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Harris-type results on geometric and subgeometric convergence to equilibrium for stochastic semigroups



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ABSTRACT

We provide simple and constructive proofs of Harris-type theorems on the existence and uniqueness of an equilibrium and the speed of equilibration of discrete-time and continuous-time stochastic semigroups. Our results apply both to cases where the relaxation speed is exponential (also called geometric) and to those with no spectral gap, with non-exponential speeds (also called subgeometric). We give constructive estimates in the subgeometric case and discrete-time statements which seem both to be new. The method of proof also differs from previous works, based on semigroup and interpolation arguments, valid for both geometric and subgeometric cases with essentially the same ideas. In particular, we present very simple new proofs of the geometric case.

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1. Introduction

Aim of the work The study of convergence to equilibrium of continuous or discrete semigroups which preserve mass and positivity is central in the theory of Markov processes and partial differential equations (PDEs). Several results ensuring geometric (i.e. exponential) or subgeometric (for instance, polynomial) convergence to equilibrium in weighted total variation norms for a broad family of processes are known as *Harris-type theorems* (or also sometimes as *Meyn-Tweedie-type theorems*). They have been widely developed during the last three decades and have seen a broad range of applications to probability and PDEs problems.

Harris-type theorems concern the trajectories of this kind of semigroups. In both the geometric and subgeometric cases, they establish the existence of an equilibrium (often called stationary state or invariant measure depending on the context) and a speed of convergence of trajectories to it. The main assumptions of this type of theorems are

(i) a *strong positivity, irreducibility or coupling* condition,

as well as

(ii) a *confinement or Foster-Lyapunov* condition.

The latter condition determines whether the speed of convergence is geometric or subgeometric.

Our purpose in this paper is to establish some theorems of this type in both the geometric and the subgeometric situations using elementary semigroup tools, and avoiding some usual probabilistic arguments such as estimates of the return time of a process to a given set.

Let us describe our results a bit more precisely by considering the typical case of a discrete stochastic process associated to an operator S , which must be positive and mass-preserving, defined on a space of measures with a weighted total variation norm.

Under both conditions (i) and (ii), we will be able to exhibit two convenient norms $\|\cdot\|$, $\|\cdot\|_*$ and a scalar $\alpha > 0$ such that

$$\|S\nu\| \leq \|\nu\| - \alpha\|\nu\|_*, \quad (1.1)$$

for any measure ν with vanishing mass. The strict contraction estimate (1.1), or a variant of it, is then used in order to prove the existence of a positive equilibrium μ^* with unit mass associated to the operator S such that $\|\mu^*\|_* < \infty$. The same estimate can also be used in order to prove the uniqueness of this equilibrium under slightly stronger assumptions, for example that S is of Feller-type. When $\|\cdot\|_*$ is equivalent to $\|\cdot\|$, which holds true when the confinement condition (ii) is strong enough, then one easily deduces from (1.1) a geometric convergence of the sequence $(S^n\nu)$ to 0 for any ν with vanishing mass. Under weaker confinement conditions, the norm $\|\cdot\|_*$ is strictly dominated by $\|\cdot\|$ and only a subgeometric convergence of the sequence $(S^n\nu)$ to 0 is established. A version of these ideas for continuous semigroups will be also deduced from the analysis of the discrete case.

Our approach is inspired by the proof of Harris' result in the geometric case by Hairer and Mattingly [22], which uses mass transportation metrics. Our proof is a simplification of these ideas which avoids the use of mass transportation arguments, and can be adapted to the subgeometric case as well. Our result gives an alternative proof to the geometric decay estimate of Hairer and Mattingly [22] and can be adapted to give subgeometric decay rates for discrete semigroups under weaker confinement conditions. In the continuous-time case we can recover similar subgeometric decay rates as in Douc et al. [15], Hairer [20]. We emphasize that the statements apply to mass-preserving semigroups and give explicit versions for them, since this is a common setting in PDE.

Previous contributions The ergodicity and stability theory of Markov processes has been widely developed since the pioneering works of Doblin [13] and Harris [23], the last one giving name to these results, though it considers only the existence of equilibrium and does not mention speed of convergence towards it. A good exposition of this type of results is given in Meyn and Tweedie [31], and a nice introduction in the setting of Markov chains can be found in Stroock [32, Chapter 2].

An important development of the theory is due to Meyn and Tweedie [27,28,29,30]. A simplified statement and a proof using mass transportation distances was given by Hairer and Mattingly [22], motivated by the application to stochastic PDEs [21]. Recent related results for non-conservative semigroups have been reported by Bansaye et al. [2], and applications to models for the electrical activity of groups of neurons can be found in Dumont and Gabriel [16], Cañizo and Yoldaş [8]. Recent works dealing with applications to Fokker-Planck equations and related models are due to Hu and Wang [24], Eberle et al. [17], Cao [11] and Lafleche [26]. We also mention applications to the study of hypocoercivity for kinetic equations and fragmentation-type equations (Cañizo et al. [10, 9]).

On the other hand, this type of theorems has been extended in several works to the case of semigroups with no spectral gap, for which the speed of convergence to equilibrium is subgeometric (slower than exponential). Probabilistic results of this kind can be found in Tuominen and Tweedie [33], Douc et al. [14]. We highlight Douc et al. [15], which deals with the continuous-time case and serves as a model for the type of results we wish to obtain in the present paper. An exposition of this same result which also uses probabilistic arguments can be found in unpublished notes by Hairer [20], see also Bernou [4]. Subgeometric convergence rates have also been studied for classical models as the Fokker-Planck equation, and the Boltzmann equation and its relatives; for this we refer to Kaviani et al. [25], Carrapatoso and Mischler [12] and the references therein, and the classical papers by Caffisch [7,6]. We also mention the recent works by Bernou and Fournier [5] and Bernou [3], where convergence to equilibrium for a collisionless model of a gas is investigated, the last one using techniques related to the present paper.

Definitions and notation We fix a measurable space (Ω, \mathcal{E}) throughout. We denote by \mathcal{M} the set of finite signed measures on Ω , and by \mathcal{P} the set of probability measures on Ω . We also call \mathcal{N} the linear subspace of \mathcal{M} consisting of zero mean measures (that is, $\nu \in \mathcal{N}$ if $\nu \in \mathcal{M}$ and $\nu(\Omega) = 0$).

We usually denote by $\int f \mu$ the integral of a function f with respect to a measure $\mu \in \mathcal{M}$, omitting the domain of integration Ω , and preferring this notation to the also common $\int f d\mu$. The positive and negative parts of a measure $\mu \in \mathcal{M}$ (with the usual Hahn-Jordan decomposition) are denoted respectively by μ_+ , μ_- , so that $\mu = \mu_+ - \mu_-$ and $|\mu| := \mu_+ + \mu_-$. The total variation norm of a measure $\mu \in \mathcal{M}$ is denoted by $\|\mu\| := \int |\mu|$.

A *stochastic operator* is a linear operator $S: \mathcal{M} \rightarrow \mathcal{M}$ which leaves \mathcal{P} invariant (that is, a linear operator which preserves mass and positivity). A *stochastic semigroup* is a family $(S_t)_{t \in [0, +\infty)}$ of stochastic operators $S_t: \mathcal{M} \rightarrow \mathcal{M}$ such that $S_0 = I$ and $S_t \circ S_s = S_{t+s}$ for all $s, t \geq 0$. It is worth emphasizing that we do not impose here any continuity assumption on the trajectory $t \mapsto S_t \mu$ for a given $\mu \in \mathcal{M}$, so this definition of stochastic semigroup is quite weak. Our results on the Harris theorem for geometric decay require no further regularity conditions on the semigroup; see Section 3.

These objects are dual to the more classical definition of Markov-Feller operators and semigroups. Whenever we need to consider Markov-Feller semigroups we will always assume that Ω is a locally compact and separable metric space, and call $C_0(\Omega)$ the space of continuous functions which converge to 0 at infinity (the completion in the supremum norm of $C_c(\Omega)$, the set of continuous compactly supported functions on Ω). The space $C_0(\Omega)$ is a Banach space when endowed with the supremum norm, and its dual is \mathcal{M} (with the total variation norm) by the Markov-Riesz representation theorem.¹ In this

¹ For any locally compact Hausdorff topological space Ω , the dual of $C_0(\Omega)$ is the set of finite Radon measures on Ω ; in our setting in which Ω is additionally a separable metric space, the set of Radon measures is just the set of finite measures—see for example Folland [18, Theorem 7.8].

setting, a *Markov-Feller operator* P is a linear and continuous operator on $C_0(\Omega)$ which is positive ($P\varphi \geq 0$ if $\varphi \geq 0$) and preserves constants ($P\varphi_n \nearrow 1$ if $\varphi_n \nearrow 1$, with convergence understood in a pointwise sense). A *Markov-Feller semigroup* $(P_t)_{t \geq 0}$ is a strongly continuous semigroup of Markov-Feller operators on $C_0(\Omega)$. If P is a Markov-Feller operator then its dual $S := P^*$ is a stochastic operator, and in that case we will say that S is of *Feller type*. Similarly, if $(P_t)_{t \geq 0}$ is a Markov-Feller semigroup then the semigroup $(S_t)_{t \geq 0}$ defined by $S_t := P_t^*$ is a stochastic semigroup. In that case, we say that $(S_t)_{t \geq 0}$ is of Feller type and we denote by L the generator of (P_t) in the sense of semigroups. We note that for a Feller type stochastic semigroup $(S_t)_{t \geq 0}$ the trajectory $t \mapsto S_t\mu$ is weakly- $*$ continuous for any given $\mu \in \mathcal{M}$ (that is, the trajectory is continuous in the weak- $*$ topology of \mathcal{M} , viewed as the dual of $C_0(\Omega)$), but it does not need to be continuous in the total variation norm. Let us emphasize that definitions of “Feller” for an operator or a semigroup vary slightly in the literature; in some references a Markov-Feller operator is defined as an operator on $C_b(\Omega)$, the set of continuous and bounded functions [20, Definition 1.9], defined through an integral formula involving a transition kernel. We will not consider this latter case here, but rather we will use the concept of Feller-type stochastic operators and semigroups defined by duality from $C_0(\Omega)$, for which some simplifications occur. However, many of the results we state work with the minimal assumption of a stochastic operator or semigroup.

For a measurable (weight) function $V : \Omega \rightarrow [1, +\infty)$, we denote by \mathcal{M}_V the subspace of finite signed measures μ on Ω such that

$$\|\mu\|_V := \int_{\Omega} V|\mu| < \infty,$$

and write $\mathcal{P}_V := \mathcal{M}_V \cap \mathcal{P}$ for the set of probability measures for which $\|\mu\|_V < +\infty$, and similarly $\mathcal{N}_V := \mathcal{M}_V \cap \mathcal{N}$ is the set zero-mean measures with $\|\mu\|_V < +\infty$. We say that S is a *stochastic operator on \mathcal{M}_V* if it is a stochastic operator on \mathcal{M} , one can restrict $S : \mathcal{M}_V \rightarrow \mathcal{M}_V$, and this restriction is bounded in the $\|\cdot\|_V$ norm. Similarly, we say that $(S_t)_{t \geq 0}$ is a *stochastic semigroup on \mathcal{M}_V* if it is a stochastic semigroup on \mathcal{M} and satisfies a growth estimate

$$\|S_t\mu\|_V \leq C_V e^{\omega_V t} \|\mu\|_V, \tag{1.2}$$

for all $\mu \in \mathcal{M}_V$ and all $t \geq 0$, and for some constants $C_V \geq 1, \omega_V \geq 0$.

Plan of the paper The paper is organized as follows. We first prove in Section 2 a simple statement, sometimes known as Doeblin’s theorem. The statement and proof of the geometric version of Harris’ theorem is next given in Section 3. Sections 4 and 5 are then devoted to our versions of Harris’ theorem in the case of subgeometric operators and semigroups respectively. In the final section 6, the proof of the existence of an equilibrium (but not its uniqueness nor its stability) is established only assuming a Lyapunov condition (but without any irreducibility assumption).

2. Doeblin's theorem

In this section, we present a basic and well-known result in the theory of Markov processes sometimes known as *Doeblin's theorem*, which is a particular case of the Harris theorem presented in the next section. We include it since the proof is extremely simple and contains ideas that are used in later proofs. The argument is widely known, and we were made aware of it through Gabriel [19].

It is well known that stochastic operators are non-expansive mappings (or contractions in the non-strict sense) in the measure space \mathcal{M} , namely

$$\|S\mu\| \leq \|\mu\|, \quad (2.1)$$

for all measures $\mu \in \mathcal{M}$. The proof of this fact is simple and instructive. We introduce the Hahn-Jordan decomposition $\mu = \mu_+ - \mu_-$, $0 \leq \mu_{\pm} \in \mathcal{M}$, which ensures $|\mu| = \mu_+ + \mu_-$, and we write

$$|S\mu| = |S\mu_+ - S\mu_-| \leq S\mu_+ + S\mu_- = S|\mu|,$$

by using the linearity and the positivity of S . We then immediately deduce (2.1) by integrating the last inequality and by using that S is mass preserving.

Doeblin's Theorem states that under some very strong positivity or irreducibility condition the above non-expansive property becomes a (strict) contraction property on the set \mathcal{N} of zero mean measures:

Theorem 2.1 (*Doeblin's theorem*). *Let $S: \mathcal{M} \rightarrow \mathcal{M}$ be a stochastic operator satisfying that there exist $0 < \alpha < 1$ and $\eta \in \mathcal{P}$ such that*

$$S\mu \geq \alpha\eta, \quad \text{for all } \mu \in \mathcal{P}. \quad (2.2)$$

Then S has a unique stationary state $\mu^ \in \mathcal{P}$, which is exponentially stable. More generally*

$$\|S^n\nu\| \leq \gamma^n\|\nu\| \quad \text{for all } \nu \in \mathcal{N} \text{ and } n \in \mathbb{N}, \quad (2.3)$$

with $\gamma := 1 - \alpha \in (0, 1)$.

We point out that for any $\mu \in \mathcal{P}$, we deduce from (2.3) that

$$\|S^n\mu - \mu^*\| \leq \gamma^n\|\mu - \mu^*\|, \quad \text{for all } n \in \mathbb{N},$$

and thus the exponential asymptotic stability of the equilibrium μ^* .

Proof of Theorem 2.1. The proof is based on an improvement of (2.1) which writes

$$\|S\nu\| \leq \gamma\|\nu\| \quad \text{for all } \nu \in \mathcal{N}, \tag{2.4}$$

with $\gamma := 1 - \alpha$. In order to prove (2.4), we observe that because of the Doeblin condition (2.2) applied to $S(\nu_{\pm}/\|\nu_{\pm}\|)$ and the fact that the integrals of ν_+ and ν_- are equal for $\nu \in \mathcal{N}$, it holds

$$S\nu_{\pm} \geq \alpha\eta \int \nu_{\pm} = r\eta, \quad r := \alpha\|\nu\|/2.$$

Similarly as in the proof of (2.1), we may deduce

$$\begin{aligned} |S\nu| &= |S\nu_+ - r\eta - S\nu_- + r\eta| \\ &\leq |S\nu_+ - r\eta| + |S\nu_- - r\eta| \\ &= S\nu_+ - r\eta + S\nu_- - r\eta = S|\nu| - 2r\eta, \end{aligned}$$

and integrating this, we get

$$\|S\nu\| \leq \|S|\nu|\| - 2r\|\eta\| = \|\nu\| - 2r = (1 - \alpha)\|\nu\|.$$

That is exactly inequality (2.4), from which (2.3) immediately follows.

In order to prove the existence and uniqueness of an equilibrium, we fix $\mu_0 \in \mathcal{P}$, and we define recursively $\mu_k := S\mu_{k-1}$ for any $k \geq 1$. Thanks to (2.4), we get

$$\sum_{k=1}^{\infty} \|\mu_k - \mu_{k-1}\| \leq \sum_{k=0}^{\infty} \gamma^k \|\mu_1 - \mu_0\| < \infty,$$

so that (μ_k) is a Cauchy sequence in \mathcal{P} . We set $\mu^* := \lim \mu_k \in \mathcal{P}$ which is a stationary state, as seen by passing to the limit in the equation $\mu_k = S\mu_{k-1}$, and which is unique in \mathcal{P} thanks to (2.4). \square

3. Harris’s theorem

We extend Doeblin’s results presented in the previous section to the case when only a weaker version of Doeblin’s positivity condition (2.2) holds, together with a *Lyapunov condition*. An important motivation is that the Doeblin condition (2.2) is indeed too restrictive and somehow limited to compact spaces. As a matter of fact, when $\Omega = \mathbb{R}^d$ for instance and S is a stochastic operator of Feller type, there exists a sequence (μ_{0n}) in \mathcal{P} such that $\mu_{0n} \rightarrow 0$ (in the weak-* sense of measures; take for example $\mu_{0n} := \delta_{x_n}$ with $|x_n| \rightarrow +\infty$). Since S is continuous in the weak-* topology due to S being of Feller type, also $S\mu_{n0} \rightarrow 0$ and the Doeblin condition (2.2) cannot hold. However, the condition (2.2) may still hold if the semigroup is not of Feller type in our sense; an example on

$\Omega = [0, +\infty)$ is the renewal equation found in Gabriel [19]. For many applications, one must thus weaken the positivity condition (2.2). This forces us to add a localization or confinement condition and work in a weighted space.

The following assumptions will be used in Harris’s theorem below. In all of this section, $V : \Omega \rightarrow [1, +\infty)$ denotes a measurable function, that we will call in the sequel a *Lyapunov or weight function*.

Hypothesis 1 (*Operator Lyapunov condition*). An operator S satisfies an *operator Lyapunov condition* with Lyapunov function V if there exist $0 < \gamma_L < 1$ and $K \geq 0$ such that

$$\|S\mu\|_V \leq \gamma_L \|\mu\|_V + K\|\mu\|, \quad \text{for } \mu \in \mathcal{M}_V. \tag{3.1}$$

Hypothesis 2 (*Harris condition*). An operator S satisfies a *Harris condition* on a set $\mathcal{C} \subseteq \Omega$ if there exist $0 < \alpha < 1$ and $\eta \in \mathcal{P}$ such that

$$S\mu \geq \alpha\eta \int_{\mathcal{C}} \mu, \quad \text{for all } 0 \leq \mu \in \mathcal{M}. \tag{3.2}$$

In other words, Hypothesis 2 states that the Doeblin condition (2.2) holds, but only for measures μ supported on the set \mathcal{C} .

Hypothesis 3 (*Local coupling condition*). An operator S satisfies a *local coupling condition* with Lyapunov function V if there exist $0 < \gamma_H < 1$ and $A > 0$ such that

$$\left(\nu \in \mathcal{N}_V, \|\nu\|_V \leq A\|\nu\| \right) \text{ implies } \|S\nu\| \leq \gamma_H\|\nu\|. \tag{3.3}$$

The term *local coupling condition* comes from the fact that it implies that (and is in fact equivalent to)

$$\left(x, y \in \Omega, V(x) + V(y) \leq A \right) \text{ implies } \|S(\delta_x - \delta_y)\| \leq 2\gamma_H,$$

so that the distance between $S\delta_x$ and $S\delta_y$ is strictly less than the distance between δ_x and δ_y under a localisation condition on x and y . The following lemma shows that, roughly speaking, the Harris Hypothesis 2 implies the Local coupling Hypothesis 3.

Lemma 3.1 (*Harris implies local coupling*). *If S satisfies the Harris condition (Hypothesis 2) on the set $\mathcal{C} = \{x \in \Omega \mid V(x) \leq R\}$ for some $R > 0$ and $0 < \alpha < 1$ then it satisfies the local coupling condition (Hypothesis 3) with any $A \in (0, R/2)$ and $\gamma_H := 1 - \alpha(1 - 2A/R) \in (0, 1)$.*

Proof of Lemma 3.1. Under the Harris condition (3.2) and when ν satisfies the LHS hypotheses of condition (3.3), a sizeable part of the mass of ν_+ and ν_- is in $\mathcal{C} := \{x \in \Omega \mid V(x) \leq R\}$, as can be seen from the bound

$$\int_{\Omega \setminus \mathcal{C}} \nu_{\pm} \leq \frac{1}{R} \int V|\nu| \leq \frac{A}{R} \int |\nu| = \frac{2A}{R} \int \nu_{\pm},$$

where we have used in a fundamental way that the masses of ν_+ and ν_- are equal in the last line. That implies

$$\int_{\mathcal{C}} \nu_{\pm} \geq \left(1 - \frac{2A}{R}\right) \int \nu_{\pm}.$$

Because of the *Harris condition* (3.2) and the fact that the mass of ν_+ and ν_- are equal, it holds

$$S\nu_{\pm} \geq \alpha\eta \left(1 - \frac{2A}{R}\right) \int \nu_{\pm} =: r\eta,$$

with

$$r := \alpha \left(1 - \frac{2A}{R}\right) \int \nu_{\pm} = \frac{1 - \gamma_H}{2} \|\nu\|.$$

Repeating the proof of Theorem 2.1, it holds then

$$\|S\nu\| \leq \|\nu\| - 2r = \gamma_H \|\nu\|.$$

That is exactly inequality (3.3) with $\gamma_H := 1 - \alpha(1 - 2A/R)$. \square

Theorem 3.2 (*Harris’s Theorem*). Consider $S: \mathcal{M}_V \rightarrow \mathcal{M}_V$ a stochastic operator which satisfies the operator Lyapunov condition (Hypothesis 1) and the local coupling condition (Hypothesis 3) with $K/A < 1 - \gamma_L$, both with the same weight function V . Then S has a unique stationary state $\mu^* \in \mathcal{P}_V$, which is exponentially stable. More generally, there exist $\gamma \in (0, 1)$ and $C \in [1, \infty)$ such that

$$\|S^n \nu - \mu^*\|_V \leq C \gamma^n \|\nu - \mu^*\|_V, \quad \text{for all } \nu \in \mathcal{N}_V \text{ and } n \in \mathbb{N}. \tag{3.4}$$

Due to Lemma 3.1, the conclusion of Theorem 3.2 applies also if S satisfies the Lyapunov condition (3.1) and the Harris condition (3.2) with $2K/R \leq 1 - \gamma_L$. With this result, we recover the main result of Hairer and Mattingly [22] with a similar approach, except that we work on the stochastic operator side rather than on the dual Markov operator side. In particular, and as in Doeblin’s framework of Section 2, we deduce the exponential asymptotic stability of the equilibrium ν^* in \mathcal{M}_V , namely

$$\|S^n \mu - \mu^*\|_V \leq C \gamma^n \|\mu - \mu^*\|_V, \quad \text{for all } \mu \in \mathcal{P}_V \text{ and } n \in \mathbb{N}.$$

The theorem doesn’t exclude the existence of other equilibria with infinite V -moment.

Proof of Theorem 3.2. We introduce a new norm $\|\cdot\|_V$ on \mathcal{M}_V defined by

$$\|\mu\|_V := \|\mu\| + \beta\|\mu\|_V, \tag{3.5}$$

for some $\beta > 0$ to be chosen later. Note that $\|\cdot\|_V$ and $\|\cdot\|_V$ are equivalent norms, with

$$(1 + \beta)^{-1}\|\mu\|_V \leq \|\mu\| \leq \beta^{-1}\|\mu\|_V.$$

We claim that there exist $\beta > 0$ small enough and $\gamma \in (0, 1)$ such that

$$\|S\nu\|_V \leq \gamma\|\nu\|_V, \quad \text{for all } \nu \in \mathcal{N}_V. \tag{3.6}$$

Using (3.6), we may then straightforwardly adapt the proof of Theorem 2.1 in order to conclude to the existence and uniqueness of a stationary state $\mu^* \in \mathcal{P}_V$ of S and to the geometrical decay (3.4) with $C := (1 + \beta)/\beta$.

We may then focus on the proof of the contraction estimate (3.6). For that purpose, we take any $\nu \in \mathcal{N}$ and estimate the norm $\|S\nu\|_V$ in two alternative cases:

First case. *Contractivity for small V-moment.* When

$$\|\nu\|_V < A\|\nu\|, \tag{3.7}$$

the local coupling condition (3.3) implies

$$\|S\nu\| \leq \gamma_H\|\nu\|.$$

Together with the Lyapunov condition (3.1), we have

$$\begin{aligned} \|S\nu\|_V &= \|S\nu\| + \beta\|S\nu\|_V \\ &\leq (\gamma_H + \beta K)\|\nu\| + \beta\gamma_L\|\nu\|_V \leq \gamma_1\|\nu\|_V, \end{aligned}$$

with

$$\gamma_1 := \max\{\gamma_H + \beta K, \gamma_L\}.$$

Choosing $\beta > 0$ small enough such that $\beta K < 1 - \gamma_H$, we get $\gamma_1 < 1$ and that gives the contractivity property (3.6) in this case.

Second case. *Contractivity for large V-moment.* Assume on the contrary that

$$\|\nu\|_V \geq A\|\nu\|. \tag{3.8}$$

From (3.1) we deduce then

$$\|S\nu\|_V \leq \gamma_L\|\nu\|_V + K\|\nu\| \leq (\gamma_L + K/A)\|\nu\|_V,$$

with $\gamma_L + K/A < 1$ by assumption. Together with (2.1), we deduce

$$\begin{aligned} \|S\nu\|_V &= \|S\nu\| + \beta\|S\nu\|_V \\ &\leq \|\nu\| + \beta(\gamma_L + K/A)\|\nu\|_V \\ &\leq (1 - \beta\delta_0)\|\nu\| + \beta(\gamma_L + K/A + \delta_0)\|\nu\|_V, \end{aligned}$$

for any $\delta_0 \geq 0$, by using that $V \geq 1$ in the last inequality above. We thus get

$$\|S\nu\|_V \leq \gamma_2\|\nu\|_V,$$

with $\gamma_2 := \max(1 - \beta\delta_0, \gamma_L + K/A + \delta_0)$. We get the contractivity property (3.6) in this case by choosing $\delta_0 > 0$ small enough (and keeping the choice of $\beta > 0$ made in the previous case) so that $\gamma_2 \in (0, 1)$. The proof of (3.6) is completed by setting $\gamma := \max\{\gamma_1, \gamma_2\}$. \square

Remark 3.3. By following the above proof one can give an explicit expression of the constants. Because

$$\gamma_L < \gamma_1 := 1 - \frac{\beta}{1 + \beta}(1 - \gamma_L - K/A),$$

we have

$$\gamma = \max\{\gamma_H + \beta K, 1 - \frac{\beta}{1 + \beta}(1 - \gamma_L - K/A)\}.$$

We see then that the best choice of β is the (uniquely defined) positive zero of the following second order polynomial equation

$$K\beta^2 + (K + b - a)\beta - a = 0,$$

with $a := 1 - \gamma_H > 0$, $b := 1 - \gamma_L - K/A > 0$.

We end the section by presenting a different proof of Theorem 3.2. The outcome is essentially the same, but we do not obtain as part of the argument the contractivity of a modified weighted total variation norm as in the previous proof. On the other hand, the result has an extremely short proof which makes the role of the assumptions very clear!

Alternative proof of Theorem 3.2. Given $\nu \in \mathcal{N}_V$, we call

$$v_n := \|S^n\nu\|_V, \quad m_n := \|S^n\nu\|,$$

for integer $n \geq 0$. The Lyapunov condition (3.1) shows that

$$v_{n+1} \leq \gamma_L v_n + K m_n. \tag{3.9}$$

The local coupling condition (3.3) and the non-expansive mapping property (2.1) together imply

$$\|S\nu\| \leq \begin{cases} \gamma_H \|\nu\| & \text{whenever } \|\nu\|_V \leq A\|\nu\|, \\ \|\nu\| & \text{always.} \end{cases}$$

In particular,

$$\|S\nu\| \leq \gamma_H \|\nu\| + \frac{1 - \gamma_H}{A} \|\nu\|_V \quad \text{always,}$$

since the inequality can be checked to be true in the two cases $\|\nu\|_V \leq A\|\nu\|$ and $\|\nu\|_V > A\|\nu\|$. Applying this to $S^n\nu$ we get $m_{n+1} \leq \gamma_H m_n + \frac{1 - \gamma_H}{A} v_n$. Together with (3.9), this gives the system

$$\begin{aligned} v_{n+1} &\leq \gamma_L v_n + K m_n, \\ m_{n+1} &\leq \frac{1 - \gamma_H}{A} v_n + \gamma_H m_n, \end{aligned}$$

whose associated matrix is

$$M := \begin{pmatrix} \gamma_L & K \\ \frac{1 - \gamma_H}{A} & \gamma_H \end{pmatrix}.$$

One can easily see that the condition for the eigenvalues of this matrix to be both strictly less than 1 is that $1 - \gamma_L > K/A$, so that both v_n and m_n decay exponentially in n . Existence and uniqueness of an equilibrium in \mathcal{P}_V follow as before. \square

4. Subgeometric convergence for discrete-time semigroups

We now extend Harris’s Theorem to cases in which a weaker form of Lyapunov condition (3.1) holds true, with a slowing of the speed of decay as a drawback.

In all of this section, $V: \Omega \rightarrow [1, +\infty)$ is a measurable weight function, still referred to as a Lyapunov or just weight function and $\varphi: [1, +\infty) \rightarrow [1, +\infty)$ a concave function with $\varphi(1) = 1$ and $\lim_{v \rightarrow +\infty} \varphi(v)/v = 0$. The following assumption generalizes the Lyapunov condition from Hypothesis 1 and will be used in the subgeometric version of Harris’s theorem below.

Hypothesis 4 (*Weak operator Lyapunov condition*). A stochastic operator S satisfies a *weak Lyapunov condition* for V and φ if there exist $K > 0$ and $0 < \varsigma < 1$ such that

$$\|S\mu\|_V + \varsigma \|\mu\|_{\varphi(V)} \leq \|\mu\|_V + K\|\mu\|, \quad \text{for all } \mu \in \mathcal{M}_V. \tag{4.1}$$

Let us make some observations.

Remark 4.1. Because $\varphi: [1, +\infty) \rightarrow [1, +\infty)$ is a concave function, then φ must be continuous and nondecreasing. The continuity of φ ensures $\varphi(V) \equiv \varphi \circ V$ is measurable. The asymptotic condition $\lim_{v \rightarrow +\infty} \varphi(v)/v = 0$ ensures that we are not in the framework of Section 3 since Hypothesis 1 does not need to hold.

Remark 4.2. If S is a Feller-type stochastic operator and P is the associated Markov-Feller operator on $C_0(\Omega)$ such that $P^* = S$ then, by duality, Hypothesis 4 is equivalent to the property

$$PV + \varsigma\varphi(V) \leq V + K,$$

which is perhaps more often found in the literature (see for instance, Douc et al. [15, Theorem 3.3 (i)]). We remark that a possible definition of the function PV is $PV := \lim P(V\varphi_n) \in [0, \infty]$, where (φ_n) is a nonnegative sequence of $C_0(\Omega)$ such that $\varphi_n \nearrow 1$, which belongs to $\mathcal{L}_{loc}^\infty(\Omega)$ because of the above Lyapunov property.

4.1. Existence of an equilibrium

We now show that under weak Lyapunov and coupling conditions one can build a norm $\|\cdot\|_V$ equivalent to $\|\cdot\|_V$ for which our stochastic operator S is a contraction, in a quantitative sense. This will be used for existence and uniqueness results, and later for obtaining decay rates.

Lemma 4.3. *Consider a stochastic operator S such that*

1. *S satisfies a weak Lyapunov condition (Hypothesis 4) associated to functions V, φ and constants K, ς .*
2. *For some integer $N \geq 1$, the operator S^N satisfies a local coupling condition (Hypothesis 3) for $\varphi(V)$, with constant $A > K/\varsigma$.*

Then for any $\nu \in \mathcal{N}_V$, there exists an integer n with $N \leq n \leq 2N - 1$ such that

$$\|S^n \nu\|_V + \alpha \sum_{k=0}^{n-1} \|S^k \nu\|_{\varphi(V)} \leq \|\nu\|_V, \quad \text{for all } \nu \in \mathcal{N}_V, \tag{4.2}$$

where

$$\|\mu\|_V := \|\mu\| + \beta\|\mu\|_V, \quad \text{for } \mu \in \mathcal{M}_V,$$

and with

$$\beta := (1 - \gamma_H)/(KN), \quad \alpha := \beta(\varsigma - K/A) > 0. \tag{4.3}$$

Proof of Lemma 4.3. We fix $\nu \in \mathcal{N}_V$ and denote $\nu_k := S^k \nu$ for all integer $k \geq 0$ and we set $V_0 := \varphi(V)$. We observe that if for a given k , we have

$$\|\nu_k\|_{V_0} \geq A\|\nu_k\|, \tag{4.4}$$

then this inequality and the weak Lyapunov condition in Hypothesis 4 imply

$$\|\nu_{k+1}\|_V \leq \|\nu_k\|_V - \left(\varsigma - \frac{K}{A}\right)\|\nu_k\|_{V_0},$$

where the quantity $\varsigma - K/A > 0$ by hypothesis, which allows us to carry out the argument. Multiplying by β , using that $\alpha = \beta(\varsigma - K/A)$, and the contractivity $\|\nu_{k+1}\| \leq \|\nu_k\|$, we have

$$\beta\|\nu_{k+1}\|_V + \|\nu_{k+1}\| \leq \beta\|\nu_k\|_V - \alpha\|\nu_k\|_{V_0} + \|\nu_k\|,$$

that is

$$\|\|\nu_{k+1}\|\|_V \leq \|\|\nu_k\|\|_V - \alpha\|\nu_k\|_{V_0}. \tag{4.5}$$

Now we have two cases:

Case 1. If (4.4) holds for all integer k with $0 \leq k \leq n - 1$, then we directly obtain (4.2) by iterating the difference inequality (4.5).

Case 2. If (4.4) fails for some k in $\{0, \dots, n - 1\}$, then take k^* the smallest integer in this range in which the condition fails. Then we may use (4.5) for $0 \leq k < k^*$ and obtain

$$\|\|\nu_{k^*}\|\|_V \leq \|\|\nu\|\|_V - \alpha \sum_{k=0}^{k^*-1} \|\nu_k\|_{V_0}. \tag{4.6}$$

Define now $n := N + k^*$ in this case. Using that S^N satisfies the coupling condition, we have

$$\|\nu_n\| \leq \gamma_H \|\nu_{k^*}\|. \tag{4.7}$$

On the other hand, we may use the weak Lyapunov condition and the fact that $k \mapsto \|\nu_k\|$ is nonincreasing to get

$$\|\nu_{k+1}\|_V \leq \|\nu_k\|_V - \varsigma\|\nu_k\|_{V_0} + K\|\nu_{k^*}\|,$$

for all $k = k^*, \dots, n - 1$. Summing the inequality in this range, we get

$$\|\nu_n\|_V \leq \|\nu_{k^*}\|_V - \varsigma \sum_{k=k^*}^{n-1} \|\nu_k\|_{V_0} + NK\|\nu_{k^*}\|.$$

Multiplying by β and adding $\|\nu_n\|$ to complete $\|\nu_n\|$ on the left hand side, we deduce

$$\beta\|\nu_n\|_V + \|\nu_n\| \leq \beta\|\nu_{k^*}\|_V - \beta\varsigma \sum_{k=k^*}^{n-1} \|\nu_k\|_{V_0} + \beta NK\|\nu_{k^*}\| + \|\nu_n\|.$$

Using (4.7) and reorganising terms, we conclude with

$$\begin{aligned} \|\nu_n\|_V + \alpha \sum_{k=k^*}^{n-1} \|\nu_k\|_{V_0} &\leq \beta\|\nu_{k^*}\|_V + (\beta NK + \gamma_H)\|\nu_{k^*}\| \\ &= \beta\|\nu_{k^*}\|_V + \|\nu_{k^*}\| = \|\nu_{k^*}\|_V \leq \|\nu\|_V - \alpha \sum_{k=0}^{k^*-1} \|\nu_k\|_{V_0}, \end{aligned}$$

where in the last inequality we have used (4.6). This shows the result. \square

Theorem 4.4 (*Existence of equilibrium*). *Consider a stochastic operator S satisfying the same conditions as in Lemma 4.3. Then there exists an equilibrium $\mu^* \in \mathcal{P}_{\varphi(V)}$.*

Proof of Theorem 4.4. Take any $\mu_0 \in \mathcal{P}_V$ and define

$$\nu_0 := S\mu_0 - \mu_0, \quad \nu_k := S^k\nu_0, \quad k \geq 1.$$

From Lemma 4.3, we can find an increasing sequence $(n_i)_{i \geq 0}$ with $n_0 = 0$, $N \leq n_{i+1} - n_i \leq 2N - 1$ and

$$\|\nu_{n_{i+1}}\|_V + \alpha \sum_{k=n_i}^{n_{i+1}-1} \|\nu_k\|_{\varphi(V)} \leq \|\nu_0\|_V, \quad i \geq 0.$$

Summing this for all i , we get

$$\alpha \sum_{k=0}^{\infty} \|\nu_k\|_{\varphi(V)} \leq \|\nu_0\|_V, \quad i \geq 0.$$

This shows that the sequence of probability measures $(S^k\mu_0)_{k \geq 0}$ is a Cauchy sequence in the norm $\|\cdot\|_{\varphi(V)}$, and hence converges to a certain probability measure $\mu^* \in \mathcal{P}_{\varphi(V)}$ which must satisfy $S\mu^* = \mu^*$ by construction. \square

4.2. Uniqueness of equilibrium

Another consequence of Hypotheses 3 and 4 is the uniqueness of equilibrium, that we present in two different frameworks.

Corollary 4.5 (*Uniqueness of equilibrium*). Let S be a stochastic operator which satisfies the Lyapunov condition (Hypothesis 4) and the local coupling condition (Hypothesis 3) of the existence Theorem 4.4 for two couples of weight and sublinear functions (V_1, φ_1) and (V_2, φ_2) such that $\varphi_2(V_2) \geq V_1$. Then S has at most one equilibrium in $\mathcal{P}_{\varphi_2(V_2)}$.

Proof of Corollary 4.5. Let us consider two equilibria $\mu_1^*, \mu_2^* \in \mathcal{P}_{\varphi_2(V_2)}$ and let us set $\nu := \mu_2^* - \mu_1^* \in \mathcal{N}_{\varphi_2(V_2)} \subset \mathcal{N}_{V_1}$. From Lemma 4.3 applied to (V_1, φ_1) and because $S^k \nu = \nu$ for any $k \geq 0$, we get

$$\|\nu\|_{V_1} + \alpha \sum_{k=0}^{n-1} \|\nu\|_{\varphi_1(V_1)} \leq \|\nu\|_{V_1},$$

for some equivalent norm $\|\cdot\|_{V_1}$, some integer $n \geq 1$ and some constant $\alpha > 0$. That implies $\|\nu\|_{\varphi_1(V_1)} = 0$, and thus $\mu_2^* = \mu_1^*$. \square

We now consider the case when S is a Feller-type stochastic operator.

Corollary 4.6 (*Uniqueness of equilibrium*). Let S be a Feller-type stochastic operator which satisfies the hypotheses of the existence Theorem 4.4. Then S has a unique equilibrium $\mu \in \mathcal{P}_{\varphi(V)}$.

Before we prove that uniqueness result we will show that the weak Lyapunov Hypothesis 4 implies similar inequalities for $\psi(V)$, where ψ is a concave function:

Lemma 4.7. Let S be a Feller-type stochastic operator which satisfies the weak Lyapunov Hypothesis 4 for V . From Remark 4.2, that is, $S = P^*$ and

$$PV \leq V - \varsigma\varphi(V) + K. \tag{4.8}$$

Then for any concave function $\psi: [1, +\infty) \rightarrow [1, +\infty)$, we have

$$P\psi(V) \leq \psi(V) - \varsigma\psi'(V)\varphi(V) + K\psi'(V).$$

Proof of Lemma 4.7. Notice that because ψ is a concave function, there holds

$$\psi(v) = \inf_{\ell \in \mathcal{U}_\psi} \ell(v), \quad \forall v \in \mathbb{R},$$

where $\mathcal{U}_\psi := \{\ell : \mathbb{R} \rightarrow \mathbb{R}, \ell(v) := av + b, a, b \in \mathbb{R}, \ell \leq \psi\}$. Using that P is a positive operator, we deduce that

$$P\psi(V) \leq P\ell(V) = \ell(PV), \quad \forall \ell \in \mathcal{U}_\psi,$$

and then the Jensen's inequality

$$P\psi(V) \leq \psi(PV). \tag{4.9}$$

Using (4.9), the nondecreasing property of ψ (as emphasized in Remark 4.1), (4.8) and the fact that ψ is concave again, we get

$$P\psi(V) \leq \psi(V - \varsigma\varphi(V) + K) \leq \psi(V) - \psi'(V)(\varsigma\varphi(V) - K),$$

which gives the inequality in the statement. \square

Proof of Corollary 4.6. The existence of an equilibrium is given by Theorem 4.4. Assume there are two equilibria $\mu_1^*, \mu_2^* \in \mathcal{P}_{\varphi(V)}$, and call $\nu := \mu_1^* - \mu_2^*$, so that in particular $\nu \in \mathcal{N}_{\varphi(V)}$ and $S\nu = \nu$. Similarly as in the proof of Corollary 4.5, we would like to use the weak Lyapunov condition from Hypothesis 4 in order to get

$$\varsigma\|\nu\|_{\varphi(V)} \leq K\|\nu\|, \tag{4.10}$$

but this is not allowed because ν is not necessarily in \mathcal{N}_V and we cannot justify cancelling the term $\|\mu\|_V$ on both sides. Hence we carry out an approximation procedure in order to deduce (4.10). Since S is of Feller-type, $S = P^*$ for a Markov-Feller operator P . Take $\psi: [1, +\infty) \rightarrow [1, +\infty)$ a bounded concave function such that $\psi'(v) \leq 1$ for all $v \geq 1$, so that

$$P\psi(V) \leq \psi(V) - \varsigma\psi'(V)\varphi(V) + K,$$

from Lemma 4.7. After integration and by duality, for any $0 \leq \mu \in \mathcal{M}_{\varphi(V)}$, we have

$$\int \psi(V)S\mu \leq \int \psi(V)\mu - \varsigma \int \psi'(V)\varphi(V)\mu + K \int \mu.$$

Applying this to $\mu := |\nu| = |\mu_1^* - \mu_2^*| \in \mathcal{M}_{\varphi(V)}$, we get

$$\int \psi(V)|S\nu| \leq \int \psi(V)S|\nu| \leq \int \psi(V)|\nu| - \varsigma \int \psi'(V)\varphi(V)|\nu| + K \int |\nu|,$$

and since $S\nu = \nu$, we deduce

$$\varsigma \int \psi'(V)\varphi(V)|\nu| \leq K \int |\nu|. \tag{4.11}$$

Taking for example $\psi_n(v) := n \arctan(\frac{\pi}{2} + v/n)$, so that $\psi'_n(v) \nearrow 1$ as $n \rightarrow \infty$, and passing to the limit as $n \rightarrow +\infty$ in (4.11), the dominated convergence theorem shows that (4.10) holds true. Since (4.10) holds, the iterated coupling condition (Hypothesis 2 in Lemma 4.3) gives that

$$\|\nu\| = \|S^N \nu\| \leq \gamma_H \|\nu\|,$$

which implies $\|\nu\| = 0$ and hence $\mu_1^* = \mu_2^*$. \square

4.3. *Subgeometric decay rates*

For a nonempty interval $I \subseteq \mathbb{R}_+$ and a function $\xi : I \rightarrow \mathbb{R}$, we recall that the associated Legendre transform $\xi^* : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$$\xi^*(u) := \sup_{\lambda \in I} (\lambda u - \xi(\lambda)),$$

is an increasing and convex function, and in particular it is continuous on the interior of the interval $D(\xi^*) := \{u \in \mathbb{R}; \xi^*(u) < +\infty\}$. We also define the closely related transform

$$\xi_*(u) := \sup_{\lambda \in I} (\xi(\lambda) - \lambda u) = (-\xi)^*(-u),$$

also defined (and possibly $+\infty$) at all $u \in \mathbb{R}$.

Our main theorem in the subgeometric case is the following, which involves two different weight functions V_1 and V_2 :

Theorem 4.8 (*Subgeometric Harris, interpolated version*). *Consider a stochastic operator S such that:*

1. *S satisfies a weak Lyapunov condition (Hypothesis 4) for two couples of weight and sublinear functions (V_1, φ_1) , (V_2, φ_2) and constants $K_1, \varsigma_1, K_2, \varsigma_2$, respectively, and such that $V_1 \leq V_2$.*
2. *There exists an integer $N \geq 1$ such that S^N satisfies a local coupling condition (Hypothesis 3) for both $\varphi_1(V_1)$ and $\varphi_2(V_2)$, with constants $A_1 > K_1/\varsigma_1$, $A_2 > K_2/\varsigma_2$ and same constant γ_H .*
3. *The following interpolation condition holds: there is a function $\xi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ which is increasing and satisfies $\xi(\lambda)/\lambda \rightarrow 0$ as $\lambda \rightarrow 0$ and such that*

$$\lambda V_1 \leq \varphi_1(V_1) + \xi(\lambda)V_2, \quad \text{for all } \lambda > 0. \tag{4.12}$$

Then there exist constructive constants $C > 0$ and $0 < r < 1$ (depending only on ξ , K_i , ς_i , A_i for $i = 1, 2$, and on γ_H) such that

$$\|S^n \nu\|_{V_1} \leq C \Theta(rn) \|\nu\|_{V_2}, \quad \text{for all } n \geq 1, \tag{4.13}$$

and

$$\|S^n \nu\| \leq \frac{C}{n} \Theta(rn) \|\nu\|_{V_2}, \quad \text{for all } n \geq 1, \tag{4.14}$$

for any $\nu \in \mathcal{N}_{V_2}$, where for $t > 0$

$$\Theta(t) := F^{-1}(t), \quad F(\lambda) := \int_{\lambda}^1 \frac{1}{\xi^*(s)} ds.$$

Remark 4.9. (1) Under the assumptions of Theorem 4.8 and when $V_1 \leq \varphi_2(V_2)$ or S is of Feller-type, we get the existence of an equilibrium (Theorem 4.4), its uniqueness (Corollary 4.5 or Corollary 4.6) and a decay rate of convergence to zero (Theorem 4.8).

(2) Let us emphasize that, in contrast with Theorem 2.1 and Theorem 3.2, here in principle there is no reason that μ^* belongs to \mathcal{P}_V , and thus, we cannot apply Theorem 4.8 to $\mu - \mu^*$ and deduce $S^n \mu \rightarrow \mu^*$ as $n \rightarrow \infty$ (with or even without rate!). We will come back on that issue in Remark 4.14 below.

In the rest of this section we prove Theorem 4.8. We start with a finite difference inequality which is at the basis of the estimates we carry out in the proof:

Lemma 4.10. *Let $(u_n)_{n \geq 0}$ be a nonnegative sequence which satisfies*

$$u_{n+1} - u_n \leq -g(u_n) \quad \text{for all integers } n \geq 0, \tag{4.15}$$

for some continuous, increasing function $g: (0, u_0] \rightarrow (0, +\infty)$ such that $v \mapsto 1/g(v)$ is not integrable on $(0, u_0)$. Then

$$u_n \leq H^{-1}(n) \quad \text{for all integers } n \geq 0,$$

where

$$H(u) := \int_u^{u_0} \frac{1}{g(v)} \, dv \quad \text{for } u \in (0, u_0].$$

Proof of Lemma 4.10. Let $u = u(t)$ be the solution for $t \geq 0$ to the ordinary differential equation

$$u'(t) = -g(u(t)), \quad u(0) = u_0,$$

which is precisely $u(t) = H^{-1}(t)$. We prove by induction that $u_n \leq u(n)$ for all $n \geq 0$. It is indeed true for $n = 0$. If we assume $u(n) \geq u_n$ for some $n \geq 0$, then

$$u(n + 1) \geq \tilde{u}(n + 1),$$

where \tilde{u} is the solution to

$$\tilde{u}' = -g(\tilde{u}), \quad \tilde{u}(n) = u_n.$$

Since \tilde{u} is decreasing and g is increasing, we then have

$$\tilde{u}(n + 1) = \tilde{u}(n) - \int_n^{n+1} g(\tilde{u}(t)) \, dt \geq \tilde{u}(n) - g(\tilde{u}(n)) = u_n - g(u_n) \geq u_{n+1},$$

which shows $u_{n+1} \leq u(n + 1)$. \square

Lemma 4.11 (*Difference inequality*). *Take $M > 0$, $0 < \delta \leq +\infty$, and $\zeta: (0, \delta) \rightarrow (0, +\infty)$ a nonnegative function satisfying $\lim_{\lambda \rightarrow 0} \zeta(\lambda)/\lambda = 0$. If a sequence $(u_n)_{n \geq 0}$ of nonnegative numbers satisfies $u_0 \leq M$ and*

$$u_{n+1} \leq (1 - \lambda)u_n + M\zeta(\lambda) \quad \text{for all } n \geq 0 \text{ and all } \lambda \in (0, \delta), \tag{4.16}$$

then

$$u_n \leq MF^{-1}(n) \quad \text{for all integers } n \geq 0,$$

where

$$F(u) := \int_u^1 \frac{1}{\zeta^*(v)} \, dv \quad \text{for } u \in (0, 1],$$

and ζ^* denotes the Legendre transform of ζ .

Proof of Lemma 4.11. Call $v_n := u_n/M$ for $n \geq 0$. Minimising (4.16) in λ we obtain

$$v_{n+1} - v_n \leq -\zeta^*(v_n) \quad \text{for } n \geq 0.$$

In particular $\zeta^*(v_n)$ must always be finite on $(0, v_0]$, since the sequence (v_n) is assumed to be a sequence of nonnegative numbers, and ζ^* is a nondecreasing function. Notice also that ζ^* is continuous since it is convex, and that the condition $\lim_{\lambda \rightarrow 0} \zeta(\lambda)/\lambda = 0$ ensures $\zeta^*(v) > 0$ for all $v \in (0, v_0]$. Then Lemma 4.10 with $g \equiv \zeta^*$ gives

$$v_n \leq \tilde{F}^{-1}(n), \tag{4.17}$$

where

$$\tilde{F}(v) := \int_v^{v_0} \frac{1}{\zeta^*(s)} \, ds \quad \text{for } v \in (0, v_0].$$

Since $v_0 = u_0/M \leq 1$ by assumption, this shows

$$\tilde{F}(v) \leq \int_v^1 \frac{1}{\zeta^*(s)} \, ds =: F(v) \quad \text{for } v \in (0, v_0],$$

where we understand $1/\zeta^*(s) = 0$ is $\zeta^*(s) = +\infty$. Hence (4.17) gives

$$u_n \leq MF^{-1}(n)$$

for all $n \geq 0$, as required. \square

We also need a technical lemma which will be used to simplify the bounds in our main results.

Lemma 4.12. *Let $g: (0, +\infty) \rightarrow (0, +\infty)$ be a positive, nondecreasing function with $\lim_{s \rightarrow 0} g(s)/s = 0$, and define*

$$F(\lambda) := \int_{\lambda}^1 \frac{1}{g(s)} \, ds \quad \text{for } 0 < \lambda \leq 1.$$

(We notice that $F^{-1}: [0, +\infty) \rightarrow (0, 1]$ is well defined, continuous and strictly decreasing, since F is strictly decreasing and $\lim_{\lambda \rightarrow 0} F(\lambda) = +\infty$.) Then for any $k > 0$, there exists a constant $C > 1$ which depends only on k and g , such that

$$F^{-1}(t - k) \leq CF^{-1}(t) \quad \text{for all } t \geq k.$$

Proof of Lemma 4.12. We first notice that for any $C > 1$ and $0 < \lambda < 1/C$,

$$F(C\lambda) = \int_{C\lambda}^1 \frac{1}{g(s)} \, ds = F(\lambda) - \int_{\lambda}^{C\lambda} \frac{1}{g(s)} \, ds \leq F(\lambda) - (C - 1) \frac{\lambda}{g(C\lambda)}.$$

Using that $\lim_{\lambda \rightarrow 0} \lambda/g(C\lambda) = +\infty$, we may take $\lambda_0 < 1/C$ small enough so that

$$(C - 1) \frac{\lambda}{g(C\lambda)} \geq k \quad \text{for all } 0 < \lambda < \lambda_0,$$

so

$$F(C\lambda) \leq F(\lambda) - k \quad \text{for all } 0 < \lambda < \lambda_0,$$

so setting $\lambda := F^{-1}(t)$ for some $t > F(\lambda_0)$ we get

$$F(CF^{-1}(t)) \leq t - k \quad \text{for all } t > F(\lambda_0),$$

which after applying F^{-1} gives the inequality in the lemma whenever $t > F(\lambda_0)$. The inequality is also clearly true, with some other constant C , for all $t \in [k, F(\lambda_0)]$, since this is a compact interval. \square

We are now ready to give the proof of Theorem 4.8:

Proof of Theorem 4.8. Let $\|\cdot\|_{V_1}$ and $\|\cdot\|_{V_2}$ denote the norms from Lemma 4.3, equivalent to $\|\cdot\|_{V_1}$ and $\|\cdot\|_{V_2}$ respectively, defined by

$$\begin{aligned} \|\mu\|_{V_1} &:= \|\mu\| + \beta_1\|\mu\|_{V_1}, \quad \forall \mu \in \mathcal{M}_{V_1}, \\ \|\mu\|_{V_2} &:= \|\mu\| + \beta_2\|\mu\|_{V_2}, \quad \forall \mu \in \mathcal{M}_{V_2}, \end{aligned}$$

with

$$\beta_1 := (1 - \gamma_H)/(K_1N), \quad \beta_2 := (1 - \gamma_H)/(K_2N).$$

We also take

$$\alpha := \min\{\beta_1(\varsigma_1 - K_1/A_1), \beta_2(\varsigma_2 - K_2/A_2)\} > 0.$$

Take $\nu \in \mathcal{N}_{V_2}$, and denote $\nu_k := S^k\nu$ for integer $k \geq 0$.

Step 1. Uniform bound on the V_2 norm. Using Lemma 4.3 for V_2 , we can recursively define an increasing sequence of integers $(n_i)_{i \geq 0}$ such that $n_0 = 0$, $N \leq n_{i+1} - n_i \leq 2N - 1$ for all i and which satisfy

$$\|\nu_{n_{i+1}}\|_{V_2} \leq \|\nu_{n_i}\|_{V_2}, \quad i \geq 1. \tag{4.18}$$

On the other hand, using the weak Lyapunov condition (Hypothesis 4) and the total variation non-expansive property (2.1), we have

$$\|\nu_{k+1}\|_{V_2} \leq \|\nu_k\|_{V_2} + \beta_2K_2\|\nu_k\| \leq (1 + \beta_2K_2)\|\nu_k\|_{V_2},$$

and thus

$$\|\nu_k\|_{V_2} \leq C_2\|\nu\|_{V_2}, \quad \text{for all } k \geq 0, \tag{4.19}$$

with $C_2 := (1 + \beta_2K_2)^N$.

Step 2. Decay along a subsequence. Using again Lemma 4.3, now for V_1 , we can recursively define a (possibly different) increasing sequence of integers $(n_i)_{i \geq 0}$ such that $n_0 = 0$, $N \leq n_{i+1} - n_i \leq 2N - 1$ for all i and which satisfy

$$\|\nu_{n_{i+1}}\|_{V_1} + \alpha \sum_{k=n_i}^{n_{i+1}-1} \|\nu_k\|_{\varphi_1(V_1)} \leq \|\nu_{n_i}\|_{V_1}, \quad i \geq 1, \tag{4.20}$$

and in particular

$$\|\nu_{n_{i+1}}\|_{V_1} + \alpha\|\nu_{n_i}\|_{\varphi_1(V_1)} \leq \|\nu_{n_i}\|_{V_1}, \quad i \geq 1, \tag{4.21}$$

where we have ignored all terms in the sum except for the first one. From (4.21) and the interpolation condition (4.12) we deduce, for any $\lambda > 0$,

$$\|\nu_{n_{i+1}}\|_{V_1} + \lambda\alpha\|\nu_{n_i}\|_{V_1} \leq \|\nu_{n_i}\|_{V_1} + \xi(\lambda)\alpha\|\nu_{n_i}\|_{V_2}.$$

We now use that the norms $\|\cdot\|_{V_1}$ and $\|\cdot\|_{V_2}$ are equivalent, respectively, to $\|\|\cdot\|\|_{V_1}$ and $\|\|\cdot\|\|_{V_2}$: from the definition of $\|\|\cdot\|\|_{V_1}$,

$$\lambda\alpha\|\nu_{n_i}\|_{V_1} \geq \lambda\kappa\|\|\nu_{n_i}\|\|_{V_1} \quad \text{with } \kappa := \frac{\alpha}{1 + \beta_1},$$

for any $\lambda > 0$. Also, from the definition of the $\|\|\cdot\|\|_{V_2}$ norm and (4.19),

$$\|\nu_{n_i}\|_{V_2} \leq \frac{1}{\beta_2}\|\|\nu_{n_i}\|\|_{V_2} \leq \frac{C_2}{\beta_2}\|\|\nu\|\|_{V_2}.$$

The three previous estimates together imply

$$\|\|\nu_{n_{i+1}}\|\|_{V_1} \leq (1 - \lambda\kappa)\|\|\nu_{n_i}\|\|_{V_1} + \frac{\xi(\lambda)C_2\alpha}{\beta_2}\|\|\nu\|\|_{V_2},$$

for any $i \geq 0$ and for any $\lambda > 0$. Using that $V_1 \leq V_2$, so $\|\|\nu\|\|_{V_1} \leq \|\|\nu\|\|_{V_2}$, Lemma 4.11 with $\zeta(\lambda) := \xi(\lambda/\kappa)$ and $M := \|\|\nu\|\|_{V_2} \max\{1, C_2\alpha/\beta_2\}$ then implies

$$\|\|\nu_{n_i}\|\|_{V_1} \leq \frac{m}{\kappa}\|\|\nu\|\|_{V_2}\Theta(\kappa i) \quad \text{for all } i \geq 1, \tag{4.22}$$

where $m := \max\{1, C_2\alpha/\beta_2\}$ and Θ is the decay rate function defined in the statement. Notice that we have used that $\zeta^*(s) = \xi^*(\kappa s)$ for all $s \in \mathbb{R}$, and that

$$\int_{\lambda}^1 \frac{1}{\zeta^*(s)} ds = \int_{\lambda}^1 \frac{1}{\xi^*(\kappa s)} ds = \frac{1}{\kappa} \int_{\kappa\lambda}^{\kappa} \frac{1}{\xi^*(s)} ds \leq \frac{1}{\kappa} F(\kappa\lambda),$$

so the decay rate in Lemma 4.11 is bounded by the one given in (4.22).

Step 3. Decay along the full sequence. Now we have proved this decay rate along the sequence $(n_i)_{i \geq 0}$. In order to extend this to all indices k , we observe that proceeding exactly as in the proof of (4.19), we get

$$\|\|S^i \nu_k\|\|_{V_1} \leq C_1^i \|\|\nu_k\|\|_{V_1}, \quad \text{for all } k, i \geq 0, \tag{4.23}$$

with $C_1 := 1 + \beta_1 K_1$. For any $k \geq 0$, choose $j \geq 0$ such that $n_j \leq k < n_{j+1}$. Due to the spacing of the terms n_i , it must hold that

$$\lfloor k/(2N - 1) \rfloor \leq j \leq \lfloor k/N \rfloor. \tag{4.24}$$

Writing $k = n_j + i$ for some $0 \leq i \leq 2N - 2$, we have, using (4.22), (4.23) and (4.24),

$$\begin{aligned} \|\nu_k\|_{V_1} &= \|S^i \nu_{n_j}\|_{V_1} \leq C_1^i \|\nu_{n_j}\|_{V_1} \\ &\leq C_1^{2N-2} \frac{m}{\kappa} \Theta(\kappa j) \|\nu\|_{V_2} \leq C \Theta\left(\kappa \left\lfloor \frac{k}{2N-1} \right\rfloor\right) \|\nu\|_{V_2}, \end{aligned}$$

where the constant C is given by

$$C := C_1^{2N-2} \frac{m}{\kappa} = C_1^{2N-2} \frac{1}{\kappa} \max\{1, C\alpha/\beta_2\}.$$

Step 4. Simplification of the decay rate. We notice that

$$\left\lfloor \frac{k}{2N-1} \right\rfloor \geq \frac{k}{2N-1} - 1 \quad \text{for all } k \geq 0,$$

so we may use Lemma 4.12 to obtain

$$\Theta\left(\kappa \left\lfloor \frac{k}{2N-1} \right\rfloor\right) \leq \Theta\left(\frac{\kappa k}{2N-1} - \kappa\right) \leq C \Theta\left(\frac{\kappa k}{2N-1}\right)$$

for all $k \geq 2N - 1$ and some constant $C > 0$. Since the inequality

$$\Theta\left(\kappa \left\lfloor \frac{k}{2N-1} \right\rfloor\right) \leq C \Theta\left(\frac{\kappa k}{2N-1}\right)$$

is clearly also true for some (other) $C \geq 1$ and the finite set of integers $0 \leq k \leq 2N - 1$, we obtain the form of the decay rate given in the statement.

Step 5. Decay in total variation norm. In order to deduce the second estimate (4.14), we come back to the first inequality in (4.20) that we iterate and sum up in order to obtain, for any $0 \leq j < i$,

$$\|\nu_{n_i}\|_{V_1} + \alpha \sum_{k=n_j}^{n_i-1} \|\nu_k\|_{\varphi_1(V_1)} \leq \|\nu_{n_j}\|_{V_1}.$$

Together with the non expansion inequality

$$\|\nu_{n_i}\| \leq \|\nu_k\| \leq \|\nu_k\|_{\varphi_1(V_1)}, \quad \forall k \leq n_i,$$

and the decay proved in (4.22), we deduce

$$(n_i - n_j)\alpha \|\nu_{n_i}\| \leq M \|\nu\|_{V_2} \Theta(j).$$

Choosing $j = \lfloor i/2 \rfloor$ and using that $n_i - n_j \geq N(j - i)$,

$$\|\nu_{n_i}\| \leq \frac{2M}{\alpha^i} \|\nu\|_{V_2} \Theta(\lfloor i/2 \rfloor).$$

Carrying out a similar argument as above to extend this to all indices k , we obtain

$$\|\nu_k\| \leq \frac{C_3}{k} \Theta\left(\left\lfloor \frac{k}{4N-2} \right\rfloor\right) \|\nu\|_{V_2},$$

for all $k \geq 1$, for some other constant $C_3 > 0$. A similar reasoning as in the previous step gives the simpler form of the decay rate given in the statement. \square

4.4. Subgeometric decay rates for Feller type stochastic operator

As a consequence of Theorem 4.8, we can prove the following theorem adapted to Feller type stochastic operator and which is closer to the continuous-time framework developed in Douc et al. [15], see also Hairer [20].

Theorem 4.13 (Discrete subgeometric Harris theorem for Feller type stochastic operator). *Consider a stochastic operator S of Feller type such that*

1. S satisfies a weak Lyapunov condition (Hypothesis 4) with functions V , φ and constants ς , K .
2. For some $N \geq 1$, S^N satisfies a Harris condition (Hypothesis 2) on the set $\mathcal{C} := \{\varphi(V) \leq 2R\}$ for some $R > 2K/\varsigma$.

Then there exists a unique equilibrium $\mu^* \in \mathcal{P}_{\varphi(V)}$. Moreover, for any strictly concave function $\psi: [1, +\infty) \rightarrow [1, +\infty)$ with $\psi(1) = \psi'(1) = 1$, $\lim_{v \rightarrow +\infty} \psi(v) = +\infty$, and such that $v \mapsto \psi'(v)\varphi(v)$ is nondecreasing and satisfies $\psi'(v)\varphi(v) > R$ whenever $\varphi(v) > 2R$, we have

$$\|S^n \nu\| \leq \frac{C}{n} \Theta_\psi(rn) \|\nu\|_V \quad \text{for all } n \geq 1, \tag{4.25}$$

for any $\nu \in \mathcal{N}_V$. Here $C \geq 1$ and $0 < r < 1$ are constructive constants and the decay rate Θ_ψ is given by

$$\Theta_\psi(n) := F_\psi^{-1}(n),$$

where

$$F_\psi(v) := \int_v^1 \frac{1}{h(u)} du, \quad h(u) := g \circ f^{-1}(u),$$

$$f(v) := \frac{\psi(v)}{v}, \quad g(v) := \frac{\psi'(v)\varphi(v)}{v}.$$

Remark 4.14. Provided that $\varphi'(v)\varphi(v) \geq \Phi(\varphi(v))$ for any $v \geq 1$ for some concave function $\Phi : [1, \infty) \rightarrow [1, \infty)$ and S^N satisfies a Harris condition (Hypothesis 2) on the set $\{\varphi(V) \leq 2R\}$ for any $R > 2K/\varsigma$, the techniques developed here make possible to establish that μ^* is asymptotically stable: there exists a decay rate function $\tilde{\Theta}$ such that for any $\mu \in \mathcal{P}_{\varphi(V)}$ there holds

$$\|S^n \mu - \mu^*\| \leq \tilde{\Theta}(n) \|\mu - \mu^*\|_{\varphi(V)}, \quad \forall n \geq 1. \tag{4.26}$$

Defining indeed the weight function $W := \varphi(V)$, Lemma 4.7 implies that

$$PW + \varsigma\Phi(W) \leq W + K,$$

which is nothing but saying that S satisfies a weak Lyapunov condition (Hypothesis 4 and Remark 4.2) with functions W , Φ and constants ς , K . Because of the above strong Harris condition, Theorem 4.13 holds with W for a family of rate functions $\tilde{\Theta}$. For any on them, we may apply Theorem 4.13 with $\nu := \mu - \mu^* \in \mathcal{P}_W$ and deduce (4.26).

Proof of Theorem 4.13. First, we notice that f is invertible since $v \mapsto \psi(v)/v$ is strictly decreasing, as can be seen from

$$\frac{d}{dv} \frac{\psi(v)}{v} = \frac{v\psi'(v) - \psi(v)}{v^2} < 0 \quad \text{for } v > 1,$$

since $v\psi'(v) < \psi(v)$ for $v > 1$ due to the strict concavity of ψ and the fact that $\psi'(1) = 1$.

In order to show the result we use Theorem 4.8 with

$$V_2 := V, \quad V_1 := \psi(V).$$

Let us check the assumptions of Theorem 4.8. First, the weak Lyapunov conditions (4.1) and (4.8) are satisfied for $V_2 = V$ by assumption. In order to see that a weak Lyapunov condition holds also for $V_1 = \psi(V)$, use that ψ is concave to write, with Jensen’s inequality (4.9) and (4.8),

$$\begin{aligned} P\psi(V) &\leq \psi(V - \varsigma\varphi(V) + K) \leq \psi(V) - \varsigma\psi'(V)\varphi(V) + \psi'(V)K \\ &\leq \psi(V) - \varsigma\psi'(V)\varphi(V) + K = \psi(V) - \varsigma\varphi_1(\psi(V)) + K, \end{aligned}$$

where we make the choice

$$\varphi_1(\psi(v)) := \psi'(v)\varphi(v).$$

Notice that φ_1 is a nondecreasing function with $\varphi_1(w) \geq \varphi_1(1) = 1$. Observing that

$$P\psi(V \wedge n) \leq P\psi(V) \leq \psi(V) - \varsigma\varphi_1(\psi(V)) + K,$$

by duality for any $0 \leq \mu \in \mathcal{M}_{\psi(V)}$, we have

$$\|S\mu\|_{\psi(V \wedge n)} = \int \mu P(V \wedge n) \leq \int \mu(\psi(V) - \varsigma\varphi_1(\psi(V)) + K).$$

By Beppo Levi theorem, we may pass to the limit in the above inequality and we obtain that the weak Lyapunov condition (4.1) also holds for $V_1 = V$. Notice that both weak Lyapunov conditions for V_1 and V_2 hold with the same constants ς and K .

The Harris condition for S^N is satisfied on the set

$$\tilde{\mathcal{C}} = \{x \in \Omega \mid \varphi_1(\psi(v)) \leq R\},$$

since by hypothesis the condition $\varphi_1(\psi(v)) = \psi'(v)\varphi(v) \leq R$ implies $\varphi(v) \leq 2R$, so $\tilde{\mathcal{C}} \subseteq \mathcal{C}$. Lemma 3.1 shows that S^N satisfies the local coupling condition (Hypothesis 3) for $\varphi_1(\psi(V))$ (and hence for $\varphi(V)$, which is larger), both with any constant $A < R/2$. Since $R > 2K/\varsigma$ we may take $A > K/\varsigma$, and the hypotheses of Theorem 4.8 are met. In order to express the conclusion of Theorem 4.8, we observe that the interpolation function ξ in (4.12) can be written more explicitly in the present case where $V_1 = \psi(V_2)$ for some function $\psi: [1, +\infty) \rightarrow [1, +\infty)$ with $\psi(v)/v$ strictly decreasing. Indeed, in that case, the interpolation is equivalent to

$$\xi(\lambda) \geq \lambda f(v) - g(v) \quad \text{for all } v \geq 1, \tag{4.27}$$

where

$$f(v) := \psi(v)/v, \quad g(v) := \varphi_1(\psi(v))/v.$$

Substituting $v = f^{-1}(z)$ in (4.27), $\xi(\lambda)$ must satisfy

$$\xi(\lambda) \geq \lambda z - g(f^{-1}(z)) \quad \text{for all } 0 < z \leq 1,$$

so we can choose

$$\xi(\lambda) := h^*(\lambda), \quad \text{where } h: (0, 1] \rightarrow \mathbb{R} \text{ is given by } h(z) := g(f^{-1}(z)).$$

Thus we have $\xi^* = h$ and we obtain $F = F_\psi$ in the conclusion of Theorem 4.8. \square

There remains the question of choosing the function ψ which gives an optimal decay rate $\tilde{\Theta}_\psi$. There are two “extreme” choices for ψ : one can take ψ asymptotically like

$$H(u) := \int_1^u \frac{1}{\varphi(v)} \, dv$$

(so that $\psi'(v)\varphi(v)$ behaves like a constant as $v \rightarrow +\infty$); or one can take $\psi(v)$ almost equal to v (but still strictly concave). These are both useful in different cases, as we show now in examples:

Polynomial decay Let us take

$$\psi(u) := 1 + \int_1^u \frac{m(v)}{\varphi(v)} \, dv, \quad u \geq 1,$$

for some continuous, nondecreasing $m: [1, +\infty) \rightarrow [1, +\infty)$ such that $v \mapsto m(v)/\varphi(v)$ is strictly decreasing, $m(1) = 1$, and with

$$m(v) > R \quad \text{whenever} \quad \varphi(v) > 2R.$$

It is possible to find such m , since one may take

$$m(v) := \begin{cases} \varphi(v)^{1-\epsilon} & \text{if } \varphi(v) < 2R, \\ (2R)^{1-\epsilon} & \text{if } \varphi(v) \geq 2R, \end{cases}$$

for small enough $\epsilon > 0$. The quantities in Theorem 4.13 can then be bounded as follows:

$$g(u) = \frac{1}{u} \psi'(u) \varphi(u) = \frac{m(u)}{u} \geq \frac{1}{u}.$$

$$F_\psi(\zeta) = \int_\zeta^1 \frac{1}{h(u)} \, du \leq \int_\zeta^1 f^{-1}(\xi) \, d\xi.$$

For example, if $\varphi(v) = v^{1-\alpha}$, $\alpha \in (0, 1)$, we obtain (with C standing for a positive constant)

$$\psi(u) \leq 1 + Cu^\alpha, \quad f(u) = \frac{\psi(u)}{u} \leq Cu^{\alpha-1}, \quad f^{-1}(\xi) \leq C\xi^{\frac{1}{\alpha-1}},$$

$$F_\psi(\zeta) \leq C(\zeta^{\frac{\alpha}{\alpha-1}} - 1), \quad F_\psi^{-1}(t) \leq (1 + Ct)^{1-\frac{1}{\alpha}}.$$

As a conclusion, in that case, the rate of convergence (4.25) is

$$\|S^n \nu\| \leq \frac{C}{n^{1/\alpha}} \|\nu\|_V \quad \text{for all } n \geq 1, \tag{4.28}$$

for any $\nu \in \mathcal{N}_V$ and some explicitly computable constant $C \geq 1$.

Exponential decay On the other hand, when $\varphi(u) := u/(\log u)^\alpha$, $\alpha > 0$, we take $\psi(u) = u^\kappa$, $0 < \kappa < 1$. We next compute

$$f(v) = v^{\kappa-1}, \quad g(v) = \kappa v^{\kappa-1}(\log v)^{-\alpha}, \quad h(u) = C_1 u(\log u^{-1})^{-\alpha},$$

for a constant $C_1 = C_1(\alpha, \kappa) \in (0, \infty)$, and finally

$$F(v) = C_2(\log v^{-1})^{\alpha+1}, \quad F^{-1}(u) = e^{-C_3 u^{\frac{1}{\alpha+1}}},$$

for some constants $C_i = C_i(\alpha, \kappa) \in (0, \infty)$. As a conclusion, in that case, the rate of convergence (4.25) is

$$\|S^n \nu\| \leq C e^{-\lambda n^{\frac{1}{\alpha+1}}} \|\nu\|_V \quad \text{for all } n \geq 1, \tag{4.29}$$

for any $\nu \in \mathcal{N}_V$ and some explicitly computable constants $C \geq 1$, $\lambda \in (0, \infty)$.

5. Results for continuous-time semigroups

In this section we again address the speed of relaxation to equilibrium, this time in the framework of continuous-time semigroups. The most straightforward results are obtained by applying the discrete-time results in the previous sections to any stochastic semigroup $(S_t)_{t \geq 0}$ as long as S_T satisfies the needed assumptions for some $T > 0$. We will state these results first. Then, in the setting of continuous-time semigroups it is perhaps more natural to look for similar Foster-Lyapunov-type conditions on the generator of the semigroup instead of conditions on S_T for a given $T > 0$. Our main aim in this section is to prove results of this type. We notice that they can be obtained as consequences of our discrete-time results both in the geometric and subgeometric cases.

5.1. Geometric convergence

First we state Doeblin’s Theorem 2.1, as applied to a continuous semigroup, with a straightforward proof:

Theorem 5.1 (*Semigroup version of Doeblin’s theorem*). *Let $(S_t)_{t \geq 0}$ be a stochastic semigroup in \mathcal{M} . If there exists $T > 0$ such that S_T satisfies the Doeblin condition (2.2) then the semigroup $(S_t)_{t \geq 0}$ has a unique equilibrium μ^* in \mathcal{P} , and*

$$\|S_t \nu\| \leq \frac{1}{1 - \alpha} e^{-\lambda t} \|\nu\|, \quad \text{for all } t \geq 0, \tag{5.1}$$

for all $\nu \in \mathcal{N}$, where

$$\lambda := -\frac{\log(1 - \alpha)}{T} > 0.$$

Proof of Theorem 5.1. Theorem 2.1 shows that the operator S_T has a unique stationary state in \mathcal{P} , which we call μ^* . In fact, μ^* is a stationary state of the whole semigroup since, for all $s \geq 0$, we have

$$S_T S_s \mu^* = S_s S_T \mu^* = S_s \mu^*,$$

which shows that $S_s \mu^*$ (which is again a probability measure) is also a stationary state of S_T . Due to uniqueness, we deduce

$$S_s \mu^* = \mu^* \quad \text{for all } s \geq 0.$$

This stationary state is clearly unique in \mathcal{P} , since any stationary state of $(S_t)_{t \geq 0}$ is in particular a stationary state of S_T .

In order to show (5.1), for any $\nu \in \mathcal{N}$ and any $t \geq 0$ we write $k := \lfloor t/T \rfloor$, where $\lfloor \cdot \rfloor$ is the floor function, so that

$$\frac{t}{T} - 1 < k \leq \frac{t}{T}.$$

Then,

$$\begin{aligned} \|S_t \nu\| &= \|S_{t-kT} S_{kT} \nu\| \leq \|S_{kT} \nu\| \\ &\leq (1 - \alpha)^k \|\nu\| \leq \frac{1}{1 - \alpha} \exp\left(\frac{t \log(1 - \alpha)}{T}\right) \|\nu\|, \end{aligned}$$

which is nothing but (5.1). \square

As above, we could write the immediate counterpart of Harris’s Theorem 3.2, as applied to a continuous semigroup. We rather present a version more adapted to a semigroup setting. Indeed, in the continuous time setting, it is natural to consider Foster-Lyapunov conditions on the generator Λ of the semigroup $(S_t)_{t \geq 0}$. A natural assumption that replaces the operator Lyapunov condition in Hypothesis 1 is

$$LV \leq -\sigma V + b, \tag{5.2}$$

for some constants $\sigma, b > 0$ and some continuous weight (Lyapunov) function $V: \Omega \rightarrow [1, +\infty)$, where L is dual to the generator Λ . When $(S_t)_{t \geq 0}$ is of Feller type, we have $\Lambda = L^*$ and L is the generator of the associated Markov-Feller semigroup $(P_t)_{t \geq 0}$ on $C_0(\Omega)$. This faces the technical problem that the generator L may not be defined on the particular functions V we wish to consider. However, for $0 \leq \mu \in \mathcal{M}_V$, we may compute at least formally

$$\frac{d}{dt} \int V S_t \mu = \int L V S_t \mu \leq \int (-\sigma V + b) S_t \mu. \tag{5.3}$$

After time integration, we thus get (still formally)

$$\int V S_t \mu \leq \int V \mu + \int_0^t \int (-\sigma V + b) S_s \mu \, ds, \tag{5.4}$$

for all $t \geq 0$ and all $0 \leq \mu \in \mathcal{M}_V$. Equation (5.4) is a common way to understand the generator Lyapunov condition (5.2) and thus to avoid the difficulty of defining LV . Let us also notice that this problem has been circumvented in different ways in other works: for diffusion semigroups, the generator is a local operator which is naturally defined on arbitrary C^2 functions V [1]; for general semigroups, one may define (5.2) to mean that the process $V(X_t) - \int_0^t (-\sigma V(X_s) + b) \, ds$ is a supermartingale for every starting condition x_0 (where $(X_t)_{t \geq 0}$ is the process associated to the semigroup $(S_t)_{t \geq 0}$), a path which is taken for example in Douc et al. [15], Hairer [20]. This probabilistic formulation is equivalent to saying that the associated Markov-Feller semigroup $(P_t)_{t \geq 0}$ satisfies

$$P_t V \leq V + \int_0^t P_s (-\sigma V + b) \, ds, \quad \text{for } t \geq 0, \tag{5.5}$$

or equivalently, that the associated stochastic semigroup $(S_t)_{t \geq 0}$ satisfies (5.4). Another possible alternative formulation, which we will not use in this work, is to require that (5.3) effectively holds for any positive μ which belongs to the domain $D_V(L)$ defined by

$$D_V(L) := \left\{ \mu \in \mathcal{M}_V; \lim_{t \searrow 0} \int \frac{S_t \mu - \mu}{t} \phi \text{ exists, for any } \phi \in C(\Omega), \phi/V \text{ bounded} \right\},$$

provided that this one is dense in \mathcal{M}_V .

One can take this a step further by observing that, again at least formally, one may use Gronwall’s lemma on (5.4) in order to deduce the bound

$$\|S_t \mu\|_V \leq e^{-\sigma t} \|\mu\|_V + \frac{b}{\sigma} (1 - e^{-\sigma t}) \|\mu\| \quad \text{for all } t \geq 0 \text{ and all } \mu \in \mathcal{M}_V. \tag{5.6}$$

In a first result, we choose to avoid these technical problems altogether and state as an assumption the specific consequence we need from either (5.2) or (5.5), which is the following:

Hypothesis 5 (*Semigroup Lyapunov*). Let $V : \Omega \rightarrow [1, +\infty)$ be a measurable function and $(S_t)_{t \geq 0}$ a stochastic semigroup on \mathcal{M}_V . We say the semigroup $(S_t)_{t \geq 0}$ satisfies the *semigroup Lyapunov condition* with function V when there exist constants $\sigma, b > 0$ such that (5.6) holds.

We will later give a specific hypothesis on the dual generator L which ensures this condition holds (see Hypothesis 6). However, we believe it is useful to state the basic

condition in Hypothesis 5, since in concrete applications it may well happen that (5.6) can be proved in some other way.

Theorem 5.2 (*Semigroup version of Harris’s theorem*). *Let $V: \Omega \rightarrow [1, +\infty)$ be a measurable (weight) function and let $(S_t)_{t \geq 0}$ be a stochastic semigroup in \mathcal{M}_V . Assume that*

1. *The semigroup $(S_t)_{t \geq 0}$ satisfies the semigroup Lyapunov condition (Hypothesis 5).*
2. *For some $T > 0$, S_T satisfies the local coupling condition (Hypothesis 3) with $b/A < \sigma$.*

Then the semigroup has an invariant probability measure $\mu^ \in \mathcal{P}_V$ which is unique within \mathcal{P}_V , and there exist $\lambda, C > 0$ such that*

$$\|S_t \nu\|_V \leq C e^{-\lambda t} \|\nu\|_V, \quad \text{for } t \geq 0, \tag{5.7}$$

for all $\nu \in \mathcal{N}_V$.

Proof of Theorem 5.2. Hypothesis 5 shows that the operator Lyapunov condition (Hypothesis 1) holds for S_T , since

$$\|S_T \mu\|_V \leq e^{-\sigma T} \|\mu\|_V + \frac{b}{\sigma} (1 - e^{-\sigma T}) \|\mu\|,$$

for all $\mu \in \mathcal{M}_V$. The condition $b/A < \sigma$ hence ensures S_T is in the conditions of Theorem 3.2.

With the same reasoning as in the proof of Theorem 5.1, we see that $(S_t)_{t \geq 0}$ has a unique stationary state in \mathcal{P}_V , which we call μ^* . We know from Theorem 3.2 that there exist a new norm $\|\cdot\|_V$ (defined through (3.5) and a parameter $\beta > 0$, which is equivalent to the norm $\|\cdot\|_V$) and $0 < \gamma < 1$ such that

$$\|S_T \nu\|_V \leq \gamma \|\nu\|_V,$$

for all measures $\nu \in \mathcal{M}_V$. In order to show (5.7), we follow a similar reasoning as in the proof of Theorem 5.1. Notice first that due to (2.1) and (1.2) we have, for $0 \leq t \leq T$,

$$\|S_t \mu\|_V \leq C_V e^{\omega_V T} \|\mu\|_V, \tag{5.8}$$

for all measures $\mu \in \mathcal{M}_V$. We conclude that (5.7) holds with

$$C := \frac{C_V e^{\omega_V T}}{\gamma} \frac{1 + \beta}{\beta}, \quad \lambda := -\frac{\log \gamma}{T} > 0,$$

and γ, β are the constants in Theorem 3.2 as applied to the operator S_T . \square

We end this section by noticing that in the case of a Feller-type semigroup, Hypothesis 5 is a consequence of the following condition on the dual generator L of the associated Markov-Feller semigroup $(P_t)_{t \geq 0}$ on $C_0(\Omega)$. Hence Theorem 5.2 also holds if $(S_t)_{t \geq 0}$ is a Feller-type stochastic semigroup which satisfies the following hypothesis instead of Hypothesis 5:

Hypothesis 6 (*Generator Lyapunov*). Let $(S_t)_{t \geq 0}$ be a Feller-type stochastic semigroup on \mathcal{M}_V . We say the semigroup $(S_t)_{t \geq 0}$ satisfies the *generator Lyapunov condition* with function V when there exist constants $\sigma, b > 0$ such that (5.4) holds.

The fact that in the case of a Feller-type stochastic semigroup, Hypothesis 6 implies Hypothesis 5 is a straightforward consequence of the following version of Gronwall’s lemma:

Lemma 5.3 (*Gronwall lemma*). Consider $(S_t)_{t \geq 0}$ a Feller-type stochastic semigroup in \mathcal{M}_V which satisfies the generator Lyapunov condition (Hypothesis 6) associated to V and some constants $\sigma, b > 0$. Then $(S_t)_{t \geq 0}$ satisfies the corresponding semigroup Lyapunov condition (Hypothesis 5) with the same function V and constants $\sigma, b > 0$.

We observe that the difficulty in proving this result is that there is no reason why the function $t \mapsto \int \mu_t V$ should be continuous, which makes it difficult to apply standard results on differential inequalities, which usually require a continuous function.

Proof of Lemma 5.3. We fix $0 \leq \mu_0 \in \mathcal{M}_V$ and we set $\mu_t := S_t \mu_0$. We split the proof into four steps.

Step 1. We first observe that the Lyapunov condition (5.4) is equivalent to

$$\int \mu_{t_2} V + \sigma \int_{t_1}^{t_2} \int \mu_s V ds \leq \int \mu_{t_1} V + b \int_{t_1}^{t_2} \int \mu_s ds, \tag{5.9}$$

for any $t_2 > t_1 \geq 0$ and all $0 \leq \mu \in \mathcal{M}_V$. The inequality (5.4) is indeed a particular case of (5.9) and the reciprocal implication is an immediate consequence of the semigroup property of $(S_t)_{t \geq 0}$.

Step 2. We claim that

$$t \mapsto \int V \mu_t \text{ is right-continuous.} \tag{5.10}$$

We recall that because $(S_t)_{t \geq 0}$ is of Feller-type, there holds

$$t \mapsto \int \mu_t \chi \in C(\mathbb{R}_+; \mathbb{R}_+), \tag{5.11}$$

for any $\chi \in C_c(\Omega)$, and even for any $\chi \in C_b(\Omega)$. On the one hand, as a consequence of (5.11), for any $\chi \in C_c(\Omega)$, $\chi \leq V$, there holds

$$\int \mu_t \chi = \lim_{s \rightarrow t} \int \mu_s \chi \leq \liminf_{s \rightarrow t} \int \mu_s V.$$

Choosing $\chi_n \nearrow V$, the monotone convergence theorem implies

$$\int \mu_t V = \lim_{n \rightarrow \infty} \int \mu_t \chi_n \leq \liminf_{s \rightarrow t} \int \mu_s V. \tag{5.12}$$

On the other hand, the semigroup Lyapunov condition (5.9) implies

$$\int V \mu_s \leq \int V \mu_t + b(s - t) \int \mu_0, \quad \forall s > t.$$

We deduce that

$$\limsup_{s \searrow t} \int V \mu_s \leq \lim_{s \searrow t} \left\{ \int V \mu_t + b(s - t) \int \mu_0 \right\} = \int V \mu_t. \tag{5.13}$$

Equations (5.12) and (5.13) together imply (5.10).

Step 3. We claim that the Lyapunov condition (5.4) (or equivalently (5.9)) is equivalent to the fact that (5.10) holds together with

$$\frac{d}{dt} \int V \mu_t \leq -\sigma \int V \mu_t + b \int \mu_t, \tag{5.14}$$

in the sense of $\mathcal{D}'(0, +\infty)$, the space of distributions on $(0, +\infty)$. On the one hand, if we assume (5.9) holds then (5.10) holds from Step 2. Multiplying equation (5.9) by a nonnegative test function $\varphi \in \mathcal{D}(0, +\infty)$, dividing it by $t_2 - t_1$, integrating and passing to the limit as $t_2 \searrow t_1$ (and using (5.10) to do this), we deduce (5.14).

On the other hand, if we assume that both (5.10) and (5.14) hold, in particular this means that

$$\int_0^\infty \int V \mu_s \phi'_s \, ds + \sigma \int_0^\infty \phi_s \int V \mu_s \, ds \leq b \int_0^\infty \phi_s \int \mu_s \, ds, \tag{5.15}$$

for any $0 \leq \phi \in \mathcal{D}(0, +\infty)$. For $t > 0$ and a given function $0 \leq \rho \in \mathcal{D}(\mathbb{R}_+)$ with integral 1 and $\text{supp } \rho \subset (0, 1)$, define the sequence (ϕ_n) by $\phi_n(0) := 0$ and $\phi'_n(s) := n\rho(s/n) - n\rho((s-t)/n)$. We may pass to the limit $n \rightarrow \infty$ in (5.15) by taking advantage of (5.10), and we conclude (5.4).

Step 4. We introduce a mollifier (ρ_ε) with $\text{supp}(\rho_\varepsilon) \subset (-\varepsilon, 0)$ and the function

$$u_\varepsilon(t) = (\|\mu\|_V * \rho_\varepsilon)(t) = \int_{\mathbb{R}} \|\mu_s\|_V \rho_\varepsilon(t - s) \, ds,$$

which clearly satisfies $u_\varepsilon \in C^1$. From (5.14), u_ε also clearly satisfies

$$u'_\varepsilon \leq -\sigma u_\varepsilon + K \int \mu_0,$$

pointwise on $(0, \infty)$. From the classical version of the Gronwall lemma, we deduce that

$$u_\varepsilon(t_2) \leq e^{-\sigma(t_2-t_1)}u_\varepsilon(t_1) + \frac{K}{\sigma}(1 - e^{-\sigma(t_2-t_1)}) \int \mu_0,$$

for any $t_2 > t_1 > 0$ and any $\varepsilon > 0$. Observing that $u_\varepsilon(t) \rightarrow \|\mu_t\|_V$ as $\varepsilon \rightarrow 0$ for any $t \geq 0$ because of (5.10), we obtain that the semigroup Lyapunov condition (5.6) holds for any $t = t_2 > 0$, by passing to the limit $\varepsilon \rightarrow 0$ and next $t_1 \rightarrow 0$. \square

5.2. Subgeometric convergence

As we have just done for the geometric case, one may state analogous results to Theorems 4.8 or 4.13 in the case of a continuous semigroup $(S_t)_{t \geq 0}$, as long as the conditions of the theorems are satisfied by S_T for some time $T > 0$. However, the conditions in Theorems 4.8 and 4.13 are not so natural for a continuous semigroup, since they involve estimates for S_T and powers of S_T .

We will avoid these statements and give more convenient conditions in terms of the semigroup, in the spirit of Hypothesis 5, and in terms of the generator of the semigroup, in the spirit of Hypothesis 6.

A natural weak counterpart of the Lyapunov condition (5.2) consists in assuming that

$$LV \leq -\sigma\varphi(V) + b, \tag{5.16}$$

for some measurable weight (Lyapunov) function $V: \Omega \rightarrow [1, +\infty)$, some concave function $\varphi: [1, +\infty) \rightarrow [1, +\infty)$ and some constants $\sigma, b > 0$, where L is adjoint to the generator Λ of $(S_t)_{t \geq 0}$. This runs into the same technical problems discussed before Hypothesis 5, so we will again use the consequence we would like to extract as an assumption, and leave it to be checked in each specific application. Proceeding similarly from (5.16) as for (5.2), we may formally compute

$$\|S_t\mu\|_V \leq \|\mu\|_V + \int_0^t (b\|S_u\mu\| - \sigma\|S_u\mu\|_{\varphi(V)}) \, du, \tag{5.17}$$

for any $0 \leq \mu \in \mathcal{M}_V$ and any $t \geq 0$. For a Feller-type semigroup, this condition is equivalent to conditions (3.1) or (3.2) in Douc et al. [15]. We can take one more step and deduce from (5.17) (still formally) the weak confinement counterpart of (5.6). As we will establish for a Feller-type stochastic semigroup (see Corollary 5.8 below), a natural consequence is

$$\|S_t\mu\|_V + \sigma t\|S_t\mu\|_{\varphi(V)} \leq \|\mu\|_V + K_t\|\mu\|, \tag{5.18}$$

for all $t \geq 0$ and $\mu \in \mathcal{M}_V$, with $K_t := tb(1 + \sigma t/2)$. We then take this last property as the assumption we impose on the semigroup:

Hypothesis 7 (*Weak semigroup Lyapunov condition*). We say that a stochastic semigroup $(S_t)_{t \geq 0}$ satisfies the weak semigroup Lyapunov condition for a weight function $V: \Omega \rightarrow [1, +\infty)$ and a concave function $\varphi: [1, +\infty) \rightarrow [1, +\infty)$, $\varphi(v) \leq v$ for any $v \geq 1$, if there exist constants $b, \sigma > 0$ such that (5.18) holds.

The following continuous-time analogue of Theorem 4.8 is our main result in this setting:

Theorem 5.4 (*subgeometric Harris, interpolated version*). Consider two measurable weight functions $V_1, V_2: \Omega \rightarrow [1, +\infty)$, $V_1 \leq V_2$, and a stochastic semigroup $(S_t)_{t \geq 0}$ on \mathcal{M}_{V_2} such that:

1. the weak semigroup Lyapunov condition (Hypothesis 7) holds for both weights V_1 and V_2 , with functions and constants φ_1, b_1, σ_1 and φ_2, b_2, σ_2 .
2. For some time $T > 0$, S_T satisfies the local coupling condition (Hypothesis 3) for both $\varphi_1(V_1)$ and $\varphi_2(V_2)$, with constants $A_1 > K_1/\sigma_1$ and $A_2 > K_2/\sigma_2$.
3. The interpolation condition (4.12) holds for some ξ .

Then there exists a unique equilibrium $\mu^* \in \mathcal{P}_{\varphi_2(V_2)}$, and there exists some $C > 0$ depending only on the constants in the assumptions such that

$$\|S_t\nu\|_{V_1} \lesssim C\Theta(t)\|\nu\|_{V_2}, \quad \forall t \geq 0,$$

and

$$\|S_t\nu\| \lesssim C\tilde{\Theta}(t)\|\nu\|_{V_2}, \quad \forall t \geq 0,$$

for any $\nu \in \mathcal{N}_V$, where

$$\Theta(t) := F^{-1}(t), \quad \tilde{\Theta}(t) := \frac{1}{t}F^{-1}\left(\frac{t}{2}\right), \quad F(v) := \int_v^1 \frac{1}{\xi^*(u)} du.$$

Before coming to the proof of Theorems 5.4, we present a technical result.

Lemma 5.5. Let $V: \Omega \rightarrow [1, +\infty)$ be a weight function and $\varphi: [1, +\infty) \rightarrow [1, +\infty)$ be a continuous and increasing function with $\varphi(1) = 1$. Let S be a stochastic operator which satisfies the following “implicit” Lyapunov-type condition:

$$\|S\mu\|_V + \sigma\|S\mu\|_{\varphi(V)} \leq \|\mu\|_V + K\|\mu\|, \quad \text{for all } \mu \in \mathcal{P} \cap \mathcal{M}_V \tag{5.19}$$

for some $K, \sigma > 0$. If we define $\tilde{V} : \Omega \rightarrow [1, +\infty)$ and $\tilde{\varphi} : [1, +\infty) \rightarrow [1, +\infty)$ by

$$\tilde{V} := \frac{1}{1 + \sigma}(V + \sigma\varphi(V)), \quad \tilde{\varphi}(\tilde{V}) := \varphi(V)$$

then $\tilde{\varphi}$ is increasing, $\tilde{\varphi}(1) = 1$, and S satisfies a usual weak explicit Lyapunov condition for \tilde{V} and $\tilde{\varphi}$, namely

$$\|S\mu\|_{\tilde{V}} + \frac{\sigma}{1 + \sigma}\|\mu\|_{\tilde{\varphi}(\tilde{V})} \leq \|\mu\|_{\tilde{V}} + \frac{K}{1 + \sigma}\|\mu\|, \quad \text{for all } \mu \in \mathcal{M}_V. \tag{5.20}$$

Proof of Lemma 5.5. First, note that $\tilde{\varphi}$ is well defined and $\tilde{\varphi}(1) = 1$, since $V \mapsto \frac{1}{1 + \sigma}(V + \sigma\varphi(V))$ is a strictly increasing function which takes the value 1 for $V = 1$. Using (5.19) and the definition of \tilde{V} we have, for any $\mu \in \mathcal{M}_V$,

$$\begin{aligned} (1 + \sigma)\|S\mu\|_{\tilde{V}} &= \|S\mu\|_V + \sigma\|S\mu\|_{\varphi(V)} \\ &\leq \|\mu\|_V + K\|\mu\| = (1 + \sigma)\|\mu\|_{\tilde{V}} - \sigma\|\mu\|_{\varphi(V)} + K\|\mu\|. \end{aligned}$$

Due to the definition of $\tilde{\varphi}$, this is precisely (5.20). \square

Proof of Theorem 5.4. Let us show that the conditions of Theorem 4.8 are met by S_{t_0} for a certain $t_0 > 0$. First, for any $t_0 > 0$, we have the following implicit Lyapunov-type inequalities by assumption:

$$\|S_{t_0}\mu\|_{V_i} + \sigma_i t_0 \|S_{t_0}\mu\|_{\varphi_i(V_i)} \leq \|\mu\|_{V_i} + K_i t_0 (1 + \sigma_i t_0 / 2) \|\mu\|,$$

for $i = 1, 2$ and all $\mu \in \mathcal{P} \cap \mathcal{M}_{V_i}$. We may define

$$\tilde{V}_i := \frac{1}{1 + \sigma_i}(V_i + \sigma_i\varphi_i(V_i)), \quad \tilde{\varphi}_i(\tilde{V}_i) := \varphi_i(V_i),$$

and we know from Lemma 5.5 that we also have the weak Lyapunov condition:

$$\|S_{t_0}\mu\|_{\tilde{V}_i} + \frac{\sigma_i t_0}{1 + \sigma_i t_0} \|\mu\|_{\tilde{\varphi}_i(\tilde{V}_i)} \leq \|\mu\|_{\tilde{V}_i} + \frac{K_i t_0}{1 + \sigma_i t_0} (1 + t_0 / 2) \|\mu\|. \tag{5.21}$$

Choose an integer $N > 0$ and take $t_0 := T/N$. We can choose N large enough so that

$$\frac{K_i}{\sigma_i} (1 + \sigma_i t_0 / 2) < A_i, \quad \text{for } i = 1, 2, \tag{5.22}$$

and then all hypotheses of Theorem 4.8 are satisfied by the operator S_{t_0} , since

1. S_{t_0} satisfies the weak Lyapunov condition (5.21) for $\tilde{V}_1, \tilde{\varphi}_1$ and $\tilde{V}_2, \tilde{\varphi}_2$.

2. $S_{t_0}^N = S_T$ satisfies the local coupling condition for both $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$, with constants which satisfy the appropriate inequality thanks to (5.22).
3. The interpolation condition (4.12) and the assumption that $\varphi_1(V_1) \leq V_1$ show that

$$\lambda \tilde{V}_1 = \frac{\lambda}{1 + \sigma_1}(V_1 + \sigma_1 \varphi_1(V_1)) \leq \lambda V_1 \leq \varphi_1(V_1) + \xi(\lambda)V_2 \leq \tilde{\varphi}_1(\tilde{V}_1) + (1 + \sigma_2)\xi(\lambda)\tilde{V}_2.$$

Hence the interpolation condition is satisfied for $\tilde{\xi}(\lambda) := (1 + \sigma_2)\xi(\lambda)$.

Applying Theorem 4.8 gives an estimate of the decay of $\|S_t \mu\|_V$ and $\|S_t \mu\|$ for $t = nt_0$. The same technique used before in the proof of Theorem 5.1 allows us to extend the decay to the whole semigroup and obtain the result. \square

We end this section by specifying Harris’ theorem to the case of a Feller-type semigroup for which some simplifications occur. In this setting, the relevant confinement condition writes:

Hypothesis 8 (*Weak generator Lyapunov condition*). We say that a Feller-type stochastic semigroup $(S_t)_{t \geq 0}$ satisfies the weak generator Lyapunov condition for a weight continuous function $V : \Omega \rightarrow [1, +\infty)$ if there exist constants $b, \sigma > 0$ and a continuous function $\varphi : [1, +\infty) \rightarrow [1, +\infty)$ such that (5.17) holds.

In much the same way as in Section 4, using Theorem 5.4 we can prove the following result, which is a close relative of the main result in Douc et al. [15]:

Theorem 5.6 (*Subgeometric Harris*). Consider a Feller-type stochastic semigroup $(S_t)_{t \geq 0}$ on \mathcal{M}_V which satisfies both the weak generator Lyapunov condition for a continuous weight function V (Hypothesis 8) and the Harris irreducibility condition (Hypothesis 2) on the set $\mathcal{C} := \{x \in \Omega \mid V(x) \leq R\}$, for large enough R . Then, there exists a unique equilibrium $\mu^* \in \mathcal{P}_{\varphi(V)}$, and there exist some constructive constant $C > 0$ and a decay rate function $\tilde{\Theta}$ such that

$$\|S_t \nu\| \leq \tilde{\Theta}(t)\|\nu\|_V, \quad \forall t \geq 0,$$

for any $\nu \in \mathcal{N}_V$, where $\tilde{\Theta}(t) := C\Theta_\psi(rt)/t$ with the notations of Theorem 4.13.

The proof of Theorem 5.6 is given in the rest of this section.

Lemma 5.7. Let $V : \Omega \rightarrow [1, +\infty)$ be a continuous weight function and $\varphi : [1, +\infty) \rightarrow [1, +\infty)$ a concave function with $\varphi(1) = 1$. If a Feller-type stochastic semigroup (S_t) satisfies the weak generator Lyapunov condition (5.16) then it satisfies

$$L\psi(V) \leq -\psi'(V)\varphi(V) + \psi'(V)b, \tag{5.23}$$

for any concave function $\psi: [1, +\infty) \rightarrow [1, +\infty)$. Both conditions have to be understood when integrated along the semigroup flow and thus L denotes the generator of the associated Feller-Markov semigroup $(P_t)_{t \geq 0}$ such that $S_t = P_t^*$.

Proof of Lemma 5.7. For the same reason as in the geometric case, we have (5.10). As a consequence, we have at least

$$t \mapsto \int \psi(V)\mu_t \text{ is càd,}$$

or even it is continuous when $\psi(s)/s \rightarrow 0$ as $s \rightarrow \infty$. On the other hand, for any $0 \leq \mu \in \mathcal{M}_V$, we have

$$\int \mu_0 \left\{ P_{t_2} V + \sigma \int_{t_1}^{t_2} P_s \varphi(V) ds \right\} \leq \int \mu_0 \left\{ P_{t_1} V + b(t_2 - t_1) \right\},$$

which is nothing but the dual form of (5.17), so that

$$P_{t_2} V + \sigma \int_{t_1}^{t_2} P_s \varphi(V) ds \leq P_{t_1} V + b(t_2 - t_1). \tag{5.24}$$

Using Jensen’s inequality (4.9) and (5.24), for any $h > 0$ we have

$$\begin{aligned} P_h \psi(V) &\leq \psi(P_h V) \leq \psi\left(V + bh - \sigma \int_0^h P_s \varphi(V) ds\right) \\ &\leq \psi(V) + \psi'(V)\left(bh - \sigma \int_0^h P_s \varphi(V) ds\right). \end{aligned}$$

By duality, for any $0 \leq \mu \in \mathcal{M}_V$, we deduce

$$\int (S_h \mu_t) \psi(V) - \int \mu_t \psi(V) \leq b \int_0^h \int (S_s(\psi'(V)\mu_t)) ds - \sigma \int_0^h \int (S_s(\psi'(V)\mu_t)) \varphi(V) ds,$$

for any $h > 0$ and $t \geq 0$. Dividing by $h > 0$ and passing to the limit $h \rightarrow 0$, we get

$$\frac{d}{dt} \int \mu_t \psi(V) + \sigma \int \mu_t \psi'(V) \varphi(V) \leq b \int \psi'(V) \mu_0,$$

which is the rigorous definition of the weak generator Lyapunov condition (5.23). \square

Corollary 5.8. *If (S_t) is a Feller-type stochastic semigroup which satisfies the weak generator Lyapunov condition (5.16) then it satisfies*

$$\|S_t\mu\|_V + \sigma t \|S_t\mu\|_{\varphi(V)} \leq \|\mu\|_V + bt(1 + \sigma t/2)\|\mu\|. \tag{5.25}$$

Proof of Corollary 5.8. Because of the weak generator Lyapunov condition (5.16) and the non-expansive mappings property (2.1), we have

$$\|S_t\mu\|_V + \sigma \int_0^t \|S_u\mu\|_{\varphi(V)} \, du \leq \|\mu\|_V + bt\|\mu\|,$$

for any $t \geq 0$. On the other hand, because of Lemma 5.7 applied to $\psi := \varphi$, we have

$$\|S_t\mu\|_{\varphi(V)} + \sigma \int_u^t \|S_u\mu\|_{\varphi'(V)\varphi(V)} \, du \leq \|S_u\mu\|_{\varphi(V)} + b(t - u)\|\mu\|.$$

After time integration of that last estimate and throwing away the second term at the LHS, we get

$$t\|S_t\mu\|_{\varphi(V)} \leq \int_0^t \|S_u\mu\|_{\varphi(V)} \, du + b\frac{t^2}{2}\|\mu\|.$$

Together with the first inequality, this allows us to conclude. \square

Proof of Theorem 5.6. Thanks to Corollary 5.8, we see that the hypotheses of Theorem 5.6 are met for $V_2 = V$ and $V_1 = \psi(V)$ for any ψ as in the statement of Theorem 4.13. We may then apply Theorem 5.4 and conclude. \square

The above result has to be compared with the already known following convergence result.

Theorem 5.9 (subgeometric Harris). *Consider a Feller type stochastic semigroup $(S_t)_{t \geq 0}$ on \mathcal{M}_V which satisfies both the generator Lyapunov condition (Hypothesis 6) and the Harris irreducibility condition (Hypothesis 2). There holds*

$$\|S_t\nu\| \lesssim \frac{1}{H^{-1}(t)}\|\nu\|_V, \quad \forall t \geq 0, \forall \nu \in \mathcal{N}_V, \tag{5.26}$$

where H is defined by $H(u) := \int_1^u \frac{ds}{\varphi(s)}$. It is worth observing that

$$\frac{1}{H^{-1}(t)} \simeq t^{-k/\delta} \quad \text{when } m = \langle x \rangle^k, \varphi(u) = u^{1-\delta/k}, 0 < \delta < k;$$

$$\frac{1}{H^{-1}(t)} \simeq e^{-\lambda t^{\sigma/(\sigma+\delta)}} \quad \text{when } m = e^{\langle x \rangle^\sigma}, \varphi(u) = \frac{u}{(\log u)^{\delta/\sigma}}, \delta, \sigma > 0,$$

when $\Omega := \mathbb{R}^d$ and $\langle x \rangle := (1 + |x|^2)^{1/2}$.

It is worth emphasizing that the above rates of convergence are precisely the same as those obtained by our method for the same two examples presented at the end of the Section 4 as made explicit in (4.28) and (4.29).

Theorem 5.9 has been established in Douc et al. [15] and an alternative proof has been proposed in Hairer [20]. Both are based on non constructive probabilistic arguments that we do not present here. We mention however that the proof of Theorem 5.9 as found in Douc et al. [15], Hairer [20] consists in establishing

$$\int_0^\infty \varphi(H^{-1}(s)) \|\nu_s\| ds \leq C \|\nu_0\|_V,$$

for any $\nu_0 \in \mathcal{N}_V$ (in fact for $\nu_0 = \delta_x - \delta_y$). Because $s \mapsto \|\nu_s\|$ is decreasing and $(H^{-1})' = \varphi(H^{-1})$, one deduces

$$\begin{aligned} H^{-1}(t) \|\nu_t\| &\leq H^{-1}(t) \|\nu_t\| - \int_0^t H^{-1}(s) \left(\frac{d}{ds} \|\nu_s\|\right) ds \\ &= H^{-1}(0) \|\nu_0\| + \int_0^t \varphi(H^{-1}(s)) \|\nu_s\| ds \\ &\leq H^{-1}(0) \|\nu_0\| + C \|\nu_0\|_V, \end{aligned}$$

which is nothing but (5.26).

We have not been able to give a constructive deterministic proof of Theorem 5.9. However, our analysis makes it possible to recover Theorem 5.9 for some specific but common examples, as explained at the very end of Section 4. We obtain constructive constants in all our results, which is an improvement in all subgeometric cases.

6. Existence of an equilibrium under a subgeometric Lyapunov condition

We give here a quite general result about existence of an equilibrium for a Feller-type stochastic semigroup which is independent of our previous results and in particular does not need a coupling or Harris condition.

We thus consider hereafter a Feller type stochastic semigroup $(S_t)_{t \geq 0}$ and we assume that the weak generator Lyapunov condition (Hypothesis 7) holds for a weight function

$V: \Omega \rightarrow [1, +\infty)$, a concave function $\varphi: [1, +\infty) \rightarrow [1, +\infty)$, for which we may assume $\varphi' \leq 1$ without loss of generality, and some constants $b, \sigma > 0$. Introducing the constant $R := \sup V \in [1, \infty]$, we furthermore assume that

$$\varphi(R) > b/\varsigma \quad \text{and} \quad \{V \leq \rho\} \text{ is compact for any } \rho \in [1, R),$$

the last condition being fundamental in the present approach which is based on the use of the Prokhorov theorem about compactness of tight sequences. More precisely, from the last condition and the Prokhorov theorem, we may claim that any sequence (μ_n) of \mathcal{P} with uniformly (in n) bounded $\varphi(V)$ -moment is relatively compact in \mathcal{P} .

By fixing $\rho \in [1, R)$ large enough and $\varepsilon > 0$ small enough such that $(\varsigma - \varepsilon)\varphi(\rho) \geq b$, we deduce that

$$LV_3 \leq -V_2 + b \mathbf{1}_{\mathcal{C}},$$

where L is the generator of the associated Markov-Feller semigroup (P_t) on $C_0(\Omega)$, $V_3 := V$, $V_2 := \varepsilon\varphi(V)$ and $\mathcal{C} := \{x \in \Omega \mid V_2(x) \leq \rho\}$.

The above Foster-Lyapunov condition provides a sufficient condition for the existence of an equilibrium.

Theorem 6.1. *Any stochastic semigroup (S_t) on \mathcal{M}_V which fulfills the above Lyapunov condition has at least one invariant probability measure $\mu^* \in \mathcal{M}_{\varphi(V)}$.*

Proof of Theorem 6.1. *Step 1.* We prove that (S_t) is bounded in the sense of Cesàro in \mathcal{M}_{V_2} . We define

$$\mathcal{A} := b\chi, \quad \mathcal{B} := L - \mathcal{A},$$

with $\chi \in C_0(\Omega)$ such that $\mathbf{1}_{\mathcal{C}} \leq \chi \leq 1$. Since \mathcal{B} is a bounded perturbation of L , we classically know that \mathcal{B} generates a semigroup $S_{\mathcal{B}}$ on the same space $C_0(\Omega)$ and furthermore

$$\mathcal{B} \geq L - b, \quad \mathcal{B}V_3 \leq -V_2 \leq 0.$$

From the first inequality, we have $S_{\mathcal{B}}(t) \geq e^{-bt}S_L(t) \geq 0$ for any $t \geq 0$, so that both $S_{\mathcal{B}}$ and $S_{\mathcal{B}}^*$ are positive semigroups. Because of the Duhamel formula

$$S_{\mathcal{B}}^* = S + S_{\mathcal{B}}^* * (-\mathcal{A})S \leq S,$$

and $S_{\mathcal{B}}^*$ is a semigroup of contraction on \mathcal{M} . In particular $S_{\mathcal{B}} \in L_t^\infty(\mathcal{B}(\mathcal{M}))$, where here and below, $L_t^\infty(\mathcal{X})$ denotes the space of bounded function from \mathbb{R}_+ into \mathcal{X} . From the same Duhamel formula, we see that $S_{\mathcal{B}}^*$ is well defined on \mathcal{M}_{V_3} and has at least

exponential growth rate. We can get a more accurate information. For $0 \leq \mu_0$ in the domain of S_B^* (defined in \mathcal{M}_{V_3}) and denoting $\mu_t := S_B^*(t)\mu_0$, we may compute

$$\frac{d}{dt} \int \mu_t V_3 \leq \int \mu_t \mathcal{B}V_3 \leq - \int \mu_t V_2,$$

so that

$$\int \mu_t V_3 + \int_0^t \int \mu_s V_2 ds \leq \int \mu_0 V_3, \quad \forall t \geq 0.$$

We deduce that

$$S_B^* \in L_t^\infty(\mathcal{B}(\mathcal{M}_{V_3})); \quad \int_0^\infty \|S_B^*(t)\mu_0\|_{\mathcal{M}_{V_2}} dt \leq \|\mu_0\|_{\mathcal{M}_{V_3}}, \quad \forall \mu_0 \in \mathcal{M}_{V_3}.$$

We thus obtain $S_B^* \in L_t^\infty(\mathcal{B}(\mathcal{M}_{V_2}))$, by interpolation together with the previous estimate $S_B \in L_t^\infty(\mathcal{B}(\mathcal{M}))$. Alternatively, we could have used Lemma 5.7, in order to get

$$\mathcal{B}\varphi(V) \leq (-\varsigma\varphi(V) + b)\varphi'(V) - b\chi\varphi(V) \leq b(\mathbf{1}_C - \chi) \leq 0,$$

next to compute directly

$$\frac{d}{dt} \int (S_B^*(t)\mu_0) V_2 \leq 0,$$

for $0 \leq \mu_0$ in the domain (in \mathcal{M}_{V_2}) of S_B^* , and finally to deduce that (S_B^*) is a semigroup of contractions in \mathcal{M}_{V_2} . We next come back the splitting of the semigroup through the Duhamel formula

$$S = S_B^* + S_B^* * \mathcal{A}S,$$

and we introduce the associated Cesàro means

$$U_T := \frac{1}{T} \int_0^T S(t) dt, \quad V_T := \frac{1}{T} \int_0^T S_B^*(t) dt, \quad W_T := \frac{1}{T} \int_0^T (S_B^* * \mathcal{A}S)(t) dt.$$

We obviously have

$$\|V_T\|_{\mathcal{B}(\mathcal{M}_{V_2})} \leq \frac{1}{T} \int_0^T \|S_B^*(t)\|_{\mathcal{B}(\mathcal{M}_{V_2})} dt \leq 1.$$

On the other hand, for $0 \leq \mu_0 \in \mathcal{M}_{V_2}$, we have

$$S_{\mathcal{B}}^*(\tau) \int_0^{T-\tau} \mathcal{A}S(s) \mu_0 ds \leq S_{\mathcal{B}}^*(\tau) \int_0^T \mathcal{A}S(s) \mu_0 ds, \quad \forall T > \tau > 0,$$

by positivity of the three operators involved in this integral formula, and then

$$\begin{aligned} \|W_T \mu_0\|_{\mathcal{M}_{V_2}} &= \left\| \frac{1}{T} \int_0^T S_{\mathcal{B}}^*(\tau) \int_0^{T-\tau} \mathcal{A}S(s) \mu_0 d\tau ds \right\|_{\mathcal{M}_{V_2}} \\ &\leq \frac{1}{T} \int_0^\infty \left\| S_{\mathcal{B}}^*(\tau) \int_0^T \mathcal{A}S(s) ds \mu_0 \right\|_{\mathcal{M}_{V_2}} d\tau \\ &\leq \frac{1}{T} \left\| \int_0^T \mathcal{A}S(s) ds \mu_0 \right\|_{\mathcal{M}_{V_3}} \leq \|\mathcal{A}\|_{\mathcal{B}(\mathcal{M}; \mathcal{M}_{V_3})} \|\mu_0\|_{\mathcal{M}}, \end{aligned}$$

so that W_T is uniformly bounded in $L_t^\infty(\mathcal{B}(\mathcal{M}_{V_2}))$. We then deduce that $U_T = V_T + W_T$ is also uniformly bounded in $L_t^\infty(\mathcal{B}(\mathcal{M}_{V_2}))$.

Step 2. Existence of an invariant measure $\mu^ \in \mathcal{M}_{V_2}$.* We define $\mathbb{K} := \mathcal{M}_{V_2} \cap \mathcal{P}$ and we fix $\mu_0 \in \mathbb{K}$ arbitrary. Because of Step 1, the sequence $\mu_T = U_T \mu_0$ is bounded in \mathbb{K} . By Prokhorov’s theorem the embedding $\mathcal{M}_{V_2} \subset \mathcal{M}$ is compact, and hence there exists a subsequence (μ_{T_k}) and $\mu^* \in \mathbb{K}$ such that $\mu_{T_k} \rightharpoonup \mu^*$ in the weak- $*$ sense $\sigma(\mathcal{M}, C_0)$ as $k \rightarrow \infty$. For any fixed $s > 0$, we observe that

$$\begin{aligned} S(s)\mu^* - \mu^* &= \lim_{k \rightarrow \infty} \left\{ \frac{1}{T_k} \int_0^{T_k} S(s)S(t)\mu_0 - \frac{1}{T_k} \int_0^{T_k} S(t)\mu_0 dt \right\} \\ &= \lim_{k \rightarrow \infty} \left\{ \frac{1}{T_k} \int_{T_k}^{T_k+s} S(t)\mu_0 - \frac{1}{T_k} \int_0^s S(t)\mu_0 dt \right\} = 0, \end{aligned}$$

so that μ^* is an invariant measure. \square

Data availability

No data was used for the research described in the article.

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