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Tense Logic and Ontology of Time

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Tense Logic and Ontology of Time

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Abstract
This work aims to make tense logic a more robust tool for ontologists, philosophers, knowledge engineers and programmers by outlining a fusion of tense logic and ontology of time. In order to make tense logic better understandable, the central formal primitives of standard tense logic are derived as theorems from an informal and intuitive ontology of time. In order to make formulation of temporal propositions easier, temporal operators that were introduced by Georg Henrik von Wright are developed, and mapped to the ontology of time.

Keywords
first-order tense logic, branching time logic, temporal logic, ontology of time, temporal operators, georg henrik von wright, unification of ontology and logic, computable tense logic, programmable tense logic

1. Introduction

A firm understanding of tense logic is indispensable for ontologists, philosophers, knowledge engineers and programmers who need to formulate exact temporal propositions about the actual world, viz., propositions qualified by time. The standard approach to tense logic has two major drawbacks: it is not easily understandable for non-specialists; standard temporal operators are too coarse-grained for many purposes. This work aims to overcome the drawbacks by founding a system of tense logic and compact temporal operators on an intuitive ontology of time.

The underlying methodology is called *the method of unification* (Styrman [1][2, §4]). The central idea and goal here is to bring deep intuition to tense logic and to make time-related concepts and semantics understandable, by founding them on an intuitive ontology of time. This is in dire contrast with the standard approach, that is devoid of openly explicated ontological commitments. Yet, every system of tense logic formalizes *some* ontology of time. It is therefore intelligible to ask how the basic principles, concepts and semantics of tense logic can be founded on ontology, i.e., how can a primitive principle be derived as a theorem from ontology, how is a concept defined in terms of ontology, and how is semantics mapped to ontology. Conversely, there is *some* logic in every consistent ontology of time. Explicated mappings between an ontology of time and a system of tense logic exactify the applied ontology and secure that it is a sufficient foundation for the system of tense logic. By studying a unified whole of tense logic and ontology of time, a non-expert may acquire a better understanding of both disciplines than by studying each of them separately.

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Temporal propositions are formulated by applying temporal operators. Standard temporal operators are too coarse-grained for many purposes. They can express propositions such as "It is possible from the aspect of the present time that an event will take place at some time in the future as a whole" and "It is necessary from the aspect of the present time that an event took place at some time in the past as a whole", but they cannot express more specific propositions such as "It is possible from the aspect of the present time that it will rain tomorrow in Helsinki" or that "It is necessary from the aspect of the present time that it was raining last year in Helsinki." Point-accessibility (PA) operators are intuitive and sufficiently fine-grained for formalizing such propositions. PA operators are inspired by Georg Henrik von Wright [3].

Some reservations are in place. This work is not a finalized account of the relations of tense logic and ontology of time, nor of the associated semantics and terminology, but an attempt of building a simple prototype which shows that it is in the first place possible to formulate their coherent fusions. Relatedly, a presentist ontology of time is applied because it is straightforward and conforms to common intuitions;[1] a discrete prototype is formulated, because this is easier than formulation of a continuous one.

The article is organized into a form of a cumulative buildup, where virtually every earlier step is applied directly or indirectly in every later step. In §2, a generic fusion of *causal presentism* (CP) and instant-based tense logic is formulated. In §3 *time* and PA operators are founded on the generic fusion of CP and tense logic. In §4, it is pointed out that PA operators save the functionality of standard temporal operators. In §5, *interaction axioms* of standard tense logic are derived as theorems from CP, by applying the PA operators. In §6 the concluding remarks are given. The below abbreviations and logical notation are used.

CP	causal presentism, independently of determinism vs. indeterminism.
LCP	linear causal presentism: CP + determinism.
BCP	branching causal presentism: CP + indeterminism.
PA	point-accessibility.
$\mathcal{T}$	temporal state of the Universe.
$\mathcal{P}$	the present temporal state of the Universe.
x, y, z, v	variables for an individual $\mathcal{T}$ or a set of $\mathcal{T}$s.
t, t', t'', t'''	points of time on a linear timeline.

$x \wedge y$	x and y.	$x \vee y$	x or y.
$\neg x$	not x.	$:=$	is defined as.
$\forall x$	for every x.	$\exists x$	there exists x.
$x \rightarrow y$	if x, then y.	$x \leftrightarrow y$	x if and only if y. x iff y.
$x \subseteq y$	x is a subset of y.	$x \subset y$	x is a proper subset of y.
$x = y \cup z$	x is the union of y and z.	$x = y \cap z$	x is the intersection of y and z.
$x = y \backslash z$	x is the difference of y and z.	$t \mathrel{-\!\!\in} t'$	$\mathcal{T}$ or $\mathcal{T}$s accessible from t at t'.
$\overrightarrow{x}$	causal successors of x.	$\overleftarrow{x}$	causal predecessors of x.

[1] A relativistic ontology of time is not a feasible starting point in exemplifying an understandable fusion of tense logic and ontology of time. It is in the first place challenging to decide which kind of a relativistic ontology should be applied, and the task of fitting together tense logic and relativistic simultaneity presents difficult questions. See Suntola [4] for a system of physics that saves relativistic phenomena while sustaining absolute simultaneity.

2. Causal Presentism and the Kripke Frame

A generic fusion of causal presentism (CP) and instant-based tense logic (IBTL) is outlined by formulating axioms of CP, by deriving primitives of IBTL from CP, and by defining primitive (and other) concepts of IBTL in terms of CP. First, an intuitive picture of CP is given. Second, the primitives of IBTL are characterized. Third, the steps of the buildup of the fusion of CP and IBTL are listed. Fourth, their fusion is formulated.

Fig. 1 illustrates linear causal presentism (LCP) and branching causal presentism (BCP). The big dots represent the present temporal state of the universe ($\mathcal{P}$) which is the only temporal state of the Universe ($\mathcal{T}$) that *exists*, i.e., *is realized*. Stars represent causal predecessors of $\mathcal{P}$, i.e., past $\mathcal{T}$s, which *did exist* and *were realized* in the past but exist no longer. Dashes represent causal successors of $\mathcal{P}$, i.e., possible future $\mathcal{T}s$ that do not exist at $\mathcal{P}$ but *are realizable* in the future. In LCP there is only one realizable future, i.e., one possible future. Therefore, in LCP every assigned future possibility *will be realized*, i.e., *will necessarily come into existence* by becoming present; after a $\mathcal{T}$ has become present, it becomes past and remains past forever. In BCP there are several possible futures, and each assigned future possibility *may be realized* and *may come into existence* by becoming present, i.e., realization of a future possibility is contingent, not necessary. In BCP, an assigned possible future $\mathcal{T}$ either becomes the present and then becomes forever past, or it becomes disconnected from the present (def. 6). The dots in BCP represent $\mathcal{T}s$ which were future possibilities in the past but were not realized, i.e., $\mathcal{T}s$ that became disconnected from $\mathcal{P}$.[2] The passage of time is the process of transitions from one

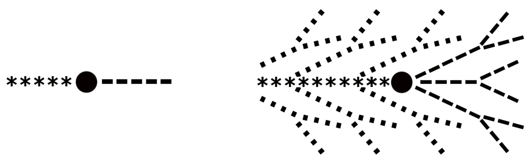

Figure 1: LCP on the left. BCP on the right.

present into a causally succeeding present. These transitions are called forward-directed, i.e., the passage of time takes time forward. Fig. 2 illustrates the passage of time as the process of causally successive $\mathcal{T}$s coming into existence and ceasing to exist.

IBTL starts from possible worlds semantics where the primitive Kripke frame $(\mathbb{U}, >)$ consist of a set of possible worlds $\mathbb{U}$ and an accessibility relation $>$ which may hold between two worlds in $\mathbb{U}$. The possible worlds are interpreted as instants of time, and $>$ as a temporal succession relation between the instants. $>$ is virtually always considered transitive to express the passage and/or direction of time, and irreflexive and asymmetric when circular time is ruled out.

The fusion of CP and IBTL is built up as follows. *Presentism* is postulated (ax. 1). *Positive duration* of $\mathcal{T}$s is postulated (ax. 2). *Causality* (ax. 3) is postulated.[3] The relation of direct causal succession ($<$) is defined (def. 1). The transitory aspect of time, or the passage of time, is derived

[2] Dummett [5, p. 73-4] and Putnam [6, p. 240] give congenial characterizations of presentism.

[3] Axioms 1-3 are expressed in natural language. The definitions, theorems and axiom 4 are expressed in first-order predicate logic that is complemented by rank 1 sets and set theoretic relations, time qualifications and modalities. The set theoretic expressions could be replaced by expressions of discrete mereology (Styrman and Halko [7, §5.1]).

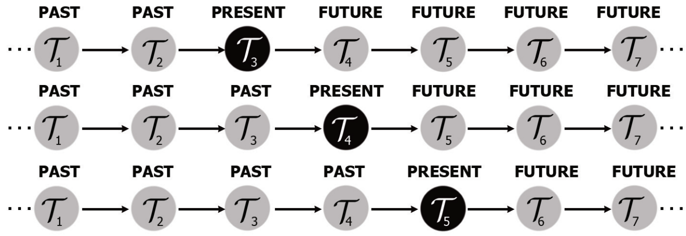

Figure 2: Causal succession of $\mathcal{T}$s. When one $\mathcal{T}$ becomes from a non-existing (gray) future possibility into existence (black) by becoming present, another $\mathcal{T}$ becomes from present into a non-existing past $\mathcal{T}$. In the first row, only $\mathcal{T}_3$ exists. In the second row, $\mathcal{T}_3$ has ceased to exist and $\mathcal{T}_4$ has come into existence. In the third row, $\mathcal{T}_4$ has ceased to exist and $\mathcal{T}_5$ has come into existence.

(th. 1). The set $\overrightarrow{\mathcal{P}}$ of causal successors of $\mathcal{P}$ (cor. 1), the set $\overleftarrow{\mathcal{P}}$ of causal predecessors of $\mathcal{P}$ (cor. 2), and the set $\mathbb{U}$ of causal successors of causal predecessors of $\mathcal{P}$ (th. 2) are derived. The relation of causal succession ($<$) is defined (def. 2). Causal successors (def. 3), causal predecessors (def. 4), $\mathcal{T}$s connected to (def. 5) and $\mathcal{T}$s disconnected from (def. 6) a point of evaluation (an arbitrary $\mathcal{T}$) are defined. Transitivity of $<$ is derived (th. 3). Irreflexivity of $<$ is postulated (ax. 4) and asymmetricity of $<$ is derived (th. 4).

Axiom 1. Presentism and absolute simultaneity. A single instantial $\mathcal{T}$ exists, and it is all that exists. It is called the present $\mathcal{T}$, abbreviated as $\mathcal{P}$. *Is present* and *exists* are the same property, which belongs only to $\mathcal{P}$ and its proper parts. $\mathcal{P}$ consists of various proper parts which exist absolutely simultaneously, and may be in different states of motion and gravitation, such as the Moon, the Earth and distant galaxies.

Axiom 2. Positive duration. The duration of $\mathcal{P}$ is positive, i.e., s seconds, where s is a real number greater than zero.[4] Together with the other axioms and definitions, positive duration of $\mathcal{T}$s entails[5] discrete motion and time, i.e., that a positive period of time consists of finitely many positive instants of time, and that every history (def. 18) is ordered discretely like the integers: an arbitrary $\mathcal{T}$ in a history can be considered as 0, its causal predecessors as $-1, -2, -3, \ldots$ and its causal successors as $1, 2, 3, \ldots$.

Axiom 3. Causality. $\mathcal{P}$ is the effect of exactly one $\mathcal{T}$: the *direct causal predecessor* of $\mathcal{P}$. In the context of determinism $\mathcal{P}$ causes the possibility of coming into existence of exactly one *direct causal successor*. In the context of indeterminism $\mathcal{P}$ causes the possibility of coming into existence of several direct causal successors, i.e., an irreducible element of randomness is involved. In both cases exactly one direct causal successor of $\mathcal{P}$ will come into existence (th. 1).

Definition 1. Direct causal successor. $x \lessdot \mathcal{P} \coloneqq \mathcal{P}$ is a direct causal successor of x, and x is the direct causal predecessor of $\mathcal{P}$.

[4] Van Bendegem [8] contemplates positive duration.

[5] When it is supposed that there are no gaps between consecutive $\mathcal{T}$s, and that there are no higher transfinite orderings of histories than the ordering of the natural numbers.

Theorem 1. The passage of time as the process of transitions. It is proved that CP entails an active process of transitions from one present to a causally successive present. Once time has been defined (def. 7) this process may called the passage of time. (1) $\mathcal{P} = \mathcal{T}_A$ exists (ax. 1). (2) $\mathcal{P}$ causes the possibility of realization of its direct causal successor(s) $\mathcal{T}_{B1}, \ldots, \mathcal{T}_{Bn}$ (ax. 3). (3) $\mathcal{T}_A$ will cease to exist, as the duration of $\mathcal{P}$ is finite (ax. 2). (4) Once $\mathcal{T}_A$ has ceased to exist, one of the only available alternatives $\mathcal{T}_{B1}, \ldots, \mathcal{T}_{Bn}$ has come into existence. ■

Corollary 1. $\overrightarrow{\mathcal{P}}$ **Future possibilities as causal successors of** $\mathcal{P}$. (1) $\mathcal{P}$ causes realizability of one (LCP) or several (BCP) direct causal successors; these are elements of the set of $\mathcal{P}$'s direct causal successors $\mathcal{P}^{+1} = \{x | \mathcal{P} \lessdot x\}$. (2) Realizability of any element x of $\mathcal{P}^{+1}$ includes *realizability of x causing realizability of its direct causal successors*; these are elements of the set of direct causal successors of $\mathcal{P}$'s direct causal successors $\mathcal{P}^{+2} = \{y | \exists x(\mathcal{P} \lessdot x \lessdot y)\}$. (3) By induction, $\mathcal{P}$ causes realizability of all elements of $\mathcal{P}^{+1}, \mathcal{P}^{+2}, \mathcal{P}^{+3}, \ldots$, where $\mathcal{P}^{+3} = \{y | \exists x, z(\mathcal{P} \lessdot x \lessdot z \lessdot y)\}$. The set of future possibilities $\overrightarrow{\mathcal{P}} = \mathcal{P}^{+1} \cup \mathcal{P}^{+2} \cup \mathcal{P}^{+3} \cup \ldots$. ■

Corollary 2. $\overleftarrow{\mathcal{P}}$ **Causal predecessors of** $\mathcal{P}$. (1) $\mathcal{P}$ was caused by its direct causal predecessor $\mathcal{T}_{-1}$, i.e., $\mathcal{T}_{-1} \lessdot \mathcal{P}$ holds. (2) $\mathcal{T}_{-1}$ was the present before $\mathcal{P}$ (th. 1). Thus, $\mathcal{T}_{-1}$ was caused by $\mathcal{T}_{-2}$, i.e., $\mathcal{T}_{-2} \lessdot \mathcal{T}_{-1}$ holds. (3) By induction, we get the chain $\ldots \mathcal{T}_{-3} \lessdot \mathcal{T}_{-2} \lessdot \mathcal{T}_{-1} \lessdot \mathcal{P}$, and $\overleftarrow{\mathcal{P}} = \{\ldots, \mathcal{T}_{-3}, \mathcal{T}_{-2}, \mathcal{T}_{-1}\}$. ■

Theorem 2. $\mathbb{U}$ is the set of causal successors of causal predecessors of $\mathcal{P}$, i.e., the set of $\mathcal{T}$s that have ever been future possibilities.[6] In LCP $\mathbb{U} = \overleftarrow{\mathcal{P}} \cup \{\mathcal{P}\} \cup \overrightarrow{\mathcal{P}}$ (cors. 1 and 2). In BCP we must consider $\mathcal{T}$s that are disconnected from $\mathcal{P}$ (def. 6). (1) $\mathcal{P}$ causes all future possibilities (cor. 1). (2) When $\mathcal{T}_{-1}$ was the present, it caused its future possibilities, that are elements of the set $\overrightarrow{\mathcal{T}_{-1}}$. (3) When $\mathcal{T}_{-2}$ was the present, it caused its future possibilities, that are elements of the set $\overrightarrow{\mathcal{T}_{-2}}$. (4) By induction, $\mathbb{U}$ is the set of causal successors of causal predecessor of $\mathcal{P}$: $\mathbb{U} = \overrightarrow{\mathcal{T}_{-1}} \cup \overrightarrow{\mathcal{T}_{-2}} \cup \overrightarrow{\mathcal{T}_{-3}} \cup \ldots$. ■

Definition 2. Causal successor. $x < y := x \lessdot y \vee \exists z_1, \ldots, z_n \in \mathbb{U}(x \lessdot z_1 \lessdot \ldots \lessdot z_n \lessdot y)$, where $x, y \in \mathbb{U}$. When y is a causal successor of x (x is a causal predecessor of y), either y is a direct causal successor of x, or there is a longer finite chain of direct causal successors from x to y. The relation $>$ is the converse of $<$: $\forall x, y \in \mathbb{U}(x < y \leftrightarrow y > x)$.

Definition 3. Causal successors of a point of evaluation. $\overrightarrow{x} = \{y | x < y\}$, where $x, y \in \mathbb{U}$. $\overrightarrow{x}$ is the set of all causal successors of x.

Definition 4. Causal predecessors of a point of evaluation. $\overleftarrow{x} = \{y | y < x\}$, where $x, y \in \mathbb{U}$. $\overleftarrow{x}$ is the set of all causal predecessors of x.

Definition 5. $\mathcal{T}$**s connected to a point of evaluation**. $con(x) = \overleftarrow{x} \cup \{x\} \cup \overrightarrow{x}$, where $x \in \mathbb{U}$. $con(x)$ is the union of the set of causal predecessors of x, the singleton set of x, and the set of causal successors of x.

[6]Compare to Belnap [9, p. 387] who in the contex of Special Relativity accommodates indeterminism "by including those point events that either are now future possibilities or were future possibilities."

Definition 6. $\mathcal{T}$s disconnected from a point of evaluation. $dis(x) = \mathbb{U} \backslash con(x)$, where $x \in \mathbb{U}$. $dis(x)$ is the difference of $\mathbb{U}$ and the set of all $\mathcal{T}$s connected to x.

Theorem 3. Transitivity of causal succession. $\forall x, y, z \in \mathbb{U}(x < y < z \rightarrow x < z)$. For all x, y, z that are elements of $\mathbb{U}$, if y is a causal successor of x, and z is a causal successor of y, then z is a causal successor of x. That $x < z$ holds (def. 2) is trivial in CP, for $x < y < z$ means that there is a chain of direct causal successors from x to z (that contains y in the middle). Consider a proof that CP and $x < y < z$ imply $x < z$, for all $x, y, z \in \mathbb{U}$. (1) CP and $x < y$ entail that $\overrightarrow{y} \subset \overrightarrow{x}$ (def. 3). (2) CP and $y < z$ entail that $z \in \overrightarrow{y}$. (3) $z \in \overrightarrow{y}$ and $\overrightarrow{y} \subset \overrightarrow{x}$ entail $z \in \overrightarrow{x}$, which entails that $x < z$. ■

Axiom 4. Irreflexivity of causal succession. $\forall x \in \mathbb{U} \neg(x < x)$. For all elements x of $\mathbb{U}$, x is not a causal successor of itself. Together with the other axioms, irreflexivity excludes causal cycles and entails e.g. that a present $\mathcal{T}$ is not realizable in the future.

Theorem 4. Asymmetricity of causal succession. $\forall x, y \in \mathbb{U}(x < y \rightarrow \neg(y < x))$. For all elements x and y of $\mathbb{U}$, if x a causal predecessor of y, then y is not a causal predecessor of x. Proof by contradiction: causal chains of the type $x < y \wedge y < x$ violate ax. 4. ■

3. Time and Point-Accessibility

Time and point-accessibility (PA) are founded on the generic fusion of CP and tense logic in the following order. Time is defined as linear, relational and superimposed (def. 7). A PA schema (def. 8) and a PA operator schema (def. 9) are formulated. It is shown how CP yields truth values of forward-directed (def. 10), backward-directed and synchronic PA propositions (def. 11). Quantified PA operators are formulated (defs. 12-13).

Definition 7. Time: linear, relational and superimposed. Points of time are linearly ordered, i.e., elements of a linear timeline: for all points of time t and t', either $t = t'$ or $t < t'$ or $t > t'$. Time is not absolute nor primitive but relational, i.e., all references to time are references to one or more $\mathcal{T}$s, viz., elements of the linear timeline are mapped to $\mathcal{T}$s.[7] In CP, the *present time* is $\mathcal{P}$, and the *past times* are causal predecessors of $\mathcal{P}$ (cor. 2). In LCP, each *future time* is an element of the linear sequence of causal successors of $\mathcal{P}$ (cor. 1). In BCP each *future time* denotes several mutually disconnected (def. 6) causal successors of $\mathcal{P}$, i.e., mutually disconnected possible future $\mathcal{T}$s. In McDermott's [12] terminology, a future time is *superimposed* by several $\mathcal{T}$s. The expressions 't denotes several $\mathcal{T}$s' and 't is superimposed by several $\mathcal{T}$s' are interchangeable.[8] The expression $t = t'$ is read as: *t is the same time as t'*. $t = t'$ is true when t and t' denote the same $\mathcal{T}$ or the same $\mathcal{T}$s. The expression $t < t'$ is read as: *t is earlier than t'*. $t < t'$ is true when each $\mathcal{T}$ denoted by t' is causal successor of some $\mathcal{T}$ denoted by t, and each $\mathcal{T}$ denoted by t is a

[7] See Ballard [10, p. 61] and Galton [11, p. 185] for the Leibnizean notion of relational time.

[8] Linear time enables applying all PA operators in LCP and BCP. Superimposition and linear time are a natural pair in BCP. Also branching time requires superimposition in practice, as it requires (a) the ontology of $\mathcal{T}$s that are possible in the future within a specific distance from $\mathcal{P}$, (b) 'individual times' for each of these $\mathcal{T}$s, and (c) a superimposed 'extra time' that denotes all the individual times.

causal predecessor of some $\mathcal{T}$ denoted by t'. Formally, where s(t) and s(t') denote sets of $\mathcal{T}$s denoted by t and t', respectively:

$$t < t' := \forall x \exists y(x \in s(t') \to y \in s(t) \land y < x) \land \forall z \exists v(z \in s(t) \to v \in s(t') \land z < v).$$

Distance between points of time can be defined in terms of superimposition. For instance, when $\mathcal{P} < t$ holds and t is associated with a distance of one second in Earth's geoid (sea level), t is superimposed by the last $\mathcal{T}$s of all chains of causally successive $\mathcal{T}$s that start from $\mathcal{P}$ and end at a $\mathcal{T}$ where a caesium-133 atom in Earth's geoid has oscillated 9192631770 times. Similarly, when $\mathcal{P} < t$ holds and t is associated with a distance of one Earth day or one Earth year, t is superimposed by the last $\mathcal{T}$s of all chains of causally successive $\mathcal{T}$s that start from $\mathcal{P}$ and end at a $\mathcal{T}$ where the Earth has rotated once around its own axis or once around the Sun, respectively.

Definition 8. PA schema: $t \multimap\!\!\in t'$, where t is the *aspect time*, $\multimap\!\!\in$ is the *point-accessibility relation*, and t' is the *target time*. In a temporal proposition (def. 9), t and t' are assigned and $t \multimap\!\!\in t'$ is replaced by a set of one or more $\mathcal{T}$s that are, were or will be accessible at t' from the aspect of t. In def. 9 modalities are defined as quantifiers over $t \multimap\!\!\in t'$. In the following, it is show how CP determines the elements of $t \multimap\!\!\in t'$ with all combinations of $t, t', \mathcal{P}$, where $t \leq \mathcal{P}$.[9] Combinations where $t = \mathcal{P}$ are represented first. Second, combinations where $t < \mathcal{P}$ are represented.

$\mathcal{P}$-forward: $\mathcal{P} \multimap\!\!\in t'$, where $t' > \mathcal{P}$. In LCP $\mathcal{P} \multimap\!\!\in t'$ is a set of exactly one causal successor of $\mathcal{P}$, which *is realizable* at t' from the aspect of $\mathcal{P}$. In BCP $\mathcal{P} \multimap\!\!\in t'$ is a set of several mutually disconnected causal successors of $\mathcal{P}$ that *are realizable* at t' from the aspect of $\mathcal{P}$ (Fig. 3).

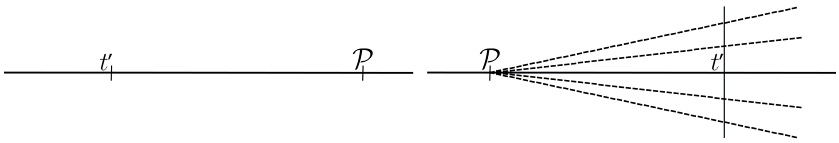

Figure 3: $\mathcal{P}$-backward on the left. $\mathcal{P}$-forward on the right (BCP).

$\mathcal{P}$-backward: $\mathcal{P} \multimap\!\!\in t'$, where $t' < \mathcal{P}$. The only element of $\mathcal{P} \multimap\!\!\in t'$ is a single causal predecessor of $\mathcal{P}$ that *was realized* at t' from the aspect of $\mathcal{P}$ (Fig. 3).

$\mathcal{P}$-synchronic: $\mathcal{P} \multimap\!\!\in t'$, where $t' = \mathcal{P}$. The only element of $\mathcal{P} \multimap\!\!\in \mathcal{P}$ is $\mathcal{P}$, which *is realized* from the aspect of $\mathcal{P}$.

$t < \mathcal{P} < t'$: $t \multimap\!\!\in t'$ is a set of one $\mathcal{T}$ that *was realizable* (LCP) or several $\mathcal{T}$s that *were realizable* (BCP) at t' from the aspect of t. From the aspect of $\mathcal{P}$, the same $\mathcal{T}$ *is realizable* at t' (LCP), or the elements of $\mathcal{P} \multimap\!\!\in t' \subset t \multimap\!\!\in t'$ *are realizable* at t' (BCP).

$t < t' = \mathcal{P}$: $t \multimap\!\!\in t'$ is a set of one $\mathcal{T}$ that *was realizable* (LCP) or several $\mathcal{T}$s which *were realizable* (BCP) at t' from the aspect of t. Exactly one of them *is realized* at $t' = \mathcal{P}$.

$t < t' < \mathcal{P}$: $t \multimap\!\!\in t'$ is a set of one $\mathcal{T}$ that *was realizable* (LCP) or several $\mathcal{T}$s which *were realizable* (BCP) at t' from the aspect of t. From the aspect of $\mathcal{P}$, exactly one of them *was realized* at t'.

[9] LCP fixes the elements of $t \multimap\!\!\in t'$ also when $t > \mathcal{P}$. In contrast, when $t > \mathcal{P}$ in BCP, t denotes several mutually disconnected $\mathcal{T}$s. Such t cannot function as an aspect time, because the aspect time must be a single $\mathcal{T}$. However, $t \multimap\!\!\in t'$ where $t > \mathcal{P}$ can be considered to have the same members as $\mathcal{P} \multimap\!\!\in t'$. For, as $\mathcal{P}$ is the ultimate viewpoint, $t \multimap\!\!\in t'$ where e.g. $\mathcal{P} < t < t'$, can be interptered to be equivalent with $\mathcal{P} \multimap\!\!\in t \multimap\!\!\in t'$, which is in turn equivalent with $\mathcal{P} \multimap\!\!\in t'$.

Definition 9. PA operator schema. Length 1 PA (PA-1) operator schema $\mathcal{M}_t\phi_{t'}$ applies a PA-1 chain $t \mathrel{-\!\!\in} t'$. $\mathcal{M}_t\phi_{t'}$ is read as: It (is, was, will be) $\mathcal{M}$ from the aspect of t that ϕ (is, was, will be) realized at t', where $\mathcal{M}$ is a modality, t is the aspect time, t' is the target time, and ϕ is a property or a disjunction of properties, or in von Wright's [3, pp. 96-8] terms "a grammatically complete sentence [such as "It rains in Helsinki."] which, however, does not express a true of false proposition unless it is qualified with respect to time."

A PA-1 operator is an instance of a PA-1 operator schema where $\mathcal{M}$ is assigned a modality: `possible` (`pos`), `contingent` (`con`), `necessary` (`nec`), `impossible` (`imp`), or `neutral` (`neu`); see defs. 10-11.[10] A PA-1 proposition is an instance of a PA-1 operator schema where t, t', ϕ and $\mathcal{M}$ are assigned. In LCP a PA-1 proposition is either true or false, with any t, t' combination. In BCP a PA-1 proposition is either true, false or indeterminate with any t, t' combination where $t \leq \mathcal{P}$.[11] Formally, a PA-1 proposition states that a specific number of elements of the set $t \mathrel{-\!\!\in} t'$ conform to ϕ, i.e., modalities are considered as quantifiers over $t \mathrel{-\!\!\in} t'$.[12] $\texttt{pos}_t\phi_{t'}$ is true iff at least one element of $t \mathrel{-\!\!\in} t'$ conforms to ϕ. $\texttt{nec}_t\phi_{t'}$ is true iff every element of $t \mathrel{-\!\!\in} t'$ conforms to ϕ. $\texttt{con}_t\phi_{t'}$ is true iff at least one but not every element of $t \mathrel{-\!\!\in} t'$ conforms to ϕ. $\texttt{imp}_t\phi_{t'}$ is true iff no element of $t \mathrel{-\!\!\in} t'$ conforms to ϕ. $\texttt{neu}_t\phi_{t'}$ is true iff $\texttt{nec}_t\phi_{t'}$ is true, false if $\texttt{imp}_t\phi_{t'}$ is true, and indeterminate iff $\texttt{con}_t\phi_{t'}$ is true. The following rules hold for PA-1 propositions:
$\mathcal{M}_t\phi_{t'} \wedge \mathcal{M}_t\phi_{t''} \equiv \mathcal{M}_t\phi_{t' \wedge t''}$.
$\mathcal{M}_t\phi_{t'} \vee \mathcal{M}_t\phi_{t''} \equiv \mathcal{M}_t\phi_{t' \vee t''}$.
$\mathcal{M}_t\phi_{t'} \vee \mathcal{M}'_t\phi_{t'} \equiv \mathcal{M} \vee \mathcal{M}'_t\phi_{t'}$.

Length n PA (PA-n) operator schema $\mathcal{M}_{t_0} \ldots \mathcal{M}'_{t_{n-1}}\phi_{t_n}$ applies a PA-n chain: It (is, was, will be) $\mathcal{M}$ from the aspect of t_0 that ... it (is, was, will be) $\mathcal{M}'$ from the aspect of t_{n-1} that ϕ (is, was, will be) realized at t_n. PA-n propositions are PA-n operator schemas where ϕ, $\mathcal{M}, \ldots, \mathcal{M}'$ and $t_0, \ldots, t_n$ are assigned. In LCP a PA-n proposition is either true or false. In BCP a PA-n proposition where $t_0 \leq \mathcal{P}$ is either true, false or indeterminate.

A PA proposition of the type $\mathcal{M}_{t_0} \ldots \mathcal{M}'_{t_{n-1}}\phi_{t_n}$ is a *single-chain* proposition as it applies a single chain of accessibility $t_0 \mathrel{-\!\!\in} \ldots \mathrel{-\!\!\in} t_{n-1} \mathrel{-\!\!\in} t_n$. Exactly one $\mathcal{T}$ suffices in making a single-chain proposition true, false or indeterminate. It is the earliest of $t_0, \ldots, t_n$; as $t_0 \leq \mathcal{P}$, the earliest of $t_0, \ldots, t_n$ is earlier than or equal to $\mathcal{P}$. This is intelligible as in CP the past and $\mathcal{P}$ are fixed, and the earliest of $t_0, \ldots, t_n$ causes all possibilities that are relevant to the focal proposition. A *multi-chain* proposition has two or more chains of accessibility; accordingly, more than one $\mathcal{T}$ may be required in making a multi-chain proposition true, false or indeterminate.

Definition 10. Forward-directed PA-1 propositions. The direction of a PA-1 proposition is the direction of its accessibility relation $t \mathrel{-\!\!\in} t'$. In forward-directed PA-1 propositions $t < t'$. Forward-directed PA-1 propositions of the type $\mathcal{M}_{\mathcal{P}}\phi_{t>\mathcal{P}}$ are present propositions about the

[10] The modalities are written in natural language instead of the traditional one-character symbols to improve readability. Note that there is no standard symbol for neutrality.

[11] Analogously to def. 8, in BCP e.g. $\texttt{pos}_t\phi_{t'}$ where $\mathcal{P} < t < t'$ is not a PA-1 proposition. Its genuine accessibility chain can be interpreted as $\mathcal{P} \mathrel{-\!\!\in} t \mathrel{-\!\!\in} t'$, but $\mathcal{M}$ in the corresponding PA operator schema $\mathcal{M}_{\mathcal{P}}\texttt{pos}_t\phi_{t'}$ is unknown. $\texttt{pos}_t\phi_{t'}$ where $\mathcal{P} < t < t'$ could be interpreted as the PA-2 proposition $\texttt{nec}_{\mathcal{P}}\texttt{pos}_t\phi_{t'}$, which is read as "It is necessary from the aspect of $\mathcal{P}$ that it will be possible from the aspect of t that ϕ is realizable at t'."

[12] PA operators can be considered to deploy *hybrid temporal logic* (Goranko and Rumberg [13, ch. 7.1]), as they express propositions that have a specific truth value at exactly one instant of time.

future, and $\mathcal{P} \multimap\!\!\in t$ is a set of one $\mathcal{T}$ that is (LCP) or several $\mathcal{T}$s that are (BCP) from the aspect of $\mathcal{P}$ realizable at t. `rains` is an abbreviation of "It rains in Helsinki."

A: $\texttt{possible}_{\mathcal{P}}\texttt{rains}_{t'>\mathcal{P}} \equiv$ "It is possible from the aspect of $\mathcal{P}$ that it will rain in Helsinki at t'." True iff it rains in Helsinki in at least one element of $\mathcal{P} \multimap\!\!\in t'$. Otherwise false.

B: $\texttt{necessary}_{\mathcal{P}}\texttt{rains}_{t'>\mathcal{P}} \equiv$ "It is necessary from the aspect of $\mathcal{P}$ that it will rain in Helsinki at t'." True iff it rains in Helsinki in every element of $\mathcal{P} \multimap\!\!\in t'$. Otherwise false.

C: $\texttt{impossible}_{\mathcal{P}}\texttt{rains}_{t'>\mathcal{P}} \equiv$ "It is impossible from the aspect of $\mathcal{P}$ that it will rain in Helsinki at t'." True iff it rains in Helsinki in no element of $\mathcal{P} \multimap\!\!\in t'$. Otherwise false.

D: $\texttt{contingent}_{\mathcal{P}}\texttt{rains}_{t'>\mathcal{P}} \equiv$ "It is contingent from the aspect of $\mathcal{P}$ that it will rain in Helsinki at t'." True iff it rains in Helsinki in at least one but not in every element of $\mathcal{P} \multimap\!\!\in t'$. Otherwise false. In other words, true iff $\texttt{imp}_{\mathcal{P}}\texttt{rains}_{t'}$ and $\texttt{nec}_{\mathcal{P}}\texttt{rains}_{t'}$ are false, and false iff $\texttt{imp}_{\mathcal{P}}\texttt{rains}_{t'}$ or $\texttt{nec}_{\mathcal{P}}\texttt{rains}_{t'}$ is true.

E: $\texttt{neutral}_{\mathcal{P}}\texttt{rains}_{t'>\mathcal{P}} \equiv$ "It is neutral from the aspect of $\mathcal{P}$ that it will rain in Helsinki at t'" $\equiv$ "It will rain in Helsinki at t'." True iff it rains in Helsinki in every element of $\mathcal{P} \multimap\!\!\in t'$, i.e., iff $\texttt{nec}_{\mathcal{P}}\texttt{rains}_{t'}$ is true. False iff it rains in no element of $\mathcal{P} \multimap\!\!\in t'$, i.e., iff $\texttt{imp}_{\mathcal{P}}\texttt{rains}_{t'}$ is true. Indeterminate iff it rains in Helsinki in at least one but not in every element of $\mathcal{P} \multimap\!\!\in t'$, i.e., iff $\texttt{con}_{\mathcal{P}}\texttt{rains}_{t'}$ is true. E is thereby a *future contingent* when D is true.[13]

Each row below represents a consistent combination of the truth values of A-E.

⟨A true⟩	⟨B true⟩	⟨C false⟩	⟨D false⟩	⟨E true⟩
⟨A false⟩	⟨B false⟩	⟨C true⟩	⟨D false⟩	⟨E false⟩
⟨A true⟩	⟨B false⟩	⟨C false⟩	⟨D true⟩	⟨E indeterminate⟩

Definition 11. Backward-directed and synchronic PA-1 propositions. PA-1 propositions $\mathcal{M}_{\mathcal{P}}\phi_{t'}$ where $t' < t$ are backward-directed. In synchronic PA-1 propositions $t = t'$. Backward-directed PA-1 propositions of the type $\mathcal{M}_{\mathcal{P}}\phi_{t'<\mathcal{P}}$ are present propositions about the past. Synchronic PA-1 propositions of the type $\mathcal{M}_{\mathcal{P}}\phi_{\mathcal{P}}$ are present propositions about the present. In backward-directed and synchronic PA-1 propositions possibility, necessity and neutrality are equivalent, and contingency statements are false. Von Wright [17, p. 25] calls this "a modal logic of a universe of propositions which has no room for contingent propositions but in which every truth is a necessity and every falsehood is an impossibility." The past and $\mathcal{P}$ are unchanging *from the aspect of* $\mathcal{P}$, even if they could have been realized differently. Therefore, propositions of the types $\texttt{pos}_{\mathcal{P}}\phi_{t'\leq\mathcal{P}}$, $\texttt{nec}_{\mathcal{P}}\phi_{t'\leq\mathcal{P}}$ and $\texttt{neu}_{\mathcal{P}}\phi_{t'\leq\mathcal{P}}$ can be written as $\phi_{t'}$, and read as *ϕ was/is the case at t'*. Backward-directed and synchronic propositions of the type $\texttt{imp}_{\mathcal{P}}\phi_{t'\leq\mathcal{P}}$ can be written as $\neg\phi_t$, and read as *ϕ was/is not the case at t'*.

A: $\texttt{possible}_{\mathcal{P}}\texttt{rains}_{t'\leq\mathcal{P}} \equiv \phi_{t'\leq\mathcal{P}}$. True iff it rains in Helsinki in at least one element of $\mathcal{P} \multimap\!\!\in t'$, i.e., in its only element.

B: $\texttt{necessary}_{\mathcal{P}}\texttt{rains}_{t'\leq\mathcal{P}} \equiv \phi_{t'\leq\mathcal{P}}$. True iff it rains in Helsinki in every element of $\mathcal{P} \multimap\!\!\in t'$, i.e., in its only element.

C: $\texttt{impossible}_{\mathcal{P}}\texttt{rains}_{t'\leq\mathcal{P}} \equiv \neg\phi_{t'\leq\mathcal{P}}$. True iff it rains in Helsinki in no element of $\mathcal{P} \multimap\!\!\in t'$, i.e., not in its only element.

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[13] Compare to Briggs and Forbes [14, ch. 2] who analyze proposition (1) "Exactly one day into the future, there will be a sea battle" as follows: "there appear to be circumstances in which (1) is true, which can usefully be contrasted with circumstances in which (1) is false (e.g., circumstances in which no one has any ships, and it is not physically possible to make any by tomorrow)." See Knuuttila [15] for historical and Akama et al. [16] for modern approaches to future contingents.

D: $\texttt{contingent}_{\mathcal{P}}\texttt{rains}_{t'\leq\mathcal{P}}$. True iff it rains in Helsinki in at least one but not in every element of $\mathcal{P} \multimap\!\in t'$, i.e., never true.

E: $\texttt{neutral}_{\mathcal{P}}\texttt{rains}_{t'\leq\mathcal{P}} \equiv \phi_{t'<\mathcal{P}}$. True iff it rains in Helsinki in every element of $\mathcal{P} \multimap\!\in t'$. False iff it rains in Helsinki in no element of $\mathcal{P} \multimap\!\in t'$. Indeterminate iff it rains in Helsinki in at least one but not in every element of $\mathcal{P} \multimap\!\in t'$, i.e., never indeterminate.

Each row below represents a consistent combination of the truth values of A-E. The truth values of A, B, E are equivalent and contrary to C; D is always false.

⟨A true⟩ ⟨B true⟩ ⟨C false⟩ ⟨D false⟩ ⟨E true⟩
⟨A false⟩ ⟨B false⟩ ⟨C true⟩ ⟨D false⟩ ⟨E false⟩

Definition 12. Quantified PA-1 operators. Table 1 represents PA-1 operators that quantify over intervals of time. Henceforth, operators 1-10 are referred to as TB1.1-10. The clause "From the aspect of *t*" is to be added in front of the natural language definitions of TB1.1-10. TB1.1 is expressed also as a conjunction and TB1.2 also as a disjunction of unquantified PA-1 operators. TB1.1' transforms TB1.1 into a version that quantifies over a finite interval of time. Similarly for all operators in the table. TB1.11-12 are examples from von Wright [3, pp. 96-8]. Von Wright's notation $\exists t' < t(\mathcal{M}_t\phi_{t'})$ is modified into $\mathcal{M}_t\phi_{\exists t'<t}$.

Table 1
Quantified PA-1 operators. In LCP: 1 ≡ 3 ≡ 5; 2 ≡ 4 ≡ 6; 8 is always false.

1	$\text{pos}_t\phi_{\exists t'>t}$	it is possible that ϕ will hold at least once after *t*.
1	$\text{pos}_t\phi_{\exists t'>t}$	$\equiv \text{pos}_t\phi_{t+1} \vee \text{pos}_t\phi_{t+2} \vee \text{pos}_t\phi_{t+3} \vee \ldots.$
1'	$\text{pos}_t\phi_{\exists t''(t<t''\leq t')}$	it is possible that ϕ will hold at least once in the interval $]t\, t']$.
2	$\text{pos}_t\phi_{\forall t'>t}$	it is possible that ϕ will hold at any time after *t*.
2	$\text{pos}_t\phi_{\forall t'>t}$	$\equiv \text{pos}_t\phi_{t+1} \wedge \text{pos}_t\phi_{t+2} \wedge \text{pos}_t\phi_{t+3} \wedge \ldots.$
3	$\text{neu}_t\phi_{\exists t'>t}$	it is neutral that ϕ will hold at least once after *t*.
4	$\text{neu}_t\phi_{\forall t'>t}$	it is neutral that ϕ will hold at any time after *t*.
5	$\text{nec}_t\phi_{\exists t'>t}$	it is necessary that ϕ will hold at least once after *t*.
6	$\text{nec}_t\phi_{\forall t'>t}$	it is necessary that ϕ will hold at any time after *t*.
7	$\text{imp}_t\phi_{\forall t'>t}$	it is impossible that ϕ will hold at any time after *t*.
8	$\text{con}_t\phi_{\exists t'>t}$	it is contingent that ϕ will hold at least once after *t*.
9	$\text{nec}_t\phi_{\exists t'<t}$	it is necessary that ϕ did hold at least once before *t*.
10	$\text{nec}_t\phi_{\forall t'<t}$	it is necessary that ϕ did hold always before *t*.
11	$\text{pos}_{\exists t'<t}\phi_t$	at least once before *t*, it was possible that ϕ holds at *t*.
12	$\text{nec}_{\forall t'<t}\phi_t$	always before *t*, it was necessary that ϕ holds at *t*.

Definition 13. Asymptotic determinism was recognized by Aristotle in *Metaphysics* 1027b10-14: "it is necessary that he who lives shall one day die…But whether he dies by disease or by violence, is not yet determined." That $\mathcal{P}$ asymptotically determines ϕ is written as $\texttt{pa-asd}_{\mathcal{P}}\phi$.[14] The 'event of death of an individual' may be interpreted to last for exactly one instant, or for a longer finite period of time. Yet, we could contemplate a property such as *being dead* that lasts for a boundless period of time. In the below formulation ϕ may have any of these durations. The basic idea of $\texttt{pa-asd}_{\mathcal{P}}\phi$ can be stated as follows: when $\mathcal{P} < t < t'$ holds, $\mathcal{P}$ determines the

[14] $\texttt{pa-asd}_{\mathcal{P}}\phi$ is always false in LCP. $\texttt{pa-asd}_{\mathcal{P}}\phi$ is an *open game quantifier*, which states that a game ends after a finite number of steps, but it is not known at which step (cf. Kolaitis [18, p. 368]).

interval $[tt']$ where ϕ will be instantiated for the first time. $\texttt{pa-asd}_{\mathcal{P}}\phi$ can be formalized as the conjunction of propositions (a-d).

(a) From the aspect of $\mathcal{P}$, it is necessary that from the aspect of every $t'' \geq t'$ it will be necessary that ϕ is/was instantiated at least once in the interval $[tt']$: $\texttt{nec}_{\mathcal{P}}\texttt{nec}_{\forall t'' \geq t'}\phi_{\exists t'''(t \leq t''' \leq t')}$.

(b) From the aspect of $\mathcal{P}$, the instantiation of ϕ is contingent at t and t': $\texttt{con}_{\mathcal{P}}\phi_{t \wedge t'}$.

(c) From the aspect of $\mathcal{P}$, the instantiation of ϕ is contingent or impossible at each time in the interval $]tt'[$: $(\texttt{con} \vee \texttt{imp})_{\mathcal{P}}\phi_{\forall t''(t<t''<t')}$.

(d) From the aspect of $\mathcal{P}$, the instantiation of ϕ is impossible before t: $\texttt{imp}_{\mathcal{P}}\phi_{\forall t''<t}$.

4. Standard Temporal Operators

PA operators save the functionality of standard temporal operators for linear and branching systems. For linear systems, Prior [19, ch. 2] originally formulated the pair of operators F and P, and later the pair of G and H (Prior [20, ch. 10]). F, P, G, H are defined and their PA analogs (§3) are given in table 2.[15]

Table 2
Prior's temporal operators for linear systems and their PA analogs.

	title	formal definition	informal definition	PA
Fϕ:	future possibility	$\exists t(\mathcal{P} < t \wedge \phi_t)$	ϕ will sometimes be true	TB1.1,3,5
Gϕ:	future necessity	$\forall t(\mathcal{P} < t \rightarrow \phi_t)$	ϕ will always be true	TB1.2,4,6
Pϕ:	past possibility	$\exists t(t < \mathcal{P} \wedge \phi_t)$	ϕ was sometimes true	TB1.9
Hϕ:	past necessity	$\forall t(t < \mathcal{P} \rightarrow \phi_t)$	ϕ was always true	TB1.10

One may start with Fϕ and Pϕ, and define Gϕ as $\neg$F$\neg\phi$ ≡ "not sometimes after $\mathcal{P}$ not ϕ", and H as $\neg$P$\neg\phi$ ≡ "not sometimes before $\mathcal{P}$ not ϕ". Alternatively, one may start with Gϕ and Hϕ, and define Fϕ as $\neg$G$\neg\phi$, i.e., "not always after $\mathcal{P}$ not ϕ", and P as $\neg$H$\neg\phi$, i.e., "not always before $\mathcal{P}$ not ϕ". The backward-directed P and H are applicable also in forward-branching and backward-linear systems such as BCP, whereas the forward-directed F and G need to be reinterpreted in branching systems (§5). All PA operators are applicable in linear and branching systems.

The study of temporal operators for branching systems has been centered on details of Prior's [24, ch. VII] *Peircean* and *Ockhamist* operators (cf. Müller [25] and Rumberg [26]). The Priorian operators are infeasible from the aspect of computability, as they quantify over maximal linear subsets (defs. 14-15) of $\mathbb{U}$. *Complete future* operators (def. 16) quantify over maximal linear subsets of the set $\vec{t}$ of causal successors of t, and thus do the job of the Priorian operators in a simpler way. It is shown that *partial future* operators (def. 17) do the job of the complete future operators. Thereby, it is also shown that the partial future operators do the job of the

[15] The formal definitions are from Hodkinson and Reynolds [21, p. 672]. The informal definitions are from Gabbay et al. [22, p. 24]. F, G, P, H appear as *statistical modalities*, complemented by a point and a direction of evaluation. Knuuttila [23, p. 163] characterizes the statistical modalities, that were familiar in the antiquity and to the scholastics: "a temporally indefinite sentence is necessarily true if it is true whenever uttered, possibly true if it is true sometimes, and impossible if it is always false."

Priorian operators. In effect, it suffices to explain the basic idea of Peircean operators (def. 18). It is notable that the partial future operators are first-order PA operators, whereas the complete future operators and the Priorian operators are second-order, as they quantify over transfinite sets.

Definition 14. Linear subset. $lin(x, y) := x \subseteq y \wedge \forall z, v(z, v \in x \rightarrow z > v \vee z < v \vee z = v)$, where $x, y \in \mathbb{U}$. When x is a linear subset of y, x is a subset of y, and for every z and v that are elements of x, z is a causal successor or a causal predecessor of v, or $z = v$.

Definition 15. Maximal linear subset. $mlin(x, y) := lin(x, y) \wedge \nexists z(x \subseteq z \wedge lin(z, y) \wedge x \neq z)$, where $x, y \in \mathbb{U}$. When x is a maximal linear subset of y, x is a linear subset of y, and x is not a subset of any other linear subset of y.

Definition 16. Complete future operators and their PA analogs are represented in table 3.[16] Complete future operators quantify over futures f_t of the point of evaluation t, i.e., over maximal linear subsets of $\vec{t}$. E.g. $\mathtt{asd}_t\phi$ is read as: in all futures f_t of t, there is a $\mathcal{T}y$ that instantiates ϕ. Note that $\mathtt{pa}\text{-}\mathtt{asd}_{\mathcal{P}}\phi$ (def. 13) is not equivalent with $\mathtt{asd}_{\mathcal{P}}\phi$, and that ¬TB1.5 entails ¬TB1.6.[17]

Table 3
Complete future operators and their PA analogs.

	complete future op.	**PA**
$\mathtt{pos}_t\phi :=$	$\exists f_t \exists y \in f_t(\phi_y)$	$\mathtt{pos}_t\phi_{\exists t'>t}$ (TB1.1)
$\mathtt{asd}_t\phi :=$	$\forall f_t \exists y \in f_t(\phi_y)$	$\mathtt{pa}\text{-}\mathtt{asd}_t\phi$ (def. 13)
$\mathtt{con}_t\phi :=$	$\mathtt{pos}_t\phi \wedge \neg\mathtt{asd}_t\phi$	$\mathtt{pos}_t\phi_{\exists t'>t} \wedge \neg\mathtt{pa}\text{-}\mathtt{asd}_t\phi \wedge \neg\mathtt{nec}_t\phi_{\exists t'>t}$ (TB1-5)
$\mathtt{nec}_t\phi :=$	$\forall f_t \forall y \in f_t(\phi_y)$	$\mathtt{nec}_t\phi_{\forall t'>t}$ (TB1-6)
$\mathtt{imp}_t\phi :=$	$\forall f_t \nexists y \in f_t(\phi_y)$	$\mathtt{imp}_t\phi_{\forall t'>t}$ (TB1-7)

Definition 17. Partial future operators. Complete future operators can be transformed into operators that quantify over partial futures $f_{]tt']}$, i.e., over maximal linear subsets of a finite stretch of future possibilities $\overrightarrow{tt'} := \{x | x \in t \multimap t'' \wedge t < t'' \leq t'\}$. $\overrightarrow{tt'}$ is the set of all $\mathcal{T}$s that are accessible from t at $t + 1 \vee t + 2 \vee ... \vee t'$. E.g. $\mathtt{pos}_t\phi$ can be formulated as a partial future operator as $\exists f_{]tt']} \exists y \in f_{]tt']}(\phi_y)$, and as TB1.1'. Similarly for the other complete future operators.

Definition 18. Peircean operators quantify over histories, i.e., maximal linear subsets of $\mathbb{U}$, viz., complete world-lines or complete courses of events. In LCP $\mathbb{U}$ is a single history. In BCP $\mathbb{U}$ consists of several histories. History h_t that passes through point of evaluation t is defined as a maximal linear subset of $\mathbb{U}$ that contains t (in BCP $t \leq \mathcal{P}$). The Peircean version of $\mathtt{asd}\phi$ may be formulated as: $\forall h_t \exists y \in h_t(t < y \wedge \phi_y)$. It is read as: In all histories h_t that pass through t, there is a causal successor y of t that instantiates ϕ.

[16] The complete future operators conform to Galton's [11, p. 202] informal definitions. $\mathtt{asd}_{\mathcal{P}}\phi$ is close to Müller's [25, p. 361] and Rumberg's [26, p. 91] versions.

[17] TB1.5&6 entail $\mathtt{asd}_{\mathcal{P}}\phi$. TB1.6 is mutually exclusive with $\mathtt{pa}\text{-}\mathtt{asd}_{\mathcal{P}}\phi$. The compatibility of TB1.5 and $\mathtt{pa}\text{-}\mathtt{asd}_{\mathcal{P}}\phi$ depends on the nature of ϕ.

5. Interaction Theorems

In standard tense logic, *interaction-, connection-* or *converse axioms* GP, FH, PG, HF govern interaction of the past operators P, H and the future operators F, G.[18] The interaction axioms are implications. They are derived as theorems by showing that CP and an antecedent of an implication entail its consequent. The interaction axioms were originally applied in linear systems where "will be" has a deterministic meaning. To avoid falsity and indeterminacy in BCP, "will be" in GP, FH and PG is interpreted as PA necessity, and in HF as PA possibility.

Theorem 5. GP. Garson [27] writes GP as $A \to GPA$: "that what is the case (A), will at all future times, be in the past (GPA)." In PA format, GP is written as $\phi_{\mathcal{P}} \to \mathtt{nec}_{\forall t > \mathcal{P}} \phi_{\exists t' < t}$: If ϕ is the case at $\mathcal{P}$, then it will be necessary from the aspect of every $t > \mathcal{P}$ that ϕ was the case at least once before t.[19] It is proved that CP and $\phi_{\mathcal{P}}$ entail $\mathtt{nec}_{\forall t > \mathcal{P}} \phi_{\exists t' < t}$. (1) Every element of $\overrightarrow{\mathcal{P}}$ is forward-accessible from $\mathcal{P}$ (cor. 1), which entails that $\mathcal{P}$ is backward-accessible from every element of $\overrightarrow{\mathcal{P}}$. (2) Therefore, when $\phi_{\mathcal{P}}$ is true, it will be necessary from the aspect of every element of $\overrightarrow{\mathcal{P}}$ that ϕ holds for at least one backward-accessible $\mathcal{T}$, namely, for $\mathcal{P}$. ■

Theorem 6. FH. Girle [28, p. 123] writes FH as $FHp \to p$: "*If at some time in the future it always had been the case that p, then p is the case now*." In PA format, FH is written as $\mathtt{nec}_{\exists t > \mathcal{P}} \phi_{\forall t' < t} \to \phi_{\mathcal{P}}$: If it will be necessary from the aspect of at least one $t > \mathcal{P}$ that ϕ is the case at every $t' < t$, then ϕ is the case at $\mathcal{P}$.[20] It is proved that CP and $\mathtt{nec}_{\exists t > \mathcal{P}} \phi_{\forall t' < t}$ entail $\phi_{\mathcal{P}}$. (1) CP and $\mathtt{nec}_{\exists t > \mathcal{P}} \phi_{\forall t' < t}$ entail that every future (def. 16) has an element x, such that every $\mathcal{T}$ that is backward-accessible from x instantiates ϕ. (2) $\mathcal{P}$ is backward-accessible from every such x, and thus $\phi_{\mathcal{P}}$ is true. ■

Theorem 7. PG. Girle [28, p. 123] writes PG as $PGp \to p$: "*If from some time in the past it is always going to be the case that p, then p is the case now*." In PA format, PG is written as $\mathtt{nec}_{\exists t < \mathcal{P}} \phi_{\forall t' > t} \to \phi_{\mathcal{P}}$: If it was necessary from the aspect of at least one time before $\mathcal{P}$ that it will always be the case that ϕ, then ϕ is the case at $\mathcal{P}$. It is proved that CP and $\mathtt{nec}_{\exists t < \mathcal{P}} \phi_{\forall t' > t}$ entail $\phi_{\mathcal{P}}$. (1) CP and $\mathtt{nec}_{\exists t < \mathcal{P}} \phi_{\forall t' > t}$ entail that every element of $\vec{t}$ instantiates ϕ. (2) $\mathcal{P}$ is an element of $\vec{t}$, and thus $\phi_{\mathcal{P}}$ is true. ■

Theorem 8. HF. Garson [27] writes HF as $A \to HFA$: "that what is true now (A) has always been such that it will occur in the future (HFA)". The interpretation of F (will be the case at least once) as any version of necessity (def. 13, def. 16, TB1.5&6) renders HF false in BCP, where e.g. the fact that person a is the president at $\mathcal{P}$, does not imply that it has always been necessary that a will be the president. The interpretation of F as contingency (TB1.8) renders HF false in LCP and BCP. For instance, if a market crash at t was necessary some time before t, it was not always contingent that the crash will take place. The interpretation of F as neutrality (TB1.3) renders HF indeterminate in BCP, i.e., $A \to HFA$ appears sometimes as `true` → `indeterminate`, which is

[18] Hodkinson and Reynolds [21, p. 697], Garson [27] and Gabbay et al. [22, p. 29] apply GP and HF. Girle [28, p. 123] applies PG and FH.

[19] Recall that in BCP the initial aspect time must be $\mathcal{P}$ or earlier. Therefore, $\phi_{\mathcal{P}} \to \mathrm{nec}_{\forall t > \mathcal{P}} \phi_{\exists t' < t}$ can be seen as an abbreviation of $\phi_{\mathcal{P}} \to \mathrm{nec}_{\mathcal{P}} \mathrm{nec}_{\forall t > \mathcal{P}} \phi_{\exists t' < t}$.

[20] $\mathrm{nec}_{\exists t > \mathcal{P}} \phi_{\forall t' < t} \to \phi_{\mathcal{P}}$ is an abbreviation of $\mathrm{nec}_{\mathcal{P}} \mathrm{nec}_{\exists t > \mathcal{P}} \phi_{\forall t' < t} \to \phi_{\mathcal{P}}$.

indeterminate in the systems of Lukasiewicz and Kleene (Akama et al. [29]). For instance, if particle p goes through slit s at $\mathcal{P}$, and it has always been contingent that p will go through s, the proposition "It has always been neutral that p will go through s" is indeterminate (def. 10). Consequently, Surowik [30, p. 93] suggests that F in HF should be replaced by 'will possibly be' (TB1.1). This reading is applied below.

Corollary 3. $\mathrm{pos}_{\exists t \leq \mathcal{P}} \phi_{\exists t' > t} \rightarrow \mathrm{pos}_{\forall t'' < t} \phi_{t'}$: If it is/was possible from the aspect of a $t \leq \mathcal{P}$ that ϕ will be realized at a $t' > t$, then it was possible from the aspect of every $t'' < t$ that ϕ will be realized at t'.[21] It is proved that CP and $\mathrm{pos}_{\exists t \leq \mathcal{P}} \phi_{\exists t' > t}$ entail $\mathrm{pos}_{\forall t'' < t} \phi_{t'}$. (1) CP and $\mathrm{pos}_{\exists t \leq \mathcal{P}} \phi_{\exists t' > t}$ entail that there is such $t \leq \mathcal{P}$ and such $t' > t$ that at least one element x of $t \mathrel{-\!\!\in} t'$ instantiates ϕ. (2) $t \mathrel{-\!\!\in} t' \subset (t-1) \mathrel{-\!\!\in} t' \subset (t-2) \mathrel{-\!\!\in} t' \subset (t-3) \mathrel{-\!\!\in} t' \subset \ldots$ holds, i.e., x is an element of $(t-n) \mathrel{-\!\!\in} t'$, for all $n \geq 0$, and thus $\mathrm{pos}_{\forall t'' < t} \phi_{t'}$ is true. ■

In PA format, HF is written as $\phi_{\mathcal{P}} \rightarrow \mathrm{pos}_{\forall t < \mathcal{P}} \phi_{\exists t' > t}$: If ϕ is the case at $\mathcal{P}$, then it was possible from the aspect of every $t < \mathcal{P}$ that ϕ will be the case at least once after t. It is proved that CP and $\phi_{\mathcal{P}}$ entail $\mathrm{pos}_{\forall t < \mathcal{P}} \phi_{\exists t' > t}$. (1) CP and $\phi_{\mathcal{P}}$ entail that $\phi_{\mathcal{P}}$ is made possible by $\mathcal{P}$, i.e., that $\mathrm{pos}_{\mathcal{P}} \phi_{\mathcal{P}}$ is true. (2) If $\mathrm{pos}_{\mathcal{P}} \phi_{\mathcal{P}}$ is true, $\mathrm{pos}_{\forall t < \mathcal{P}} \phi_{\mathcal{P}}$ is true (cor. 3). Thus $\mathrm{pos}_{\forall t < \mathcal{P}} \phi_{\exists t' > t}$ is true. ■

6. Conclusions

The central primitives, concepts and semantics of tense logic have been founded on a commonsense ontology of time. The firm connection between tense logic and ontology of time shows that the two disciplines can be seen as a unified whole and studied as one rather than as two separate lines of inquiry. The given fusion sets a precedent for more thorough or different fusions. For instance, it would be interesting to see a comprehensive fusion of tense logic and a relativistic ontology of time.

Point-accessibility operators compose a more comprehensive and comprehensible toolset than standard modal operators. They provide ontologists, philosophers, knowledge engineers and programmers a better basis for formulating and programming temporal propositions, and for dealing with time-related issues. For instance, point-accessibility operators provide naturalist philosophers better chances of explicating truthmakers and assessing truth values of propositions about this world.

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[21] Von Wright [3, p. 98] gives an analogous characterization: "Assume now that at some time t' it is antecedently possible that p at a t. Then, regardless of whether this proposition comes true or not at t, it must have been antecedently possible already at any time before t'."

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