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2021-04-05

Kuusi , T , Misawa , M & Nakamura , K 2021 , ' Global existence for the p-Sobolev flow ' , Journal of Differential Equations , vol. 279 , pp. 245-281 . https://doi.org/10.1016/j.jde.2021.01.018

http://hdl.handle.net/10138/354086
https://doi.org/10.1016/j.jde.2021.01.018

Global existence for the p-Sobolev flow

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Keywords— p-Sobolev flow, Nonlinear Intrinsic Scaling Transformation, Expansion of Positivity.

MSC2010— Primary: 35B45, 35B65, Secondary: 35D30, 35K61.

Abstract

In this paper, we study a doubly nonlinear parabolic equation arising from the gradient flow for p-Sobolev type inequality, referred as p-Sobolev flow. In the special case $p = 2$ our theory includes the classical Yamabe flow on a bounded domain in Euclidean space. Our main aim is to prove the global existence of the p-Sobolev flow together with its qualitative properties.

Contents

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ be a bounded domain with smooth boundary $\partial\Omega$. For any positive $T \leq \infty$, let $\Omega_T := \Omega \times (0, T)$ be the space-time cylinder. Throughout the paper we fix $p \in [2, n)$ and set $q := p^* - 1$, where $p^* := \frac{np}{n-p}$ is the Sobolev conjugate of p. We consider the following doubly nonlinear parabolic system

$$\begin{cases} \partial_t(|u|^{q-1}u) - \Delta_p u = \lambda(t)|u|^{q-1}u & \text{in} \quad \Omega_\infty \\ \|u(t)\|_{L^{q+1}(\Omega)} = 1 & \text{for all } t > 0 \\ u = 0 & \text{on} \quad \partial\Omega \times (0, \infty) \\ u(\cdot, 0) = u_0(\cdot) & \text{in} \quad \Omega. \end{cases} \tag{1.1}$$

Here the unknown function $u = u(x, t)$ is a real-valued function defined for $(x, t) \in \Omega_\infty$, and the initial data u_0 is assumed to be in the Sobolev space $W_0^{1,p}(\Omega)$, positive, bounded in Ω and satisfy $\|u_0\|_{L^{q+1}(\Omega)} = 1$, as usual, $\partial_t := \partial/\partial t$ and $\partial_\alpha := \partial/\partial x_\alpha, \alpha = 1, \dots, n$ are the partial derivatives on time and space, respectively, and $\nabla := (\partial_1, \dots, \partial_n)$ is the gradient on space, and $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian. The condition imposed in the second line of (1.1) is called the *volume constraint* and $\lambda(t)$ is Lagrange multiplier stemming from this volume constraint. Indeed, multiplying (1.1) by u and integrating by parts, we find by a formal computation that $\lambda(t) = \int_\Omega |\nabla u(x,t)|^p \, dx$ (See [21, Proposition 5.2] for its proof). We call the system (1.1) as *p-Sobolev flow*.

Our main result in this paper is the following theorem.

Theorem 1.1 *Assume that the initial value u_0 belongs to the Sobolev space $W_0^{1,p}$, positive, bounded in Ω, and satisfies $\|u_0\|_{L^{q+1}(\Omega)} = 1$. Then there exists a global weak solution to the equation (1.1), which is positive and bounded in Ω_∞ and is, together with its spatial gradient, locally Hölder continuous in Ω_∞.*

In our forthcoming work we will proceed further with the analysis of the p-Sobolev flow. Especially, we will classify the limits as time tends to infinity.

The doubly nonlinear equations have been considered by Vespri [34], Porzio and Vespri [24], and Ivanov [14, 15]. See also [10, 35, 19, 18]. The regularity proofs for doubly nonlinear equations are based on the intrinsic scaling method, originally introduced by DiBenedetto, and they have to be arranged in some way depending on the particular form of the equation. In [21] we have already treated the very fast diffusive doubly nonlinear equation such as the p-Sobolev flow (1.1), and obtained the positivity, boundedness and regularity of weak solutions. In particular, the expansion of positivity for (1.1) is shown by the De Giorgi's iteration based on local energy estimates in the intrinsic scaling setting (refer to [8]). The solution to (2.4) remains positive for all finite times by the volume constraint. This is here applied for the global existence of the p-Sobolev flow (1.1), as explained later.

In compact manifold setting with $p = 2$, our p-Sobolev flow (2.4) is exactly the classical Yamabe flow equation in the Euclidean space. The classical Yamabe flow was originally introduced by Hamilton in his study of the so-called Yamabe problem ([36, 2, 3]), asking the existence of a conformal metric of constant curvature on $n(\geq 3)$-dimensional closed Riemannian manifolds ([12]). Let $(\mathcal{M}, g_0)$ be a $n(\geq 3)$-dimensional smooth, closed Riemannian manifold with scalar curvature $R_0 = R_{g_0}$. The classical Yamabe flow is given by the heat flow equation

$$u_t = (s - R)u = u^{-\frac{4}{n-2}}(c_n \Delta_{g_0} u - R_0 u) + su, \tag{1.2}$$

where $u = u(t), t \geq 0$, is a positive smooth function on $\mathcal{M}$ such that $g(t) = u(t)^{\frac{4}{n-2}} g_0$ is a conformal change of a Riemannian metric g_0, with volume constraint $\mathrm{Vol}(\mathcal{M}) = \int_{\mathcal{M}} dvol_g = \int_{\mathcal{M}} u^{\frac{2n}{n-2}} dvol_{g_0} = 1$, having total curvature

$$s := \int_{\mathcal{M}} (c_n |\nabla u|^2_{g_0} + R_0 u^2)\, dvol_{g_0} = \int_{\mathcal{M}} R\, dvol_g, \quad c_n := \frac{4(n-1)}{n-2}.$$

Note that the condition for volume above naturally corresponds to the volume constraint in (1.1). Hamilton ([12]) proved a convergence of the Yamabe flow as $t \longrightarrow \infty$ under some geometric conditions. Under the assumption that $(\mathcal{M}, g_0)$ is of positive scalar curvature and locally conformal flat, Ye ([37]) showed the global existence of the Yamabe flow and its convergence as $t \longrightarrow \infty$ to a metric of constant scalar curvature. Schwetlick and Struwe ([25]) established the asymptotic convergence of the Yamabe flow for an initial positive scalar curvature in the case $3 \leq n \leq 5$, under an appropriate condition of Yamabe invariance $Y(\mathcal{M}, g_0)$, which is given by infimum of Yamabe energy $E(u) = \int_{\mathcal{M}} (c_n |\nabla u|^2_{g_0} + R_0 u^2)\, dvol_{g_0}$ among all positive smooth function u on $\mathcal{M}$ with $\mathrm{Vol}(\mathcal{M}) = 1$. In Euclidean case, since $R_{g_0} = 0$ their curvature assumptions are not verified. In above outstanding results concerning the Yamabe flow, the equation is equivalently transformed to the scalar curvature equation, and this is crucial for obtaining many properties for the Yamabe flow. In contrast to their methods, we are forced to take a direct approach dictated by the structure of the p-Laplacian leading to the degenerate or singular parabolic equation of the p-Sobolev flow. Let us remark that our results cover those of the classical Yamabe flow in the Euclidian setting.

Our global existence result for the p-Sobolev flow (1.1) is established by applying a nonlinear intrinsic scaling transformation to the following prototype doubly nonlinear parabolic equation

$$\begin{cases} \partial_s(|v|^{q-1}v) - \Delta_p v = 0 & \text{in} \quad \Omega_S \\ v = 0 & \text{on} \quad \partial\Omega \times (0, S) \\ v(\cdot, 0) = v_0(\cdot) & \text{in} \quad \Omega. \end{cases} \tag{1.3}$$

Here $0 < S \leq \infty$, the unknown function $v = v(x, s)$ is real-valued function defined for $(x, s) \in \Omega_S$, and the given function v_0 is in the Sobolev space $W_0^{1,p}(\Omega)$, nonnegative and bounded in Ω. In [23] the existence result is obtained from the backward difference quotient on time and Galerkin's procedure. For the global existence we crucially use the expansion of positivity for the p-Sobolev flow, as stated before. Based on the positivity estimates in [21], here we present the refinement for the expansion of positivity for (1.1) with its precise proof (See Appendix A). Combining the nonlinear intrinsic scaling applied for (1.3), and the refined expansion of positivity, we establish the global existence of a regular weak solution to the p-Sobolev flow.

The structure of this paper is as follows. In Section 2, we prepare some notation and give the definition of a weak solution of (2.4). In Section 3, we recall the global existence and regularity estimates for (1.3) obtained in [23, 21]. Starting from positive initial data, the solution of (1.3) is positive up to a finite time, that is, the positivity expands and, furthermore, the solution vanishes at a finite time, that is verified by the comparison principle. In Section 4, we present the nonlinear intrinsic scaling transformation from (1.3) to (2.4), which is justified via mollifier argument in Appendix C. In Section 5 we give the proof of Theorem 1.1. Here the expansion of positivity by the volume constraint for the p-Sobolev flow (1.1) is crucially applied for extending the life span of the solution and yielding the global existence. In Appendix A, we present the refined expansion of positivity for the p-Sobolev flow type equation by use of a stretching time transformation with its precise proof, and also prove the key propositions used in the proof of Theorem 1.1. In Appendix B we give the elementary convergence result with its proof, which is used in the next appendix. In Appendix C, we demonstrate that the nonlinear intrinsic scaling rigorously works.

Acknowledgments

The nonlinear intrinsic scaling transformation in Section 4 was kindly suggested to us by Professor J. L. Vazquez ([33]) in 2013. T. Kuusi is supported by the Academy of Finland (grant 323099) and the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 818437).

The work by M. Misawa was partially supported by the Grant-in-Aid for Scientific Research (C) Grant number No.18K03375 at Japan Society for the Promotion of Science.

K. Nakamura is supported by Foundation of Research Fellows, The Mathematical Society of Japan.

2 Preliminaries

We prepare some notation and fundamental tools, which are used throughout this paper.

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ be a bounded domain with smooth boundary $\partial\Omega$. Let us define the parabolic boundary of the space-time cylinder $\Omega_T := \Omega \times (0,T)$ by

$$\partial_p \Omega_T := \partial\Omega \times [0,T) \cup \Omega \times \{t = 0\}.$$

Let $B_R(x_0) := \{x \in \mathbb{R}^n \,:\, |x - x_0| < R\}$ denote the open ball with radius $R > 0$ centered at some $x_0 \in \mathbb{R}^n$. We denote the positive part of $a \in \mathbb{R}$ by $a_+ := \max\{a, 0\}$.

In what follows, we denote by C, C_1, $C_2, \cdots$ different positive constants in a given context. Relevant dependencies on parameters will be emphasized using parentheses. For instance $C = C(n, p, \Omega, \cdots)$ means that C depends on $n, p, \Omega \cdots$. As customary, the equation number $(\,\cdot\,)_n$ denotes the n-th line of the Eq. $(\,\cdot\,)$.

We next prepare some function spaces, defined on space-time region. For two indices $1 \leq p, q \leq \infty$, $L^q(t_1, t_2\,;\, L^p(\Omega))$ denotes the space of measurable real-valued functions on a space-time region $\Omega \times (t_1, t_2)$ with a finite norm

$$\|v\|_{L^q(t_1,t_2\,;\,L^p(\Omega))} := \begin{cases} \left(\displaystyle\int_{t_1}^{t_2} \|v(t)\|^q_{L^p(\Omega)}\,dt \right)^{1/q} & (1 \leq q < \infty) \\ \operatorname*{ess\,sup}_{t_1 \leq t \leq t_2} \|v(t)\|_{L^p(\Omega)} & (q = \infty), \end{cases}$$

where

$$\|v(t)\|_p = \|v(t)\|_{L^p(\Omega)} := \begin{cases} \left(\displaystyle\int_{\Omega} |v(x,t)|^p\,dx \right)^{1/p} & (1 \leq p < \infty) \\ \operatorname*{ess\,sup}_{x \in \Omega} |v(x,t)| & (p = \infty). \end{cases}$$

When $p = q$, we write $L^p(\Omega \times (t_1, t_2)) = L^p(t_1, t_2\,;\, L^p(\Omega))$ for brevity. For $1 \leq p < \infty$ the Sobolev space $W^{1,p}(\Omega)$ consists of measurable real-valued functions in Ω which are weakly differentiable and of which weak derivatives are p-th integrable on Ω, with the norm

$$\|v\|_{W^{1,p}(\Omega)} := \left(\int\limits_{\Omega} |v|^p + |\nabla v|^p \, dx \right)^{1/p}$$

and let $W_0^{1,p}(\Omega)$ be the closure of $C_0^\infty(\Omega)$, the smooth functions with a compact support, with respect to the norm $\| \cdot \|_{W^{1,p}}$. The space $L^q(t_1, t_2 \,;\, W_0^{1,p}(\Omega))$ comprises all measurable real-valued functions on space-time domain with a finite norm

$$\|v\|_{L^q(t_1,t_2\,;\,W_0^{1,p}(\Omega))} := \left(\int\limits_{t_1}^{t_2} \|v(t)\|^q_{W^{1,p}(\Omega)} \, dt \right)^{1/q}.$$

Additionally, for an interval $I \subset \mathbb{R}$, the space $C(I; L^q(\Omega))$ consists of all continuous functions $I \ni t \mapsto u(t) \in L^q(\Omega)$. The function space $C(I; W_0^{1,p}(\Omega))$ is defined analogously to the above.

We next need the following fundamental algebraic inequality, associated with the p -Laplace operator (see [5, 7]).

Lemma 2.1 (Algebraic inequality) *For every $p \in (1, \infty)$ there exist positive constants $C_1(p, n)$ and $C_2(p, n)$ such that for all ξ, $\eta \in \mathbb{R}^n$*

$$||\xi|^{p-2}\xi - |\eta|^{p-2}\eta| \leq C_1(|\xi| + |\eta|)^{p-2}|\xi - \eta| \tag{2.1}$$

and

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta) \cdot (\xi - \eta) \geq C_2(|\xi| + |\eta|)^{p-2}|\xi - \eta|^2, \tag{2.2}$$

where dot $\cdot$ denotes the inner product in $\mathbb{R}^n$. In particular, if $p \geq 2$, then

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta) \cdot (\xi - \eta) \geq C_2|\xi - \eta|^p. \tag{2.3}$$

Following [21, Definition 3.2], we present the definition of a weak solution to the p-Sobolev flow equation (1.1).

Definition 2.2 (weak solution of the p-Sobolev flow) Let $0 < T \leq \infty$. A measurable function u defined on Ω_T is called a *weak solution* of (1.1) if the following (D1)-(D4) are satisfied.

(D1) $u \in L^\infty(0, T\,;\, W^{1,p}(\Omega))$; $\quad \partial_t(|u|^{q-1}u) \in L^2(\Omega_T)$.

(D2) There exists a function $\lambda(t) \in L^1_{\mathrm{loc}}(0, T)$ such that, for every $\varphi \in C_0^\infty(\Omega_T)$,

$$-\int\limits_{\Omega_T} |u|^{q-1}u\varphi_t \, dz + \int\limits_{\Omega_T} |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \, dz = \int\limits_{\Omega_T} \lambda(t)|u|^{q-1}u\varphi dz.$$

(D3) $\|u(t)\|_{L^{q+1}(\Omega)} = 1$ for all positive $t < T$.

(D4) $u = 0$ on $\partial\Omega \times (0, T)$ and $u(0) = u_0$ in Ω in the trace sense: $u(t) \in W_0^{1,p}(\Omega)$ for almost every $t \in (0, T)$;

$$\|u(t) - u_0\|_{W^{1,p}(\Omega)} \to 0 \quad \text{as} \quad t \searrow 0.$$

Proposition 2.3 *Let u be a weak solution of (1.1). If the initial value u_0 is nonnegative and bounded in Ω then so is u. Furthermore, the local L^1-function $\lambda(t)$ in the first line of (1.1) is given by*

$$\lambda(t) = \int_\Omega |\nabla u(t)|^p \, dx.$$

Proof. See [21, Proposition 5.1, Proposition 5.2] for more details. □

In what follows, under the assumption that the initial data u_0 is in the Sobolev space $W_0^{1,p}(\Omega)$, positive and bounded in Ω and satisfies $\|u_0\|_{L^{q+1}(\Omega)} = 1$, we address the following equation (2.4) in place of (1.1):

$$\begin{cases} \partial_t(u^q) - \Delta_p u = \lambda(t) u^q & \text{in} \quad \Omega_\infty \\ \|u(t)\|_{L^{q+1}(\Omega)} = 1 & \text{for all } t \geq 0 \\ u = 0 & \text{on} \quad \partial\Omega \times (0, \infty) \\ u(\cdot, 0) = u_0(\cdot) & \text{in} \quad \Omega. \end{cases} \tag{2.4}$$

3 Prototype Doubly Nonlinear Equation

In this section, we study the nonlinear scaling for the following doubly nonlinear equation (1.3). Firstly, we recall the definition and the global existence result of weak solutions to (1.3):

Definition 3.1 Let $0 < S \leq \infty$. A measurable function v, defined on $\Omega_S := \Omega \times (0, S)$, is a *weak supersolution (subsolution)* of (1.3) if the following conditions are satisfied:

(i) $v \in L^\infty(0, S\,; W^{1,p}(\Omega))$, $\partial_s(|v|^{q-1}v) \in L^2(\Omega_S)$.

(ii) For every nonnegative $\eta \in C_0^\infty(\Omega_S)$

$$-\int_{\Omega_S} |v|^{q-1} v \cdot \partial_s \eta \, dxds + \int_{\Omega_S} |\nabla v|^{p-2} \nabla v \cdot \nabla \eta \, dxds \geq (\leq) 0.$$

(iii) $v = 0$ on $\partial\Omega \times (0, S)$ and $v(0) = v_0$ in Ω in the trace sense: $v(s) \in W_0^{1,p}(\Omega)$ for almost every $s \in (0, S)$;

$$\|v(s) - v_0\|_{W^{1,p}(\Omega)} \to 0 \quad \text{as} \quad s \searrow 0.$$

Furthermore, a measurable function v defined on Ω_S is called a *weak solution* to (1.3) if it is simultaneously a weak super and subsolution, i.e.,

$$-\int\limits_{\Omega_S} |v|^{q-1}v \cdot \partial_s \eta \, dxds + \int\limits_{\Omega_S} |\nabla v|^{p-2}\nabla v \cdot \nabla \eta \, dxds = 0.$$

holds for every $\eta \in C_0^\infty(\Omega_S)$.

In [23, Theorem 1.1] and [21], we proved the global existence of a weak solution to (1.3) and it's regularity estimates as follows:

Theorem 3.2 (Global existence of (1.3) [21, 23]) *Assume the initial value v_0 be in $W_0^{1,p}(\Omega)$, nonnegative and bounded in Ω. Then there exists a global in time weak solution v of (1.3), which is nonnegative and bounded in Ω_∞ that is,*

$$0 \le v \le \|v_0\|_\infty \quad \text{in } \Omega_\infty. \tag{3.1}$$

In addition, v satisfies the following energy equality, for $0 \le s_1 < s_2 < \infty$,

$$\|v(s_2)\|_{q+1}^{q+1} + \frac{q+1}{q}\int\limits_{s_1}^{s_2} \|\nabla v(s)\|_p^p \, ds = \|v(s_1)\|_{q+1}^{q+1} \tag{3.2}$$

and, the integral inequalities hold true, for any nonnegative $s < \infty$,

$$\|v(s)\|_{q+1} \le \|v_0\|_{q+1}, \tag{3.3}$$

$$\|\nabla v(s)\|_p \le \|\nabla v_0\|_p, \tag{3.4}$$

$$\|\partial_s v^q\|_2^2 \le C\|v_0\|_\infty^{q-1}\|\nabla v_0\|_p^p, \tag{3.5}$$

where $C = C(n,p)$ is a positive constant, and the L^p-norm on space of v is denoted by $\|v\|_p := \|v\|_{L^p(\Omega)}$ for brevity.

Proof. Eq. (3.1) follows from [21, Propositions 3.4, 3.5]. Using a similar argument to [21, Appendix B], one can prove (3.2) and from this, (3.3) immediately follows. By the same way as in [23, Lemma 3.2, (3.7); Lemma 4.1; Proof of Theorem 1.1], (3.4) is actually verified. Finally, (3.5) is proved via [21, Lemmas 3.4, 4.1]. □

Remark 3.3 The existence of a weak solution v to (1.3) in Theorem 3.2 is proved by a time discretization and weak convergence of bounded approximating solutions in a reflexive Banach space. In our preceding work [23], we showed the existence on $[0, S]$ of a weak solution for any positive $S < \infty$. As seen from the proof [23, Section 5, pp.167–168], we can choose $S = +\infty$.

3.1 Extinction of Solutions

We study the finite-time extinction of a solution to (1.3). Firstly, the extinction time is defined as follows:

Definition 3.4 Let v be a nonnegative weak solution to (1.3) in Ω_∞. We call a positive number S^* as the *extinction time* of v provided that the following conditions hold:

(i) $v(x,s)$ is nonnegative and not identically zero on $\Omega \times (0, S^*)$

(ii) $v(x,s) = 0$ for any $x \in \overline{\Omega}$ and all $s \geq S^*$.

Now we will show a finite time extinction of a solution of (1.3). For this purpose we apply the comparison theorem [21, Theorem 3.6], which is originally provided by Alt-Luckhaus ([1, Theorem 2.2, p.325]).

Theorem 3.5 (Comparison theorem) *Let $0 < S \leq \infty$ and, let v_1 and v_2 be a weak supersolution and subsolution to (1.3) in Ω_S, respectively. Suppose that $v_1 \geq v_2$ on $\partial_p \Omega_S$. Then*

$$v_1 \geq v_2 \quad \textit{in} \quad \Omega_S.$$

For the construction of an appropriate comparison function we use a special solution to the elliptic type equation associated with (1.3). This special solution is called *Talenti function* [28], defined as

$$Y_{a,b,y}(x) := \left(a + b|x-y|^{\frac{p}{p-1}}\right)^{-\frac{n-p}{p}}, \quad x, y \in \mathbb{R}^n, \tag{3.6}$$

where a and b are positive numbers. In his seminal paper [28], Talenti showed that this function realizes the best constant in the Sobolev inequality. Moreover, a direct computation shows that $Y_{a,b}$ solves the equation

$$-\Delta_p Y_{a,b,z} = n\left(\tfrac{n-p}{p-1}\right)^{p-1} ab^{p-1} Y_{a,b,z}^q \quad \text{in } \mathbb{R}^n.$$

As Sciunzi showed in [26], there is a one-parameter family of functions classifying the solutions, up to translations, of $-\Delta_p Y_\lambda = Y_\lambda^q$. Indeed, one chooses a and b so that $n\left(\frac{n-p}{p-1}\right)^{p-1} ab^{p-1} = 1$ and then, by [26], the solution of $-\Delta_p Y = Y^q$ is necessarily of the form

$$Y(x) = Y_{\lambda,y}(x) := \frac{1}{\lambda}\left(n\left(\frac{n-p}{p-1}\right)^{p-1}\right)^{\frac{1}{p}} \left(1 + \left(\frac{|x-y|}{\lambda}\right)^{\frac{p}{p-1}}\right)^{-\frac{n-p}{p}} \tag{3.7}$$

with a parameter $\lambda > 0$. We show that a solution of (1.3) vanishes in finite time.

Proposition 3.6 (Finite time extinction of solutions) *Let v be a nonnegative weak solution to (1.3) in Ω_∞. Then there exists a extinction time $S^* > 0$ for v, which is bounded as follows :*

$$S^* \le \frac{q}{q+1-p}\left(\frac{\max_\Omega u_0}{\min_\Omega Y}\right)^{q+1-p},$$

where Y is Talenti's function defined by (3.7).

Proof. By translation, we may assume the origin $0 \in \Omega$. Let $v = v(x,s)$ be a nonnegative weak solution to (1.3). By the nonnegativity v is a weak solution of

$$\partial_s v^q - \Delta_p v = 0 \quad \text{in} \quad \Omega_\infty.$$

Next, let $W(x,s) = X(x)Z(s)$ be a nonnegative separable solution of

$$\partial_s W^q - \Delta_p W = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty).$$

Then $X(x)Z(s)$ satisfies

$$\begin{cases} (Z(s)^q)' = \mu Z(s)^{p-1} & \text{in } (0,\infty) \\ \Delta_p X = \mu X^q & \text{in } \mathbb{R}^n, \end{cases} \tag{3.8}$$

where μ is a separation constant. By an integration by parts we see that $\mu < 0$. Set $X := (-\mu)^{-\frac{1}{q+1-p}} Y$ to obtain

$$-\Delta_p Y = Y^q \quad \text{in } \mathbb{R}^n. \tag{3.9}$$

As discussed before the proof, an energy-finite solution to (3.9) is given by (3.7). By a straightforward computation, we find that

$$Z(s) = Z(0)\left(1 + \mu\frac{q+1-p}{q} Z(0)^{p-(q+1)} s\right)_+^{\frac{1}{q+1-p}}$$

solves the first equation in (3.8), where $Z(0)$ is the initial data. Thus the vanishing time Z_0 of $Z(s)$ is given by

$$Z_0 = (-\mu)^{-1}\frac{q}{q+1-p} Z(0)^{q+1-p}.$$

Let $V(x,s)$ be $(-\mu)^{-\frac{1}{q+1-p}} Y(x)Z(s)$. Then

$$0 = v(x,s) \le V(x,s) \quad \text{on } \partial\Omega \times [0,\infty)$$

We choose the initial data for the ODE in (3.8) as

$$Z(0) = \frac{\max_\Omega u_0}{\min_\Omega Y}(-\mu)^{\frac{1}{q+1-p}} \tag{3.10}$$

and therefore, we find that

$$u_0(x) \le V(x,0) \quad \text{in } \ \Omega.$$

According to the comparison theorem [21, Theorem 3.6], we have

$$v(x,s) \le V(x,s) \quad \text{in } \ \Omega_S \ \text{ for any positive } \ S < \infty$$

and thus, the vanishing time S^* of $v(x,s)$ is estimated as

$$S^* \le S_0 = \frac{q}{q+1-p}\left(\frac{\max_\Omega u_0}{\min_\Omega Y}\right)^{q+1-p},$$

where (3.10) is used. The proof is complete. □

4 Nonlinear Intrinsic Scaling Transformation

In this section we will introduce a scaling transformation, which transforms the prototype equation (1.3) into the p-Sobolev equation (2.4). Hereafter we choose the initial data v_0 in (1.3) as u_0 in (2.4). We suppose that the initial data u_0 is in $W_0^{1,p}(\Omega)$, positive and bounded in Ω, and $\|u_0\|_{q+1} = 1$. As in Theorem 3.2, by [21, Proposition 3.4], the solution v of (1.3) must be nonnegative and thus, we can consider (1.3) as

$$\begin{cases} \partial_s(v^q) - \Delta_p v = 0 & \text{in } \ \Omega_S \\ v = 0 & \text{on } \ \partial\Omega \times (0,S) \\ v(\cdot,0) = u_0(\cdot) & \text{in } \ \Omega. \end{cases} \tag{4.1}$$

From now on, we will mainly consider (4.1) instead of (1.3).

Let us consider the following nonlinear intrinsic scaling transforming (4.1) to (2.4).

Proposition 4.1 (Nonlinear intrinsic scaling) *Let v be a nonnegative weak solution to the equation (4.1) in Ω_∞ and let $S^* < +\infty$ be a finite extinction time of v. There exist unique $\Lambda \in C^1[0,\infty)$ solving*

$$\begin{cases} \Lambda'(\tau) = (S^*)^{-1}\left(\displaystyle\int_\Omega v^{q+1}\left(x, S^*\left(1-e^{-\Lambda(\tau)}\right)\right)\,dx\right)^{\frac{p}{n}} \\ \Lambda(0) = 0 \end{cases} \tag{4.2}$$

and, subsequently, $g \in C^1[0,\infty)$ solving

$$\begin{cases} g'(t) = e^{\Lambda(g(t))} \\ g(0) = 0 \end{cases} \tag{4.3}$$

such that the following is valid: Let

$$s(t) = S^* \left(1 - e^{-\Lambda(g(t))}\right) \tag{4.4}$$

and set

$$u(x,t) := \frac{v\,(x, s(t))}{\gamma(t)}, \quad \gamma(t) := \left(\int_\Omega v^{q+1}\,(x, s(t))\; dx \right)^{\frac{1}{q+1}}. \tag{4.5}$$

Then u is a nonnegative weak solution of the p-Sobolev flow (2.4) on Ω_∞. More precisely, u satisfies the conditions (D1)–(D4) of Definition 2.2, where $\lambda(t) := -q\dfrac{\gamma'(t)}{\gamma(t)} = \displaystyle\int_\Omega |\nabla u(x,t)|^p\, dx$.

Proof of Proposition 4.1. Here, we will make a formal computation and show the relevance of intrinsic scaling above to the p-Sobolev flow. The rigorous argument will be given in Appendix C.

Firstly, let us verify that u satisfies $(2.4)_1$. Noticing (4.4)

$$s(t) = S^*(1 - e^{-\Lambda(g(t))}) \quad \Longleftrightarrow \quad \Lambda(g(t)) = \log \left(\frac{S^*}{S^* - s(t)} \right),$$

we compute as

$$\begin{aligned}
\frac{d}{dt}\Lambda(g(t)) &= \Lambda'(g(t))g'(t) \\
&= (S^*)^{-1} e^{\Lambda(g(t))} \left(\int_\Omega v^{q+1}(x, s(t))\, dx \right)^{\frac{p}{n}} \\
&= (S^*)^{-1} e^{\Lambda(g(t))} \gamma(t)^{(q+1)\frac{p}{n}}
\end{aligned}$$

and thus,

$$s_t = \frac{ds}{dt} = S^* e^{-\Lambda(g(t))} \frac{d}{dt}\Lambda(g(t)) = \gamma(t)^{(q+1)\frac{p}{n}}. \tag{4.6}$$

By (3.2) and (3.4) in Theorem 3.2, $s \mapsto \left(\displaystyle\int_\Omega v^{q+1}(x, s)\, dx \right)^{\frac{1}{q+1}}$ is Lipschitz continuous. This together with (4.6) provides

$$\begin{aligned}
\partial_t u^q &= \partial_s v^q s_t \gamma^{-q} + v^q(-q)\gamma^{-q-1}\gamma'(t) \\
&= \partial_s v^q \gamma^{-q+(q+1)\frac{p}{n}} - q u^q \gamma^{-1}\gamma'(t).
\end{aligned} \tag{4.7}$$

Multiplying $(2.4)_1$ by v and integration by parts give

$$\frac{q}{q+1}\frac{d}{ds} \int_\Omega v^{q+1}(s)\, dx + \int_\Omega |\nabla v(s)|^p\, dx = 0, \tag{4.8}$$

that is the same reasoning as (3.2) in Theorem 3.2. Furthermore

$$\begin{aligned}
\gamma'(t) &= \frac{1}{q+1}\left(\int\limits_\Omega v^{q+1}(x,s(t))\,dx\right)^{\frac{1}{q+1}-1}\frac{d}{ds}\int\limits_\Omega v^{q+1}(s)\,dx\Bigg|_{s=s(t)} s_t \\
&\overset{(4.8)}{=} -\frac{1}{q}\gamma^{1-(q+1)}\int\limits_\Omega |\nabla v(s(t))|^p\,dx\cdot s_t \\
&= -\frac{1}{q}\gamma\int\limits_\Omega |\nabla u(t)|^p\,dx \qquad (4.9)
\end{aligned}$$

and

$$\Delta_p u = \gamma^{1-p}\Delta_p v. \qquad (4.10)$$

From (4.7) and (4.9) it follows that

$$\begin{aligned}
\partial_t u^q &= \partial_s v^q\gamma^{1-p} + \left(\int\limits_\Omega |\nabla u(t)|^p\,dx\right)u^q \qquad (4.11)\\
&= \left[\partial_s v^q + \left(\int\limits_\Omega |\nabla v(s(t))|^p\,dx\right)v^q\gamma^{-(q+1)}\right]\gamma^{1-p}. \qquad (4.12)
\end{aligned}$$

Eq. (4.11) together with (4.10) and Eq. $(2.4)_1$ yield that

$$\partial_t u^q - \Delta_p u = \left(\int\limits_\Omega |\nabla u(t)|^p\,dx\right)u^q,$$

which is exatcly $(2.4)_1$.

We will verify that u satisfies the condition (D1) in Definition 2.2. Let $t_0 < \infty$ be any positive number and set $s_0 = S^*(1-e^{-\Lambda(g(t_0))})$. We shall notice the fact : As shown later in (C.6) in Lemma C.2, we find that there is a positive number c_0 such that $c_0 := \min\limits_{0\le s\le s_0}\|v(s)\|_{q+1} > 0$. From $\gamma(t)\ge c_0>0$ and (3.4), it follows that

$$\begin{aligned}
\int\limits_\Omega |\nabla u(t)|^p\,dx &= \frac{1}{\gamma(t)^p}\int\limits_\Omega |\nabla v(s(t))|^p\,dx \\
&\le c_0^{-p}\|\nabla u_0\|_p^p < \infty \qquad (4.13)
\end{aligned}$$

and thus, $u\in L^\infty(0,t_0\,;W^{1,p}(\Omega))$. By changing of variable $s = S^*(1-e^{-\Lambda(g(t))})$ and (4.6), and merging (4.12),(C.6) in Lemma C.2, (3.1), (3.3), (3.4) and (3.5) we get

$$\int\limits_0^{t_0}\!\!\int\limits_\Omega (\partial_t u^q)^2\,dxdt = \int\limits_0^{s_0}\!\!\int\limits_\Omega (\partial_t u^q)^2\gamma^{-(q+1)\frac{p}{n}}\,dxds$$

$$
\begin{aligned}
&\le 2\int_0^{s_0}\!\!\int_\Omega \left[(\partial_s v^q)^2 + \|\nabla v(s(t))\|_p^{2p}\, v^{2q}\gamma^{-2(q+1)}\right]\gamma^{2(1-p)}\cdot\gamma^{-(q+1)\frac{p}{n}}\,dxds \\
&\le 2\|u_0\|_{q+1}^{(q+1)\frac{p}{n}} c_0^{-2q}\int_0^{s_0}\!\!\int_\Omega (\partial_s v^q)^2\,dxds \\
&\quad + 2\|u_0\|_{q+1}^{(q+1)\frac{p}{n}} c_0^{-4q-2}\|u_0\|_\infty^{2q}\|\nabla u_0\|_p^{2p}|\Omega| s_0 < \infty,
\end{aligned}
$$

which yields (D1) for any positive $T < \infty$.

By the very definition of u as in (4.5), $\displaystyle\int_\Omega u^{q+1}(x,t)\,dx = 1$ for any $t \in [0,\infty)$, that is (D3) with $T = \infty$.

Since $v(s) \in W_0^{1,p}(\Omega)$ for a.e. $s > 0$, $u(t) = v(s(t))/\gamma(t) \in W_0^{1,p}(\Omega)$ for a.e. $t \in [0, t_0]$. In addition,

$$
\begin{aligned}
\|u(t) - u_0\|_{W^{1,p}(\Omega)} &= \left\|\frac{v(s(t))}{\gamma(t)} - u_0\right\|_{W^{1,p}(\Omega)} \\
&\le \frac{1}{\gamma(t)}\left\{\|v(s(t)) - u_0\|_{W^{1,p}(\Omega)} + \|u_0\|_{W^{1,p}(\Omega)}|\gamma(t) - \gamma(0)|\right\},
\end{aligned}
\tag{4.14}
$$

where $\gamma(0) = \|u_0\|_{q+1} = 1$. Remark that $s(t) = S^*(1 - e^{-\Lambda(g(t))}) \searrow 0 \iff t \searrow 0$ because $\Lambda(\tau)$, $0 \le \tau < \infty$ and $g(t)$, $0 \le t < \infty$, are monotone increasing and $\Lambda(0) = g(0) = 0$ by (4.3) and (4.2). By the Minkowski and Sobolev inequalities, we get, as $t \searrow 0$,

$$
\begin{aligned}
|\gamma(t) - \gamma(0)| &\le \left|\,\|v(s(t))\|_{q+1} - \|u(0)\|_{q+1}\right| \\
&\le C\,\|v(s(t)) - u_0\|_{W^{1,p}(\Omega)} \longrightarrow 0
\end{aligned}
\tag{4.15}
$$

since $\|v(s) - u_0\|_{W^{1,p}(\Omega)} \longrightarrow 0$ as $s \searrow 0$. Merging $\gamma(t) \ge c_0 > 0$, (4.14) and (4.15), we obtain $\|u(t) - u_0\|_{W^{1,p}(\Omega)}$ as $t \searrow 0$, that gives (D4).

Therefore we finish the proof. □

5 Proof of Theorem 1.1

In this section, we shall prove our main theorem,Theorem 1.1.

The scheme of our proof is the following: Firstly, by Theorem 3.2, we will solve the prototype equation (1.3) with the initial data u_0 and then, by Proposition 4.1, we transform the solution v to the desired solution u of the p-Sobolev flow (2.4), that can be possible up to any finite time, since the extinction time of solution v is converted to the infinity. Here, the expansion of positivity of the solution of (2.4) on the domain is

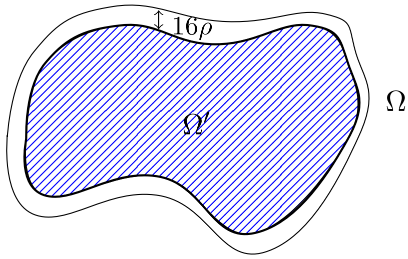

Figure 5.1: Domain and subdomain

used. In particular, the time-length of expansion of positivity is estimated only by the volume, $\|u_0\|_{L^{q+1}(\Omega)} = 1$, the boundedness, and the positive lower bound of the initial data u_0 in the interior of domain. See Proposition 5.4 for details. The solution u is actually bounded at any finite time as in (5.1) of Proposition 5.1. In this way, we have the global existence of solution of the p-Sobolev flow (2.4).

We have the boundedness of weak solutions of p-Sobolev flow (2.4). Here we use by the fact that by Proposition 2.3 we have that $\lambda(t) = \|\nabla u(t)\|_p^p$ in (2.4).

Proposition 5.1 (Boundedness of the p-Sobolev flow) *Let u be a nonnegative weak solution of (2.4) in Ω_T. Then u is bounded from above in Ω_T and*

$$\|u(t)\|_\infty \leq e^{\frac{1}{q}\int_0^T \|\nabla u(t)\|_p^p\,dt}\|u_0\|_\infty \quad \textit{for every} \quad 0 \leq t < T. \tag{5.1}$$

In [21] we proved *the expansion of positivity* of a solution of the doubly nonlinear equation such as (2.4) and (1.3). In particular, the convexity of domain is not needed by virtue of the so-called Harnack chain argument. See [21, Theorem 4.7, Corollary 4.8] for detail and its proof. We are going to deduce the refined assertion of them.

Before stating, we set the notation as below. Let Ω' be a subdomain contained compactly in Ω. Let ρ be any positive number satisfying $\rho \leq \frac{1}{16}\mathrm{dist}(\Omega', \partial\Omega)$.

Now, we state the refinement of expansion of positivity with a waiting time (cf. [21, Theorem 4.9]).

Theorem 5.2 (Expansion of positivity with a waiting time) *Let u be a nonnegative weak solution of (2.4) in Ω_T. Let Ω' be a subdomain contained compactly in Ω. Let ρ be any positive number satisfying $\rho \leq \mathrm{dist}(\Omega', \partial\Omega)/16$. Let $t_0 \in (0, T]$. Suppose that*

$$\left|\Omega' \cap \{u(t_0) \geq L\}\right| \geq \alpha|\Omega'| \tag{5.2}$$

holds for some $L > 0$ and $\alpha \in (0, 1]$. Then there exist positive integer $N = N(\Omega', \rho)$, positive real number families $\delta_0, \delta_N, \eta_N, \eta_{N+1}, \sigma_N \in (0, 1)$, J_N, $I_N \in \mathbb{N}$ depending on α, N, n, p and independent of L, and a time $t_N > t_0$ such that

$$u \geq \eta_{N+1}L$$

almost everywhere in

$$\Omega' \times \left(t_N + (1-\sigma_N)\delta_N(\eta_N L)^{q+1-p}\rho^p,\, t_N + \delta_N(\eta_N L)^{q+1-p}\rho^p\right),$$

where $\sigma_N = e^{-(\tau_N + 2e^{\tau_N})}$, $e^{\tau_N} = C2^{I_N+J_N}$ *with* $C = C(n,p) > 0$, *and* t_N *is written as*

$$t_N = t_0 + (\delta_0 - \delta_N \eta_N^{q+1-p})L^{q+1-p}\rho^p$$

and thus, the terminal time of the time interval above is

$$t_0 + \delta_0 L^{q+1-p}\rho^p.$$

Proof. The proof of this theorem is postponed, and will be given in Appendix A.2. □

We also state the refined expansion of positivity without a waiting time (cf. [21, Corollary 4.10]).

Proposition 5.3 (Expansion of positivity without a waiting time) *Let* u *be a nonnegative weak solution of* (2.4) *in* Ω_T. *Let* Ω' *be a subdomain contained compactly in* Ω. *Let* ρ *be any positive number satisfying* $\rho \le \mathrm{dist}(\Omega', \partial\Omega)/16$. *Suppose that* $u(t_0) > 0$ *in* Ω *for some* $t_0 \in [0,T)$. *Then there exist positive numbers* η_0 *and* τ_0 *such that*

$$u \ge \eta_0 \quad a.e. \quad in \quad \Omega' \times (t_0, t_0 + \tau_0).$$

We also give the proof of this proposition in Appendix A.2.

Applying Theorem 5.2 and Proposition 5.3 with $t_0 = 0$, we have another refinement of the interior positivity by the volume constraint (cf. [21, Proposition 5.4]).

Proposition 5.4 (Interior positivity by the volume constraint) *Let the initial data* $u_0 \in W_0^{1,p}(\Omega)$ *be positive, bounded in* Ω *and satisfy* $\|u_0\|_{q+1} = 1$. *Let* u *be a nonnegative weak solution of* (2.4) *in* Ω_T *with* $T > 0$. *Put* $M := e^{\frac{1}{q}\int_0^T \|\nabla u(t)\|_p^p\,dt}\|u_0\|_\infty$ *and let* Ω' *be a subdomain compactly contained in* Ω *satisfying* $|\Omega \setminus \Omega'| \le \frac{1}{4M^{q+1}}$. *Then there exists a positive constant* η *such that*

$$u(x,t) \ge \eta L \quad in \quad \Omega' \times [0,T].$$

Here $0 < L \le \min\left\{\left(\frac{1}{4|\Omega'|}\right)^{\frac{1}{q+1}}, \inf_{\Omega''} u_0\right\}$, *where* Ω'' *is compactly contained in* Ω *and compactly containing* Ω', *and the positive constant* η *depends only on* p, n, Ω', M *and* N, *where* N *is the number of chain balls of* Ω'. *The constant* η *is given as a non-increasing positive function in* T *and* $\|u_0\|_\infty$.

We present the proof of this proposition in Appendix A.3.

Finally, we give the positivity near the boundary for the solutions of p-Sobolev flow (2.4). See [21, Propositions 5.5, 4.9] for details.

Proposition 5.5 (Positivity around the boundary) *Suppose that $u_0 > 0$ in Ω. Let u be a nonnegative weak solution u to (2.4) in Ω_T. Then u is positive in Ω_T near the boundary.*

Under the above preliminaries, we now prove Theorem 1.1.

Proof of Theorem 1.1. We divide the proof in three steps.

Step 1: We choose the initial data v_0 as u_0, that of the p-Sobolev flow (2.4) and solve the prototype equation (1.3) with the initial data $v_0 = u_0$. Let v be a nonnegative weak solution of (1.3) in Ω_∞ with the initial data $v_0 = u_0$. Let S^* be the extinction time of v.

Step 2: Here we will verify the positivity for the p-Sobolev flow (2.4). From Propositions C.5 and 5.5, for any positive $T < \infty$

$$u > 0 \quad \text{in} \quad \Omega \times [0, T]. \tag{5.3}$$

From the global existence result for the prototype doubly nonlinear equation (1.3) in Theorem 3.2 and the nonlinear intrinsic scaling transformation in Proposition 4.1, we plainly get the global existence for the p-Sobolev flow (2.4).

Step 3: Finally, we will show the local Hölder regularity for the p-Sobolev flow (2.4).

Following [21, Section 5.2], we recall the result of Hölder and gradient Hölder continuity of the solution to p-Sobolev flow (2.4) with respect to space-time variable.

Suppose the initial value $u_0 > 0$ in Ω. Then by Propositions 5.4 and 5.1, for any Ω' compactly contained in Ω and $T \in (0, \infty)$, we can take a positive constant $\tilde{c}$ such that

$$0 < \tilde{c} \le u \le M =: e^{\frac{1}{q}\int_0^T \|\nabla u(t)\|_p^p \, dt} \|u_0\|_\infty \quad \text{in} \quad \Omega' \times [0, T]. \tag{5.4}$$

As discussed in [21, Section 5.2], by (5.4), we can rewrite the first equation of (2.4) as follows : Set $v := u^q$, which is equivalent to $u = v^{\frac{1}{q}}$ and put $g := \frac{1}{q} v^{1/q-1}$ and then, it is easily seen that the first equation of (2.4) is equivalent to

$$\partial_t v - \operatorname{div}\left(|\nabla v|^{p-2} g^{p-1} \nabla v\right) = \lambda(t) v \quad \text{in} \quad \Omega' \times [0, T] \tag{5.5}$$

and thus, v is a positive and bounded weak solution of the evolutionary p-Laplacian equation (5.5). By (5.4) g is uniformly elliptic and bounded in Ω'_T.

The following Hölder continuity is proved via the local energy inequality for a local weak solution v to (5.5) ([21, Lemma C.1]) and standard iterative real analysis methods. See also[7, Chapter III] or [30, Section 4.4, pp.44–47] for more details.

Theorem 5.6 (Hölder continuity [21, Theorem 5.6]) *Let v be a positive and bounded weak solution to (5.5). Then v is locally Hölder continuous in Ω'_T with a Hölder exponent $\beta \in (0,1)$ on a parabolic metric $|x| + |t|^{1/p}$.*

By a positivity and boundedness as in (5.4) and a Hölder continuity in Theorem 5.6, we see that the coefficient g^{p-1} is Hölder continuous and thus, obtain a Hölder continuity of its spatial gradient.

Theorem 5.7 (Gradient Hölder continuity [21, Theorem 5.7]) *Let v be a positive and bounded weak solution to (5.5). Then, there exist a positive constant C depending only on $n, p, \tilde{c}, M, \lambda(0), \beta, \|\nabla v\|_{L^p(\Omega'_T)}, [g]_{\beta,\Omega'_T}$ and $[v]_{\beta,\Omega'_T}$ and a positive exponent $\alpha < 1$ depending only on n, p and β such that ∇v is locally Hölder continuous in Ω'_T with an exponent α on the usual parabolic metric. Furthermore, its Hölder constant is bounded above by C, where $[f]_\beta$ denote the Hölder semi-norm of a Hölder continuous function f with a Hölder exponent β.*

By using an elementary algebraic estimate and a interior positivity, boundedness and a Hölder regularity of v and its gradient ∇v in Theorems 5.6 and 5.7, we also obtain a local Hölder regularity of the weak solution u to (2.4) and its gradient ∇u, which gives our final assertion in Theorem 1.1.

Theorem 5.8 (Hölder regularity for the p-Sobolev flow [21, Theorem 5.7]) *Let u be a positive and bounded weak solution to* (2.4). *Then, there exist a positive exponent $\gamma < 1$ depending only on n, p, β, α and a positive constant C depending only on $n, p, \tilde{c}, M, \lambda(0), \beta, \alpha, \|\nabla u\|_{L^p(\Omega'_T)}, [g]_{\beta,\Omega'_T}$ and $[v]_{\beta,\Omega'_T}$ such that, both u and ∇u are locally Hölder continuous in Ω'_T with an exponent γ on a parabolic metric $|x| + |t|^{1/p}$ and on the parabolic one, respectively. The Hölder constants are bounded above by C, where $[f]_\beta$ denote the Hölder semi-norm of a Hölder continuous function f with a Hölder exponent β.*

From Steps 1 to 3, the proof of Theorem 1.1 is concluded.

$\square$

Appendix A Refined Expansion of Positivity

This section is devoted to the refinement of the expansion of positivity which is proven in [21, Section 4]. Firstly, we give the transformation stretching the time-interval, which is needed for the proof of Theorem 5.2 and Proposition 5.3.

In this section, following [21, Sections 3 and 4], we consider the *doubly nonlinear equations of p-Sobolev flow type*:

$$\begin{cases} \partial_t u^q - \Delta_p u = c u^q & \text{in } \Omega_T \\ 0 \le u \le M & \text{on } \partial_p \Omega_T, \end{cases} \tag{pST}$$

where $T \in (0, \infty)$, $u = u(x,t) : \Omega_T \longrightarrow \mathbb{R}$ be a nonnegative real valued function, and c and M are nonnegative constant and positive one, respectively. Here the initial value u_0 is in the Sobolev space $W_0^{1,p}(\Omega)$, positive and bounded in Ω.

As mentioned in [21, Remark 3.3], a nonnegative weak solution of p-Sobolev flow (2.4) is a weak supersolution of (pST) with $c = 0$.

Here we recall the fundamental positivity results, proved in [21, Section 4], which are referred later.

Proposition A.1 ([21, Proposition 4.1]) *Let u be a nonnegative weak supersolution of (pST). Let $B_\rho(x_0) \subset \Omega$ with center $x_0 \in \Omega$ and radius $\rho > 0$, and $t_0 \in (0,T]$. Suppose that*

$$\left|B_\rho(x_0) \cap \{u(t_0) \geq L\}\right| \geq \alpha |B_\rho| \tag{A.1}$$

holds for some $L > 0$ and $\alpha \in (0,1]$. Then there exist positive numbers δ_0, $\varepsilon_0 \in (0,1)$ depending only on p, n and α and independent of L such that

$$\left|B_\rho(x_0) \cap \{u(t) \geq \varepsilon L\}\right| \geq \frac{\alpha}{2} |B_\rho| \tag{A.2}$$

holds for any positive $\delta \leq \delta_0$, any positive $\varepsilon \leq \varepsilon_0$ and all $t \in [t_0,\, t_0 + \delta L^{q+1-p}\rho^p]$. Here, if t_0 is very close to T, then $\delta > 0$ is chosen so small that $\delta L^{q+1-p}\rho^p = T - t_0$.

Proof. This proposition is proved by combination of the De Giorgi iteration method and the following Caccioppoli type estimate. See [21, Proposition 4.1] for detailed proof. ☐

Proposition A.2 ([21, Proposition 3.8]) *Let $k \geq 0$. Let u be a nonnegative weak supersolution of (pST). Let K be a subset compactly contained in Ω, and $0 < t_1 < t_2 \leq T$. Here we use the notation $K_{t_1,t_2} = K \times (t_1, t_2)$. Let ζ be a Lipschitz function such that $\zeta = 0$ outside K_{t_1,t_2}. Then, there exists a positive constant C depending only on p, n such that*

$$\begin{aligned}
&\operatorname*{ess\,sup}_{t_1<t<t_2} \int_{K\times\{t\}} (k-u)_+^{q+1}\zeta^p\,dx + \int_{K_{t_1,t_2}} |\nabla(k-u)_+\zeta|^p dxdt \\
&\quad \leq C \int_{K\times\{t_1\}} k^{q-1}(k-u)_+^2\zeta^p\,dx + C\int_{K_{t_1,t_2}} (k-u)_+^p|\nabla\zeta|^p\,dxdt \\
&\qquad + C\int_{K_{t_1,t_2}} k^{q-1}(k-u)_+^2|\zeta_t|\,dxdt.
\end{aligned} \tag{A.3}$$

We further recall the following crucial lemma.

Lemma A.3 ([21, Lemma 4.2]) *Let u be a nonnegative weak supersolution of (pST). Suppose further (A.1). Let $Q_{4\rho}(z_0) := B_{4\rho}(x_0) \times (t_0, t_0 + \delta L^{q+1-p}\rho^p) \subset \Omega_T$, where δ is selected in Proposition A.1. Then for any $\nu \in (0,1)$ there exists a positive number ε_ν depending only on $p, n, \alpha, \delta, \nu$ such that*

$$\left|Q_{4\rho}(z_0) \cap \{u < \varepsilon_\nu L\}\right| < \nu \left|Q_{4\rho}\right|.$$

Proof. This lemma is also shown by the above Caccioppoli type estimate and De Giorgi's inequality. See [21, Lemma 4.2] for detailed proof. □

As a corollary of [21, Theorem 4.4], if a solution is positive at some time t_0, its positivity expands in space-time without "waiting time". This corollary is used in the proof of Proposition 5.3. See Appendix A.2.

Proposition A.4 ([21, Corollary 4.6]) *Let u be a nonnegative weak supersolution of (pST). Assume that $u(t_0) > 0$ in $B_{4\rho}(x_0) \subset \Omega$. Then there exist positive numbers η_0 and τ_0 such that*

$$u \geq \eta_0 \quad a.e. \quad in \quad B_{2\rho}(x_0) \times (t_0, t_0 + \tau_0).$$

Proof. The proof is done by combination of letting $L = \inf_{B_{4\rho}(x_0)} u(t_0)$ in Proposition A.1 and De Giorgi's iteration method. See [21, Corollary 4.6] for detailed proof. □

A.1 Expansion of Positivity via Transformation Stretching the Time-interval

We study the expansion of interior positivity under a changing of variables stretching the time-interval. We choose $\rho > 0$ such that

$$Q_{16\rho}(z_0) := B_{16\rho}(x_0) \times (t_0, t_0 + \delta L^{q+1-p}\rho^p) \subset \Omega_T,$$

where the a positive $\delta \leq \delta_0$ is selected in Proposition A.1. By translation, we may assume $z_0 = (x_0, t_0) = (0, 0)$. Following [8, Section 5.1, pp.73–78], we consider the following changing of variables stretching the time-interval:

$$y = \frac{x}{\rho}, \quad -e^{-\tau} = \frac{t - \delta L^{q+1-p}\rho^p}{\delta L^{q+1-p}\rho^p}. \tag{A.4}$$

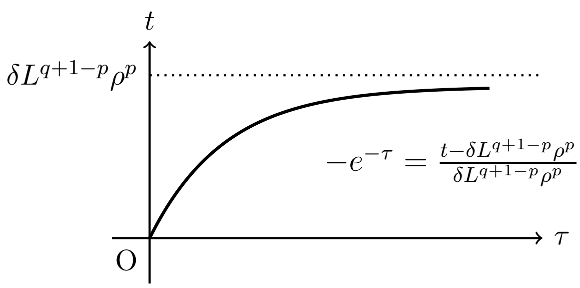

Figure A.1: Stretching the time-interval

Note that this transformation $(x, t) \mapsto (y, \tau)$ maps $Q_{16\rho}(0)$ to $B_{16} \times (0, \infty)$. For a nonnegative weak solution u to (pST) we set

$$v(y, \tau) := \frac{1}{L} u(x, t) e^{\frac{\tau}{q+1-p}} \tag{A.5}$$

and thus, by simple calculation, we find that v satisfies the following equation in the weak sense:

$$\partial_\tau v^q = \delta\Delta_p v + \left(c\delta L^{q+1-p}\rho^p e^{-\tau} + \frac{q}{q+1-p}\right) v^q. \tag{A.6}$$

Using (A.5), we write the conclusion (A.2) in Proposition A.1 as

$$\left|B_1 \cap \{v(\tau) \geq \varepsilon e^{\frac{\tau}{q+1-p}}\}\right| \geq \frac{1}{2}\alpha|B_1| \tag{A.7}$$

for every $\tau \in (0,\infty)$. Letting

$$k_0 := \varepsilon e^{\frac{\tau_0}{q+1-p}}, \qquad k_j := \frac{k_0}{2^j}, \quad j = 0,1,\ldots$$

with the parameter $\tau_0 > 0$ determined later, we obtain from (A.7) that

$$\left|B_8 \cap \{v(\tau_0) \geq k_j\}\right| \geq \frac{\alpha}{2}8^{-n}|B_8|. \tag{A.8}$$

We define the following two space-time cylinders:

$$\begin{aligned} Q &:= B_8 \times (\tau_0 + k_0^{q+1-p}, \tau_0 + 2k_0^{q+1-p}), \\ Q' &:= B_{16} \times (\tau_0, \tau_0 + 2k_0^{q+1-p}). \end{aligned}$$

From (A.6) and similar calculation as [21, Proposition 3.8], we obtain the *Caccioppoli type inequality* of v

$$\begin{aligned} &\operatorname*{ess\,sup}_{\tau_1<\tau<\tau_2} \int_{K\times\{\tau\}} (k-v)_+^{q+1}\zeta^p\,dy + \int_{K_{\tau_1,\tau_2}} |\nabla(k-v)_+\zeta|^p dy d\tau \\ &\quad\leq \frac{C}{\delta}\int_{K\times\{\tau_1\}} k^{q-1}(k-v)_+^2\zeta^p\,dx + C\int_{K_{\tau_1,\tau_2}} (k-v)_+^p|\nabla\zeta|^p\,dy d\tau \\ &\qquad + \frac{C}{\delta}\int_{K_{\tau_1,\tau_2}} k^{q-1}(k-v)_+^2|\zeta_\tau|\,dy d\tau, \end{aligned} \tag{A.9}$$

where $k \geq 0$, $K_{\tau_1,t_2} = K \times (\tau_1,\tau_2)$ for a compact set $K \subset B_{16}$ and $\tau_2 > \tau_1 > 0$, and ζ is a smooth function such that $0 \leq \zeta \leq 1$ and $\zeta = 0$ outside K_{τ_1,τ_2}. By the Caccioppoli type inequality (A.9) and the very similar argument as the proof of [21, Lemma 4.2], for every $\nu > 0$, there exists a natural number $J \geq \left(\frac{C(n,p)}{\nu\alpha\delta^{\frac{1}{p}}}\right)^{\frac{p}{p-1}}$ such that

$$\left|Q \cap \{v < k_J\}\right| < \nu|Q|. \tag{A.10}$$

As mentioned in [21, Remark 4.3] , we can choose ε such that

$$\varepsilon = \left(\frac{\delta}{2^{I_1}}\right)^{\frac{1}{q+1-p}} \tag{A.11}$$

for some large natural number I_1 depending only on n, p and α. We also choose k_j as

$$k_j = \left(\frac{\delta e^{\tau_0}}{2^{I_1+j}}\right)^{\frac{1}{q+1-p}} \quad \text{for} \quad j = 0, 1, \ldots, J. \tag{A.12}$$

Under such choice as above we note that $k_0^{q+1-p}/k_J^{q+1-p} = 2^J$ is a positive integer. We divide Q along time direction into parabolic cylinders of number 2^J with each time-length k_J^{q+1-p}, and set

$$Q^{(\ell)} := B_8 \times \left(\tau_0 + k_0^{q+1-p} + \ell k_J^{q+1-p},\ \tau_0 + k_0^{q+1-p} + (\ell+1)k_J^{q+1-p}\right)$$

for $\ell = 0, 1, \ldots, 2^J - 1$. By (A.10) there is a $Q^{(\ell)}$ such that

$$\left|Q^{(\ell)} \cap \{v < k_J\}\right| < \nu |Q^{(\ell)}|. \tag{A.13}$$

As a result, we obtain the expansion of interior positivity of time-stretched solution v

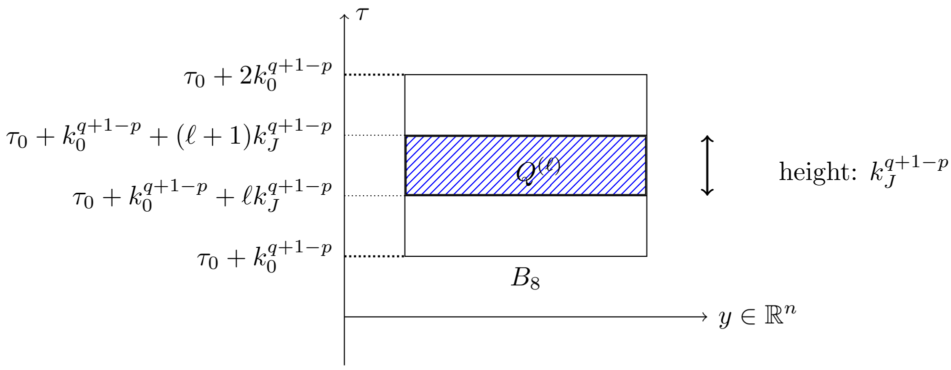

Figure A.2: Image of $Q^{(\ell)}$

as follows.

Proposition A.5 Let v be a nonnegative weak solution of (A.6) in $B_{16} \times (0, \infty)$ defined by (A.5). Then

$$v \geq \frac{k_J}{2} \quad \text{a.e. in } B_4 \times \left(\tau_0 + k_0^{q+1-p} + \ell k_J^{q+1-p},\ \tau_0 + k_0^{q+1-p} + (\ell+1)k_J^{q+1-p}\right).$$

Proof. The proof is done by the same argument as [21, Theorem 4.4]. □

Under preliminaries as above, we are in position to state the main theorem in this subsection.

Theorem A.6 (Expansion of interior positivity up to end time) *Let u be a nonnegative weak solution to (pST). Let $B_{16\rho}(x_0) \times (t_0, t_0 + \delta L^{q+1-p}\rho^p) \subset \Omega_T$ with $x_0 \in \Omega$, $\rho > 0$, and $t_0 \in (0, T]$. Suppose (A.1). Then there exist positive numbers $\eta < 1$ and $\sigma < 1$ depending only on n, p, α and independent of L such that*

$$u \geq \eta L \quad \textit{a.e. in } B_{2\rho}(x_0) \times \left(t_0 + (1-\sigma)\delta L^{q+1-p}\rho^p, t_0 + \delta L^{q+1-p}\rho^p\right), \tag{A.14}$$

where the $\delta = \delta(n, p, \alpha) > 0$ is selected in Proposition A.1.

Proof. The assertion is verified by Proposition A.5, the scaling back $(x,t) \leftrightarrow (y,\tau)$ and De Giorgi's iteration method as in [21, Theorem 4.4]. Again, by translation, we may consider $(x_0,t_0) = (0,0)$. It follows from Proposition A.5 and (A.12) that for almost every time τ_1, $\tau_0 + k_0^{q+1-p} + \ell k_J^{q+1-p} < \tau_1 < \tau_0 + k_0^{q+1-p} + (\ell+1)k_J^{q+1-p}$

$$v(y,\tau_1) \geq \frac{1}{2}\left(\frac{\delta e^{\tau_0}}{2^{I_1+J}}\right)^{\frac{1}{q+1-p}} \quad \text{a.e. in } B_4,$$

which, letting t_1 by (A.4) with τ_1 as

$$-e^{-\tau_1} = \frac{t_1 - \delta L^{q+1-p}\rho^p}{\delta L^{q+1-p}\rho^p}, \tag{A.15}$$

leads to

$$u(x,t_1) \geq \frac{1}{2}\left(\frac{\delta e^{-(\tau_1-\tau_0)}}{2^{I_1+J}}\right)^{\frac{1}{q+1-p}} L \quad \text{a.e. in } B_{4\rho}. \tag{A.16}$$

For brevity, we set

$$L_0 := \frac{1}{2}\left(\frac{\delta e^{-(\tau_1-\tau_0)}}{2^{I_1+J}}\right)^{\frac{1}{q+1-p}} L. \tag{A.17}$$

Since it follows from (A.16) that $\big|B_{4\rho} \cap \{u(t_1) \geq L_0\big| = |B_{4\rho}|$, by Proposition A.1, there exist positive numbers $\tilde{\delta} < 1$ and $\tilde{\varepsilon} < 1$ depending only on n and p and independent of L_0 such that

$$\big|B_{4\rho} \cap \{u(t) \geq \tilde{\varepsilon} L_0\}\big| \geq \frac{1}{2}\big|B_{4\rho}\big| \tag{A.18}$$

holds for every $t \in [t_1, t_1 + \tilde{\delta} L_0^{q+1-p}\rho^p]$. For a positive $\theta \leq \tilde{\delta} L_0^{q+1-p}$ let $Q_{4\rho}^\theta := B_{4\rho} \times (t_1, t_1 + \theta\rho^p)$. By Lemma A.3, for every $\tilde{\nu} \in (0,1)$ there exists a positive $\tilde{\varepsilon}_{\tilde{\nu}}$ such that

$$\big|Q_{4\rho}^\theta \cap \{u < \tilde{\varepsilon}_{\tilde{\nu}} L_0\big| < \tilde{\nu}\big|Q_{4\rho}^\theta\big|. \tag{A.19}$$

As in the proof of [21, Theorem 4.4], let

$$\begin{aligned} &\rho_m := \left(2 + \frac{1}{2^{m-1}}\right)\rho, \qquad Q_m := B_{\rho_m} \times (t_1, t_1 + \theta\rho^p), \\ &\theta := \tilde{\delta} L_0^{q+1-p}, \qquad \kappa_m := \left(\frac{1}{2} + \frac{1}{2^{m+1}}\right)\tilde{k}_{I_2}. \end{aligned}$$

where $\tilde{k}_{I_2} := \dfrac{\tilde{\varepsilon}_{\tilde{\nu}} L_0}{2^{\frac{I_2}{q+1-p}}}$ with $\tilde{\varepsilon}_{\tilde{\nu}}$ in (A.19) and a natural number I_2 satisfying $I_2 \geq \left(\frac{C(n,p)}{\tilde{\nu}\tilde{\delta}^{\frac{1}{p}}}\right)^{\frac{p}{p-1}}$. It then plainly holds that

$$\begin{cases} 4\rho = \rho_0 \geq \rho_m \searrow \rho_\infty = 2\rho \\ Q_{4\rho}^\theta = Q_0 \supset Q_m \searrow Q_\infty = B_{2\rho} \times (t_1, t_1 + \theta\rho^p) \\ \tilde{k}_{I_2} = \kappa_0 \geq \kappa_m \searrow \kappa_\infty = \tilde{k}_{I_2}/2. \end{cases}$$

Letting $Y_m := \fint_{Q_m} \chi_{\{(\kappa_m - u)_+ > 0\}}\,dz$ and using the Caccioppoli type inequality (A.3) with the cut-off function $\zeta = \zeta(x)$ in Q_m as the proof of [21, Corollary 4.6], we obtain

$$Y_{m+1} \le C b^m Y_m^{1+\frac{p}{n}}, \quad m = 0, 1, \dots,$$

where $b := 2^{p(1+\frac{p}{n})+p\frac{n+q+1}{n}} > 1$. By the fast geometric convergence lemma (see [21, Lemma 2.3] and also [7, Lemma 4.1, p12]), if

$$Y_0 \le C^{-\frac{n}{p}} b^{-(\frac{n}{p})^2} =: \tilde{\nu}_0, \tag{A.20}$$

then

$$Y_m \longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty. \tag{A.21}$$

Eq. (A.20) follows from (A.19) with $\tilde{\nu} = \tilde{\nu}_0$ and thus, (A.21) gives that

$$u \ge \frac{\tilde{k}_{I_2}}{2} \quad \text{in } B_{2\rho} \times (t_1, t_1 + \theta\rho^p),$$

that is,

$$u \ge \frac{\tilde{\varepsilon}}{4} \left(\frac{\delta e^{-(\tau_1 - \tau_0)}}{2^{I+J}} \right)^{\frac{1}{q+1-p}} L \quad \text{in } B_{2\rho} \times (t_1, t_1 + \tilde{\delta} L_0^{q+1-p} \rho^p), \tag{A.22}$$

where we put $I := I_1 + I_2 \in \mathbb{N}$, which depends only on $n, p, \alpha, \varepsilon$ and δ and independent of L and $\tilde{\varepsilon} = \tilde{\varepsilon}_{\tilde{\nu}_0}$. Here we note that

$$\tilde{\delta} L_0^{q+1-p} = \tilde{\delta} \frac{\delta e^{-(\tau_1 - \tau_0)}}{2^{I_1+J}} \left(\frac{L}{2} \right)^{q+1-p}.$$

Thus, the inequality (A.22) is written as

$$u(x,t) \ge \frac{\tilde{\varepsilon}}{4} \left(\frac{\delta e^{-(\tau_1 - \tau_0)}}{2^{I+J}} \right)^{\frac{1}{q+1-p}} L \quad \text{in } B_{2\rho} \tag{A.23}$$

for all times $t_1 < t < t_1 + \tilde{\delta} \frac{\delta e^{-(\tau_1-\tau_0)}}{2^{I_1+J}} \left(\frac{L}{2}\right)^{q+1-p} \rho^p$. The transformed τ_0 is still free of choice, and it will be selected as follows. By the change of variables (A.4) with (A.15), τ_0 is chosen as

$$\begin{aligned} & \delta L^{q+1-p} \rho^p - t_1 = \tilde{\delta} \frac{\delta e^{-(\tau_1 - \tau_0)}}{2^{I_1+J}} \left(\frac{L}{2} \right)^{q+1-p} \rho^p \\ \iff & \tau_0 = \log \left(\frac{2^{I_1+J+q+1-p}}{\tilde{\delta}} \right). \end{aligned} \tag{A.24}$$

This τ_0 depends only on $n, p, \varepsilon, \delta$ and α because $\tilde{\delta}$ and $\tilde{\varepsilon}$ depend only on n, p. Therefore, (A.23) holds for all times

$$t_1 = \delta L^{q+1-p} \rho^p - \tilde{\delta} \frac{\delta e^{-(\tau_1 - \tau_0)}}{2^{I_1+J}} \left(\frac{L}{2} \right)^{q+1-p} \rho^p < t \le \delta L^{q+1-p} \rho^p. \tag{A.25}$$

Lastly, we will estimate the above "left edge time" t_1. Since

$$\begin{aligned} \tau_1 \le \tau_0 + 2k_0^{q+1-p} &\iff e^{-\tau_1} \ge e^{-(\tau_0+2k_0^{q+1-p})}, \\ k_0^{q+1-p} = \varepsilon^{q+1-p}e^{\tau_0} \le e^{\tau_0} &\iff e^{-2k_0^{q+1-p}} \ge e^{-2e^{\tau_0}}, \end{aligned} \tag{A.26}$$

it follows from (A.24) and (A.26) that

$$\begin{aligned} t_1 &= \delta L^{q+1-p}\rho^p - \tilde{\delta}\frac{\delta e^{-(\tau_1-\tau_0)}}{2^{I_1+J}}\left(\frac{L}{2}\right)^{q+1-p}\rho^p \\ &= (1-e^{-\tau_1})\delta L^{q+1-p}\rho^p \\ &\le \left(1-e^{-(\tau_0+2k_0^{q+1-p})}\right)\delta L^{q+1-p}\rho^p \\ &\le \left(1-e^{-(\tau_0+2e^{\tau_0})}\right)\delta L^{q+1-p}\rho^p. \end{aligned}$$

Here we set $\sigma := e^{-(\tau_0+2e^{\tau_0})} \in (0,1)$. This together with (A.23) implies that

$$u \ge \frac{\tilde{\varepsilon}}{4}\left(\frac{\delta e^{-2\tau_0}}{2^{I+J}}\right)^{\frac{1}{q+1-p}} L \quad \text{in } B_{2\rho}\times((1-\sigma)\delta L^{q+1-p},\, \delta L^{q+1-p}\rho^p)$$

and thus, letting $\eta := \frac{\tilde{\varepsilon}}{4}\left(\frac{\delta e^{-2\tau_0}}{2^{I+J}}\right)^{\frac{1}{q+1-p}}$, we complete the proof. $\square$

A.2 Proof of Theorem 5.2 and Proposition 5.3

This subsection is devoted to the detailed proof of Theorem 5.2 and Proposition 5.3.

We shall prove Theorem 5.2 by using Theorem A.6 and a method of chain of finitely many balls as used in Harnack's inequality for harmonic functions, which is so-called *Harnack chain* (see [9, Theorem 11, pp.32–33] and [4, 17] in the p-parabolic setting). Here we use the special choice of parameters, as explained in Theorem A.6 (see also [21, Theorem 4.4]).

Proof of Theorem 5.2. We follow a similar argument as [21, Section 4.3].

We will prove the assertion in four steps.

Step 1: Since $\overline{\Omega'}$ is compact, it is covered by finitely many balls $\{B_\rho(x_j)\}_{j=1}^N$ ($x_j \in \Omega'$, $j = 1, 2, \ldots, N$) with $N = N(\Omega', \rho)$, such that

$$\Omega' \subset \bigcup_{j=1}^N B_\rho(x_j), \quad \rho < |x_i - x_{i+1}| < 2\rho, \;\; B_{16\rho}(x_i) \subset \Omega, \;\text{ for all } 1 \le i \le N,$$

where we put $x_{N+1} = x_1$. For brevity we denote $B_\rho(x_j)$ by B_j for each $j = 1, 2, \ldots, N$.

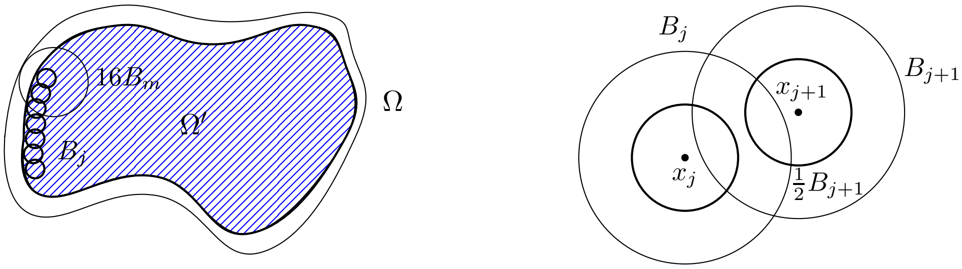

Figure A.3: Harnack's chain argument

By (5.2), there exists at least one $B_j = B_\rho(x_j)$, denoted by $x_1 = x_j$ and $B_1 = B_j$, such that

$$|B_1 \cap \{u(t_0) \geq L\}| \geq \frac{\alpha}{2^n}|B_1|.$$

Thus, by Theorem A.6, there exists positive numbers $\delta_0, \varepsilon_0, \sigma_0, \eta_1 \in (0,1)$ depending only on p, n and $\alpha_0 = \alpha$ and independent of L such that

$$u \geq \eta_1 L \quad \text{a.e. in} \quad B_1 \times \left(t_0 + (1-\sigma_0)\delta_0 L^{q+1-p}\rho^p,\ t_0 + \delta_0 L^{q+1-p}\rho^p\right), \tag{A.27}$$

where $\sigma_0 := e^{-(\tau_0 + 2e^{\tau_0})}$, $e^{\tau_0} = C(n,p)2^{I_0+J_0}$ for some $I_0,\ J_0 \in \mathbb{N}$ depending only on n, p and α_0, and $\eta_1 = \dfrac{\tilde{\varepsilon}_0}{4}\left(\dfrac{\delta_0 e^{-2\tau_0}}{2^{I_0+J_0}}\right)^{\frac{1}{q+1-p}}$ for some $\tilde{\varepsilon}_0 = \tilde{\varepsilon}_0(n,p) \in (0,1)$. We put $\eta_0 = 1$ for later reference.

Step 2: By $\rho < |x_1 - x_2| < 2\rho$,

$$D_1 := B_1 \cap B_2 \neq \varnothing.$$

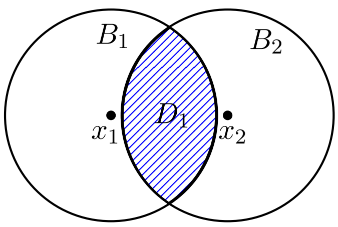

Figure A.4: Intersection of two balls

Via (A.27), we have

$$u \geq \eta_1 L \quad \text{a.e. } D_1 \times \mathcal{I}_0, \tag{A.28}$$

where let $\mathcal{I}_0 := \left(t_0 + (1-\sigma_0)\delta_0 L^{q+1-p}\rho^p,\ t_0 + \delta_0 L^{q+1-p}\rho^p\right)$. By (A.28), for any $t_1 \in \mathcal{I}_0$,

$$|D_1 \cap \{u(t_1) \geq \eta_1 L\}| = |D_1|,$$

which is, setting $\alpha_1 := \dfrac{|D_1|}{|B_2|} \in (0,1)$,

$$\big|B_2 \cap \{u(t_1) \geq \eta_1 L\}\big| \geq \alpha_1 \big|B_2\big|.$$

By the very same argument as Step 1, there exist positive numbers $\delta_1, \sigma_1, \eta_2 \in (0,1)$ depending only on p, n and α_1 and independent of L such that

$$u \geq \eta_2 L \quad \text{a.e. in} \quad B_2 \times \big(t_1 + (1-\sigma_1)\delta_1(\eta_1 L)^{q+1-p}\rho^p,\ t_1 + \delta_1(\eta_1 L)^{q+1-p}\rho^p\big), \tag{A.29}$$

where $\sigma_1 := e^{-(\tau_1 + 2e^{\tau_1})}$, $e^{\tau_1} = C(n,p)2^{I_1+J_1}$ for some large $I_1,\ J_1 \in \mathbb{N}$ depending only on n, p and α_1, and $\eta_2 = \dfrac{\tilde{\varepsilon}_1}{4}\left(\dfrac{\delta_1 e^{-2\tau_1}}{2^{I_1+J_1}}\right)^{\frac{1}{q+1-p}}\eta_1$ for some $\tilde{\varepsilon}_1 = \tilde{\varepsilon}_1(n,p) \in (0,1)$. Here we choose $t_1 \in \mathcal{I}_0$ as

$$\begin{aligned} &t_1 := t_0 + (\delta_0 - \delta_1\eta_1^{q+1-p})L^{q+1-p}\rho^p > 0 \\ \iff\ &t_1 + \delta_1(\eta_1 L)^{q+1-p}\rho^p = t_0 + \delta_0 L^{q+1-p}\rho^p. \end{aligned}$$

Note that this choice of $t_1 \in \mathcal{I}_0$ is admissible by

$$\begin{aligned} &t_1 - \big(t_0 + (1-\sigma_0)\delta_0 L^{q+1-p}\rho^p\big) \\ &= \left(\sigma_0 - \delta_1\left(\frac{\tilde{\varepsilon}_0}{4}\right)^{q+1-p}\frac{e^{-2\tau_0}}{2^{I_0+J_0}}\right)\delta_0 L^{q+1-p}\rho^p > 0 \end{aligned}$$

and δ_1 can be chosen as small.

Step 3: We will proceed by induction on m. Suppose that for some $m \in \{1, 2. \ldots, N\}$

$$u \geq \eta_m L \quad \text{a.e. in } B_m \times \mathcal{I}_{m-1}. \tag{A.30}$$

Here let

$$\mathcal{I}_{m-1} := \big(t_{m-1} + (1-\sigma_{m-1})\delta_{m-1}(\eta_{m-1}L)^{q+1-p}\rho^p,\ t_{m-1} + \delta_{m-1}(\eta_{m-1}L)^{q+1-p}\rho^p\big)$$

with $t_{m-1} := t_0 + (\delta_0 - \delta_{m-1}\eta_{m-1}^{q+1-p})L^{q+1-p}\rho^p$, where $\delta_{m-1}, \sigma_{m-1} \in (0,1), \tau_{m-1} > 0$ are determined inductively as follows: $\sigma_{m-1} := e^{-(\tau_{m-1}+2e^{\tau_{m-1}})}$, $e^{\tau_{m-1}} = C(n,p)2^{I_{m-1}+J_{m-1}}$ for some large $I_{m-1},\ J_{m-1} \in \mathbb{N}$ depending only on n, p and $\alpha_{m-1} = \frac{|D_{m-1}|}{|B_m|} = \frac{|D_1|}{|B_2|} = \alpha_1$ as before, and $\eta_m = \dfrac{\tilde{\varepsilon}_{m-1}}{4}\left(\dfrac{\delta_{m-1}e^{-2\tau_{m-1}}}{2^{I_{m-1}+J_{m-1}}}\right)^{\frac{1}{q+1-p}}\eta_{m-1}$ for some $\tilde{\varepsilon}_{m-1} = \tilde{\varepsilon}_{m-1}(n,p) \in (0,1)$.

By $\rho < |x_m - x_{m+1}| < 2\rho$ again,

$$D_m := B_m \cap B_{m+1} \neq \varnothing$$

and thus, (A.30) yields that

$$u \geq \eta_m L \quad \text{a.e. in } D_m \times \mathcal{I}_{m-1}. \tag{A.31}$$

By (A.31), for any $t_m \in \mathcal{I}_{m-1}$,

$$\big|B_{m+1} \cap \{u(t_m) \geq \eta_m L\}\big| \geq \alpha_m \big|B_{m+1}\big|,$$

where let $\alpha_m := \dfrac{|D_m|}{|B_{m+1}|} = \dfrac{|D_1|}{|B_2|} = \alpha_1 \in (0,1)$. Again, similarly as in Steps 1 and 2, there exist positive numbers $\delta_m, \sigma_m, \eta_{m+1} \in (0,1)$ depending only on p, n and α_1 and independent of L such that

$$u \geq \eta_{m+1} L \quad \text{a.e. in} \quad B_{m+1} \times \big(t_m + (1-\sigma_m)(\delta_m L)^{q+1-p}\rho^p,\, t_m + (\delta_m L)^{q+1-p}\rho^p\big), \tag{A.32}$$

where $\sigma_m := e^{-(\tau_m + 2e^{\tau_m})}$, $e^{\tau_m} = C(n,p)2^{I_m+J_m}$ for some large I_m, $J_m \in \mathbb{N}$ depending only on n, p and α_1, and $\eta_{m+1} = \dfrac{\tilde{\varepsilon}_m}{4}\left(\dfrac{\delta_m e^{-2\tau_m}}{2^{I_m+J_m}}\right)^{\frac{1}{q+1-p}} \eta_m$ for some $\tilde{\varepsilon}_m = \tilde{\varepsilon}_m(n,p) \in (0,1)$. Again, we choose $t_m \in \mathcal{I}_{m-1}$ as

$$\begin{aligned} &t_m := t_0 + (\delta_0 - \delta_m \eta_m^{q+1-p})L^{q+1-p}\rho^p > 0 \\ &\iff t_m + \delta_m(\eta_m L)^{q+1-p}\rho^p = t_0 + \delta_0 L^{q+1-p}\rho^p, \end{aligned}$$

because

$$\begin{aligned} &t_m - \big(t_{m-1} + (1-\sigma_{m-1})\delta_{m-1}(\eta_{m-1}L)^{q+1-p}\rho^p\big) \\ &= \left(\sigma_{m-1} - \delta_m \left(\frac{\tilde{\varepsilon}_{m-1}}{4}\right)^{q+1-p} \frac{e^{-2\tau_{m-1}}}{2^{I_{m-1}+J_{m-1}}}\right) \delta_{m-1}(\eta_{m-1}L)^{q+1-p}\rho^p > 0 \end{aligned}$$

and δ_m can be taken to be small enough. Thus our induction on m is done.

Step 4: By Step 3, we have, for all $m = 1, 2, \ldots, N$,

$$u \geq \eta_{m+1} L \quad \text{a.e. in } B_{m+1} \times \mathcal{I}_m, \tag{A.33}$$

where let $B_{N+1} := B_1$. Since, by construction, $\{\eta_m\}_{m=1}^{N+1}$ is decreasing, it follows from (A.33) that, for all $m = 1, 2, \ldots, N$,

$$u \geq \eta_{N+1} L \quad \text{a.e. in } B_{m+1} \times \mathcal{I}_N,$$

where we set

$$\mathcal{I}_N := \big(t_N + (1-\sigma_N)(\delta_N L)^{q+1-p}\rho^p,\, t_N + (\delta_N L)^{q+1-p}\rho^p\big)$$

with $t_N = t_0 + (\delta_0 - \delta_N \eta_N^{q+1-p})L^{q+1-p}\rho^p$. Therefore we complete the proof. □

Lastly, we will prove Proposition 5.3.

Proof of Proposition 5.3. Since $\overline{\Omega'}$ is compact, it is covered by finitely many balls $\{B_\rho(x_j)\}_{j=1}^N$ ($x_j \in \Omega'$, $j = 1, 2, \ldots, N$), where $N = N(\Omega', \rho)$, such that

$$\Omega' \subset \bigcup_{j=1}^{N} B_\rho(x_j), \quad \rho < |x_i - x_{i+1}| < 2\rho, \;\; B_{16\rho}(x_i) \subset \Omega, \;\text{for all } 1 \le i \le N,$$

where we put $x_{N+1} = x_1$. For brevity we denote $B_\rho(x_j)$ by B_j for each $j = 1, 2, \ldots, N$ and let $4B_j := B_{4\rho}(x_j)$. By assumption, $u(t_0) > 0$ in each ball $4B_j$, $j = 1, \ldots, N$. Let $L_1 = \inf_{4B_1} u(t_0)$. By Corollary A.4 with $L = L_1$ (see the proof of [21, Corollary 4.6]) there exist positive numbers η_1 and τ_1 depending on n and p such that

$$u \ge \eta_1 L_1 \quad \text{a.e. in } B_1 \times (t_0, t_0 + \tau_1 L_1^{q+1-p} \rho^p).$$

Letting $L_2 := \inf_{4B_2} u(t_0)$ and applying Corollary A.4 with $L = L_2$, there exists positive numbers η_2 and τ_2 depending on n and p such that

$$u \ge \eta_2 L_2 \quad \text{a.e. in } B_2 \times (t_0, t_0 + \tau_2 L_2^{q+1-p} \rho^p).$$

Repeating this argument finitely, there exist positive numbers η_N and τ_N depending on n and p such that, letting $L_N = \inf_{4B_N} u(t_0)$,

$$u \ge \eta_N L_N \quad \text{a.e. in } B_j \times (t_0, t_0 + \tau_N L_N^{q+1-p} \rho^p).$$

for all $j = 1, \ldots, N$.

Finally, we define the subdomain of Ω as $\Omega'' := \{x \in \Omega : \operatorname{dist}(x, \Omega') < 4\rho\}$, and put $L := \inf_{\Omega''} u(t_0) > 0$. Then $L \le L_i$ for $i = 1, \ldots, N$. Thus, putting $\eta_0 := \min\limits_{i=1,\ldots,N} \eta_i L$ and $\tau_0 := \min\limits_{i=1,\ldots,N} \tau_i L^{q+1-p} \rho^p$, we complete the proof.

□

A.3 Proof of Proposition 5.4

We will prove Proposition 5.4 here.

Proof of Proposition 5.4. Following [21, Proposition 5.4] with a minor change, we give the proof.

Note that a nonnegative weak solution u of (2.4) is a weak supersolution to (pST) with $c = 0$. By the volume constraint together with the boundedness, letting $M := e^{\frac{1}{q}\int_0^T \|\nabla u(t)\|_p^p \, dt} \|u_0\|_\infty$, we have, for any $t \in [0, T]$

$$\begin{aligned} 1 = \int_\Omega u^{q+1}(t)\, dx &= \int_{\Omega' \cap \{u(t) \ge L\}} u^{q+1}(t)\, dx + \int_{\Omega' \cap \{u(t) < L\}} u^{q+1}(t)\, dx + \int_{\Omega \setminus \Omega'} u^{q+1}(t)\, dx \\ &\le M^{q+1} \big|\Omega' \cap \{u(t) \ge L\}\big| + L^{q+1} |\Omega'| + M^{q+1} |\Omega \setminus \Omega'|\, ; \end{aligned}$$

i.e.,

$$\frac{1-L^{q+1}|\Omega'|-M^{q+1}|\Omega\setminus\Omega'|}{M^{q+1}} \le \big|\Omega'\cap\{u(t)\ge L\}\big|.$$

Under the choice of Ω' and L in Proposition 5.4, we find that, for any $t\in[0,T]$,

$$\alpha|\Omega'| \le \big|\Omega'\cap\{u(t)\ge L\}\big|, \tag{A.34}$$

where $\alpha := \dfrac{1}{2M^{q+1}|\Omega'|}$. Let $\rho>0$ be arbitrarily taken and fixed, satisfying $\rho\le \mathrm{dist}(\Omega',\partial\Omega)/16$. We choose subdomain Ω'' as $\Omega''=\{x\in\Omega:\mathrm{dist}(x,\Omega')<4\rho\}$. By Theorem 5.2, there are positive integer $N=N(\Omega',\rho)$, positive real number families $\delta_0,\delta_N,\ \eta_N,\eta_{N+1},\sigma_N\in(0,1),\ J_N,\ I_N\in\mathbb{N}$ depending on α,N,n,p and independent of L and a time $t_N>t_0$ such that

$$u\ge\eta_{N+1}L \quad \text{a.e. in} \quad \Omega'\times\mathcal{I}_N(t),$$

where

$$\mathcal{I}_N(t) := \left(t_N+(1-\sigma_N)\delta_N(\eta_N L)^{q+1-p}\rho^p,\ t_N+\delta_N(\eta_N L)^{q+1-p}\rho^p\right),$$

with $\sigma_N=e^{-(\tau_N+2e^{\tau_N})}$, $e^{\tau_N}=C(n,p)2^{I_N+J_N}$ and t_N given by

$$t_N=t+(\delta_0-\delta_N\eta_N^{q+1-p})L^{q+1-p}\rho^p.$$

Notice that the terminal time of $\mathcal{I}_N(t)$ is $t+\delta_0L^{q+1-p}\rho^p$. Meanwhile, it follows from $u_0>0$ in Ω and Proposition 5.3 with $t_0=0$ that, there exist positive number η and τ depending only on N,n and p such that

$$u\ge\eta L \quad \text{a.e. in} \quad \Omega'\times(0,\tau L^{q+1-p}\rho^p).$$

Furthermore, if t is very close to T, then we can choose $\delta_0>0$ so small that $t+\delta_0L^{q+1-p}\rho^p=T$. Since $t\in[0,T]$ is arbitrary, choosing $\bar\eta:=\min\{\eta_{N+1},\eta\}$, we have that

$$u(x,t)\ge\bar\eta L \quad \text{a.e. in} \quad \Omega'\times[0,T],$$

which is our assertion of Proposition 5.4. □

Appendix B Convergence result

We will recall the fundamental convergence result, used in Appendix C. A weak convergent sequence in L^r, $1<r<\infty$, satisfying the norm convergence is strong convergent in L^r, that was originally proved by Clarkson ([6]) and Hanner ([13]).

Lemma B.1 *Let $r\in(1,\infty)$. Let $\{f_j\}_{j=1}^{\infty}$ be a sequence in $L^r(\Omega)$. Assume that $f_j\longrightarrow f$ weakly in $L^r(\Omega)$ and $\|f_j\|_r\longrightarrow\|f\|_r$. Then we have $f_j\longrightarrow f$ strongly in $L^r(\Omega)$.*

Proof. It is a consequence of the uniform convexity of L^p-space. We omit the details of proof. □

Appendix C Proof of Proposition 4.1

In this appendix, we shall prove Proposition 4.1. From now on, we will show that the function u defined by (4.5) satisfies the conditions (D1)–(D4) in Definition 2.2. Firstly, we introduce the mollifier, which is used later. For a function $f \in L^1(\mathbb{R}^{n+1})$, we denote the *mollifier* of f by

$$f_\varepsilon(z) := (f * \rho_\varepsilon)(z) = \int_{\mathbb{R}^{n+1}} f(w)\rho_\varepsilon(z-w)\,dw.$$

Here, $\varepsilon > 0$ and $z = (x,t), w = (y,s) \in \mathbb{R}^{n+1}$ are space-time points, and let $\rho_\varepsilon(z) := \dfrac{1}{\varepsilon^{n+1}}\rho\left(\dfrac{z}{\varepsilon}\right)$, where $\rho(z)$ is a smooth symmetric function in the following sense:

$$\rho(x,t) = \rho(|x|,|t|) \quad \text{for} \quad z = (x,t) \in \mathbb{R}^{n+1}$$

and satisfies

$$\int_{\mathbb{R}^{n+1}} \rho(z)\,dz = 1, \quad \operatorname{supp}\rho \subset \{(x,t) : |x| < 1, |t| < 1\}, \quad \rho \geq 0.$$

In what follows, let v be a weak solution of (1.3) with the initial data $v_0 = u_0$, obtained from Theorem 3.2. We extend v as $v = u_0$ for $t \leq 0$, and $v = 0$ outside Ω and then the extended function is also written by the same notation. Note that this extension is Lipschitz extension of v. Let us denote the mollification of v by v_ε.

Lemma C.1 *We have the following uniform convergences: As $\varepsilon \searrow 0$,*

$$v_\varepsilon \longrightarrow v \quad \textit{locally uniformly in} \quad C([0,\infty)\,;\,L^{q+1}(\Omega)) \tag{C.1}$$

and

$$(v^q)_\varepsilon \longrightarrow v^q \quad \textit{locally uniformly in} \quad C([0,\infty)\,;\,L^{\frac{q+1}{q}}(\Omega)). \tag{C.2}$$

Proof. From the energy equality (3.2) and the continuity of $x^{\frac{1}{q+1}}$ for $x \geq 0$, we find that $\|v(s)\|_{q+1}$ is locally continuous in $s \in [0,\infty)$, in fact, locally absolutely continuous on $[0,\infty)$. This together with Lemma B.1 and (3.2) implies that $v \in C([0,\infty)\,;\,L^{q+1}(\Omega))$. Via the fundamental property of mollifier, (C.1) immediately follows. By the same argument as above, (C.2) readily follows. $\square$

Let S^* be the extinction time of this solution v of (4.1). Define $\Lambda(\tau)$ by a C^1-solution to the following ODE

$$\begin{cases} \dfrac{d}{d\tau}\Lambda(\tau) = \left(\displaystyle\int_\Omega v^{q+1}(x,\Lambda(\tau))\,dx\right)^{\frac{p}{n}} \\ \Lambda(0) = 0. \end{cases} \tag{C.3}$$

Let $g(t)$ be a C^1-solution of the ODE on $[0, \infty)$

$$\begin{cases} g'(t) = e^{\Lambda(g(t))} \\ g(0) = 0. \end{cases} \tag{C.4}$$

For unique $\Lambda \in C^1(0, \infty)$ solving (C.3) and, subsequently, $g \in C^1(0, \infty)$ solving (C.4), let

$$s(t) = S^* \left(1 - e^{-\Lambda(g(t))}\right)$$

and set

$$u(x, t) := \frac{v\,(x, s(t))}{\gamma(t)}, \quad \gamma(t) := \left(\int_\Omega v^{q+1}\,(x, s(t))\; dx \right)^{\frac{1}{q+1}}. \tag{C.5}$$

The ODE (C.3) is actually solvable, since the integral of the right hand side of the ODE (4.2) is (locally absolutely) continuous on t in $[0, \infty)$, by the energy equality (3.2). We further remark that the following relation holds:

$$s \nearrow S^* \iff t \nearrow \infty.$$

Let $t_0 < \infty$ be any positive number and set $s_0 := S^* \left(1 - e^{-\Lambda(g(t_0))}\right)$. We now deduce the positivity of $\|v(s)\|_{q+1}$ for any nonnegative $s \leq s_0$.

Lemma C.2 *There exists a positive number c_0 such that, for every nonnegative $s \leq s_0$,*

$$\|v(s)\|_{q+1} \geq c_0. \tag{C.6}$$

Proof. Since $\|v(s)\|_{q+1}$ is continuous and positive on $[0, s_0]$ there exists a positive number c_0 satisfying $c_0 := \min\limits_{0 \leq s \leq s_0} \|v(s)\|_{q+1} > 0$, yielding (C.6). The proof is done. $\square$

We have to verify the regularity of the scaled solution u, defined by (4.2) and (C.5).

Lemma C.3 (The regularity of a composite function and its chain rule) *Let u be defined by (C.5). Then there holds that*

$$u \in C([0, \infty); L^{q+1}(\Omega)), \quad \nabla u \in L^\infty_{\mathrm{loc}}([0, \infty); L^p(\Omega)). \tag{C.7}$$

Furthermore, the function $\gamma(t)$ is Lipschitz on $[0, t_0]$ and the weak derivative on time $\partial_t v^q(x, S^* \left(1 - e^{-\Lambda(g(t))}\right))$ is in $L^2(\Omega_\infty)$. In addition, there exists $\partial_t u^q \in L^2(\Omega_{t_0})$ for any $t_0 < \infty$ such that

$$\partial_t u^q = \partial_t v^q \cdot \gamma^{-q} - q v^q \gamma^{-q-1} \gamma'(t) \tag{C.8}$$

in a weak sense.

Proof. The first part of this proof deals with $u \in C([0,\infty); L^{q+1}(\Omega))$. From (3.2) $v\left(x, S^*\left(1-e^{-\Lambda(g(t))}\right)\right) \in C([0,\infty); L^{q+1}(\Omega))$ and $\gamma(t) \in C([0,\infty))$ and thus, by the very definition (C.5), $u(x,t) \in C([0,\infty); L^{q+1}(\Omega))$.

Note that

$$s(t) = S^*\left(1-e^{-\Lambda(g(t))}\right) \qquad \Longleftrightarrow \qquad t = (\Lambda\circ g)^{-1}\left(\log\left(\frac{S^*}{S^*-s}\right)\right)$$

and, by (C.3) and (C.4),

$$\begin{aligned} s_t = \frac{ds}{dt} &= S^* e^{-\Lambda(g(t))}\frac{d}{dt}\Lambda(g(t)) \\ &= S^* e^{-\Lambda(g(t))}\Lambda'(g(t))g'(t) = S^*\Lambda'(g(t)) = \|v(s)\|_{q+1}^{(q+1)\frac{p}{n}}. \end{aligned}$$

Thus, by the changing of variable $s = s(t) = S^*\left(1-e^{-\Lambda(g(t))}\right)$ and integration by parts, one has for any $\varphi \in C_0^\infty(\Omega_\infty)$,

$$\begin{aligned} &\int_0^\infty\!\!\int_\Omega u(x,t)\nabla\varphi(x,t)\,dxdt \\ &= \int_0^{S^*} \frac{v(x,s)}{\|v(s)\|_{q+1}}\nabla\varphi\left(x,(\Lambda\circ g)^{-1}\left(\log\left(\tfrac{S^*}{S^*-s}\right)\right)\right)\,dx\,\|v(s)\|_{q+1}^{-(q+1)\frac{p}{n}}\,ds \\ &= \int_0^{S^*} \|v(s)\|_{q+1}^{-1-(q+1)\frac{p}{n}}\left(-\int_\Omega \nabla v(x,s)\varphi\left(x,(\Lambda\circ g)^{-1}\left(\log\left(\tfrac{S^*}{S^*-s}\right)\right)\right)\,dx\right)\,ds \\ &= \int_0^{S^*}\!\!\int_\Omega \frac{\nabla v(x,s)\varphi\left(x,(\Lambda\circ g)^{-1}\left(\log\left(\tfrac{S^*}{S^*-s}\right)\right)\right)}{\|v(s)\|_{q+1}}\,dx\,\|v(s)\|_{q+1}^{-(q+1)\frac{p}{n}}ds, \end{aligned}$$

since $\nabla v \in L^\infty((0,\infty); L^p(\Omega))$ by (3.4) and $\operatorname{supp}\varphi\left(x,(\Lambda\circ g)^{-1}\left(\log\left(\tfrac{S^*}{S^*-s}\right)\right)\right) \Subset \Omega_{S^*}$. Again, the changing of variable $t = (\Lambda\circ g)^{-1}\left(\log\left(\dfrac{S^*}{S^*-s}\right)\right)$ yields

$$\int_0^\infty\!\!\int_\Omega u(x,t)\nabla\varphi(x,t)\,dxdt = -\int_0^\infty\!\!\int_\Omega \frac{\nabla v\left(x,s(t)\right)}{\gamma(t)}\varphi(x,t)\,dxdt$$

and thus, there exists a weak derivative $\nabla u(x,t)$ such that, for any nonnegative $t_0 < \infty$,

$$\nabla u(x,t) = \frac{\nabla v(x,s(t))}{\gamma(t)} \in L^\infty([0,\infty); L^p(\Omega)). \tag{C.9}$$

In order to prove (C.8) being valid in the weak sense, we verify that $\gamma(t)$ and $v^q(x, s(t))$ are weak differentiable in t and their weak derivatives are integrable on $(0, t_0)$ and Ω_∞, respectively.

Firstly, we show $\gamma(t)$ is weak differentiable in $(0, \infty)$. Now, set $f(s) := \|v(s)\|_{q+1}^{q+1}$. From (3.2) $f(s)$ is a locally absolutely continuous function on $s \in [0, \infty)$ and for a.e. $s \in [0, \infty)$,

$$\frac{d}{ds} f(s) = \frac{d}{ds} \|v(s)\|_{q+1}^{q+1} = -\frac{q+1}{q} \|\nabla v(s)\|_p^p \tag{C.10}$$

and $\frac{d}{ds} f(s)$ is bounded on $(0, \infty)$ by (3.4) and thus, $f(s)$ is actually Lipschitz function. Now, $f_h(s)$ and $\frac{d}{ds} f_h(s)$ denote the mollification with respect to time variable s of $f(s)$ and $\frac{d}{ds} f(s)$, respectively. According to the fundamental property of mollifier, we have, as $h \searrow 0$,

$$f_h(s) \longrightarrow f(s) \quad \text{locally uniformly} \quad \text{on } [0, \infty) \tag{C.11}$$

and, for all $r \geq 1$,

$$\frac{d}{ds} f_h(s) \longrightarrow \frac{d}{ds} f(s) \quad \text{strongly} \quad \text{in } L^r(0, \infty). \tag{C.12}$$

By integration by parts we see that, for every $\phi(t) \in C_0^\infty((0, \infty))$,

$$\int_0^\infty \phi(t) \left[f_h\left(s(t)\right) \right]' dt = -\int_0^\infty \phi'(t) f_h\left(s(t)\right)\, dt, \tag{C.13}$$

where $' := \frac{d}{dt}$. By changing of the variable $s = s(t) = S^*\left(1 - e^{-\Lambda(g(t))}\right)$, the integration in the left hand side of (C.13) is computed as

$$\begin{aligned}
\int_0^\infty \phi(t) \left[f_h(s(t)) \right]' dt &= \int_0^\infty \phi(t) \frac{d}{ds} f_h\left(s(t)\right) \underbrace{s_t\, dt}_{=ds} \\
&= \int_0^{S^*} \phi\left((\Lambda \circ g)^{-1} \left(\log\left(\tfrac{S^*}{S^* - s} \right) \right) \right) \frac{d}{ds} f_h(s)\, ds,
\end{aligned}$$

which converges to

$$\int_0^{S^*} \phi\left((\Lambda \circ g)^{-1} \left(\log\left(\tfrac{S^*}{S^* - s} \right) \right) \right) \frac{d}{ds} f(s)\, ds = \int_0^\infty \phi(t) \frac{d}{ds} f\left(s(t)\right) s_t\, dt \tag{C.14}$$

as $h \searrow 0$, by (C.12). On the other hand, the integration in the right hand side of (C.13) converges to

$$-\int_0^\infty \phi'(t) f_h\left(s(t)\right)\, dt \longrightarrow -\int_0^\infty \phi'(t) f\left(s(t)\right)\, dt \tag{C.15}$$

as $h \searrow 0$, by (C.11). From (C.13), (C.14) and (C.15) we obtain

$$\int_0^\infty \phi(t)\, s_t \frac{d}{ds} f\left(s(t)\right)\, dt = -\int_0^\infty \phi'(t) f\left(s(t)\right)\, dt$$

and thus, there exists a weak derivative $\frac{d}{dt} f(s(t))$ in $(0,\infty)$ such that

$$\frac{d}{dt} f\left(s(t)\right) = s_t \frac{d}{ds} f\left(s(t)\right) \tag{C.16}$$

and therefore, by (3.3) and (4.2), $f\left(s(t)\right)$ is Lipschitz on $[0,\infty)$. From (C.16), $\gamma(t)$ is weak differentiable in $(0, t_0)$ and

$$\begin{aligned}
\gamma'(t) &= \frac{d}{dt} \left\|v\left(s(t)\right)\right\|_{q+1} \\
&= \frac{1}{q+1} \left\|v\left(s(t)\right)\right\|_{q+1}^{-q} \frac{d}{dt} f\left(s(t)\right) \\
&= \frac{1}{q+1} \left\|v\left(s(t)\right)\right\|_{q+1}^{-q}\, s_t \frac{d}{ds} f\left(s(t)\right),
\end{aligned} \tag{C.17}$$

because $\gamma^{\frac{1}{q+1}}$ is Lipschitz for $\gamma \geq c_0$. From (3.3), (3.4), (4.2), (C.6), (C.10) and (C.17), it follows that, for every $t \in (0, t_0)$,

$$|\gamma'(t)| \leq \frac{c_0^{-q}}{q} \|u_0\|_{q+1}^{(q+1)\frac{p}{n}} \|\nabla u_0\|_p^p \tag{C.18}$$

and thus, $\gamma(t)$ is surely Lipschitz on $[0, t_0]$.

Next, we will verify that the weak derivative on time of $v^q\left(x, S^*\left(1 - e^{-\Lambda(g(t))}\right)\right)$ is in $L^2(\Omega_\infty)$. From Definition 3.1 it follows that $\partial_s v^q(x,s) \in L^2(\Omega_\infty)$ and thus,

$$\partial_s (v^q)_\varepsilon \longrightarrow \partial_s v^q \quad \text{strongly} \quad \text{in } L^2(\Omega_\infty). \tag{C.19}$$

Again, by integration by parts we see that, for every $\varphi \in C_0^\infty(\Omega_\infty)$,

$$\int_0^\infty\!\!\int_\Omega \varphi\, \partial_t (v^q)_\varepsilon\, dxdt = -\int_0^\infty\!\!\int_\Omega \partial_t \varphi\, (v^q)_\varepsilon\, dxdt. \tag{C.20}$$

By $s = s(t) = S^*\left(1 - e^{-\Lambda(g(t))}\right)$ and (C.19), the integration in the left hand side of (C.20) is computed as

$$\int_0^\infty\!\!\int_\Omega \varphi\, \partial_t (v^q)_\varepsilon\, dxdt = \int_0^\infty\!\!\int_\Omega \varphi(x,t) \partial_s (v^q)_\varepsilon \left(x, f\left(s(t)\right)\right) s_t\, dxdt$$

$$= \int_0^{S^*}\!\!\int_\Omega \varphi\left(x, (\Lambda\circ g)^{-1}\left(\log\left(\tfrac{S^*}{S^*-s}\right)\right)\right)\,\partial_s(v^q)_\varepsilon(x,s)\,dxds,$$

which converges to

$$\int_0^{S^*}\!\!\int_\Omega \varphi\left(x, (\Lambda\circ g)^{-1}\left(\log\left(\tfrac{S^*}{S^*-s}\right)\right)\right)\,\partial_s(v^q)(x,s)\,dxds$$

$$= \int_0^{\infty}\!\!\int_\Omega \varphi(x,t)\,\partial_s v^q\,(x,s(t))\,s_t\,dxdt \tag{C.21}$$

as $\varepsilon \searrow 0$. The integration in the right hand side of (C.20) converges as

$$-\int_0^{\infty}\!\!\int_\Omega \partial_t\varphi\,(v^q)_\varepsilon\,dxdt \;\longrightarrow\; -\int_0^{\infty}\!\!\int_\Omega \partial_t\varphi\,v^q\,dxdt \tag{C.22}$$

as $\varepsilon \searrow 0$, by (C.2). By (C.20), (C.21) and (C.22) we have

$$\int_0^{\infty}\!\!\int_\Omega \varphi(x,t)\partial_s v^q\,(x,s(t))\,s_t\,dxdt = -\int_0^{\infty}\!\!\int_\Omega \partial_t\varphi\,v^q\,dxdt$$

and thus, there exists a weak derivative $\partial_t v^q\,(x,s(t))$ in $\Omega\times(0,\infty)$ such that

$$\partial_t v^q = \partial_s v^q\,(x,s(t))\,s_t. \tag{C.23}$$

By (3.3), (4.2) and (C.23)

$$|\partial_t v^q| \le \|u_0\|_{q+1}^{(q+1)\frac{p}{n}}\left|\partial_s v^q\left(x, S^*\left(1-e^{-\Lambda(g(t))}\right)\right)\right| \tag{C.24}$$

and thus, the weak derivative on time of $v^q\,(x,s(t))$ is in $L^2(\Omega_\infty)$.

Note that $\gamma\mapsto\gamma^{-q}$ is locally Lipschitz on $\{\gamma\ge c_0\}$ and $\gamma(t)^{-q}$ is the composite function of γ^{-q} with $\gamma(t)$. This together with (4.5), (C.17), (C.18), (C.23) and (C.24) yields that there exists $\partial_t u^q \in L^2(\Omega_{t_0})$ for any $t_0<\infty$ such that

$$\partial_t u^q = \partial_t v^q\cdot\gamma^{-q} - qv^q\gamma^{-q-1}\gamma'(t)$$

in a weak sense, which is the desired result (C.8).

□

Remark C.4 Let $t_0<\infty$ be any positive number. By the very definition (4.5) of u, one has $\|u(t)\|_{q+1}=1$ for $t\in[0,t_0]$ and $0\le u\le c_0^{-1}\|u_0\|_\infty$ for every $(x,t)\in\Omega\times[0,t_0]$ via (C.6), (3.1) and the nonnegativity of v. Applying the same argument as the proof of [21, Proposition 5.2], we readily get $\lambda(t) = \|\nabla u(t)\|_p^p$ for every $t \in [0, t_0]$ and thus, by (C.6), (C.9) and (3.4), $\lambda(t) \le c_0^{-p}\|\nabla u_0\|_p^p < \infty$. Therefore the following equation holds true in a weak sense, and almost everywhere in Ω_{t_0}:

$$\partial_t u^q - \Delta_p u = \lambda(t) u^q \quad \text{in } \Omega_{t_0} \tag{C.25}$$

and thus, it plainly holds that $\Delta_p u \in L^2(\Omega_{t_0})$.

Here we obtain the interior positivity with c_0 in (C.6) by the volume constraint.

Proposition C.5 (Interior positivity with c_0 in (C.6) by the volume constraint) *Let the initial data $u_0 \in W_0^{1,p}(\Omega)$ be positive, bounded in Ω and satisfy $\|u_0\|_{q+1} = 1$. Let u be a nonnegative weak solution of (C.25) in Ω_{t_0} with any positive $t_0 < \infty$. Put $\widetilde{M} := e^{c_0^{-p}\|\nabla u_0\|_p^p t_0/q}\|u_0\|_\infty$ and let Ω' be a subdomain compactly contained in Ω satisfying $|\Omega \setminus \Omega'| \le \frac{1}{4\widetilde{M}^{q+1}}$. Then there exists a positive constant $\widetilde{\eta}$ such that*

$$u(x,t) \ge \widetilde{\eta} L \quad \text{in} \quad \Omega' \times [0, t_0].$$

Here $0 < L \le \min\left\{\left(\frac{1}{4|\Omega'|}\right)^{\frac{1}{q+1}}, \inf_{\Omega''} u_0\right\}$, where Ω'' is compactly contained in Ω and compactly containing Ω', and the positive constant $\widetilde{\eta}$ depends only on p, n, Ω', M and N, where N is the number of chain balls of Ω'. The constant $\widetilde{\eta}$ also depends on the positive constant c_0.

Proof. The proof of this proposition is done by the same argument as Proposition 5.4. □

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