PHYSICAL REVIEW A **104**, L050204 (2021)

Letter

Time-energy uncertainty relation for quantum events

Matteo Fadel [1,2,*] and Lorenzo Maccone [3]

[1] *Department of Physics, ETH Zürich, 8093 Zürich, Switzerland*

[2] *Department of Physics, University of Basel, Klingelbergstrasse 82, 4056 Basel, Switzerland*

[3] *Dipartimento Fisica and INFN Pavia, University of Pavia, via Bassi 6, I-27100 Pavia, Italy*

(Received 7 September 2021; accepted 22 October 2021; published 24 November 2021)

Textbook quantum mechanics treats time as a classical parameter and not as a quantum observable with an associated Hermitian operator. For this reason, to make sense of usual time-energy uncertainty relations such as $\Delta t \Delta E \gtrsim \hbar$, the term Δt must be interpreted as a time interval and not as a time measurement uncertainty due to quantum noise. However, quantum clocks allow for a measurement of the "time at which an event happens" by conditioning the system's evolution on an additional quantum degree of freedom. Within this framework we derive here two uncertainty relations that relate the uncertainty in the quantum measurement of the time at which a quantum event happens on a system to its energy uncertainty.

DOI: 10.1103/PhysRevA.104.L050204

While classical mechanics allows for the simultaneous assignment of exact values of any set of physical observables, quantum mechanics predicts situations in which this is fundamentally forbidden. Concrete examples include position and momentum, and different angular momentum or spin components.

For physical observables associated to Hermitian operators, uncertainty relations provide quantitative lower bounds on the measurement uncertainties of their values: if the system is prepared in a state where a property A is defined with a precision ΔA, this bounds the precision ΔB of another property B and vice versa. The Heisenberg-Robertson [1,2] uncertainty relation $\Delta A \Delta B \geqslant |\langle [A,B]\rangle|/2$ expresses a tradeoff between the root-mean-square-error (RMSE) $\Delta X = \sqrt{\langle X^2\rangle - \langle X\rangle^2}$ of two observables A and B in terms of their commutator. By considering also the anticommutator, a tighter version of this inequality was found by Schrödinger [3] to be $\Delta A^2 \Delta B^2 \geqslant |\langle [A,B]\rangle/2|^2 + |\langle \{A,B\}_+\rangle/2 - \langle A\rangle\langle B\rangle|^2$. In both these relations, uncertainties originate from the stochastic nature of quantum measurements and the bound from quantum complementarity: it is impossible to simultaneously assign incompatible observables with arbitrary precision.

It is tempting, but wrong, to give a similar interpretation to uncertainty relations between time and energy [4]. In fact, in textbook quantum mechanics time is not a quantum observable but a classical parameter with no intrinsic quantum uncertainty. For this reason, any term Δt has to be understood as a time interval. For example, the correct interpretation of the Mandelstamm-Tamm relation [5] is the smallest time *interval* Δt required for a system with energy spread ΔE to evolve into an orthogonal state is lower bounded by $\Delta E \Delta t \gtrsim \hbar$. This interpretation follows immediately from the (classical) inequality between time *duration* Δt and bandwidth $\Delta\omega$ of a signal, $\Delta t \Delta\omega \gtrsim 1$, together with Planck's relation $E = \hbar\omega$. Interestingly, a similar bound was also given in terms of the average energy by Margolus and Levitin [6]: the smallest time interval required for a system with average energy $\langle E\rangle$ (above the ground state) to evolve into an orthogonal state is lower bounded by $\Delta t \geqslant \pi\hbar/2\langle E\rangle$. Both these relations can also be extended to the case in which the evolved state has arbitrary overlap with the initial state [7–9]. In essence, rather than quantum uncertainty relations, these inequalities are more properly "quantum speed limits" [10,11], bounding the "speed" of a dynamical evolution. Other uncertainty relations assign to Δt the minimum time *interval* required to estimate the energy of a system with precision ΔE. While this last interpretation is in general incorrect [12], it is valid if the system's Hamiltonian is unknown [13,14]. Therefore, in all the aforementioned relations (Mandelstamm-Tamm, Margolus-Levitin, Aharonov-Massar-Popescu) the quantity Δt is never a RMSE due to quantum noise but a time interval. Energy-time uncertainties also were studied for weak values [15] and in quantum gravity for clocks of limited dimensions [16]; a high precision of the clock requires a large energy devoted to it, but if that energy is compressed inside its Schwarzschild radius, a gravitational collapse occurs and the clock ceases to keep time. Similar effects are due to time dilation [17]. Finally, uncertainty relations that connect parameters (such as the time of textbook quantum mechanics) to observables can be found via the quantum Cramér-Rao bound [18,19]. All these arguments are not obtained from quantum complementarity of observables.

Instead, in this paper we prove the RMSE time-energy uncertainty relation (due to incompatible observables)

$$\Delta t_{ev} \Delta E_{ev} \geqslant \hbar/2, \tag{1}$$

where t_{ev} is the time at which some event happens in a quantum system, and E_{ev} is the system's energy conditioned on the event happening. In contrast to the time-energy uncertainty relations mentioned before, the quantity Δt_{ev} refers now to the uncertainty (RMSE) in the measurement of a quantum time observable due to quantum noise. The Hermitian time operator T_c, from which t_{ev} can be obtained through the Born

*fadelm@phys.ethz.ch

rule, is constructed here by considering an ancillary quantum system that serves as a clock [20]. Crucially, the energy of the system conditioned on the event happening, E_{ev}, can be connected to a system observable only if the projector Π that tests whether the event has happened is compatible (i.e., commutes) with the system's Hamiltonian H_s, namely, the two can be measured jointly. In this case $E_{ev} = \Pi H_s \Pi$. Otherwise, the "conditioned energy" is not even a well-defined concept due to quantum complementarity. In this case, when $[\Pi, H_s] \neq 0$, we propose a definition of conditional energy, which recovers the expected form in the commuting case thanks to a constraint equation between the system's and the clock's energies.

Time-energy relations considering the RMSE of a time measurements have also been introduced in Refs. [21–23]. These, however, refer only to uncertainties in the measurement of time (or proper time) with a clock of given energy, rather than the measurement of the *time at which an event happens*, as we do here.

Quantum time measurement. To give a quantum description of a time measurement, we first need to treat the clock that is used for this purpose as a quantum system [24–26]. Then the quantum time measurement is obtained from the observable $T_c \equiv \int dt\, t\, |t\rangle\langle t|$, with $|t\rangle$ the clock state associated to what happens to the system at time t [20]. To see this, let us define the timeless state $|\Psi\rangle\rangle$ that contains the full dynamics of the system's state $|\psi(t)\rangle$ by correlating it to the clock's state $|t\rangle$ as

$$|\Psi\rangle\rangle = \frac{1}{\sqrt{T}} \int_{-T/2}^{T/2} dt\, |t\rangle|\psi(t)\rangle. \tag{2}$$

Here, the double ket notation is just a reminder that $|\Psi\rangle\rangle$ is a joint clock-system state, and T is a regularization parameter that represents the total time interval we consider ($T \to \infty$ considers the full evolution of the system).

The state $|\Psi\rangle\rangle$ is an eigenstate of the Dirac-type [27] constraint operator $(H_c \otimes \mathbb{1}_s + \mathbb{1}_c \otimes H_s)|\Psi\rangle\rangle = 0$, where $H_{s(c)}$ is the system (clock) Hamiltonian. In order for $|\psi(t)\rangle$ to satisfy the Schrödinger equation, H_c must coincide with the clock's momentum, namely, $[T_c, H_c] = i\hbar$ [24,26]. The time observable T_c can thus be interpreted as the "position" observable for the clock, conjugate to its energy (such that the clock's energy is the generator of time translations). In this framework, time is an internal degree of freedom (a clock observable) instead of being an external parameter that labels the states as in the conventional formulation. Further details can be found in Refs. [26,28].

By "the event happens" one means that some property of the system acquires a value that is connected with the event. For example, if one wants to measure "the time at which the spin is up," then one must measure the time at which the spin value is "up." Each value of a system property is connected to some eigenvalue of a system observable. The observable's eigenvectors span orthogonal subspaces of the system Hilbert space, and the projector Π refers to the subspace where the system observable takes the value that refers to the event having happened, e.g., for the above example, $\Pi = |\uparrow\rangle\langle\uparrow|$ is the projector on the system's "spin-up" state.

Consider the observable $T_\pi = T_c \otimes \Pi$, where Π is the projector relative to the value of the property that indicates that the event happens. The Born rule tells us that the *joint* probability that the clock shows time t *and* that the event has happened is

$$p(t, \Pi) = \mathrm{Tr}[|t\rangle\langle t| \otimes \Pi|\Psi\rangle\rangle\langle\langle\Psi|], \tag{3}$$

(e.g., $p(t, \Pi) = |\langle\langle\Psi|t\rangle|\uparrow\rangle|^2$ for the "spin-up event" example). Similarly, the probability that the event happens at any time is $p(\Pi) = \langle\langle\Psi|\mathbb{1}_c \otimes \Pi|\Psi\rangle\rangle$, which follows from $\int dt|t\rangle\langle t| = \mathbb{1}$. From these two probabilities, using Bayes' rule, we can calculate the conditional probability that the clock shows time t given that the event happened as $p(t|\Pi) = p(t, \Pi)/p(\Pi)$. This probability allows us to compute the expectation value for the time at which the event happened as

$$\langle t_{ev}\rangle = \int dt\, t\, p(t|\Pi) = \frac{\langle T_\pi\rangle}{\langle\Pi\rangle} = \alpha T\langle T_\pi\rangle, \tag{4}$$

where all expectation values are calculated on $|\Psi\rangle\rangle$ and $\alpha^{-1} \equiv \int dt \langle\psi(t)|\Pi|\psi(t)\rangle = \langle\Pi\rangle T$, namely, $(\alpha T)^{-1} = p(\Pi)$ is the probability that the event happened at any time. The same conditional probability distribution allows us to calculate also the variance,[1]

$$\Delta t_{ev}^2 = \langle t_{ev}^2\rangle - \langle t_{ev}\rangle^2 = \alpha T\langle T_\pi^2\rangle - \alpha^2 T^2\langle T_\pi\rangle^2, \tag{5}$$

which expresses the uncertainty in the measurement of the time t_{ev} at which the event happens. Whenever $\langle t_{ev}\rangle = 0$ it is clear that $\langle T_\pi\rangle = 0$, which implies $\Delta t_{ev}^2 = \alpha T \Delta T_\pi^2$. The latter equality, however, is true also in all other cases where $\langle t_{ev}\rangle \neq 0$, since a shift of the averages will not affect the variances, and a shift t_0 of the average value of t_{ev} corresponds to a shift $\alpha T t_0$ of the average value of T_π, since $\langle t_{ev}\rangle = \langle T_\pi\rangle\alpha T$. In conclusion, we always have $\Delta t_{ev}^2 = \alpha T \Delta T_\pi^2$.

Conditional energy. In order to prove Eq. (1), we now need to evaluate ΔH_{ev}, namely, the RMSE of the energy conditioned on the event having happened. This term can be calculated from the clock energy, thanks to the constraint $(H_c \otimes \mathbb{1}_s + \mathbb{1}_c \otimes H_s)|\Psi\rangle\rangle = 0$, which guarantees that the clock and the system have (in modulo) equal energy.[2] To show this, note first that for time-independent H_s the average energy calculated on the system state $\langle\psi(t)|H_s|\psi(t)\rangle$ is independent of t because of energy conservation, and therefore it can also be calculated on $|\Psi\rangle\rangle$, giving the same result. In fact, we have

$$\begin{aligned}\langle\langle\Psi|\mathbb{1}_c \otimes H_s|\Psi\rangle\rangle &= \int_{-T/2}^{T/2} \frac{dt dt'}{T}\langle t'|t\rangle\langle\psi(t')|H_s|\psi(t)\rangle \\ &= \langle\psi(t)|H_s|\psi(t)\rangle \int_{-T/2}^{T/2} \frac{dt}{T} \\ &= \langle\psi(t)|H_s|\psi(t)\rangle = \langle H_s\rangle.\end{aligned} \tag{6}$$

[1]This formula is written incorrectly in [20], where the T factors are missing.

[2]Note here that any clock Hamiltonian H_c' can be rescaled by a constant k, such that $(kH_c' + H_s)|\Psi\rangle\rangle = 0$. In fact, this only results in a rescaling of the time parameter (a change of time units) appearing in the Schrödinger equation. Also, one can add an arbitrary additive constant $(H_c' + H_s + k')|\Psi\rangle\rangle = 0$ without any observable consequences.

Analogously, we find $(\Delta H_s)_{|\psi(t)\rangle} = (\Delta H_s)_{|\Psi\rangle\rangle}$. We can now use the constraint equation $H_c|\Psi\rangle\rangle = -H_s|\Psi\rangle\rangle$ to show that the system's average energy is the same as the clock's:

$$\begin{aligned}\langle\psi(t)|H_s|\psi(t)\rangle &= \langle\langle\Psi|\mathbb{1}_c\otimes H_s|\Psi\rangle\rangle\\ &= -\langle\langle\Psi|H_c\otimes\mathbb{1}_s|\Psi\rangle\rangle\\ &= -\mathrm{Tr}[H_c\otimes\mathbb{1}_s|\Psi\rangle\rangle\langle\langle\Psi|] = -\langle H_c\rangle. \quad (7)\end{aligned}$$

This directly implies that $(\Delta H_s)_{|\Psi\rangle\rangle} = (\Delta H_c)_{|\Psi\rangle\rangle}$.

If $[\Pi, H_s] = 0$, we can condition the energy of the system on the event and define the conditioned energy observable as $\Pi H_s \Pi$. Its expectation value is

$$\begin{aligned}\langle E_{ev}\rangle &= \int dE\, E\, p(E|\Pi) = \int dE\, E\, \frac{p(E,\Pi)}{p(\Pi)}\\ &= \langle\psi(t)|\Pi H_s\Pi|\psi(t)\rangle/p(\Pi)\\ &= \langle\psi(t)|\Pi H_s\Pi|\psi(t)\rangle\alpha T, \quad (8)\end{aligned}$$

where $p(E|\Pi)$ and $p(E,\Pi)$ are the conditional and the joint probability distributions for the energy E and the event Π happening. Analogously, one can calculate the variance

$$\Delta E_{ev}^2 = \alpha T\langle\Pi H_s^2\Pi\rangle - (\alpha T)^2\langle\Pi H_s\Pi\rangle^2. \quad (9)$$

Similarly to what we saw before, both the expectation value $\langle E_{ev}\rangle$ and the RMSE ΔE_{ev} can be calculated on the state $|\Psi\rangle\rangle$, obtaining the same result,

$$\begin{aligned}\langle\langle\Psi|E_{ev}|\Psi\rangle\rangle &= \int\frac{dtdt'}{T}\langle t'|t\rangle\langle\psi(t')|\Pi H_s\Pi|\psi(t)\rangle/p(\Pi)\\ &= \langle\psi(t)|\Pi H_s\Pi|\psi(t)\rangle/p(\Pi)\int_{-T/2}^{T/2}\frac{dt}{T}, \quad (10)\end{aligned}$$

where the last integral is equal to 1. A similar relation holds for computing $(\Delta E_{ev})_{|\Psi\rangle\rangle}$. Again, thanks to the constraint $H_s|\Psi\rangle\rangle = -H_c|\Psi\rangle\rangle$, the above relations written in terms of the system's energy H_s can be also written equivalently in terms of the clock energy H_c using

$$\langle\langle\Psi|\mathbb{1}_c\otimes\Pi H_s\Pi|\Psi\rangle\rangle = -\langle\langle\Psi|H_c\otimes\Pi|\Psi\rangle\rangle, \quad (11)$$

where $[H_s,\Pi] = 0$ was used.

If, instead, $[H_s,\Pi] \neq 0$, the property "the event has happened" and the system energy are incompatible observables, so they cannot be jointly defined. Nonetheless, a conditional energy (the energy conditioned to the event having happened) can still be defined using the constraint equation (7). In fact, the constraint ensures that the clock energy is equal to the system energy (rather, proportional, see footnote 2), so that one can estimate the system energy from a measurement of the clock energy, which is compatible with the system observable Π: it commutes with it (and can be measured jointly). In essence, we can use Eq. (7) to argue that a conditioned energy can be obtained by looking at the clock energy, conditioned on the event having happened on the system. We then define $\langle E_{ev}\rangle = -\langle\langle\Psi|H_c\otimes\Pi|\Psi\rangle\rangle\alpha T$ and $\Delta E_{ev}^2 = \alpha T\langle(H_c\otimes\Pi)^2\rangle - (\alpha T\langle H_c\otimes\Pi\rangle)^2$, where the αT terms come from the Bayes rule, as in Eqs. (8) and (9).

In both cases just considered, defining $H_\pi \equiv H_c\otimes\Pi$ we have

$$\Delta E_{ev}^2 = \alpha T\langle\langle\Psi|H_\pi|\Psi\rangle\rangle - (\alpha T)^2\langle\langle\Psi|H_\pi|\Psi\rangle\rangle^2. \quad (12)$$

As for the conditioned time, also the conditioned energy satisfies $\Delta E_{ev}^2 = \alpha T\Delta H_\pi^2$ because a shift E_0 in the average value of E_{ev} corresponds to a shift $\alpha T E_0$ of the average value of H_π.

Uncertainty relation. We can now derive Eq. (1) as

$$\begin{aligned}\Delta t_{ev}^2\Delta E_{ev}^2 &= (\alpha T)^2\Delta T_\pi^2\Delta H_\pi^2 \geqslant (\alpha T)^2|\langle[T_\pi, H_\pi]\rangle|^2/4\\ &= (\alpha T)^2\hbar^2\langle\Pi\rangle^2/4 = \hbar^2/4, \quad (13)\end{aligned}$$

where the inequality follows from the Robertson uncertainty relation for T_π and H_π calculated on $|\Psi\rangle\rangle$, and the second row is obtained from $[T_\pi, H_\pi] = [T_c, H_c]\otimes\Pi^2 = i\hbar\mathbb{1}_c\otimes\Pi$.

In a similar way, we can also find an uncertainty relation for the unconditioned system energy:

$$\begin{aligned}\Delta t_{ev}^2\Delta H_s^2 &= \alpha T\Delta T_\pi^2\Delta H_s^2\\ &\geqslant \frac{\alpha T}{4}|\langle\langle\Psi|[T_\pi, \mathbb{1}_c\otimes H_s]|\Psi\rangle\rangle|^2, \quad (14)\end{aligned}$$

where the inequality sign comes again from the Robertson uncertainty relation between T_π and H_s, with both variances calculated on $|\Psi\rangle\rangle$. Using the constraint equation and the condition $[T_c, H_c] = i\hbar$, we find

$$\langle[T_\pi, \mathbb{1}_c\otimes H_s]\rangle = \langle-(T_cH_c\otimes\Pi) + (H_cT_c\otimes\Pi)\rangle \quad (15)$$

$$= -\langle[T_c, H_c]\otimes\Pi\rangle = -i\hbar/\alpha T, \quad (16)$$

which, joined together with (14), gives

$$\Delta t_{ev}\Delta H_s \geqslant \frac{\hbar}{2}\sqrt{p(\Pi)}, \quad (17)$$

where $p(\Pi)$ is the overall probability that the event happens. Unfortunately, it appears that this last relation is always trivial: if the event does not happen an infinite number of times, then in the limit $T\to\infty$ we have $p(\Pi)\to 0$; if it does happen an infinite number of times, then clearly $\Delta t_{ev}\to\infty$. In both cases, the inequality (17) is satisfied trivially.

Examples. As a first example of the above inequalities, consider a photon with spectral amplitude $\varphi(\omega)$, namely, the state $|\psi\rangle = \int\frac{d\omega}{2\pi}\varphi(\omega)|1\rangle_\omega$, where $|1\rangle_\omega = a_{\vec{k}}^\dagger|0\rangle$ is a single photon at frequency $\omega = |\vec{k}|c$ (assuming a mode with fixed spatial direction $\vec{k}/|\vec{k}|$). The free evolution of $|\psi\rangle$ governed by the electromagnetic field Hamiltonian $H_s = \int d\omega\hbar\omega a_\omega^\dagger a_\omega$ induces the phase shift $\varphi(\omega)\to\varphi(\omega)\,e^{-i\omega t}$, so that the timeless state reads

$$|\Psi\rangle\rangle = \int\frac{dt}{\sqrt{T}}|t\rangle\int\frac{d\omega}{\sqrt{2\pi}}\,\varphi(\omega)e^{-i\omega t}|1\rangle_\omega. \quad (18)$$

The phase factor appearing in Eq. (18) is equivalent to a translation z along the propagation direction $\vec{k}/|\vec{k}|$: the photon wave packet propagates at a speed c determined by the Klein-Gordon equation $\Box(a_{\vec{k}}e^{-i(\omega t-\vec{k}\cdot\vec{x})} + \mathrm{H.c.}) = 0$ for all $\vec{k}$. Consider now a screen placed perpendicularly to the propagation at the position z_0. The projector associated to the detection of a photon by the screen is

$$\Pi_{z_0} = |1\rangle_{t_0}\langle 1| \equiv \int d\omega d\omega' e^{it_0(\omega'-\omega)}(|1\rangle_\omega)(_{\omega'}\langle 1|), \quad (19)$$

with $t_0 = z_0/c$. Then, as expected, the probability amplitude that the photon arrives at time t is the Fourier transform $\tilde{\varphi}(t)\propto$

$\int d\omega\, e^{-i\omega t}\varphi(\omega)$. In fact, we have

$$p(t|\Pi) = \langle\langle\Psi|(|t\rangle\langle t| \otimes \Pi_{z_0}|\Psi\rangle\rangle/p(\Pi_{z_0}) = |\tilde{\varphi}(t-t_0)|^2, \tag{20}$$

assuming ${}_{\omega'}\langle 1|1\rangle_\omega = \delta(\omega-\omega')$ (which follows from the Dirac δ commutators of the $a_{\vec{k}}$, restricting to positive energies) and with the normalization $\int d\omega|\varphi(\omega)|^2/2\pi = \int dt|\tilde{\varphi}(t)|^2 = 1$. Note that the regularization factor T does not appear in Eq. (20) thanks to $p(\Pi) = 1/T$. Then, Δt_{ev} is clearly the width Δt of the probability distribution $|\tilde{\varphi}(t)|^2$. The energy of the photon conditioned on having arrived at position z_0 can be calculated as

$$\begin{aligned}\langle E_{ev}\rangle &= -\langle\langle\Psi|H_c\otimes\Pi_{z_0}|\Psi\rangle\rangle/p(\Pi_{z_0})\\ &= -\int\frac{dt\,dt'}{2\pi}\langle t'|H_c|t\rangle\tilde{\varphi}^*(t'-t_0)\tilde{\varphi}(t-t_0)\\ &= -\int\frac{dp}{2\pi}\,\hbar p|\varphi(p)|^2.\end{aligned} \tag{21}$$

Similarly, $\Delta E_{ev}/\hbar$ can be shown to coincide with the width $\Delta\omega$ of the spectrum $|\varphi(\omega)|^2$. The Parseval inequalities, expressing a lower bound to the time-bandwidth product, imply that the widths Δt and $\Delta\omega$ of the probability distributions $|\tilde{\varphi}(t)|^2$ and $|\varphi(\omega)|^2/2\pi$ satisfy $\Delta t\Delta\omega \geqslant 1/2$. This then directly implies the validity of inequality (1) for this example.

Another example is the case in which the event consists in observing a photon of frequency ω_0, namely, the projector $\Pi_{\omega_0} = |1\rangle_{\omega_0}\langle 1|$. In this case we find $p(\Pi) = |\varphi(\omega_0)|^2$. Therefore, if ω_0 is in the support of φ, $p(t|\Pi_{\omega_0}) = 1/T$ is a constant (where the regularization T cannot be eliminated). This implies that $\Delta t_{ev} = \infty$, while ΔE_{ev} is finite (the width of φ), so inequality (1) is trivially satisfied.

Conclusion. Time-energy uncertainty relations commonly found in the literature relate the energy spread of a system to the length of a time interval and not to the uncertainty in the measurement of time due to quantum noise. In this sense, they are better understood as quantum speed limits in the time evolution of a state. In contrast, we presented here uncertainty relations that relate the uncertainty in the quantum measurement of the time at which a quantum event happens on a system to its energy uncertainty.

Our results are of foundational interest, as they clarify the connection between time intervals and time measurements: if the system takes an interval at least Δt to evolve to an orthogonal state, then an event cannot even be defined on a shorter timescale (in accordance to quantum speed limits). This means that the measurement outcome of when the event happens will have at least that quantum uncertainty associated to it [29]. In addition, our results might also be of practical relevance and find verification in state-of-the-art experiments [30,31].

[1] H. P. Robertson, The uncertainty principle, Phys. Rev. **34**, 163 (1929).

[2] D. A. Trifonov, Generalizations of Heisenberg uncertainty relation, Europ. Phys. J. B **29**, 349 (2002).

[3] E. Schrödinger, Zum Heisenbergschen Unschärfeprinzip, Sitzungsberichte der Preussischen Akademie der Wissenschaften, Phys.-Math. Kl. **14**, 296 (1930).

[4] P. Busch, The time-energy uncertainty relation, in *Time in Quantum Mechanics*, edited by J. G. Muga, R. Sala Mayato, and I. L. Egusquiza (Springer-Verlag, Berlin, 2002), pp. 69–98, 2nd rev. ed. 2008, pp. 73–105.

[5] L. Mandelstam and I. G. Tamm, The uncertainty relation between energy and time in nonrelativistic quantum mechanics, J. Phys. USSR **9**, 249 (1945).

[6] N. Margolus and L. B. Levitin, The maximum speed of dynamical evolution, Physica D **120**, 188 (1998).

[7] K. Bhattacharyya, Quantum decay and the Mandelstam-Tamm-energy inequality, J. Phys. A **16**, 2993 (1983).

[8] P. Pfeifer, How Fast Can a Quantum State Change with Time? Phys. Rev. Lett. **70**, 3365 (1993).

[9] V. Giovannetti, S. Lloyd, and L. Maccone, Quantum limits to dynamical evolution, Phys. Rev. A **67**, 052109 (2003).

[10] M. M. Taddei, B. M. Escher, L. Davidovich, and R. L. de Matos Filho, Quantum Speed Limit for Physical Processes, Phys. Rev. Lett. **110**, 050402 (2013).

[11] S. Deffner and S. Campbell, Quantum speed limits: From Heisenberg's uncertainty principle to optimal quantum control, J. Phys. A: Math. Theor. **50**, 453001 (2017).

[12] Y. Aharonov and D. Bohm, Time in the quantum theory and the uncertainty relation for time and energy, Phys. Rev. **122**, 1649 (1961).

[13] Y. Aharonov, S. Massar, and S. Popescu, Measuring energy, estimating Hamiltonians, and the time-energy uncertainty relation, Phys. Rev. A **66**, 052107 (2002).

[14] I. L. Paiva, A. C. Lobo, and E. Cohen, Flow of time during energy measurements and the resulting time-energy uncertainty relations, arXiv:2106.00523.

[15] E. Pollak and S. Miret-Artés, Uncertainty relations for time-averaged weak values, Phys. Rev. A **99**, 012108 (2019).

[16] M. P. Bronstein, Quantentheorie schwacher Gravitationsfelder, Phys. Z. Sowjetunion **9**, 140 (1936), as cited in [32].

[17] R. Gambini and J. Pullin, Fundamental bound for time measurements and minimum uncertainty clocks, J. Phys. Commun. **4**, 065008 (2020).

[18] S. L. Braunstein and C. M. Caves, Statistical Distance and the Geometry of Quantum States, Phys. Rev. Lett. **72**, 3439 (1994).

[19] S. L. Braunstein, C. M. Caves, and G. J. Milburn, Generalized uncertainty relations: Theory, examples, and Lorentz invariance, Ann. Phys. **247**, 135 (1996).

[20] L. Maccone and K. Sacha, Quantum Measurements of Time, Phys. Rev. Lett. **124**, 110402 (2020).

[21] P. J. Coles, V. Katariya, S. Lloyd, I. Marvian, and M. M. Wilde, Entropic Energy-Time Uncertainty Relation, Phys. Rev. Lett. **122**, 100401 (2019).

[22] A. R. H. Smith and M. Ahmadi, Quantum clocks observe classical and quantum time dilation, Nat. Commun. **11**, 5360 (2020).

[23] C. Foti, A. Coppo, G. Barni, A. Cuccoli, and P. Verrucchi, There is only one time, Nat. Commun. **12**, 1787 (2021).

[24] D. N. Page and W. K. Wootters, Evolution without evolution: Dynamics described by stationary observers, Phys. Rev. D **27**, 2885 (1983).

[25] Y. Aharonov and T. Kaufherr, Quantum frames of reference, Phys. Rev. D **30**, 368 (1984).

[26] V. Giovannetti, S. Lloyd, and L. Maccone, Quantum time, Phys. Rev. D **92**, 045033 (2015).

[27] P. A. M. Dirac, Generalized Hamiltonian dynamics, Proc. R. Soc. London A, Math. Phys. **246**, 326 (1958).

[28] P. A. Höhn, A. R. H. Smith, and M. P. E. Lock, Trinity of relational quantum dynamics, Phys. Rev. D **104**, 066001 (2021).

[29] M. Fadel, L. Ares, A. Luis, and Qiongyi He, Number-phase entanglement and Einstein-Podolsky-Rosen steering, Phys. Rev. A **101**, 052117 (2020).

[30] E. Moreva, G. Brida, M. Gramegna, V. Giovannetti, L. Maccone, and M. Genovese, Time from quantum entanglement: An experimental illustration, Phys. Rev. A **89**, 052122 (2014).

[31] E. Moreva, G. Brida, M. Gramegna, L. Maccone, and M. Genovese, Quantum time: Experimental multi-time correlations, Phys. Rev. D **96**, 102005 (2017).

[32] C. Rovelli and F. Vidotto, *Covariant Loop Quantum Gravity, An Elementary Introduction to Quantum Gravity and Spinfoam Theory* (Cambridge University Press, Cambridge, England, 2015).