 4.6, we have that

$$v\colon \mathbb{R} \to \Sigma_c, \qquad v(t) := \sqrt{2c}(\sin t, \cos t, \cos t, -\sin t)$$

is a τ-periodic twisted characteristic for all periods $\tau \in 2\pi\mathbb{Z} + \frac{\pi}{2}$ and $c > 0$. Thus if we choose $v^- \in \mathscr{C}_\varphi(\Sigma_c)$ of period $\tau > 0$, then we compute for $v \in \mathscr{C}_\varphi(\Sigma_c)$ of period $\tau + 2\pi$

$$\Omega(v) - \Omega(v^-) = c(\tau + 2\pi) - c\tau = 2\pi c = e(\Sigma_c)$$

by Example 4.6 and Example 2.50. Hence, we have verified the statement of the Forcing Theorem 4.11 for the displaceable twisted stable hypersurface Σ_c in the symplectically aspherical and geometrically bounded symplectic manifold $(T^*\mathbb{T}^2, \omega_\sigma)$. Indeed, $(T^*\mathbb{T}^2, \omega_\sigma)$ is geometrically bounded by Example 4.10, and symplectically aspherical as

$$\pi_2(T^*\mathbb{T}^n) \cong \pi_2(\mathbb{T}^n) \times \pi_2(\mathbb{R}^n) \cong 0.$$

Chapter 5
Further Steps in Twisted Rabinowitz–Floer Homology

In this final chapter we discuss some possible further research in twisted Rabinowitz–Floer homology for future work. One can of course try to find a twisted version of every result provided by standard Rabinowitz–Floer homology. Following the survey article [8], major results relate Rabinowitz–Floer homology to symplectic homology.

In the first section, we point out that one can combine Theorems 1.1 and 4.11 to yield a partial multiplicity result as in Theorem 1.3.

In the second section, we discuss an invariance result under isotopies of twisted Rabinowitz–Floer homology.

In the third section, we briefly outline a further computation of twisted Rabinowitz–Floer homology, where the hypersurface is not displaceable.

In the last section, we explain an important physical setting where the Forcing Theorem 4.11 and Theorem 1.1 might be applicable.

5.1 Forcing

Combining Theorem 1.1 and Theorem 4.11 yields the following prototypical result. Recall, that a parametrised periodic orbit γ of a Hamiltonian function H is called *(transversely) nondegenerate*, if $\mathfrak{K}(\gamma) = \langle (X_H(\gamma(0)), 0) \rangle$ by [23, Definition 7.3.1].

Theorem 5.1. *Let $\Sigma \subseteq \mathbb{C}^n$, $n \geq 2$, be a compact and connected star-shaped hypersurface invariant under the rotation*

$$\varphi\colon \mathbb{C}^n \to \mathbb{C}^n, \quad \varphi(z_1, \dots, z_n) := \left(e^{2\pi i k_1/m} z_1, \dots, e^{2\pi i k_n/m} z_n\right)$$

for some even $m \geq 2$ and $k_1, \dots, k_n \in \mathbb{Z}$ coprime to m. Suppose that there exists a nondegenerated simple noncontractible periodic Reeb orbit γ_- on $\Sigma/\mathbb{Z}_m$. Then there exists a second noncontractible periodic Reeb orbit γ_+ on $\Sigma/\mathbb{Z}_m$ such that

$$\int_0^1 \gamma_+^* \lambda - \int_0^1 \gamma_-^* \lambda \leq e(\Sigma).$$

In particular, if γ_+ is an n-fold iteration of γ_-, then we get the estimate

$$0 < \int_0^1 \gamma_-^* \lambda \leq \frac{1}{n-1} e(\Sigma).$$

5.2 Invariance

Let (W, λ) be a Liouville domain. Given $\varphi, \psi \in \mathrm{Aut}(W, \lambda)$ contained in the same connected component of $\mathrm{Aut}(W, \lambda)$, we expect a similar invariance statement to hold as in [54, Theorem 2.34]. More precisely, we expect that there exists an isomorphism

$$\mathrm{RFH}^{\varphi}(\partial W, M) \cong \mathrm{RFH}^{\psi}(\partial W, M),$$

where (M, λ) denotes the completion of (W, λ). This was already used in the proof of Theorem 4.1 for the special case of spheres and rotations. However, there the Reeb flow is periodic and one can explicitly write down an isomorphism for the generators of the corresponding chain complexes. The general case is assumed to be similar but technically more challenging. Moreover, this isomorphism might imply a contact version of the Seidel representation [42, Section 11.4]. This was suggested by Will Merry.

5.3 Cotangent Bundles

Let (M, g) be a compact connected Riemannian manifold and let (S^*M, pdq) be the spherisation of M as in Example 2.36. By [3] or [18, Theorem 1.10], we have

$$\mathrm{RFH}_k(S^*M, T^*M) \cong \begin{cases} \mathrm{H}^{-k+1}(\mathscr{L}M) & k < 0, \\ \mathrm{H}_k(\mathscr{L}M) & k > 1. \end{cases}$$

In the degrees $k = 0, 1$ the answer is known and depends on the Euler class. The proof uses a relation between Rabinowitz–Floer homology and symplectic homology, respectively symplectic cohomology. If $\varphi \in \mathrm{Aut}(D^*M, pdq|_{D^*M})$ is a Liouville automorphism, then it is plausible to expect

$$\mathrm{RFH}_k^{\varphi}(S^*M, T^*M) \cong \begin{cases} \mathrm{H}^{-k+1}(\mathscr{L}_\varphi M) & k < 0, \\ \mathrm{H}_k(\mathscr{L}_\varphi M) & k > 1. \end{cases}$$

However, it would be also interesting to study the loop space homology $\mathrm{H}_*(\mathscr{L}_\varphi M)$ itself, because usually one computes the free loop space homology via Morse theory. For details see [37, Chapter 2].

5.4 Stark–Zeeman Systems

Following [20] we introduce Stark–Zeeman systems. For $\mu_\pm > 0$ define the potential functions

$$V_\pm\colon \mathbb{C} \setminus \{\pm 1\} \to \mathbb{R}, \qquad V_\pm(z) := -\frac{\mu_\pm}{|z \pm 1|}.$$

Let $U_0 \subseteq \mathbb{C}$ be open and star-shaped with respect to the origin such that $\pm 1 \in U_0$. Choose $V_0 \in C^\infty(U_0)$. Moreover, set

$$V := V_+ + V_- + V_0 \in C^\infty(U)$$

for $U := U_0 \setminus \{\pm 1\}$. For a function $B \in C^\infty(U_0)$, let $\sigma_B := B\,dq_1 \wedge dq_2$ and abbreviate by (T^*U, ω_B) the associated magnetic cotangent bundle. Fix a Riemannian metric g on U_0 which is conformal to the standard metric. A ***Stark–Zeeman system*** is the magnetic Hamiltonian system (T^*U, ω_B, H) with

$$H(q,p) := \frac{1}{2}\,\|p\|^2_{g^*} + V(q) \qquad \forall (q,p) \in T^*U.$$

For $c \in \mathbb{R}$ a regular value of H, we consider a connected component $\Sigma_c \subseteq H^{-1}(c)$ such that $\mathfrak{K}_c \cup \{\pm 1\}$ is bounded and simply connected, where the Hill's region $\mathfrak{K}_c$ is defined by

$$\mathfrak{K}_c := \pi(\Sigma_c) \subseteq \{q \in U : V(q) \leq c\}.$$

For example, the planar circular restricted three-body problem is a Stark–Zeeman system. In order to deal with collisions, we regularise Σ_c.

Definition 5.2 (Regularisation, [23, p. 48]). Let (Σ, ω) be a noncompact Hamiltonian manifold. A ***regularisation of*** $\boldsymbol{(\Sigma, \omega)}$ is defined to be a compact Hamiltonian manifold $(\bar{\Sigma}, \bar{\omega})$ such that there exists an embedding $\iota\colon \Sigma \hookrightarrow \bar{\Sigma}$ with $\iota^*\bar{\omega} = \omega$.

Consider the ***Birkhoff regularisation map***

$$\varphi\colon \mathbb{C}^* \to \mathbb{C}, \qquad B(z) := \frac{1}{2}\left(z + \frac{1}{z}\right).$$

By Example 2.17, the cotangent lift $D\varphi^\dagger$ of φ is given by

$$D\varphi^\dagger\colon T^*\mathbb{C}^* \to T^*\mathbb{C}, \qquad D\varphi^\dagger(z,w) = \left(\frac{z^2+1}{2z}, \frac{2\bar{z}^2 w}{\bar{z}^2 - 1}\right) = (q,p).$$

The regular energy surface Σ_c gives rise to a regular energy surface $\Sigma_c^B \subseteq K^{-1}(0)$, where the rescaled Hamiltonian $K := H \circ D\varphi^\dagger$ is given by

$$K(z,w) = \frac{\|w\|^2_{g^*}}{2} - \frac{\mu_+\,|z+1|^2}{2\,|z|^3} - \frac{\mu_-\,|z-1|^2}{2\,|z|^3} + \frac{(V_0(q) - c)\,|z^2-1|^2}{4\,|z|^4}.$$

The compact regular energy surface Σ_c^B is called the ***Birkhoff regularisaton of Σ_c***. This regularisation is invariant under the cotangent lift

$$\Phi \colon T^*\mathbb{C}^* \to T^*\mathbb{C}^*, \qquad \Phi(z,w) := \left(\frac{1}{z}, -\overline{z}^2 w\right)$$

As the induced action of Φ on Σ_c^B is free, we obtain the cover

$$\Sigma_c^B \to \Sigma_c^B/\mathbb{Z}_2.$$

Explicitly, there exist diffeomorphisms such that

$$\Sigma_c^B \cong \mathbb{S}^1 \times \mathbb{S}^2 \qquad \text{and} \qquad \Sigma_c^B/\mathbb{Z}_2 \cong \mathbb{RP}^3 \# \mathbb{RP}^3.$$

Therefore, it may be possible to apply the ideas developed in the proof of Theorem 1.1 or the Forcing Theorem 4.11 to these hypersurfaces. The analysis of these hypersurfaces is already quite delicate in the special case of the planar circular restricted three-body problem. Indeed, it requires some work to show that the regularised energy hypersurface is fibrewise star-shaped for energy values below the first critical value. For details, see [23, Theorem 5.7.2]. Hence we cannot expect stability of the hypersurfaces in a general Stark–Zeeman system.

Appendix A
Twisted Loop Spaces

In this appendix, we will consider the category of topological manifolds rather than the category of smooth manifolds, because smoothness does not add much to the discussion. Free and based loop spaces are fundamental objects in Algebraic Topology, for a vast treatment of the geometry and topology of based as well as free loop spaces see for example [37]. But so-called twisted loop spaces are not considered that much.

Theorem A.1 (Twisted Loops in Universal Covering Manifolds). *Let (M, x) be a connected pointed topological manifold and $\pi\colon \widetilde{M} \to M$ the universal covering.*

(a) *Fix $[\eta] \in \pi_1(M, x)$ and denote by $U_\eta \subseteq \mathscr{L}(M, x)$ the path component corresponding to $[\eta]$ via the bijection $\pi_0(\mathscr{L}(M, x)) \cong \pi_1(M, x)$. For every $e, e' \in \pi^{-1}(x)$ and $\varphi \in \mathrm{Aut}_\pi(\widetilde{M})$ such that $\varphi(e) = \widetilde{\eta}_e(1)$, where $\widetilde{\eta}_e$ denotes the unique lift of η with $\widetilde{\eta}_e(0) = e$, we have a commutative diagram of homeomorphisms*

$$\begin{array}{ccc} \mathscr{L}_\varphi(\widetilde{M}, e) & \xrightarrow{\;L_\psi\;} & \mathscr{L}_{\psi\circ\varphi\circ\psi^{-1}}(\widetilde{M}, e') \\ & {\scriptstyle\Psi_e}\nwarrow \qquad \nearrow{\scriptstyle\Psi_{e'}} & \\ & U_\eta, & \end{array} \tag{A.1}$$

where $\psi \in \mathrm{Aut}_\pi(\widetilde{M})$ is such that $\psi(e) = e'$,

$$L_\psi\colon \mathscr{L}_\varphi(\widetilde{M}, e) \to \mathscr{L}_{\psi\circ\varphi\circ\psi^{-1}}(\widetilde{M}, e'), \qquad L_\psi(\gamma) := \psi \circ \gamma,$$

and

$$\begin{aligned} \Psi_e\colon U_\eta &\to \mathscr{L}_\varphi(\widetilde{M}, e), & \Psi_e(\gamma) &:= \widetilde{\gamma}_e, \\ \Psi_{e'}\colon U_\eta &\to \mathscr{L}_{\psi\circ\varphi\circ\psi^{-1}}(\widetilde{M}, e'), & \Psi_{e'}(\gamma) &:= \widetilde{\gamma}_{e'}. \end{aligned}$$

Moreover, $U_{c_x} \cong \mathscr{L}_\varphi(\widetilde{M}, e)$ via Ψ_e if and only if $\varphi = \mathrm{id}_{\widetilde{M}}$, where c_x denotes the constant loop at x.

(b) *For every* $\varphi \in \mathrm{Aut}_\pi(\widetilde{M})$ *and* $e, e' \in \pi^{-1}(x)$ *we have a commutative diagram of isomorphisms*

$$\begin{array}{ccc} \mathrm{Aut}_\pi(\widetilde{M}) & \xrightarrow{\quad C_\psi \quad} & \mathrm{Aut}_\pi(\widetilde{M}) \\ & {}_{\Phi_e}\nwarrow \qquad \nearrow_{\Phi_{e'}} & \\ & \pi_1(M, x), & \end{array}$$

where for $\psi \in \mathrm{Aut}_\pi(\widetilde{M})$ *sucht that* $\psi(e) = e'$

$$C_\psi\colon\ \mathrm{Aut}_\pi(\widetilde{M}) \to \mathrm{Aut}_\pi(\widetilde{M}), \qquad C_\psi(\varphi) := \psi \circ \varphi \circ \psi^{-1},$$

and

$$\Phi_e\colon \pi_1(M, x) \to \mathrm{Aut}_\pi(\widetilde{M}), \qquad \Phi_e([\gamma]) := \varphi^e_{[\gamma]},$$
$$\Phi_{e'}\colon \pi_1(M, x) \to \mathrm{Aut}_\pi(\widetilde{M}), \qquad \Phi_{e'}([\gamma]) := \varphi^{e'}_{[\gamma]},$$

with $\varphi^e_{[\gamma]}(e) = \widetilde{\gamma}_e(1)$ *and* $\varphi^{e'}_{[\gamma]}(e') = \widetilde{\gamma}_{e'}(1)$.

(c) *The projection*

$$\widetilde{\pi}_x\colon \coprod_{\substack{\varphi \in \mathrm{Aut}_\pi(\widetilde{M}) \\ e \in \pi^{-1}(x)}} \mathcal{L}_\varphi(\widetilde{M}, e) \to \mathcal{L}(M, x)$$

defined by $\widetilde{\pi}_x(\gamma) := \pi \circ \gamma$ *is a covering map with number of sheets coinciding with the cardinality of* $\pi_1(M, x)$. *Moreover,* $\widetilde{\pi}_x$ *restricts to define a covering map*

$$\widetilde{\pi}_x|_{\mathrm{id}_{\widetilde{M}}}\colon \coprod_{e \in \pi^{-1}(x)} \mathcal{L}(\widetilde{M}, e) \to U_{c_x},$$

and $\widetilde{\pi}_x$ *gives rise to a principal* $\mathrm{Aut}_\pi(\widetilde{M})$*-bundle. If* M *admits a smooth structure, then this bundle is additionally a bundle of smooth Banach manifolds.*

Proof. For proving part (a), fix a path class $[\gamma] \in \pi_1(M, x)$. As any topological manifold is Hausdorff, paracompact and locally metrisable by definition, the Smirnov Metrisation Theorem [46, Theorem 42.1] implies that M is metrisable. Let d be a metric on M and $\overline{d}$ be the standard bounded metric corresponding to d, that is,

$$\overline{d}(x, y) = \min\{d(x, y), 1\} \qquad \forall x, y \in M.$$

The metric $\overline{d}$ induces the same topology on M as d by [46, Theorem 20.1]. Topologise the based loop space $\mathcal{L}(M, x) \subseteq \mathcal{L}M$ as a subspace of the free loop space on M, where $\mathcal{L}M$ is equipped with the topology of uniform convergence, that is, with the supremum metric

$$\overline{d}_\infty(\gamma,\gamma') = \sup_{t\in\mathbb{S}^1} \overline{d}\left(\gamma(t),\gamma'(t)\right) \qquad \forall \gamma,\gamma' \in \mathcal{L}M.$$

There is a canonical pseudometric on the universal covering manifold $\widetilde{M}$ induced by $\overline{d}$ given by $\overline{d} \circ \pi$. As every pseudometric generates a topology, we topologise the based twisted loop space $\mathcal{L}_\varphi(\widetilde{M}, e) \subseteq \mathcal{P}\widetilde{M}$ as a subspace of the free path space on $\widetilde{M}$ for every $e \in \pi^{-1}(x)$ via the supremum metric $\widetilde{d}_\infty$ corresponding to $\overline{d} \circ \pi$. In fact, $\widetilde{d}_\infty$ is a metric as if $\widetilde{d}_\infty(\gamma,\gamma') = 0$, then by definition of $\widetilde{d}_\infty$ we have that $\pi(\gamma) = \pi(\gamma')$. But as $\gamma(0) = e = \gamma'(0)$, we conclude $\gamma = \gamma'$ by the unique lifting property of paths [38, Corollary 11.14]. Note that the resulting topology of uniform convergence on $\mathcal{L}_\varphi(\widetilde{M}, e)$ coincides with the compact-open topology by [46, Theorem 46.8] or [32, Proposition A.13]. In particular, the topology of uniform convergence does not depend on the choice of a metric (see [46, Corollary 46.9]). It follows from [38, Theorem 11.15 (b)], that Ψ_e and $\Psi_{e'}$ are well-defined. Moreover, it is immediate by the fact that the projection $\pi\colon \widetilde{M} \to M$ is an isometry with respect to the above metric, that Ψ_e and $\Psi_{e'}$ are continuous with continuous inverse given by the composition with π. It is also immediate that L_ψ is continuous with continuous inverse $L_{\psi^{-1}}$.

Next we show that the diagram (A.1) commutes. Note that

$$\pi \circ L_\psi \circ \Psi_e = \pi \circ \Psi_e = \mathrm{id}_{U_\eta} = \pi \circ \Psi_{e'},$$

thus by

$$(L_\psi \circ \Psi_e(\gamma))(0) = \psi(\widetilde{\gamma}_e(0)) = \psi(e) = e' = \widetilde{\gamma}_{e'}(0) = \Psi_{e'}(\gamma)(0)$$

and by uniqueness it follows that

$$L_\psi \circ \Psi_e = \Psi_{e'}.$$

In particular

$$\Psi_{e'}(1) = (L_\psi \circ \Psi_e)(1) = \psi(\varphi(e)) = (\psi \circ \varphi \circ \psi^{-1})(e'),$$

and thus $\Psi_{e'}(\gamma) \in \mathcal{L}_{\psi\circ\varphi\circ\psi^{-1}}(\widetilde{M}, e')$. Consequently, the homeomorphism $\Psi_{e'}$ is well-defined.

Recall, that by the Monodromy Theorem [38, Theorem 11.15 (b)]

$$\gamma \simeq \gamma' \qquad \Leftrightarrow \qquad \Psi_e(\gamma)(1) = \Psi_e(\gamma')(1)$$

for all paths γ and γ' in M starting at x and ending at the same point. Note that the statement of the the Monodromy Theorem is an if-and-only-if statement since $\widetilde{M}$ is simply connected.

Suppose $\gamma \in \mathcal{L}(M, x)$ is contractible. Then $\gamma \simeq c_x$, implying $e \in \mathrm{Fix}(\varphi)$. But the only deck transformation of π fixing any point of $\widetilde{M}$ is $\mathrm{id}_{\widetilde{M}}$ by [38, Proposition 12.1 (a)].

Conversely, assume that $\gamma \in \mathcal{L}(M,x)$ is not contractible. Then we have that $\Psi_e(\gamma)(1) \neq e$. Indeed, if $\Psi_e(\gamma)(1) = e$, then $\gamma \simeq c_x$ and consequently, γ would be contractible. As normal covering maps have transitive automorphism groups by [38, Corollary 12.5], there exists $\psi \in \mathrm{Aut}_\pi(\widetilde{M}) \setminus \{\mathrm{id}_{\widetilde{M}}\}$ such that $\Psi_e(\gamma)(1) = \psi(e)$.

For proving part (b), observe that Φ_e and $\Phi_{e'}$ are isomorphisms follows from [38, Corollary 12.9]. Moreover, it is also clear that C_ψ is an isomorphism with inverse $C_{\psi^{-1}}$. Let $[\gamma] \in \pi_1(M,x)$. Then using part (a) we compute

$$
\begin{aligned}
(C_\psi \circ \Phi_e)[\gamma](e') &= (\psi \circ \Phi_e[\gamma] \circ \psi^{-1})(e') \\
&= \psi\left(\varphi^e_{[\gamma]}(e)\right) \\
&= \psi(\widetilde{\gamma}_e(1)) \\
&= (L_\psi \circ \Psi_e)(\gamma)(1) \\
&= \Psi_{e'}(\gamma)(1) \\
&= \widetilde{\gamma}_{e'}(1) \\
&= \varphi^{e'}_{[\gamma]}(e') \\
&= \Phi_{e'}[\gamma](e').
\end{aligned}
$$

Thus by uniqueness [38, Proposition 12.1 (a)], we conclude

$$C_\psi \circ \Phi_e = \Phi_{e'}.$$

Finally for proving (c), define a metric $\widetilde{d}_\infty$ on

$$E := \coprod_{\substack{\varphi \in \mathrm{Aut}_\pi(\widetilde{M}) \\ e \in \pi^{-1}(x)}} \mathcal{L}_\varphi(\widetilde{M}, e)$$

by

$$\widetilde{d}_\infty(\gamma,\gamma') := \begin{cases} \overline{d}_\infty\left(\pi(\gamma),\pi(\gamma')\right) & \gamma,\gamma' \in \mathcal{L}_\varphi(\widetilde{M},e), \\ 1 & \text{else.} \end{cases}$$

Then the induced topology coincides with the disjoint union topology and with respect to this topology, $\widetilde{\pi}_x$ is continuous. So left to show is that $\widetilde{\pi}_x$ is a covering map. Surjectivity is clear. So let $\gamma \in \mathcal{L}(M,x)$. Then $\gamma \in U_\eta$ for some $[\eta] \in \pi_1(M,x)$. Now note that U_η is open in $\mathcal{L}(M,x)$ and by part (a) we conclude

$$\widetilde{\pi}_x^{-1}(U_\eta) = \coprod_{\psi \in \mathrm{Aut}_\pi(\widetilde{M})} \mathcal{L}_{\psi\circ\varphi\circ\psi^{-1}}(\widetilde{M}, \psi(e)) \tag{A.2}$$

for some fixed $e \in \pi^{-1}(x)$ and $\varphi \in \mathrm{Aut}_\pi(\widetilde{M})$ such that $\varphi(e) = \widetilde{\eta}_e(1)$.

As the cardinality of the fibre $\pi^{-1}(x)$ and of $\mathrm{Aut}_\pi(\widetilde{M})$ coincides with the cardinality of the fundamental group $\pi_1(M,x)$ by [38, Corollary 11.31] and part (b), we conclude that the number of sheets is equal to the cardinality of the fundamental group $\pi_1(M, x)$.

Equip $\mathrm{Aut}_\pi(\widetilde{M})$ with the discrete topology. As the fundamental group of every topological manifold is countable by [38, Theorem 7.21], we have that $\mathrm{Aut}_\pi(\widetilde{M})$ is a discrete topological Lie group. Now label the distinct path classes in $\pi_1(M, x)$ by $\beta \in B$ and for fixed $e \in \pi^{-1}(x)$ define local trivialisations

$$(\widetilde{\pi}_x, \alpha_\beta)\colon \widetilde{\pi}_x^{-1}(U_\beta) \xrightarrow{\cong} U_\beta \times \mathrm{Aut}_\pi(\widetilde{M}),$$

making use of (A.2) by

$$\alpha_\beta(\gamma) := \psi^{-1},$$

whenever $\gamma \in \mathscr{L}_{\psi\circ\varphi\circ\psi^{-1}}(\widetilde{M}, \psi(e))$. Consequently, $\widetilde{\pi}_x$ is a fibre bundle with discrete fibre $\mathrm{Aut}_\pi(\widetilde{M})$ and bundle atlas $(U_\beta, \alpha_\beta)_{\beta\in B}$. Define a free right action

$$E \times \mathrm{Aut}_\pi(\widetilde{M}) \to E, \qquad \gamma \cdot \xi := \xi^{-1} \circ \gamma.$$

Then α_β is $\mathrm{Aut}_\pi(\widetilde{M})$-equivariant with respect to this action for all $\beta \in B$. Indeed, using again the commutative diagram (A.1) we compute

$$\alpha_\beta(\gamma \cdot \xi) = \alpha_\beta(\xi^{-1} \circ \gamma) = \left(\xi^{-1} \circ \psi\right)^{-1} = \psi^{-1} \circ \xi = \alpha_\beta(\gamma) \circ \xi$$

for all $\xi \in \mathrm{Aut}_\pi(\widetilde{M})$ and $\gamma \in \mathscr{L}_{\psi\circ\varphi\circ\psi^{-1}}(\widetilde{M}, \psi(e))$. Note, that here we use again the fact that $\mathrm{Aut}_\pi(\widetilde{M})$ acts transitively on the fibre $\pi^{-1}(x)$.

Suppose that M admits a smooth structure. Then for every compact smooth manifold N we have that the mapping space $C(N, M)$ admits the structure of a smooth Banach manifold by [56]. By [37, Theorem 1.1 p. 24], there is a smooth fibre bundle, called the ***loop-loop fibre bundle***,

$$\mathscr{L}(M, x) \hookrightarrow \mathscr{L}M \xrightarrow{\mathrm{ev}_0} M$$

where

$$\mathrm{ev}_0\colon \mathscr{L}M \to M, \qquad \mathrm{ev}_0(\gamma) := \gamma(0).$$

Thus the based loop space $\mathscr{L}(M, x) = \mathrm{ev}_0^{-1}(x)$ on M is a smooth Banach manifold by the implicit function theorem [42, Theorem A.3.3] for all $x \in M$. Likewise, by [37, Theorem 1.2 p. 25], there is a smooth fibre bundle, called the ***path-loop fibre bundle***,

$$\mathscr{L}(\widetilde{M}, e) \hookrightarrow \mathscr{P}(\widetilde{M}, e) \xrightarrow{\mathrm{ev}_1} \widetilde{M},$$

where

$$\mathscr{P}(\widetilde{M}, e) := \{\gamma \in C(I, \widetilde{M}) : \gamma(0) = e\}$$

denotes the based path space and

$$\mathrm{ev}_1\colon \mathscr{P}(\widetilde{M}, e) \to \widetilde{M}, \qquad \mathrm{ev}_1(\gamma) := \gamma(1).$$

Therefore, the twisted loop space $\mathcal{L}_\varphi(\widetilde{M}, e) = \mathrm{ev}_1^{-1}(\varphi(e))$ is also a smooth Banach manifold for all $\varphi \in \mathrm{Aut}_\pi(\widetilde{M})$ and $e \in \pi^{-1}(x)$ by the implicit function theorem [42, Theorem A.3.3]. As the fundamental group $\pi_1(M, x)$ is countable, the topological space E has only countably many connected components being smooth Banach manifolds and thus the total space itself is a smooth Banach manifold. Finally, $\mathrm{Aut}_\pi(\widetilde{M})$ is trivially a Banach manifold with $\dim \mathrm{Aut}_\pi(\widetilde{M}) = 0$ as a discrete Lie group. □

Corollary A.2. *Let (M, x) be a connected pointed topological manifold and denote by $\pi\colon \widetilde{M} \to M$ the universal covering of M. Assume that $\pi_1(M, x)$ is abelian.*

(a) *Fix a path class $[\eta] \in \pi_1(M, x)$. For every $e, e' \in \pi^{-1}(x)$ and deck transformation $\varphi \in \mathrm{Aut}_\pi(\widetilde{M})$ such that $\varphi(e) = \widetilde{\eta}_e(1)$, we have a commutative diagram of homeomorphisms*

$$\begin{array}{ccc} \mathcal{L}_\varphi(\widetilde{M}, e) & \xrightarrow{\quad L_\psi \quad} & \mathcal{L}_\varphi(\widetilde{M}, e') \\ & {\scriptstyle \Psi_e}\nwarrow \qquad \nearrow{\scriptstyle \Psi_{e'}} & \\ & U_\eta, & \end{array}$$

where $\psi \in \mathrm{Aut}_\pi(\widetilde{M})$ is such that $\psi(e) = e'$.

(b) *For every $\varphi \in \mathrm{Aut}_\pi(\widetilde{M})$ we have that $\Phi_e = \Phi_{e'}$ for all $e, e' \in \pi^{-1}(x)$.*

Lemma 1.5 now follows from part (a) of Theorem A.1. Indeed, by assumption $\varphi \in \mathrm{Aut}_\pi(\Sigma) \setminus \{\mathrm{id}_\Sigma\}$ and using the long exact sequence of homotopy groups of a fibration [32, Theorem 4.41], there is a short exact sequence

$$0 \longrightarrow \pi_1(\Sigma, x) \longrightarrow \pi_1(\Sigma/\mathbb{Z}_m, \pi(x)) \longrightarrow \pi_0(\mathbb{Z}_m) \longrightarrow 0.$$

In particular, by [38, Corollary 12.9] we conclude

$$\mathrm{Aut}_\pi(\Sigma) \cong \pi_1(\Sigma/\mathbb{Z}_m, \pi(x)) \cong \mathbb{Z}_m \cong \{\mathrm{id}_\Sigma, \varphi, \ldots, \varphi^{m-1}\}.$$

Finally, we discuss a smooth structure on the continuous free twisted loop space of a smooth manifold.

Lemma A.3. *Let M be a smooth manifold and $\varphi \in \mathrm{Diff}(M)$. Then the continuous free twisted loop space $\mathcal{L}_\varphi M$ is the pullback of*

$$(\mathrm{ev}_0, \mathrm{ev}_1)\colon \mathcal{P}M \to M \times M, \qquad \gamma \mapsto (\gamma(0), \gamma(1)),$$

where we abbreviate $\mathcal{P}M := C(I, M)$, along the graph of φ

$$\Gamma_\varphi\colon M \to M \times M, \qquad \Gamma_\varphi(x) := (x, \varphi(x)),$$

in the category of smooth Banach manifolds. Moreover, we have that

$$T_\gamma \mathcal{L}_\varphi M = \{X \in \Gamma^0(\gamma^* TM) : X(1) = D\varphi(X(0))\}$$

for all $\gamma \in \mathcal{L}_\varphi M$.

Proof. Write $f := (\mathrm{ev}_0, \mathrm{ev}_1)$. Then

$$\mathcal{L}_\varphi M = f^{-1}(\Gamma_\varphi(M)).$$

Thus in order to show that the free twisted loop space $\mathcal{L}_\varphi M$ is a smooth Banach manifold, it is enough to show that f is transverse to the properly embedded smooth submanifold $\Gamma_\varphi(M) \subseteq M \times M$. By [36, Proposition 2.4] we need to show that the composition

$$\Phi_\gamma \colon T_\gamma \mathcal{P} M \xrightarrow{Df_\gamma} T_{(x,\varphi(x))}(M \times M) \to T_{(x,\varphi(x))}(M \times M)/T_{(x,\varphi(x))}\Gamma_\varphi(M)$$

is surjective and $\ker \Phi_\gamma$ is complemented for all $\gamma \in f^{-1}(\Gamma_\varphi(M))$, where we abbreviate $x := \gamma(0)$. Note that we have a canonical isomorphism

$$T_{(x,\varphi(x))}(M \times M)/T_{(x,\varphi(x))}\Gamma_\varphi(M) \to T_{\varphi(x)}M, \quad [(v,u)] := u - D\varphi(v).$$

Under this canonical isomorphism, the linear map Φ_γ is given by

$$\Phi_\gamma(X) = X(1) - D\varphi(X(0)), \qquad \forall X \in \Gamma^0(\gamma^* TM).$$

Fix a Riemannian metric on M and let $X_v \in \Gamma(\gamma^* TM)$ be the unique parallel vector field with $X_v(1) = v \in T_{\varphi(x)}M$. Fix a cutoff function $\beta \in C^\infty(I)$ such that $\operatorname{supp} \beta \subseteq \left[\frac{1}{2}, 1\right]$ and $\beta = 1$ in a neighbourhood of 1. Then $\Phi_\gamma(\beta X_v) = v$ and consequently, Φ_γ is surjective. Moreover

$$\ker \Phi_\gamma = \{X \in \Gamma^0(\gamma^* TM) : X(1) = D\varphi(X(0))\}$$

is complemented by the finite-dimensional vector space

$$V := \{\beta X_v \in \Gamma(\gamma^* TM) : v \in T_{\varphi(x)}M\}.$$

Indeed, any $X \in \Gamma^0(\gamma^* TM)$ can be decomposed uniquely as

$$X = X - \beta X_v + \beta X_v, \qquad v := X(1) - D\varphi(X(0)).$$

Abbreviating $Y := X - \beta X_v \in \Gamma^0(\gamma^* TM)$, we have that

$$Y(1) = D\varphi(X(0)) = D\varphi(Y(0)),$$

implying $Y \in \ker \Phi_\gamma$. Thus $\mathcal{L}_\varphi M$ is a smooth Banach manifold.

Now note that $\mathcal{L}_\varphi M$ can be identified with the pullback

$$f^* \mathcal{P} M = \{(x, \gamma) \in M \times \mathcal{P} M : (\gamma(0), \gamma(1)) = (x, \varphi(x))\},$$

making the diagram

$$\begin{array}{ccc} f^*\mathcal{P}M & \xrightarrow{\mathrm{pr}_2} & \mathcal{P}M \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle f} \\ M & \xrightarrow[\Gamma_\varphi]{} & M \times M \end{array}$$

commute, via the homeomorphism

$$\mathcal{L}_\varphi M \to f^*\mathcal{P}M, \qquad \gamma \mapsto (\gamma(0), \gamma).$$

Finally, one computes

$$T_{(x,\gamma)} f^*\mathcal{P}M = \{(v, X) \in T_x M \times T_\gamma \mathcal{P}M : Df_\gamma X = D\Gamma_\varphi|_x(v)\}$$

for all $(x, \gamma) \in f^*\mathcal{P}M$. □

Remark A.4. Using Lemma A.3 one should be able to prove similar results as in Theorem A.1 in the case of free twisted loop spaces. However, in the non-abelian case the situation gets much more complicated as in general it is not true, that lifts of conjugated elements of the fundamental group lie in the same free twisted loop space by [37, Theorem 1.6 (i)].

Appendix B
On the Nonexistence of the Gradient of the Twisted Rabinowitz Action Functional

Let (M, λ) be an exact symplectic manifold and $\varphi \in \mathrm{Symp}(M, d\lambda)$ a symplectomorphism of finite order. For $H \in C^\infty(M)$ such that $H \circ \varphi = H$, one can define the twisted Rabinowitz action functional

$$\mathcal{A}_\varphi^H : \mathcal{L}_\varphi M \times \mathbb{R} \to \mathbb{R}, \qquad \mathcal{A}_\varphi^H(\gamma, \tau) := \int_0^1 \gamma^*\lambda - \tau \int_0^1 H(\gamma(t))dt. \tag{B.1}$$

Let J be a $d\lambda$-compatible almost complex structure such that $\varphi^* J = J$. Then one can consider the gradient of $\mathcal{A}_\varphi^H$ with respect to the L^2-metric

$$\langle (X, \eta), (Y, \sigma) \rangle_J := \int_0^1 d\lambda(JX(t), Y(t))dt + \eta\sigma \tag{B.2}$$

for all $(X, \eta), (Y, \sigma) \in T_\gamma \mathcal{L}_\varphi M \times \mathbb{R}$ and $(\gamma, \tau) \in \mathcal{L}_\varphi M \times \mathbb{R}$.

Theorem B.1 (Nonexistence Gradient). *Let (M, λ) be a connected exact symplectic manifold, $\varphi \in \mathrm{Symp}(M, d\lambda)$ of finite order and $H \in C^\infty(M)$ such that $H \circ \varphi = H$. If $\varphi^*\lambda \neq \lambda$, then the gradient of the twisted Rabinowitz action functional* (B.1) *with respect to the L^2-metric* (B.2) *does not exist.*

Proof. Assume that the gradient $\operatorname{grad} \mathcal{A}_\varphi^H \in \mathfrak{X}(\mathcal{L}_\varphi M \times \mathbb{R})$ exists. We write

$$\operatorname{grad} \mathcal{A}_\varphi^H = \begin{pmatrix} J(\dot{\gamma} - \tau X_H(\gamma)) \\ -\displaystyle\int_0^1 H(\gamma(t))dt \end{pmatrix} + V$$

for some $V \in \mathfrak{X}(\mathcal{L}_\varphi M)$ and for all $(\gamma, \tau) \in \mathcal{L}_\varphi M \times \mathbb{R}$. Indeed, this follows from

$$d\mathcal{A}_\varphi^H|_{(\gamma,\tau)}(X, \eta) = \int_0^1 d\lambda(X, \dot{\gamma}(t) - \tau X_H(\gamma(t)))dt - \eta \int_0^1 H(\gamma(t))dt + (\varphi^*\lambda - \lambda)(X(0)) \tag{B.3}$$

for all $(X, \eta) \in T_\gamma \mathscr{L}_\varphi M \times \mathbb{R}$. By assumption, there exists $x \in M$ and $v \in T_x M$ with $(\varphi^*\lambda)_x(v) \neq \lambda_x(v)$. As by assumption M is connected, there exists a smooth path $u \in C^\infty([0,1], M)$ from x to $\varphi(x)$. Fix a smooth cutoff function $\beta \in C^\infty([0,1],[0,1])$ such that $\beta = 0$ in a neighbourhood of 0 and $\beta = 1$ in a neighbourhood of 1. Then we can extend u by

$$\gamma(t) := \varphi^k(u(\beta(t-k))) \qquad \forall t \in [k, k+1], k \in \mathbb{Z}.$$

Clearly, $\gamma \in \mathscr{L}_\varphi M$ by construction. Extend $v \in T_{\gamma(0)} M$ to $X_v \in T_\gamma \mathscr{L}_\varphi M$ by

$$X_v(t) := (1-\beta(t-k)) P^{\varphi^k \circ u}_{0,\beta(t-k)}(D\varphi^k(v)) + \beta(t-k) P^{\varphi^k \circ u}_{1,\beta(t-k)}(D\varphi^{k+1}(v)),$$

for all $t \in [k, k+1]$ and $k \in \mathbb{Z}$, where P denotes the parallel transport system induced by the Levi–Civita connection associated with the metric m_J. Choose a sequence $(\beta_j) \subseteq C^\infty(\mathbb{S}^1, [0,1])$ with $\beta_j = 1$ on $\left[0, \frac{1}{2j}\right] \cup \left[1 - \frac{1}{2j}, 1\right]$ and such that $\operatorname{supp} \beta_j \subseteq \left[0, \frac{1}{j}\right] \cup \left[1 - \frac{1}{j}, 1\right]$ for all $j \in \mathbb{N}$. Using (B.3) we compute

$$\langle V, \beta_j X_v \rangle_J = (\varphi^*\lambda - \lambda)(\beta_j(0) X_v(0)) = (\varphi^*\lambda - \lambda)(v)$$

for all $j \in \mathbb{N}$, implying

$$(\varphi^*\lambda - \lambda)(v) = \lim_{j \to \infty} \langle V, \beta_j X_v \rangle_J = \lim_{j \to \infty} \int_0^1 d\lambda(JV(t), \beta_j(t) X_v(t)) dt = 0$$

by dominated convergence. $\square$

Appendix C
Bubbling Analysis

In this section we prove the main result about the compactness of the moduli space of negative gradient flow lines of the symplectic action functional.

Definition C.1 (Symplectic Asphericity). A connected symplectic manifold (M, ω) is said to be ***symplectically aspherical***, if

$$\int_{\mathbb{S}^2} f^*\omega = 0 \qquad \forall f \in C^\infty(\mathbb{S}^2, M).$$

Remark C.2. A symplectic manifold (M, ω) is symplectically aspherical, if and only if $[\omega]|_{\pi_2(M)} = 0$, where $[\omega] \in \mathrm{H}^2_{\mathrm{dR}}(M; \mathbb{R})$ denotes the cohomology class of ω.

Theorem C.3 (Bubbling). *Let (M, ω) be a compact symplectically aspherical symplectic manifold and let (u_k) be a sequence of negative gradient flow lines of the symplectic action functional $\mathcal{A}_H$ for some $H \in C^\infty(M \times \mathbb{T})$ with uniformly bounded energy*

$$E_J(u_k) := \int_{-\infty}^{+\infty} \|\partial_s u_k\|_J^2$$

for some, and hence every, ω-compatible almost complex structure J. Then the derivatives of (u_k) are uniformly bounded.

The main idea of the proof is to assume that the derivatives (Du_k) explode and then to construct a nonconstant J-holomorphic sphere. Indeed, assume that there exists a sequence (s_k, t_k) in $\mathbb{R} \times \mathbb{T}$ such that

$$\lim_{k \to \infty} \|\partial_s u_k(s_k, t_k)\| \to +\infty.$$

Then we rescale the sequence (u_k), see Figure C.1. Set

$$m_k := \|\partial_s u_k(s_k, t_k)\| \qquad \text{and} \qquad v_k(\sigma, \tau) := u_k\left(\frac{\sigma}{m_k} + s_k, \frac{\tau}{m_k} + t_k\right)$$

for all $(\sigma, \tau) \in \mathbb{C}$.

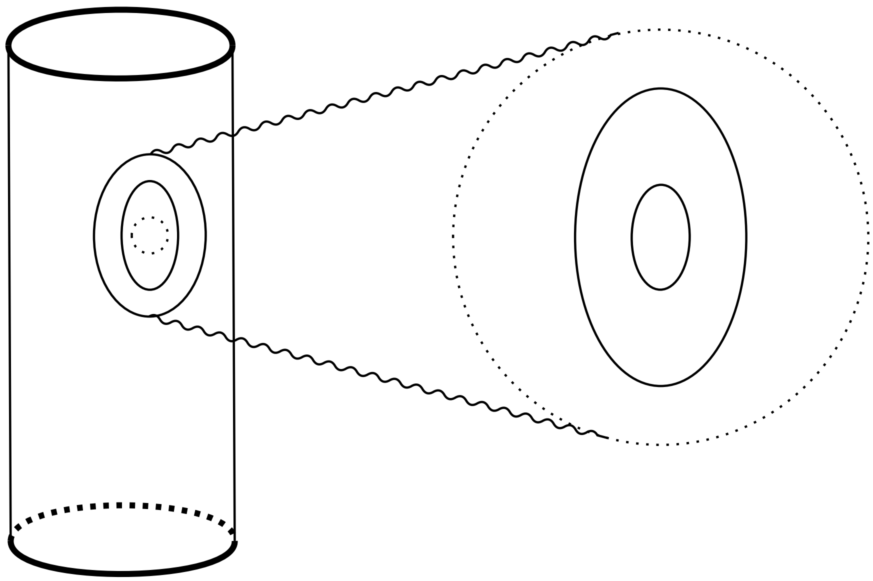

Fig. C.1: Looking at the sequence (u_k) of negative gradient flow lines of the symplectic action functional with a magnifying glass via rescaling.

Then we compute

$$m_k \partial_\sigma v_k(\sigma, \tau) = \partial_s u_k \left(\frac{\sigma}{m_k} + s_k, \frac{\tau}{m_k} + t_k \right),$$
$$m_k \partial_\tau v_k(\sigma, \tau) = \partial_t u_k \left(\frac{\sigma}{m_k} + s_k, \frac{\tau}{m_k} + t_k \right).$$

In particular $\|\partial_\sigma v_k(0,0)\| = 1$ for all $k \in \mathbb{N}$. Moreover, every v_k solves

$$\partial_\sigma v_k(\sigma, \tau) + J \partial_\tau v_k(\sigma, \tau) = \frac{1}{m_k} J X_{H_{\frac{\tau}{m_k} + t_k}}(v_k(\sigma, \tau)) \qquad \forall (\sigma, \tau) \in \mathbb{C}$$

as u_k satisfies the Floer equation (2.4). If there exists $v_\infty \in C^\infty(\mathbb{C}, M)$ such that

$$v_k \xrightarrow{C^\infty_{\text{loc}}} v_\infty, \qquad k \to \infty,$$

modulo subsequences, then v_∞ satisfies

$$\partial_\sigma v_\infty(\sigma, \tau) + J \partial_\tau v_\infty(\sigma, \tau) = 0 \qquad \forall (\sigma, \tau) \in \mathbb{C}.$$

Consequently, v_∞ is a J-holomorphic plane. Using the assumption that the energy of the sequence (u_k) is uniformly bounded, one can extend v_∞ to a J-holomorphic sphere $v \in C^\infty(\mathbb{S}^2, M)$ such that $v|_{\mathbb{C}} = v_\infty$ via the identification $\mathbb{S}^2 \cong \mathbb{C} \cup \{\infty\}$.

C.1 Rescaling

Lemma C.4 (Hofer, [1, Lemma 4.4.4]). *Let* (X, d) *be a complete metric space and*

$$f\colon X \to [0, +\infty)$$

continuous. Given $\varepsilon > 0$ *and* $x \in X$*, there exist* $0 < \delta \leq \varepsilon$ *and* $y \in X$ *such that*

$$d(x, y) \leq 2\delta, \qquad \delta f(y) \geq \varepsilon f(x), \qquad \text{and} \qquad \sup_{z \in \overline{B}_\delta(y)} f(z) \leq 2 f(y).$$

Lemma C.5 (Rescaling). *Let* (u_k) *be a sequence of negative gradient flow lines of the symplectic action functional* $\mathcal{A}_H$ *for some* $H \in C^\infty(M \times \mathbb{T})$ *and* J *some* ω*-compatible almost complex structure. Then there exists a sequence* (R_k) *in* $(0, +\infty)$ *with* $R_k \to +\infty$ *and a sequence* $v_k \in C^\infty(B_{R_k}(0), M)$ *such that*

$$\lim_{k\to\infty} \|\partial_\sigma v_k + J \partial_\tau v_k\|_\infty = 0 \qquad \text{and} \qquad 1 \leq \|\partial_\sigma v_k\|_\infty \leq 2.$$

Proof. Abbreviate $z_k := (s_k, t_k) \in \mathbb{R} \times \mathbb{T}$. Apply Hofer's Lemma C.4 with

$$f : \mathbb{R} \times \mathbb{T} \to [0, +\infty), \qquad f(s, t) := \|\partial_s u_k(s, t)\|,$$

as well as

$$\varepsilon_k := \frac{1}{\sqrt{m_k}}, \qquad \text{and} \qquad x_k := z_k.$$

Thus there exists a nearby sequence $z'_k = (s'_k, t'_k) \in \mathbb{R} \times \mathbb{T}$ and $0 < \delta_k \leq \varepsilon_k$ with

$$\delta_k \|\partial_s u_k(z'_k)\| \geq \varepsilon_k \|\partial_s u_k(z_k)\| = \sqrt{m}_k$$

and

$$\sup_{z \in B_{\delta_k}(z'_k)} \|\partial_s u_k(z)\| \leq 2 \|\partial_s u_k(z'_k)\|.$$

Rescale $m'_k := \|\partial_s u_k(z'_k)\|$, set $R_k := \delta_k m'_k$ and

$$v_k(\sigma, \tau) := u_k\left(\frac{\sigma}{m'_k} + s'_k, \frac{\tau}{m'_k} + t'_k\right) \qquad \forall (\sigma, \tau) \in B_{R_k}(0).$$

As u_k is a solution to the Floer equation (2.4), v_k satisfies the equation

$$\partial_\sigma v_k(\sigma, \tau) + J \partial_\tau v_k(\sigma, \tau) = \frac{1}{m'_k} J X_{H_{\frac{\tau}{m'_k} + t'_k}}(v_k(\sigma, \tau)) \tag{C.1}$$

for all $(\sigma, \tau) \in B_{R_k}(0)$. As $m'_k \geq m_k \to +\infty$, we conclude

$$\lim_{k\to\infty} \|\partial_\sigma v_k + J \partial_\tau v_k\|_\infty = 0.$$

Finally, we compute

$$\begin{aligned}\|\partial_\sigma v_k\|_\infty &= \sup_{z\in B_{R_k}(0)} \|\partial_\sigma v_k(z)\| \\ &= \frac{1}{m_k'} \sup_{z\in B_{\delta_k}(z_k')} \|\partial_s u_k(z)\| \\ &\le \frac{2}{m_k'} \|\partial_s u_k(z_k')\| \\ &= 2,\end{aligned}$$

and

$$m_k' \|\partial_\sigma v_k(0,0)\| = \|\partial_s u_k(z_k')\| = m_k'.$$

This concludes the proof of the lemma. □

Lemma C.6. *Let (v_k) be the sequence constructed in the Rescaling Lemma C.5. Then there exists $v_\infty \in C^0(\mathbb{C}, M)$ such that*

$$v_k \xrightarrow{C^0_{\text{loc}}} v_\infty, \qquad k \to \infty,$$

up to a subsequence.

Proof. By the Rescaling Lemma C.5 and compactness of M there exists $C > 0$ such that

$$\|\partial_\tau v_k\|_\infty \le C \qquad \forall k \in \mathbb{N}.$$

Fix $R > 0$ and choose $K_R \in \mathbb{N}$ such that $R_k \ge R$ for all $k \ge K_R$ and consider the restrictions $v_k|_{B_R(0)}$ for all $k \ge K_R$. As the derivatives of v_k are uniformly bounded, we conclude that $v_k|_{B_R(0)}$ is of Sobolev class

$$W^{1,\infty}(B_R(0)) := \{f \in W^{1,\infty}(B_R(0), \mathbb{R}^{4n+1}) : f(B_R(0)) \subseteq M\}$$

for all $k \ge K_k$ where we consider $M^{2n} \hookrightarrow \mathbb{R}^{4n+1}$ via the Whitney embedding Theorem. Thus by Morrey's inequality [12, Corollary 9.14], every $v_k|_{B_R(0)}$ is of Hölder class $C^{0,1}(B_R(0))$, and hence $v_k|_{B_R(0)}$ is equicontinuous. As M is compact, Ascolis Theorem implies the existence of $v^R \in C^0(B_R(0), M)$ with

$$v_k|_{B_R(0)} \xrightarrow{C^0} v^R, \qquad k \to \infty,$$

up to a subsequence. Choose a subsequence (k_j^1) with $k_j^1 \ge K_1$ for all $j \in \mathbb{N}$ and such that there exists $v^1 \in C^0(B_1(0), M)$ with

$$v_{k_j^1}|_{B_1(0)} \xrightarrow{C^0} v^1, \qquad j \to \infty.$$

Inductively, choose a subsequence $(k_j^{\mu+1})$ of (k_j^{μ}) for all $\mu \in \mathbb{N}$ with $k_j^{\mu+1} \geq K_{\mu+1}$ for all $j \in \mathbb{N}$ and such that there exists $v^{\mu+1} \in C^0(B_{\mu+1}(0), M)$ with

$$v_{k_j^{\mu+1}}|_{B_{\mu+1}(0)} \xrightarrow{C^0} v^{\mu+1}, \qquad j \to \infty.$$

Finally, taking the diagonal subsequence yields

$$v_{k_j^j} \xrightarrow{C^0_{\text{loc}}} v_\infty \in C^0(\mathbb{C}, M), \qquad j \to \infty.$$

□

C.2 Elliptic Bootstrapping

Lemma C.7 (Elliptic Bootstrapping). *Denote by $v_\infty \in C^0(\mathbb{C}, M)$ the map constructed in Lemma C.6. Then $v_\infty \in C^\infty(\mathbb{C}, M)$ is a nonconstant J-holomorphic plane.*

Proof. Fix sequences $(z_j) \subseteq \mathbb{C}$ and $r_j \subseteq (0, +\infty)$ such that

- $v_\infty|_{B_{4r_j}(z_j)}$ is contained in a chart U_j of M for all $j \in \mathbb{N}$.
- $\bigcup_{j\in\mathbb{N}} B_{r_j}(z_j) = \mathbb{C}$.
- for all $j \in \mathbb{N}$ there exists K_j such that $v_k|_{B_{2r_j}(z_j)} \subseteq U_j$ for all $k \geq K_j$.

Fix smooth bump functions $\beta_j \in C^\infty(\mathbb{C}, [0, 1])$ for $\overline{B}_{r_j}(z_j)$ supported in $B_{2r_j}(z_j)$ and define

$$\overline{v}_k^j := \beta_j v_k \in C_c^\infty(\mathbb{C}, \mathbb{R}^{2n})$$

for all $k, j \in \mathbb{N}$. We compute

$$\Delta\overline{v}_k^j = (\Delta\beta_j)v_k + 2\partial_\sigma\beta_j\partial_\sigma v_k + 2\partial_\tau\beta_j\partial_\tau v_k + \beta_j\Delta v_k. \qquad \text{(C.2)}$$

To compute Δv_k, we differentiate (C.1) in charts. Applying ∂_σ we get that

$$\partial_\sigma^2 v_k(\sigma,\tau) + DJ\partial_\sigma v_k(\sigma,\tau)\partial_\tau v_k(\sigma,\tau) + J\partial_\sigma\partial_\tau v_k(\sigma,\tau)$$

is equal to

$$\frac{1}{m_k'} D\,\text{grad}_J\, H_{\frac{\tau}{m_k'}+t_k'}(v_k(\sigma,\tau))\partial_\sigma v_k(\sigma,\tau),$$

where $\text{grad}_J H_t$ denotes the gradient of H_t with respect to the Riemannian metric induced by J, that is, the Riemannian metric $\omega(J\cdot,\cdot)$. Applying ∂_τ to (C.1) we get that

$$\partial_\tau\partial_\sigma v_k(\sigma,\tau) + J\partial_\tau^2 v_k(\sigma,\tau) + DJ(\partial_\tau v_k(\sigma,\tau))^2$$

is equal to

$$\frac{1}{m'_k} D \operatorname{grad}_J H_{\frac{\tau}{m'_k}+t'_k}(v_k(\sigma,\tau))\partial_\tau v_k(\sigma,\tau) + \frac{1}{m'_k}\partial_\tau \operatorname{grad}_J H_{\frac{\tau}{m'_k}+t'_k}(v_k(\sigma,\tau)).$$

Hence

$$\Delta v_k = P(\partial_\sigma v_k, \partial_\tau v_k),$$

where P is a polynomial of degree 2 with C^∞-coefficients and so

$$\Delta \overline{v}_k^j = P(\partial_\sigma \overline{v}_k^j, \partial \overline{v}_k^j)$$

by (C.2). As the derivatives $\partial_\sigma v_k$ and $\partial_\tau v_k$ are uniformly bounded, we conclude that $\Delta \overline{v}_k^j$ is also uniformly bounded. Thus $\|\Delta \overline{v}_k^j\|_{L^p}$ is uniformly bounded for all exponents $p \in [1, +\infty]$ as $\Delta \overline{v}_k^j$ is compactly supported by construction. Therefore, the Calderon–Zygmund inequality [42, Corollary B.2.7] implies that $\|\overline{v}_k^j\|_{W^{2,p}}$ is uniformly bounded for all $1 < p < +\infty$. In particular,

$$\|\partial_\sigma \overline{v}_k^j\|_{W^{1,p}} \qquad \text{and} \qquad \|\partial_\tau \overline{v}_k^j\|_{W^{1,p}}$$

are uniformly bounded. Thus again by Morrey's inequality, $\partial_\sigma \overline{v}_k^j$ and $\partial_\tau \overline{v}_k^j$ belong to the Hölder class $C^{0,\alpha}$ for all $0 < \alpha < 1$. By Ascolis Theorem, $\partial_\sigma \overline{v}_k^j$ and $\partial_\tau \overline{v}_k^j$ admit convergent subsequences and

$$\overline{v}_\infty^j := \beta_j v_\infty \in C_c^1(\mathbb{C}, \mathbb{R}^{2n})$$

satisfies

$$v_k|_{B_{r_j}(z_j)} \xrightarrow{C^1} \overline{v}_\infty^j$$

for all $j \in \mathbb{N}$. Hence we have showed that $v_\infty \in C^1(\mathbb{C}, M)$ and

$$v_k \xrightarrow{C^1_{\text{loc}}} v_\infty$$

up to subsequences. This procedure can be generalised to higher derivatives and is referred to as ***elliptic bootstrapping***. Note that taking higher order derivatives makes only sense locally in a chart if we do not refer to a particular connection. Thus we get $v_\infty \in C^\infty(\mathbb{C}, M)$ and

$$v_k \xrightarrow{C^\infty_{\text{loc}}} v_\infty, \qquad k \to \infty,$$

up to subsequences. Finally, v_∞ is nonconstant as

$$\|\partial_\sigma v_\infty\|_\infty = \lim_{k\to\infty} \|\partial_\sigma v_k\|_\infty \geq 1$$

and satisfies

$$\partial_\sigma v_\infty + J\partial_\tau v_\infty = 0$$

by the Rescaling Lemma C.5. □

C.3 Removal of Singularities

Consider the conformal diffeomorphism

$$\varphi\colon \mathbb{C}\setminus\{0\}\to\mathbb{R}\times\mathbb{T},\qquad \varphi(re^{2\pi i\theta}):=\mathrm{Log}(re^{2\pi i\theta}),$$

and define

$$w\colon \mathbb{R}\times\mathbb{T}\to M,\qquad w:=v_\infty\circ\varphi^{-1},$$

where $v_\infty\in C^\infty(\mathbb{C},M)$ denotes the nonconstant J-holomorphic plane constructed in Lemma C.6. Then

$$\lim_{s\to-\infty} w(s,\cdot)=v_\infty(0)$$

and w is of finite energy. Indeed, we compute

$$E_J(w)=\int_{\mathbb{R}\times\mathbb{T}} w^*\omega=\int_{-\infty}^{+\infty}\int_0^1 \omega(J\partial_s w(s,t),\partial_s w(s,t))dtds\le\sup_{k\in\mathbb{N}} E_J(u_k)$$

by [42, Lemma 2.2.1]. Moreover, w is a negative gradient flow line of the symplectic area functional

$$\mathcal{A}\colon \Lambda M\to\mathbb{R},\qquad \mathcal{A}(\gamma):=\int_{\mathbb{D}}\overline{\gamma}^*\omega.$$

Lemma C.8. *For every sequence $(r_k)\subseteq(0,+\infty)$ with $r_k\to+\infty$ as $k\to\infty$ define a sequence (w_k) by*

$$w_k(s,t):=w(s+r_k,t)\qquad\forall(s,t)\in\mathbb{R}\times\mathbb{T}.$$

Then there exists a point $w_\infty\in M$ such that

$$w_k\xrightarrow{C^\infty_{\mathrm{loc}}} w_\infty,\qquad k\to\infty,$$

up to a subsequence.

Proof. By [42, p. 89–90] there exists a constant $a\ge 0$ such that

$$\Delta e\ge -ae^2,$$

where

$$e\colon \mathbb{R}\times\mathbb{T}\to[0,+\infty),\qquad e(s,t):=\|\partial_s w(s,t)\|^2$$

denotes the energy density. As $E_J(w)<+\infty$, there exists $R>0$ such that

$$\int_R^{+\infty}\int_0^1 e(s,t)dtds\le\min\left\{\frac{\pi}{8a},1\right\}$$

and

$$\int_{-\infty}^{-R}\int_0^1 e(s,t)dtds\le\min\left\{\frac{\pi}{8a},1\right\}.$$

Let $z = (s, t) \in \mathbb{R} \times \mathbb{T}$ such that $|s| \geq R + 1$. Then

$$\int_{B_1(z)} e < \frac{\pi}{8a}.$$

By [42, Lemma 4.3.3] we conclude

$$e(z) \leq \frac{8}{\pi} \int_{B_1(z)} e \leq \frac{8}{\pi}.$$

Consequently, e is uniformly bounded, as $e|_{[-R-1,R+1]\times\mathbb{T}}$ is uniformly bounded by continuity of e. Hence $\|\partial_s w\|$ is uniformly bounded by definition of e and $\|\partial_t w\|$ is uniformly bounded as w is a J-holomorphic curve. By an eliptic bootrstrapping argument as in Lemma C.7, we conclude

$$w_k \xrightarrow{C^\infty_{\text{loc}}} w_\infty, \qquad k \to \infty,$$

up to a subsequence, where w_∞ is a negative gradient flow line of the symplectic area functional $\mathcal{A}$. Then w_∞ is constant. Indeed, assume that w_∞ is not constant. Then there exists $s < s'$ such that

$$\varepsilon := \mathcal{A}(w_\infty(s)) - \mathcal{A}(w_\infty(s')) > 0.$$

Moreover, there exists $K \in \mathbb{N}$ such that

$$\mathcal{A}(w_k(s)) - \mathcal{A}(w_k(s')) = \mathcal{A}(w(s + r_k)) - \mathcal{A}(w(s' + r_k)) \geq \frac{\varepsilon}{2}$$

for all $k \geq K$. Define a subsequence (r_{k_j}) of (r_k) recursively by

$$k_0 := K \qquad \text{and} \qquad k_j := \min\{l \in \mathbb{N} : s' + r_{k_{j-1}} \leq s + r_l\}.$$

This works as $r_k \to +\infty$ as $k \to \infty$. Fix $l \in \mathbb{N}$. Then we compute

$$\begin{aligned} E(w) &= \sup_{s\in\mathbb{R}} \mathcal{A}(w(s)) - \inf_{s\in\mathbb{R}} \mathcal{A}(w(s)) \\ &\geq \mathcal{A}(w(s + r_{k_0})) - \mathcal{A}(w(s + r_{k_l})) \\ &= \sum_{\nu=1}^{l} \big(\mathcal{A}(w(s + r_{k_{\nu-1}})) - \mathcal{A}(w(s + r_{k_\nu}))\big) \\ &\geq \sum_{\nu=1}^{l} \big(\mathcal{A}(w(s + r_{k_{\nu-1}})) - \mathcal{A}(w(s' + r_{k_{\nu-1}}))\big) \\ &\geq \frac{\varepsilon l}{2}. \end{aligned}$$

As $l \in \mathbb{N}$ was arbitrary, we conlcude that $E(w) = +\infty$. □

Proof (of Theorem C.3). As M is compact, there exists a finite open cover $U_1, \dots, U_m$ of M such that $\bar{U}_j$ is contained in a Darboux chart. Define

$$\mathcal{U} := \{\gamma \in \Lambda M : \gamma(\mathbb{T}) \subseteq U_j \text{ for some } j = 1, \dots, m\}.$$

Step 1: There exists $s_0 \in \mathbb{R}$ such that $w(s) \in \mathcal{U}$ for all $s \geq s_0$. Assume that there exists a sequence $(r_k) \subseteq (0, +\infty)$ with $r_k \to +\infty$ as $k \to \infty$ and $w(r_k) \notin \mathcal{U}$ for all $k \in \mathbb{N}$. By Lemma C.8, there exists $w_\infty \in M$ such that

$$w_k \xrightarrow{C^\infty_{\mathrm{loc}}} w_\infty, \qquad k \to \infty,$$

up to a subsequence. But $w_\infty \in U_j$ for some j and thus there exists $K \in \mathbb{N}$ such that $w(r_k, t) \in U_j$ for all $t \in \mathbb{T}$ and $k \geq K$. Consequently, $w(r_k) \in \mathcal{U}$ for all $k \geq K$.

Step 2: There exist constants $C > 0$ and $\kappa > 0$ such that

$$d_{L^2}(w(s), w_\infty) \leq C e^{-\kappa s} \qquad \forall s \geq s_0,$$

where w_∞ is the limit of $w(k) = w_k(0)$ as $k \to \infty$ up to a subsequence (k_j). By [24, Proposition 6.4] and [24, Lemma 6.3], we have the action-energy inequality

$$|\mathcal{A}(\gamma)| \leq C_0 \, \|\mathrm{grad}_J\, \mathcal{A}(\gamma)\|_J^2 \qquad \forall \gamma \in \mathcal{U}$$

for some constant $C_0 > 0$ as the symplectic area functional $\mathcal{A}$ is Morse–Bott. See Remark 2.2. Let $s \geq s_0$. Choose j_0 such that $k_{j_0} > s$. We estimate

$$\mathcal{A}(w(s)) > \mathcal{A}(w(k_{j_0})) \geq \lim_{j \to \infty} \mathcal{A}(w(k_j)) = \lim_{j \to \infty} \mathcal{A}(w_{k_j}(0)) = \mathcal{A}(w_\infty) = 0.$$

Let $s_2 > s_1 > s_0$. Using [24, Lemma 6.5] and [24, Lemma 6.6] we compute

$$\begin{aligned} d(w(s_1), w(s_2)) &\leq \frac{2}{\sqrt{C_0}} \left(\sqrt{\mathcal{A}(w(s_1))} - \sqrt{\mathcal{A}(w(s_2))} \right) \\ &\leq \frac{2}{\sqrt{C_0}} \sqrt{\mathcal{A}(w(s_1))} \\ &\leq \frac{2}{\sqrt{C_0}} \sqrt{\mathcal{A}(w(s_0))} e^{\frac{1}{2C_0}(s_0 - s_1)} \end{aligned}$$

Choose j_1 such that $k_{j_1} \geq s_1$. Then for all $j \geq j_1$, we have that

$$d_{L^2}(w(s_1), w(k_j)) \leq C e^{-\kappa s_1},$$

where

$$C := \frac{2}{\sqrt{C_0}} \sqrt{\mathcal{A}(w(s_0))} e^{\frac{1}{2C_0} s_0} \qquad \text{and} \qquad \kappa := \frac{1}{2C_0}.$$

Thus

$$d_{L^2}(w(s_1), w_\infty) = \lim_{j \to \infty} d_{L^2}(w(s_1), w(k_j)) \leq C e^{-\kappa s_1}.$$

Step 3: There exists a unique point $w_\infty \in M$ such that

$$w(s) \xrightarrow{C^\infty} w_\infty, \qquad s \to +\infty.$$

Uniqueness follows immediately from the exponential decay established in Step 2. Indeed, a priori, w_∞ does depend on the choice of subsequence (k_j). Let $w'_\infty \in M$ be the limit of a different subsequence. Then we compute

$$d_{L^2}(w_\infty, w'_\infty) \le d_{L^2}(w(s), w_\infty) + d_{L^2}(w(s), w'_\infty) \le Ce^{-\kappa s} + C'e^{-\kappa' s} \to 0$$

as $s \to +\infty$. That the limit can also be taken with respect to the C^∞-topology follows from [10, Proposition 6.5.3]. □

Corollary C.9. *Let (M, ω) be a symplectically aspherical symplectic manifold and let (u_k, τ_k) be a sequence of solutions $(u_k, \tau_k) \in C^\infty(\mathbb{R} \times \mathbb{T}, M) \times C^\infty(\mathbb{R}, \mathbb{R})$ of*

$$\begin{cases} \partial_s u_k(s,t) + J(\partial_t u_k(s,t) - \tau_k(s) X_H(u_k(s,t))) = 0, \\ \partial_s \tau_k(s) = \displaystyle\int_0^1 H(u_k(s,t))dt, \end{cases}$$

for all $s \in \mathbb{R}$ and $k \in \mathbb{N}$ for some $H \in C^\infty(M)$, with uniformly bounded energy. If there exists a compact subset $K \subseteq M \times \mathbb{R}$ such that

$$\operatorname{im}(u_k, \tau_k) \subseteq K \qquad \forall k \in \mathbb{N},$$

then the derivatives of (u_k) are uniformly bounded.

Proof. Crucial is the observation that under the above assumptions, the limit of (C.1) is still a J-holomorphic curve. □

Appendix D
M-Polyfolds

The classical approach for establishing generic transversality results in Floer theories is via a suitable version of the Sard–Smale Theorem [42, Theorem A.5.1]. The idea is to represent the moduli space of negative gradient flow lines as the zero set of an appropriate Fredholm section. Unfortunately, this does not work for the moduli space of unparametrised negative gradient flow lines as the reparametrisation action is not smooth. Moreover, the transversality results usually require perturbing the given metric to a generic one. There is a more abstract approach for proving transversality results via polyfold theory. This theory was and is still developed by Hofer–Wysocki–Zehnder [33] primarily having symplectic field theory in mind. Another more algebraic approach to abstract perturbations is via Kuranishi structures developed by Fukaya–Oh–Ohta–Ono [27].

In the first section we introduce the basic terminology of polyfold theory, namely the notion of scale smoothness on scale Banach spaces.

In the second section we formulate a prototypical result for Morse–Bott homology following the brilliant lecture notes [15].

D.1 Scale Calculus

Definition D.1 (Scale Structure, [33, Definition 1.1.1]). A ***scale structure*** on a Banach space E is a decreasing sequence

$$E =: E_0 \supseteq E_1 \supseteq E_2 \supseteq \dots$$

of Banach spaces such that the inclusion $E_{k+1} \hookrightarrow E_k$ is compact for every $k \in \mathbb{N}_0$ and such that $E_\infty := \cap_{k=0}^{\infty} E_k$ is dense in every E_k. A Banach space E together with a scale structure (E_k) is called a ***scale Banach space***.

Example D.2 (Shifted Scale Banach Space, [26, Definition 3.3]). Let $(E, (E_k))$ be a scale Banach space and $m \in \mathbb{N}_0$. Then $\left(E^k, (E_k^m)\right)$ is a scale Banach space where $E_k^m := E_{m+k}$ for all $k \in \mathbb{N}_0$.

Example D.3 (Scale Direct Sum, [26, Definition 3.4]). Let $(E, (E_k))$ and $(F, (F_k))$ be scale Banach spaces. Then $(E \oplus F, (E_k \oplus F_k))$ is also a scale Banach space.

The following example underlies Morse and Floer theory.

Example D.4 (Weighted Sobolev Spaces, [26, Example 3.9]). Fix a monotone cutoff function $\beta \in C^\infty(\mathbb{R}, [-1, 1])$ with

$$\beta(s) = \begin{cases} 1 & s \geq 1, \\ -1 & s \leq -1, \end{cases}$$

and $\delta > 0$. Define

$$\gamma_\delta \colon \mathbb{R} \to \mathbb{R}, \qquad \gamma_\delta(s) := e^{\delta\beta(s)s}.$$

For $p \in (1, +\infty)$ and $k \in \mathbb{N}_0$ define the ***Sobolev spaces with weight*** $\boldsymbol{\delta}$ by

$$W^{k,p}_\delta(\mathbb{R}, \mathbb{R}^n) := \{u \in W^{k,p}(\mathbb{R}, \mathbb{R}^n) : \gamma_\delta u \in W^{k,p}(\mathbb{R}, \mathbb{R}^n)\}.$$

Choose a strictly increasing sequence (δ_k) with $\delta_0 = 0$. Then $(E, (E_k))$ is a scale Banach space with

$$E_k := W^{k,p}_{\delta_k}(\mathbb{R}, \mathbb{R}^n) \qquad \forall k \in \mathbb{N}_0.$$

Definition D.5 (Scale Continuity, [26, Definition 4.1]). Let $(E, (E_k))$ and $(F, (F_k))$ be two scale Banach spaces. A map $f : E \to F$ is called ***scale continuous***, iff $f(E_k) \subseteq F_k$ for all $k \in \mathbb{N}_0$ and the restriction $f|_{E_k} : E_k \to F_k$ is continuous.

Definition D.6 (Tangent Bundle, [33, Definition 1.1.14]). For a scale Banach space $(E, (E_k))$ define its ***tangent bundle*** by $TE := E^1 \oplus E$.

Definition D.7 (Scale Differentiability, [26, Definition 4.2]). A scale continuous map $f : E \to F$ between scale Banach spaces $(E, (E_k))$ and $(F, (F_k))$ is called ***scale differentiable***, iff for every $x \in E_1$ there exists a bounded linear operator

$$Df(x) \colon E_0 \to F_0,$$

called the ***scale differential of*** $\boldsymbol{f}$, such that the restriction $f|_{E_1} : E_1 \to F_0$ is Fréchet differentiable with derivative $Df|_{E_1}$ and the ***tangent map***

$$Tf : TE \to TF, \qquad Tf(x, h) := (f(x), Df(x)h)$$

is scale continuous.

Using the iterated notion of scale differentiability one can define higher scale regularity. We say that a scale differentiable map is ***scale smooth***, iff its tangent map is infinitely scale differentiable. The following example motivated the development of scale calculus.

Example D.8 (Weighted Sobolev Spaces, [26, Theorem 6.2]). Let E_k be the scale of weighted Sobolev spaces introduced in Example D.4 on the scale Banach space $E = L^p(\mathbb{R}, \mathbb{R}^n)$. Then the shift map

$$\mathbb{R} \oplus E \to E, \qquad (r, x) \mapsto x(\cdot + r)$$

is scale smooth.

One important property of scale differentiability is that the chain rule remains valid in this general setting.

Proposition D.9 (Chain Rule, [33, Theorem 1.3.1]). *Uppose that $f : E \to F$ and $g : F \to G$ are scale differentiable maps for scale Banach spaces E, F and G. Then the composition $g \circ f$ is scale differentiable with tangent map*

$$T(g \circ f) = Tg \circ Tf : TE \to TG.$$

The proof of the chain rule heavily relies on the compactness of the embeddings of the scales. Using the chain rule one is able to define the notion of scale manifolds and scale differential geometry in analogy to the finite-dimensional case. For details see [15, Chapter 5]. Thus with the theory developed so far, one can make sense of the smooth moduli space of unparametrised negative gradient flow lines. However, for broken negative gradient flow lines one needs an even more general notion, including scale manifolds.

Definition D.10 (Retraction, [33, Definition 2.1.1]). Let E be scale Banach space. A ***retraction on E*** is a scale smooth map $r : E \to E$ such that $r^2 = r$.

Remark D.11. If X is a smooth Banach manifold, then $\mathrm{Fix}(r) = r(M)$ is a smooth submanifold for every smooth retraction $r : X \to X$ by a result of Cartan [33, Proposition 2.1.2].

It is in general not true that the fixed point set of a scale smooth retraction of a scale manifold is a scale submanifold. This lead Hofer–Wysocki–Zehnder to the generalised notion of an ***M-polyfold***, where the "M" stands for "manifold flavoured". Heuristically, an M-polyfold is locally the fixed point set of a scale smooth retraction of a scale smooth manifold. The main aspect for us is that Fredholm theory still is valid in M-polyfolds in some sense. For an extensive treatment see [33, Part I].

D.2 M-Polyfold Setup for Morse–Bott Homology

In this section we explain how an M-polyfold Fredholm setup for Morse–Bott homology can be defined. We follow [15, Section 8.4]. The essential arguments are contained in [22, Appendix A]. We assume the following setup. Let (M, g) be a compact Riemannian manifold and $f \in C^\infty(M)$ a Morse–Bott function. Choose an additional Morse function $h \in C^\infty(\mathrm{Crit}\, f)$ and a Riemannian metric g_0 on Crit f such that (h, g_0) is a Morse–Smale pair, that is, the stable and unstable manifolds intersect transversally. Pick two connected critical components $C^\pm \subseteq \mathrm{Crit}\, f$. For

$$0 < \delta < \min\{|\lambda| : \lambda \in \sigma(\mathrm{Hess}_x\, f)) \setminus \{0\}, x \in \mathrm{Crit}\, f\}$$

and a positive strictly increasing sequence $0 < \delta_0 < \delta_1 < \cdots < \delta$, consider the Banach manifold $\widetilde{\mathcal{E}}_k(C^-, C^+)$ consisting of all maps $u \in H^{k+2}_{\delta_k}(\mathbb{R}, M)$ converging exponentially to $C^\pm$, that is, there exist $x^\pm \in C^\pm$ as well as

$$\xi^- \in H^{k+2}_{\delta_k}((-\infty, -T]\,, T_{x^-}M) \quad \text{and} \quad \xi^+ \in H^{k+2}_{\delta_k}([T, +\infty)\,, T_{x^+}M)$$

for some $T \in \mathbb{R}$ with

$$u(s) = \exp_{x^\pm}(\xi^\pm(s)) \qquad \forall \pm s \geq T.$$

See Figure D.1. Local charts on $\widetilde{\mathcal{E}}_k(C^-, C^+)$ are constructed by exponential neighbourhoods around smooth paths [49, Appendix A]. Similar to [9], one can construct scale smooth charts using those exponential neighbourhoods, equipping the Banach manifold $\widetilde{\mathcal{E}}(C_-, C_+) := \widetilde{\mathcal{E}}_0(C_-, C_+)$ with a scale smooth structure. Moreover, there are two natural scale smooth evaluation maps

$$\mathrm{ev}^\pm \colon \widetilde{\mathcal{E}}(C^-, C^+) \to \mathrm{Crit}\, f, \qquad \mathrm{ev}^\pm(u) = x^\pm.$$

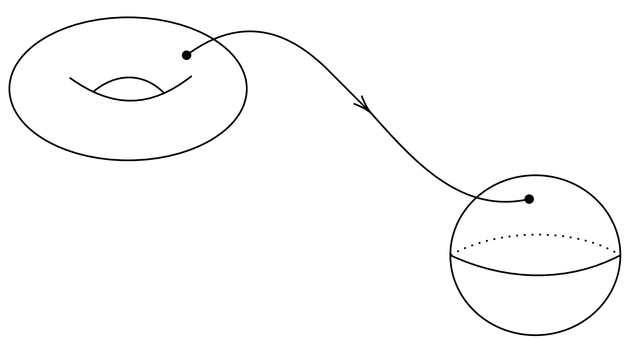

Fig. D.1: An asymptotically exponential Sobolev path connecting two critical components of a Morse–Bott function.

Fix $x^\pm \in C^\pm \cap \mathrm{Crit}\, h$ and define the stable and unstable manifolds by

$$W^\pm(x^\pm) := \left\{ x \in \mathrm{Crit}\, f : \lim_{s \to \pm\infty} \phi_s(x) = x^\pm \right\},$$

where $\phi\colon \mathbb{R} \times \operatorname{Crit} f \to \operatorname{Crit} f$ denotes the negative gradient flow of h with respect to the Morse–Smale metric m_0. Define

$$\widetilde{\mathcal{E}}^1_k(x^-, x^+) := \{u \in \widetilde{\mathcal{E}}_k(C^-, C^+) : \mathrm{ev}^\pm(u) \in W^\pm(x^\pm)\}$$

for all $k \in \mathbb{N}$. See Figure D.2. Again, $\widetilde{\mathcal{E}}^1(x_-, x_+) := \widetilde{\mathcal{E}}^1_0(x_-, x_+)$ is a scale manifold. For details, see the proof of [22, Theorem A.12] and [22, Theorem A.14].

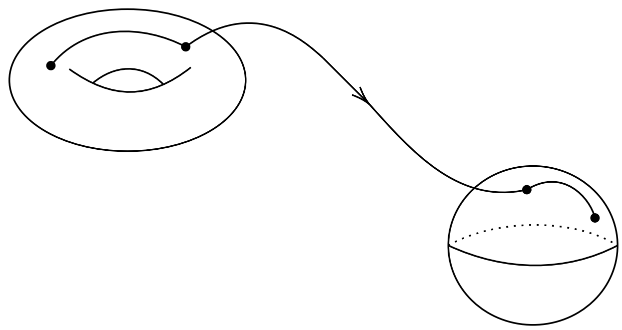

Fig. D.2: A weighted Sobolev path with one cascade.

There exists a canonical scale smooth bundle

$$\widetilde{\mathcal{F}}^1(x^-, x^+) \to \widetilde{\mathcal{E}}^1(x^-, x^+),$$

where the fibre over $u \in \widetilde{\mathcal{E}}^1_k(C^-, C^+)$ is given by

$$\widetilde{\mathcal{F}}^1_k(x^-, x^+)|_u = H^{k+1}_{\delta_k}(\mathbb{R}, u^*TM)$$

for all $k \in \mathbb{N}$. Then

$$\widetilde{\partial}\colon \widetilde{\mathcal{E}}^1(x^-, x^+) \to \widetilde{\mathcal{F}}^1(x^-, x^+), \qquad \widetilde{\partial}(u) := \dot{u} + \operatorname{grad}_m f(u)$$

is a scale Fredholm operator. If $x^- \neq x^+$, the scale smooth reparametrisation action

$$\mathbb{R} \oplus \widetilde{\mathcal{E}}^1(x^-, x^+) \to \widetilde{\mathcal{E}}^1(x^-, x^+), \qquad (r, u) \mapsto u(\cdot + r),$$

as in Example D.8 is free, giving rise to a scale smooth bundle

$$\widetilde{\mathcal{F}}^1(x^-, x^+)/\mathbb{R} \to \widetilde{\mathcal{E}}^1(x^-, x^+)/\mathbb{R}$$

to which $\widetilde{\partial}$ descends to a scale Fredholm section ∂. This construction can be generalised to m cascades and thus yields the scale smooth bundle

$$\mathcal{F}^m(x^-, x^+) \to \mathcal{E}^m(x^-, x^+)$$

of unparametrised paths from x^- to x^+ with m cascades. Now we consider broken paths. For $x^0, \dots, x^l \in \operatorname{Crit} h$ with

$$f(x^0) \geq f(x^1) \geq \cdots \geq f(x^{l-1}) \geq f(x^l),$$

we define the space of broken paths from x^0 to x^l by

$$\mathcal{E}(x^0, \dots, x^l) := \coprod_{(\alpha_0, \dots, \alpha_l) \in \mathbb{N}_0^{l+1}} \mathcal{E}^{\alpha_0}(x^0, x^1) \times \cdots \times \mathcal{E}^{\alpha_l}(x^{l-1}, x^l).$$

Then $\mathcal{E}(x^0, \dots, x^l)$ is a scale manifold over which we have the scale bundle

$$\mathcal{F}(x^0, \dots, x^l) := \coprod_{(\alpha_0, \dots, \alpha_l) \in \mathbb{N}_0^{l+1}} \mathcal{F}^{\alpha_0}(x^0, x^1) \times \cdots \times \mathcal{F}^{\alpha_l}(x^{l-1}, x^l)$$

and the scale Fredholm section

$$\partial \times \cdots \times \partial \colon \mathcal{E}(x^0, \dots, x^l) \to \mathcal{F}(x^0, \dots, x^l).$$

For $x^\pm \in \operatorname{Crit} h$ with $f(x^-) \geq f(x^+)$ we define the space of broken paths from x^- to x^+ by

$$X(x^-, x^+) := \coprod_{l \in \mathbb{N}} \coprod_{\substack{x^1, \dots, x^{l-1} \in \operatorname{Crit} h \\ f(x^-) \geq f(x^1) \geq \cdots \geq f(x^{l-1}) \geq f(x^+)}} \mathcal{E}(x^-, x^1, \dots, x^{l-1}, x^+).$$

Similarly, one defines

$$Y(x^-, x^+) := \coprod_{l \in \mathbb{N}} \coprod_{\substack{x^1, \dots, x^{l-1} \in \operatorname{Crit} h \\ f(x^-) \geq f(x^1) \geq \cdots \geq f(x^{l-1}) \geq f(x^+)}} \mathcal{F}(x^-, x^1, \dots, x^{l-1}, x^+).$$

One can show that $X(x^-, x^+)$ carries the natural structure of an M-polyfold and

$$Y(x^-, x^+) \to X(x^-, x^+)$$

is a bundle. Moreover, the scale Fredholm section above induces a scale Fredholm section

$$\partial \colon X(x^-, x^+) \to Y(x^-, x^+)$$

of index

$$\operatorname{ind} \partial = \operatorname{ind}(x^-) - \operatorname{ind}(x^+) - 1,$$

where either

$$\operatorname{ind} = \operatorname{ind}_f + \operatorname{ind}_h \qquad \text{or} \qquad \operatorname{ind} = -\frac{1}{2} \operatorname{sgn} \operatorname{Hess} f - \frac{1}{2} \operatorname{sgn} \operatorname{Hess} h.$$

For details see [17, Appendix A]. One can show that the level set $\partial^{-1}(0)$ of unparametrised broken negative gradient flow lines is compact and that there exists an abstract perturbation of ∂, that is, a scale smooth section $\sigma\colon X \to Y$, where

$$X := \coprod_{\substack{x^{\pm}\in \operatorname{Crit} h\\ f(x^-)\geq f(x^+)}} X(x^-, x^+) \qquad \text{and} \qquad Y := \coprod_{\substack{x^{\pm}\in \operatorname{Crit} h\\ f(x^-)\geq f(x^+)}} Y(x^-, x^+),$$

with the property that for all $x^{\pm} \in \operatorname{Crit} h$ the moduli space

$$\mathcal{M}(x^-, x^+) := \{u \in X(x^-, x^+) : (\partial + \sigma)(u) = 0\}$$

is compact and $\partial+\sigma$ is transverse to the zero section. Hence we arrive at a prototypical result for Morse–Bott homology, compare [15, Corollary 8.9].

Theorem D.12 (M-Polyfold Setup for Morse–Bott Homology). *Let (M, g) be a compact Riemannian manifold without boundary and $f \in C^{\infty}(M)$ a Morse–Bott function. Choose an additional Morse function $h \in C^{\infty}(\operatorname{Crit} f)$ and an additional Riemannian metric g_0 on $\operatorname{Crit} f$ such that (h, g_0) is a Morse–Smale pair. For all critical points $x^{\pm} \in \operatorname{Crit} h$, the moduli space*

$$\mathcal{M}(x^-, x^+) := \{u \in X(x^-, x^+) : (\partial + \sigma)(u) = 0\}$$

is a smooth compact manifold with corners of dimension

$$\dim \mathcal{M}(x^-, x^+) = \operatorname{ind}(x^-) - \operatorname{ind}(x^+) - 1.$$

Moreover, there is a canoncial diffeomorphism

$$\partial\mathcal{M}(x^-, x^+) = \mathcal{M}(x^-, x^+) \cap \partial X \cong \coprod_{x\in \operatorname{Crit} h} \mathcal{M}(x^-, x) \times \mathcal{M}(x, x^+).$$

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