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THE JACOBSON RADICAL OF A PROPOSITIONAL THEORY

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A Contribution to Ninety Years of Glivenko's Theorem

Abstract. Alongside the analogy between maximal ideals and complete theories, the Jacobson radical carries over from ideals of commutative rings to theories of propositional calculi. This prompts a variant of Lindenbaum's Lemma that relates classical validity and intuitionistic provability, and the syntactical counterpart of which is Glivenko's Theorem. The Jacobson radical in fact turns out to coincide with the classical deductive closure. As a by-product we obtain a possible interpretation in logic of the axioms-as-rules conservation criterion for a multi-conclusion Scott-style entailment relation over a single-conclusion one.

§1. Introduction. Glivenko's theorem from 1929 says that if a propositional formula φ is provable in classical logic, then its double negation $\neg\neg\varphi$ is provable in intuitionistic logic. In 1933 Gödel extended this to predicate logic, which move required to admit on the intuitionistic side the scheme of double negation shift. With Gödel's and Gentzen's negative translation in place of double negation, both from 1933, one can even get by with minimal logic in place of intuitionistic logic. More than one related proof translation saw the light of the day, e.g., Kolmogorov's (1925) and Kuroda's (1951).

Glivenko's theorem thus stood right at the beginning of a fundamental change of perspective: that classical logic can be embedded into intuitionistic or even minimal logic, rather than the latter being a limited version of the former. Together with the revision of Hilbert's Programme ascribed to Kreisel and Feferman, this has led to the much broader quest for the computational content of classical proofs, today culminating in agile areas such as dynamical algebra, formal topology, program extraction from proofs, proof analysis, proof mining and proof translations. The growing success of these approaches suggests that customary mathematics, with

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This note emerged from the the first and third author's M.Sc and Ph.D thesis, respectively [38, 113]. For a preprint superseded by the present note see [40].

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classical logic and set theory, might eventually prove to be much more constructive than widely thought.

In 1930 Tarski ascribed to Lindenbaum the theorem that in classical logic any given theory T equals the intersection of all the complete theories containing T. Its typical use for Gödel's Completeness Theorem aside, this Lindenbaum Lemma is one of several theorems from that period which describe the intersection of all the ideal objects extending a given concrete object. Those intersection theorems, in their full generality recognised as forms of the Axiom of Choice (AC), are often put by contraposition as extension or separation theorems. Apart from Lindenbaum's, prominent cases are known by the names of Artin–Schreier, Hahn–Banach, Krull and Szpilrajn. The case in algebra closest to Lindenbaum's Lemma, however, gained prominence only in 1945, when Jacobson pointed out the relevance of the intersection of all the maximal ideals of a given ring, i.e., of what is now known as the Jacobson radical.

In the present note we follow the analogy between maximal (proper) ideals and complete (consistent) theories to carry over the Jacobson radical from ideals of commutative rings to theories of propositional calculi (Section 5.2), where it turns out to coincide with the stable closure or with the closure with respect to classical logic (Proposition 3 and Corollary 1). This prompts a variant of Lindenbaum's Lemma that relates classical validity and intuitionistic provability (Proposition 2), and the syntactical counterpart of which happens to be Glivenko's Theorem in the form recalled above (Theorem 2).

As a by-product we obtain a possible interpretation in logic (Theorem 3) of the axioms-as-rules conservation criterion (Theorem 1) for a multi-conclusion Scott-style entailment relation $\vdash$ over a single-conclusion one $\rhd$. This criterion has proved to be the common core of many a syntactical counterpart of a semantic conservation theorem corresponding to one of the aforementioned intersection theorems. Typically any such case of conservation means reduction to a special case characterised by additional axioms with (possibly empty) disjunctions in positive position. Applying the criterion means to eliminate the additional axioms for $\vdash$ by way of the corresponding disjunction elimination rules for $\rhd$. The latter equally suffice for proof practice, and have proved admissible in all mathematical instances yet considered.

Our interpretation of the conservation criterion in propositional logic (Theorem 3) is tantamount to Glivenko's Theorem (Theorem 2). As for the latter, disjunction elimination plays a central role in the proof of the former, together with some notorious features of (double) negation in intuitionistic logic and of provability in classical propositional logic (Lemmas 1 and 2).

§2. Preliminaries. Unless specified otherwise, we work in a suitable fragment of Aczel's *Constructive Zermelo–Fraenkel Set Theory* (**CZF**) [1–5] based on intuitionistic first-order predicate logic. While in general the concepts of this paper are elementary and the proofs are direct anyway, we still pin down **CZF** as metatheory if only rather for convenience's sake; in fact much less might suffice. Likewise, when we occasionally need to invoke a fragment of the principle of Excluded Middle or even a form of the AC, and thus go beyond **CZF**, we simply switch to **ZF** and **ZFC**, respectively, and indicate this accordingly.

For example, the *Restricted Law of Excluded Middle* (REM) is not a principle of **CZF**. This REM means $\varphi \vee \neg\varphi$ for every set-theoretic formula φ that is, *bounded* in the sense that only set-bounded quantifiers of the types $\forall x \in y$ and $\exists x \in y$ occur in φ. As is common in this context, negation is a defined connective: $\neg\varphi \equiv \varphi \to \bot$.

By a *finite set* we understand a set that can be written as $a_1, \dots, a_n$ for some $n \geq 0$. Given any set S, let $\mathrm{Pow}(S)$ (respectively, $\mathrm{Fin}(S)$) consist of the (finite) subsets of S. We refer to [92] for further provisos to carry over to the present note.[1]

Convention. By an *intermediate logic* we mean an intermediate propositional calculus obtained by adding to the axioms of intuitionistic logic some classically valid propositional formulas [37].

We write $\rhd$ to denote (deducibility in) any such intermediate logic in a propositional language S.

The subsequent properties of (double) negation are due to Brouwer for intuitionistic logic [11, 12, 45, 109, 110] and carry over to an arbitrary $\rhd$:

Lemma 1. *For any given intermediate logic $\rhd$ in a propositional language S,*

$$\frac{}{\neg\neg\neg\psi \rhd \neg\psi} \qquad \frac{\Gamma, \varphi \rhd \neg\psi}{\Gamma, \neg\neg\varphi \rhd \neg\psi} \qquad \frac{}{\rhd \neg\neg(\psi \vee \neg\psi)},$$

for every $\Gamma \in \mathrm{Pow}(S)$ and all $\varphi, \psi \in S$.

We refer to [55] and [77, p. 27] for a deeper discussion with earlier references of the following:

Lemma 2. *Let $\vdash$ and $\rhd$ stand for classical logic and an intermediate logic, respectively, in a propositional language S. If $\Gamma \in \mathrm{Pow}(S)$ and $\psi \in S$, then $\Gamma \vdash \psi$ if and only if $\Gamma, \Delta \rhd \psi$ for a suitable finite subset Δ of*

$$\mathrm{TND}_0(\Gamma, \psi) = \{\varphi \vee \neg\varphi : \varphi \text{ propositional variable occurring in } \Gamma \text{ or } \psi\},$$

i.e., the set of relevant instances of tertium non datur for propositional variables.

A *theory* of an intermediate logic $\rhd$ is a subset T of the underlying propositional language S that is, *deductively closed* with respect to $\rhd$:

$$\forall \varphi \in S(T \rhd \varphi \Rightarrow T \ni \varphi).$$

As usual, a theory T of $\rhd$ is

[1] For example, we deviate from the terminology prevalent in constructive mathematics and set theory [4, 5, 8, 9, 64, 69]: to reserve the term 'finite' to sets which are in *bijection* with $\{1, \dots, n\}$ for a necessarily unique $n \geq 0$. Those exactly are the sets which are finite in our sense and are *discrete* too, i.e., have decidable equality [69].

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— *consistent* if $\bot \notin T$, which is to say that $T \neq S$;
— *complete* if

$$\forall \varphi \in S(T \ni \varphi \vee \neg T \ni \varphi);$$

— *stable* if

$$\forall \varphi \in S(T \ni \neg\neg\varphi \Rightarrow T \ni \varphi).$$

As an immediate consequence of Lemmas 1 and 2 we have the following:

LEMMA 3. *Let $\rhd$ be an intermediate logic in a propositional language S. The following statements are equivalent for any given subset T of S:*

1. *T is deductively closed with respect to classical logic.*
2. *T is a stable theory of $\rhd$.*
3. *T is a theory of $\rhd$ that contains all instances of excluded middle $\varphi \vee \neg\varphi$ with $\varphi \in S$.*
4. *T is a theory of $\rhd$ that contains all $\varphi \vee \neg\varphi$ where φ is a propositional variable of S.*

In particular, if a theory T of an intermediate logic $\rhd$ is complete, then T is stable.

§3. Entailment relations. Entailment relations, both in their single- and multi-conclusion variant, are at the heart of this note. We briefly recall the basic notions, to which end we closely follow [91, 92].

3.1. Consequence. Let S be a set and $\rhd \subseteq \mathrm{Pow}(S) \times S$. Once abstracted from the context of logical formulas, all but one of Tarski's axioms of *consequence* [106] can be put as

$$\frac{U \ni a}{U \rhd a}\ (\mathrm{R}) \qquad \frac{\forall b \in U(V \rhd b) \quad U \rhd a}{V \rhd a}\ (\mathrm{T}) \qquad \frac{U \rhd a}{\exists U_0 \in \mathrm{Fin}(U)(U_0 \rhd a)}\ (\mathrm{A})$$

where $U, V \subseteq S$ and $a \in S$. These axioms also characterise a finitary covering or Stone covering in formal topology [95];[2] see further [17, 19, 72, 73, 97, 98]. The notion of consequence has allegedly been described first by Hertz [49–51]; see also [7, 59].

We do not employ the one of Tarski's axioms by which he required that S be countable. This aside, Tarski has rather characterised the set of consequences of a set of propositions, which corresponds to the *algebraic closure operator* $U \mapsto U^{\rhd}$ on $\mathrm{Pow}(S)$ of a relation $\rhd$ as above where

$$U^{\rhd} \equiv \{a \in S : U \rhd a\}.$$

3.2. Single-conclusion entailment. Rather than with Tarski's notion, we henceforth work with its (tantamount) restriction to finite subsets, i.e., a

[2]This is from where we have taken the symbol $\rhd$, used also [16, 112] to denote a 'consecution' [88].

single-conclusion entailment relation. This is, a relation $\rhd \subseteq \mathrm{Fin}(S) \times S$ such that

$$\frac{U \ni a}{U \rhd a}\ (\mathrm{R}) \qquad \frac{V \rhd b \quad V', b \rhd a}{V, V' \rhd a}\ (\mathrm{T}) \qquad \frac{U \rhd a}{U, U' \rhd a}\ (\mathrm{M})$$

for all finite $U, U', V, V' \subseteq S$ and $a, b \in S$, where as usual $U, V \equiv U \cup V$ and $V, b \equiv V \cup \{b\}$. Our focus thus is on *finite* subsets of S, for which we henceforth reserve the letters $U, V, W, \ldots$; we also sometimes write $a_1, \ldots, a_n$ in place of $\{a_1, \ldots, a_n\}$ even if $n = 0$. Redefining

$$T^{\rhd} \equiv \{a \in S : \exists U \in \mathrm{Fin}(T)(U \rhd a)\}, \tag{1}$$

for *arbitrary* subsets T of S gives back an algebraic closure operator on $\mathrm{Pow}(S)$. By writing $T \rhd a$ in place of $a \in T^{\rhd}$, the single-conclusion entailment relations thus correspond exactly to the relations satisfying Tarski's axioms above.

3.3. Multi-conclusion entailment. Let S be a set and $\vdash \subseteq \mathrm{Fin}(S) \times \mathrm{Fin}(S)$. Scott's [102] axioms of entailment can be put as

$$\frac{U \between W}{U \vdash W}\ (\mathrm{R}) \qquad \frac{V \vdash W, b \quad V', b \vdash W'}{V, V' \vdash W, W'}\ (\mathrm{T}) \qquad \frac{U \vdash W}{U, U' \vdash W, W'}\ (\mathrm{M})$$

for finite $U, U', V, V', W, W' \subseteq S$ and $b \in S$, where $U \between W$ means that U and W have an element in common [97]. Any such $\vdash$ is a *multi-conclusion entailment relation*, where 'multi' includes 'empty'. In practice, $\rhd$ and $\vdash$ are *inductively generated* by the axioms of the intended models, which procedure we here take for granted [14, 31]; see also [3, 87, 92, 94].

This fairly general notion of entailment has been introduced by Scott [101–103], building on Hertz's and Tarski's work (see above), and of course on Gentzen's sequent calculus [43, 44]. Shoesmith and Smiley [104] trace multi-conclusion entailment relations back to Carnap [13], and Popper apparently had related ideas [85, 86].[3] Before Scott, Lorenzen had developed analogous concepts formally [65–68]; he even listed [66, pp. 84–85] counterparts of the axioms (R), (T) and (M) for single- and multi-conclusion entailment [27, 81].[4] As compared with Gentzen's and Lorenzen's approaches, Scott's entailment relation follow the contexts-as-sets paradigm, which has caused reservations [78, 79]. The relevance of the notion of entailment relation to point-free topology and constructive algebra has been pointed out in [14], and has been used very widely, e.g., in [20–22, 24, 25, 29, 32, 80, 89, 93, 100, 114, 115]. Consequence and entailment have further caught interest from various other angles [6, 35, 41, 52–54, 83, 99, 104, 117].

§4. Conservation. Again following [91, 92], we sketch the concept of conservative extension of a multi-conclusion entailment relation $\vdash$ over

[3] David Binder has kindly hinted us at Popper's work.

[4] Stefan Neuwirth has kindly pointed this out to us.

a single-conclusion entailment relation $\rhd$ on the same set S. After that we extract from [90]—based on [18]—possible interpretations limited to classical logic.

4.1. Conservation in syntax and semantics. Let S be a set, and let $a, b, c, \ldots$ and $U, V, W, \ldots$ range over the elements of S and $\mathrm{Fin}(S)$, respectively. Given a multi-conclusion entailment relation $\vdash$ and a single-conclusion entailment relation $\rhd$ on the same set S, we throughout assume *Extension*:

$$\text{Ext} \qquad \frac{U \rhd a}{U \vdash a}$$

Of major interest to us is the converse, alias *Conservation*:

$$\text{Con} \qquad \frac{U \vdash a}{U \rhd a}$$

The *trace* of any given $\vdash$ is the single-conclusion entailment relation $\rhd_\vdash$ defined by

$$U \rhd_\vdash a \equiv U \vdash a,$$

for which Ext and Con are tantamount to $\rhd \subseteq \rhd_\vdash$ and $\rhd \supseteq \rhd_\vdash$, respectively.

An *arbitrary* subset P of S is a *model of* $\vdash$ if

$$P \supseteq U \Rightarrow V \between P \quad \text{whenever } U \vdash V.$$

The notion of model carries over to single-conclusion $\rhd$ in the apparent manner, such that the *models of* $\rhd$ are exactly the $P \in \mathrm{Pow}(S)$ which are *closed* under $\rhd$, i.e., for which $P^{\rhd} = P$. Let $\mathrm{Mod}(\vdash)$ and $\mathrm{Mod}(\rhd)$ consist of the models of $\vdash$ and $\rhd$, respectively. By Extension, $\mathrm{Mod}(\vdash) \subseteq \mathrm{Mod}(\rhd)$, which in **ZFC** is equivalent to Extension [92, Lemma 9].

Now Con follows from the *Generalised Krull–Lindenbaum* (GKL) *Lemma*, viz.

$$\text{GKL} \qquad \forall P \in \mathrm{Mod}(\vdash)(P \supseteq U \Rightarrow a \in P) \implies U \rhd a,$$

the converse of which holds by Extension. Again by Extension, GKL implies the *Trace Completeness Theorem* (TCT), viz.

$$\text{TCT} \qquad \forall P \in \mathrm{Mod}(\vdash)(P \supseteq U \Rightarrow a \in P) \implies U \vdash a,$$

the converse of which holds by the definition of a model of $\vdash$. This TCT is a fragment of AC that implies REM [92, Corollary 5].[5]

In **ZFC**, GKL and Con are equivalent [92, Theorem 6]. In **CZF** we can make this more precise:

Remark 1. In the presence of Ext, GKL is equivalent to the conjunction of Con and TCT.

In all, GKL is semantic conservation, and Con is its syntactical counterpart.

[5] The proof of [92, Proposition 4] goes equally through with TCT in place of full CT.

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4.2. Conservation in proof practice. In proof practice, GKL is useful for reductions to special cases, by making possible to use $\vdash$ in proofs about $\rhd$, but GKL is of semantic nature, entails REM and requires some AC. In comparison, Con is equally sufficient for that kind of reduction, is syntactical and has elementary proofs. Many such cases are known in point-free topology such as locale theory and formal topology [14, 15, 20, 23, 70, 71]; in constructive algebra, especially with dynamical methods [26, 33, 61–64, 116, 118, 119]; and in the proof theory of order relations [78, 80]. Most of those cases concern algebra at large. But what about logic? One may think of Gentzen's classical multi-succedent sequent calculus as extending his intuitionistic single-succedent variant [43, 44, 77, 105]. As we will see, this thought goes in the right direction.

A typical situation is as follows: Let the single-conclusion entailment relation $\rhd$ on a set S be generated by axioms. Then the multi-conclusion entailment relation $\vdash$ on the same set S is generated by the axioms of $\rhd$, of course with $\vdash$ in place of $\rhd$, and by *additional axioms*

$$a_1, \dots, a_k \vdash b_1, \dots, b_\ell,$$

where $k, \ell \geq 0$. In any such situation we say that $\vdash$ *extends* $\rhd$, and list the additional axioms if needed. This is legitimate inasmuch as if $\vdash$ extends $\rhd$, then Ext is satisfied. What about Con?

The following most versatile *conservation criterion* [91, 92], which in fact gathers together many of the cases of Con mentioned before, will also help to understand Con for logic:

Theorem 1. *Let $\vdash$ extend $\rhd$ with certain additional axioms of the form*

$$a_1, \dots, a_k \vdash b_1, \dots, b_\ell, \tag{2}$$

where $k, \ell \geq 0$. Then $\vdash$ and $\rhd$ satisfy Con *if and only if*

$$\frac{W, b_1 \rhd c \quad \cdots \quad W, b_\ell \rhd c}{W, a_1, \dots, a_k \rhd c} \tag{3}$$

for every additional axiom (2), all $c \in S$ and every $W \in$ Fin(S).

This swiftly follows [92, Theorem 2] from a sandwich criterion for conservation given by Scott [102], and also is a corollary of cut elimination for entailment relations [94] as related to cut elimination in the presence of axioms [76].

Quite a few instances of GKL can be classified by the two cases named *Universal Krull* (UK) and *Universal Lindenbaum* (UL) in [92], for which S is a set with

UK : a distinguished element e of S and a binary operation $*$ on S.

UL : a unary operation $\sim$ on S.

The additional axioms for $\vdash$ extending $\rhd$ are

$$\begin{aligned} \text{UK}: &\quad e \vdash \qquad a * b \vdash a, b, \\ \text{UL}: &\quad a, {\sim}a \vdash \qquad \vdash a, {\sim}a, \end{aligned}$$

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where $a, b \in S$. The corresponding conservation criteria (Theorem 1) read

$$\text{UK}: \quad \frac{}{W, e \rhd c} \qquad \frac{W, a \rhd c \quad W, b \rhd c}{W, a * b \rhd c}$$

$$\text{UL}: \quad \frac{}{W, a, {\sim}a \rhd c} \qquad \frac{W, a \rhd c \quad W, {\sim}a \rhd c}{W \rhd c}$$

where $W \in \text{Fin}(S)$ and $a, b, c \in S$.[6] We refer to [91, 92] for details and references.

4.3. The case of classical logic. Building upon [18], in [90] the instances of GKL in the cases UK and UL have been considered for the following data: S consists of the sentences of a logical language, $\rhd$ stands for deducibility with *classical* logic, e is absurdity $\bot$, the operator $*$ is disjunction $\vee$, and $\sim$ is negation $\neg$. While the models of $\rhd$ are the stable theories of $\rhd$, the models of $\vdash$ are the complete consistent theories in S. Hence GKL is Lindenbaum's Lemma [106], and Con is provable but little interesting, simply because $\rhd$ is classical logic already. Let's try to get more by relativising $\rhd$.

Now let $\rhd$ denote (deducibility in) an intermediate logic in a propositional language S; whence the models of $\rhd$ are the theories of $\rhd$.[7] Let $\vdash$ extend $\rhd$ with the following additional axioms:

$$\bot \vdash \qquad \vdash \varphi, \neg\varphi \quad (\varphi \in S).$$

The models of $\vdash$ are exactly the complete consistent theories of $\rhd$, and the corresponding conservation criteria (Theorem 1) read

$$\frac{}{\Gamma, \bot \rhd \psi} \qquad \frac{\Gamma, \varphi \rhd \psi \quad \Gamma, \neg\varphi \rhd \psi}{\Gamma \rhd \psi}$$

with $\Gamma \in \text{Fin}(S)$ and $\varphi, \psi \in S$. While the first criterion holds for any given intermediate logic $\rhd$, the second one amounts to $\rhd$ satisfying $\rhd\varphi \vee \neg\varphi$, which is to say that $\rhd$ be classical logic. Hence Con in this case simply means that conservatively adding $\vdash \varphi, \neg\varphi$ is equivalent to requiring $\rhd\varphi \vee \neg\varphi$. This of course is well known and of relatively little interest either. Can't we do better?

§5. Jacobson radicals.

5.1. The Jacobson radical in algebra. Let $S = R$ be a commutative ring with 1, and let $\rhd$ stand for generation in R, i.e., $U \rhd a$ means that a is a linear combination with coefficients from R of the elements of U. A model of $\rhd$ is nothing but an *ideal* of R, i.e., a subset closed under linear combination. An ideal J of R is

— *proper* if $1 \notin J$, which is tantamount to $J \neq R$ and

[6] The criteria for UK have occurred [90] as 'e is *convincing* for $\rhd$' and '$\rhd$ satisfies *Encoding*'.

[7] For the related covering of formulas [30, 96], the saturated sets rather are the *complements* of the theories.

— *complete* if modulo J any given $r \in R$ is either 0 or invertible, that is,

$$\forall r \in R(J \ni r \vee J, r \rhd 1). \tag{4}$$

Every proper complete ideal is a *maximal ideal*, i.e., maximal among the proper ideals, and vice versa in **ZF**. With the current notation, the *Jacobson radical of an ideal* J can be defined as

$$\mathrm{Jac}(J) = \{a \in R : \forall b \in R\,(a, b \rhd 1 \Rightarrow J, b \rhd 1)\}. \tag{5}$$

We thus carry over to commutative rings the first-order definition of the Jacobson radical for distributive lattices [10, 28, 58] rather than using the more common one for commutative rings present e.g., in [64]. The latter reads

$$\mathrm{Jac}(J) = \{a \in R : \forall b \in R\,\exists c \in R\,(1-(1-ab)c \in J)\}, \tag{6}$$

which is to say that any given $a \in R$ belongs to $\mathrm{Jac}(R)$ precisely when $1 - ab$ is invertible modulo J for every $b \in R$. We give precedence to (5) over (6) because the former, unlike the latter, can be transferred to logic without further ado (Section 5.2).

Just as (6), the first-order definition we employ (5) is anyway equivalent in **ZFC** to the following more customary second-order characterisation of the Jacobson radical [57]. Although the proof is of course similar to the one with (6) in place of (5) and for maximal rather than complete ideals, see e.g., [64], we detail this one because it carries over to logic (Proposition 2).

Proposition 1 (**ZFC**). *For every ideal J of a commutative ring R,*

$$\bigcap \mathrm{Com}_J(R) = \mathrm{Jac}(J)$$

where $\mathrm{Com}_J(R)$ *consists of the complete ideals* $\mathfrak{c}$ *in* R *with* $J \subseteq \mathfrak{c}$.

Proof. Let $a \in \mathrm{Jac}(J)$, and let $\mathfrak{c}$ be a complete ideal such that $\mathfrak{c} \supseteq J$. Either $\mathfrak{c} \ni a$ or $\mathfrak{c}, a \rhd 1$. In the former case we are done. In the latter case there is $b \in R$ such that $\mathfrak{c} \rhd b$ (in particular, $b \in \mathfrak{c}$) and $a, b \rhd 1$. Since $a \in \mathrm{Jac}(J)$, we get $J, b \rhd 1$. As $J \subseteq \mathfrak{c}$ and $b \in \mathfrak{c}$, this implies $\mathfrak{c} \rhd 1$. Hence $\mathfrak{c} = R$, by which again $\mathfrak{c} \ni a$.

Conversely, if $a \notin \mathrm{Jac}(J)$, then there is $b \in R$ for which $a, b \rhd 1$ holds but $J, b \rhd 1$ fails, and thus $(J, b)^{\rhd}$ lacks a. Zorn's Lemma yields an ideal $\mathfrak{c}$ maximal among the ones that contain $(J, b)^{\rhd}$ yet miss a. Any such $\mathfrak{c}$ is complete: if $\mathfrak{c} \not\ni b$, then $\mathfrak{c}, b \rhd a$ by maximality, and thus $\mathfrak{c}, b \rhd 1$ by $a, b \rhd 1$. ⊣

It obviously is irrelevant whether the intersection ranges over the only improper ideal R as well.

5.2. The Jacobson radical in logic. Let again $\rhd$ stand for (deducibility in) an intermediate logic in a propositional language S. That a (consistent) theory T of $\rhd$ be *complete* can equivalently be put as

$$\forall \varphi \in S(T \ni \varphi \vee T, \varphi \rhd \bot). \tag{7}$$

This move makes fully evident the analogy between complete ideals (4) and complete theories (7), which rests upon the following brief (and necessarily incomprehensive) dictionary:

propositional logic	commutative algebra
language	ring
deduction $\triangleright$	generation $\triangleright$
theory	ideal
absurdity $\bot$	unit 1
consistent theory	proper ideal
complete theory	complete ideal

With this at hand we translate (5) into a definition of the *Jacobson radical of a theory T*:

$$\mathrm{Jac}(T) = \{\alpha \in S : \forall \beta \in S\, (\alpha, \beta \triangleright \bot \Rightarrow T, \beta \triangleright \bot)\}.$$

This is obviously equivalent to the following characterisation:

$$\mathrm{Jac}(T) = \{\alpha \in S \colon \forall \beta \in S (\alpha \triangleright \neg\beta \Rightarrow T \triangleright \neg\beta)\}.$$

Mutatis mutandis the proof of Proposition 1 proves what we would like to provisionally call the *Intermediate Lindenbaum Lemma*:

PROPOSITION 2 (**ZFC**). *For every theory T of an intermediate logic $\triangleright$ in a propositional languageS,*

$$\text{ILL} \qquad \bigcap \mathrm{Com}_T(S) = \mathrm{Jac}(T),$$

where $\mathrm{Com}_T(S)$ consists of the complete (consistent) theories C in S with $T \subseteq C$.

As for Proposition 1, it is irrelevant whether the intersection includes the only inconsistent theory S.

Since every complete theory is stable (Lemma 3), the left-hand side of ILL is as for the original *Lindenbaum Lemma* [106] in the form

$$\bigcap \mathrm{Com}_T(S) = T,$$

for every *stable* theory T in S. Hence the left-hand side of ILL equals in **ZFC** the classical deductive closure of T. What about the right-hand side of ILL?

PROPOSITION 3. *For every theory T of an intermediate logic $\triangleright$ in a propositional language S,*

$$\mathrm{Jac}(T) = \{\alpha \in S : T \ni \neg\neg\alpha\}.$$

PROOF. Let $\alpha \in \mathrm{Jac}(T)$. Since $\alpha \triangleright \neg\neg\alpha$, we get $T \triangleright \neg\neg\alpha$. Conversely, let $\alpha \in S$ be such that $T \ni \neg\neg\alpha$. If $\beta \in S$ is such that $\alpha \triangleright \neg\beta$, then $\neg\neg\alpha \triangleright \neg\beta$ and thus $T \triangleright \neg\beta$. ⊣

With Lemma 3 at hand we obtain the following:

COROLLARY 1. $\mathrm{Jac}(T)$ *is the least stable theory of* $\rhd$ *which contains the given theory* T *of* $\rhd$*; in other words,* $\mathrm{Jac}(T)$ *equals the deductive closure of* T *with respect to classical logic.*

So ILL for any intermediate logic $\rhd$ whatsoever is nothing but the original Lindenbaum Lemma!

Now let $\rhd$ be intuitionistic logic $\rhd_i$, and write $\rhd_c$ for classical logic, always in the given propositional language S. In this case and by the above, Lindenbaum's Lemma in the form of ILL is the semantics of *Glivenko's Theorem* [46], which in turn is well known as purely syntactical:

THEOREM 2 (Glivenko 1929). *Let* S *be a propositional language. For all* $\Gamma \in \mathrm{Fin}(S)$ *and* $\varphi \in S$,

$$\Gamma \rhd_c \varphi \Rightarrow \Gamma \rhd_i \neg\neg\varphi.$$

For example, this follows from Corollary 1. We hasten to add that the latter rests upon Lemmas 1 and 2, which of course are the main ingredients of a very common proof of Glivenko's theorem. Recent literature about Glivenko's Theorem includes [36, 39, 42, 48, 56, 60, 74, 75, 82, 84].[8]

§6. Glivenko's theorem as syntactical conservation. Once more let $\rhd_i$ and $\rhd_c$ stand for intuitionistic and classical logic in a propositional language S. For $\Gamma, \Delta \in \mathrm{Fin}(S)$ and $\varphi \in S$, set

$$\Gamma \rhd_g \varphi \equiv \Gamma \rhd_i \neg\neg\varphi \qquad \text{and} \qquad \Gamma \vdash_c \Delta \equiv \Gamma \rhd_c \bigvee \Delta,$$

which defines a single- and a multi-conclusion entailment relation, respectively. Of course the trace of $\vdash_c$ is nothing but $\rhd_c$; so Glivenko's Theorem (Theorem 2) can be rephrased as the following syntactical conservation:

THEOREM 3. *The extension* $\vdash_c$ *of* $\rhd_g$ *is conservative, that is,*

$$\text{Gli} \qquad \Gamma \rhd_c \varphi \Rightarrow \Gamma \rhd_g \varphi,$$

for all $\Gamma \in \mathrm{Fin}(S)$ *and* $\varphi \in S$.

To see how Theorem 1 applies in this context, we now prove Theorem 3 in detail. Clearly this proof will otherwise have the main ingredients of any proof of Glivenko's Theorem (Theorem 2). By Lemma 2, $\vdash_c$ extends $\rhd_i$, and thus $\rhd_g$, with the following additional axioms:

$$\bot \vdash_c \qquad \vdash_c \varphi, \neg\varphi \quad (\varphi \in S). \tag{8}$$

The corresponding conservation criteria (3) read

$$\frac{}{\Gamma, \bot \rhd_g \psi} \qquad \frac{\Gamma, \varphi \rhd_g \psi \quad \Gamma, \neg\varphi \rhd_g \psi}{\Gamma \rhd_g \psi} \tag{9}$$

with $\Gamma \in \mathrm{Fin}(S)$ and $\varphi, \psi \in S$.

[8]This list is by no means meant exhaustive.

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To prove Theorem 3, in view of Theorem 1 it thus is (necessary and) sufficient to verify (9). Writing $\rhd_i\neg\neg$ for $\rhd_g$ this goes as follows. Needless to say, $\Gamma, \bot \rhd_i \neg\neg\psi$. If both $\Gamma, \varphi \rhd_i \neg\neg\psi$ and $\Gamma, \neg\varphi \rhd_i \neg\neg\psi$, then $\Gamma, \varphi \vee \neg\varphi \rhd_i \neg\neg\psi$ by disjunction elimination. By Lemma 1 we get $\Gamma, \neg\neg(\varphi \vee \neg\varphi) \rhd_i \neg\neg\psi$ and thus $\Gamma \rhd_i \neg\neg\psi$ as desired.

As the models of $\vdash_c$ are exactly the complete consistent theories, ILL for $T \equiv \Gamma^{\rhd_i}$ with $\Gamma \in \mathrm{Fin}(S)$ is to Gli just as GKL is to Con for $\vdash \equiv \vdash_c$ and $\rhd \equiv \rhd_g$. Although $\vdash_c$ equally extends $\rhd_i$ with the same additional axioms (8), and the first conservation criterion of (9) also holds for $\rhd_i$ in place of $\rhd_g$, this of course is not the case in general for the second one, e.g., if $\psi \equiv \varphi \vee \neg\varphi$.

We conclude by a relativised version of Glivenko's Theorem as syntactical conservation. Let $\Gamma \subseteq \mathrm{Fin}(S)$ and $\psi \in S$. With $\mathrm{TND}_0(\Gamma, \psi)$ as in Lemma 2, for every propositional variable $\varphi \in S$ consider the conservation criterion from Theorem 1 for $\vdash_c \varphi, \neg\varphi$ over $\rhd_i$:

$$\mathrm{Cri}_\varphi(\Gamma, \psi) : \frac{\Gamma, \Delta, \varphi \rhd_i \psi \quad \Gamma, \Delta, \neg\varphi \rhd_i \psi}{\Gamma, \Delta \rhd_i \psi} \text{ or, equivalently, } \frac{\Gamma, \Delta, \varphi \vee \neg\varphi \rhd_i \psi}{\Gamma, \Delta \rhd_i \psi}$$
$$\text{for all finite subsets } \Delta \text{ of } \mathrm{TND}_0(\Gamma, \psi).$$

PROPOSITION 4. *For arbitrary but fixed* $\Gamma \subseteq \mathrm{Fin}(S)$ *and* $\psi \in S$, *the following items are equivalent*:

1. $\mathrm{Cri}_\varphi(\Gamma, \psi)$ *for all propositional variables* $\varphi \in S$ *occurring in* Γ *or* ψ.
2. $\Gamma \rhd_c \psi \Rightarrow \Gamma \rhd_i \psi$, *i.e.*, $\vdash_c$ *is conservative over* $\rhd_i$ *for the given* Γ *and* ψ.

While the first implies the second item by Lemma 2 and Theorem 1, the converse is evident.

Glivenko's Theorem 2 is the case of the second item in which Γ is arbitrary but ψ is a *negated* formula, in which case the first item obtains by Lemma 1. Other cases include the one in which $\Gamma \cup \{\psi\}$ is made of *negative* formulas only; see e.g., [34, 107, 108].

§7. Complements. We briefly review some related observations recently made about double negation [39] in the more general context of a nucleus j in place of $\neg\neg$.

First, for any given intermediate *propositional* logic $\rhd$ the following are equivalent:

A. $\Gamma \vdash_c \varphi \Rightarrow \Gamma \rhd \neg\neg\varphi$ for all Γ, φ and
B. $\varphi \to \neg\neg\psi \rhd \neg\neg(\varphi \to \psi)$ for all φ, ψ.

Now B is well-known to hold whenever $\rhd$ is intuitionistic logic $\rhd_i$ (see, e.g., [111, Lemma 6.2.2]), in which case A becomes Glivenko's theorem. *A posteriori* B holds for any intermediate logic $\rhd$ whatsover, as any such $\rhd$ extends $\rhd_i$.

Next, if $\rhd$ is an intermediate *predicate* logic, then A is equivalent to B in conjunction with the *double negation shift* for $\rhd$:

C. $\forall x \neg\neg\varphi \rhd \neg\neg\forall x \varphi$ for all formulae φ.

Again if $\rhd$ is intuitionistic logic $\rhd_i$, this yields Gödel's extension of Glivenko's theorem [37, 47].

Now C trivially holds for any existential logic, i.e., without $\forall$ altogether, for which A with $\rhd_i$ as $\rhd$ is [107, Corollary to Proposition 2.3.8]. For A to hold it suffices to refrain from using the $\forall$–introduction rule or right rule R$\forall$, which in fact is the only rule that can cause issues in this setting [39]. To be able to avoid R$\forall$ it is enough that the sequent under consideration have no positive occurrences of $\forall$, because derivations of such sequents by classical logic can be cleared from that rule (see, e.g., [74]).

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