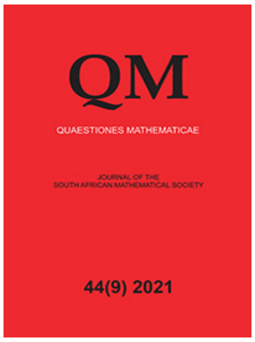

Quaestiones Mathematicae

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/tqma20

Equivalent almost periodic functions in terms of the new property of almost equality

Juan Matías Sepulcre & Tomás Vidal

To cite this article: Juan Matías Sepulcre & Tomás Vidal (2022): Equivalent almost periodic functions in terms of the new property of almost equality, Quaestiones Mathematicae, DOI: 10.2989/16073606.2021.2012291

To link to this article: https://doi.org/10.2989/16073606.2021.2012291

Published online: 16 Jan 2022.

Quaestiones Mathematicae 2021: 1–14.
© 2021 NISC (Pty) Ltd
https://doi.org/10.2989/16073606.2021.2012291

This is the final version of the article that is
published ahead of the print and online issue

EQUIVALENT ALMOST PERIODIC FUNCTIONS IN TERMS OF THE NEW PROPERTY OF ALMOST EQUALITY

Juan Matías Sepulcre*

Department of Mathematics, University of Alicante, P.O. Box 99, 03080 Alicante, Spain.
E-Mail JM.Sepulcre@ua.es

Tomás Vidal

Universidad Nacional de Educación a Distancia (UNED), CL. Candalix s/n,
03202 Elche, Alicante, Spain.
E-Mail tmvg@alu.ua.es

Abstract. In this paper we introduce the notion of almost equality (or, more specifically, almost equality by translations) of complex functions of an unrestricted real variable in terms of the new concept of ε-translation number of a function with respect to other one, which is inspired by Bohr's notion of ε-translation number associated with an almost periodic function. We develop the main properties of this new class of functions and obtain a characterization through a very important equivalence relation which we introduced in previous papers in the context of the almost periodicity.

Mathematics Subject Classification (2020): Primary: 42A75, 46Axx; Secondary: 42A16, 43A60, 42Axx.

Key words: Almost equal functions, almost periodic functions, *-equivalence, SV-equivalence, almost equality by translations, ε-translation numbers.

1. Introduction. The theory of almost periodic functions was introduced in its main features by H. Bohr during the 1920's and it shortly acquired numerous applications to various areas of mathematics, from harmonic analysis to differential equations. Some important references on the basic ingredients of this theory are [2, 3, 4, 5, 6, 7, 9, 10].

We emphasize that the definition given by Harald Bohr of an almost periodic function is based upon two properly generalized concepts: the periodicity to the so-called almost periodicity, and the periodic distribution of periods to the so-called relative density of ε-almost periods or ε-translation numbers. Specifically, let $f:\mathbb{R}\mapsto\mathbb{C}$ be a complex function of an unrestricted real variable, the notion of *almost periodicity* involves the fact that $f(x)$ must be continuous, and for every $\varepsilon>0$ there corresponds a number $l=l(\varepsilon)>0$ such that any interval of length l

*Corresponding author.

contains a number τ satisfying

$$\sup\{|f(x+\tau)-f(x)| : x \in \mathbb{R}\} \leq \varepsilon \tag{1}$$

(or also $|f(x+\tau)-f(x)| < \varepsilon$ for all x). In fact, a number τ satisfying (1) is called an $\varepsilon-$*almost period* or a ε-*translation number* of the function f. We will denote as $AP(\mathbb{R},\mathbb{C})$ the space of almost periodic functions in the sense of this definition. As in classical Fourier analysis, every function in $AP(\mathbb{R},\mathbb{C})$ is bounded and is associated with a Fourier series with real frequencies.

In this framework, we established in 2018 a new equivalence relation on the classes of almost periodic functions (and more generally in the context of the functions which can be represented by certain exponential sums, as in the case of the Besicovitch space $B(\mathbb{R},\mathbb{C})$) which led to refining Bochner's result that characterizes these spaces of functions and to an extension of Bohr's equivalence theorem (see for instance [11, 12, 13, 14, 15, 16]). This equivalence relation, which we will here call *-equivalence or SV-equivalence, is defined in the following terms:

DEFINITION 1.1. Given an arbitrary countable set $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_j, \ldots\}$ of distinct real numbers (which we will call exponents or frequencies), consider $A_1(p)$ and $A_2(p)$ two exponential sums of the form

$$A_1(p) = \sum_{j\geq 1} a_j e^{\lambda_j p} \quad \text{and} \quad A_2(p) = \sum_{j\geq 1} b_j e^{\lambda_j p}, \text{ with } a_j, b_j \in \mathbb{C},\ \lambda_j \in \Lambda,$$

where p is a parameter (which could also adopt the form ip, with i the imaginary unit). We will say that A_1 is *-*equivalent* to A_2 (or A_1 is *SV-equivalent* to A_2) if for each integer value $n \geq 1$, with $n \leq \sharp\Lambda$ ($\sharp\Lambda$ denotes the cardinal of the set Λ), there exists a $\mathbb{Q}$-linear map $\psi_n : V_n \to \mathbb{R}$, where V_n is the $\mathbb{Q}$-vector space generated by $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, such that

$$b_j = a_j e^{i\psi_n(\lambda_j)},\ j = 1, \ldots, n.$$

We will write $A_1 \overset{*}{\sim} A_2$.

Also, let f_1 and f_2 be two almost periodic functions in $AP(\mathbb{R},\mathbb{C})$ whose associated Fourier series have the same set of frequencies or exponents. We will say that f_1 is *-*equivalent* to f_2 (or f_1 is *SV-equivalent* to f_2) when their associated Fourier series satisfy the previous definition.

In comparison with the concept of ε-translation number and almost periodicity, in this paper we introduce the notion of almost equality by translations (or simply almost equality) which is defined in terms of the relative density of the set of ε-translation numbers of a function with respect to other one (see Definitions 2.1 and 2.4).

We develop the main basic properties of this new class of functions (see for example Lemma 2.6 and Proposition 4.4). In particular, we show that the almost equality connects the property of uniform continuity and almost periodicity of the underlying functions (see Lemma 4.1). In fact, under the assumption of the continuity of at least one of the involving functions, we show that the almost equal functions are uniformly continuous and bounded on $\mathbb{R}$ (see Proposition 4.2) and we demonstrate that the properties of SV-equivalence and almost equality are equivalent, which is our main result (see Theorem 4.6). This means that the SV-equivalence in $AP(\mathbb{R},\mathbb{C})$ can be characterized in terms which are more practical than those of Definition 1.1 (in fact, in terms which are similar to those used by Bohr to define the almost periodicity). Finally, as a consequence of our main result, we prove the *-equivalence of the product by scalars, the conjugates or the powers of *-equivalent functions in $AP(\mathbb{R},\mathbb{C})$ (see Corollary 4.8) or the *-equivalence of the uniform limits of two sequences formed by functions in $AP(\mathbb{R},\mathbb{C})$ which are *-equivalent pairwise (see Corollary 4.10).

2. Definition of ε-translation numbers and almost equal functions. Inspired by the previous research work, we next consider a condition which is an adaptation of the concept of an ε-translation number (or an ε-almost period) associated with a unique function.

DEFINITION 2.1. (ε-translation number of f_2 with respect to f_1) Consider $f_1, f_2 : \mathbb{R} \mapsto \mathbb{C}$ to be two complex functions, and consider $\varepsilon > 0$. A real number τ satisfying

$$\sup_{x\in\mathbb{R}} |f_2(x+\tau) - f_1(x)| \leq \varepsilon \tag{2}$$

will be called an ε-translation number of f_2 with respect to f_1.

The existence of a relatively dense set of ε-translation numbers of f_2 with respect to f_1 would yield the existence of a positive number $l > 0$ such that all the intervals in $\mathbb{R}$ of length $l > 0$ contain at least one real value τ satisfying (2).

REMARK 2.2. Note that τ is an ε-translation number of f_2 with respect to f_1 if and only if $-\tau$ is an ε-translation number of f_1 with respect to f_2. Indeed, if we take $t = x + \tau$, it is clear that

$$\sup_{x\in\mathbb{R}} |f_2(x+\tau) - f_1(x)| = \sup_{t\in\mathbb{R}} |f_1(t-\tau) - f_2(t)|.$$

REMARK 2.3. Fix $c \in \mathbb{C} \setminus \{0\}$. If τ is an ε-translation number of f_2 with respect to f_1, then it is clear that:

i) τ is also an $\varepsilon \cdot |c|$-translation number of cf_2 with respect to cf_1.

ii) τ is also an ε-translation number of $\overline{f_2}$ with respect to $\overline{f_1}$ (the conjugate functions).

For the next key definition in this paper, we recall that a set $\mathcal{T}$ of real numbers is said to be *relatively dense* if there exists $l > 0$ such that any interval $(\alpha, \alpha + l)$ intersects with $\mathcal{T}$.

DEFINITION 2.4. (Almost equal functions) Let $f_1, f_2 : \mathbb{R} \mapsto \mathbb{C}$ be two complex functions. We will say that f_1 and f_2 are almost equal by translations (or, simply, almost equal) if to every positive number ε there corresponds a relatively dense set of ε-translation numbers of f_2 with respect to f_1 (or, equivalently, a relatively dense set of ε-translation numbers of f_1 with respect to f_2).

Remark 2.5. The notion of almost equality between two complex functions f_1 and f_2 is well defined in view of the property of symmetry motivated by Remark 2.2. That is, f_1 is almost equal to f_2 if and only if f_2 is almost equal to f_1. That is why we will say that f_1 and f_2 are almost equal.

Moreover, the property of transitivity also holds in the sense that if f_1 and f_2 are almost equal and if f_2 and f_3 are also almost equal, then f_1 and f_3 are almost equal. In fact, if τ_1 is an $\frac{\varepsilon}{2}$-translation number of f_2 with respect to f_1 and if τ_2 is an $\frac{\varepsilon}{2}$-translation number of f_3 with respect to f_2, then $\tau_1+\tau_2$ is an ε-translation number of f_3 with respect to f_1. Indeed, we have

$$\begin{aligned}&\sup_{x\in\mathbb{R}}|f_3(x+\tau_1+\tau_2)-f_1(x)|\\ &\qquad\le \sup_{x\in\mathbb{R}}|f_3(x+\tau_1+\tau_2)-f_2(x+\tau_1)|+\sup_{x\in\mathbb{R}}|f_2(x+\tau_1)-f_1(x)|.\end{aligned}$$

Finally, it is worth noting that the reflexivity holds when we handle complex functions for which to every $\varepsilon>0$ there corresponds a relatively dense set of ε-almost periods. Hence $f\in AP(\mathbb{R},\mathbb{C})$ is almost equal to itself in virtue of Bohr's notion of almost periodicity. This means that, under this hypothesis of existence of relatively dense sets of ε-almost periods, the notion of almost equality could also be treated as an equivalence relation.

The following basic result will be useful for the next sections of our paper.

Lemma 2.6. *Let $f_1,f_2:\mathbb{R}\mapsto\mathbb{C}$ be two almost equal functions. Then it is satisfied that:*

i) *If $f_2(x)$ is continuous (on the whole $\mathbb{R}$), then $f_1(x)$ is bounded (on the whole $\mathbb{R}$).*

ii) *If $f_1(x)$ is continuous (on the whole $\mathbb{R}$), then $f_2(x)$ is bounded (on the whole $\mathbb{R}$).*

Proof. Fix $\varepsilon=1$. By hypothesis, we know that every interval of a certain length $l>0$ contains at least one real value τ satisfying (6) or, equivalently, it satisfies the condition $\sup_{x\in\mathbb{R}}|f_1(x-\tau)-f_2(x)|<1$.

i) Suppose first that $f_2(x)$ is continuous on $\mathbb{R}$, and denote by M the maximum of $|f_2(x)|$ in the interval $[0,l]$. If $x\in\mathbb{R}$, we can assure the existence of $\tau_x\in[-x,-x+l)$ satisfying (6), which yields that $x+\tau_x\in[0,l)$ and consequently $|f_2(x+\tau_x)|\le M$. Hence

$$|f_1(x)|\le|f_1(x)-f_2(x+\tau_x)|+|f_2(x+\tau_x)|\le 1+M,$$

for all values of x.

ii) If we suppose second that $f_1(x)$ is continuous on $\mathbb{R}$, then we deduce by symmetry that the function $f_2(x)$ is bounded. In fact, if we denote by K the maximum of $|f_1(x)|$ in the interval $[0,l]$, then any $x\in\mathbb{R}$ can be associated with a value $\tau_x\in(x-l,x]$ satisfying $\sup_{x\in\mathbb{R}}|f_1(x-\tau_x)-f_2(x)|<1$ and with $x-\tau_x\in[0,l)$. Consequently $|f_1(x-\tau)|\le K$ and

$$|f_2(x)|\le|f_2(x)-f_1(x-\tau_x)|+|f_1(x-\tau_x)|\le 1+K,$$

for all values of x. □

3. First connection between *-equivalent almost periodic functions and almost equal functions. We next show that the *-equivalence of almost periodic functions leads to the property of almost equality.

LEMMA 3.1. *Let $f_1, f_2 : \mathbb{R} \mapsto \mathbb{C}$ be two complex functions such that at least one of them is in $AP(\mathbb{R},\mathbb{C})$ and $f_1 \overset{*}{\sim} f_2$. Then f_1 and f_2 are almost equal functions.*

Proof. Suppose first that $f_2 \in AP(\mathbb{R},\mathbb{C})$. Then f_2 is associated with a Fourier series with some set Λ of frequencies or exponents. Let $\mathcal{F}_\Lambda$ be the class of functions in $AP(\mathbb{R},\mathbb{C})$ whose Fourier exponents coincide with Λ, and let $\mathcal{G}$ be the equivalence class in $\mathcal{F}_\Lambda/\overset{*}{\sim}$ such that $f_2 \in \mathcal{G}$. Also, consider $f_1 \overset{*}{\sim} f_2$. Since the set of translates of f_2 is dense in $\mathcal{G}$ (see [11, Theorem 2 and Corollary 3]), there exists an increasing unbounded sequence $\{\delta_n\}_{n\geq 1}$ of positive numbers such that the sequence of functions $\{f_2(x+\delta_n)\}_{n\geq 1}$ converges uniformly on $\mathbb{R}$ to $f_1(x)$. Equivalently, given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$|f_2(x+\delta_n) - f_1(x)| < \varepsilon/2 \ \ \forall n \geq n_0, \ \forall x \in \mathbb{R}.$$

Moreover, since $f_2(x)$ is in $AP(\mathbb{R},\mathbb{C})$, there exists $l = l(\varepsilon) > 0$ such that any interval $(a, a+l)$ contains a number γ satisfying $|f_2(x+\gamma) - f_2(x)| < \varepsilon/2$ $\forall x \in \mathbb{R}$. Hence any interval $(a, a+l)$ contains a number γ satisfying

$$\begin{aligned}&|f_2(x+\delta_n+\gamma) - f_1(x)| \\ &\quad \leq |f_2(x+\delta_n+\gamma) - f_2(x+\delta_n)| + |f_2(x+\delta_n) - f_1(x)| < \varepsilon \ \forall n \geq n_0, \ \forall x \in \mathbb{R},\end{aligned}$$

which proves the existence of a relatively dense set of positive numbers τ (of the form $\gamma + \delta_n$, with $n \geq n_0$) such that

$$\sup_{x\in\mathbb{R}} |f_2(x+\tau) - f_1(x)| \leq \varepsilon.$$

In the case that $f_1 \in AP(\mathbb{R},\mathbb{C})$ and $f_2 \overset{*}{\sim} f_1$, we analogously prove the existence of an increasing unbounded sequence $\{\delta_n\}_{n\geq 1}$ of positive numbers such that the sequence of functions $\{f_1(x+\delta_n)\}_{n\geq 1}$ converges uniformly on $\mathbb{R}$ to $f_2(x)$. This means that, given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$|f_1(x+\delta_n) - f_2(x)| < \varepsilon/2 \ \ \forall n \geq n_0, \ \forall x \in \mathbb{R}.$$

Moreover, since $f_1(x)$ is in $AP(\mathbb{R},\mathbb{C})$, there exists $l = l(\varepsilon) > 0$ such that any interval $(a, a+l)$ contains a number γ satisfying $|f_1(x+\gamma) - f_1(x)| < \varepsilon/2$ $\forall x \in \mathbb{R}$. Hence any interval $(a, a+l)$ contains a number γ satisfying

$$\begin{aligned}&|f_1(x+\delta_n+\gamma) - f_2(x)| \\ &\quad \leq |f_1(x+\delta_n+\gamma) - f_1(x+\delta_n)| + |f_1(x+\delta_n) - f_2(x)| < \varepsilon \ \forall n \geq n_0, \ \forall x \in \mathbb{R},\end{aligned}$$

which proves the existence of a relatively dense set of positive numbers τ (of the form $\gamma+\delta_n$, with $n \geq n_0$) such that

$$\sup_{x\in\mathbb{R}} |f_1(x+\tau)-f_2(x)| \leq \varepsilon,$$

i.e. τ is an ε-translation number of f_1 with respect to f_2 or, equivalently, $-\tau$ is an ε-translation number of f_2 with respect to f_1. This proves the result. □

We next prove that the converse of the lemma above is also true, which leads to a characterization of the property of almost equality (and the *-equivalence) under the hypothesis that at least one of the functions is in $AP(\mathbb{R},\mathbb{C})$ (we will improve this result in Theorem 4.6).

THEOREM 3.2. *Let $f_1, f_2 : \mathbb{R} \mapsto \mathbb{C}$ be two complex functions such that at least one of them is in $AP(\mathbb{R},\mathbb{C})$. Then f_1 is *-equivalent to f_2 if and only if f_1 and f_2 are almost equal.*

Proof. If $f_1 \overset{*}{\sim} f_2$ and at least one of them is in $AP(\mathbb{R},\mathbb{C})$, the direct implication is a consequence of Lemma 3.1.
Conversely, suppose first that $f_2 \in AP(\mathbb{R},\mathbb{C})$, and let $\mathcal{G}$ be the equivalence class in $\mathcal{F}_\Lambda/\overset{*}{\sim}$ such that $f_2 \in \mathcal{G}$, where $\mathcal{F}_\Lambda$ is the class of functions in $AP(\mathbb{R},\mathbb{C})$ whose Fourier exponents are given by a set Λ. Also, suppose the existence of a relatively dense set of real numbers τ satisfying

$$\sup_{x\in\mathbb{R}} |f_2(x+\tau)-f_1(x)| \leq \varepsilon.$$

Hence we can extract a sequence $\{\tau_n\}_n$ such that $\{g_n(x) := f_2(x+\tau_n)\}_n$ tends to $f_1(x)$ with respect to the topology of the uniform convergence on $\mathbb{R}$. This means that f_1 is a limit point of the set of translates of f_2 and, consequently, $f_1 \in \mathcal{G}$ (in particular, it is in $AP(\mathbb{R},\mathbb{C})$), i.e. $f_1 \overset{*}{\sim} f_2$ (see [11, Corollary 2]).
In the case that $f_1 \in AP(\mathbb{R},\mathbb{C})$, the condition to every $\varepsilon > 0$ there corresponds a relatively dense set of real numbers τ satisfying

$$\sup_{x\in\mathbb{R}} |f_2(x+\tau)-f_1(x)| \leq \varepsilon$$

is equivalent to

$$\sup_{t\in\mathbb{R}} |f_1(t-\tau)-f_2(t)| \leq \varepsilon,$$

which also leads to the result in an analogous way. □

4. Main characterization and properties of the almost equal functions.

As we next show, the almost equality connects the property of uniform continuity and almost periodicity of the underlying functions.

LEMMA 4.1. *Let $f_1, f_2 : \mathbb{R} \mapsto \mathbb{C}$ be two almost equal functions. Then it is satisfied that:*

i) *f_1 is uniformly continuous on $\mathbb{R}$ if and only if f_2 is uniformly continuous on $\mathbb{R}$.*

ii) *$f_1 \in AP(\mathbb{R}, \mathbb{C})$ if and only if $f_2 \in AP(\mathbb{R}, \mathbb{C})$.*

Proof. i) Fix $\varepsilon > 0$. By hypothesis, there corresponds a relatively dense set of real numbers τ satisfying

$$\sup_{x\in\mathbb{R}} |f_2(x+\tau) - f_1(x)| \leq \frac{\varepsilon}{3}. \tag{3}$$

If f_2 is uniformly continuous on $\mathbb{R}$, then so is f_1. Indeed, given $\varepsilon > 0$, there exists $\delta_2 > 0$ such that $|f_2(x) - f_2(y)| < \frac{\varepsilon}{3}$ if $|x - y| < \delta_2$. In this way, if $x, y \in \mathbb{R}$ with $|x - y| < \delta_2$ then

$$|f_1(x)-f_1(y)| \leq |f_1(x)-f_2(x+\tau)|+|f_2(x+\tau)-f_2(y+\tau)|+|f_2(y+\tau)-f_1(y)| < \varepsilon, \tag{4}$$

where τ is a real number satisfying (3). Conversely, suppose that f_1 is uniformly continuous on $\mathbb{R}$, which yields the existence of $\delta_1 > 0$ such that $|f_1(x) - f_1(y)| < \frac{\varepsilon}{3}$ if $|x - y| < \delta_1$ (for a prefixed $\varepsilon > 0$). In this way, if $x, y \in \mathbb{R}$ with $|x - y| < \delta_1$ then

$$|f_2(x)-f_2(y)| \leq |f_2(x)-f_1(x-\tau)|+|f_1(x-\tau)-f_1(y-\tau)|+|f_1(y-\tau)-f_2(y)| < \varepsilon, \tag{5}$$

where τ is a real number satisfying $\sup_{t\in\mathbb{R}} |f_1(t-\tau)-f_2(t)| \leq \varepsilon$, which is equivalent to the condition (3).

ii) If $f_1 \in AP(\mathbb{R}, \mathbb{C})$, then Theorem 3.2 shows that $f_1 \overset{*}{\sim} f_2$, which yields that $f_2 \in AP(\mathbb{R}, \mathbb{C})$ (see also [11, Lemma 2]). The converse is analogous. □

We next show that the continuity of one of the two functions which are almost equal leads to the uniform continuity and boundedness of both of them.

PROPOSITION 4.2. *Let $f_1, f_2 : \mathbb{R} \mapsto \mathbb{C}$ be two almost equal functions. If at least one of the functions $f_1(x)$ or $f_2(x)$ is continuous (on $\mathbb{R}$), then $f_1(x)$ and $f_2(x)$ are uniformly continuous and bounded (on $\mathbb{R}$).*

Proof. Suppose first that $f_2(x)$ is continuous (on the whole $\mathbb{R}$), and fix $\varepsilon > 0$. Let $l > 0$ be the positive number for which every interval in $\mathbb{R}$ of length l contains at least one $\frac{\varepsilon}{3}$-translation number of f_2 with respect to f_1. We know that $f_2(x)$ is uniformly continuous on the compact set $[-1, 1 + l]$, which means that there is a positive number δ (which we assume to be less than 1) such that $|f_2(x_1)-f_2(y_1)| < \frac{\varepsilon}{3}$ whenever $|x_1 - y_1| < \delta$ with $x_1, y_1 \in [-1, 1+l]$. Let now x and y be any two real numbers such that $|x - y| < \delta$ (that is, $-\delta < x - y < \delta$). Thus we can assure the existence of a real value $\tau \in [-x, -x + l)$ satisfying $\sup_{x\in\mathbb{R}} |f_2(x+\tau) - f_1(x)| \leq \frac{\varepsilon}{3}$. Moreover, it is also satisfied $y+\tau \in [-x+y, -x+y+l)$, which yields $-1 < -\delta < y+\tau < \delta+l < 1+l$. Hence the real values $x+\tau$ and $y+\tau$ are in the interval $[-1, 1+l]$ and we have $|f_2(x+\tau) - f_2(y+\tau)| < \frac{\varepsilon}{3}$. Consequently,

$$|f_1(x)-f_1(y)| \leq |f_1(x)-f_2(x+\tau)|+|f_2(x+\tau)-f_2(y+\tau)|+|f_2(y+\tau)-f_1(y)| \leq 3\frac{\varepsilon}{3} = \varepsilon.$$

Suppose second that $f_1(x)$ is continuous (on $\mathbb{R}$). Then we deduce by symmetry that the function $f_2(x)$ is uniformly continuous. Indeed, by repeating the same argument as above, we know that $f_1(x)$ is also uniformly continuous on the compact set $[-1, 1+l]$, i.e., given $\varepsilon > 0$, there is a positive number δ (which we assume to be less than 1) such that $|f_1(x_1) - f_1(y_1)| < \frac{\varepsilon}{3}$ whenever $|x_1 - y_1| < \delta$ with $x_1, y_1 \in [-1, 1+l]$. Let now x and y be any two real numbers such that $|x-y| < \delta$ (that is, $-\delta < x - y < \delta$). Thus we can assure the existence of a real value $\tau \in (x-l, x]$ (that is, $-\tau \in [-x, -x+l)$) satisfying $\sup_{x\in\mathbb{R}} |f_2(x) - f_1(x-\tau)| \le \frac{\varepsilon}{3}$. Moreover, it is also satisfied $x - \tau \in [0, l]$ and $y - \tau \in [-x+y, -x+y+l)$, which yields $-1 < -\delta < y - \tau < \delta + l < 1 + l$. Hence the real values $x - \tau$ and $y - \tau$ are in the interval $[-1, 1+l]$ and we have $|f_1(x-\tau) - f_1(y-\tau)| < \frac{\varepsilon}{3}$. Consequently,

$$|f_2(x)-f_2(y)| \le |f_2(x)-f_1(x-\tau)|+|f_1(x-\tau)-f_1(y-\tau)|+|f_1(y-\tau)-f_2(y)| \le 3\frac{\varepsilon}{3} = \varepsilon.$$

Finally, we deduce from Lemma 4.1, point i), that both functions $f_1(x)$ and $f_2(x)$ are uniformly continuous (on $\mathbb{R}$). Moreover, by Lemma 2.6, they are also bounded (on the whole $\mathbb{R}$). □

As a consequence of the proposition above, we can prove that the set of the ε-translation numbers of a function with respect to the another one contains intervals of a certain length centered at every point which is an ε_1-translation number with $0 < \varepsilon_1 < \varepsilon$.

COROLLARY 4.3. *Let $f_1, f_2 : \mathbb{R} \mapsto \mathbb{C}$ be two almost equal functions, and suppose that at least one of them is continuous (on $\mathbb{R}$). Given $\varepsilon_2 > 0$, consider a positive number $\varepsilon_1 < \varepsilon_2$. Then for any ε_1-translation number of f_2 with respect to f_1, say τ_{ε_1}, there exists $\delta_{\varepsilon_1,\varepsilon_2} > 0$ (which only depends on ε_1 and ε_2) such that every point in the open interval $(\tau_{\varepsilon_1} - \delta_{\varepsilon_1,\varepsilon_2}, \tau_{\varepsilon_1} + \delta_{\varepsilon_1,\varepsilon_2})$ is an ε_2-translation number of f_2 with respect to f_1.*

Proof. By Proposition 4.2, we know that $f_1(x)$ and $f_2(x)$ are uniformly continuous on the whole $\mathbb{R}$. In particular, given $\varepsilon = \varepsilon_2 - \varepsilon_1 > 0$ there exists $\delta > 0$ such that $|f_2(x) - f_2(y)| < \varepsilon$ whenever $|x - y| < \delta$ with $x, y \in \mathbb{R}$. Now, consider a real number τ satisfying $\sup_{x\in\mathbb{R}} |f_2(x+\tau) - f_1(x)| \le \varepsilon_1$. If $\eta \in \mathbb{R}$ is so that $|\eta - \tau| < \delta$ (that is, $\eta \in (-\delta + \tau, \delta + \tau)$), then

$$\begin{aligned}&\sup_{x\in\mathbb{R}} |f_2(x+\eta) - f_1(x)| \\ &\quad \le \sup_{x\in\mathbb{R}} |f_2(x+\eta) - f_2(x+\tau)| + \sup_{x\in\mathbb{R}} |f_2(x+\tau) - f_1(x)| \le \varepsilon + \varepsilon_1 = \varepsilon_2.\end{aligned}$$

This means that η is an ε_2-translation number of f_2 with respect to f_1. □

Some basic properties of the set of almost equal functions are now immediately deduced from Remark 2.3 and Proposition 4.2.

PROPOSITION 4.4. *Let $f_1, f_2 : \mathbb{R} \mapsto \mathbb{C}$ be two almost equal functions. Then it is satisfied that:*

i) *Given $c \in \mathbb{C}$, the functions cf_1 and cf_2 are almost equal.*

ii) *The functions $\overline{f_1}$ and $\overline{f_2}$ are almost equal.*

iii) *Let $k \in \mathbb{N}$. If at least one of the functions $f_1(x)$ or $f_2(x)$ is continuous (on $\mathbb{R}$), then the functions f_1^k and f_2^k are almost equal.*

iv) *If $\inf\{|f_1(x)| : x \in \mathbb{R}\} > 0$ and $\inf\{|f_2(x)| : x \in \mathbb{R}\} > 0$, then the functions $\frac{1}{f_1}$ and $\frac{1}{f_2}$ are almost equal.*

Proof. The two first statements are a consequence of Remark 2.3.
The property iii) derives from Proposition 4.2 and the fact that, given $\varepsilon > 0$, the real values τ which are ε-translation numbers of f_2 with respect to f_1 satisfy

$$\begin{aligned}
&\sup_{x\in\mathbb{R}} |f_2^k(x+\tau) - f_1^k(x)| \\
&= \sup_{x\in\mathbb{R}} |f_2(x+\tau)^{k-1} + f_2(x+\tau)^{k-2} f_1(x) + \dots \\
&\qquad \dots + f_2(x+\tau) f_1(x)^{k-2} + f_1(x)^{k-1}||f_2(x+\tau) - f_1(x)| \\
&\le (M_2^{k-1} + M_1 M_2^{k-2} + \dots + M_1^{k-2} M_2 + M_1^{k-1})\varepsilon,
\end{aligned}$$

where $M_1 = \sup\{|f_1(x)| : x \in \mathbb{R}\} < \infty$ and $M_2 = \sup\{|f_2(x)| : x \in \mathbb{R}\} < \infty$ (see also Proposition 4.2).
Finally, concerning the property iv), we note that the real values τ which are ε-translation numbers of f_2 with respect to f_1 satisfy

$$\sup_{x\in\mathbb{R}} \left| \frac{1}{f_2(x+\tau)} - \frac{1}{f_1(x)} \right| = \sup_{x\in\mathbb{R}} \left| \frac{f_1(x) - f_2(x+\tau)}{f_1(x) f_2(x+\tau)} \right| \le \frac{\varepsilon}{m_1 m_2},$$

where $m_1 = \inf\{|f_1(x)| : x \in \mathbb{R}\} > 0$ and $m_2 = \inf\{|f_2(x)| : x \in \mathbb{R}\} > 0$. This proves that the set of ε-translation numbers of $\frac{1}{f_2}$ with respect to $\frac{1}{f_1}$ is relatively dense. □

PROPOSITION 4.5. *Let $\{f_k\}_{k\ge1}$ and $\{g_k\}_{k\ge1}$, with $f_k, g_k : \mathbb{R} \mapsto \mathbb{C}$, be two sequences of complex functions converging uniformly on $\mathbb{R}$ to $f(x)$ and $g(x)$, respectively. If the functions f_k and g_k are almost equal for each $k \ge 1$, then f and g are also almost equal.*

Proof. By hypothesis, given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $|f(x) - f_n(x)| < \frac{\varepsilon}{3}$ and $|g(x) - g_n(x)| < \frac{\varepsilon}{3}$ for each $n \ge n_0$ and all $x \in \mathbb{R}$. Now, let τ be a real number satisfying

$$\sup_{x\in\mathbb{R}} |f_{n_0}(x+\tau) - g_{n_0}(x)| \le \frac{\varepsilon}{3}.$$

Thus

$$\begin{aligned}
\sup_{x\in\mathbb{R}} |f(x+\tau) - g(x)| &\le \sup_{x\in\mathbb{R}} |f(x+\tau) - f_{n_0}(x+\tau)| + \\
&\quad + \sup_{x\in\mathbb{R}} |f_{n_0}(x+\tau) - g_{n_0}(x)| + \\
&\quad + \sup_{x\in\mathbb{R}} |g_{n_0}(x) - g(x)| \\
&\le 3\frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

This proves the result. □

Under very weak conditions on the underlying functions, we next improve Theorem 3.2 by obtaining a characterization of the almost equality of two complex functions through the *-equivalence in $AP(\mathbb{R},\mathbb{C})$. Therefore, at the same time, we obtain a characterization of the *-equivalence in $AP(\mathbb{R},\mathbb{C})$ in terms of the property of almost equality of two complex functions (or, equivalently, through the set of ε-translation numbers of a function with respect to the another one, i.e. in terms which are similar to Bohr's notion of almost periodicity).

Theorem 4.6. (Main result) *Let $f_1, f_2 : \mathbb{R} \mapsto \mathbb{C}$ be two complex functions such that at least one of them is continuous. Then f_1 is *-equivalent to f_2, with $f_1, f_2 \in AP(\mathbb{R},\mathbb{C})$, if and only if f_1 and f_2 are almost equal.*

Proof. The direct implication is a consequence of Lemma 3.1.
Conversely, suppose that to every $\varepsilon > 0$ there corresponds a relatively dense set of real numbers τ satisfying

$$\sup_{x\in\mathbb{R}} |f_2(x+\tau) - f_1(x)| \leq \frac{\varepsilon}{3}, \tag{6}$$

which yields that any interval of a certain length $l > 0$ contains at least one real value τ satisfying (6). By Proposition 4.2, we know that both functions $f_1(x)$ and $f_2(x)$ are uniformly continuous on $\mathbb{R}$, i.e., given $\varepsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that $|f_1(x) - f_1(y)| < \frac{\varepsilon}{3}$ if $|x-y| < 2\delta_1$ and $|f_2(x) - f_2(y)| < \frac{\varepsilon}{3}$ if $|x-y| < 2\delta_2$. Now, consider a sequence of real numbers $\{h_j\}_{j\geq 1}$. Since any interval $(h_j - l, h_j)$ contains a real value τ_j satisfying (6), then every h_j can be represented in the form $h_j = \tau_j + r_j$, where $0 \leq r_j \leq l$. In this way, by Bolzano-Weierstrass theorem, take r a limit point of the bounded set $\{r_1, r_2, \ldots\}$. Now, consider the set of all $h_j = \tau_j + r_j$ for which $r - \delta_1 < r_j < r + \delta_1$ (which form a subsequence of the initial sequence $\{h_j\}_{j\geq 1}$). If h_j and h_k are two such values, then $|r_j - r_k| < 2\delta_2$ and

$$\begin{aligned}
&\sup\{|f_2(x+h_j) - f_2(x+h_k)| : x \in \mathbb{R}\} \\
&= \sup\{|f_2(t+h_j-h_k) - f_2(t)| : t \in \mathbb{R}\} \\
&= \sup\{|f_2(t+r_j-r_k+\tau_j-\tau_k) - f_2(t)| : t \in \mathbb{R}\} \\
&\leq \sup\{|f_2(t+r_j-r_k+\tau_j-\tau_k) - f_2(t+\tau_j-\tau_k)| : t \in \mathbb{R}\} \\
&\qquad\qquad + \sup\{|f_2(t+\tau_j-\tau_k) - f_2(t)| : t \in \mathbb{R}\} \\
&\leq \frac{\varepsilon}{3} + \sup\{|f_2(t+\tau_j-\tau_k) - f_1(t-\tau_k)| : t \in \mathbb{R}\} + \sup\{|f_1(t-\tau_k) - f_2(t)| : t \in \mathbb{R}\} \\
&\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Consequently, given any sequence of real numbers $\{h_j\}_{j\geq 1}$, it is clear that there exists a subsequence $\{h_{j_k}\}_{k\geq 1}$ such that the sequence of functions $\{f_2(x+h_{j_k})\}_{k\geq 1}$ is uniformly convergent on $\mathbb{R}$ (see also the proof of [2, p. 11, 3°. Theorem]). This means that f_2 satisfies the property of normality and hence it is in $AP(\mathbb{R},\mathbb{C})$. Finally, the fact that f_1 is *-equivalent to f_2 is a consequence of Theorem 3.2. □

REMARK 4.7. It is worth noting that if $f_1, f_2 : \mathbb{R} \mapsto \mathbb{C}$ are two complex functions such that at least one of them is continuous, then the converse of our main result states that the almost equality of both functions yields particularly that f_1 and f_2 are almost periodic, i.e. $f_1, f_2 \in AP(\mathbb{R}, \mathbb{C})$ (and hence both functions are also uniformly continuous and bounded). Furthermore, under the same requirements, f_1 and f_2 are also *-equivalent in $AP(\mathbb{R}, \mathbb{C})$.

As a consequence of Theorem 4.6, we can directly prove the following result.

COROLLARY 4.8. *Let $f_1, f_2 : \mathbb{R} \mapsto \mathbb{C}$ be two complex functions. If f_1 is *-equivalent to f_2, with $f_1, f_2 \in AP(\mathbb{R}, \mathbb{C})$, then*

i) *cf_1 is *-equivalent to cf_2 for any $c \in \mathbb{C}$.*

ii) *$\overline{f_1}$ is *-equivalent to $\overline{f_2}$.*

iii) *f_1^k is *-equivalent to f_2^k for each $k \in \mathbb{N}$.*

iv) *If $\inf\{|f_1(x)| : x \in \mathbb{R}\} > 0$ and $\inf\{|f_2(x)| : x \in \mathbb{R}\} > 0$, then f_1^k is *-equivalent to f_2^k for each $k \in \mathbb{Z}$.*

Proof. By hypothesis, we know that both functions f_1 and f_2 are in $AP(\mathbb{R}, \mathbb{C})$, which yields that they (and also the powers f_1^k and f_2^k) are continuous. Hence the result is a direct consequence of Theorem 4.6 and Proposition 4.4. □

For the following remark, we recall that it was demonstrated in [11, Proposition 1' (mod.)] or [12, Proposition 1] that the *-equivalence can be characterized in terms of a basis $G_\Lambda = \{g_1, g_2, \ldots, g_k, \ldots\}$ of the $\mathbb{Q}$-vector space generated by a set $\Lambda = \{\lambda_1, \lambda_2, \ldots\}$ of exponents, i.e. G_Λ is linearly independent over the rational numbers and each λ_j is expressible as a finite linear combination of terms of G_Λ, say

$$\lambda_j = \sum_{k=1}^{i_j} r_{j,k} g_k, \text{ for some } r_{j,k} \in \mathbb{Q},\ i_j \in \mathbb{N}. \tag{7}$$

Indeed, two exponential sums $A_1(p) = \sum_{j\geq 1} a_j e^{\lambda_j p}$ and $A_2(p) = \sum_{j\geq 1} b_j e^{\lambda_j p}$ are *-equivalent if and only if for each integer value $n \geq 1$, with $n \leq \sharp\Lambda$, there exists $\mathbf{x}_n = (x_{n,1}, x_{n,2}, \ldots, x_{n,k}, \ldots) \in \mathbb{R}^{\sharp G_\Lambda}$ such that $b_j = a_j e^{<\mathbf{r}_j, \mathbf{x}_n>i}$ for $j = 1, 2, \ldots, n$, where $\mathbf{r}_j \in \mathbb{R}^{\sharp G_\Lambda}$ is the vector of rational components satisfying (7). In particular, note that the modulus of the coefficients a_j and b_j of *-equivalent exponential sums are equal pairwise.

REMARK 4.9. If f_1 is *-equivalent to f_2 and g_1 is *-equivalent to g_2, with $f_1, f_2, g_1, g_2 \in AP(\mathbb{R}, \mathbb{C})$, then it is not necessarily true that $f_1 \cdot g_1$ is *-equivalent to $f_2 \cdot g_2$ (and the same for the sum, i.e. $f_1 + g_1$ is not necessarily *-equivalent to $f_2 + g_2$). For example, consider the complex functions

$$f_1(x) = e^{ix \ln 2} + e^{ix \ln 3} + e^{ix \ln 4} \text{ and } f_2(x) = -e^{ix \ln 2} + e^{ix \ln 3} + e^{ix \ln 4}.$$

Then f_1 and f_2 are *-equivalent (indeed, take $\Lambda = \{\ln 2, \ln 3, \ln 4\}$, $G_\Lambda = \{\ln 2, \ln 3\}$, $\mathbf{r}_1 = (1,0)$, $\mathbf{r}_2 = (0,1)$, $\mathbf{r}_3 = (2,0)$ and $\mathbf{x}_0 = (\pi, 2\pi)$ in the characterization above). Also, consider the complex functions

$$g_1(x) = g_2(x) = -e^{ix\ln 2} - e^{ix\ln 3} + e^{ix\ln 4},$$

which are of course *-equivalent. However, it is clear that

$$(f_1 \cdot g_1)(x) = -e^{ix\ln 4} - 2e^{ix\ln 6} - e^{ix\ln 9} + e^{ix\ln 16}$$

is not *-equivalent to

$$(f_2 \cdot g_2)(x) = e^{ix\ln 4} - 2e^{ix\ln 8} - e^{ix\ln 9} + e^{ix\ln 16}.$$

Moreover, it is also clear that

$$(f_1 + g_1)(x) = 2e^{ix\ln 4}$$

is not *-equivalent to

$$(f_2 + g_2)(x) = -2e^{ix\ln 2} + 2e^{ix\ln 4}.$$

Finally, the following result is a direct consequence of Theorem 4.6 and Proposition 4.5 (recall that the functions in $AP(\mathbb{R}, \mathbb{C})$ are uniformly continuous).

COROLLARY 4.10. *Let $\{f_k\}_{k\geq 1}$ and $\{g_k\}_{k\geq 1}$, with $f_k, g_k : \mathbb{R} \mapsto \mathbb{C}$, be two sequences of complex functions converging uniformly on $\mathbb{R}$ to $f(x)$ and $g(x)$, respectively. Suppose that f_k is *-equivalent to g_k, with $f_k, g_k \in AP(\mathbb{R}, \mathbb{C})$, for each $k \in \mathbb{N}$. Then f is *-equivalent to g.*

REMARK 4.11. (Some extensions) The main definitions and results in this paper have been formulated for the case of complex functions of an unrestricted real variable and with respect to the topology of the uniform convergence (which leads to the spaces of Bohr's almost periodic functions $AP(\mathbb{R}, \mathbb{C})$). That is why we use the absolute value (or the modulus) in the initial definition of ε-translation number of a function with respect to other one (see Definition 2.1). However, we could go further by generalizing this development to the context of several spaces of locally integrable maps from $\mathbb{R}$ to $\mathbb{C}$ satisfying certain characteristics. In particular, this extension can be formulated for the case of the spaces of almost periodic functions in Stepanov's sense $S^p(\mathbb{R}, \mathbb{C})$, $1 \leq p < \infty$, where we must replace the supremum or uniform norm by the norm

$$\|f\|_{S^p} := \sup \left\{ \left(\int_r^{r+1} |f(t)|^p \, dt \right)^{1/p} : r \in \mathbb{R} \right\},$$

or, more generally, to the context of almost periodic functions in Weyl's sense $W^p(\mathbb{R}, \mathbb{C})$ ($\supset S^p(\mathbb{R}, \mathbb{C})$), $1 \leq p < \infty$, where it is considered the seminorm

$$\|f\|_{W^p} := \lim_{l\to\infty} \sup \left\{ \left(\frac{1}{l} \int_r^{r+l} |f(t)|^p \, dt \right)^{1/p} : r \in \mathbb{R} \right\}.$$

Furthermore, another extension can be made to the context of the almost periodic functions in Besicovitch's sense $B^p(\mathbb{R},\mathbb{C})$, $1 \leq p < \infty$, where it is considered the seminorm

$$\|f\|_{B^p} := \left(\limsup_{l\to\infty} (2l)^{-1} \int_{-l}^{l} |f(t)|^p \, dt \right)^{1/p},$$

and, in addition to an extra condition, the property of relative density is replaced by that of a satisfactorily uniform set of real numbers, which means that there exists $l > 0$ such that the ratio of the maximum number of terms in this set which are included in an interval of length l to the minimum number is less than 2. In particular, the space $B(\mathbb{R},\mathbb{C})$ contains all spaces of almost periodic functions in Bohr's, Stepanov's, Weyl's and Besicovitch's sense. See for instance [1, 15] for more details and properties concerning these spaces of functions.

REMARK 4.12. (Final remark) During the refereeing process, a reviewer informed us of the connection between our Definition 2.4 and the notion of ρ-almost periodicity for continuous functions introduced in the ArXiv paper [8, Definition 2.1] shortly after the submission of this work (with the choice $I' = I = \mathbb{R}$, $\mathcal{B} = \{X\}$, $Y = R(f_2)$ and $\rho : R(f_2) \mapsto R(f_1)$ defined as $\rho(f_2(x)) := f_1(x)$ for all $x \in \mathbb{R}$, where $R(f_j) = \{y \in \mathbb{C} : \exists x \in \mathbb{R}$ such that $y = f_j(x)\}$, $j = 1, 2)$. Moreover, the reviewer indicated us that [8, Proposition 2.2] could also be used to prove the first statement of Remark 4.7.

Acknowledgements. The first author was supported by PGC2018-097960-B-C22 (MCIU/AEI/ERDF, UE).

REFERENCES

1. J. ANDRES, A.M. BERSANI, AND A.M. GRANDE, Hierarchy of almost-periodic function spaces, *Rend. Mat. Appl.* **26** (2006), 121–188.
2. A.S. BESICOVITCH, *Almost periodic functions,* Dover, New York, 1954.
3. S. BOCHNER, A new approach to almost periodicity, *Proc. Nat. Acad. Sci.* **48** (1962), 2039–2043.
4. H. BOHR, *Almost periodic functions,* Chelsea, New York, 1951.
5. C. CORDUNEANU, *Almost periodic functions,* Interscience publishers, New York/London/Sydney/Toronto, 1968.
6. ———, *Almost periodic oscillations and waves,* Springer, New York, 2009.
7. T. DIAGANA, *Almost automorphic type and almost periodic type functions in abstract spaces,* Springer-Verlag, New York, 2013.
8. M. FEČKAN, M.T. KHALLADI, M. KOSTIĆ, AND A. RAHMANI, Multi-dimensional ρ-almost periodic type functions and applications, submitted. arXiv:2109.10223.
9. M. KOSTIĆ, *Almost periodic and almost automorphic type solutions to integro-differential equations,* W. de Gruyter, Berlin, 2019.

10. M. Levitan, *Almost periodic functions,* G.I.T.T.L., Moscow, 1959 (in Russian).

11. J.M. Sepulcre and T. Vidal, Almost periodic functions in terms of Bohr's equivalence relation, *Ramanujan J.* **46**(1) (2018), 245–267; Corrigendum, *ibid*, **48**(3) (2019), 685–690.

12. ________________ , Bohr's equivalence relation in the space of Besicovitch almost periodic functions, *Ramanujan J.* **49**(3) (2019), 625–639.

13. ________________ , A generalization of Bohr's equivalence theorem, *Complex Anal. Oper. Theory* **13**(4) (2019), 1975–1988.

14. ________________ , Sets of values of equivalent almost periodic functions, *Ramanujan J.* **56**(1) (2021), 87–102.

15. ________________ , Bochner-type property on spaces of generalized almost periodic functions, *Mediterr. J. Math.* **17** (2021), DOI: 10.1007/s00009-020-01628-x.

16. ________________ , On the real projections of zeros of analytic almost periodic functions, *Carpathian J. Math.,* to appear, 2022.

Received 22 August, 2021 and in revised form 29 October, 2021.