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Calculus of Variations

Local boundedness for solutions of a class of nonlinear elliptic systems

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Abstract

In this paper we are concerned with the regularity of solutions to a nonlinear elliptic system of m equations in divergence form, satisfying p growth from below and q growth from above, with $p \leq q$; this case is known as p, q-growth conditions. Well known counterexamples, even in the simpler case $p = q$, show that solutions to systems may be singular; so, it is necessary to add suitable structure conditions on the system that force solutions to be regular. Here we obtain local boundedness of solutions under a componentwise coercivity condition. Our result is obtained by proving that each component u^α of the solution $u = (u^1, ..., u^m)$ satisfies an improved Caccioppoli's inequality and we get the boundedness of u^α by applying De Giorgi's iteration method, provided the two exponents p and q are not too far apart. Let us remark that, in dimension $n = 3$ and when $p = q$, our result works for $\frac{3}{2} < p \leq 3$, thus it complements the one of Bjorn whose technique allowed her to deal with $p \leq 2$ only. In the final section, we provide applications of our result.

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1 Introduction

In this paper we are concerned with the regularity of solutions to a nonlinear elliptic system of m equations in divergence form

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(A_i^\alpha(x, Du(x)) \right) = 0, \quad 1 \le \alpha \le m, \tag{1.1}$$

where $x \in \Omega$ and Ω is a bounded open set in $\mathbb{R}^n$, $n \ge 2$. The function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$, has components $(u^1, ..., u^m)$; then $Du(x)$ is the $m \times n$ matrix $\left(\frac{\partial u^\alpha}{\partial x_i}(x) \right)_{i=1,\dots,n}^{\alpha=1,\dots,m}$.

We assume that $A_i^\alpha : \Omega \times \mathbb{R}^{m\times n} \to \mathbb{R}$, $1 \le i \le n$, $1 \le \alpha \le m$, are Carathéodory functions satisfying for every $x \in \Omega$ and for every $z = (z^1, \dots, z^m)^T \in \mathbb{R}^{m\times n}$ the following p, q-growth assumptions:

$$\nu |z^\alpha|^p - a(x) \le \sum_{i=1}^{n} A_i^\alpha(x, z) z_i^\alpha \qquad \forall \alpha \in \{1, \cdots, m\}, \tag{1.2}$$

$$\sum_{i=1}^{n} |A_i^\alpha(x, z)| \le M \left(|z|^{q-1} + b(x) \right), \tag{1.3}$$

where $1 < p \le q$, $p \le n$, $\nu, M > 0$, $a \in L^{\tau_1}_{\mathrm{loc}}(\Omega)$ and $b \in L^{\tau_2}_{\mathrm{loc}}(\Omega)$ are non-negative functions, with $1 < \tau_i \le +\infty$, $i = 1, 2$, and $\tau_2 \ge \frac{q}{q-1}$.

Let us recall that $u \in W^{1,q}_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$ is a weak solution of (1.1) if

$$\int_B \sum_{\beta=1}^{m} \sum_{i=1}^{n} A_i^\beta(x, Du(x)) D_i \psi^\beta(x)\, dx = 0, \tag{1.4}$$

for every open set $B \Subset \Omega$ and for every $\psi \in W^{1,q}_0(B; \mathbb{R}^m)$.

As usual the Sobolev exponent is $p^* = \frac{np}{n-p}$ if $p < n$, and p^* is any real number $\mu > p$ if $p = n$. The Hölder conjugate exponent of p is $p' = \frac{p}{p-1}$. We use the position $\frac{1}{+\infty} = 0$. Our regularity result is the following.

Theorem 1.1 *Assume that* (1.2) *and* (1.3) *hold, with* $1 < p \le n$, $p \le q$ *and* $1 < \tau_1, \tau_2 \le +\infty$, *satisfying*

$$q < p^* \frac{n}{p(n+1)}, \qquad \tau_1 > \frac{n}{p}, \qquad \tau_2 \ge \frac{q}{q-1}. \tag{1.5}$$

Then any weak solution $u \in W^{1,q}_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$ *of* (1.1) *is locally bounded.*

In the vector-valued case, as suggested by well known counterexamples [11, 13, 14, 18–20, 23, 24, 28, 38, 43, 45, 49, 50], special structures on the operator are required for everywhere regularity, even under reasonable assumptions on the coefficients; see also the surveys [40, 41] and [25].

In the literature there are still few contributions about the boundedness of weak solutions to elliptic systems. Ladyzhenskaya and Ural'tseva ([27], Chapter 7) first proposed the local boundedness of solutions $u = \left(u^1, u^2, \ldots, u^m\right)$ to the *linear* elliptic system

$$\begin{aligned}\sum_{i=1}^{n} \frac{\partial}{\partial x_i} &\left(\sum_{j=1}^{n} a_{ij}(x)\, u^{\alpha}_{x_j} + \sum_{\beta=1}^{m} b_i^{\alpha\beta}(x)\, u^{\beta} + f_i^{\alpha}(x) \right) \\ &+ \sum_{i=1}^{n}\sum_{\beta=1}^{m} c_i^{\alpha\beta}(x)\, u^{\beta}_{x_i} + \sum_{\beta=1}^{m} d^{\alpha\beta}(x)\, u^{\beta} = f^{\alpha}(x)\,, \qquad \forall\, \alpha = 1, 2, \ldots, m,\end{aligned} \tag{1.6}$$

with bounded measurable coefficients a_{ij} , $b_i^{\alpha\beta}$, $c_i^{\alpha\beta}$, $d^{\alpha\beta}$ and given functions f_i^{α} , f^{α}. Here the *structure condition* is stated in terms of the positive definite $n \times n$ matrix $\left(a_{ij}\right)$, which *does not depend* on α, β. In [39] Meier extended these results to nonlinear elliptic systems of the form

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(A_i^{\alpha}(x, u, Du)\right) = 0, \tag{1.7}$$

under the following p-growth conditions, $1 < p \leq n$,

$$\sum_{i=1}^{n}\sum_{\alpha=1}^{m} A_i^{\alpha}(x, u, z) z_i^{\alpha} \geq |z|^p - d(x)|u|^p - g(x) \tag{1.8}$$

$$|A^{\alpha}(x, u, z)| \leq a|z|^{p-1} + b(x)|u|^{p-1} + e(x) \tag{1.9}$$

for $a > 0$ and under suitable integrability assumptions on the nonnegative functions b, e, d, g. Meier introduces the so-called *indicator function* of the operator

$$I_A(x, u, Du) := \sum_{\alpha,\beta,i} A_i^{\alpha}(x, u, Du) D_i u^{\beta} \frac{u^{\alpha} u^{\beta}}{|u|^2} \tag{1.10}$$

and a pointwise assumption turns out to be crucial in Meier's techniques, indeed a weak solution u of (1.7) is locally bounded if

$$I_A(x, u, Du) \geq 0 \tag{1.11}$$

holds for large values of $|u|$. Notice that (1.11) is satisfied in linear case (1.6). Assumption (1.11) is satisfied also by some nonlinear operators. For example:

$$A_i^{\alpha}(Du) = \sigma(Du) D_i u^{\alpha}, \tag{1.12}$$

when $0 \leq \sigma$, like in the case of Euler's system of the functional

$$\int F(|Du|) dx, \tag{1.13}$$

where F increases and we take $\sigma(Du) = \frac{F'(|Du|)}{|Du|}$. A third example, for which (1.11) holds true, is given when considering Euler's system of the anisotropic integral

$$\int \sum_{i=1}^{n} g_i(|D_i u|) dx, \tag{1.14}$$

where g_i increases and we take

$$A_i^{\alpha}(Du) = \frac{g_i'(|D_i u|)}{|D_i u|} D_i u^{\alpha}, \tag{1.15}$$

see section 4 in [30]. Let us look at another example: we set $m = n$ and we consider the polyconvex integral

$$\int (|Du|^p + h(\det Du))dx, \tag{1.16}$$

where h is convex, C^1, bounded from below. In this case Euler's system gives

$$A_i^\alpha(Du) = p|Du|^{p-2} D_i u^\alpha + h'(\det Du)(\mathrm{Cof}\, Du)_i^\alpha, \tag{1.17}$$

where $(\mathrm{Cof}\, Du)_i^\alpha$ is the determinant of the $(n-1)\times(n-1)$ matrix obtained from the $n \times n$ matrix Du by deleting row α and column i, with the sign given by $(-1)^{\alpha+i}$. It turns out that

$$I_A(x, u, Du) = p|Du|^{p-2} \sum_{i=1}^n \left(\sum_{\alpha=1}^n \frac{u^\alpha}{|u|} D_i u^\alpha \right)^2 + h'(\det Du) \det Du \geq \inf_{\mathbb{R}} h - h(0), \tag{1.18}$$

then we get (1.11), provided $h(0) = \inf_{\mathbb{R}} h$; see Sect. 3, later in the present paper; see also [31].

The previous examples show that Meier's condition allows us to deal with quite a large class of nonlinear systems. Boundedness results for weak solutions to nonlinear elliptic systems are proved by Krömer [26] under assumptions similar to (1.11), see also Landes [29].

Actually Meier's regularity result is obtained under a weaker assumption, since I_A can be allowed to be negative, but not too much.

More precisely, under (1.7) and (1.8), there exist positive constants λ and L such that

$$I_A(x, u, z) := \sum_{\alpha,\beta,i} A_i^\alpha(x, u, z) z_i^\beta \frac{u^\alpha u^\beta}{|u|^2} \geq -\left\{ \delta|z|^p + \left(\frac{1}{\delta}\right)^\lambda [d(x)|u|^p + g(x)] \right\} \tag{1.19}$$

for every $\delta \in (0, 1)$, for all $(x, u, z) \in \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n}$, with $|u| > L$.

Let us observe that the following linear decoupled system does not verify (1.19), see [32] and Sect. 3, later in the present paper:

$$A_i^\alpha(x, Du) = \sigma^\alpha(x) D_i u^\alpha, \tag{1.20}$$

where $m = 2$,

$$\sigma^1(x) = 18 + 2\sin(|x|^2) \quad \text{and} \quad \sigma^2(x) = 2 + \sin(|x|^2). \tag{1.21}$$

Now we consider another example, see [31], in which the equations are coupled and Meier's condition (1.19) is not satisfied: it is Euler's system of

$$\int \left[|Du|^2 + h(D_1 u^1 D_1 u^2) \right] dx \tag{1.22}$$

where $m = 2$, h is convex, C^1, bounded from below, so that

$$A_i^\alpha(Du) = 2D_i u^\alpha + h'(D_1 u^1 D_1 u^2) D_1 u^{\hat\alpha} \delta_{i1}, \tag{1.23}$$

where

$$\hat\alpha = 2 \ \text{ if } \alpha = 1 \ \text{ and } \ \hat\alpha = 1 \ \text{ if } \alpha = 2; \quad \text{moreover,} \ \delta_{i1} = 1 \ \text{ if } \ i = 1$$
$$\text{and } \delta_{i1} = 0 \ \text{ otherwise.}$$

Meier's condition (1.19) is not satisfied, provided $h'(0) \leq -8$: for instance, $h(t) = 16\sqrt{1 + (t-1)^2}$; see Sect. 3 later in the present paper.

Combining coefficients $\sigma^\alpha(x)$ similar to (1.21) with the nonlinear part of (1.23), we are able to build an example with p growth that does not satisfy Meier's condition (1.19). Indeed,

$$A_i^\alpha(Du) = \sigma^\alpha(x) p |Du|^{p-2} D_i u^\alpha + h'(D_1 u^1 D_1 u^2) D_1 u^{\hat{\alpha}} \delta_{i1}, \tag{1.24}$$

where $2 \leq p$, h is convex, C^1, bounded from below; $\hat{\alpha}$ and δ_{i1} are defined as before. Moreover, $m = 2$ and

$$\sigma^1(x) = 48 + 3\sin(|x|^2) \text{ and } \sigma^2(x) = 2 + \sin(|x|^2). \tag{1.25}$$

Meier's condition (1.19) is not satisfied, provided $h'(0) \leq 0$: for instance, $h(t) = \left(1 + t^2\right)^{p/4}$; see Sect. 3 for the details.

In [2] Bjorn obtained boundedness of solutions u of systems without considering the indicator function but assuming componentwise coercivity:

$$\nu |z^\alpha|^p - a(x) - b(x)|u|^p \leq \sum_{i=1}^n A_i^\alpha(x, u, z) z_i^\alpha, \quad \text{with} \quad \nu > 0. \tag{1.26}$$

Previous assumption (1.26) says that, even if row α of the system contains all the components of $z = Du$, after multiplying this row by component α of $z = Du$, from below we only see the α component of $z = Du$ and none of other components.

(1.26) is satisfied in system (1.20), provided $\sigma^\alpha(x) \geq \nu$ for some positive constant ν. Furthermore, the structure in (1.12) guarantees (1.26), provided $\sigma(Du) \geq \nu |Du|^{p-2}$, for some constants $p \geq 2$ and $\nu > 0$. Let us mention that polyconvex structure (1.17) enjoys (1.26), provided $p \geq 2$, see Sect. 3. Finally, systems in (1.23) and (1.24) satisfy (1.26): details are in Sect. 3.

Let us observe that the interesting Bjorn's technique allows to deal only with the subquadratic case $1 < p \leq 2$. When A_i^α does not depend on u, in Theorem 1.1, we are able to deal with the case $p_0 < p \leq n$, for a suitable $p_0 = p_0(n)$; in the three dimensional case $n = 3$, $p_0 = 3/2$, so our result complements the one of Bjorn and we get boundedness of solutions of elliptic systems under componentwise coercivity, see details at the end of this introduction.

It is worth pointing out that we study system satisfing p, q-growth, according to Marcellini [35]. Regularity in this case is obtained when q is not far from p, see the survey [40] and, more recently, [36, 37, 42]; inequality $p \leq q < p^* \frac{n}{p(n+1)}$ tells us that q cannot be too far from p.

We underline that the strategy for proving our vectorial regularity result is De Giorgi's elegant and powerful method, see [10]. Precisely, we prove separately that each component u^α satisfies a suitable Caccioppoli-type inequality, a decay of the "excess" on super-(sub-) level sets of u^α that allow to apply iteration arguments and, eventually, the local boundedness of the α-th component of u. A similar strategy has been successfully applied in [6] to prove the boundedness of local minimizers of polyconvex functionals satisfying a non-standard growth, see also [3, 4, 47]. Local boundedness of weak solutions to some elliptic systems with anisotropic or p, q growth has been proved in [7] by using Moser's iteration technique. In [31, 48], a kind of maximum principle has been proved for systems verifying a condition similar to (1.26); see also [46]. Recent results on the regularity of minimizers of variational integrals or equations in the scalar framework are in [1, 21]; see also [22], where the boundedness of scalar local minimizers of variational integrals is proved under a sharp bound on the exponents p, q, in the light of the counterexamples in [16, 33, 34]. We also cite the interesting paper

[9], where both the scalar and the vectorial case are considered, this last case under the so called Uhlenbeck assumption, i.e. the radial structure $f(x, Du) = \tilde{f}(x, |Du|)$, where f is the energy density of the variational integral. Under this assumption, the usual condition on the exponents p and q to have the local boundedness of vectorial minimizers of functionals, or of weak solutions to systems, is $q < p^*$, see e.g. [7, 8]. Our gap condition on p and q is more restrictive, but we do not require the strong Uhlenbeck assumption.

We try to explain why we are able to consider values of p larger than the ones considered in [2]. Bjorn uses Caccioppoli inequality on superlevel sets $\{v > k\}$ with the *same* exponent p both for Dv and $v - k$. We use Caccioppoli inequality on superlevel sets with *different* exponents: p for Dv and p^* for $v - k$. When p is close to n, then p^* is, by far, larger than p, and this helps a lot. Let us also mention that Bjorn takes $v = \max\{|u^1|, ..., |u^m|\}$, where $u = (u^1, ..., u^m)$ is the solution of the system; on the contrary, we take $v = u^\alpha$, the component α of u.

Let us discuss inequalities $1 < p \leq n$, $p \leq q < p^* \frac{n}{p(n+1)}$, as required in (1.5) of our Theorem 1.1. If $p = n$ the condition on q is trivially satisfied. We have to solve $p < p^* \frac{n}{p(n+1)}$ when $1 < p < n$. This means that $0 < (n+1)p^2 - n(n+1)p + n^2$; when $n = 2$ this is satisfied for every p; when $n = 3$, it is true for $p \neq \frac{3}{2}$; when $n \geq 4$ the inequality is satisfied for $1 < p < p_-$ or $p_+ < p < n$, where

$$p_\pm = \frac{n}{2}\left(1 \pm \sqrt{\frac{n-3}{n+1}}\right). \tag{1.27}$$

Note that

$$1 < p_- < 2 < p_+ < n. \tag{1.28}$$

If we confine ourselves to the case $p = q$, it is possible to make a comparison with Bjorn [2]. When $n = 2$, we recover Bjorn's boundedness result for every $1 < p \leq n = 2$. When $n = 3$, Bjorn's result is limited to $1 < p \leq 2$ and we complement it, since we are able to deal with $2 < p \leq n = 3$. When $n \geq 4$, Bjorn's result holds true for $1 < p \leq 2$, our result is valid when $p_+ < p \leq n$, so it remains open the case $2 < p \leq p_+$.

We conclude by observing that in the definition of weak solution of an elliptic equation or system with p, q-growth, the solution is assumed to be in $W^{1,q}_{\mathrm{loc}}$ and not in $W^{1,p}_{\mathrm{loc}}$, see e.g. [35] and [37]. Enforcing the assumptions on the structure of the nonlinear operator, it is possible to prove the existence of a solution in $W^{1,q}_{\mathrm{loc}} \cap (W^{1,p}_0 + u_0)$ of a Dirichlet problem with a sufficiently regular boundary datum u_0, see [5]. On this topic we also refer to Theorem 4.1 in [35], where the scalar case is considered.

Our paper is organized as follows. In the next section we present the proof of Theorem 1.1. In Sect. 3 we give details for some of the previous examples.

2 Proof of Theorem 1.1

The proof of Theorem 1.1 is based on the DeGiorgi method, see [10], suitable for dealing with equations. Nevertheless we apply it in the vectorial framework, since we can apply it to each component u^α of a weak solution u separately.

In what follows we limit ourselves to consider the case $p < n$. The remaining case, $p = n$, can be obtained by the previous one. Indeed, by using the inequality $|z|^{n-\epsilon} - 1 \leq |z|^n$, that holds true for any positive $\epsilon \leq n$, we get that the n, q-growth implies a $n - \epsilon, q$-growth and that the assumptions on the exponents, see (1.5), are easily satisfied by choosing $0 < \epsilon < \epsilon_0$, with $\epsilon_0 = \min\left\{\frac{n^2}{q(n+1)}, n\left(1-\frac{1}{\tau_1}\right)\right\}$.

STEP 1. Caccioppoli inequality

The particular growth conditions (1.2) and (1.3) guarantee a Caccioppoli inequality for any component u^α of u on every superlevel set $\{u^\alpha > k\}$.

Proposition 2.1 *Let us consider the system (1.1) and assume that (1.2), (1.3) hold. Let $u \in W^{1,q}_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$ be a weak solution of (1.1). Let $B_R(x_0) \Subset \Omega$ with $|B_R(x_0)| \le 1$; for $k \in \mathbb{R}$, $\alpha = 1, ..., m$ and $0 < \tau \le R$, denote*

$$A^\alpha_{k,\tau} := \{x \in B_\tau(x_0) \,:\, u^\alpha(x) > k\}.$$

If $q \le p^*$ then, there exists $c = c(n, p, \nu, M) > 0$ such that, for every s, t with $0 < s < t \le R$, for every $k \in \mathbb{R}$ and for every $\alpha = 1, ..., m$ we have

$$\begin{aligned}\int_{A^\alpha_{k,s}} |Du^\alpha|^p\,dx \le\; & c\int_{A^\alpha_{k,t}} \left(\frac{u^\alpha - k}{t-s}\right)^{p^*} dx \\ & + c\left\{\|Du\|^{(q-1)(p^*)'}_{L^q(B_R(x_0))} + \|a\|_{L^{\tau_1}(B_R(x_0))}\right. \\ & \left. + \|b\|^{(p^*)'}_{L^{\tau_2}(B_R(x_0))}\right\} |A^\alpha_{k,t}|^\vartheta, \end{aligned} \tag{2.1}$$

where

$$\vartheta := \min\left\{1 - \frac{(p^*)'}{q'}, 1 - \frac{1}{\tau_1}, 1 - \frac{(p^*)'}{\tau_2}\right\}.$$

We can take $c = \frac{1+M2^{1+p^*}}{\nu}$.

Proof Fix $\alpha \in \{1, \dots, m\}$. Consider a cut-off function $\eta \in C^1_0(B_t(x_0))$ satisfying the following assumptions:

$$0 \le \eta \le 1, \quad \eta \equiv 1 \text{ in } B_s(x_0), \quad |D\eta| \le \frac{2}{t-s}. \tag{2.2}$$

Define the test function $\psi = (\psi^1, ..., \psi^m) \in W^{1,q}_0(B_t(x_0);\mathbb{R}^m)$, where $\psi^\beta = 0$ if $\beta \ne \alpha$ and $\psi^\alpha = (u^\alpha - k)_+\,\eta$, where $\tau_+ = \max\{\tau, 0\}$. Notice that

$$\psi^\alpha_{x_i} = \chi_{\{u^\alpha > k\}} u^\alpha_{x_i}\eta + \eta_{x_i}(u^\alpha - k)_+,$$

where $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ otherwise; moreover, $f_{x_i} = D_i f = \frac{\partial f}{\partial x_i}$.

We insert such a ψ into (1.4) and we get

$$\sum_{i=1}^n \int_{\{u^\alpha > k\}} A^\alpha_i(x, Du)u^\alpha_{x_i}\eta\,dx = -\sum_{i=1}^n \int_{\{u^\alpha > k\}} A^\alpha_i(x, Du)(u^\alpha - k)\eta_{x_i}\,dx. \tag{2.3}$$

By (1.2) and (1.3)

$$\begin{aligned}\nu \int_{\{u^\alpha > k\}} |Du^\alpha|^p \eta \, dx \leq & \int_{\{u^\alpha > k, \eta > 0\}} a(x) \eta \, dx \\ & + M \int_{\{u^\alpha > k\}} (u^\alpha - k) |Du|^{q-1} |D\eta| \, dx \\ & + M \int_{\{u^\alpha > k\}} (u^\alpha - k) b(x) |D\eta| \, dx \\ =: & \, J_1 + J_2 + J_3. \end{aligned} \tag{2.4}$$

It is easy to estimate J_1, indeed, using Hölder inequality

$$J_1 \leq \|a\|_{L^{\tau_1}(B_R(x_0))} |A^\alpha_{k,t}|^{1-\frac{1}{\tau_1}}. \tag{2.5}$$

In order to estimate J_2, we first use Young inequality with exponents p^* and $(p^*)'$.

$$J_2 \leq M \int_{A^\alpha_{k,t}} (u^\alpha - k)^{p^*} |D\eta|^{p^*} \, dx + M \int_{A^\alpha_{k,t}} |Du|^{(q-1)(p^*)'} \, dx.$$

Since $q < p^*$ then $(q-1)(p^*)' < q$. Therefore we can use Hölder inequality with first exponent $\frac{q'}{(p^*)'} > 1$ to estimate the last integral, obtaining

$$M \int_{A^\alpha_{k,t}} |Du|^{(q-1)(p^*)'} \, dx \leq M \left(\int_{A^\alpha_{k,t}} |Du|^q \, dx \right)^{\frac{(p^*)'}{q'}} \left| A^\alpha_{k,t} \right|^{1-\frac{(p^*)'}{q'}}. \tag{2.6}$$

Thus, if we keep in mind that $|D\eta| \leq 2/(t-s)$, then

$$J_2 \leq M 2^{p^*} \int_{A^\alpha_{k,t}} \left(\frac{u^\alpha - k}{t-s} \right)^{p^*} dx + M \left(\int_{A^\alpha_{k,t}} |Du|^q \, dx \right)^{\frac{(p^*)'}{q'}} \left| A^\alpha_{k,t} \right|^{1-\frac{(p^*)'}{q'}}. \tag{2.7}$$

In order to estimate J_3, we first use Young inequality with exponents p^* and $(p^*)'$:

$$M \int_{\{u^\alpha > k\}} (u^\alpha - k) b(x) |D\eta| \, dx \leq M \int_{A^\alpha_{k,t}} (u^\alpha - k)^{p^*} |D\eta|^{p^*} \, dx + M \int_{A^\alpha_{k,t}} b^{(p^*)'} \, dx;$$

note that $\tau_2 \geq q' > (p^*)'$; so, we can use Hölder inequality with first exponent $\frac{\tau_2}{(p^*)'} > 1$ and we get

$$M \int_{A^\alpha_{k,t}} b(x)^{(p^*)'} \, dx \leq M \left(\int_{A^\alpha_{k,t}} b(x)^{\tau_2} \, dx \right)^{\frac{(p^*)'}{\tau_2}} \left| A^\alpha_{k,t} \right|^{1-\frac{(p^*)'}{\tau_2}}.$$

Once again we use that $|D\eta| \leq 2/(t-s)$, then

$$J_3 \leq M 2^{p^*} \int_{A^\alpha_{k,t}} \left(\frac{u^\alpha - k}{t-s} \right)^{p^*} dx + M \left(\int_{B_R(x_0)} b(x)^{\tau_2} \, dx \right)^{\frac{(p^*)'}{\tau_2}} \left| A^\alpha_{k,t} \right|^{1-\frac{(p^*)'}{\tau_2}}. \tag{2.8}$$

Collecting (2.4), (2.5), (2.7), (2.8), we get

$$\begin{aligned}
\nu \int_{A^\alpha_{k,t}} |Du^\alpha|^p \eta\, dx \le\ & M 2^{1+p^*} \int_{A^\alpha_{k,t}} \left(\frac{u^\alpha - k}{t-s}\right)^{p^*} dx + \|a\|_{L^{\tau_1}(B_R(x_0))} |A^\alpha_{k,t}|^{1-\frac{1}{\tau_1}} \\
& + M \left(\int_{A^\alpha_{k,t}} |Du|^q\, dx\right)^{\frac{(p^*)'}{q'}} \left|A^\alpha_{k,t}\right|^{1-\frac{(p^*)'}{q'}} \\
& + M \left(\int_{B_R(x_0)} b(x)^{\tau_2}\, dx\right)^{\frac{(p^*)'}{\tau_2}} \left|A^\alpha_{k,t}\right|^{1-\frac{(p^*)'}{\tau_2}} \\
\le\ & M 2^{1+p^*} \int_{A^\alpha_{k,t}} \left(\frac{u^\alpha - k}{t-s}\right)^{p^*} dx + \|a\|_{L^{\tau_1}(B_R(x_0))} |A^\alpha_{k,t}|^{1-\frac{1}{\tau_1}} \\
& + M \|Du\|^{(q-1)(p^*)'}_{L^q(B_R(x_0))} \left|A^\alpha_{k,t}\right|^{1-\frac{(p^*)'}{q'}} + M \|b\|^{(p^*)'}_{L^{\tau_2}(B_R(x_0))} \left|A^\alpha_{k,t}\right|^{1-\frac{(p^*)'}{\tau_2}} .
\end{aligned} \tag{2.9}$$

We keep in mind that $\eta = 1$ on $B_s(x_0)$ and $|A^\alpha_{k,t}| \le |B_R(x_0)| \le 1$: inequality (2.1) follows by taking $c = \frac{1+M2^{1+p^*}}{\nu}$. □

STEP 2: Decay of the "excess" on superlevel sets

In this step we consider a *scalar* Sobolev function $v : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, $n \ge 2$.

Let us assume that Ω is an open set in $\mathbb{R}^n$ and v is a *scalar* function $v \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R})$, $p \ge 1$. Fix $B_{R_0}(x_0) \Subset \Omega$, with $R_0 < 1$ small enough so that

$$|B_{R_0}(x_0)| < 1 \quad \text{and} \quad \int_{B_{R_0}} |v|^{p^*}\, dx < 1. \tag{2.10}$$

Here $p^* = \frac{np}{n-p}$, since $p < n$.

For every $R \in (0, R_0]$ we define the decreasing sequences

$$\rho_h := \frac{R}{2} + \frac{R}{2^{h+1}} = \frac{R}{2}\left(1 + \frac{1}{2^h}\right), \qquad \bar\rho_h := \frac{\rho_h + \rho_{h+1}}{2} = \frac{R}{2}\left(1 + \frac{3}{4\cdot 2^h}\right).$$

Fixed a positive constant $d \ge 1$, define the increasing sequence of positive real numbers

$$k_h := d\left(1 - \frac{1}{2^{h+1}}\right), \quad h \in \mathbb{N}.$$

Moreover, define the sequence $(J_{v,h})$,

$$J_{v,h} := \int_{A_{k_h,\rho_h}} (v - k_h)^{p^*}\, dx,$$

where $A_{k,\rho} = \{v > k\} \cap B_\rho$. The following result holds (see [6,Proposition 2.4], [15, 44]).

Proposition 2.2 *Let $v \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R})$, $p \ge 1$. Fix $B_{R_0}(x_0) \Subset \Omega$, with $R_0 < 1$ small enough such that (2.10) holds. If there exists $0 \le \vartheta \le 1$ and $c_0 > 0$ such that for every $0 < s < t \le R_0$ and for every $k \in \mathbb{R}$*

$$\int_{A_{k,s}} |Dv|^p\, dx \le c_0 \left\{ \int_{A_{k,t}} \left(\frac{v-k}{t-s}\right)^{p^*} dx + |A_{k,t}|^\vartheta \right\}, \tag{2.11}$$

then, for every $R \in (0, R_0]$,

$$J_{v,h+1} \leq c(\vartheta, R) \left(2^{\frac{p^* p^*}{p}} \right)^h J_{v,h}^{\vartheta \frac{p^*}{p}},$$

with the positive constant c *independent of* h.

STEP 3: Iteration and proof of Theorem 1.1

We now resume the proof of Theorem 1.1.

We need the following classical result, see e.g. [17].

Lemma 2.3 *Let* $\gamma > 0$ *and let* (J_h) *be a sequence of real positive numbers, such that*

$$J_{h+1} \leq A \, \lambda^h J_h^{1+\gamma} \qquad \forall h \in \mathbb{N} \cup \{0\}, \tag{2.12}$$

with $A > 0$ *and* $\lambda > 1$. *If* $J_0 \leq A^{-\frac{1}{\gamma}} \lambda^{-\frac{1}{\gamma^2}}$, *then* $J_h \leq \lambda^{-\frac{h}{\gamma}} J_0$ *and* $\lim_{h \to \infty} J_h = 0$.

Fix $B_{R_0}(x_0) \Subset \Omega$, with $R_0 < 1$ small enough such that $|B_{R_0}(x_0)| < 1$ and $\int_{B_{R_0}} |u|^{p^*} \, dx < 1$. By Proposition 2.1 we have that u^α satisfies (2.1); i.e. for every $0 < s < t \leq R_0$ and every $k \in \mathbb{R}$,

$$\begin{aligned} \int_{A^\alpha_{k,s}} |Du^\alpha|^p \, dx \leq c \int_{A^\alpha_{k,t}} \left(\frac{u^\alpha - k}{t - s} \right)^{p^*} dx \\ + c \left\{ \|Du\|^{(q-1)(p^*)'}_{L^q(B_{R_0}(x_0))} + \|a\|_{L^{\tau_1}(B_{R_0}(x_0))} \right. \\ \left. + \|b\|^{(p^*)'}_{L^{\tau_2}(B_{R_0}(x_0))} \right\} |A^\alpha_{k,t}|^\vartheta, \end{aligned}$$

where

$$\vartheta := \min \left\{ 1 - \frac{(p^*)'}{q'}, 1 - \frac{1}{\tau_1}, 1 - \frac{(p^*)'}{\tau_2} \right\}$$

and $c = \frac{1 + M 2^{1+p^*}}{\nu}$.

Therefore the scalar function u^α satisfies (2.11) of Proposition 2.2 with constant c_0 depending on

$$\|Du\|^{(q-1)(p^*)'}_{L^q(B_{R_0}(x_0))}, \quad \|a\|_{L^{\tau_1}(B_{R_0}(x_0))} \quad \text{and} \quad \|b\|^{(p^*)'}_{L^{\tau_2}(B_{R_0}(x_0))}.$$

Note that these integrals are finite.

As above, let us define

$$k_h := d \left(1 - \frac{1}{2^{h+1}} \right), \quad h \in \mathbb{N}$$

with $d \geq 1$ (d will be fixed later) and, for every $R \in (0, R_0]$, define

$$\rho_h := \frac{R}{2} + \frac{R}{2^{h+1}} = \frac{R}{2} \left(1 + \frac{1}{2^h} \right), \qquad \bar{\rho}_h := \frac{\rho_h + \rho_{h+1}}{2} = \frac{R}{2} \left(1 + \frac{3}{4 \cdot 2^h} \right)$$

and

$$J_{u^\alpha,h} := \int_{A^\alpha_{k_h,\rho_h}} (u^\alpha - k_h)^{p^*} \, dx.$$

Proposition 2.2, applied to u^α, gives

$$J_{u^\alpha,h+1} \le c(\vartheta, R) \left(2^{\frac{p^* p^*}{p}}\right)^h J_{u^\alpha,h}^{\vartheta \frac{p^*}{p}}, \tag{2.13}$$

with the positive constant c independent of h and, by (1.5), with the exponent $\vartheta \frac{p^*}{p}$ greater than 1. Indeed, we notice that $q < p^* \frac{n}{p(n+1)}$ is equivalent to $\frac{q}{q-1} > \frac{p^*}{p^*-1} \frac{n}{p}$; therefore (1.5) implies

$$\frac{p}{p^*} < \min \left\{ 1 - \frac{(p^*)'}{q'}, 1 - \frac{1}{\tau_1}, 1 - \frac{(p^*)'}{\tau_2} \right\} = \vartheta,$$

so we get $1 < \vartheta \frac{p^*}{p}$.

Moreover, since

$$J_{u^\alpha,0} = \int_{A^\alpha_{\frac{d}{2},R}} \left(u^\alpha - \frac{d}{2}\right)^{p^*} dx \to 0 \quad \text{as } d \to +\infty,$$

we can choose $d \ge 1$ large enough, so that

$$J_{u^\alpha,0} < c(\vartheta, R)^{-\frac{1}{\vartheta \frac{p^*}{p} - 1}} \left(2^{\frac{p^* p^*}{p}}\right)^{-\frac{1}{(\vartheta \frac{p^*}{p} - 1)^2}}.$$

Therefore, by Lemma 2.3, $\lim_{h \to +\infty} J_{u^\alpha,h} = 0$. Thus, $u^\alpha \le d$ a.e. in $B_{\frac{R}{2}}(x_0)$. We have so proved that u^α is locally bounded from above.

To prove that u^α is locally bounded from below, we can observe that $\tilde{u} = -u$ is a weak solution for

$$\sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\tilde{A}_i^\alpha(x, D\tilde{u}(x)) \right) = 0, \quad 1 \le \alpha \le m,$$

where $\tilde{A}(x, z) = -A(x, -z)$. It is easy to check that $\tilde{A}$ satisfies assumptions analogous to (1.2) and (1.3). Therefore, by what previously proved, there exists d' such that $\tilde{u}^\alpha = -u^\alpha \le d'$ a.e. in $B_{\frac{R}{2}}(x_0)$. We have so proved that $u^\alpha \in L^\infty(B_{\frac{R}{2}}(x_0))$. Due to the arbitrariness of x_0 and R_0, we get $u^\alpha \in L^\infty_{\mathrm{loc}}(\Omega)$.

3 Examples

Example 1 We consider example (1.17) that we rewrite for the convenience of the reader:

$$A_i^\alpha(z) = p|z|^{p-2} z_i^\alpha + h'(\det z)(\mathrm{Cof}\, z)_i^\alpha, \tag{3.1}$$

where $m = n$, $z \in \mathbb{R}^{n \times n}$, $\det z = \sum_{i=1}^n z_i^\alpha (\mathrm{Cof}\, z)_i^\alpha$; moreover, h is convex, bounded from below and C^1. Exploiting the convexity of h, we get

$$h(0) \ge h(t) + h'(t)(0 - t), \tag{3.2}$$

so that

$$h'(t)t \ge h(t) - h(0) \ge \inf_{\mathbb{R}} h - h(0). \tag{3.3}$$

Let us compute the indicator function for this choice of A: we get

$$
\begin{aligned}
I_A(x,u,z) &= \sum_{i,\alpha,\beta} A_i^\alpha(z) z_i^\beta \frac{u^\alpha u^\beta}{|u|^2} = \sum_{i,\alpha,\beta} p|z|^{p-2} z_i^\alpha z_i^\beta \frac{u^\alpha u^\beta}{|u|^2} \\
&\quad + \sum_{i,\alpha,\beta} h'(\det z)(\operatorname{Cof} z)_i^\alpha z_i^\beta \frac{u^\alpha u^\beta}{|u|^2} \\
&= p|z|^{p-2} \sum_i \sum_\alpha z_i^\alpha \frac{u^\alpha}{|u|} \sum_\beta z_i^\beta \frac{u^\beta}{|u|} + h'(\det z) \sum_{\alpha,\beta} \frac{u^\alpha u^\beta}{|u|^2} \sum_i (\operatorname{Cof} z)_i^\alpha z_i^\beta \\
&= p|z|^{p-2} \sum_i \left(\sum_\alpha z_i^\alpha \frac{u^\alpha}{|u|} \right)^2 + h'(\det z) \sum_\alpha \frac{u^\alpha u^\alpha}{|u|^2} \sum_i (\operatorname{Cof} z)_i^\alpha z_i^\alpha \\
&= p|z|^{p-2} \sum_i \left(\sum_\alpha z_i^\alpha \frac{u^\alpha}{|u|} \right)^2 + h'(\det z) \det z \geq \inf_{\mathbb{R}} h - h(0),
\end{aligned}
$$

where we used the property $\sum_i (\operatorname{Cof} z)_i^\alpha z_i^\beta = 0$ if $\beta \neq \alpha$. When $h(0) = \inf_{\mathbb{R}} h$, then strong Meier's condition (1.11) is satisfied; if $h(0) > \inf_{\mathbb{R}} h$, then weak Meier's condition (1.19) is verified with $\lambda = 1$, $d(x) = 0$ and $g(x) = h(0) - \inf_{\mathbb{R}} h$. Now, let us verify componentwise coercivity (1.26). We have

$$
\begin{aligned}
\sum_i A_i^\alpha(z) z_i^\alpha &= \sum_i p|z|^{p-2} z_i^\alpha z_i^\alpha + \sum_i h'(\det z)(\operatorname{Cof} z)_i^\alpha z_i^\alpha \\
&= p|z|^{p-2}|z^\alpha|^2 + h'(\det z)\det z \geq p|z^\alpha|^p + \inf_{\mathbb{R}} h - h(0),
\end{aligned}
$$

provided $p \geq 2$; then (1.26) is verified with $\nu = p$, $a(x) = h(0) - \inf_{\mathbb{R}} h$ and $b(x) = 0$.

Example 2 We consider example (1.20) that we rewrite for the convenience of the reader:

$$
A_i^\alpha(x,z) = \sigma^\alpha(x) z_i^\alpha, \tag{3.4}
$$

where $m = 2$, $\sigma^1(x) = 18 + 2\sin(|x|^2)$ and $\sigma^2(x) = 2 + \sin(|x|^2)$. Since $\sigma^\alpha(x) \geq 1$, it is easy to check (1.26):

$$
\sum_i A_i^\alpha(x,z) z_i^\alpha = \sum_i \sigma^\alpha(x) z_i^\alpha z_i^\alpha = \sigma^\alpha(x)|z^\alpha|^2 \geq |z^\alpha|^2;
$$

so, (1.26) is verified with $p = 2$, $\nu = 1$, $a(x) = 0$ and $b(x) = 0$. We are going to show that (1.19) is not fulfilled. Indeed, we take $u^1 = u^2 = s > 0$ with s large enough (see (3.5) later); moreover, we take $z_i^\alpha = 0$ if $i \geq 2$, $z_1^1 = -s^2$, $z_1^2 = 2s^2$. Then $|z|^2 = 5s^4$, $|u|^2 = 2s^2$, $\frac{u^\alpha u^\beta}{|u|^2} = \frac{1}{2}$ and

$$
\begin{aligned}
\sum_{i,\alpha,\beta} A_i^\alpha(x,z) z_i^\beta \frac{u^\alpha u^\beta}{|u|^2} &= \frac{1}{2} \sum_{\alpha,\beta} A_1^\alpha(x,z) z_1^\beta = \frac{1}{2} \sum_\alpha A_1^\alpha(x,z) \sum_\beta z_1^\beta \\
&= \frac{1}{2}(\sigma^1(x) z_1^1 + \sigma^2(x) z_1^2)(z_1^1 + z_1^2) = \frac{1}{2}(-\sigma^1(x) + 2\sigma^2(x))s^4 = -7s^4 \underbrace{<}_{0<\delta<1} -\delta 7 s^4
\end{aligned}
$$

$$= -\delta s^4 \{5+1+1\} \underbrace{\le}_{(*)} -\delta s^4 \left\{5 + \left(\frac{1}{\delta}\right)^{\lambda+1} \frac{2d(x)}{s^2} + \left(\frac{1}{\delta}\right)^{\lambda+1} \frac{g(x)}{s^4}\right\}$$
$$= -\left\{\delta|z|^2 + \left(\frac{1}{\delta}\right)^{\lambda} \left[d(x)|u|^2 + g(x)\right]\right\},$$

where (*) is guaranteed by the choice of s as follows

$$s = \max\left\{L; \left[\left(\frac{1}{\delta}\right)^{\lambda+1} 2d(x)\right]^{1/2}; \left[\left(\frac{1}{\delta}\right)^{\lambda+1} g(x)\right]^{1/4}\right\}. \tag{3.5}$$

Example 3 Let us consider example (1.23) that we rewrite for the convenience of the reader:

$$A_i^\alpha(z) = 2z_i^\alpha + h'(z_1^1 z_1^2) z_1^{\hat\alpha} \delta_{i1}, \tag{3.6}$$

where $m = 2$, $\hat\alpha = 2$ if $\alpha = 1$ and $\hat\alpha = 1$ if $\alpha = 2$; moreover, $\delta_{i1} = 1$ if $i = 1$ and $\delta_{i1} = 0$ otherwise. Here, h is convex, C^1, bounded from below and $h'(0) \le -8$. For instance,

$$h(t) = 16\sqrt{1 + (t-1)^2}. \tag{3.7}$$

Let us first check (1.26):

$$\sum_i A_i^\alpha(z) z_i^\alpha = \sum_i 2z_i^\alpha z_i^\alpha + \sum_i h'(z_1^1 z_1^2) z_1^{\hat\alpha} \delta_{i1} z_i^\alpha$$
$$= 2|z^\alpha|^2 + h'(z_1^1 z_1^2) z_1^{\hat\alpha} z_1^\alpha = 2|z^\alpha|^2 + h'(z_1^1 z_1^2) z_1^1 z_1^2 \ge 2|z^\alpha|^2 + \inf_{\mathbb{R}} h - h(0),$$

since $z_1^{\hat\alpha} z_1^\alpha = z_1^1 z_1^2$; then (1.26) is verified with $\nu = 2$, $p = 2$, $a(x) = h(0) - \inf_{\mathbb{R}} h$ and $b(x) = 0$. We are going to show that (1.19) is not fulfilled. Indeed, we take $u^1 = u^2 = s > 0$ with s large enough (see (3.5) as before); moreover, we take $z_1^2 = s^2$ and $z_i^\alpha = 0$ otherwise. Then $|z|^2 = s^4$, $|u|^2 = 2s^2$, $\frac{u^\alpha u^\beta}{|u|^2} = \frac{1}{2}$ and

$$\sum_{i,\alpha,\beta} A_i^\alpha(z) z_i^\beta \frac{u^\alpha u^\beta}{|u|^2} = \frac{1}{2}\sum_{\alpha,\beta} A_1^\alpha(z) z_1^\beta = \frac{1}{2}\sum_\alpha A_1^\alpha(z) \sum_\beta z_1^\beta$$
$$= \frac{1}{2}(2z_1^2 + h'(0) z_1^2)(z_1^2) = \frac{1}{2}(2 + h'(0)) s^4 \underbrace{\le}_{h'(0)\le -8} \frac{1}{2}(2-8)s^4 = -3s^4 \underbrace{<}_{0<\delta<1} -\delta 3 s^4$$
$$= -\delta s^4 \{1+1+1\} \underbrace{\le}_{(*)} -\delta s^4 \left\{1 + \left(\frac{1}{\delta}\right)^{\lambda+1} \frac{2d(x)}{s^2} + \left(\frac{1}{\delta}\right)^{\lambda+1} \frac{g(x)}{s^4}\right\}$$
$$= -\left\{\delta|z|^2 + \left(\frac{1}{\delta}\right)^{\lambda} \left[d(x)|u|^2 + g(x)\right]\right\},$$

where (*) is guaranteed by the choice of s (3.5) as before. In order to show that we can use Theorem 1.1, we use formula (3.7) and we select $n = 3$. Then $|h'(t)| \le 16$ and we get

$$\sum_{i=1}^3 \left|A_i^\alpha(z)\right| \le 54|z|, \tag{3.8}$$

so (1.3) is satisfied with $q = 2$, $M = 54$ and $b(x) = 0$, $\tau_2 = +\infty$. Note that previous calculations checked the validity of (1.2) with $p = 2$, $\nu = 2$, $a(x) = h(0) - \inf_{\mathbb{R}} h = 16(\sqrt{2} - 1)$ and $\tau_1 = +\infty$. Since we selected $n = 3$, $q = p = 2$, then $\frac{3}{2} = p_0(n) < 2 = q = p < p^* \frac{n}{p(n+1)}$; this implies that (1.5) is satisfied and we can use our Theorem 1.1 and we get the following

Corollary 3.1 *If Ω is a bounded open subset of $\mathbb{R}^3$, then all solutions $u \in W^{1,2}_{loc}(\Omega; \mathbb{R}^2)$ of system (1.1), with $n = 3$, $m = 2$, (3.6) and (3.7), are locally bounded in Ω.*

Example 4 Let us consider example (1.24) that we rewrite for the convenience of the reader:

$$A_i^\alpha(x, z) = \sigma^\alpha(x) p |z|^{p-2} z_i^\alpha + h'(z_1^1 z_1^2) z_1^{\hat\alpha} \delta_{i1}, \tag{3.9}$$

where $m = 2$, $\sigma^1(x) = 48 + 3\sin(|x|^2)$ and $\sigma^2(x) = 2 + \sin(|x|^2)$, $\hat\alpha = 2$ if $\alpha = 1$ and $\hat\alpha = 1$ if $\alpha = 2$; moreover, $\delta_{i1} = 1$ if $i = 1$ and $\delta_{i1} = 0$ otherwise. Here, $2 \le p$, h is convex, C^1, bounded from below and $h'(0) \le 0$. For instance,

$$h(t) = \left(1 + t^2\right)^{p/4}. \tag{3.10}$$

Let us first check (1.26); since $\sigma^\alpha(x) \ge 1$ and $z_1^{\hat\alpha} z_1^\alpha = z_1^1 z_1^2$,

$$\begin{aligned}
\sum_i A_i^\alpha(x, z) z_i^\alpha &= \sum_i \sigma^\alpha(x) p |z|^{p-2} z_i^\alpha z_i^\alpha + \sum_i h'(z_1^1 z_1^2) z_1^{\hat\alpha} \delta_{i1} z_i^\alpha \\
&= \sigma^\alpha(x) p |z|^{p-2} |z^\alpha|^2 + h'(z_1^1 z_1^2) z_1^{\hat\alpha} z_1^\alpha = \sigma^\alpha(x) p |z|^{p-2} |z^\alpha|^2 + h'(z_1^1 z_1^2) z_1^1 z_1^2 \\
&\ge p |z^\alpha|^p + \inf_{\mathbb{R}} h - h(0),
\end{aligned}$$

then (1.26) is verified with $\nu = p$, $a(x) = h(0) - \inf_{\mathbb{R}} h$ and $b(x) = 0$. We are going to show that (1.19) is not fulfilled. Indeed, we take $u^1 = u^2 = s > 0$ with s large enough (see (3.11) later); moreover, we take $z_1^1 = s^2$, $z_1^2 = 0$, $z_2^1 = -2s^2$, $z_2^2 = 3s^2$ and $z_i^\alpha = 0$ otherwise. Then $|z|^2 = 14 s^4$, $|u|^2 = 2s^2$, $\frac{u^\alpha u^\beta}{|u|^2} = \frac{1}{2}$ and

$$\begin{aligned}
&\sum_{i,\alpha,\beta} A_i^\alpha(x, z) z_i^\beta \frac{u^\alpha u^\beta}{|u|^2} = \frac{1}{2} \sum_{i,\alpha,\beta} A_i^\alpha(x, z) z_i^\beta = \frac{1}{2} \sum_i \sum_\alpha A_i^\alpha(x, z) \sum_\beta z_i^\beta \\
&= \frac{1}{2} \sum_i \sum_\alpha \sigma^\alpha(x) p |z|^{p-2} z_i^\alpha \sum_\beta z_i^\beta + \frac{1}{2} \sum_i \sum_\alpha h'(z_1^1 z_1^2) z_1^{\hat\alpha} \delta_{i1} \sum_\beta z_i^\beta \\
&= \frac{1}{2} \sum_i \sum_\alpha \sigma^\alpha(x) p |z|^{p-2} z_i^\alpha \sum_\beta z_i^\beta + \frac{1}{2} \sum_\alpha h'(z_1^1 z_1^2) z_1^{\hat\alpha} \sum_\beta z_1^\beta \\
&= \frac{p}{2} |z|^{p-2} \sum_i \left[\sigma^1(x) z_i^1 + \sigma^2(x) z_i^2\right] \left[z_i^1 + z_i^2\right] + \frac{1}{2} h'(z_1^1 z_1^2) \left[z_1^2 + z_1^1\right] \left[z_1^1 + z_1^2\right] \\
&= \frac{p}{2} |z|^{p-2} \left\{\left[\sigma^1(x) z_1^1 + \sigma^2(x) z_1^2\right] \left[z_1^1 + z_1^2\right] + \left[\sigma^1(x) z_2^1 + \sigma^2(x) z_2^2\right] \left[z_2^1 + z_2^2\right]\right\} \\
&\quad + \frac{1}{2} h'(z_1^1 z_1^2) \left[z_1^1 + z_1^2\right]^2 \\
&= \frac{p}{2} |z|^{p-2} \left\{\sigma^1(x) s^4 + \left[-2\sigma^1(x) + 3\sigma^2(x)\right] s^4\right\} + \frac{1}{2} h'(0) s^4 \\
&\underbrace{\le}_{h'(0) \le 0} \frac{p}{2} |z|^{p-2} s^4 \left\{-\sigma^1(x) + 3\sigma^2(x)\right\} = \frac{p}{28} |z|^p \left\{-\sigma^1(x) + 3\sigma^2(x)\right\} = -3 \frac{p}{2} |z|^p
\end{aligned}$$

$$\underbrace{\leq}_{2\leq p} -3|z|^p \underbrace{<}_{0<\delta<1} -\delta|z|^p 3 = -\delta|z|^p \{1+1+1\}$$

$$\underbrace{\leq}_{(**)} -\delta|z|^p \left\{1 + \left(\frac{1}{\delta}\right)^{\lambda+1} d(x)\frac{1}{(\sqrt{7}s)^p} + \left(\frac{1}{\delta}\right)^{\lambda+1} \frac{g(x)}{(\sqrt{14}s^2)^p}\right\}$$

$$= -\left\{\delta|z|^p + \left(\frac{1}{\delta}\right)^{\lambda} \left[d(x)|u|^p + g(x)\right]\right\},$$

where (**) is guaranteed by the choice (3.11) of s as follows.

$$s = \max\left\{L; \left[\left(\frac{1}{\delta}\right)^{\lambda+1} \frac{d(x)}{(\sqrt{7})^p}\right]^{1/p}; \left[\left(\frac{1}{\delta}\right)^{\lambda+1} \frac{g(x)}{(\sqrt{14})^p}\right]^{1/(2p)}\right\}. \tag{3.11}$$

In order to show that we can use Theorem 1.1, we use formula (3.10), we select $n = 3$ and we require $p < 3 = n$. Then $|h'(t)| \leq \frac{p}{2}\left(1+t^2\right)^{\frac{p-2}{4}}$ and we get

$$\sum_{i=1}^{3} \left|A_i^{\alpha}(x,z)\right| \leq 3\sigma^{\alpha}(x)p|z|^{p-1} + 3\frac{p}{2}\left(1+|z|^4\right)^{\frac{p-2}{4}}|z| \leq 153p|z|^{p-1}$$
$$+3\frac{p}{2}\left(1+|z|^4\right)^{\frac{p-1}{4}}$$
$$\leq 153p|z|^{p-1} + 3\frac{p}{2}2^{\frac{p-1}{4}}\left(1+|z|^{p-1}\right) \leq p(153+2^{1+\frac{p-1}{4}})(|z|^{p-1}+1),$$

so (1.3) is satisfied with $q = p$, $M = p(153+2^{1+\frac{p-1}{4}})$ and $b(x) = 1$, $\tau_2 = +\infty$. Note that previous calculations checked the validity of (1.2) with $\nu = p$, $a(x) = h(0) - \inf_{\mathbb{R}} h = 0$ and $\tau_1 = +\infty$. Since we selected $n = 3$, $q = p \in [2,3)$, then $\frac{3}{2} = p_0(n) < 2 \leq q = p < p^* \frac{n}{p(n+1)}$; this implies that (1.5) is satisfied and we can use our Theorem 1.1 and we get the following

Corollary 3.2 *If Ω is a bounded open subset of $\mathbb{R}^3$, then all solutions $u \in W^{1,p}_{loc}(\Omega;\mathbb{R}^2)$ of system (1.1), with $2 \leq p < 3 = n$, $m = 2$, (3.9) and (3.10), are locally bounded in Ω.*

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