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ASYMPTOTICS OF THE SPECTRUM OF THE MIXED BOUNDARY VALUE PROBLEM FOR THE LAPLACE OPERATOR IN A THIN SPINDLE-SHAPED DOMAIN.

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ABSTRACT. We examine the asymptotics of solutions to the spectral problem for the Laplace operator in a d-dimensional thin, of diameter $O(h)$, spindle-shaped domain Ω^h with the Dirichlet condition on small, of size $h \ll 1$, terminal zones $\Gamma^h_\pm$ and the Neumann condition on the remaining part of the boundary $\partial\Omega^h$. In the limit as $h \to +0$ an ordinary differential equation on the axis $(-1,1) \ni z$ of the spindle appears with a coefficient degenerating at the points $z = \pm 1$ and besides without any boundary condition because the requirement on boundness of eigenfunctions makes the limit spectral problem correct. We derive error estimates of the one-dimensional model but in the case $d = 3$ it is necessary to construct boundary layers near the sets $\Gamma^h_\pm$ and in the case $d = 2$ to deal with self-adjoint extensions of the differential operator. The extension parameters depend linearly on $\ln h$ so that its eigenvalues imply analytical functions in the variable $1/|\ln h|$. As a result, in all dimensions the one-dimensional model gets the power-law accuracy $O(h^{\delta_d})$ with an exponent $\delta_d > 0$. First (the smallest) eigenvalues, positive in $\Omega^h$and null in $(-1,1)$ require individual treatment. We also discuss infinite asymptotic series, the static problem (without the spectral parameter) and alike shapes of thin domains.

Dedicated to Vasilii Mikhailovich Babich,
who knows everything about boundary layers.

1. Introduction.

1.1. Problem statement. Let ω be a domain in the Euclidean space $\mathbb{R}^{d-1}$ with $(d-2)$-dimensional boundary $\partial\omega$ (which is assumed to be C^∞-smooth for simplicity, cf. Section 2 § 6) and compact closure $\overline{\omega} = \omega \cup \partial\omega$, and let Ω^h be a thin spindle-shaped body as in Fig. 1,a

$$\Omega^h = \{x = (y,z) \in \mathbb{R}^{d-1} \times \mathbb{R} : \eta := h^{-1}H(z)^{-1}y \in \omega, z \in (-1,1)\}, \tag{1.1}$$

which is simply called a spindle in the following. Here, $d \geqslant 2$, h is a small positive parameter and $H \in C^\infty[-1,1]$ is a profile function fulfilling the relations

$$\begin{aligned} &H(\pm 1) = 0, \quad H(z) > 0 \text{ for } z \in (-1,1), \\ &H(z) = (1 \mp z)(H_\pm + \widetilde{H}_\pm(1 \mp z)) \text{ for } \pm z \in [0,1], \end{aligned} \tag{1.2}$$

where $H_\pm$ are positive numbers and the function $t \mapsto \widetilde{H}_\pm(t)$ belongs to $C^\infty[0,1]$ with $\widetilde{H}_\pm(0) = 0$; for technical reasons we assume that the functions $\widetilde{H}_\pm$ are real analytic in some neighborhoods of $t = 0$, i.e., for some $t_0 > 0$ and $\{a_{n,\pm}\}_{n=1}^\infty \in \mathbb{R}$ we have

$$\widetilde{H}_\pm(t) = \sum_{n=1}^\infty a_{n,\pm} t^n, \quad t \in [0, t_0]. \tag{1.3}$$

Key words and phrases. Spindle-shaped thin domain, asymptotics of eigenvalues, boundary layers, self-adjoint extensions.

We rescale the length of the body (1.1) in the $z = x_d$-axis direction to be two, hence, the Cartesian coordinates $x = (x_1, \ldots, x_d)$ and geometric parameters are made dimensionless and, in particular, the meaning of the restriction $h \ll 1$ becomes clear.

Also, fixing the numbers $\rho_\pm > 0$ we denote the top and middle parts of the boundary $\partial\Omega^h$, respectively, by

$$\Gamma^h_\pm = \{x \in \partial\Omega^h : 1 > \pm z > 1 - h\rho_\pm\} \tag{1.4}$$

and $\Gamma^h_\circ = \partial\Omega^h \setminus (\overline{\Gamma^h_+ \cup \Gamma^h_-})$. In the domain (1.1) we consider the following spectral mixed boundary value problem for the Laplace operator $\Delta_x = \nabla_x \cdot \nabla_x$ with the Dirichlet condition on $\Gamma^h_\pm$ and the Neumann one on $\Gamma^h_\circ$:

$$-\Delta_x u^h(x) = \lambda^h u^h(x), \quad x \in \Omega^h, \tag{1.5}$$

$$u^h(x) = 0, \quad x \in \Gamma^h_\pm, \tag{1.6}$$

$$\partial_\nu u^h(x) = 0, \quad x \in \Gamma^h_\circ. \tag{1.7}$$

Here, $\nabla_x = \text{grad}$, $\partial_\nu = \nu(x) \cdot \nabla_x$, the central dot stands for the scalar product in the space $\mathbb{R}^m$, $\nu(x)$ is the unit outward normal vector on the surface $\partial\Omega^h \setminus \{P^+, P^-\}$ and $P^\pm = \{0, \ldots, 0, \pm1\}$ are tips of the conical (angular in the case $d = 2$) sharpenings.

The variational formulation of the problem (1.5)–(1.7) consists of the integral identity [1]

$$(\nabla_x u^h, \nabla_x \psi^h)_{\Omega^h} = \lambda^h (u^h, \psi^h)_{\Omega^h} \quad \forall \psi^h \in H^1_0(\Omega^h; \Gamma^h_\pm), \tag{1.8}$$

where $(\ ,\ )_{\Omega^h}$ is the natural scalar product in the Lebesgue space $L^2(\Omega^h)$ of either scalar or vector valued functions and $H^1_0(\Omega^h; \Gamma^h_\pm)$ denotes the subspace of functions in the Sobolev space $H^1(\Omega^h)$ satisfying the Dirichlet condition (1.6). The bilinear form on the left-hand side of (1.8) is closed and positive definite in the space $H^1_0(\Omega^h; \Gamma^h_\pm)$ and, therefore, according to [2, Ch. 10] the problem (1.8) or (1.5)–(1.7) is associated with an unbounded positive definite self-adjoint operator A^h in the Hilbert space $L^2(\Omega^h)$ with the domain $\mathcal{D}(A^h) \subset H^1_0(\Omega^h; \Gamma^h_\pm)$. We emphasize that $\mathcal{D}(A^h)$ is much larger than the space $H^2(\Omega^h) \cap H^1_0(\Omega^h; \Gamma^h_\pm)$, because of the square-root singularities of the derivatives of the solutions to the mixed boundary value problems on the collision surfaces $\{x \in \partial\Omega^h : \pm z = 1 - h\rho_\pm\}$, which are lines in the case $d = 3$ and points, if $d = 2$; see [3, 4] [5, Ch. 2] etc. Also, owing to the compactness of the embedding $H^1(\Omega^h) \subset L^2(\Omega^h)$ (see, e.g., [1]) the spectrum of the operator A^h is discrete and forms a monotone unbounded positive sequence

$$0 < \lambda^h_1 < \lambda^h_2 \leqslant \lambda^h_3 \leqslant \cdots \leqslant \lambda^h_n \leqslant \cdots \to +\infty \tag{1.9}$$

composed by taking into account the multiplicities of the eigenvalues (see, e.g., [2, Thm. 10.1.5 and 10.2.2]). The corresponding eigenfunctions $u^h_1, u^h_2, u^h_3, \ldots, u^h_n, \cdots \in H^1_0(\Omega^h; \Gamma^h_\pm)$ can be subject to the normalization and orthogonality conditions

$$(u^h_n, u^h_m)_{\Omega^h} = \delta_{n,m}, \quad n, m \in \mathbb{N}, \tag{1.10}$$

where $\delta_{n,m}$ is the Kronecker symbol and $\mathbb{N} = \{1, 2, 3, \ldots\}$ the set of natural numbers.

1.2. Motivation. Thin elastic structures like bodies, plates, shells and rods appear everywhere in biotic and abiotic nature as well as in mechanics and civil engineering, and consequently the amount of research and publications on deformations of such objects, starting from the nineteenth century, cannot be evaluated. These studies represent different levels of rigor and various methods, like asymptotic ones. Here, it is usually essential to perform a dimension reduction procedure leading to some differential equation: for example, the two-dimensional image of a thin flattened solid is called the Kirchhoff plate and computation of its bend is based on the Sophie Germain equation (see, e.g., the book [6]).

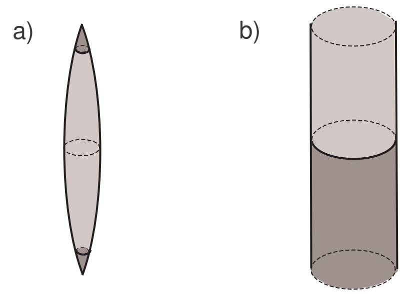

FIGURE 1.1. The orginal spindle (a) and the artificially constructed cylinder (b) for the description of the boundary layer. The Dirichlet zones are shaded.

In the introductory presentations on the theory of plates and rods, asymptotic analysis of boundary value problems for scalar second-order equations is usually regarded as elementary preliminary material. For example, in cylindrical domains, the Laplace and Helmholtz equations can be simply treated by separation of variables, which makes the dimension reduction procedure elementary. Thus, it may seem that hardly any new scalar problems in thin domains are worth scientific attention; however, the authors present here such a novel case.

Thin *spindle-shaped objects* (Fig. 1, a) are not so common as thin cylindrical ones, but examples of them are shuttles in sewing machines and looms; also, this sharpened cigar-shape facilitates the liquid flow and therefore it is typical for marine reptiles and fishes. The shape of the spindle is fixed just by the conical tops (cf. definition in (1.4)). The spectral problem (1.5)–(1.7) in Ω^h and the corresponding stationary one are related to the distribution of heat (a metal skewer inside chopped meat could be considered as a spindle, but to model this setting one needs a serious modification of the problem statement which we skip, since one is hardly interested in warming a spindle while processing kebab). Finally, we point out that we do not know precisely the asymptotic structures for the corresponding, much more complicated vectorial problem of the linear elasticity. We mention that in the papers [7, 8], rods with paraboloidal instead of conical ends were considered (see Section 1 § 6).

The interest in the asymptotic analysis of the problem (1.5)–(1.7) is due to a specific boundary layer phenomenon. Namely, although ε^{-1} seems at the first sight to be the correct coordinate dilation coefficient, it turns out the the proper one is ε^{-2}. Also, the limit problem is posed in a cylinder instead of a cone (see Fig. 1, b and Section 1 § 4). In the most interesting situations $d = 3$ and $d = 2$ the boundary layer affects the derivation of the error estimates for the one-dimensional model of the spindle, but it also has effects on the model itself. In the case $d = 2$ the model consists of a self-adjoint extension of a differential operator with degenerating coefficients at the points $z = \pm 1$. Exclusivity of the two-dimensional case $d = 2$ is explained in Section 5 § 6, in particular, we discuss analytic dependence of the main asymptotic terms of eigenvalues in the variable $1/|\ln h|$.

1.3. Content and structure of the paper. First of all, in § 2 we perform the formal asymptotic analysis of the eigenpairs $\{\lambda_n^h, u_n^h\}$ of the mixed boundary value problem (1.5)–(1.7) and present all basic properties of the derived limit problem, which is a degenerate differential equation (2.9) with no boundary conditions at the points $z = \pm 1$. Its variational formulation (2.14) is posed in a weighted Sobolev-type space .

The third section starts with the verification of the simplest assertion about asymptotics, that is, the traditional convergence theorem (Lemma 3.1). We then prove asymptotic formulas for the eigenvalues λ_n^h and eigenfunctions u_n^h (Theorems 3.3 and 3.7) but only under the restriction $d \geqslant 4$. As was mentioned, the spatial case $d = 3$ requires the construction of the boundary layer and we do this in § 4 (see, in particular, Theorem 4.1). Furthermore, Theorem 4.5 refines the asymptotic expression of the first eigenvalue $\lambda_1^h > 0$, (1.9). This is necessary since the smallest eigenvalue μ_1 of the limit equation is null, and the result of Theorem 3.3 thus does not give any useful information on the behaviour of $\lambda_1^h > 0$ as $h \to +0$.

In § 5 we proceed with studying the unbounded symmetric operator

$$-H^{1-d}\partial_z H^{d-1}\partial_z$$

in the weighted Lebesgue space. We show that, for $d \geqslant 4$, this operator itself is self-adjoint, but, for $d = 3$ and $d = 2$, it requires a self-adjoint extension. One such extension and its parameters will be found by an analysis of the boundary layer. This leads to an appropriate one-dimensional model of the planar spindle $\Omega^h \subset \mathbb{R}^2$ and provides the accuracy $O(h^{1/2})$ for the eigenvalues λ_n^h (see Theorem 5.8). We emphasize that the usual variational formulation of the limit problem (Section 2 § 2) is associated with the Friedrichs extension but, for $d = 2$, generates too large approximation errors $O((1 + |\ln h|)^{-1})$ (cf., Proposition 5.7).

In the last § 6 we discuss several possible generalisations of the results, in particular, the inhomogeneous stationary ($\lambda^h = 0$) boundary value problem in Ω^h whose operator norm grows unboundedly as $h \to +0$ (Proposition 4.6). Furthermore, we outline iterative processes to construct infinite asymptotic series for the eigenpairs of the spectral problem (1.5)–(1.7) and we describe modifications of the asymptotic procedures in the cases of rounded or truncated ends of spindle. Here, we do not prove precise assertions but only sketch asymptotic structures.

2. Formal derivation and examination of the limit equation.

2.1. Asymptotic analysis. We accept the standard ansätze (see, e.g., [9, § 1 Ch. 7]) for the eigenpairs of problem (1.5)–(1.7)

$$\lambda^h = \mu + \dots \tag{2.1}$$

and

$$u^h(x) = \mathbf{v}(z) + h^2 V(\eta, z) + \dots, \tag{2.2}$$

where $\eta = h^{-1}H(z)^{-1}y$ are stretched transversal coordinates, the number μ and the functions $\mathbf{v}, V$ are to be determined and dots stand for higher-order terms of no importance for the formal analysis.

We insert the ansätze (2.1) and (2.2) into equation (1.5) and collect coefficients of the same powers of the small parameter h. The main term on the right-hand side of (2.2) depends only on the "slow" variable z and the changes $y \mapsto \eta$, $z \mapsto \zeta = z$ lead to the following differentiation formulas for the composite functions,

$$\begin{aligned} &\nabla_y V(h^{-1}H(z)^{-1}y, z) = h^{-1}H(\zeta)^{-1}\nabla_\eta V(\eta,\zeta),\\ &\partial_z V(h^{-1}H(z)^{-1}y, z) = \partial_\zeta V(\eta,\zeta) - H(\zeta)^{-1}\partial_\zeta H(\zeta)\,\eta\cdot\nabla_\eta V(\eta,\zeta), \end{aligned} \tag{2.3}$$

so that we obtain the equation

$$-H(z)^{-2}\Delta_\eta V(\eta, z) = F(\eta, z) := \partial_z^2\mathbf{v}(z) + \mu\mathbf{v}(z), \quad \eta \in \omega. \tag{2.4}$$

The unit outward normal vector on the surface $\Gamma_\circ^h$ takes the form

$$\left(1 + h^2|\partial_z H(z)|^2|\eta\cdot\nu'(\eta)|^2\right)^{-1/2}(\nu'(\eta), -h\partial_z H(z)\,\eta\cdot\nu'(\eta)), \tag{2.5}$$

where ν' is the $(d-1)$-dimensional unit outward normal vector on the boundary $\partial\omega \subset \mathbb{R}^{d-1}$. Thus,

$$\partial_\nu(\mathbf{v}(z) + h^2 V(\eta, z)) = h\left(H(z)^{-1}\partial_{\nu'(\eta)}V(\eta, z) - \partial_z H(z)\eta\cdot\nu'(\eta)\partial_z\mathbf{v}(z)\right) + \dots,$$

and, therefore, equation (2.4) is supplied with the boundary condition

$$H(z)^{-2}\partial_{\nu'(\eta)}V(\eta,z)=G(\eta,z):=H(z)^{-1}\partial_z H(z)\eta\cdot\nu'(\eta)\partial_z\mathbf{v}(z),\quad \eta\in\partial\omega. \tag{2.6}$$

Now according to the obvious formula

$$\int\limits_{\partial\omega}\eta\cdot\nu'(\eta)\,ds_\eta=\int\limits_{\omega}\nabla_\eta\cdot\eta\,d\eta=(d-1)|\omega|,$$

where $|\omega|=\mathrm{mes}_{d-1}\omega$ is the $(d-1)$-dimensional area of the cross-section ω, the compatibility condition in the Neumann problem (2.4), (2.6) converts into

$$\begin{aligned}0&=\int\limits_{\omega}F(\eta,z)d\eta+\int\limits_{\partial\omega}G(\eta,z)\,ds_\eta\\&=|\omega|(\partial_z^2\mathbf{v}(z)+\mu\mathbf{v}(z))+(d-1)|\omega|H(z)^{-1}\partial_z H(z)\partial_z\mathbf{v}(z).\end{aligned} \tag{2.7}$$

The solution V is fixed by the orthogonality condition

$$\int\limits_{\omega}V(\eta,z)d\eta=0,\quad z\in(-1,1). \tag{2.8}$$

We multiply the left and right sides of the relation (2.7) by $|\omega|^{-1}H(z)^{d-1}$ and finally arrive at the ordinary differential equation

$$-\partial_z(H(z)^{d-1}\partial_z\mathbf{v}(z))=\mu H(z)^{d-1}\mathbf{v}(z),\quad z\in(-1,1). \tag{2.9}$$

Due to assumptions (1.2) the coefficient of the second order derivative degenerates at the endpoints $P^\pm$ of the interval $(-1,1)$.

2.2. On the spectrum of the degenerate equation. The assertions collected in this section are quite simple and mainly known but we give short proofs for them, because the notation will be used later.

The weighted Sobolev and Lebesgue spaces $\mathbf{V}^1_\beta(-1,1)$ and $L^2_\beta(-1,1)$ are obtained as the completion of the space $C_c^\infty(-1,1)$ (infinitely differentiable and compactly supported functions) with respect to the norms

$$\|\mathbf{v};\mathbf{V}^1_\beta(-1,1)\|=\Big(\int\limits_{-1}^{1}(1-|z|)^{2\beta}(|\partial_z\mathbf{v}(z)|^2+|\mathbf{v}(z)|^2)\,dz\Big)^{1/2}, \tag{2.10}$$

$$\|\mathbf{f};L^2_\beta(-1,1)\|=\Big(\int\limits_{-1}^{1}(1-|z|)^{2\beta}|\mathbf{f}(z)|^2\,dz\Big)^{1/2}. \tag{2.11}$$

Lemma 2.1. *For $d\geqslant 2$, the space $\mathbf{V}^1_{(d-1)/2}(-1,1)$ contains all smooth functions on the closed interval $[-1,1]\ni z$.*

Proof. Let $\mathbf{v}\in C^\infty[-1,1]$ and $\mathbf{v}^\varepsilon=\chi^\varepsilon\mathbf{v}$ where $\varepsilon\in(0,1)$,

$$\chi^\varepsilon(z)=\chi\Big(\frac{1}{\varepsilon}(1-|z|)\Big)\text{ for }d\geqslant 3\ \text{ and }\ \chi^\varepsilon(z)=1-\chi\Big(\frac{\ln(1-|z|)}{\ln\varepsilon}\Big)\text{ for }d=2, \tag{2.12}$$

and $\chi\in C^\infty(\mathbb{R})$ is the reference cut-off function,

$$\chi(t)=1\text{ for }t\geqslant 1,\quad \chi(t)=0\text{ for }t\leqslant\frac{1}{2},\quad 0\leqslant\chi\leqslant 1. \tag{2.13}$$

For $\varepsilon \to +0$, we have

$$\|\mathbf{v}^\varepsilon - \mathbf{v}; \mathbf{V}^1_{(d-1)/2}(-1,1)\|^2 = \int_{-1}^{1} (1-|z|)^{d-1} \Big(\Big| \frac{d}{dz}(1-\chi^\varepsilon(z))\mathbf{v}(z)\Big|^2 + |(1-\chi^\varepsilon(z))\mathbf{v}(z)|^2 \Big)\, dz \leqslant c_{\mathbf{v}} \delta(\varepsilon),$$

where the factor $c_{\mathbf{v}}$ depends on $\mathbf{v}$ while $\delta(\varepsilon)$ vanishes on the limit, $\delta(\varepsilon) = O(\varepsilon^{d-2})$ for $d > 2$ and $\delta(\varepsilon) = O(|\ln \varepsilon|^{-1})$ for $d = 2$. This, exactly, was to be checked. ⊠

Thus, equation (2.9) does not need any boundary condition at the points $z = \pm 1$ (cf., the book [6, § 36] as well as the papers [10, 11, 12] and others about degenerate elliptic equations). Hence, according to formulas (1.2), we state the variational formulation of the limit degenerate equation in the space $\mathbf{H} = \mathbf{V}^1_{(d-1)/2}(-1,1)$, and it reduces to the integral identity

$$(H^{d-1}\partal_z \mathbf{v}, \partial_z \varphi) = \mu(H^{d-1}\mathbf{v}, \varphi) \qquad \forall \varphi \in \mathbf{H}, \tag{2.14}$$

where $(\ ,\ )$ is the extension of the scalar product $L^2(-1,1)$ up to the duality between appropriate weighted spaces $L^2_\beta(-1,1)$ and $L^2_{-\beta}(-1,1)$.

Lemma 2.2. *The norm* (2.10) *with the weight exponent* $\beta = (d-1)/2$ *is equivalent to*

$$\|\mathbf{v}; \mathbf{H}\| = \Big(\int_{-1}^{1} (1-|z|)^{d-1} \Big(|\partial_z \mathbf{v}(z)|^2 + (1-|z|)^{-2}\big(1 + \delta_{d,2}\big(1 + |\ln(1-|z|)|\big)\big)^{-2} |\mathbf{v}(z)|^2 \Big)\, dz \Big)^{1/2}. \tag{2.15}$$

Proof. It suffices apply to the products $U(t) = (1-\chi(t))\mathbf{v}(\pm 1 \mp t)$ the following variants of the one-dimensional Hardy inequality:

$$\int_0^1 t^{2\alpha-1} |U(t)|^2\, dt \leqslant \frac{1}{\alpha^2} \int_0^1 t^{2\alpha+1} |\partial_t U(t)|^2\, dt \ \ \forall U \in C_c^\infty[0,1),\ \alpha = \frac{d}{2} - 1 > 0 \text{ for } d \geqslant 3,$$

$$\int_0^1 t^{-1} |\ln t|^{-2} |U(t)|^2\, dt \leqslant 4 \int_0^1 t |\partial_t U(t)|^2\, dt \ \ \forall\ U \in C_c^\infty[0,1) \text{ for } d = 2. \tag{2.16}$$

In the both cases the required relation $U(1) = 0$ follows from the definition (2.12) of the cut-off function χ. ⊠

Lemma 2.3. *The embedding of the space* $\mathbf{H}$ *into the space* $\mathbf{L} = L^2_{(d-1)/2}(-1,1)$ *with the norm* $\|\mathbf{v}; \mathbf{L}\| = \|H^{(d-1)/2}\mathbf{v}; L^2(-1,1)\|$ *is compact.*

Proof. The embedding operator $\mathbf{H} \subset \mathbf{L}$ is approximated in the operator norm as $\varepsilon \to +0$ by compact operators of multiplication with the cut-off functions (2.12) because the weight multiplier of $|\mathbf{v}|^2$ in the norm (2.15) is larger than the weight multiplier $(1-|z|)^\beta = (1-|z|)^{d-1}$ in the norm (2.11) of the space $\mathbf{L}$. ⊠

The left-hand side of the integral identity (2.14) includes a positive and closed bilinear form in $\mathbf{H}$ which according to [2, § 10.1] is associated with an unbounded positive self-adjoint operator $\mathbf{A}$ in $\mathbf{L}$ with the domain $\mathcal{D}(\mathbf{A}) \subset \mathbf{H}$. Finally, by Theorems 10.1.5 and 10.2.2 [2] the spectrum of the operator $\mathbf{A}$ consists of the monotone non-negative sequence of eigenvalues

$$0 = \mu_1 < \mu_2 < \mu_3 < \cdots < \mu_n \leqslant \cdots \to +\infty. \tag{2.17}$$

The corresponding eigenfunctions $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n, \dots \in \mathbf{H}$ can be subject to the normalization and orthogonality conditions

$$(H^{d-1}\mathbf{v}_m, \mathbf{v}_n) = \delta_{m,n}, \quad m, n \in \mathbb{N}. \tag{2.18}$$

The first eigenvalue is null and the corresponding eigenfunction is constant. We will prove in Lemma 2.4) that all eigenvalues are simple.

The model equation, which describes the behaviour of eigenfunctions near the endpoints $z = \pm 1$, is obtained by the changes $z \mapsto t = 1 \mp z$ and $H(z) \mapsto tH_\pm$ in the operator on left-hand side of(2.9) and looks as follows:

$$-\frac{d}{dt}\Big((H_\pm t)^{d-1}\frac{d\mathcal{V}}{dt}(t)\Big) = 0, \quad t \in (0, +\infty). \tag{2.19}$$

In addition to the constant solution $\mathcal{V}^0(t) = 1$, this equation has the singular solution

$$\mathcal{V}^1(t) = t^{2-d} \text{ for } d \geqslant 2 \quad \text{and} \quad \mathcal{V}^1(t) = \ln t \text{ for } d = 2. \tag{2.20}$$

We set $\mathcal{V}^1_\pm(z) = \mathcal{V}^1(1 \mp z)$. Functions with the singularities $O(\mathcal{V}^1_\pm(z))$ do not belong to the "energy" class $\mathbf{H}$.

Lemma 2.4. *The eigenfunctions $\mathbf{v}_n$ are infinitely many times differentiable on the closed interval $[-1, 1]$ and*

$$\mathbf{v}_n(\pm 1) \neq 0, \quad \partial_z \mathbf{v}_n(\pm 1) = 0 \tag{2.21}$$

so that the eigenvalue μ_n cannot be multiple.

Proof. Let us fix an eigenvalue μ_n in the sequence (2.17). Taking into account (1.2) and (1.3), equation (2.9) for the eigenfunction $\mathbf{v}_n$ can be written in a neighborhood of $z = 1$ as

$$P(z)\partial_z^2 \mathbf{v}_n(z) + Q(z)\partial_z \mathbf{v}_n(z) + R_n(z)\mathbf{v}_n(z) = 0 \tag{2.22}$$

where, according to the assumptions made in §1, the functions P, Q and R have absolutely convergent series

$$P(z) = H(z)^{d-1} = H_+^{d-1}(1-z)^{d-1} + \sum_{j=d}^{\infty} b_{j,1}(1-z)^j,$$

$$Q(z) = (d-1)H(z)^{d-2}H'(z) = -(d-1)H_+^{d-1}(1-z)^{d-2} + \sum_{j=d-1}^{\infty} b_{j,2}(1-z)^j,$$

$$R_n(z) = \mu_n H_+^{d-1}(1-z)^{d-1} + \sum_{j=d}^{\infty} b_{j,3}(1-z)^j$$

with some $b_{j,k} \in \mathbb{R}$. There holds

$$p(z) := \frac{Q(z)}{P(z)} = \frac{1}{1-z}\sum_{j=0}^{\infty} p_j(1-z)^j, \quad q(z) := \frac{R_n(z)}{P(z)} = \frac{1}{(1-z)^2}\sum_{j=0}^{\infty} q_j(1-z)^j$$

for the coefficients $p_j, q_j \in \mathbb{R}$ with $p_0 = d-1$, $q_0 = q_1 = 0$, $q_2 = \mu_n$ so that $z = 1$ is a regular singular point for the equation (2.22). Moreover, the Frobenius ansatz

$$\mathbf{v}(z; s) = (z-1)^s \sum_{j=0}^{\infty} r_j(s)(z-1)^j, \tag{2.23}$$

where $s \in \mathbb{R}$ and $r_0 \neq 0$, leads to the indicial equation

$$I(s) := s(s-1) + (d-1)s = 0, \tag{2.24}$$

with the indicial polynomial I having zeros at $s = 0$ and $s = -d+2$. According to the general theory (see [13, Theorem 11.6.46] or other textbooks on singular linear second order ordinary differential equations), the power series on the right of (2.23) converges near the point 1 and the expression (2.23) with $s = 0$ is always a (bounded) solution of (2.22) in a neighborhood of $z = 1$. Moreover, in the case $d \geqslant 3$, the function (2.23) with $s = -d+2$ is also a solution having the singularity $(1-z)^{2-d}$ at $z = 1$, and in the case $d = 2$ the second solution of (2.22) is known to have the singularity $\ln|1-z|$. As it was already remarked, a function $\mathbf{v}$ with such irregular behaviour cannot belong to $\mathbf{H}$. Since the eigenfunction $\mathbf{v}_n$ lives in $\mathbf{H}$, it must coincide with (2.23) for $s = 0$. Hence, $\mathbf{v}_n$ is smooth in a neighborhood of $z = 1$.

The first relation (2.21) holds at $z = 1$ because $r_0 \neq 0$ in the ansatz (2.23). According to formula (11.6.48) in [13], the coefficient r_1 is obtained from the relation

$$I(s+1)r_1(s) = -(p_1 s + q_1)r_0$$

for the indicial polynomial (2.24), where the parameter value is set to $s = 0$. Since $I(1) \neq 0$ and $q_1 = 0$, this yields $r_1 = 0$ and thus the second relation in (2.21) holds true, too.

The same argument applies also in a neighborhood of $z = -1$, thus, $\mathbf{v}_n \in C^\infty[-1,1]$. Furthermore, the existence of two linearly independent solutions implies that their linear combination vanishes at the point $z = 1$ and the series (2.23) for it is null on $[1-t_0, 1]$ and therefore everywhere on [-1,1]. In other words, eigenvalues (2.17) are simple. ⊠

Lemma 2.4 and relations (1.2) yield the following assertion.

Corollary 2.5. *The solution of the problem* (2.4), (2.6) *which corresponds to* $\mathbf{v}_n$ *and satisfies* (2.8), *meets the estimate*

$$H(z)^{-1}|V_n(\eta,z)| + |\nabla_{(\eta,z)}V_n(\eta,z)| + |\nabla^2_{(\eta,z)}V(\eta,z)| \leqslant c_n. \quad ⊠ \tag{2.25}$$

3. Limit passage in the spectral problem.

3.1. The convergence theorem. Let λ_n^h be an entry in the sequence (1.9) with the fixed number $n \in \mathbb{N}$ and let u_n^h be the corresponding eigenfunction of problem (1.5)–(1.7) normalized in $L^2(\Omega^h)$ according to (1.10). We will show in Remark 3.5 that there exist positive $h^{(n)}$ and $c^{(n)}$ such that

$$\lambda_n^h \leqslant c^{(n)} \quad \text{for} \quad h \in (0, h^{(m)}]. \tag{3.1}$$

Thus, one finds a sequence $\{h_j\}_{j\in\mathbb{N}}$ convergent to null such that

$$\lambda_n^{h_j} \ \to \ \widehat{\mu}_n \geqslant 0 \ \text{ as } j \to +\infty. \tag{3.2}$$

In this section we omit the indexes j and n in h_j and $\lambda_n^{h_j}$, $u_n^{h_j}$. From formulas (1.8), (1.10) and (3.1) we derive the relation

$$\|\nabla_x u^h; L^2(\Omega^h)\|^2 = \lambda^h \|u^h; L^2(\Omega^h)\|^2 \leqslant c. \tag{3.3}$$

We set

$$\overline{u}^h(z) = \frac{1}{|\omega^h(z)|} \int\limits_{\omega^h(z)} u^h(y,z)\,dy, \tag{3.4}$$

$$u_\perp^h(yz) = u^h(y,z) - \overline{u}^h(z), \qquad \int\limits_{\omega^h(z)} u_\perp^h(y,z)\,dy = 0. \tag{3.5}$$

Here,

$$\omega^h(z) = \{y : (y,z) \in \Omega^h\} \tag{3.6}$$

and

$$|\omega^h(z)| = h^{d-1}H(z)^{d-1}|\omega| \tag{3.7}$$

are the cross-section of the spindle (1.1) and its $(d-1)$-dimensional area. Recalling the definition of the weighted norm in Section 2 § 2 and taking into account formulas (3.4), (3.3) and (1.10), we write down the inequalities

$$\|\overline{u}^h; \mathbf{L}\|^2 = \int_{-1}^{1} H(z)^{d-1}|\overline{u}^h(z)|^2\,dz = \frac{1}{h^{2(d-1)}|\omega|^2}\int_{-1}^{1}\frac{1}{H(z)^{d-1}}\Big(\int_{\omega^h(z)} u^h(y,z)\,dy\Big)^2 dz \leqslant$$

$$\leqslant \frac{1}{h^{d-1}|\omega|}\int_{-1}^{1}\int_{\omega^h(z)} |u^h(y,z)|^2\,dydz = \frac{1}{|\omega|}h^{1-d} \tag{3.8}$$

and

$$\|\partial_z\overline{u}^h; \mathbf{L}\|^2 = \frac{1}{|\omega|^2}\int_{-1}^{1} H(z)^{d-1}\Big|\frac{\partial}{\partial z}\int_{\omega} u^h(hH(z)\eta, z)\,d\eta\Big|^2 dz$$

$$\leqslant c\int_{-1}^{1} H(z)^{d-1}\Big|\int_{\omega}\big(|\partial_z u^h(hH(z)\eta,z)| + h|\partial_z H(z)|\,|\eta\cdot\nabla_y u^h(hH(z)\eta,z)|\big)\,d\eta\Big|^2 dz$$

$$\leqslant ch^{2(1-d)}\int_{-1}^{1} H(z)^{1-d}\Big(\int_{\omega^h(z)}\big(|\partial_z u^h(y,z)| + h|\nabla_y u^h(y,z)|\big)\,dy\Big)^2 dz$$

$$\leqslant \frac{c}{h^{d-1}}\|\nabla_x u^h; L^2(\Omega^h)\|^2 \leqslant ch^{1-d}.$$

Thus, we can pass to another subsequence such that, with a redefinition of the notation $\{h_l\}_{j\in\mathbb{N}}$, there holds the weak convergence

$$h_j^{(d-1)/2}\overline{u}_n^{h_j} \ \to\ \widehat{\mathbf{v}}_n \in \mathbf{H} = \mathbf{V}^1_{(d-1)/2}(-1,1) \ \text{ as } j\to+\infty, \tag{3.9}$$

which implies strong convergence in the space $\mathbf{L} = L^2_{(d-1)/2}(-1,1)$ in view of Lemma 2.3.

Finally, due to the orthogonality condition (3.5) we can apply the Poincaré inequality in the domain (3.6) of diameter $O(hH(z))$ to obtain the relation

$$h^{-2}\int_{\Omega^h} H(z)^{-2}|u_\perp^h(y,z)|^2\,dydz \leqslant c_\omega\int_{\Omega^h}|\nabla_y u_\perp^h(y,z)|^2\,dx$$

$$= c_\omega\int_{\Omega^h}|\nabla_y(u_\perp^h(y,z) + \overline{u}^h(z))|^2\,dx \leqslant c_\omega\|\nabla_x u^h; L^2(\Omega^h)\|^2 \leqslant C. \tag{3.10}$$

Hence, owing to this, we have

$$1 = \int_{\Omega^h}|\overline{u}^h(z) + u_\perp^h(y,z)|^2\,dx, \quad \int_{\Omega^h}|u_\perp^h(y,z)|^2\,dx \leqslant C_\Omega h^{2(d-1)} \tag{3.11}$$

$$\Rightarrow \Big|h^{d-1}\int_{-1}^{1} H(z)^{d-1}|\overline{u}^h(z)|^2\,dz - \frac{1}{|\omega|}\Big| \leqslant ch^2 \quad\Rightarrow\quad \|\widehat{\mathbf{v}}_n; \mathbf{L}\| = |\omega|^{-1/2}.$$

Let $\varphi \in C_c^\infty(-1,1)$. According to formula (1.4), the function φ of the longitudinal variable z satisfies the Dirichlet condition for a small $h>0$ and can be inserted in the integral identity (1.8) as a test function. Taking definition (3.4) into account, we obtain that

$$h^{1-d}\int_{-1}^{1}\varphi(z)\int_{\omega^h(z)} u^h(y,z)\,dydz = |\omega|\int_{-1}^{1} H(z)^{d-1}\varphi(z)\overline{u}^h(z)\,dz \tag{3.12}$$

and

$$h^{1-d}\int_{-1}^{1}\partial_z\varphi(z)\int_{\omega^h(z)}\partial_z u^h(y,z)\,dydz$$

$$\int_{-1}^{1} H(\zeta)^{d-1}\partial_\zeta\varphi(\zeta)\int_{\omega}(\partial_\zeta u^h(hH(\zeta)\eta,\zeta)+h\partial_\zeta H(\zeta)\eta\cdot\nabla_y u^h(hH(\zeta)\eta,\zeta))d\eta d\zeta =$$

$$= |\omega|\int_{1}^{1} H(z)^{d-1}\partial_z\varphi(z)\partial_z\overline{u}^h(z)\,dz + h^{2-d}\int_{\Omega^h}\partial_z\varphi(z)\partial_z H(z)\eta\cdot\nabla_y u^h(y,z)\,dydz. \tag{3.13}$$

The last integral J^h (including the factor h^{2-d}) can be estimated by

$$|J^h| \leqslant ch^{2-d}\|\partial_z\varphi; L^2(\Omega^h)\|\,\|\eta\cdot\nabla_y u^h; L^2(\Omega^h)\|$$
$$\leqslant c_\varphi h^{1+(1-d)/2}\|\nabla_y u^h; L^2(\Omega^h)\| \leqslant C_\varphi h^{1+(1-d)/2}. \tag{3.14}$$

Thus, the convergence (3.2) for λ^h and (3.9) for $h^{(d-1)/2}\overline{u}^h$ together with the inequality (3.14) as well as passing to the limit $h\to+0$ in the integral identity (1.8) with the test function $\psi^h(x) = h^{(d-1)/2}\varphi(z)$ lead to the relation

$$(H^{d-1}\partial_z\widehat{\mathbf{v}}_n, \partial_z\varphi) = \widehat{\mu}_n(H^{d-1}\widehat{\mathbf{v}}_n,\varphi) \qquad \forall\varphi\in C_c^\infty(-1,1). \tag{3.15}$$

By a density argument, the one-dimensional integral identity (3.15) holds for all test functions $\varphi\in\mathbf{H}$ and, therefore, $\widehat{\mu}_n$ is an eigenvalue and $\widehat{\mathbf{v}}_n$ is the corresponding eigenfunction of the problem (2.14), which is non-zero due to the last equality in (3.12).

All in all, we have proved the next assertion which is known as the "convergence theorem".

Lemma 3.1. *Given the eigenpair* $\{\lambda_n^h, u_n^h\}$ *satisfying the relations* (3.1) *and* (1.10), *the convergent sequences* (3.2), (3.9) *and formula* (3.4) *define the spectral pair* $\{\widehat{\mu}_n, \widehat{\mathbf{v}}_n\}$ *of the equation* (2.9).

We emphasize that the coinciding of the eigenvalues $\widehat{\mu}_n$ and μ_n has not been proved yet.

3.2 Abstract formulation of the problem. We endow the Hilbert space $\mathcal{H}^h = H_0^1(\Omega^h;\Gamma^h)$ with the natural scalar product

$$\langle u^h,\varphi^h\rangle_h = (\nabla_x u^h, \nabla_x\psi^h)_{\Omega^h} + (u^h,\psi^h)_{\Omega^h} \tag{3.16}$$

and define the operator $\mathcal{K}^h$,

$$\langle\mathcal{K}^h u^h,\psi^h\rangle_h = (u^h,\psi^h)_{\Omega^h} \qquad \forall u^h,\psi^h\in\mathcal{H}^h \tag{3.17}$$

which is positive, symmetric, continuous and thus self-adjoint. Formulas (3.16) and (3.17) show that the variational formulation (1.8) of the problem (1.5)–(1.7) is equivalent to the abstract equation

$$\mathcal{K}^h u^h = \kappa^h u^h \quad \text{in} \quad \mathcal{H}^h$$

with the new spectral parameter

$$\kappa^h = (1+\lambda^h)^{-1}. \tag{3.18}$$

In view of the compactness of the embedding $H^1_0(\Omega^h;\Gamma^h) \subset L^2(\Omega^h)$, the operator $\mathcal{K}^h$ is compact so that by [2, Theorems 10.1.5 and 10.2.1] its essential spectrum consists of the single point $\kappa = 0$ and its discrete spectrum forms a positive sequence convergent to null,

$$1 > \kappa_1^h > \kappa_2^h \geqslant \kappa_3^h \geqslant \cdots \geqslant \kappa_n^h \geqslant \cdots \to +0 \tag{3.19}$$

which comes from the sequence (1.9) via (3.18).

The next assertion follos from the spectral decomposition of resolvent (see, e.g., [2, Ch. 6]) is known as the lemma on "almost eigenvalues and eigenvectors", [14].

Lemma 3.2. *Let* $\mathbf{U}^h \in \mathcal{H}^h$ *and* $\mathbf{k}^h > 0$ *be such that*

$$\|\mathbf{U}^h;\mathcal{H}^h\| = 1, \quad \|\mathcal{K}^h\mathbf{U}^h - \mathbf{k}^h\mathbf{U}^h;\mathcal{H}^h\| = t^h \in [0,\kappa^h).$$

Then, there exists an element of the sequence (3.19) *which fulfils*

$$|\mathbf{k}^h - \kappa_p^h| \leqslant t^h. \tag{3.20}$$

Moreover, for any $T^h \in (t^h, \mathbf{k}^h)$, *one finds a coefficient vector* $c^h = (c^h_{N^h}, \dots c^h_{N^h+K^h-1}) \in \mathbb{R}^{K^h}$ *satisfying the relations*

$$\Big\|\mathbf{U}^h - \sum_{q=N^h}^{N^h+K^h-1} c_q^h \mathcal{U}_q^h;\mathcal{H}^h\Big\| \leqslant 2\frac{t^h}{T^h}, \quad \sum_{q=N^h}^{N^h+K^h-1} |c_q^h|^2 = 1. \tag{3.21}$$

Here, $\mathcal{U}^h_{N^h}, \dots, \mathcal{U}^h_{N^h+K^h-1} \in \mathcal{H}^h$ *is the set of all eigenvectors of the operator* $\mathcal{K}^h$ *which correspond to the eigenvalues in the interval* $[\mathbf{k}^h - T^h, \mathbf{k}^h + T^h]$ *and are subject to the orthogonality and normalization conditions* $\langle \mathcal{U}^h_q, \mathcal{U}^h_p\rangle_h = \delta_{q,p}$.

3.3. Almost eigenvalue and eigenvector. In this section we deal with dimensions $d \geqslant 4$ and postpone the spatial and planar cases $d = 3, 2$ to the next two sections. Taking the eigenvalue μ_n and the eigenfunction $\mathbf{v}_n \in \mathbf{H}$ of the limit problem (2.14), we set

$$\mathbf{k}_n = (1+\mu_n)^{-1}, \qquad \mathbf{U}^h_n = \|\mathbf{u}^h_n;\mathcal{H}^h\|^{-1}\mathbf{u}^h_n,\ \mathbf{u}^h_n = X^h\mathbf{v}_n \tag{3.22}$$

where we introduced the cut-off function

$$X^h(z) = \chi(h^{-2}(1-h\rho_- + z))\,\chi(h^{-2}(1-h\rho_+ - z)) \tag{3.23}$$

and χ is the reference cut-off function (2.13). Moreover,

$$\begin{aligned}&|\partial_z X^h(z)| \leqslant ch^{-2}, \quad \operatorname{supp}(\partial_z X^h) \subset \Omega^h_\ddagger = \big\{x \in \overline{\Omega^h} : \mp z + 1 - h\rho_\pm \in [h^2/2, h^2]\big\},\\ &|\Omega^h_\ddagger| \leqslant ch^{2d} \quad \text{and} \quad |\mathbf{v}_n(z)| \leqslant c_n,\ |\partial_z \mathbf{v}_n(z)| \leqslant c_n h \text{ for } x \in \Omega^h_\ddagger\end{aligned} \tag{3.24}$$

by definitions (1.2), (1.1) and Lemma 2.4.

Thus, the function $\mathbf{u}^h_n \in C^\infty_c(-1,1)$ vanishes for $\pm z \in (1-h\rho_\pm, 1)$ and, therefore, satisfies the Dirichlet condition (1.6) on the sets (1.4) and belongs to the space $\mathcal{H}^h$. In Lemma 3.6 we will show that

$$\|\mathbf{u}^h_n;\mathcal{H}^h\| \geqslant \mathbf{c}h^{(d-1)/2},\ \mathbf{c} > 0. \tag{3.25}$$

Taking into account formulas (3.16), (3.17) yields

$$\begin{aligned}t^h_n &:= \|\mathcal{K}^h\mathbf{U}^h_n - \mathbf{k}_n\mathbf{U}^h_n;\mathcal{H}^h\| = \sup|\langle \mathcal{K}^h\mathbf{U}^h_n - \mathbf{k}^h_n\mathbf{U}^h_n, \psi^h\rangle| =\\ &= \|\mathbf{u}^h_n;\mathcal{H}^h\|^{-1}(1+\mu_n)^{-1}\sup\big|\mu_n(\mathbf{u}^h_n,\psi^h)_{\Omega^h} - (\nabla_x\mathbf{u}^h_n, \nabla_x\psi^h)_{\Omega^h}\big|,\end{aligned} \tag{3.26}$$

where the supremum is computed over the unit ball of the space $\mathcal{H}^h$, hence,

$$\|\psi^h;\mathcal{H}^h\|^2 = \|\nabla_x\psi^h; L^2(\Omega^h)\|^2 + \|\psi^h; L^2(\Omega^h)\|^2 \leqslant 1. \tag{3.27}$$

Let us evaluate the quantity (3.26). As in (3.12), we deduce that

$$\mu_n(\mathbf{u}_n^h, \psi^h)_{\Omega^h} = h^{d-1}|\omega|\mu_n(H^{d-1}\mathbf{u}_n, X^h\overline{\psi}^h) \tag{3.28}$$

where $\overline{\psi}^h$ is the mean value (3.4) of the function ψ^h of the cross-section (3.6) of the spindle Ω^h. Furthermore,

$$\begin{aligned}(\nabla_x\mathbf{u}_n^h, \nabla_x\psi^h)_{\Omega^h} &= (\partial_z\mathbf{v}_n, \partial_z(X^h\psi^h))_{\Omega^h} + (\mathbf{v}_n\partial_z X^h, \partial_z\psi^h)_{\Omega^h} - \\ &-(\partial_z\mathbf{v}_n, \psi^h\partial_z X^h)_{\Omega^h} =: I_1^h(\psi^h) + I_2^h(\psi^h) - I_3^h(\psi^h).\end{aligned} \tag{3.29}$$

We make the substitutions $\varphi \mapsto \mathbf{v}_n$, $u^h \mapsto X^h\psi^h$ in the calulations (3.13) and (3.14) , recall (3.27) and end up with the inequality

$$\begin{aligned}&\Big|I_1^h(\psi^h) - h^{d-1}|\omega|(H^{d-1}\partial_z\mathbf{v}_n, \partial_z(X^h\overline{\psi}^h))\Big| \leqslant \\ &\leqslant c\int_{\Omega^h} |\partial_z\mathbf{v}_n(z)|\,|\partial_z H(z)|\,h\,|\eta\cdot\nabla_y(X^h(z)\psi^h(x))|\,dx \leqslant \\ &\leqslant c_H h^{(d-1)/2}\|H^{(d-1)/2}\partial_z\mathbf{v}_n; L^2(-1,1)\|\,h\,\|\nabla_y\psi^h; L^2(\Omega^h)\| \leqslant c_n h^{(d+1)/2}.\end{aligned} \tag{3.30}$$

It remains to consider the terms of (3.29) which contain derivatives of the cut-off function X^h. We denote by $\Omega_\ddagger^h$ the support of these derivatives and observe that it is finite union of sets with diameter $O(h^2)$. Thus, by formulas (3.24), we obtain

$$\begin{aligned}|I_2^h(\psi)| &\leqslant ch^{-2}|\Omega_\ddagger^h|^{1/2}\max_{x\in\Omega_\ddagger^h}|\mathbf{v}_n(z)|\,\|\nabla_x\psi^h; L^2(\Omega^h)\| \leqslant c_n h^{d-2}, \\ |I_3^h(\psi)| &\leqslant c|\Omega_\ddagger^h|^{1/2}\max_{x\in\Omega_\ddagger^h}|\partial_z\mathbf{v}_n(z)|\,h^{-2}\|\psi^h; L^2(\Omega^h)\| \leqslant c_n h^{d-1}.\end{aligned} \tag{3.31}$$

The subtrahend on the left-hand side of (3.30) coincides with the expression (3.28) due to the integral identity (2.14), so that taking into account (3.25), our calculations yield the estimate

$$t_n^h \leqslant c_n h^{(1-d)/2}(h^{(d+1)/2} + h^{d-2}) \leqslant 2c_n h^{\min\{2,d-3\}/2}. \tag{3.32}$$

This bound vanishes on the limit $h \to +0$ only, if $d \geqslant 4$: the definition (3.22) is useless in the case $d = 3$ of the spatial spindle. Nevertheless, we formulate the following theorem, the proof of which will be completed in the next section for $d \geqslant 4$ (note Remark 3.4 for $d = 4$) and by using a different construction in Sections 3 and 4 § 4. The planar case $d = 2$ is postponed to § 5, since the corresponding error estimates will become of a completely different nature (see Proposition 5.7).

Theorem 3.3. *For $d \geqslant 3$ and $n \in \mathbb{N}$, there exist positive numbers $c_d^{(n)}$ and $h_d^{(n)}$ such that the eigenvalues* (1.9) *and* (2.17) *of the problems* (1.8) *and* (2.14), *respectively, are related by*

$$|\lambda_n^h - \mu_n| \leqslant c_d^{(n)} h \quad \text{for} \quad h \in (0, h_d^{(n)}]. \tag{3.33}$$

Remark 3.4. Lemma 3.2 and inequality (3.32) for $d = 4$ already give the estimate (3.33) with the weaker bound $c_4^{(n)}h^{1/2}$, however, the full statement of Theorem 3.3 for $d = 4$ needs the construction of the boundary layers, which will be described in detail for the three-dimensional spindle (see also Section 3 §6 for the general situation). This improvement is not of principal importance because of the power-law order of smallness in the bound $c_4^{(n)}h^{1/2}$ (cf. §5 for the case $d = 2$). If one only proceeds with this weaker form of the relation (3.32) then, for $d = 4$, one has to replace the multipliers h by $h^{1/2}$ in Section 4 §3. ⊠

3.4. Asymptotics of eigenfunctions. By Lemma 3.2 we find an eigenvalue κ_p^h of the operator $\mathcal{K}^h$ such that the estimate (3.20), including (3.22) and (3.32), holds true. Thus, using (3.22) and (3.18) we deduce that

$$|\lambda_p^h - \mu_n| \leqslant c_n h(1+\lambda_p^h)(1+\mu_n) \quad \Rightarrow \quad (1+\lambda_p^h)(1-c_n h(1+\mu_n)) \leqslant 1+\mu_n \quad \Rightarrow$$
$$\Rightarrow \quad 1+\lambda_p^h \leqslant 2(1+\mu_n) \text{ for } h \leqslant h_d^{(n)} := (2c_n(1+\mu_n))^{-1}. \tag{3.34}$$

Hence, in view of the the first and last formulas in (3.34), we obtain the relation (3.33) with the coefficient $c_d^{(n)} = 2c_n(1+\mu_n)^2$, but only for some eigenvalue λ_p^h of (1.9) with an unknown index $p = p_h(n)$ in the place of λ_n^h. Since the eigenvalues (2.17) of the problem (2.14) are simple, the mapping $n \mapsto p_h(n)$ satisfies $p^h(m) < p^h(n)$ for all $m < n$ and small enough h, hence, there holds $p^h(n) \geqslant n$ for all $n \in \mathbb{N}$ and small h.

Remark 3.5. The relation (3.1) can now proved by

$$\lambda_n^h \leqslant \lambda_{p^h(n)}^h \leqslant \mu_n + c_n h(1+\mu_n)^2.$$

The same considerations and calculations will also be applied in Sections 4 § 4 and 5 § 5 so that (3.1) holds true also in the cases $d = 3$ and $d = 2$. ⊠

Let us assume that $p^h(n) > n$. Then, due to formula (3.34) the limit $\lim \lambda_{n+1}^h = \widehat{\mu}_{n+1} \leqslant \mu_n$ is an eigenvalue of equation (2.9) for a small h, by Lemma 3.1. The eigenfunction u_{n+1}^h is orthogonal to the eigenfunctions $u_1^h, \dots, u_n^h$ in $L^2(\Omega^h)$, and the strong convergence (3.9) and transformations analogous to (3.7), (3.10), (3.12) yield the following equalities in the limit as $h \to +0$ (cf., (2.18)):

$$(H^{d-1}\widehat{\mathbf{v}}_{n+1}, \widehat{\mathbf{v}}_\ell) = 0, \quad \ell = 1, \dots, n.$$

Consequently, we have found $n+1$ eigenfunctions of the problem (2.7) which are mutually orthogonal in the space $\mathbf{L}$ to each other and correspond to the eigenvalues $\mu_1, \dots, \mu_n$. This is impossible, and Theorem 3.3 is thus proved for $d \geqslant 4$.

We derive an estimate of the remainder in the asymptotic representation of the eigenfunction u_n^h for the second part of Lemma 3.2, and to this end we need another estimate which also holds for $d \geqslant 4$ only.

Lemma 3.6. *Let $\mathbf{v}_n$ and $\mathbf{v}_m$ be eigenfunctions of the equation* (2.9) *subject to the orthogonality and normalization conditions* (2.18)*. Then, the products $\mathbf{u}_n^h = X^h \mathbf{v}_n$ and $\mathbf{u}_m^h = X^h \mathbf{v}_m$ with the cut-off function* (3.23) *satisfy the inequality*

$$\left| \langle \mathbf{u}_n^h, \mathbf{u}_m^h \rangle_h - h^{d-1}|\omega|(1+\mu_n)\delta_{n,m} \right| \leqslant c_{m,n} h^{d+\min\{0,d-4\}} \tag{3.35}$$

where the bound is $o(h^{d-1})$ in the case $d \geqslant 4$.

Proof. We can directly write

$$\left| (X^h \mathbf{v}_n, X^h \mathbf{v}_m)_{\Omega^h} - h^{d-1}|\omega|(H^{d-1}\mathbf{v}_n, \mathbf{v}_m) \right| \leqslant c_{n,m} h^d$$

because the d-dimensional volume of the set $\{x \in \Omega^h : X^h(z) \neq 1\}$ equals $O(h^d)$. Moreover,

$$(\nabla_x \mathbf{u}_n^h, \nabla_x \mathbf{u}_m^h)_{\Omega^h} = h^{d-1}|\omega|(H^{d-1}\partial_z \mathbf{v}_n, \partial_z \mathbf{v}_m) - ((1-X^h)\partial_z \mathbf{v}_n, (1+X^h)\partial_z \mathbf{v}_m)_{\Omega^h} +$$
$$+ (\mathbf{v}_n \partial_z X^h, \mathbf{v}_m \partial_z X^h)_{\Omega_\ddagger^h} + (X^h \partial_z \mathbf{v}_n, \mathbf{v}_m \partial_z X^h)_{\Omega_\ddagger^h} + (\mathbf{v}_n \partial_z X^h, H^h \partial_z \mathbf{v}_m)_{\Omega_\ddagger^h}, \tag{3.36}$$

where we label the last four scalar products by $j = 1, \dots, 4$. The moduli of them can be estimated using the relations (3.24) by the bounds $c_j h^{\rho_j}$, where

$$\rho_1 = d+2, \quad \rho_2 = 2d-4, \quad \rho_3 = \rho_4 = 2d-1.$$

It remains to recall the identity (2.14) and the definition of the scalar product (3.16). ⊠

Thus, $\|\mathbf{u}_n^h;\mathcal{H}^h\| = h^{(d-1)/2}(((1+\mu_n)^{1/2}|\omega|^{1/2}+O(h))$. In Lemma 3.2 we choose $T^h = T_n > 0$ such that the interval $[\mathbf{k}_n - T_n, \mathbf{k}_n + T_n]$ contains only one eigenvalue $\kappa_n^h = (1+\mu_n)^{-1} + O(h)$ of the operator $\mathcal{K}^h$, cf. Theorem 3.3. Then $N^h = n$, $K^h = 1$ in (3.21) and, therefore, $c_n^h = \pm 1$; if necessary we change sign of $\mathcal{U}_n^h$ and obtain $c_n^h = 1$. Furthermore, the normalization of the eigenfunctions u_n^h and $\mathcal{U}_n^h$ implies that

$$\|u_n^h;\mathcal{H}^h\|^2 = (1+\lambda_n^h)\|u_n^h;L^2(\Omega^h)\|^2 = 1+\lambda_n^h \quad \Rightarrow \quad u_n^h = (1+\mu_n+O(h))^{1/2}\mathcal{U}_n^h. \tag{3.37}$$

Finally, a simple calculation gives us the last assertion in this section.

Theorem 3.7. *For $d \geqslant 4$ and any $n \in \mathbb{N}$, there exist positive numbers $C_d^{(n)}$ and $h_d^{(n)}$ such that eigenfunctions of problems* (1.8) *and* (2.14) *are related by*

$$\left\|u_n^h - h^{(1-d)/2}|\omega|^{-1/2}X^h\mathbf{v}_n; H^1(\Omega^h)\right\| \leqslant C_d^{(n)}h \quad \text{for} \quad h \in (0, h_d^{(n)}].$$

4. Spatial spindle. Boundary layer and asymptotic correction terms.

4.1. Prelude. By Lemma 2.4, the eigenfucntion $\mathbf{v}_n$ does not vanish at the points $z = \pm 1$ and, therefore, the main term of the ansatz (2.2) does certainly not satisfy the Dirichlet condition (1.6) in the sets $\Gamma_\pm^h$ of diameter $O(h)$. That is why the construction (3.22) of the almost eigenvector $\mathbf{u}_n^h$ contained the cut-off function X^h and the bounds in estimates (3.32) and (3.35) were not sufficiently small in the case $d < 4$. Thus, the asymptotic ansatz needs to be modified at least near the ends of the spindle (1.1) and as usual it is necessary to construct boundary layers (see, e.g., [14, 15], [16, Ch. 4, 5 and 15, 16], [17], [18], [19]). We will explain by formal calculations both their structure and the passing to the stretched coordinates

$$\xi^\pm = (\xi_\bullet^\pm, \xi_d^\pm) = h^{-2}(\rho_\pm H_\pm)^{-1}(y, 1 - h\rho_\pm \mp z) \tag{4.1}$$

where the positive numbers $\rho_\pm$ and $H_\pm$ are taken from (1.4) and (1.2).

First of all, in view of (1.2) and (4.1), we transform the multiplier of y in the definition (1.1) of the spindle Ω^h as follows:

$$\begin{aligned} h^{-1}H(z)^{-1} &= h^{-1}(1\mp z)^{-1}(H_\pm + O(1\mp z))^{-1} = \\ &= h^{-1}(h\rho_\pm + h^2\rho_\pm H_\pm \xi_d^\pm)^{-1}(H_\pm + O(h\rho_\pm(1 + hH_\pm\xi_d^\pm)))^{-1} = \\ &= h^{-2}(\rho_\pm H_\pm)^{-1}(1 + O(hH_\pm\xi_d^\pm(1 + hH_\pm\xi_d^\pm))). \end{aligned} \tag{4.2}$$

The dilation coefficient $h^{-2}(\rho_\pm H_\pm)^{-1}$ in (4.1) thus becomes correct because the cross-sections $\omega^h(\pm 1 \mp \rho_\pm h)$ transform into the domain ω of unit size. As a result, the change $x \mapsto \xi^\pm$ and the formal passing to $h = 0$ convert the domain Ω^h into the cylinder $\Pi = \omega \times \mathbb{R} \ni (\xi_\bullet^\pm, \xi_d^\pm)$. Moreover, the conical surface $\Gamma_\pm^h$ turns into the cylindrical surface $(\partial\Pi)_-$: the bounds in formula (1.4) turn into $-(hH_\pm)^{-1} < \xi_d^\pm < 0$ in the stretched coordinates (4.1) as the parameter h tends to zero. Besides, we have set $(\partial\Pi)_\pm = \{\xi \in \partial\Pi : \pm\xi_d > 0\}$.

Now we predict that in the case $d \geqslant 3$ the refined ansatz (2.2) looks as follows at some distance from the top zones of the spindle :

$$u^h(x) = \mathbf{v}(z) + h^{d-2}\mathbf{v}'(z) + h^2(V(\eta, z) + h^{d-2}V'(\eta, z)) + \dots. \tag{4.3}$$

Here, $\mathbf{v}'$ is an unknown function in the interval $(-1, 1)$ having the following behaviour at its endpoints:

$$\mathbf{v}'(z) = (1 \mp z)^{2-d}(c_\pm + \dots). \tag{4.4}$$

The singularity is matched with the solution (2.20) of the model equation (2.19) and the quantities $c_\pm$ must be fixed such that the sum of the first couple of the terms on the right-hand side of (4.3) vanishes at the points $z = \pm 1 \mp \rho_\pm h$; this allows us to extend the sum as null over the ends $\{x \in \Omega^h : \pm z > 1 - h\rho_\pm\}$, and hence the Dirichlet condition (1.6) is met by the sum on the surfaces $\Gamma^h_\pm$. Due to relations (4.4), (4.1) and the Taylor formula, we have

$$\mathbf{v}(z) + h^{d-2}\mathbf{v}'(z) = \mathbf{v}(\pm 1) + c_\pm h^{d-2}(h\rho_\pm + h^2\rho_\pm H_\pm \xi_d^\pm)^{2-d} + \dots = \\ = \mathbf{v}(\pm 1) + c_\pm \rho_\pm^{2-d} - hc_\pm(d-2)\rho_\pm^{2-d} H_\pm \xi_d^\pm + \dots . \tag{4.5}$$

Thus, the explicit terms of the expression (4.5) vanish at $\xi_d^\pm = 0$, if and only if

$$c_\pm = -\rho_\pm^{d-2}\mathbf{v}(\pm 1). \tag{4.6}$$

Furthermore, in the framework of the method of matched asymptotic expansions (see the monographs [20, 17], [16, Ch. 2] and others), we observe from the form of the coefficient of h on the right-hand side of (4.5) that, near the ends of the spindle, the main terms of the inner expansions of the solutions of (1.5)–(1.7) must be functions with linear growth as $\xi_d^\pm \to +\infty$.

4.2. Mixed boundary value problem in the cylinder. Formulas (2.1), (2.5) and (4.1) imply the relations

$$\Delta_x + \lambda^h = h^{-4}\Delta_{\xi^\pm} + \mu + \dots, \\ \partial_{\nu(x)} = h^{-2}\partial_{\nu'(\eta)} + \dots . \tag{4.7}$$

Taking into account the leading terms and using the geometric transformations of Section 1 § 4 yield the following boundary value problem in the cylinder Π (Fig. 1, b):

$$-\Delta_\xi w(\xi) = 0,\ \xi \in \Pi = \omega \times \mathbb{R}, \\ w(\xi) = 0,\ \xi \in (\partial\Pi)_- = \partial\omega \times \mathbb{R}_-, \\ \partial_{\nu'} w(\xi) = 0,\ \xi \in (\partial\Pi)_+ = \partial\omega \times \mathbb{R}_+. \tag{4.8}$$

It would not be difficult to find a solution[1] of the corresponding inhomogeneous problem (4.8) with a bounded Dirichlet integral (see Section 1 § 6) but in this section we only need a solution of the homogeneous problem with linear growth as $\xi_d^\pm \to +\infty$ (cf., the end of Section 1 § 4). This solution is defined by its behaviour at infinity:

$$W(\xi) = \widetilde{W}_-(\xi), \quad \xi_d < 0, \\ W(\xi) = \xi_d + K_\omega + \widetilde{W}_+(\xi), \quad \xi_d > 0. \tag{4.9}$$

Here, K_ω is a constant depending only on the cross-section ω of the cylinder Π and $\widetilde{W}_\pm$ are exponentially decaying remainders,

$$\left|\widetilde{W}_\pm(\xi)\right| \leqslant ce^{-\kappa_\pm|\xi_d|}, \quad \pm\xi_d > 0,\ \kappa_\pm > 0, \tag{4.10}$$

where κ_+^2 and κ_-^2 are the first positive eigenvalues of the Neumann and Dirichlet problems in the domain ω.

4.3. Boundary layers in the spatial case. Let us fix dimension $d = 3$, although we will still write in the following the exponents as $d-1$, $d-2$ and so on, in order to match the notation with the subsequent sections. We refine the ansatz (2.1) as follows:

$$\lambda_n^h = \mu_n + h^{d-2}\mu_n' + \dots . \tag{4.11}$$

[1]Formally self-adjoint boundary value problems for elliptic systems in domains with cylindrical outlets to infinity are studied in detail in the papers [21], [22, § 3]; these results require only some algebraic operations, which become particularly simple in the case of the Laplace operator.

The outer expansion (4.3) of the eigenfunction u_n^h of the problem (1.5)–(1.7) includes the bounded eigenfunction $\mathbf{v} = \mathbf{v}_n$ of the equation (2.9), which is normalized in $\mathbf{L}$, and the correction term $\mathbf{v}' = \mathbf{v}'_n$ with the decomposition (4.4); this and the number μ'_n on the right-hand side of (4.11) are to be determined. As in Section 1 § 2, the term $V'_n(\eta, z)$ can be found from the following problem, which is analogous to (2.4), (2.6) and formulated on the cross-section ω with the parameter $z \in (-1, 1)$, namely

$$-H(z)^{-2}\Delta_\eta V'(\eta, z) = \partial_z^2 \mathbf{v}'_n(z) + \mu_n \mathbf{v}'_n(z) + \mu'_n \mathbf{v}_n(z),\ \eta \in \omega,$$
$$H(z)^{-2}\partial_{\nu'(\eta)} V'(\eta, z) = H(z)^{-1}\partial_z H(z)\, \eta \cdot \nu'(\eta)\, \partial_z \mathbf{v}'_n(z),\ \eta \in \partial\omega.$$

The compatibility condition in this Neumann problem turns into the inhomogeneous ordinary differential equation (2.9)

$$-\partial_z(H(z)^{d-1}\partial_z \mathbf{v}'_n(z)) - \mu_n H(z)^{d-1}\mathbf{v}'_n(z) = \mu'_n H(z)^{d-1}\mathbf{v}_n(z),\ z \in (-1, 1). \tag{4.12}$$

The matching procedure for the outer expansion (4.5) uses formulas (4.9) and (4.6), and yields the inner expansion

$$u^h(x) = h(d-2)\mathbf{v}_n(\pm 1)H_\pm W(\xi^\pm) + \dots .$$

This also makes our prediction (4.4) precise:

$$\mathbf{v}'_n(z) = \widehat{\mathbf{v}}'_n(z) - \sum_\pm \chi_\pm(z)(1 \mp z)^{2-d}\rho_\pm^{d-2}\mathbf{v}_n(\pm 1), \qquad (1-|z|)^{d-2}\widehat{\mathbf{v}}'_n(z)\big|_{z=\pm 1} = 0. \tag{4.13}$$

Here and later we employ the cut-off functions defined by (2.13) and

$$\chi_\pm(z) = \chi(\pm z). \tag{4.14}$$

There exists a solution to equation (4.12) with behaviour (4.13) near the ends of the interval $(-1, 1)$, if and only if the following condition is satisfied

$$\begin{aligned}
\mu'_n &= \mu'_n \int_{-1}^{1} H(z)^{d-1}\mathbf{v}_n(z)^2 dz \\
&= -\lim_{\varepsilon \to +0} \int_{-1+\varepsilon}^{1-\varepsilon} \mathbf{v}_n(z)\left(\partial_z(H(z)^{d-1}\partial_z \mathbf{v}'_n(z)) + \mu_n H(z)^{d-1}\mathbf{v}'_n(z)\right) dz \\
&= -\lim_{\varepsilon \to +0} \sum_\pm \pm\Big(\mathbf{v}_n(\pm 1 \mp \varepsilon)H(\pm 1 \mp \varepsilon)^{d-1}\partial_z \mathbf{v}'_n(\pm 1 \mp \varepsilon) \\
&\qquad - \mathbf{v}'_n(\pm 1 \mp \varepsilon)H(\pm 1 \mp \varepsilon)^{d-1}\partial_z \mathbf{v}_n(\pm 1 \mp \varepsilon)\Big).
\end{aligned} \tag{4.15}$$

We compute the limit with the help of the formulas (2.21), (1.2), (4.13) and finally obtain that

$$\mu'_n = (d-2)\sum_\pm H_\pm^{d-1}\rho_\pm^{d-2}\mathbf{v}_n(\pm 1)^2\big|_{d=3} = \sum_\pm H_\pm^2 \rho_\pm \mathbf{v}_n(\pm 1)^2. \tag{4.16}$$

Thus, the correction term in the ansatz (4.11) has been found. Let us formulate an assertion the proof of which will be presented in the next section.

Theorem 4.1. *Let $d = 3$. For all $n \in \mathbb{N}$, there exist positive numbers $c_3^{(n)}$ and $h_3^{(n)}$ such that there holds the inequality*

$$\left|\lambda_n^h - \mu_n - h\mu'_n\right| \leqslant c_3^{(n)} h^{3/2} \text{ for } h \in (0, h_3^{(n)}], \tag{4.17}$$

where λ_n^h and μ_n are the entries of the sequences (1.9) *and* (2.17)*, and μ'_n is the quantity* (4.16).

Of course, formula (4.17) also implies the estimate (3.33).

4.4. Error estimates in the spatial case. We next derive an inequality which completes the proof of Theorem 3.3. Moreover, we found in (4.16) the correction term in the representation (4.11) for the eigenvalue λ_n^h, which led to the much more precise asymptotic formula (4.17) in comparison with (3.33). Here, we will present a couple of estimates, which together with the considerations in Section 3 and 4 § 3, also yield the proof of (4.17) and thus of Theorem 4.1. Furthermore, in section 3 § 6 we will extend the discussion to infinite asymptotic series for eigenvalues and eigenfunctions.

The quantity $\mathbf{k}_n^h = (1+\mu_n + h\mu_n')^{-1}$ differs only little from $\mathbf{k}_n$ in (3.22), but the structure of the corresponding function $\mathbf{u}_n^h$ is much more complicated:

$$\mathbf{u}_n^h(x) = X^h(z)\mathbf{v}_{n\flat}^h(z) + hX^{\sqrt{h}}(z)\widehat{\mathbf{v}}_n'(z) + h\sum_{\pm}\chi_\pm(z)w_{n\pm}(\eta,\xi_3^\pm) + h^2X^h(z)V(\eta,z). \quad (4.18)$$

Here, V is a solution of the problem (2.4), (2.6) subject to restriction (2.8) and estimate (2.25),

$$\mathbf{v}_{n\flat}^h(z) = \mathbf{v}_n(z) - h\sum_{\pm}\chi_\pm(z)(1\mp z)^{2-d}\rho_\pm^{d-2}\mathbf{v}_n(\pm 1) \quad (4.19)$$

and $\mathbf{v}_n'$ is a solution to problem (4.12), (4.13) which exists due to relation (4.15) and has the representation (4.13); it is not unique but has been fixed somehow. Moreover, we will verify in Lemma 4.2 the inequality

$$|\widehat{\mathbf{v}}_n'(z)| \leqslant c_n(1+|\ln(1-|z|)|), \quad |\partial_z\widehat{\mathbf{v}}_n'(z)| \leqslant c_n(1-|z|)^{-1} \quad (4.20)$$

for $d=3$. The cut-off functions X^h, $\chi_\pm$ and $X^{\sqrt{h}}$, $\chi_\pm^h$ are defined in formulas (3.23), (4.14) and

$$X^{\sqrt{h}}(z) = \chi(h^{-1}(1-h\rho_- + z))\chi(h^{-1}(1-h\rho_+ - z)), \quad (4.21)$$

$$\chi_\pm^h(z) = \chi\Big(\frac{3}{2}+\frac{3}{2}(h\rho_\pm)^{-1}(1-h\rho_\pm \mp z)\Big)\,\chi\Big(\frac{3}{2}-\frac{3}{2}(h\rho_\pm)^{-1}(1-h\rho_\pm\mp z)\Big). \quad (4.22)$$

Furthermore, $X^{\sqrt{h}}(z) = 1$ for $z\in(-1+h(1+\rho_-), 1-h(1+\rho_+))$ and $X^{\sqrt{h}} = 0$ in small neighborhoods of the sets $\Gamma_\pm^h$, while $\chi_\pm^h$ equals one in the $(h\rho_\pm/3)$-neighborhood of the point $z=\pm1\mp h\rho_\pm$ and null outside its $(2h\rho_\pm/3)$-neighborhood.

Finally, $\xi^\pm = (\xi_\bullet^\pm, \xi_3^\pm)\in\Pi$ and $\eta\in\omega$ are the stretched coordinates, see (4.1) and (1.1). The function

$$w_{n\pm}(\xi^\pm) = \mathbf{v}_n(\pm1)H_\pm(W(\xi^\pm) - \mathcal{X}_\pm(\xi_3^\pm)\xi_3^\pm), \text{ where } \mathcal{X}_\pm(\xi_3^\pm) = \chi(\rho_\pm H_\pm\xi_3^\pm), \quad (4.23)$$

is defined by using the special solution W of the problem (4.8) in the cylinder Π, see Section 2 § 4. The definition of W can be extended to the whole spindle Ω^h, since it is rewritten in the coordinates $(\eta,\xi_3^\pm)$ and multiplied with the cut-off function $\chi_\pm^h(z)$. Finally, due to relations (4.9) and (4.10), the function (4.23) decays exponentially as $\xi_3^\pm\to-\infty$ and converges to the constant $\mathbf{v}_n(\pm1)H_\pm K_\omega$ as $\xi_3^\pm\to+\infty$. The derivatives of (4.23) vanish at infinity with the rate $O(e^{-\kappa_0|\xi_3^\pm|})$, $\kappa_0>0$.

We next derive analogues of formulas (3.26), (3.32) and (3.35), namely we treat the quantities

$$\langle\mathbf{u}_n^h,\mathbf{u}_n^h\rangle_h = \|\nabla_x\mathbf{u}_n^h; L^2(\Omega^h)\|^2 + \|\mathbf{u}_n^h; L^2(\Omega^h)\|^2, \quad (4.24)$$

$$\sup\left|(\mu_n+h\mu_n')(\mathbf{u}_n^h,\psi^h)_{\Omega^h} - (\nabla_x\mathbf{u}_n^h,\nabla_x\psi^h)_{\Omega^h}\right|, \quad (4.25)$$

where the supremum is taken over all functions ψ^h belonging to the unit ball of $\mathcal{H}^h$, cf. (3.27).

We proceed with (4.24) and immediately find that, thanks to the boundedness of the function $w_{n\pm}$ in Π and the exponential decay of its derivatives,

$$h^2\|\nabla_x(\chi_\pm^h w_{n\pm}); L^2(\Omega^h)\|^2$$

$$\leqslant ch^2 \int\limits_{\Upsilon^h_\pm} \int\limits_{\omega^h(z)} \Big(\frac{1}{h^4}|\nabla_\xi w_{n\pm}(\eta,\xi_3^\pm)|^2 + \frac{1}{h^2}|w_{n\pm}(\eta,\xi_3^\pm)|^2\Big) dydz$$

$$\leqslant Ch^2 \bigg(h^{-4}h^6 \|e^{\kappa|\xi_3^\pm|}\nabla_{\xi^\pm} w_{n\pm}; L^2(\Pi)\| \max_{z\in\Upsilon^h_\pm} \Big(e^{-2\kappa|\xi_3^\pm|}(1+hH_\pm|\xi_3^\pm|)^2\Big)$$

$$+h^{-2}\mathrm{mes}_3\{x\in\Omega^h : z\in\Upsilon^h_\pm\} \sup_{\xi^\pm\in\Pi}|w_{n\pm}(\xi^\pm)|^2\bigg) \leqslant ch^4. \tag{4.26}$$

Here $\kappa\in(0,\kappa_0)$ and according to definitions (4.22) and (1.1), (1.2), (4.1)

$$\Upsilon^h_\pm = \{z : |\pm z - 1 + h\rho_\pm| \leqslant 2h\rho_\pm/3\} \supset \mathrm{supp}\,\chi^h_\pm, \quad dydz = (h^2\rho_\pm H_\pm)^3 d\eta d\xi_3^\pm,$$

$$|\omega^h(z)| = O(h^2(1\pm z)^2) = O(h^4\rho_\pm^2(1+hH_\pm|\xi_3^\pm|)^2) \text{ for } z\in\Upsilon^h_\pm.$$

With the help of formulas (2.25) and (3.23), (3.24) we conclude that

$$h^4\|\nabla_x(X^hV); L^2(\Omega^h)\|^2$$

$$\leqslant ch^4\bigg(\int\limits_{-1}^{1}\int\limits_{\omega^h(z)} \Big(\frac{1}{h^2H(z)^2}|\nabla_\eta V(\eta,z)|^2 + |\partial_z V(\eta,z)|^2\Big)dydz$$

$$+\sum_\pm \int\limits_{1-h\rho_\pm-ch^2}^{1-h\rho_\pm+ch^2}\int\limits_{\omega^h(\pm t)} \frac{1}{h^4}|V(\eta,\pm t)|^2 dydt\bigg)$$

$$\leqslant ch^4\left(h^{-2}h^2 + h^{-4}h^2h^4\right) \leqslant Ch^4. \tag{4.27}$$

Furthermore,

$$\|\nabla_x(X^h\mathbf{v}^h_{n\flat} + hX^{\sqrt{h}}\widehat{\mathbf{v}}'_n); L^2(\Omega^h)\|^2$$

$$= h^2|\omega|\int\limits_{-1}^{1} H(z)^2\big|X^h(z)\partial_z\mathbf{v}^h_{n\flat}(z) + \mathbf{v}^h_{n\flat}(z)\partial_z X^h(z) + h\partial_z(X^{\sqrt{h}}(z)\widehat{\mathbf{v}}'_n(z))\big|^2 dz$$

$$= h^2|\omega|\int\limits_{-1}^{1} H(z)^2|\partial_z\mathbf{v}_n(z)|^2dz + O(h^{5/2}). \tag{4.28}$$

Here, the bound $ch^{5/2}$ for the remainder follows from the next observation, based on the formulas (4.19), (4.13) and (3.23):

$$|\mathbf{v}^h_{n\flat}(z)| \leqslant c_n h \text{ for } x\in\bigcup_\pm \upsilon^h_\pm, \quad |\widehat{\mathbf{v}}'_n(z)| \leqslant c_n(1+|\ln h|) \text{ for } x\in\bigcup_\pm\Upsilon^h_\pm,$$

$$|\partial_z X^h(z)| \leqslant c_X h^{-2}, \qquad |\partial_z X^{\sqrt{h}}(z)| \leqslant c_X h^{-1}. \tag{4.29}$$

We also have

$$\upsilon^h_\pm = \{z : \pm z - 1 + h\rho_\pm \in [-h^2, -h^2/2]\}$$

and

$$h^2\int\limits_{-1}^{1} H(z)^2X^h(z)^2|\partial_z(\mathbf{v}^h_n(z) - \mathbf{v}^h_{n\flat}(z))|^2dz \leqslant ch^4\int\limits_{ch}^{C} t^{-2}dt \leqslant c_nh^3,$$

$$h^2\int_{-1}^{1} H(z)^2(1-X^h(z)^2)|\partial_z \mathbf{v}^h_{n\flat}(z)|^2dz \leqslant c_n h^7,$$
$$h^2\int_{-1}^{1} H(z)^2|\partial_z X^h(z)|^2|\mathbf{v}^h_{n\flat}(z)|^2dz \leqslant c_n h^3,$$
$$h^4\int_{-1}^{1} H(z)^2|\partial_z (X^{\sqrt{h}}(z)\widehat{\mathbf{v}}'_n(z))|^2dz \leqslant c_n h^4. \tag{4.30}$$

Similar but much simpler computations give an approximate formula, analogous to (4.28), for the $L^2(\Omega^h)$-norm of the sum $X^h\mathbf{v}^h_{n\flat} + hX^{\sqrt{h}}\widehat{\mathbf{v}}'_n + h\sum_{\pm}\chi^h_{\pm}w_{n\pm} + h^2X^hV$. This estimate together with the normalization of the eigenfunction

$$(H^2\partial_z\mathbf{v}_n, \partial_z\mathbf{v}_n) = \mu_n(H^2\mathbf{v}_n, \mathbf{v}_n) = \mu_n$$

which follows from (2.18) and (2.14), prove the desired inequality in the case $d = 3$

$$\big|\langle \mathbf{u}^h_n, \mathbf{u}^h_n\rangle_h - h^2|\omega|(1+\mu_n)\big| \leqslant c_n h^{5/2}; \tag{4.31}$$

cf. inequality (3.35) with $d \geqslant 3$.

Let us turn to the quantity (4.25) where the expression inside the modulus is the sum of the following terms:

$$\begin{aligned}
I^h_1(\psi^h) &= (\mu_n + h\mu'_n)(\mathbf{v}_n + h\mathbf{v}'_n, X^h\psi^h)_{\Omega^h} - (\partial_z\mathbf{v}_n + h\partial_z\mathbf{v}'_n, \partial_z(X^h\psi^h))_{\Omega^h}\\
&\quad -h^2(\nabla_x V, \nabla_x(X^h\psi^h))_{\Omega^h},\\
I^h_2(\psi^h) &= -(\mathbf{v}^h_{n\flat}\partial_z X^h, \partial_z\psi^h)_{\Omega^h} + (\partial_z\mathbf{v}^h_{n\flat}, \psi^h\partial_z X^h)_{\Omega^h},\\
I^h_3(\psi^h) &= h(\mu_n + h\mu'_n)((X^{\sqrt{h}} - X^h)\widetilde{\mathbf{v}}'_n, \psi^h)_{\Omega^h}\\
&\quad +h(\partial_z\widetilde{\mathbf{v}}'_n, \partial_z(X^h\psi^h))_{\Omega^h} - h(\partial_z(X^{\sqrt{h}}\widetilde{\mathbf{v}}'_n), \partial_z\psi^h)_{\Omega^h},\\
I^{h\pm}_4(\psi^h) &= h(\mu_n + h\mu'_n)(\chi^h_{\pm}w_{n\pm}, \psi^h)_{\Omega^h},\\
I^{h\pm}_5(\psi^h) &= -h(\nabla_x(\chi^h_{\pm}w_{n\pm}), \nabla_x\psi^h)_{\Omega^h}\\
I^{h\pm}_6(\psi^h) &= h^2(\mu_n + h\mu'_h)(X^hV, \psi^h)_{\Omega^h} - h^2(V\partial_z X^h, \partial_z\psi^h)_{\Omega^h} + h^2(\partial_z V, \psi^h\partial_z X^h)_{\Omega^h}.
\end{aligned} \tag{4.32}$$

We proceed with the term which is the simplest, due to the factor h^2. We repeat the calculation (4.27), with a small modification, take (2.25) into account and observe that

$$\begin{aligned}
|I^h_6(\psi^h)| &\leqslant ch^2\Bigg(\|X^hV; L^2(\Omega^h)\|\,\|\psi^h; L^2(\Omega^h)\|\\
&\quad + \frac{1}{h^2}\sum_{\pm}\int_{1-h\rho_\pm - ch^2}^{1-h\rho_\pm + ch^2}\big(|V(\eta,\pm t)|\,|\partial_z\psi^h(y,\pm t)| + |\psi^h(y,\pm t)|\,|\partial_z V(\eta,\pm t)|\big)\,dydz\Bigg)\\
&\leqslant ch^2(h^2 + h^{-4}h^2h^4)^{1/2}\|\psi^h; \mathcal{H}^h\| \leqslant Ch^3.
\end{aligned} \tag{4.33}$$

Let us now process the expression on the first line of the list (4.32):

$$\begin{aligned}
I^h_1(\psi^h) &= I^h_{11}(\psi^h) + I^h_{12}(\psi^h) + I^h_{13}(\psi^h),\\
I^h_{11}(\psi^h) &= h\mu'_n(\mathbf{v}_n, X^h\psi^h)_{\Omega^h} + h(\mu_n + h\mu'_n)(\mathbf{v}'_n, X^h\psi^h)_{\Omega^h},\\
I^h_{12}(\psi^h) &= -h(\partial_z\mathbf{v}'_n, \partial_z(X^h\psi^h))_{\Omega^h},\\
I^h_{13}(\psi^h) &= \mu_n(\mathbf{v}_n, X^h\psi^h)_{\Omega^h} - (\partial_z\mathbf{v}_n, \partial_z(X^h\psi^h))_{\Omega^h} - h^2(\nabla_x V, \nabla_x(X^h\psi^h))_{\Omega^h}.
\end{aligned} \tag{4.34}$$

As in formulas (3.4) and (3.5) we represent the test function ψ^h in the form $\overline{\psi}^h + \psi^h_\perp$. Repeating the calculations (3.8) and (3.14) yields

$$I^h_{11}(\psi^h) = h^3|\omega|\left(\mu_n'(H^2\mathbf{v}_n, X^h\overline{\psi}^h) + \mu_n(H^2\mathbf{v}_n', X^h\overline{\psi}^h)\right) + h^4|\omega|\mu_n'(H^2\mathbf{v}_n', X^h\overline{\psi}^h), \quad (4.35)$$

Repeating again with small modifications the calculations (3.13), (3.14), which are applied to the functions $\mathbf{v}_n'$, $X^h\psi^h$, we find that

$$\begin{aligned}&\left|I^h_{12}(\psi^h) + h^3|\omega|(H^2\partial_z\mathbf{v}_n', \partial_z(X^h\overline{\psi}^h))\right| \\ &\leqslant ch^2\left|\int_{\Omega^h} X^h(z)\partial_z\mathbf{v}_n'(z)\eta\cdot\nabla_y\psi^h(x)dx\right| \\ &\leqslant ch^2\|X^h\partial_z\mathbf{v}_n'; L^2(\Omega^h)\|\,\|\nabla_y\psi^h; L^2(\Omega^h)\| \leqslant ch^{5/2}.\end{aligned}$$

We emphasize that the singularity (4.1), (4.4) of the correction term $\mathbf{v}_n'$ is eliminated by the cut-off function (3.23) so that the $L^2(\Omega^h)$-norm of the product $X^h\partial_z\mathbf{v}_n'$ is $O(h^{1/2})$. The last expression in (4.34) is transformed by integration by parts as

$$\begin{aligned}I^h_{13}(\psi^h) &= (\mu_n\mathbf{v}_n + \partial_z^2\mathbf{v}_n + h^2\Delta_x V, X^h\psi^h))_{\Omega^h} - (\partial_\nu(\mathbf{v}_n + h^2V), X^h\psi^h))_{\partial\Omega^h} \\ &=: I^h_{14}(\psi^h) - I^h_{15}(\psi^h). \end{aligned}\quad (4.36)$$

The differentiation rule (2.3), estimate (2.25) and equation (2.4) show that

$$\begin{aligned}|I^h_{14}(\psi^h)| &\leqslant ch^2\left(\|\partial_z^2V; L^2(\Omega^h)\| + \|\partial_z\partial_\eta V; L^2(\Omega^h)\| + \|\partial_\eta V; L^2(\Omega^h)\|\right)\|\psi^h; L^2(\Omega^h)\| \\ &\leqslant ch^3. \end{aligned}\quad (4.37)$$

Furthermore, using the same estimate (2.25) as well as the representation of the normal vector (2.5), the boundary condition (2.6) and a simple trace inequality

$$h\int_{\partial\Omega^h} H(z)|\psi^h(x)|^2 ds_x \leqslant c\int_{\Omega^h}\left(|\nabla_x\psi^h(x)|^2 + |\psi^h(x)|^2\right)dx,$$

we conclude the relation

$$|I^h_{15}(\psi^h)| \leqslant ch^2\|H^{-1/2}\nabla_{(\eta,z)}V; L^2(\partial\Omega^h)\|\|H^{1/2}\psi^h; L^2(\partial\Omega^h)\| \leqslant ch^2hh^{-1/2} = ch^{5/2}. \quad (4.38)$$

Summing up, we observe that the product $X^h\overline{\psi}^h$ vanishes in the neighborhoods of the points $z = \pm 1$ and therefore the equation (4.12) can be reformulated as the integral identity

$$(H^2\partial_z\mathbf{v}_n', \partial_z(X^h\overline{\psi}^h)) - \mu_n(H^2\mathbf{v}_n', X^h\overline{\psi}^h) = \mu_n'(H^2\mathbf{v}_n, X^h\overline{\psi}^h).$$

Thus, we have estimated all terms of the expression $I^h_1(\psi^h)$ except the last one in (4.35) by $O(h^{5/2})$. But the remaining term meets the bound

$$h^4|\omega|\,|\mu_n'|\left|(H^2\mathbf{v}_n', X^h\overline{\psi}^h)\right| \leqslant ch^4\|\mathbf{v}_n'; \mathbf{L}\|\,\|\overline{\psi}^h; \mathbf{L}\| \leqslant ch^3,$$

because $\|\overline{\psi}^h; \mathbf{L}\| \leqslant ch^{-1}\|\psi^h; L^2(\Omega^h)\|$, where $\mathbf{L} = L^2_1(-1,1)$, can be deduced as in formula (3.8) while the function $\mathbf{v}_n'$ also lives in the weighted space $\mathbf{L}$ due the singularities (4.13).

We also mention the elementary estimate

$$|I^{h\pm}_4(\psi^h)| \leqslant ch\|\chi^h_\pm w_{n\pm}; L^2(\Omega^h)\|\,\|\psi^h; L^2(\Omega^h)\| \leqslant chh^{5/2} = ch^{7/2}, \quad (4.39)$$

where we take into account the boundedness of the $w_{n\pm}$ (see relations (4.9), (4.10) and (4.23)) and the fact that the support of the function $\chi^h_\pm$ belongs to the closure of a narrow conical set $\Omega^h_\pm = \{x \in \Omega^h : \pm z \geqslant 1 - 2h\rho_\pm\}$ of volume $O(h^5)$.

We now transform the last scalar product in (4.32) as follows:

$$I_5^{h\pm}(\psi^h) = -h(\nabla_x w_{n\pm}, \nabla_x(\chi_\pm^h \psi^h)_{\Omega^h} - h(w_{n\pm}\partial_z\chi_\pm^h, \partial_z\psi^h)_{\Omega^h} +$$
$$+h(\partial_z w_{n\pm}, \psi^h\partial_z\chi_\pm^h)_{\Omega^h} = I_{51}^{h\pm}(\psi^h) + I_{52}^{h\pm}(\psi^h) + I_{53}^{h\pm}(\psi^h).$$

In view of formulas (4.22) and (2.13), the derivative $\partial_z\chi_\pm^h$ differs from zero only for

$$|z \mp 1 \pm h\rho_\pm| \in \Big[\frac{1}{3}h\rho_\pm, \frac{2}{3}h\rho_\pm\Big] \quad \Leftrightarrow \quad |\xi_3^\pm| \in \Big[\frac{1}{3}\frac{1}{hH_\pm}, \frac{2}{3}\frac{1}{hH_\pm}\Big].$$

Hence, due to the decay rate $O(e^{-\kappa_0|\xi_3^\pm|})$ of the gradient $\nabla_{\xi^\pm} w_{n\pm}$ we obtain that

$$|I_{53}^{h\pm}(\psi^h)| \leqslant ch^{-m}e^{-\kappa_0/h}\|\psi^h; L^2(\Omega^h)\| \leqslant C_m h^{5/2} \tag{4.40}$$

for all $m \in \mathbb{R}$ and small $h > 0$ because $\kappa_0 > 0$; note that the multipliers h^{-m} with negative exponents are caused by the differentiation of the functions $w_{n\pm}$ and $\chi_\pm^h$. As in (4.18), the inequality $|w_{n\pm}(\xi^\pm)| \leqslant c_\pm$ and the smallness of the set $\mathrm{supp}\chi_\pm^h \subset \overline{\Omega}_\pm^h$ yield the relation

$$|I_{52}^{h\pm}(\psi^h)| \leqslant chh^{-1}|\mathrm{supp}\chi_\pm^h|^{1/2}\,\|\partial_z\psi^h; L^2(\Omega^h)\| \leqslant C_m h^{5/2}.$$

The immediate objective becomes to show that

$$\Big|I_{51}^{h\pm}(\psi^h) + h^3 H_\pm\rho_\pm \int_\Pi \nabla_{\xi^\pm} w_\pm(\xi^\pm)\cdot\nabla_{\xi^\pm}\Psi_\pm^h(\xi^\pm)d\xi^\pm\Big| \leqslant ch^3. \tag{4.41}$$

Here, $\Psi_\pm^h$ is the product $\chi_\pm^h\psi^h$ written in the stretched coordinates $(\eta, \xi_3^\pm)$. The estimate

$$\|\Psi_\pm^h; H^1(\Omega^h)\|^2 \leqslant c(\|\psi^h; H^1(\Omega^h)\|^2 + h^{-2}\|\psi^h; L^2(\Omega_\pm^h)\|^2) \leqslant c\|\psi^h; H^1(\Omega^h)\| \leqslant C, \tag{4.42}$$

is obtained from the formula

$$\|\psi^h; L^2(\Omega_\pm^h)\| \leqslant ch\|\nabla_x\psi^h; L^2(\Omega_\pm^h)\|, \tag{4.43}$$

which follows from the Friedrichs and Poincare inequalities

$$\begin{aligned}&\|\psi^h; L^2(\Omega_\pm^{h\prime})\|^2 \leqslant ch^4\|\nabla_y\psi^h; L^2(\Omega_\pm^{h\prime})\|^2,\\ &\|\psi^h; L^2(\Omega_\pm^h)\|^2 \leqslant c(h^2\|\nabla_x\psi^h; L^2(\Omega_\pm^h)\|^2 + \|\psi^h; L^2(\Omega_\pm^{h\prime})\|^2).\end{aligned} \tag{4.44}$$

Here, we took into account the transversal and longitudinal sizes, $O(h^2)$ and $O(h)$, respectively, of the sets $\Omega_\pm^{h\prime} = \{x \in \Omega_\pm^h : \pm z > 1 - h\rho_\pm\}$ and $\Omega_\pm^h$ as well as the Dirichlet condition for ψ^h on $\partial\Omega_\pm^{h\prime} \cap \Gamma_\pm^h$.

We can also write

$$\begin{aligned}&\nabla_y w_{n\pm}(\eta, \xi_3^\pm) = h^{-1}H(z)^{-1}\nabla_\eta w_{n\pm}(\eta, \xi_3^\pm),\\ &\partial_z w_{n\pm}(\eta, \xi_3^\pm) = \mp h^{-2}(H_\pm\rho_\pm)^{-1}\partial_{\xi_3^\pm} w_{n\pm}(\eta, \xi_3^\pm) - \partial_z H(z)H(z)^{-1}\eta\cdot\nabla_\eta w_{n\pm}(\eta, \xi_3^\pm),\\ &dydz = \mp h^4 H_\pm\rho_\pm H(z)^2 d\eta d\xi_3^\pm,\\ &\Big|\frac{1}{H(z)} - \frac{1}{hH_\pm\rho_\pm}\Big| \leqslant c_\pm(1 + |\xi_3^\pm|)(1 + h|\xi_3^\pm|) \text{ and } \mp z + 1 - h\rho_\pm \in \Big[-\frac{2}{3}h\rho_\pm, \frac{2}{3}h\rho_\pm\Big]\end{aligned} \tag{4.45}$$

The last inequality is valid on the supports of the cut-off functions $\chi_\pm^h$, which are the multipliers of the boundary layer terms in the ansatz (4.18). Under the same restrictions the relations

$$|h^2 H_\pm\rho_\pm H(z)^2 - h^6(H_\pm\rho_\pm)^3| \leqslant c_H h^7, \tag{4.46}$$

$$|h^2 H_\pm\rho_\pm\nabla_x w_\pm(\eta, \xi_3^\pm) - \nabla_{(\eta,\xi_3^\pm)} w_\pm(\eta, \xi_3^\pm)| \leqslant ch|\nabla_{\eta,\xi_3^\pm} w_\pm(\eta, \xi_3^\pm)|, \tag{4.47}$$

hold in the set $\Omega^h_\pm$, and the estimate (4.47) is valid for the function $\Psi^h_\pm$, too. Recalling the inclusion $e^{\kappa_0|\xi^\pm_3|}\nabla_{\eta,\xi^\pm_3} w_\pm \in L^2(\Pi)$ and the inequalities (4.42), (3.27) we obtain the desired estimate (4.41).

Since the test function $\Psi^h_\pm$ has a compact support, the solution (4.9) of the problem (4.8) with growth as $\xi^\pm_3 \to +\infty$ satisfies the integral identity $(\nabla_{\xi^\pm} W, \nabla_{\xi^\pm}\Psi^h_\pm)_\Pi = 0$, which is converted by means of formulas (4.23), (2.13) and simple transformations into

$$\int\limits_\Pi \nabla_{\xi^\pm} w_{n\pm}(\eta,\xi^\pm_3)\cdot\nabla_{\xi^\pm}\Psi^h_\pm(\xi^\pm)d\xi^\pm = \mathbf{v}_n(\pm1)H_\pm\int\limits_\Pi \partial_{\xi^\pm_3}(\xi^\pm_3\mathcal{X}_\pm(\xi^\pm_3))\partial_{\xi^\pm_3}\Psi^h_\pm(\xi^\pm)d\xi^\pm$$

$$= \mathbf{v}_n(\pm1)H_\pm\int\limits_\Pi \big(\Psi^h_\pm(\xi^\pm)\partial_{\xi^\pm_3}\mathcal{X}_\pm(\xi^\pm_3) - \xi^\pm_3\partial_{\xi^\pm_3}\mathcal{X}_\pm(\xi^\pm_3)\partial_{\xi^\pm_3}\Psi^h_\pm(\xi^\pm)\big)d\xi^\pm =: J_\pm(\Psi^h_\pm). \tag{4.48}$$

We now process the expression $I^h_2(\psi^h)$. Due to Lemma 2.4 and formula (4.1) the relation

$$\mathbf{v}^h_{n\flat}(z) = \mathbf{v}^h_n(\pm1)\Big(1 - h\sum_\pm \rho_\pm\chi_\pm(z)(1\mp z)^{-1}\Big) + O(h^2) =$$

$$= \mathbf{v}_n(\pm1)\Big(1 - h\sum_\pm \rho_\pm(h\rho_\pm + h^2\rho_\pm H_\pm\xi^\pm_3)^{-1}\Big) + O(h^2) = h\mathbf{v}_n(\pm1)H_\pm\xi^\pm_3 + O(h^2)$$

is valid in the set $\upsilon^h_\pm$. Definitions (3.23) and (4.1) of the cut-off function X^h and the stretched coordinates $\xi^\pm_d$, respectively, show that

$$X^h(z) = \mathcal{X}_\pm(\xi^\pm_d), \quad [\Delta_x, X^h(z)] = h^{-4}[\Delta_x,\mathcal{X}_\pm(\xi^\pm_d)] \text{ for } \pm z > 0. \tag{4.49}$$

Eventually, the coordinate change $x \mapsto (\eta,\xi^\pm_3)$ and formulas (4.45) and (4.46) yield the inequality

$$\Big|I^{h\pm}_2(\psi^h) + h^3\sum_\pm H_\pm\rho_\pm J_\pm(\Psi^h_\pm)\Big| \leqslant ch^3, \tag{4.50}$$

which contains the right-hand side of the identity (4.48). Thus, we derive from (4.41) and (4.50) the estimate

$$\sum_\pm \Big|I^{h\pm}_{51}(\psi^h) - I^{h\pm}_2(\psi^h)\Big| \leqslant ch^3. \tag{4.51}$$

It remains to process the expression $I^h_3(\psi^h)$ in (4.32) which, owing to the coincidence of the cut-off functions X^h and $X^{\sqrt{h}}$ in the set $\Omega^h\setminus(\Omega^h_+\cup\Omega^h_-)$, reduces to the sum of the corresponding scalar products in $L^2(\Omega^h_\pm)$. Hence, by means of the estimate (4.29) and, for example, the inequality

$$h|(\partial_z\widehat{\mathbf{v}}'_n, (X^h - X^{\sqrt{h}})\partial_z\psi)_{\Omega^h} \leqslant ch\|\partial_z\psi; L^2\Omega^h\|\Big(h^2\sum_\pm\int\limits_{ch}^{Ch} H(1\mp t)^2|\partial_z\widehat{\mathbf{v}}'_n(1\mp t)|^2dt\Big)^{1/2} \leqslant ch^{5/2}$$

derived from formulas (4.20), (3.23), (4.21), we conclude that

$$|I^{h\pm}_3| \leqslant ch^{5/2}. \tag{4.52}$$

Collecting the estimates (4.33)–(4.41) and (4.50)–(4.52) shows that the supremum (4.25) does not exceed $ch^{5/2}$. Thus, the calculations and considerations in Sections 3 and 4 § 3, which led to Theorem 3.3 for $d > 3$, also prove Theorem 4.1 and simultaneously Theorem 3.3 for $d = 3$.

4.5. Remarks on boundary layers and asymptotics of eigenfunctions. The main complication in the structure of the asymptotic approximation (4.18) is caused by the incomplete construction of the boundary layers, namely the function (4.23) does not decay at infinity but tends to a constant as $\xi_3^\pm \to +\infty$. Moreover, (4.18) involves the cut-off function (4.21), which is mismatched with the stretched coordinates (4.1). In Section 3 § 6 we will discuss infinite decompositions of the eigenpairs $\{\lambda_n^h, u_n^h\}$ and will show how it becomes possible to make the boundary layer to decay and simultaneously to fix the arbitrariness in the choice of the correction term $\mathbf{v}_n'$ in the ansatz (4.18). We emphasize that incompleteness of the asymptotic approximation used until now does not affect the main result, that is, Theorem 3.3 of the paper. Also, we will prove the exact bound $C_3^{(n)} h^2(1 + |\ln h|)$ in the estimate (4.17), where the logarithmic growth of the component $\widehat{\mathbf{v}}_n'$ of the function $\mathbf{v}_n'$ shows up.

Lemma 4.2. *The component $\widehat{\mathbf{v}}_n'$ of the solution* (4.13) *of the equation* (4.12) *is of the form*

$$\widehat{\mathbf{v}}_n'(z) = \widetilde{\mathbf{v}}_n'(z) - \sum_\pm \chi_\pm(z)\Big(\mathbf{c}_n^\pm + 2\rho_\pm^{d-2}\mathbf{v}_n(\pm 1)\frac{\partial_t \widetilde{H}_\pm(0)}{H_\pm}\ln(1 \mp z)\Big),$$

where $\mathbf{c}_n^\pm$ are some constants, $H_\pm$ and $\widetilde{H}_\pm$ are ingredients of the profile function H, see (1.2), *and the remainder $\widetilde{\mathbf{v}}_n' = O(1 \mp z)(1 + |\ln z|))$ is negligible. If at least one of the numbers $\partial_t \widetilde{H}(0)$ is not null, then $\widehat{\mathbf{v}}_n' \notin \mathbf{H}$.*

Proof. The term $\mathcal{V}_\pm^2(t) = \mathbf{c}_n^\pm + 2H_\pm^{-1}\partial_t \widetilde{H}_\pm(0)\ln t$ is a solution of the inhomogeneous model equation (2.19) with $d = 3$, that is,

$$-H_\pm^{d-1}\frac{d}{dt}\Big(t^{d-1}\frac{d\mathcal{V}_\pm^2}{dt}(t)\Big) = (d-1)H_\pm^{d-2}\partial_t\widetilde{H}_\pm(0)\frac{d}{dt}\Big(t^d\frac{d}{dt}\mathcal{V}^1(t)\Big), \quad t \in \mathbb{R}_+, \tag{4.53}$$

where the right-hand side is explained by the term of order $1 \mp z$ in the representation (1.2) of the profile function H near the endpoints of the interval $(-1, 1)$. ⊠

Remark 4.3. The change $x \mapsto \xi^\pm$ leads to the relation $\ln(1 \mp z) = \ln h + \ln \rho_\pm + \ln(1 + hH_\pm\xi_d^\pm)$ and, therefore, higher-order terms in the asymptotic expansions of the eigenvalues and eigenfunctions for the problem in $\Omega^h \subset \mathbb{R}^3$ get linear or polynomial dependence on the parameter $|\ln h|$. The same happens for $d > 3$ as well, because the singular solution (4.13) of equation (4.12) clearly admits the representation

$$\mathbf{v}'(z) = \widetilde{\mathbf{v}}'(z) + \sum_\pm \chi_\pm(x)\Big(\sum_{j=0}^{d-3}\mathbf{c}_j^\pm(1 \mp z)^{2-d+j} + \mathbf{c}_{d-2,0}^\pm + \mathbf{c}_{d-2,1}^\pm \ln(1 \mp z)\Big) \tag{4.54}$$

where $\mathbf{c}_j^\pm$ and $\mathbf{c}_{d-2,p}^\pm$ are certain coefficients and $\widetilde{\mathbf{v}}' \in C^\infty(-1,1) \cap C[-1,1]$, $\widetilde{\mathbf{v}}'(\pm 1) = 0$. We emphasize that the terms of order $(1 \mp z)^j$, $j \geqslant 1$, in the expansion (4.54) also depend polynomially on $\ln(1 \mp z)$. See § 5 for the planar case $d = 2$. ⊠

As in Section 3 § 4, we can now employ the above formulas for the quantities (4.24) and (3.33) in order to justify the asymptotics of the eigenfunctions.

Theorem 4.4. *For all $n \in \mathbb{N}$, there exist positive numbers $h_3^{(n)}$ and $C_3^{(n)}$ such that the eigenfunctions of the problems* (1.8) *and* (2.14), *normalized by the conditions* (1.10) *and* (2.18), *satisfy the inequality*

$$\big\|u_n^h - h^{-1}|\omega|^{-1/2}\mathbf{v}_n; H^1(\Omega^h)\big\| \leqslant C_3^{(n)}h^{1/2} \text{ for } h \in (0, h_3^{(n)}]. \tag{4.55}$$

Proof. The second part of Lemma 3.2 and the information of Theorem 4.1 about the eigenvalue λ_n^h lead to the relation

$$\|\mathcal{U}_n^h - \|\mathbf{u}_n^h;\mathcal{H}^h\|^{-1}\mathbf{u}_n^h;\mathcal{H}^h\| \leqslant C_3^{(n)} h^{5/2}. \tag{4.56}$$

Recalling formulas (3.37), (4.31) and (3.16), we turn (4.56) into the inequality

$$\|u_n^h - h^{-1}|\omega|^{-1/2}\mathbf{u}_n^h; H^1(\Omega^h)\| \leqslant C_3^{(n)} h^{1/2}.$$

We use the estimates (4.26) and (4.30) to move the "excess" terms $hX^{\sqrt{h}}\widehat{v}_n'$ and $\chi_\pm^h w_\pm$ in the definition (4.18) of the almost eigenfunction $\mathbf{u}_n^h$ to the remainder. This means that they can be omitted on the left-hand side of (4.56). The term $h^2 X^h V$ is processed by means of an evident modification of the calculations (4.27) and (4.33). Finally, due to the simple formula

$$\|X^h \chi_\pm (1 \mp z)^{-1}; H^1(\Omega^h)\|^2 \leqslant ch^2 \int_{h\rho_\pm}^{1/2} t^2(t^{-4} + t^{-2})dt \leqslant c_0 h$$

we can replace $\mathbf{v}_{n\flat}^h \mapsto \mathbf{v}_n$ (see (4.18), (4.19)) on the left-hand side of (4.56) . This yields the inequality (4.55). ⊠

4.6. Asymptotics of the first eigenvalue. In Theorem 3.3, formula (3.33) with $n = 1$ does not contain much information on the first eigenvalue $\lambda_1^h > 0$ of the problem (1.5)–(1.7), since $\mu_1 = 0$. However, in the case $d \geqslant 3$ we write the more precise asymptotic decomposition (cf. (4.11)).

$$\lambda_1^h = 0 + h^{d-2}\mu_1' + \dots,$$

and we repeat the reasoning and calculation of the previous sections in order to construct μ_1' in the correction term. In the outer expansion (4.3) of the eigenfunction u_1^h we have the null term $V = 0$ and constant term

$$\mathbf{v}(z) = \mathbf{v}_1 = \Big(\int_{-1}^{1} H(z)^{d-1} dz \Big)^{-1/2}.$$

The last formula comes from the normalization condition (2.18) for the constant eigenfunction in the space $\mathbf{L}$. The correction terms $\mathbf{v}' = \mathbf{v}_1'$ and μ_1' have to be found from equation (4.12) with $n = 1$. A solution of the form (4.13) exists, if the relation (4.16) holds, and this turns (without the restriction $d = 3$) into

$$\mu_1' = (d-2)\mathbf{v}_1^2 (H_+^{d-1}\rho_+^{d-2} + H_-^{d-1}\rho_-^{d-2}). \tag{4.57}$$

Let us formulate a result, which follows by the considerations in Sections 4 § 4 and 3 § 3.

Theorem 4.5. *If $d \geqslant 3$, there exist positive numbers $c_d^{(1)}$ and $h_d^{(1)}$ such that*

$$|\lambda_1^h - h^{d-2}\mu_1'| \leqslant c_d^{(1)} h^{d-3/2} \textit{ for } h \in (0, h_d^{(1)}]. \tag{4.58}$$

Here, λ_1^h is the first eigenvalue of problem (1.5)–(1.7) *and $\mu_1' > 0$ is the quantity* (4.57).

In the planar case $d = 2$ the asymptotic expansions of eigenvalues are essentially different, and we will discuss them in the next section, cf. Proposition 5.7. Finally, we mention that formula (4.58), and the corresponing relation (5.35) to be proven in the case $d = 2$, give an asymptotically precise estimate of the constant in the Friedrichs inequality in the space $H_0^1(\Omega^h; \Gamma_\pm^h)$. This can also be verified by the calculations in Section 1. § 3 and the Hardy inequality (2.16).

Proposition 4.6. *For all functions $u^h \in H^1_0(\Omega^h;\Gamma^h_\pm)$, there holds the inequality*

$$\|u^h; L^2(\Omega^h)\| \leqslant c_d h^{(2-d)/2}(1+|\ln h|)^{\delta_{d,2}}\|\nabla_x u^h; L^2(\Omega^h)\|,$$

where the coefficient c_d is independent of the function u^h and the parameter $h \in (0, h(d)]$ for some $h(d) > 0$.

5. Planar spindle. Self-adjoint extensions of the differential operator.

5.1. Unbounded self-adjoint operator in a weighted class. The weighted Sobolev space $V^l_\beta(-1,1)$, also called the Kondratiev space [4], is endowed with the norm

$$\|\mathbf{v}; V^l_\beta(-1,1)\| = \left(\sum_{k=0}^{l}\|\partial_z^k\mathbf{v}; L^2_{\beta-l+k}(-1,1)\|^2\right)^{1/2}, \tag{5.1}$$

where $l \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$ and $\beta \in \mathbb{R}$ are the smoothness and weight exponents and the norm of the weighted Lebesgue space $L^2_\gamma(-1,1)$ is of the form (2.11). In the next lemma we will show that the mapping

$$V^{l+2}_{l+(d+1)/2}(-1,1) \ni \mathbf{v} \ \mapsto \ \mathbf{f} \in V^l_{l+(d+1)/2}(-1,1) \tag{5.2}$$

associated with the solving of the equation

$$-\partial_z(H(z)^{d-1}\partial_z\mathbf{v}(z)) = H(z)^{d-1}\mathbf{f}(z), \quad z \in (-1,1), \tag{5.3}$$

is Fredholm, however, in the case $d \geqslant 3$ only. In order to treat the case $d = 2$ we introduce "stepwise" weighted space $V^{l,0}_\beta(-1,1)$ equipped with the norm

$$\|\mathbf{v}; V^{l,0}_\beta(-1,1)\| = \left(\sum_{k=1}^{l}\|\partial_z^k\mathbf{v}; L^2_{\beta-l+k}(-1,1)\|^2 + \|\mathbf{v}; L^2_{\beta-l+1}(-1,1)\|^2\right)^{1/2}; \tag{5.4}$$

we will soon consider also the mapping

$$V^{l+2,0}_\beta(-1,1) \ni \mathbf{v} \ \mapsto \ \mathbf{f} \in V^l_\beta(-1,1) \tag{5.5}$$

with

$$d = 2 \quad \text{and} \quad \beta \in (l+1/2, l+3/2). \tag{5.6}$$

The restrictions (5.6) on the weight index β are reasoned by the requirements $\chi_\pm \in V^{l+2,0}_\beta(-1,1)$ and $(z \mapsto \chi_\pm(z)\ln(1\mp z)) \notin V^{l+2,0}_\beta(-1,1)$ (see (4.14) and (2.20)). In other words, constants live in the space $V^{l+2,0}_\beta(-1,1)$ but logarithmic singularities do not.

Note that the dependence of the weight exponent $\beta - l + k$ on the order of the differentiation is linear in the both norms (5.1) and (5.4), except that the latter has a step at the value $k = 1$. We emphasize that the space $\mathbf{H} = \mathbf{V}^1_{(d-1)/2}(-1,1)$ coincides with $V^1_{(d-1)/2}(-1,1)$ for $d \geqslant 3$ due to the Hardy inequality (2.16) and with $V^{1,0}_{(d-1)/2}(-1,1)$ for $d = 2$, by definition (2.10). Moreover, $L^2_\beta(-1,1) = V^0_\beta(-1,1)$.

In order to study properties of the mappings (5.2) and (5.5) we apply the Kondratiev theory [4] (see also the monographs [5, 23] and others), a standard tool in the applications to elliptic boundary value problem in domains with conical boundaries. The ordinary differential equation (5.3) can of course be examined with other, much simpler methods but by interpreting the endpoints of the interval $(-1,1)$ as tops of one-dimensional cones, that is, the semi-axes $\mathbb{R}_\mp$, we can refer to general results and thus avoid any calculations.

Lemma 5.1. *The mappings (5.2) for $d \geqslant 3$ and (5.5) for $d = 2$ are Fredholm with one-dimensional kernels and co-kernels generated by constant functions.*

Proof. We first consider the case $d > 2$. For $m > 0$, the integral identity

$$(H^{d-1}\partial_z \mathbf{v}, \partial_z \varphi) + m(H^{d-1}\mathbf{v}, \varphi) = (H^{d-1}\mathbf{f}, \varphi) \qquad \forall \varphi \in \mathbf{H} \tag{5.7}$$

has a unique solution $\mathbf{v} \in \mathbf{H}$ because the left-hand side of (5.7) is a scalar product in the space $\mathbf{H}$ and the right-hand side is a continuous functional in $\mathbf{H}$, due to the relations

$$|(H^{d-1}\mathbf{f}, \varphi)| = |(H^{(d+1)/2}\mathbf{f}, H^{(d-3)/2}\varphi)| \leqslant\leqslant \|\mathbf{f}; V^0_{(d+1)/2}(-1,1)\| \, \|\varphi; V^0_{(d-3)/2}(-1,1)\|$$
$$\leqslant \|\mathbf{f}; V^l_{l+(d+1)/2}(-1,1)\| \, \|\varphi; V^{l+2}_{l+(d+1)/2}(-1,1)\|. \tag{5.8}$$

By Lemma 2.3, the scalar product $(H^{d-1}\mathbf{v}, \varphi)$ in (5.7) gives rise to a compact operator in $\mathbf{H}$ and the quadratic form

$$(H^{d-1}\partial_z \mathbf{v}, \partial_z \mathbf{v})$$

is null only for constant functions. Thus, the Fredholm alternative shows that the integral identity

$$(H^{d-1}\partial_z \mathbf{v}, \partial_z \varphi) = (H^{d-1}\mathbf{f}, \varphi) \qquad \forall \varphi \in \mathbf{H}$$

which is associated to the equation (5.3) and coincides with (5.7) for $m = 0$, gets only one compatibility condition

$$(H^{d-1}\mathbf{f}, 1) = \int_{-1}^{1} H(z)^{d-1}\mathbf{f}(z)\, dz = 0. \tag{5.9}$$

In the many-dimensional case $d \geqslant 3$ the properties of the weak solutions can be passed to solutions $\mathbf{v} \in V^{l+2}_{l+(d+1)/2}(-1,1) \subset \mathbf{H}$ of the equation (5.3) by using of the known estimate [4]

$$\|\mathbf{v}; V^{l+2}_\beta(-1,1)\| \leqslant c(\|H^{1-d}\partial_z H^{d-1}\partial_z \mathbf{v}; V^l_\beta(-1,1)\| + \|\mathbf{v}; V^0_{\beta-l-2}(-1,1)\|). \tag{5.10}$$

This holds true for any weight exponent, for instance $\beta = l + (d+1)/2$, and can be proven with the help of local elliptic estimates for equations on the intervals $\{z : \pm z \in (1 - 2^{-k+1}, 1 - 2^{-k})\}$, which collapse into the points $z = \pm 1$ (see details in [5, Ch. 3, § 5]).

In the planar case $d = 2$ it is necessary to modify the reasoning by using the stepwise weighted norms (5.4), as presented in the paper [24] and the monograph [5, Ch. 8 § 4]. Namely, inequalities (5.8) are turned into the following:

$$|(H\mathbf{f}, \varphi)| \leqslant \|\mathbf{f}; V^0_{\beta-l}(-1,1)\| \, \|H^{1+l-\beta}\varphi; L^2(-1,1)\| \leqslant \|\mathbf{f}; V^l_\beta(-1,1)\| \, \|\varphi; \mathbf{H}\|.$$

Here, we took into account formulas (5.6) and the definitions of the norms (5.1) and (2.15). Moreover, the estimate (5.10) is replaced by the estimate (5.6) which holds true for the weight index β in (5.6),

$$\|\partial_z \mathbf{v}; V^{l+1}_{l+(d+1)/2}(-1,1)\| \leqslant c(\|H^{1-d}\partial_z H^{d-1}\partial_z \mathbf{v}; V^l_{l+(d+1)/2}(-1,1)\|$$
$$+ \|\partial_z \mathbf{v}; V^0_{(d-1)/2}(-1,1)\|). \tag{5.11}$$

All other considerations remain the same. ⊠

We define the Hilbert space $\mathbf{L}$ which coincides algebraically and topologically with

$$V^0_{(d-1)/2}(-1,1)$$

and is equipped with the scalar product

$$\langle \mathbf{v}, \mathbf{u} \rangle = (H^{(d-1)/2}\mathbf{v}, H^{(d-1)/2}\mathbf{u}). \tag{5.12}$$

Let us introduce in $\mathbf{L}$ the unbounded operator $\mathbf{A}_0$ with the differential expression $-H^{1-d}\partial_z H^{d-1}\partial_z$ and the domain $\mathcal{D}(\mathbf{A}_0) = C^\infty_{(-1,1)}$. Clearly, $\mathbf{A}_0$ is a symmetric operator.

Proposition 5.2. *The domain of the closure* $\overline{\mathbf{A}}_0$ *of the operator* $\mathbf{A}_0$ *equals*

$$\mathcal{D}(\overline{\mathbf{A}}_0) = V^2_{(d-1)/2}(-1,1) \text{ for } d \neq 4,$$
$$\mathcal{D}(\overline{\mathbf{A}}_0) = V^{2,0}_{(d-1)/2}(-1,1) \text{ for } d = 4. \tag{5.13}$$

while its differential expression is the same $-H^{1-d}\partial_z H^{d-1}\partial_z$ *as that of* $\mathbf{A}_0$.

Proof. We employ another estimate [4]

$$\|\mathbf{v}; V^2_\beta(-1,1)\| \leqslant c(\|H^{1-d}\partial_z H^{d-1}\partial_z \mathbf{v}; V^0_\beta(-1,1)\| + \|\mathbf{v}; L^2(-1/2,1/2)\|) \tag{5.14}$$

which is of a different nature in comparison with the estimate (5.11). It is a consequence of the Fredholm property of the continuous mapping

$$H^{1-d}\partial_z H^{d-1}\partial_z : V^2_\beta(-1,1) \ \to \ V^0_\beta(-1,1) \tag{5.15}$$

generated by the equation (5.3) multiplied by $H(z)^{1-d}$. The Kondratiev theory guarantees that the operator (5.15) is Fredholm for any $\beta \in \mathbb{R}$ with exception of the forbidden indices

$$\beta_0(d) = 3/2 \quad \text{and} \quad \beta_1(d) = d - 1/2. \tag{5.16}$$

These are determined by the solutions $\mathcal{V}^0_\pm$ and $\mathcal{V}^1_\pm$ of the model equation (2.19) (see formulas (2.20) and, for example, [5, Ch. 3 § 1]), when one of the integrals

$$\int_{-1}^{1} H(z)^{2(\beta_p(d)-2)} |\chi_\pm(z)\mathcal{V}^p_\pm(1 \mp z)|^2 \, dz, \quad p = 0, 1, \tag{5.17}$$

diverges at the logarithmic rate. In other words, the product $\chi_\pm(z)\mathcal{V}^p_\pm(1 \mp z)$ does not belong the space $V^2_{\beta_p(d)}(-1,1)$ but falls into $V^2_{\beta_p(d)+\varepsilon}(-1,1)$ for all $\varepsilon > 0$.

Observing that $\|\mathbf{v}; L^2(-1/2,1/2)\| \leqslant c\|\mathbf{v}; V^0_{(d-1)/2}(-1,1)\|$, we apply the estimate (5.14) with the weight exponent $\beta = (d-1)/2$, which is different from (5.16) for $d = 2, 3, 5, 6, \dots$. As a result, we find that in the definition of the closed operator

$$\mathcal{D}(\mathbf{A}_0) \ni \mathbf{v}_n \ \to \ \mathbf{v}_\infty, \ \mathbf{A}_0\mathbf{v}_n \ \to \ \mathbf{f}_\infty \ \ \mathbf{L} \quad \Rightarrow \quad \mathbf{v}_\infty \in \mathcal{D}(\overline{\mathbf{A}}_0), \ \overline{\mathbf{A}}_0\mathbf{v}_\infty = \mathbf{f}_\infty$$

the limit $\mathbf{v}_\infty$ belongs to the space $V^2_{(d-1)/2}(-1,1)$. It is also very important to note that the space $V^2_{(d-1)/2}(-1,1)$ is mapped onto the space $V^0_{(d-1)/2}(-1,1) = \mathbf{L}$ by the differential operator $H^{d-1}\partial_z H^{d-1}\partial_z$.

In the exceptional case $d = 4$ the integral (5.17) diverges, if $p = 0$, i.e., for the constant solution of the model equation (2.19). This situation allows us to employ a modification[2] of the Kondratiev theory [4] (see also [5, Ch. 8 § 4 and 9 § 6]) and use the estimate

$$\|\mathbf{v}; V^{2,0}_{\beta_0(4)}(-1,1)\| \leqslant c(\|H^{1-d}\partial_z H^{d-1}\partial_z \mathbf{v}; V^0_{\beta_0(d)}(-1,1)\| + \|\mathbf{v}; L^2(-1/2,1/2)\|) \tag{5.18}$$

which follows from the Fredholm property of the operator

$$-H^{1-d}\partial_z H^{d-1}\partial_z : V^{2,0}_{\beta_0(4)}(-1,1) \to V^0_{\beta_0(4)}(-1,1). \tag{5.19}$$

This proves the last of the formulas (5.13). Let us pay attention to two points. First, the "inconvenient" constant component of the solution $\mathbf{v}$ is eliminated by the differential operator (5.19), and it also disappears from each term in the sum over $k \geqslant 1$ in (5.4). Moreover, according to definition (5.13), the exponent of the power of the weight multiplier in the norm

[2]It is worth mentioning that the last section of the paper [4] contains an outline of a possible modification of the definition of the weighted space V^ℓ_β needed for polynomial solutions of the model problem in a cone. A complete theory of the Neumann problem in the mechanics of cracks [25, 26] is presented in the paper [24] and the book [5, Ch. 8 § 4].

$\|\mathbf{v}; V^0_{\beta_0(4)-1}(-1,1)\|$ is increased by one, which makes the corresponding integral convergent for $\mathbf{v} = 1$. Second, it is possible to insert into formulas (5.18) and (5.19) the "limit" exponent $\beta = \beta_0(4)$ (compare relations (5.6) with $l = 0$ and (5.6) with $d = 2$ and $\beta_0(2) = \beta_1(2) = 3/2$), because there is no need to avoid the logarithmic singularity as it does not appear in (2.20) for $d = 4$. Finally, we emphasize that the space $V^{2,0}_{3/2}(-1,1)$ with "stepwise" norm (5.4) contains all functions of $C^\infty[-1,1]$ in contrast to the space $V^2_{3/2}(-1,1)$ with "homogeneous" norm (5.4), the elements of which must vanish at the points $z = \pm 1$. ⊠

5.2. Self-adjoint extensions of the operator in the weighted space. We aim to determine the dimensions d in which the operator $\overline{\mathbf{A}}_0$ is self-adjoint, and to this end we first describe the adjoint operator $\mathbf{A}_0^*$.

Proposition 5.3. *The domain of the adjoint operator* $\mathbf{A}_0^*$ *is as follows:*

$$\mathcal{D}(\mathbf{A}_0^*) = V^2_{(d-1)/2}(-1,1) \quad \text{for } d = 5, 6, \dots, \tag{5.20}$$

$$\mathcal{D}(\mathbf{A}_0^*) = V^{2,0}_{(d-1)/2}(-1,1) \quad \text{for } d = 4, \tag{5.21}$$

$$\mathcal{D}(\mathbf{A}_0^*) = \Big\{ \mathbf{v}(z) = \widetilde{\mathbf{v}}(z) + \sum_{\pm} \chi_\pm(z) \Big(a_\pm + \frac{b_\pm}{H_\pm^2} \Big(\frac{1}{1 \mp z} + 2 \frac{\partial_z \widetilde{H}_\pm(0)}{H_\pm} \ln(1 \mp z) \Big) \Big) \Big| \ \widetilde{\mathbf{v}} \in V^2_{(d-1)/2}(-1,1),\ a_\pm, b_\pm \in \mathbb{C} \Big\} \quad \text{for } d = 3, \tag{5.22}$$

$$\mathcal{D}(\mathbf{A}_0^*) = \Big\{ \mathbf{v}(z) = \widetilde{\mathbf{v}}(z) + \sum_{\pm} \chi_\pm(z) \Big(a_\pm + \frac{b_\pm}{H_\pm} \Big(\ln \frac{1}{1 \mp z} + \frac{\partial_z \widetilde{H}_\pm(0)}{H_\pm} (1 \mp z) \Big) \Big) \Big| \ \widetilde{\mathbf{v}} \in V^2_{(d-1)/2}(-1,1),\ a_\pm, b_\pm \in \mathbb{C} \Big\} \quad \text{for } d = 2. \tag{5.23}$$

Here, $\chi_\pm$ *are the cut-off functions* (4.14).

Proof. The adjoint operator is determined as follows:

$$\mathbf{v}, \mathbf{f} \in \mathbf{L} = V^0_{(d-1)/2}(-1,1),\ \langle \mathbf{f}, \varphi \rangle = \langle \mathbf{v}, \mathbf{A}_0 \varphi \rangle \quad \forall \varphi \in \mathcal{D}(\mathbf{A}_0) \quad \Rightarrow \quad \Rightarrow \quad \mathbf{v} \in \mathcal{D}(\mathbf{A}_0^*),\ \mathbf{f} = \mathbf{A}_0^* \mathbf{v}. \tag{5.24}$$

The definition of the operator $\mathbf{A}_0$ and relation (5.12) show that the equality in (5.24) containing the test function φ is equivalent to the integral identity

$$(H^{d-1}\mathbf{f}, \varphi) = -(\mathbf{v}, \partial_z H^{d-1} \partial_z \varphi) \quad \forall \varphi \in C_c^\infty(-1,1).$$

Hence, standard local elliptic estimates (see, e.g., [27, Ch. 2 § 5, 6]) imply that $\mathbf{v} \in H^2_{\mathrm{loc}}(-1,1)$ and the differential equation (5.3) holds. We apply the estimate (5.10) (with the exponents $l = 0$ and $\beta = (d+3)/2$) whose right-hand and, therefore, left-hand side are finite due to the assumption in (5.24) and the quite trivial inequality

$$\|\mathbf{f}; V^0_{(d+3)/2}(-1,1)\| \leqslant c \|\mathbf{f}; V^0_{(d-1)/2}(-1,1)\|.$$

Hence, we see that the function $\mathbf{v}$ belongs to the space $V^2_{(d+3)/2}(-1,1)$ which thus contains the set $\mathcal{D}(\mathbf{A}_0^*)$ as a subspace. In order to describe this set in detail we apply the Kondratiev theorem [4] (see also [5, Theorem 4.2.1]) on the asymptotics of solutions near conical points (recall that the endpoints $z = \pm 1$ of the interval $(-1,1)$ are interpreted as tops of the "cones" $\mathbb{R}_\mp$). To this end, we observe that due to formulas (2.20) there holds the relation

$$\chi_\pm \mathcal{V}^1_\pm \notin V^2_{(d+3)/2}(-1,1),\ \chi_\pm \mathcal{V}^0_\pm \in V^2_{(d-1)/2}(-1,1) \text{ for } d \geqslant 5,$$

$$\chi_\pm \mathcal{V}^1_\pm \notin V^2_{(d+3)/2}(-1,1),\ \chi_\pm \mathcal{V}^0_\pm \in V^{2,0}_{(d-1)/2}(-1,1) \text{ for } d=4.$$

According to the above-mentioned Kondratiev theorem the solutions $\mathcal{V}^1$ and $\mathcal{V}^0$ of the model equation (2.19) form the asymptotics of a solution to equation (5.3). For $d \geqslant 4$, the products $\chi_\pm \mathcal{V}^p_\pm$ lie in $V^2_{(d+3)/2}(-1,1)$ and out of $\mathcal{D}(\mathbf{A}^*_0)$ simultaneously (the latter space is defined in (5.20) and (5.21)); this proves the assertion in large dimensions $d \geqslant 4$.

If $d=2$ and $d=3$, then we have

$$\chi_\pm \mathcal{V}^p_\pm \in V^2_{(d+3)/2}(-1,1)/V^2_{(d-1)/2}(-1,1),\ p=0,1$$

due to formulas (2.20). As a result, the Kondratiev theorem predicts a bit more complicated structure for the solution $\mathbf{v} \in V^2_{(d+3)/2}(-1,1)$. Due to the representation (1.2) of the coefficient H, the main and correction singularities in the equations (2.19) and (4.53) take the form

$$\mathcal{V}^1(t) = -\ln t,\ \mathcal{V}^2_\pm(t) = H^{-1}_\pm \partial_t \widetilde{H}_\pm(0)t \quad \text{for } d=2,$$
$$\mathcal{V}^1(t) = 1/t,\ \mathcal{V}^2_\pm(t) = H^{-1}_\pm \partial_t \widetilde{H}_\pm(0)\ln t \quad \text{for } d=3,$$

hence, the functions

$$(-1,1) \ni z \mapsto H(z)^{1-d}\partial_z H(z)^{d-1}\partial_z \chi_\pm(z)(\mathcal{V}^1_\pm)(1\mp z) + \mathcal{V}^2_\pm)(1\mp z))$$

belong to the space $\mathbf{L} = V^0_{(d-1)/2}(-1,1)$. In other words, thanks to the introduction of higher-order terms in the asymptotics (see, e.g., [5, Lemma 3.5.11 and Theorem 3.5.12]), the representations in (5.22) and (5.23) hold true, while the necessary estimate

$$\|\widetilde{\mathbf{v}}; V^2_{(d-1)/2}(-1,1)\| + \sum_\pm (|a_\pm| + |b_\pm|) \leqslant c(\|\mathbf{f}; V^0_{(d-1)/2}(-1,1)\| + \|\mathbf{v}; V^2_{(d+3)/2}(-1,1)\|)$$

follows from the Kondratiev theorem. ⊠

Proposition 5.3 means that the operator $\overline{\mathbf{A}}_0$ is self-adjoint only for $d \geqslant 4$. Furthermore, the defect index of this operator for $d=2,3$ is equal to $(2:2)$. In the case $d=3$ the operator $\mathbf{A}$ of Section 2 § 2, when restricted to the subspace $V^2_{(d+1)/2}(-1,1) \subset \mathbf{H}$, is a self-adjoint unbounded operator in the Hilbert space $\mathbf{L}$. Hence, it must be interpreted as the Friedrichs extension of the operator $\mathbf{A}_0$. The same self-adjoint Friedrichs extension is obtained in the case $d=2$, too, when $\mathbf{A}$ is restricted onto the subspace $V^{2,0}_{3/2}(-1,1) \subset \mathbf{H}$.

Let us describe all possible self-adjoint extensions with the help of an approach proposed in [28] (see the publications [29, 30, 31, 32] and others, related to concrete problems in mathematical physics).

Proposition 5.4. *Let $d=3$ or $d=2$. The operator $\mathbf{A}(\mathbf{s})$ which is defined by the differential expression $-H^{1-d}\partial_z H^{d-1}\partial_z$ and has the domain*

$$\mathcal{D}(\mathbf{A}(\mathbf{s})) = \{\mathbf{v} \in \mathcal{D}(\mathbf{A}^*_0) : i(\mathbb{I}-\mathbf{s})a_\mathbf{v} = (\mathbb{I}+\mathbf{s})b_\mathbf{v}\}, \tag{5.25}$$

is a self-adjoint extension of the operator $\mathbf{A}_0$. Here, $a_v = (a_+, a_-)$ and $b_v = (b_+, b_-)$ are the coefficient columns of decompositions of the function $\mathbf{v}$ in (5.22) or (5.23) while $\mathbb{I}$ is the unit matrix and $\mathbf{s}$ a unitary matrix, both of size 2×2. The domain of any self- adjoint extension of the operator $\mathbf{A}_0$ equals (5.25) for some unitary matrix $\mathbf{s}$.

Proof. Methods developed in the above-cited papers and the definition of the scalar product (5.12) show that it suffices to prove the generalized Green formula

$$\langle \mathbf{A}^*_0\mathbf{v}, \mathbf{w}\rangle - \langle \mathbf{v}, \mathbf{A}^*_0\mathbf{w}\rangle = a_\mathbf{v}\cdot\overline{b_\mathbf{w}} - b_\mathbf{v}\cdot\overline{a_\mathbf{w}} \qquad \forall \mathbf{v},\mathbf{w} \in \mathcal{D}(\mathbf{A}^*_0) \tag{5.26}$$

where bar stands for complex conjugation and the central dot for the scalar product in $\mathbb{C}$ (we now deal with complex-valued functions). This formula can be checked by a direct calculation:

$$\langle \mathbf{A}_0^*\mathbf{v}, \mathbf{w}\rangle - \langle \mathbf{v}, \mathbf{A}_0^*\mathbf{w}\rangle = (\mathbf{v}, \partial_z H^{d-1}\partial_z \mathbf{w}) - (\partial_z H^{d-1}\partial_z \mathbf{v}, \mathbf{w}) =$$

$$= \lim_{\varepsilon\to+0}\sum_{\pm} \pm \Big(\mathbf{v}(z)H(z)^{d-1}\overline{\partial_z \mathbf{w}(z)} - \overline{\mathbf{w}(z)}H(z)^{d-1}\partial_z \mathbf{v}(z)\Big)\Big|_{z=\pm1\mp\varepsilon}$$

The decompositions of the functions $\mathbf{v}$ and $\mathbf{w}$, which are indicated in (5.22) and (5.23), show that the limit as $\varepsilon \to +0$ equals (5.26).

Our way of writing the parameters of self-adjoint extensions is non-standard, hence, we explain for the convenience of the reader why (5.25) is a domain of a self-adjoint extension. (This is the only assertion of the proposition to be used in the sequel.)

Let $\mathbb{K}^\pm = \ker(\mathbf{s} \pm \mathbb{I})$ be the eigenspace of the matrix $\mathbf{s}$ corresponding to its eigenvalue ∓ 1. We set

$$b_{\mathbf{v}} = b^0_{\mathbf{v}} + b^+_{\mathbf{v}} + b^-_{\mathbf{v}}, \quad b^\pm_{\mathbf{v}} \in \mathbb{K}^\pm,\ b^0_{\mathbf{v}} \in \mathbb{K}^0 = \mathbb{C}^2 \ominus (\mathbb{K}^+ \oplus \mathbb{K}^-)$$

and apply the same splitting to $a_{\mathbf{v}}$ and $a_{\mathbf{w}}$, $b_{\mathbf{w}}$. We have

$$a_{\mathbf{v}} \cdot \overline{b_{\mathbf{w}}} - b_{\mathbf{v}} \cdot \overline{a_{\mathbf{w}}} = \sum_{\alpha=0,\pm} (a^\alpha_{\mathbf{v}} \cdot \overline{b^\alpha_{\mathbf{w}}} - b^\alpha_{\mathbf{v}} \cdot \overline{a^\alpha_{\mathbf{w}}}).$$

Terms with the subscripts $\alpha = +$ and $\alpha = -$ disappear because $a^-_{\mathbf{v}} = a^-_{\mathbf{w}} = 0$ and $b^+_{\mathbf{v}} = b^+_{\mathbf{w}} = 0$, by formula (5.25). Furthermore, the Cayley transform of the unitary matrix $\mathbf{s}$ is the symmetric operator $\mathbf{m} = i\,(\mathbb{I} + \mathbf{s})^{-1}(\mathbb{I} - \mathbf{s})$ in the subspace $\mathbb{K}^0$. Hence,

$$a^0_{\mathbf{v}} \cdot \overline{b^0_{\mathbf{w}}} - b^0_{\mathbf{v}} \cdot \overline{a^0_{\mathbf{w}}} = a^0_{\mathbf{v}} \cdot \overline{\mathbf{m} a^0_{\mathbf{v}}} - \mathbf{m} a^0_{\mathbf{v}} \cdot \overline{a^0_{\mathbf{v}}} = 0,$$

and the operator $\mathbf{A}(\mathbf{s})$ is symmetric, too. Since the dimension of the quotient space $\mathcal{D}(\mathbf{A}_0^*)/\mathcal{D}(\mathbf{A}(\mathbf{s}))$ equals two and $(2:2)$ is the defect index of the operator $\overline{\mathbf{A}}_0$, we see that the operator $\mathbf{A}(\mathbf{s})$ with the domain (5.25) is self-adjoint, indeed. ⊠

Remark 5.5. If $\mathbf{s} = \mathbb{I}_2$, then, owing to (5.25), the coefficients $b_\pm$ of the singular components in the decompositions in (5.22) and (5.23) vanish so that the subspace $\mathcal{D}(\mathbf{A}(\mathbb{I}))$ contains only bounded functions. Hence, $\mathbf{A}(\mathbb{I})$ is nothing but the Friedrichs extension. ⊠

5.3. Choosing the parameters of the self-adjoint extension. For large dimensions $d \geqslant 4$ the operator $\overline{\mathbf{A}}_0$ is self-adjoint, and at the same time, the justification of asymptotics in § 3 did not require the construction of the boundary layer. The authors do not know a reason for this coincidence. We will show how the specially chosen self-adjoint extension $\mathbf{A}(\mathbf{s}^{|\ln h|})$ of the operator $\mathbf{A}_0$ helps to create a proper one-dimensional model of the planar ($d = 2$) spindle Ω^h. An analogous self-adjoint extension could also be defined in the spatial case $d = 3$, but in § 4 we managed to prove Theorems 3.3 and 4.4 with power order accuracy for the one-dimensional model, which is sufficient for most of goals and does not require an improvement. For $d = 2$, the error increases up to an unacceptable quantity $O((1 + |\ln h|)^{-1})$ (see the relation (5.35), below); hence, the technique to be introduced here is quite useful.

Let us proceed with the boundary layer analysis, which was skipped in Section 1 § 4 in the case $d = 2$. Instead of the refined ansatz (4.3) we take the original ansatz (2.2), where we however allow its terms $\mathbf{v}^\mathfrak{h}(z)$ and $V^\mathfrak{h}(\eta, z)$ to depend on the parameter $\mathfrak{h} = |\ln h|$. Furthermore, we include into the main term the singular component (2.20), namely

$$\mathbf{v}^\mathfrak{h}(z) = b^\mathfrak{h}_\pm H_\pm^{-1} \ln(1 \mp z) + a^\mathfrak{h}_\pm + O(1 \mp z) \quad \text{for} \quad z \to \pm 1 \mp 0. \tag{5.27}$$

Now, the calculation (4.5) using the stretched coordinates (4.1) looks as follows:

$$\mathbf{v}^\mathfrak{h}(z) = a^\mathfrak{h}_\pm + b^\mathfrak{h}_\pm H_\pm^{-1} \ln(h\rho_\pm + h^2\rho_\pm H_\pm \xi_2^\pm) + \dots$$

$$= b_\pm^{\mathfrak{h}} H_\pm^{-1}(\ln \rho_\pm - \mathfrak{h}) + a_\pm^{\mathfrak{h}} + h b_\pm^{\mathfrak{h}} \xi_2^\pm + \dots . \tag{5.28}$$

Hence, the main terms of the boundary layers are the form

$$-h b_\pm^{\mathfrak{h}} W(\xi^\pm),$$

and to fulfil the Dirichlet condition (1.6) one needs the equalities

$$a_\pm^{\mathfrak{h}} = -b_\pm^{\mathfrak{h}} H_\pm^{-1}(\ln \rho_\pm - \mathfrak{h}), \tag{5.29}$$

which coincide with the vector equality in (5.25), where $\mathbf{s}$ is the diagonal unitary matrix

$$\mathbf{s}^{\mathfrak{h}} = \operatorname{diag}\left\{ \frac{i(\ln \rho_+ - \mathfrak{h}) + H_+}{i(\ln \rho_+ - \mathfrak{h}) - H_+}, \frac{i(\ln \rho_- - \mathfrak{h}) + H_-}{i(\ln \rho_- - \mathfrak{h}) - H_-} \right\}. \tag{5.30}$$

This completes the construction of the desired self-adjoint extension $\mathbf{A}(\mathbf{s}^{\mathfrak{h}})$ of the operator $\mathbf{A}^0$.

Proposition 5.6. *The operator $\mathbf{A}(\mathbf{s}^{\mathfrak{h}})$ is positive definite and its spectrum forms the unbounded monotone positive sequence of eigenvalues*

$$0 < \mu_1^{\mathfrak{h}} \leqslant \mu_2^{\mathfrak{h}} \leqslant \dots \leqslant \mu_n^{\mathfrak{h}} \leqslant \dots \to +\infty. \tag{5.31}$$

Proof. The self-adjoint extension $\mathbf{A}(\mathbf{s}^{\mathfrak{h}})$ preserves the lower semi-boundness of $\mathbf{A}_0$, and the embedding $\mathcal{D}(\mathbf{A}(\mathbf{s}^{\mathfrak{h}})) \subset \mathbf{L}$ preserves the compactness. Thus, according to [2, Theorems 10.1.5 and 10.2.1], the spectrum of the operator $\mathbf{A}(\mathbf{s}^{\mathfrak{h}})$ is discrete and forms an unbounded and monotone sequence. The first inequality in (5.31) and the positive definiteness will be shown in the proof of Proposition 5.7. ⊠

5.4. Spectral problem with asymptotic conditions at the endpoints of the interval. By virtue of the estimate (5.10) with $l = 0$ and $\beta = 5/2$, a solution $\mathbf{v} \in \mathbf{L} = V_{1/2}^0(-1,1)$ of the equation (5.3) with a right-hand side $\mathbf{f} \in C_c^\infty(-1,1)$ falls into the space $V_{5/2}^2(-1,1)$ (if $d = 2$, the only forbidden exponent is $\beta = 3/2$; see (5.16)). For $d = 2$, $l = 0$ and $\beta = 5/2$, we consider the mapping (5.2) and the corresponding pre-image $\mathbf{D}$ of the subspace $\mathbf{L} = V_{1/2}^0(-1,1) \subset V_{3/2}^0(-1,1)$. By the above mentioned Kondratiev theorem on asymptotics [4], the space $\mathbf{D}$ coincides as a set with (5.23) and, being a weighted space with detached asymptotics, can be identified with the space $\mathfrak{D}$ of the triples $\{a_{\mathbf{v}}, b_{\mathbf{v}}, \widetilde{\mathbf{v}}\}$:

$$\mathbf{D} \approx \mathfrak{D} = \mathbb{C}^2 \times \mathbb{C}^2 \times V_{1/2}^2(-1,1).$$

This becomes a Hibert space, when equipped with the norm

$$\|\mathbf{v}; \mathfrak{D}\| = (|a_{\mathbf{v}}|^2 + |b_{\mathbf{v}}|^2 + \|\widetilde{\mathbf{v}}; V_{1/2}^2(-1,1)\|^2)^{1/2}.$$

Furthermore, due to the properties of the operator $\mathbf{A}_0^*$, which were described above, the mapping $\mathfrak{A}_0 : \mathfrak{D} \to V_{1/2}^0(-1,1)$,

$$\mathfrak{A}_0 \mathbf{v} = -H^{1-d} \partial_z H^{d-1} \partial_z \left(\widetilde{\mathbf{v}} + \sum_\pm \chi_\pm \left(a_\pm + \frac{b_\pm}{H_\pm} \ln \frac{1}{1 \mp z} \right) \right),$$

is a Fredholm epimorphism with a two-dimensional kernel. Finally, the mapping

$$\mathfrak{D} \ni \mathbf{v} \ \mapsto \ \mathfrak{A}^{\mathfrak{h}} \mathbf{v} = \{ b_\pm + H_\pm (\ln \rho_\pm - \mathfrak{h})^{-1} a_\pm, \mathfrak{A}_0 \mathbf{v} \} \in \mathfrak{L} := \mathbb{C}^2 \times V_{1/2}^0(-1,1) \tag{5.32}$$

preserves the Fredholm property with index equal to null. Due to the formulas (5.29), (5.30) and (5.25), the spectrum of the pencil

$$\mu \ \mapsto \ \mathfrak{A}^{\mathfrak{h}}(\mu) = \mathfrak{A}^{\mathfrak{h}} - \mu \mathfrak{E} \tag{5.33}$$

where $\mathfrak{E}\mathbf{v} = \{0, 0, \mathbf{v}\}$, coincides with the sequence (5.31) of the eigenvalues of the operator $\mathbf{A}(\mathbf{s}^{\mathfrak{h}})$.

Proposition 5.7. *For every $N \in \mathbb{N}$, there exist $\mathfrak{h}^{(N)} \in (0,1)$ and $c^{(N)} > 0$ such that the function*

$$[0, 1/\mathfrak{h}^{(N)}) \ni \mathfrak{h} \ \mapsto \ \mu_n^{\mathfrak{h}}, \quad n = 1, \dots, N, \tag{5.34}$$

is real analytic. Moreover, there holds the relation

$$\Big| \mu_n^{\mathfrak{h}} - \mu_n - \mathfrak{h}^{-1} \sum_{\pm} H_{\pm}^{-1} \mathbf{v}_n(\pm 1)^2 \Big| \leqslant c^{(n)} \mathfrak{h}^{-2} \quad \text{for} \quad \mathfrak{h} \in (0, 1/\mathfrak{h}^{(N)}), \tag{5.35}$$

where μ_n is an element of the sequence (2.17) *and $n = 1, \dots, N$.*

Proof. Passing to the limit $\mathfrak{h} \to +\infty$ turns[3] the operator (5.32) into the operator

$$\mathfrak{A}_{\mathbf{v}}^{\infty} = \{b_{\mathbf{v}}, \mathfrak{A}_0 \mathbf{v}\} \tag{5.36}$$

which can be interpreted as the restriction of the operator $\mathbf{A}$, Section 2 § 2, onto the subspace $\mathbf{D}_0 = \{\mathbf{v} \in \mathbf{D} : b_{\mathbf{v}} = 0 \in \mathbb{C}^2\} \subset \mathbf{H}$. Moreover, we look for the asymptotics of eigenpairs of the pencil (5.33) in the form

$$\mu_n^{\mathfrak{h}} = \mu_n + \mathfrak{h}^{-1} \mu_n' + \dots, \quad \mathbf{v}^{\mathfrak{h}} = \mathbf{v}_n + \mathfrak{h}^{-1} \mathbf{v}_n' + \dots$$

and for the coefficients of the decompositions near the points ± 1 as

$$a_{(n)\pm}^{\mathfrak{h}} = \mathbf{v}_n(\pm 1) + \mathfrak{h}^{-1} a'_{(n)\pm} + \dots, \quad b_{(n)\pm}^{\mathfrak{h}} = 0 + \mathfrak{h}^{-1} b'_{(n)\pm} + \dots;$$

we thus obtain the following problem for the correction terms:

$$\begin{aligned} &- \partial_z (H(z) \partial_z \mathbf{v}_n'(z)) - \mu_n H(z) \mathbf{v}_n'(z) = \mu_n' H(z) \mathbf{v}_n(z), \quad z \in (-1, 1), \\ &\mathbf{v}_n'(z) = -H_{\pm}^{-1} \mathbf{v}_n(\pm 1) \ln(1 \mp z) + a'_{(n)\pm} + o(1), z \to \pm 1. \end{aligned}$$

The following calculation is based on the generalized Green's formula (5.26) and Lemma 2.4 with the normalization condition (2.18), and one of the conditions for the existence of the solution $\mathbf{v}_n' \in \mathbf{D}$ is discarded:

$$\begin{aligned} \mu_n' &= \mu_n' (H \mathbf{v}_n, \mathbf{v}_n) = -(\partial_z H \partial_z \mathbf{v}_n' + \mu_n H \mathbf{v}_n', \mathbf{v}_n) = \\ &= \sum_{\pm} (a'_{(n)\pm} 0 - \mathbf{v}_n(\pm 1) b'_{(n)\pm}) = \sum_{\pm} H_{\pm}^{-1} \mathbf{v}_n(\pm 1)^2 > 0. \end{aligned} \tag{5.37}$$

Since $\mathfrak{A}^{\mathfrak{h}}$ is a regular perturbation of the operator $\mathfrak{A}^{\infty}$ (cf., (5.32) and (5.36)), general results of the perturbation theory of linear operators justify the formal calculations as well as the estimate (5.35) (cf., for, [33, Ch. 8] and [34, Ch. II §3]) Relations (5.37) and (5.35) assure that $\mu_1^{\mathfrak{h}} > 0$, if the parameter $\mathfrak{h}$ is large, hence, we have also proven the first assertion of Proposition and that the eigenvalues $\mu_1^{\mathfrak{h}}, \dots, \mu_N^{\mathfrak{h}}$ are simple. Finally, the operator (1.10) depends analytically on the variable $1/\mathfrak{h}$ and this, cf. [34, Ch. II], proves the analyticity of the functions (5.34). ⊠

5.5. Justification of asymptotics in the planar case. As in Sections 4 § 3 and 4 § 4, we apply Lemma 3.2 with the following almost eigenvalue and eigenvector of the operator $\mathcal{K}^h$

$$\mathbf{k}_n^h = (1 + \mu_n^{\mathfrak{h}})^{-1}, \quad \mathbf{U}_n^h = \|\mathbf{u}_n^h; \mathcal{H}^h\|^{-1} \mathbf{u}_n^h, \quad \mathbf{u}_n^h = X^h \mathbf{v}_n^{\mathfrak{h}}.$$

These definitions are analogous to (3.22), but they contain the eigenpair $\{\mu_n^{\mathfrak{h}}, \mathbf{v}_n^{\mathfrak{h}}\} \in \mathbb{R}_+ \times \mathcal{D}(\mathbf{A}(\mathbf{s}^{\mathfrak{h}}))$ of the self-adjoint extension $\mathbf{A}(\mathbf{s}^{\mathfrak{h}})$ of the operator $\mathbf{A}_0$ (see Proposition 5.6) instead of the eigenpair $\{\mu_n, \mathbf{v}_n\} \in \overline{\mathbb{R}_+} \times \mathbf{H}$ of the variational problem (2.14). The eigenfunction $\mathbf{v}_n^{\mathfrak{h}}$ is normalized in the space $\mathbf{L} = V_{1/2}^0(-1, 1)$, and it satisfies the abstract equation

$$\mathfrak{A}^{\mathfrak{h}} \mathbf{v}_n^{\mathfrak{h}} = \mathfrak{f}_n^{\mathfrak{h}} := (0, 0, \mu_n^{\mathfrak{h}} \mathbf{v}^{\mathfrak{h}}) \in \mathbb{R}^2 \times \mathbb{R}^2 \times V_{1/2}^0(-1, 1),$$

[3]The same limit passage shows that $\mathbf{s}^{\infty} = \mathbb{I}$ is the unit matrix and $\mathbf{A}(\mathbf{s}^{\infty})$ is the Friedrichs extension.

with the operator $\mathfrak{A}^{\mathfrak{h}}$ which is a small perturbation of the operator $\mathfrak{A}^{\infty}$. Thus, owing to Propositions 5.6 and 5.7, the terms of the representation

$$\mathbf{v}_n^{\mathfrak{h}}(z) = \widetilde{\mathbf{v}}_n^{\mathfrak{h}}(z) + \sum_{\pm} \chi_{\pm}(z)(a_{n\pm}^{\mathfrak{h}} + b_{n\pm}^{\mathfrak{h}} H_{\pm}^{-1} \ln(1 \mp z))$$

there holds the inequality

$$\|\widetilde{\mathbf{v}}_n^{\mathfrak{h}}; V_{1/2}^2(-1,1)\| + \sum_{\pm} (|a_{n\pm}^{\mathfrak{h}}| + |b_{n\pm}^{\mathfrak{h}}|) \leqslant c\,(1+\mu_n^{\mathfrak{h}})\,\|\mathbf{v}_n^{\mathfrak{h}}; \mathbf{L}\| \leqslant C_n, \tag{5.38}$$

where the bound is independent of a small $h > 0$. Furthermore, the equalities (5.29) yield the estimate

$$|b_{n\pm}^{\mathfrak{h}}| \leqslant c\mathfrak{h}^{-1}|a_{n\pm}^{\mathfrak{h}}| \leqslant C_n \mathfrak{h}^{-1}. \tag{5.39}$$

We next repeat the calculations in Section 3 § 3 with the necessary modifications: we estimate the quantity (3.26) of Lemma 3.2. The equality (3.28) is preserved, once one makes the substitutions $d = 2$ and $\{\mu_n, \mathbf{v}_n\} = \{\mu_n^{\mathfrak{h}}, \mathbf{v}_n^{\mathfrak{h}}\}$, and the same in the other formulas. We then process the terms on the right-hand side of (3.29). The estimate (3.30) remains valid with the bound $c_n h^{3/2}$ (since $d = 2$). Moreover, we have

$$|I_2^h(\psi^h)| \leqslant ch^{-2}|\Omega_{\ddagger}^h|^{1/2} \max_{x\in\Omega_{\ddagger}^h} |\mathbf{v}_n^{\mathfrak{h}}(z)|\, \|\nabla_x \psi^h; L^2(\Omega^h)\| \leqslant c_n h^{-2} h^2 h = c_n h. \tag{5.40}$$

In comparison with the first estimate (3.31), there is an additional factor h, which came into (5.40) by the following observation: using (5.27) and (5.28) we obtain that

$$\begin{aligned} &|\mathbf{v}_n^{\mathfrak{h}}(z)| = |b_{n\pm}^{\mathfrak{h}} H_{\pm}^{-1} \ln(1 - \mp z) + b_{n\pm}^{\mathfrak{h}} H_{\pm}^{-1}(\mathfrak{h} - \ln \rho_{\pm}) + \widetilde{\mathbf{v}}_n^{\mathfrak{h}}(z)| \leqslant c_n h \\ &\text{for } \ \pm z \in \varsigma_{\pm}^h := (1 - \rho_{\pm} h - h^2/2, 1 - \rho_{\pm} h + h^2) \end{aligned} \tag{5.41}$$

(cf., definition of $\Omega_{\ddagger}^h$ in (3.24)). Finally, the second estimate in (3.31) can be improved as

$$\begin{aligned} |I_3^h(\psi^h)| &\leqslant c|\Omega_{\ddagger}^h|^{1/2} \max_{x\in\Omega_{\ddagger}^h} |\partial_z \mathbf{v}_n^{\mathfrak{h}}(z)| h^{-2} \|\psi^h; L^2(\Omega_{\ddagger}^h)\| \\ &\leqslant c_n h^2 h^{-2} h^2 \|\nabla_x \psi^h; L^2(\Omega^h)\| \leqslant c_n h^2 \end{aligned} \tag{5.42}$$

by an application of the Friedrichs inequality (4.44) on sets of diameter $O(h^2)$

$$\|\psi^h; L^2(\{x \in \Omega^h : \pm z \in \varsigma_{\pm}^h\})\| \leqslant ch^2 \|\nabla_x \psi^h; L^2(\{x \in \Omega^h : \pm z \in \varsigma_{\pm}^h\})\|.$$

The treatment of the last estimate

$$|\langle \mathbf{u}_n^h, \mathbf{u}_n^h \rangle_h - h|\omega|(1 + \mu_n^{\mathfrak{h}})| \leqslant ch^{3/2} \tag{5.43}$$

differs from the proof of Lemma 3.6 because $\mathbf{v}_n^{\mathfrak{h}} \notin \mathbf{H}$. We have

$$\begin{aligned} h^{-1}|\omega|^{-1}\langle \mathbf{u}_n^h, \mathbf{u}_n^h \rangle_h &= (H\partial_z(X^h \mathbf{v}_n^{\mathfrak{h}}), \partial_z(X^h \mathbf{v}_n^{\mathfrak{h}})) + (HX^h \mathbf{v}_n^{\mathfrak{h}}, X^h \mathbf{v}_n^h) \\ = \ & (-\partial_z(H\partial_z \mathbf{v}_n^{\mathfrak{h}}), (X^h)^2 \mathbf{v}_n^2) + (H\mathbf{v}_n^{\mathfrak{h}} \partial_z X^h, \partial_z(X^h \mathbf{v}_n^{\mathfrak{h}})) \\ - \ & (HX^h \partial_z \mathbf{v}_n^{\mathfrak{h}}, \mathbf{v}_n^{\mathfrak{h}} \partial_z X^h) + (HX^h \mathbf{v}_n^{\mathfrak{h}}, X^h \mathbf{v}_n^h). \end{aligned} \tag{5.44}$$

Recalling that $-\partial_z H \partial_z \mathbf{v}_n^{\mathfrak{h}} = \mu_n^h H \mathbf{v}_n^{\mathfrak{h}}$, we transform this formula into the inequality (5.43) with the help of the estimates

$$\begin{aligned} &\|H^{1/2}(1 - X^h)\mathbf{v}_n^{\mathfrak{h}}; L^2(-1,1)\| \leqslant c_n h, \quad \|H^{1/2} \mathbf{v}^h \partial_z X^h; L^2(-1,1)\| \leqslant c_n h^{1/2}, \\ &\|H^{1/2} X^h \partial_z \mathbf{v}_n^{\mathfrak{h}}; L^2(-1,1)\| \leqslant c_n. \end{aligned}$$

In the first one we have used the decomposition of the function $\mathbf{v}_n^\natural$ near the points $z = \pm 1$ and the definition (3.23) of the cut-off function X^h, in the second one in addition the relation (5.41), and in the third one also the formula

$$|b_\pm^\natural|^2 \int_{-1+h\rho-}^{1-h\rho+} H(z)\chi_\pm(z)^2|\partial_z \ln(1\mp z)|^2\,dz \leqslant c|\ln h|^{-2}\int_{h\rho_\pm}^{1} t\,\frac{dt}{t^2} \leqslant \frac{c}{|\ln h|}.$$

The inequalities (5.38) and (5.39), of course, were applied as well.

Collecting the estimates (3.28), (3.30) and (5.40), (5.42), (5.43), we see that the quantity (3.26) does not exceed $c_n h^{-1/2} h = c_n h^{1/2}$ because $\|\mathbf{u}_n^\natural;\mathcal{H}\| \geqslant ch^{1/2}$, $c > 0$. Repeating the reasoning in Sections 3 and 4 § 3 with a minor modification, we arrive at the following analogue of Theorems 3.3 and 3.7 for the planar spindle.

Theorem 5.8. *For all $n \in \mathbb{N}$, there exist positive numbers $h_2^{(n)}$ and $c_2^{(n)}$ such that, for $h \in (0, h_2^{(n)}]$, eigenpairs of the problem* (1.5)–(1.7) *(or* (1.8) *in the variational formulation) and the pencil* (5.33) *(or the self-adjoint operator $\mathbf{A}(\mathbf{s}^\natural)$ defined by formulas* (5.25) *and* (5.30)*) are in the relationship*

$$|\lambda_n^h - \mu_n^\natural| \leqslant c_n h^{1/2} \quad\text{and}\quad \|u_n^h - h^{-1/2}|\omega|^{-1/2}X^h\mathbf{v}_n^\natural; H^1(\Omega^h)\| \leqslant c_n h^{1/2}.$$

6. Variants and generalizations.

6.1. Other shapes of the tops of the spindle. One does not need any changes in the dimension reduction procedure or in the proofs of the error estimates, when studying the asymptotics of the solutions of the spectral problem (1.5)–(1.5) in the spindle-shaped domain with broken ends (see Fig. 2, a)

$$\Omega_\ddagger^h = \{x \in \Omega^h : -1 + h\rho_- < z < 1 - h\rho_+\};$$

note that the Dirichlet condition is posed at the ends

$$\Gamma_\pm^h = \{x \in \Omega^h : z = \pm 1 \mp h\rho_\pm\}.$$

The only difference is inessential: the boundary layer is to be found from the mixed boundary value problem in the semi-infinite cylinder $\Pi_+ = \omega \times \mathbb{R}_+$ (Fig. 2, b), where the Neumann and Dirichlet conditions are posed on the lateral surface $(\partial\Pi)_+$ and the end $\partial\Pi_+ \setminus \overline{(\partial\Pi)_+}$, respectively.

We next consider the spindle with rounded ends (Fig. 2, c), namely we assume that the profile function $H \in C^\infty(-1,1)$ in the definition (1.1) of the domain Ω^h satisfies

$$\begin{aligned}
&H(\pm 1) = 0, \quad H(z) > 0 \text{ for } z \in (-1,1),\\
&H(z) = (1 \mp z)^\gamma (H_\pm + \widetilde{H}_\pm(1 \mp z)) \text{ for } \pm z \in [0,1],\\
&\gamma \in (0,1), \quad H_\pm > 0, \quad \widetilde{H}_\pm \in C^\infty[0,1],\ \widetilde{H}_\pm(0) = 0.
\end{aligned} \tag{6.1}$$

If Ω^h is an ellipsoid, relations (6.1) hold for it with $\gamma = 1/2$. We let the end zones $\Gamma_\pm^h$ still be of the form (1.4). A simple modification of the calculation (4.2) shows that

$$h^{-1}H(z)^{-1} = h^{-1-\gamma}\rho_\pm^{-\gamma}H_\pm^{-1}(1 + O(h + h^\gamma \xi_d^\pm)),$$

where the stretched coordinates

$$\xi^\pm = (\xi_\bullet^\pm, \xi_d^\pm) = h^{-1-\gamma}(H_\pm \rho_\pm^\gamma)^{-1}(y, 1 - h\rho_\pm \mp z) \tag{6.2}$$

were used. The changes $x \mapsto \xi^\pm$ and $h \mapsto 0$ lead again to the problem (4.8) in the cylinder $\Pi = \omega \times \mathbb{R}$, and this must be solved in order to describe the boundary layer phenomenon.

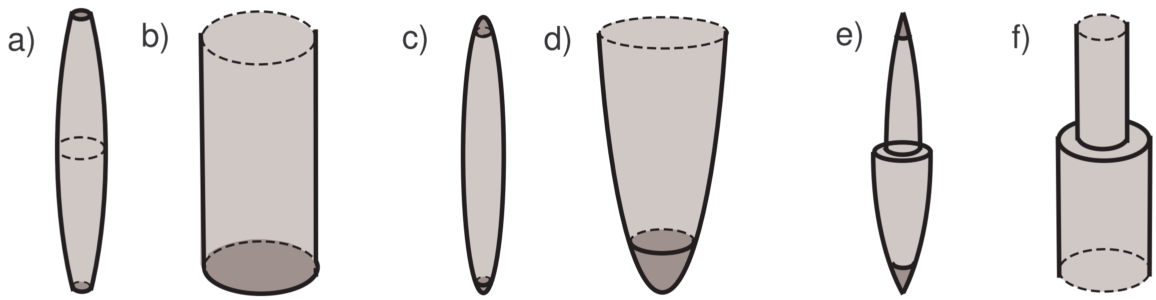

FIGURE 6.1. Spindle with broken (a) and blunted (c) ends. Domains to describe the boundary layers: smi-infinite cylinder (b) and paraboloid (d). Piecewise smooth profiles function H gives rise to the composite spindle (e) and cylinder (f).

We now assume that

$$\Gamma^h_\pm = \{x \in \partial\Omega^h : \pm z > 1 - h^\sigma \rho_\pm\}, \quad \rho_\pm > 0, \ \sigma = (1-\gamma)^{-1} > 1 + \gamma > 1, \tag{6.3}$$

that is, we diminish the Dirichlet zone, which means that the dilation coefficient must increase in comparison with (6.2):

$$\xi^\pm = (\xi^\pm_\bullet, \xi^\pm_d) = (h^{-\sigma} y, h^{-\sigma}(1 \mp z)). \tag{6.4}$$

The exponent σ of the power of the small parameter h in formulas (6.3) and (6.4) was chosen such that, due to relations (1.1) and (6.1), the factor $h^{-1-\gamma\sigma+\sigma}$ turns into one in the expression

$$h^{-1} H_\pm(\pm 1 \mp h^\sigma \rho_\pm)^{-1} y = h^{-1}(h^\sigma \rho_\pm)^{-\gamma}(H_\pm + O(h^\sigma))^{-1} h^\sigma \xi^\pm_\bullet.$$

The changes $x \mapsto \xi^\pm$ and $h \mapsto 0$ transform the sets Ω^h and $\Gamma^h_\pm$, respectively, into the paraboloid

$$\Pi^\subseteq_\pm = \{\xi^\pm : H^{-1}_\pm (\xi^\pm_d)^{-\sigma} \rho^{-\gamma}_\pm \xi^\pm_\bullet \in \omega\}$$

and the subset $\Gamma^\subseteq_\pm = \{\xi^\pm \in \Pi^\subseteq_\pm : \xi^\pm_d < \rho_\pm\}$ of its surface. Thus, the boundary layer is described by means of the solutions of the mixed boundary value problem

$$-\Delta_{\xi^\pm} w^\pm(\xi^\pm) = f^\pm(\xi^\pm), \quad \xi^\pm \in \Pi^\subseteq_\pm, \tag{6.5}$$

$$w^\pm(\xi^\pm) = 0, \quad \xi^\pm \in \Gamma^\subseteq_\pm, \tag{6.6}$$

$$\partial_{\nu(\xi^\pm)} w^\pm(\xi^\pm) = 0, \quad \xi^\pm \in \partial\Pi^\subseteq_\pm \setminus \overline{\Gamma^\subseteq_\pm}. \tag{6.7}$$

The compatibility conditions in this problem and the properties of its solutions depend on the relation between dimension d and the exponent γ in formulas (6.1). Let us present a known assertion (see, e.g., the monograph [23]), which can also be easily verified.

We introduce the space $\mathcal{H}(\Pi^\subseteq_\pm)$ as the completion of the subspace $C^\infty_c(\overline{\Pi^\subseteq_\pm} \setminus \overline{\Gamma^\subseteq_\pm})$ of functions satisfying the Dirichlet condition (6.6) with respect to the norm

$$\|w^\pm; \mathcal{H}(\Pi^\subseteq_\pm)\| = \|\nabla_{\xi^\pm} w^\pm; L^2(\Pi^\subseteq_\pm)\|.$$

We consider the variational formulation of the problem (6.5)–(6.7), namely the integral identity [1]

$$(\nabla_{\xi^\pm} w^\pm, \nabla_{\xi^\pm} \psi^\pm)_{\Pi^\subseteq_\pm} = F^\pm(\psi^\pm), \quad \psi^\pm \in \mathcal{H}(\Pi^\subseteq_\pm). \tag{6.8}$$

Here, $F^\pm \in \mathcal{H}(\Pi^\subsetneq_\pm)^*$ ia a linear continuous functional in the space $\mathcal{H}(\Pi^\subsetneq_\pm)$, for example, $F^\pm(\psi^\pm) = (f^\pm, \psi^\pm)_{\Pi^\subsetneq_\pm}$ in the case $f^\pm \in C^\infty_c(\overline{\Pi^\subsetneq_\pm})$. If

$$\gamma > \gamma_d := (d-1)^{-1} \geqslant 1, \tag{6.9}$$

the variational problem (6.8) is uniquely solvable in $\mathcal{H}(\Pi^\subsetneq_\pm)$ but the differential problem (6.5)–(6.7) has a solution with the asymptotic form

$$W^\pm(\xi^\pm) = K^\pm_\Pi + (1+\xi^\pm_d)^{1+\gamma(1-d)} \left(1 + O((1+\xi^\pm_d)^{-1})\right) \tag{6.10}$$

at infinity, where $K^\pm_\Pi$ is some constant. If condition (6.9) holds, the solution (6.10) does not belong to $\mathcal{H}(\Pi^\subsetneq_\pm)$ because it cannot be approximated by functions in $C^\infty_c(\overline{\Pi^\subsetneq_\pm})$. At the same time, the set (6.10) is contained in the space $\mathcal{H}(\Pi^\subsetneq_\pm)$ in the case

$$\gamma < \gamma_d. \tag{6.11}$$

Hence, problem (6.8) with the right-hand side $F^\pm \in \mathcal{H}(\Pi^\subsetneq_\pm)^*$ loses the property of unique solvability and and its solution $w^\pm \in \mathcal{H}(\Pi^\subsetneq_\pm)$ exists only under the orthogonality condition $F^\pm(W^\pm) = 0$.

In both cases (6.9) and (6.11) the inner expansions of the solutions of the problem (1.5)–(1.7) in the "ellipsoid" Ω^h are of the form

$$u^h(x) = c^h_\pm W^\pm(\xi^\pm) + \dots.$$

In the critical case $\gamma = \gamma_d$, the expansions of the solutions of the homogeneous problem (6.5)–(6.7) include a logarithm:

$$W^\pm(\xi^\pm) = K^\pm_\Pi + \ln(1+\xi^\pm_d) \left(1 + O((1+\xi^\pm_d)^{-1})\right).$$

The structures of the asymptotic expansions are similar to those in § 5.

The information presented above suffices for the construction of the limit problem and the asymptotics of the solutions of the spectral problem (1.5)–(1.7) in the domain Ω^h defined by formulas (1.1) and (6.1). The authors do not know existing publications on this topic, but the linear elasticity problem has been studied in [7, 8] in bodies with similar geometry under some restrictions on the exponent γ in (6.1).

In the most interesting case $\gamma = 1$ of this paper (cf., (1.2) and (6.1)) the last inequality in (6.3) does not make sense, but the Dirichlet zones $\Gamma^h_\pm$ of the problem (1.5)–(1.7) can still be defined by the first equality in (6.3) by using any exponent $\sigma > 1$. Hence, the corresponding modifications of the asymptotic structures are minor, namely the dilation coefficient h^{-2} in the coordinates (4.1) must be changed into $h^{-1-\sigma}$.

6.2. On the smoothness of the boundary. All calculations and results remain true even if the boundary $\partial\omega$ is only Lipschitz instead of the infinite smoothness required in Section 1 § 1. In order deal with classical solutions to the limit equation (2.9), it suffices to assume that the function

$$(-1,1) \ni z \;\mapsto\; (1-z^2)^{-1} H(z)$$

is in the Hölder class $C^{1,\delta}[-1,1]$ with $\delta \in (0,1)$. In the weak formulation of the equation, (2.14), much weaker restrictions on the coefficient H^{d-1} suffice, for instance, it can be a piecewise continuous function with jumps at the interior points of the interval $(-1,1)$ (Fig. 2, e). In addition, near the break points there emerges an "interior" boundary layer, which is described by the solutions of the Neumann problem in the composite cylinder (Fig. 2, f).

6.3. Complete asymptotic expansions. In Section 1 § 2 we derived the limit problem (2.4), (2.6) by extracting only the main terms from the differential operators Δ_x and $\partial_{\nu(x)}$ (see relations (2.3) and (2.5)). However, these operators can be split into infinite asymptotic series with the help of the Taylor decompositions of the functions $\widetilde{H}_\pm$ in (1.2). Analogous splittings of the operators (4.7) appeared in § 4 when constructing boundary layers related with the coordinate change $x \mapsto \xi^\pm$ of (4.1). Thus, there arises the question on the continuation of the asymptotic procedure and on the derivation of asymptotic series for the eigenvalues and eigenvectors. In § 4 and § 5 we employed the method of matched expansions (see [20, 17] etc) which is convenient for the analysis of the main asymptotic terms but the method of composite expansions (see [14, 16, 18] etc) is much simpler for the construction of the whole asymptotic series.

In this section we let $d \geqslant 3$, while the exceptional case $d = 2$ is discussed in Remark 6.2. We start by a simple assertion.

Lemma 6.1. *The equation* (2.9) *with* $\mu = \mu_n$ *has, in addition to the eigenfunction* $\mathbf{v}_n$*, also the singular solution*

$$\mathbf{V}_n(z) = \widehat{\mathbf{V}}_n(z) + \sum_\pm \pm\chi_\pm(z)\mathbf{v}_n(\pm1)^{-1}H_\pm^{1-d}(1\mp z)^{2-d}, \tag{6.12}$$

where $\chi_\pm$ *are the cut-off functions* (4.14) *and the remainder satisfies the estimate*

$$\left|\partial_z^j\widehat{\mathbf{V}}_n(z)\right| \leqslant c_j(1\mp z)^{3-d-j}(1+|\ln(1\mp z)|)^{\delta_{d,3}\delta_{j,0}}, \quad j \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}. \tag{6.13}$$

Proof. The term $\widehat{\mathbf{V}}_n(z)$ is solved from the equation

$$\begin{aligned} &-\partial_z(H(z)^{d-1}\partial_z\widehat{\mathbf{V}}_n(z)) - \mu_n H(z)^{d-1}\widehat{\mathbf{V}}_n(z) = \mathbf{F}_n(z) \\ &:= \sum_\pm \pm\frac{H_\pm^{1-d}}{\mathbf{v}_n(\pm1)}\left(\partial_z H(z)^{d-1}\partial_z + \mu_n H(z)^{d-1}\right)\left(\chi_\pm(z)(1\mp z)^{2-d}\right), \quad z\in(-1,1). \end{aligned} \tag{6.14}$$

The right-hand side $\mathbf{F}_n \in C^\infty(-1,1)$ is bounded and continuous for $z \in [-1,1]$, which follows from the representations (1.2) for the coefficient H and (2.20) for the solution $\mathcal{V}^1$ of the model equation (2.19) . The information in Section 1 § 5 on the problem operator of (6.14) implies that there exists one compatibility condition (according to the number of eigenfunctions) and it is fulfilled:

$$\begin{aligned} &\int_{-1}^{1}\mathbf{v}_n(z)\mathbf{F}_n(z)dz \\ = &\ -\sum_\pm \pm\frac{H_\pm^{1-d}}{\mathbf{v}_n(\pm1)}\lim_{\varepsilon\to+0}\int_{-1+\varepsilon}^{1-\varepsilon}\mathbf{v}_n(z)\left(\partial_z H(z)^{d-1}\partial_z + \mu_n H(z)^{d-1}\right)\left(\chi_\pm(z)(1\mp z)^{2-d}\right)dz \\ = &\ -\lim_{\varepsilon\to+0}\sum_\pm\frac{H_\pm^{1-d}}{\mathbf{v}_n(\pm1)}\Big(\mathbf{v}_n(z)H(z)^{d-1}\partial_z(1\mp z)^{2-d} \\ &\ -(1\mp z)^{2-d}H(z)^{d-1}\partial_z\mathbf{v}_n(z)\Big)\Big|_{z=\pm1\mp\varepsilon} \\ = &\ (d-2)\lim_{\varepsilon\to+0}(H_+^{d-1}\varepsilon^{d-1}H_+^{1-d}\varepsilon^{1-d} - H_-^{d-1}\varepsilon^{d-1}H_-^{1-d}\varepsilon^{1-d}) = 0. \end{aligned}$$

Thus, equation (6.13) indeed has a solution satisfying the inequalities (6.14). ⊠

For $d \geqslant 3$, we look for the formal asymptotic series for the eigenpairs $\{\lambda_n^h, u_n^h\}$ of the problem (1.5)–(1.7) in the form

$$\lambda_n^h \sim \sum_{j=0}^{\infty} h^j\mu_{nj} \tag{6.15}$$

and

$$u_n^h(x) \sim \sum_{j=0}^{\infty} \Big(X^h(z)(\mathbf{v}'_{nj}(z) + \mathbf{c}_{nj}\mathbf{v}_n(z) + h^{d-2}\mathbf{C}_{nj}\mathbf{V}_n(z)) + \sum_{\pm} h^{1+j}\chi_{\pm}^h(z) w_{nj}^{\pm}(\eta, \xi_d^{\pm}) \Big). \tag{6.16}$$

Here, X^h and $\chi_{\pm}^h$ are the cut-off functions (3.23) and (4.22), $\xi^{\pm}$ and η the stretched coordinates (see (4.1) and (1.1)) and the functions $\Pi \ni \xi^{\pm} \mapsto w_{nj}^{\pm}(\xi^{\pm})$ are exponentially decaying solutions of the mixed boundary value problem (cf., problem (4.8))

$$\begin{aligned} -\Delta_{\xi^{\pm}} w_{nj}^{\pm}(\xi^{\pm}) &= f_{nj}^{\pm}(\xi^{\pm}),\ \xi^{\pm} \in \Pi, \\ w_{nj}^{\pm}(\xi^{\pm}) &= 0, \qquad \xi \in (\partial\Pi)_-, \\ \partial_{\nu'} w_{nj}^{\pm}(\xi^{\pm}) &= g_{nj}^{\pm}(\xi^{\pm}),\ \xi^{\pm} \in (\partial\Pi)_+. \end{aligned} \tag{6.17}$$

Furthermore, $\mathbf{v}_n$ and $\mathbf{V}_n$ are the eigenfunction and the singular solution of equation (2.9) with $\mu = \mu_n$, mentioned in Lemma 6.1, and $\mathbf{v}'_{nj} \in \mathbf{H}$ is a solution of equation

$$\begin{aligned} &-\partial_z(H(z)^{d-1}\partial_z \mathbf{v}'_{nj}(z)) - \mu_n H(z)^{d-1}\mathbf{v}'_{nj}(z) \\ &= \mu_{nj} H(z)^{d-1}\mathbf{v}_n(z) + \mathbf{f}'_{nj}(z),\ z \in (-1,1), \end{aligned} \tag{6.18}$$

subject to the orthogonality condition

$$(H^{d-1}\mathbf{v}'_n, \mathbf{v}_n) = 0. \tag{6.19}$$

Thus, the constants $\mathbf{c}_{nj}$, $\mathbf{C}_{nj}$ and the coefficients μ_{nj} of the series (6.15) remain to be determined. Besides,

$$\mu_{n0} = \mu_n, \quad \mathbf{c}_{n0} = 1,$$

but we did not compute the quantities $\mathbf{C}_{n0}$ and $\mathbf{v}'_{n0}$ in § 3 and § 4 because this was neither needed for the justification of main asymptotic terms for $d > 3$, nor were exact formulas used for the correction terms in the asymptotic solution (4.18), $d = 3$.

Since the derivation of the limit problems (2.4), (2.6) and (4.8) was based on the main terms of the differential operators (2.5) and (4.7), it is not surprising that there appear inhomogeneities in problems (6.17) and (6.18). Explicit expressions for the right-hand sides $f_{nj}^{\pm}$, $g_{nj}^{\pm}$ and $\mathbf{f}'_{nj}$ can be found by multiplying the formal series and collecting coefficients of the same powers of the small parameter h, which is simple but scrupulous work (cf., the monograph [16, Ch. 4, 5, 16]) and thus left beyond the scope of this paper. Instead, we explain how the arising problems on the interval $(-1, 1)$ and in the cylinder Π are solved. As above in similar situations, the existence condition for the solution $\mathbf{v}'_{nj}$ of the equation (6.18) gives the expression of the term

$$\mu_{nj} = -(\mathbf{f}'_{nj}, \mathbf{v}_n)$$

in the series (6.15). The uniqueness of the solution $\mathbf{v}'_{nj}$ is achieved by posing the orthogonality condition (6.19). Recall that $(\, ,)$ is the natural scalar product in $L^2(-1, 1)$ and the eigenfunction $\mathbf{v}_n$ is normalized by (2.18).

It is a somewhat more complicated question to make the boundary layer terms $w_{nj}^{\pm}$ to decay. To this end we mention that the functions $f_{nj}^{\pm}$ and $g_{nj}^{\pm}$ come from the commuting of the differential operators with the cut-off function (3.23), which equals $\chi(\xi_d^{\pm})$ near the terminals $\Gamma_{\pm}^h$ of the spindle Ω^h (cf., comments to formula (4.49) in Section 4 § 4) and from the action of the differential operators to the terms $w_{n0}^{\pm}, \dots, w_{nj-1}^{\pm}$, which decay at an exponential rate as

$|\xi_d^\pm| \to \infty$. As a result, the right-hand sides of the problem (6.17) decay at the same exponential rate and therefore, this problem has a solution

$$w_{nj}^\pm(\xi^\pm) = \widetilde{w}_{nj}^\pm(\xi^\pm) + a_{nj}^\pm \chi(\xi_d^\pm) \tag{6.20}$$

with a finite Dirichlet integral $\|\nabla_{\xi^\pm} w_{nj}^\pm; L^2(\Pi)\|^2$, where $a_{nj}^\pm$ are some constants and $e^{\kappa|\xi_d^\pm|}\widetilde{w}_{nj}^\pm \in H^1(\Pi)$, $\kappa > 0$. The requirements

$$a_{nj}^\pm = 0 \tag{6.21}$$

imply the equality $w_{nj}^\pm = \widetilde{w}_{nj}^\pm$. Hence, the decay property of the boundary layer, which is an inherent property in the method of composite asymptotic expansions, is achieved by a proper choice of the coefficients $\mathbf{c}_{nj}$ and $\mathbf{C}_{nj}$ in the ansatz (6.16). Indeed, calculations in Section 1 § 4 and the equalities (6.12) show that

$$\begin{aligned} &\mathbf{c}_{nj}\mathbf{v}_n(z) + h^{d-2}\mathbf{C}_{nj}\mathbf{V}_n(z) = \mathbf{a}_{nj}^\pm + \dots \text{ for } \pm z = 1 - h\rho_\pm + O(h^2), \\ &\mathbf{a}_{nj}^\pm = \mathbf{c}_{nj}\mathbf{v}_n(\pm 1) \pm \mathbf{C}_{nj}\mathbf{v}_n(\pm 1)^{-1} H_\pm^{1-d}\rho_\pm^{2-d}. \end{aligned} \tag{6.22}$$

Thus, the right-hand side $f_{nj}^\pm$ of the Poisson equation can be written as

$$f_{nj}^\pm(\xi^\pm) = \mathbf{a}_{nj}^\pm \Delta_{\xi^\pm}\chi(\xi_d^\pm) + \widehat{f}_{nj}^\pm(\xi^\pm)$$

where the first term has a compact support, since the Laplacian acts to the cut-off function (2.13), and the other data $\widehat{f}_{nj}^\pm$ and $g_{nj}^\pm$ has been defined in the previous steps of the iterative process. Hence, the constants $a_{nj}^\pm$ of the decompositions (6.20) satisfy

$$a_{nj}^\pm = \mathbf{a}_{nj}^\pm + \widehat{a}_{nj}^\pm. \tag{6.23}$$

Here, the numbers $\widehat{a}_{nj}^\pm$ are known, hence, by formulas (6.22) and (6.23), the equalities (6.21) turn into a system of two linear equations for the unknowns $\mathbf{c}_{nj}$ and $\mathbf{C}_{nj}$; the matrix of the system with the determinant

$$\det\begin{pmatrix} \mathbf{v}_n(+1) & \mathbf{v}_n(+1)^{-1}H_+^{1-d}\rho_+^{2-d} \\ \mathbf{v}_n(-1) & -\mathbf{v}_n(-1)^{-1}H_-^{1-d}\rho_-^{2-d} \end{pmatrix} = -\sum_\pm \frac{\mathbf{v}_n(\pm 1)}{\mathbf{v}_n(\mp 1)} H_\mp^{1-d}\rho_\mp^{2-d}$$

is non-degenerate due to Lemma 2.4.

Remark 6.2. Analogous but much more complicated calculations are needed to construct infinite asymptotic series for the eigenpairs of problem (1.5)–(1.7) for the planar spindle. Also, there appear some differences: the coefficients of the series depend analytically on the variable $\mathfrak{h}^{-1} = |\ln h|^{-1}$, the terms $h^{d-2}\mathbf{C}_{nj}\mathbf{V}_n(z)$ disappear and the requirement of the decay of the boundary layer terms $w_{nj}^\pm$ leads to asymptotic conditions at the points $z = \pm 1$ for the solutions of the equation (6.18) (cf., the statement of problem (5.32) and see the end of Section 4 § 6). ⊠

6.4. Stationary problem. We consider the Poisson equation

$$-\Delta_x u^h(x) = f^h(x), \quad x \in \Omega^h, \tag{6.24}$$

with boundary conditions (1.6) and (1.7). The null spectral parameter $\lambda^h = 0$ can be included in the equation (6.24), and since it stays close to the first eigenvalue λ_1^h of the problem (1.5)–(1.7) by Theorems 4.5 and 5.8, we note that solutions to the problem (6.24), (1.6), (1.7) may have singular components. Let us now confirm this fact by using asymptotic analysis. We assume that the right-hand side of equation (6.24) is of the form

$$f^h(x) = h^{-2}H(s)^{-2}F(\eta, z) + \widetilde{f}^h(x)$$

where $\eta = h^{-1}H(s)^{-1}y$, $\widetilde{f}^h$ is a small remainder and F is a smooth function in the set $\overline{\omega}\times[-1,1]$. For simplicity, let us assume that $\widetilde{f}^h = 0$ and $F(\eta,\pm1)=0$. Repeating the dimension reduction procedure in Section 1 § 2 with small modifications, we see that the main terms of the possible asymptotic ansatz (2.2) are to be found from the Neumann problem

$$-H(z)^{-2}\Delta_\eta V(\eta,z) = \partial_z^2\mathbf{v}(z) + F(\eta,z), \quad \eta\in\omega,$$
$$H(z)^{-2}\partial_{\nu'(\eta)}V(\eta,z) = H(z)^{-1}\partial_z H(z)\eta\cdot\nu'(\eta)\partial_z\mathbf{v}(z), \quad \eta\in\partial\omega,$$

and from its compatibility condition, which turns into the ordinary differential equation

$$|\omega|\partial_z^2\mathbf{v}(z) + (d-1)|\omega|H(z)^{-1}\partial_z H(z)\partial_z\mathbf{v}(z) + \int_\omega F(\eta,z)d\eta = 0. \tag{6.25}$$

This is the same as the equation (5.3) with the right-hand side

$$\mathbf{f}(z) = \frac{1}{|\omega|}\int_\omega F(\eta,z)d\eta, \tag{6.26}$$

and the compatibility condition in (6.25) is but the equality (5.9).

If f^h is such that condition (5.9) fails, the ansatz (2.2) must be modified. To this end, in the case $d\geqslant 3$ we allow for the solution the growth

$$\mathbf{v}(z) = H_\pm^{1-d}(1\mp z)^{2-d}(\mathbf{c}_\pm + O((1\mp z))) \tag{6.27}$$

as $z\to\pm1$. Such a solution is certainly not unique, although it exists, if

$$\begin{aligned}&(H^{d-1}\mathbf{f},1) = -(\partial_z H^{d-1}\partial_z\mathbf{v},1)\\ &= -\lim_{\varepsilon\to+0}\sum_\pm \pm H(\pm1\mp\varepsilon)^{d-1}\partial_z\mathbf{v}(\pm1\mp\varepsilon) = (d-2)(\mathbf{c}_+ + \mathbf{c}_-)\end{aligned} \tag{6.28}$$

(see Lemma 6.1 and the generalized Green's formula (5.26)).

As was shown in Section 1§ 4, the singularity of the function $\mathbf{v}$ is compensated by the boundary layer near the terminals $\Gamma_\pm^h$ and requires the following ansatz in the middle part of the spindle Ω^h:

$$h^{2-d}\mathbf{c}_0 + \mathbf{v}(z). \tag{6.29}$$

In addition, the coefficients $\mathbf{c}_\pm$ in the expansions (6.27) are related to the constant $\mathbf{c}_0$ by the equations

$$\mathbf{c}_\pm = -\rho_\pm^{d-2}\mathbf{c}_0. \tag{6.30}$$

As a result, (6.28) and (6.30) yield the formula

$$\mathbf{c}_0 = -\left(\rho_+^{d-2}+\rho_-^{d-2}\right)^{-1}\frac{1}{d-2}\int_{-1}^{1} H(z)^{d-1}\mathbf{f}(z)\,dz. \tag{6.31}$$

It is clear that

$$|\mathbf{c}_0| + \sum_\pm |\mathbf{c}_\pm| \leqslant c\|\mathbf{f};\mathbf{L}\|. \tag{6.32}$$

We now consider the planar spindle. If $d=2$, equation (5.3) with the right-hand side (6.26) has the singular solution

$$\mathbf{v}(z) = \widetilde{\mathbf{v}}(z) + \sum_\pm \chi_\pm(z)H_\pm^{-1}\mathbf{b}_\pm\ln(1\mp z),$$

assuming there holds

$$(H\mathbf{f},1)=\mathbf{b}_{+}+\mathbf{b}_{-} \tag{6.33}$$

(cf., the generalized Green's formula (5.26)). To construct boundary layers near the ends $\Gamma^h_{\pm}$, we follow Section 3 § 5 and impose for the constant term in the ansatz

$$\mathbf{a}+\mathbf{v}(z) \tag{6.34}$$

the conditions

$$\mathbf{a}+\mathbf{b}_{\pm}H_{\pm}^{-1}(\ln h+\ln \rho_{\pm})=0. \tag{6.35}$$

Solving the system (6.33), (6.35) of three algebraic equations, we find that

$$\begin{aligned}\mathbf{a}:=\mathbf{a}(\ln h)&=\left(\frac{H_{+}}{|\ln h|-\ln\rho_{+}}+\frac{H_{-}}{|\ln h|-\ln\rho_{-}}\right)^{-1}(H\mathbf{f},1),\\ \mathbf{b}_{\pm}:=\mathbf{b}_{\pm}(\ln h)&=\frac{H_{\pm}\mathbf{a}(\ln h)}{|\ln h|-\ln\rho_{\pm}}.\end{aligned} \tag{6.36}$$

For a small h, the quantities (6.36) are properly defined and there holds the inequality

$$(1+|\ln h|)|\mathbf{a}|+\sum_{\pm}|\mathbf{b}_{\pm}|\leqslant c\|\mathbf{f};\mathbf{L}\|. \tag{6.37}$$

Due to the formulas (6.31), (6.32) and (6.36), (6.37) as well as (6.26), Proposition 4.6 can be applied to the ansätze (6.29) and (6.34) and it implies that the norm of the inverse operator of the problem (6.24), (1.6), (1.7). is of order $h^{2-d}(1+|\ln h|)^{\delta_{d,2}}$.

6.5. Discussing results in the case $d=2$. Formulas (6.36) show that the main term (6.34) in the asymptotics of solutions to problem (6.24), (1.6), (1.7) is a rational function of the large parameter $\mathfrak{h}=|\ln h|$. Such a dependence is also inherited by higher-order terms including the boundary layers. Similar asymptotic structures were discovered for the first time in the paper [35] devoted to the Dirichlet problem for the Poisson equation in a planar domain with a small hole of diameter $h\ll 1$; the paper only contains expansions of the solutions as series of inverse powers of the parameter $|\ln h|$. The sums of the above-mentioned series were calculated in [35]. The phenomenon of the rational dependence on logarithms has been discovered in other situatons, too (see the monographs [16, 17]) and not only for second-order equations but also for elliptic systems and higher-order equations, for example, the bi-harmonic equation describing the Kirchhoff plates (see, e.g., [6, § 30]). Moreover, in the monograph [16, Ch. 4] it was predicted that the asymptotic terms may also contain powers h^{is} with imaginary exponents (cf., the study [36] of the Laplace equation with the Steklov boundary condition in a specific geometry).

Asymptotic constructions for spectral problems are of course similar to those for stationary problems. In the case $d\geqslant 3$ the structure of the solutions of the problem (6.24), (1.6), (1.7) and the eigenfunctions of the problem (1.5)–(1.7) are quite similar and include power series of the small parameter h with coefficients which are polynomials of $|\ln h|$ (see Remark 4.3). However, the eigenpairs $\{\lambda^h_n,u^h_n\}$ for the planar spindle $\Omega^h\subset\mathbb{R}^2$ are characterized by analytic dependence on $|\ln h|^{-1}$ instead of rational (Theorem 5.8). This kind of a result was for the first time obtained in the paper [37] for the spectral Dirichlet problem in a planar domain with a small hole. This approach has become useful also in other problems in mathematical physics as well (see the book [16, Ch. 9 and 10]).

It is not difficult to show the analyticity of the functions (5.34) by analysing the perturbed pencil (5.33). On the other hand, there are certain obstacles in the examination of the spectra of operator pencils, and, therefore, our interpretation of (5.33) as a self-adjoint operator with domain (5.25), (5.30) is fundamentally important because the existing analytical and computational methods for self-adjoint operators have needed significant improvements here.

Another method of studying the spectra of problems in domains with singularly perturbed boundaries was proposed in the papers [38, 39, 40]. For the Dirichlet and Neumann problems in a d-dimensional domain with holes, it was shown that simple eigenvalues are analytic functions of the variable h ($d \geqslant 3$) or of the two variables h and $|\ln h|^{-1}$ ($d = 2$). The authors do not know if such a technique applies to the problem considered in this paper.

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