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The undersigned

SURNAME	WEN
NAME	Yuanji
PhD Registration Number	1287648

Thesis title:

Essays on Information, Learning, and Trading

PhD in	Finance
Cycle	XXIII
Candidate's tutor	Barbara Rindi
Year of discussion	2013

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SURNAME	Wen		Yuanji

ESSAYS ON
INFORMATION, LEARNING, AND TRADING

A dissertation presented

By

Yuanji Wen

In partial fulfillment of the requirements for the Degree of Doctor in Finance

Bocconi University

October 2012

Dedicated to my parents

Yulin Wen and Qunying Xie

Acknowledgments

My plan to pursue PhD took shape while I interned at a securities company during the course of my masters degree at Nankai University. The internship had ample opportunity for researching trading activities. I wrote a paper that was published in a CSSCI journal. This encouraged me to undertake research as a career. Under the encouragement of my friend Pinghan Liang and his generous help, I attended the PhD program at Bocconi University.

I want to take this unique opportunity to thank several people who were very important during my PhD study. I would like to acknowledge my advisors Barbara Rindi and Vincent Crawford for their patience, guidance, and advice. Barbara exposed me to the frontiers of market microstructure (her book is a very good text for the advanced topics in market microstructure) and aroused my interest to pursue further in this field. When I got an opportunity to visit Vincent Crawford at University of Oxford, Barbara fully supported me in my endeavor. At Oxford, Vince helped polish my raw ideas patiently. Under his guidance, I developed an experimental design and started writing the paper which is the second chapter of this thesis. Though my stay at Oxford was short but it was extremely enriching experience. It had great influence on my future and it was the turning point for me to grow up as a researcher. When I started preparing my job market paper, Barbara kept reading drafts and provided useful comments and constant encouragement. Without her gentle "nudges", I would have taken much longer to finish the thesis.

I am also thankful to my co-author Sabrina Buti. She shared the first sample code written in Mathematica from where I programmed my own codes; she guided me in solving the model when we worked on the co-authored paper and later I applied the same techniques in my job market paper.

I would like to express my gratitude to my tutor Francesco Corielli, Carlo Favero who supported me at my late stage, and Paolo Colla whom I worked for and discussed related research with. I would like to thank also all those who helped me with my experimental work. I am grateful to Jason Shachat who hosted me at Xiamen University to run the experiments and provided feedback on my research ideas. I have fond memories of the short visit at Xiamen University;

it was very precious and joyful. Prior to visitng Xiamen, Michele Belot and Wojteck Prezepiorka at Nuffield (Oxford) helped me to get hands on the experiments and discussed some of the experimental details. After Xiamen, at Bocconi I closely followed the group of experimental economists and I thank Pierpaolo Battigalli, Joshua Miller, and Debrah Meloso, who discussed my work.

I thank my friends at Bocconi Allya Koesoema (now at UNSW, Sydney), Kun Fu (now at Imperial College London), Haoxi Yang, Domenico Tarzia, Antonio Gargano, Roberto Steri, and Carlo Chiarella, with whom I shared countless meals and cups of coffee. I was fortunate to share my office with Gerry Abdesaken who helped me with English and patiently listened to my daily complaints. A group of friends Liz Liu, Jing Sun, Daniel Li, Dawei Fang, Chole Zeng, and Li Lin enriched my life when I was at Oxford; Zhenxuan Zhang, Lijia Tan, and Kuangli Xie hung out with me and are a part of sweet memories from Xiamen. In my late years of PhD life, I have been sharing my apartment with Martina Bigoni, who not only took care of most of housework but also supported me during my hard times. Personally, I am grateful to Anmol Ratan who taught me several things outside the academia and provided trust and support. He read my drafts and corrected my errors. I am happy and grateful to have him standing beside me during my late stage of the PhD life which was actually the tough part.

Finally, I will never be able to reciprocate the efforts of my parents; they provided unconditional love and support. My acknowledgements to them appear in Chinese language below. They encouraged me to pursue what I want, supported me whenever I felt lost, puzzled, or wronged, and shared with me any happiness and success. It was their love that warmed me when I felt lonely, and their words that lightened up my way and encouraged me to carry on. They are like ever-shining stars in the sky above me. I feel sorry that I could not spend enough time with them during the course of my PhD program; I am ever grateful for many things that I received from them. I am, therefore, dedicating this thesis to them!

致我的父亲母亲：感谢你们一直鼓励我追寻自己所要，感谢你们在我感到困惑、迷失抑或委屈的时候支持我，感谢你们也分享我的快乐与成功的喜悦。正是你们的爱让我在孤独的时候感到温暖，正是你们的话语点亮我前方的旅程，激励前进。

你们永远是我天空中的星星，照耀着我。这几年，我一直没有在你们身边陪伴，感到抱歉， 谨以此博士论文报答多年养育之恩！

Contents

Preface

This thesis provides theoretical analysis on trading behaviors in widely-run market structure, i.e., limit order markets, and an experimental investigation on learning behaviors. The main focus of this thesis directly addresses issues in market microstructure; the other chapters use tools from game theory and experimental economics to examine issues that are pertinent to market microstructure.

The first chapter focuses on one parameter of the market structure in financial trading and discusses a hot-debated issue: sub-penny trading. It has been presented as a joint work with Professors Barbara Rindi, Sabrina Buti, and Ingrid Werner. The previous version was titled as "Tick size regulation, intermarket competition, and sub-penny trading" and presented at various academic conferences and seminar series, including the American Finance Association Annual Meeting, Financial Management Association Annual Meeting, the 7th Annual Central Bank Workshop on the Microstructure of Financial Markets, and European

Financial Management Association Annual Meeting.

The second chapter is an experimental design which discriminates among different learning models in game theory. It studies how informed players learn from the previous experience in the laboratory. It was presented at Economic Science Association International Meeting, at NYU (June 2012).

Embracing the theme of this thesis, the third chapter builds a game theoretical model to analyze the informed and the uninformed traders' order submission strategies in limit order markets. It shares the features of the model in the first chapter while assuming that people learn and using a concept of equilibrium that can be justified by the findings in the second chapter.

Introduction

The research on market microstructure is devoted to theoretical, empirical, and experimental research on the economics of securities markets, including the role of information in the price discovery process, the definition, measurement, control and determinants of liquidity and transaction costs, and their implications for the efficiency, welfare, and regulation of alternative trading mechanisms and market structures. Information plays an important role in price discovery process, thus microstructure models extensively take into account whether and how information affects the market outcome. This is also the theme of this thesis.

Chapter 1 extends Parlour (1998)'s model and discusses how the minimum price variation, i.e., tick size, of the limit order book affects the stock trading. It shows that the effect of a reduction in the tick size is detrimental on spread, depth, and traders' welfare for illiquid stocks, whereas it benefits liquid stocks. Then a dual-market model is built to investigate the issue of sub-penny trading that is discussed in the SEC concept release on Equity Market Structure (2010). In this dual-market model, a PLB competes with an Internalization Pool (IP) which is a dark pool that works like a limit order book characterized by a smaller price grid, and in which only broker-dealers can post limit orders. When broker-dealers are able to use the IP to provide price improvements by a fraction of the tick size, the quality of the PLB deteriorates for illiquid, low priced stocks; for liquid stocks, however, the introduction of an IP promotes competition and improves both spread and depth. Consequently, total welfare always increases as all traders can demand liquidity in the IP. The model also predicts that broker-dealers would use the IP more intensively for low priced and liquid stocks. This chapter aims at providing regulators (or trading venues) with a set of guidelines for determining the optimal tick size and also informs the empirical researcher of further accessing the effects of tick size reductions. To the best of my knowledge, it is the first model that allows researchers to investigate the tick size rule within a framework that takes into account both the value and the liquidity characteristics of the stock. It is also the first one that discusses the effects of sub-penny trading by modeling two markets of different tick sizes.

Compared to Chapter 1, Chapter 2 adopts a different approach and switches the focus from trading to learning and private information. It investigates the following research question: how do informed players learn from the previous experience. This chapter examines the two-stage asymmetric information game studied by Feltovich (1997, 1999, and 2000) and compares it to a modified game in the laboratory. By tracking subjects' information acquisition, I find evidence in support of a belief-based learning model. The maximum-likelihood estimations of subjects' learning behaviors using the EWA model (Camerer and Ho 1999) show that informed and uninformed players learn at different speeds, and the informed group is more active in learning than its counterpart. Besides, the estimations of EWA suggest that, while the reduced belief-based model characterizes players' behaviors better than the reduced reinforcement model, it slightly underperforms in prediction. Finally, the cluster and individual estimations combined with information acquisition data show that the majority of seemingly fictitious plays coincide with the dominance of a belief-based updating pattern defined by information acquisition measures. Tracking subject's information acquisition can be a complementary tool to study learning. The novel design helps comparing different learning models in game theory and confirms its methodological value derived from the other branch of game theory: strategic thinking.

Chapter 3 continues with the theoretical shell in the first chapter of this thesis and embeds asymmetric information. It builds a two-period model with uninformed traders' learning from the limit order book. The Bayesian learning is slightly different from the concept of "learning" in the second chapter, but the game in the laboratory can be considered as a distilled version of the trading game in this chapter. I explore the model of limit order book with information asymmetry to explain the experimental findings by Bloomfield, O'Hara, and Saar (2005). By endogenizing the direction of trade and the choice of order type, I find that informed traders more often exploit market orders than limit orders and they are more aggressive in the presence of a larger volatility; in general, conditional on the favorable realization of liquidation value, informed buyers and sellers are more aggressive than their uninformed counterparts respectively. I also show that both the volatility and the proportion of informed traders frustrate uninformed traders' willingness to participate in the market and their willingness to provide liquidity if they desire. Total welfare of market participants is improved by informed trading but the informed traders gain at the cost of their uninformed counterpart's welfare.

Chapter 1

Tick Size Regulation and Sub-Penny Trading

1.1 Introduction

The tick size, the minimum size of an asset price variation, is one of the most relevant factors affecting the level of liquidity of financial securities traded on public limit order books (PLBs). As a consequence, it has been at the top of the regulatory agenda over the past decade (e.g., SEC 2010 and SEC 2012). Decimalization, i.e., the transition to trading and quoting securities in one penny increments, started in 2001 in the US and in 2004 in Europe and had profound consequences both for the level of liquidity, and for the business model of those institutions which support liquidity. The anticipated and unanticipated effects of tick-size variations on the quality of the markets and on the welfare of market participants have been debated extensively.[1] The rationale for tick-size reductions was to encourage trading activity by reducing transaction costs; the unanticipated consequence was that the increased competition for the provision of liquidity reduced the incentive for market participants to supply liquidity.

The comparative assessment of markets with different tick sizes is even more complex today, because the documented outcome of changes in the tick size must be appraised within a global framework in which lit markets compete with dark markets for liquidity. In the latter, more that 17% of consolidated equity volumes (SEC, 2010) are executed through broker-dealers who have access to venues that allow them to undercut the existing liquidity on regulated markets by placing quotes in sub-penny increments, i.e., at fractions of the minimum tick size.

Sub-penny trading is controversial, and the model we present in this paper aims at providing regulators and trading venues with a set of guidelines for selecting the optimal tick size. Our model also informs empirical research on how to evaluate the effects of tick size changes. We consider an environment where a double-auction trading platform competes for the provision of liquidity with a dark trading platform and in the latter market liquidity providers can trade in sub-pennies.

[1] See Section 1.2 and 1.3.

A relevant trade-off to decide the optimal tick size is between undercutting and liquidity provision. On the one hand, the smaller the tick size, the cheaper is the undercutting for all market participants, thus a smaller tick size reduces the profits made by broker-dealers from submitting sub-penny passive limit orders. To some extent, a reduction in the tick size is effectively a way to alleviate the effects of sub-penny trading. On the other hand, a smaller tick size discourages liquidity providers from posting limit orders at the top of the book due to the increased danger of being undercut. Therefore, the critical regulatory issue is to adjust the tick size for each stock to optimally balance the effect it has on sub-penny trading with the effect it has on liquidity provision, which in turn crucially depends on both the initial level of liquidity and the tick-to-price ratio (Goldstein and Kavajecz, 2000).

Starting with a one-market model, we show that the effects of having a different tick size depend both on the liquidity and on the price of the stock. For liquid stocks a smaller tick size increases competition among liquidity suppliers and hence improves both market quality and traders' welfare. When the top of the book is very deep, however, the liquidity pressure at the best bid-offer is so intense that inside depth decreases with tick size. For illiquid stocks, instead, a smaller tick size discourages liquidity provision and worsens both market quality and traders' welfare. The effects of introducing a smaller tick size are more significant when the value of the tick size is relatively large compared to the stock price (i.e., for low priced stocks). Based on these findings, we suggest that when setting the minimum price improvement, regulators and market operators should consider both the asset price level and the liquidity of the stock.

We extend the framework to a dual-market model in which a group of broker-dealers can execute customers' orders by choosing between trading on a PLB or on an internalization pool (IP). IPs are dark venues that work like opaque limit order books but have finer price grids than regulated markets. Regular traders access the liquidity posted on the IP through smart order routers which seek the best execution available on the two markets. This extension allows us to investigate the consequences of sub-penny trading in a setting where the main PLB trades in penny increments. This setup captures the essence of today's stock trading environment where market venues such as NYSE-Euronext and Nasdaq-OMX compete with internalizers and dark pools.

Our results show that competition from the IP reduces the number of both limit and market orders sent to the PLB. The reduction of limit orders causes a reduction in the provision of liquidity on the PLB and hence has a detrimental effect on both market depth and the inside spread. Conversely, the reduction of market orders leads to a decrease in the demand for liquidity. This preserves depth on the PLB, generating a positive effect on liquidity. However, it also reduces the execution probability of limit orders, reducing the incentive to supply liquidity. Our model shows that the effect on the quality of the PLB is dictated by the net result of these two forces, which in turn depends on the initial state of the PLB.

We show that for illiquid stocks the existence of an IP is detrimental for the level of liquidity on the PLB: the provision of liquidity in the PLB decreases to such a degree that both market depth and the inside spread worsen. On the other hand, when competition for the provision of liquidity on the PLB is high, because the book is liquid, the positive effect of the reduced liquidity demand on the PLB dominates. The introduction of the IP induces broker-dealers to trade intensively on the IP: aggressive market orders are intercepted by the IP away from the PLB, thus preserving liquidity on the PLB. All these effects are stronger for low priced stocks, as they are driven by the tick to price ratio rather than by the absolute value of the tick size.

Further, our results show that even though competition from a dark market with a finer price grid has mixed effects on the quality of the lit market, it makes all traders better off, be they broker-dealers who supply liquidity, or regular traders who can access this extra liquidity via smart order routers. The results also show that gains from trade are higher for low priced stocks, which explains why broker-dealers are more active in these stocks. A word of caution is warranted, though. Regular traders' welfare increases with sub-penny trading provided that smart order routers allow them to benefit from the liquidity posted on the IP.

Hence, unsophisticated retail traders, who are likely to be unable to take advantage of this optional liquidity, could be harmed when the quality of the PLB deteriorates, as we observe for illiquid stocks.

Our model contributes to the empirical literature on the impact of tick size changes by providing several new empirical predictions. By selecting a sample of stocks to account for variation in liquidity and price, this new framework could be exploited to shed light on the relationship between tick size changes, market quality, and traders' welfare, when regular exchanges compete with dark markets. In addition, the results from our model show that it is necessary to study the dynamics of order flows to understand how PLBs and IPs interact by affecting both the provision of liquidity and the demand for liquidity. Finally, a more in-depth analysis of how order submission strategies change following a tick size reduction in markets characterized by competition from dark pools should help investigating the effect of subpenny trading on liquidity and traders' welfare.

This paper is related to three strands of the existing theoretical literature, that is to intermarket competition,[2] to the optimal tick size,[3] and to the internalization of order flows by broker-dealers (Battalio and Holden, 2001). To the best of our knowledge, it is the first model that allows researchers to investigate the tick size rule within a framework that

[2] See, for example, Chowdhry and Nanda (1991), Parlour and Seppi (2003), and Foucault and Menkveld (2008).

[3] See Anshuman and Kalay (1998), Cordella and Foucault (1999), Foucault et al. (2005), Goettler et al. (2005), Kadan (2006), and Seppi (1997).

takes into account both the asset value and the liquidity of the stock. It also departs from the existing theoretical works as it embeds sub-penny trading through modelling an IP.

The remaining part of this paper is structured as follows: in Section 1.2, we discuss the regulatory debate on tick size, in Section 1.3 we overview the related literature. In Section 1.4 and 1.5, we focus on the single market model, whereas Section 1.6 contains the model with an IP. In Section 1.7, we discuss the effects of tick size changes on traders' welfare. We present the empirical implications in Section 1.8, and we draw policy conclusions in Section 1.9. All the proofs appear in the Appendix.

1.2 Regulatory Debate

As a vast body of empirical literature has shown,[4] when the tick size is reduced spread decreases but depth at the top of the book deteriorates. For this reason, regulators are concerned by those trading strategies that exploit the possibility to submit orders at fractions of the minimum tick size. In 2005 the Securities and Exchange Commission (SEC) introduced the Sub-Penny Rule [adopted Rule 612 under Regulation National Market System (NMS)]. The rule was aimed at protecting displayed limit orders from being undercut by trivial amounts. It prohibits market participants from displaying, ranking, or accepting quotations in NMS stocks that are

[4] See Section 1.3.

priced at smaller increments than the allowed minimum price variation.

In the years following the introduction of Rule 612, however, the development of dark markets deeply affected intermarket competition, and made the rule ineffective in protecting displayed limit orders. In particular, two features of the rule paved the way for sub-penny trading. First, Rule 612 prohibits market participants from quoting prices in sub-penny, but in the belief that sub-penny trading would not be as detrimental as sub-penny quoting, it expressly allows broker-dealers to provide price improvement to a customer order that resulted in a sub-penny execution, thus allowing sub-penny trading. Second, the Rule 612 prohibition of sub-penny quoting does not apply to dark markets; this means that broker-dealers can exploit IPs to jump the queue by a fraction of a penny and so preempt the National Best Bid Offer (NBBO).

Another important factor that facilitates sub-penny trading is the growing importance of fast trading facilities. Using algorithmic programs to generate replications of trading strategies, broker-dealers trading large volumes can make significant profits even though they sacrifice a fraction of a penny in order to step ahead of the PLB.[5] As a result, volumes traded on IPs have steadily increased over time as shown in Table 1.1. In August 2010 they executed 8.55% of the consolidated US equity volume -the rest being executed in public crossing networks, exchange and

[5] Jarnecic and Snape (2010) suggest that high frequency trading is negatively related to the tick size.

consortium-based pools- which is an increase of 25% over the previous year. Furthermore, the proportion of sub-penny trading (queue jumping) has dramatically increased over the past 10 years as shown in Figure 1.1.

The SEC (2010) has recently proposed a potential solution to the sub-penny issue in the form of the Trade-At Rule that would practically ban sub-penny trading by prohibiting "any trading center from executing a trade at the price of the NBBO unless the trading center was displaying that price at the time it received the incoming contra-side order." By contrast, BATS (2009) proposed to reduce the minimum price increment of publicly displayed market centers to sub-pennies, in order to level the playing field.

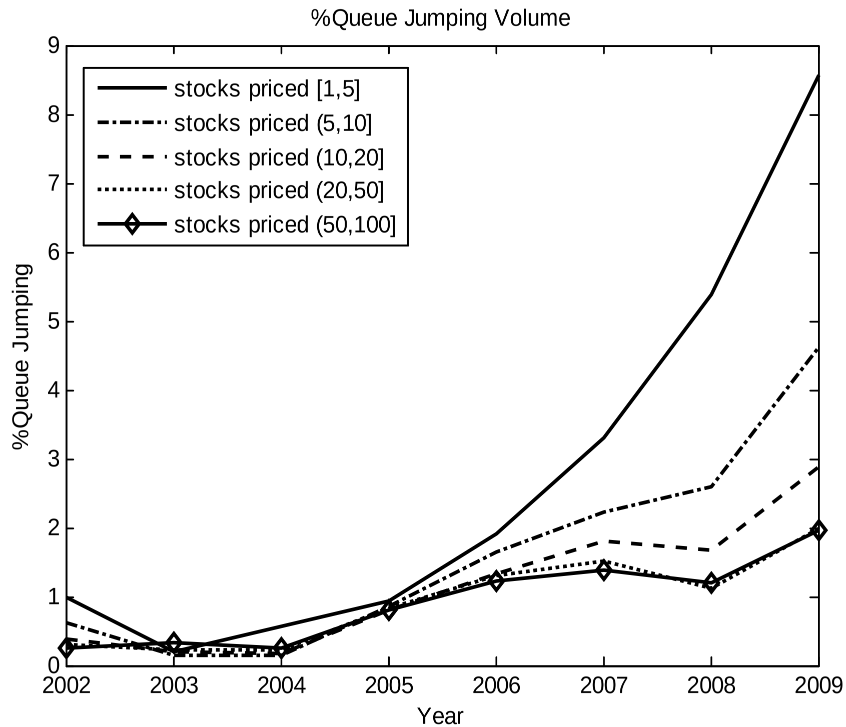
More recently, in April 2012, the US Congress passed the Jumpstart Our Business Startup (JOBS) Act which instructed the SEC to study the impact of decimalization on liquidity for small and medium capitalization companies. According to the JOBS Act, if needed, the SEC is allowed to increase the minimum trading increment of emerging growth companies. However, the conclusions of the SEC Report to Congress on Decimalization (2012) discouraged the Commission from proceeding with the rulemaking required to increase the tick size. Our model shows that an increase in the tick size for smaller stocks may be ineffective in today's trading environment. We explain why in Section 1.9 below.

Table 1.1 US Dark Pool Volume (Consolidated US Equity Volume)

	Time (Month/Year)	06/08	08/08	06/09	08/09	06/10	08/10
Internalized Pool (Broker-sponsored) (IP)	Credit Suisse Crossfinder	.81	.88	1.42	1.55	2.08	2.10
	Goldman Sachs Sigma X	1.37	3.00	1.19	1.44	1.47	1.55
	Knight Link	.98	1.17	1.05	1.23	1.71	1.38
	Getco Execution Service	.41	.64	1.06	1.11	1.03	1.28
	Morgan Stanley MS Pool	.26	.31	.53	.48	.58	.78
	Barclays LX	.47	-	.20	.25	.57	.67
	UBS Pin	.29	.58	.42	.43	.37	.41
	Citi Match	.43	.50	.35	.37	.31	.38
	%IP	5.02	7.08	6.22	6.86	8.12	8.55
Public Crossing Networks (PC)	ITG Posit	.32	.29	.21	.24	.34	.44
	Instinet CBX	.20	.27	.19	.24	.41	.41
	Liquidnet	.39	.41	.26	.28	.21	.25
	Convergex Millennium	.31	.30	.16	.16	.21	.18
	Convergex Vortex	.05	.06	.08	.05	.07	.10
	Pipeline Trading	.20	.15	.09	.12	.08	.09
	% PC	1.47	1.48	.99	1.09	1.32	1.47
Consortium-based Pools(CBP)	Level	.40	.56	.50	.10	.79	.93
	Bid Trading	.13	.10	.10	.53	.22	.31
	%CBP	.53	.66	.60	.63	1.01	1.24
	ALL	7.02	9.22	7.81	8.58	10.45	11.26

Source: Rosenblatt Securities Inc. This Table shows the recent evolution of U.S. dark pool volume. Dark pools are classified as Internalization Pools, Public Crossing Networks and Consortium-based Pools.

Figure 1.1 NASDAQ Stocks - Queue Jumping



Source: Thomson Reuters. This Figure shows the evolution of sub-penny trading over the past 10 years for different priced NASDAQ stocks.

1.3 Literature Review

There is extensive research on the relationship between the reduction of the tick size and market quality. Empirical studies from various markets around the world have found that a tick size reduction is associated with a decline in both the spread and depth, and that the spread is not equally affected across stocks.[6] These findings are confirmed by a more recent pilot program implemented by the major European platforms aimed at investigating the effect of a reduction of the tick size,[7] and are consistent with the early predictions of Angel (1997) and Harris (1994).

Theoretical models have also been developed to study the effect of a tick size variation in different market structures. Seppi (1997) investigates the optimal tick size in a market in which a specialist competes for liquidity provision against a competitive limit order book and finds that large traders may prefer a larger tick size than small traders. Cordella and Foucault (1999) study competition between dealers who arrive at the market sequentially and whose bidding strategy depends of the value of

[6] See Ahn et al. (1996 and 2007), Bacidore (1997), Bourghelle and Declerck (2004), Cai et al. (2008), Golstein and Kavajecz (2000), Griffiths et al. (1998), Harris (1994), Lau and McInish (1995), Porter and Weaver (1997), and Ronen and Weaver (2001).

[7] In December 2008, BATS Europe, in conjuction with Chi-X, Nasdaq OMX Europe and Turquoise, developed a proposal to standardize the tick size of the pan European trading platforms. Starting June 1, 2009, Chi-X, followed by Turquoise, BATS Europe, and finally the LSE and Nasdaq OMX Europe, reduced the tick size for a number of stocks. This pilot program, aimed at studying the effect of a change in the tick size based on actual market data, showed that following the reduction of the tick size, effective spread, inside spread, inside depth and average trade size decreased (BATS, 2009).

the tick size. They show that a larger tick size can increase the speed at which dealers adjust their quotes towards the competitive price, especially when monitoring costs are high. Hence transaction costs can ultimately decrease following an increase in the tick size. Similarly, competition among dealers for the provision of liquidity to an incoming market order drives the results obtained by Kadan (2006). He shows that when the number of dealers is small, liquidity benefits from a small tick size since this prevents dealers from exploiting their market power. When the number of dealers is instead large, a smaller tick size may still improve liquidity. The reason is that it allows dealers to post quotes as close as possible to their reservation value, thus transferring welfare to liquidity demanders.

Our model departs from all these protocols as we consider a pure order driven market where liquidity provision is endogenously created by market participants who choose limit as opposed to market orders. To evaluate the effects of a tick size variation in pure limit order markets, one has to consider the competitive interaction of both patient traders who supply liquidity via limit orders, and impatient traders who demand liquidity via market orders. Our framework shares this feature with the dynamic model of Foucault et al. (2005) who show that a reduction of the tick size may harm market resiliency and have adverse effects on transaction costs. In their model, however, traders cannot refrain from trading and when submitting limit orders they must provide a price improvement. Hence, because patient traders cannot join the queue at the existing best bid or offer, a larger tick size has the effect of making their orders more rather than less aggressive, resulting in an increased resiliency and a narrowed spread. By contrast, in our model traders are free to submit market and limit orders at any level on the price grid, as well as to refrain from trading.

Our protocol is closer to Goettler et al. (2005) who consider an infinite horizon version of Parlour (1998) and model a limit order book as a stochastic sequential game with rational traders arriving and choosing to submit orders at, above or below the existing best quotes. Their model is very rich as it also embeds an asset value shock at each trading period. However, because of its richness, it is analytically intractable and hence solved by numerical simulations. In contrast to Goettler et al. (2005), we focus our analysis on a limited number of periods that allows us to obtain a closed form solution. Within their framework Goettler et al. (2005) show that, by reducing the tick size, regulators achieve an increase of total investors' surplus. We show that the effects of a tick size variation depend on the liquidity of the limit order book and on the price of the security traded. We also study the effects of competition from a dark market with a smaller price grid, and finally we provide intuitive explanations for the interaction between liquidity suppliers and liquidity demanders.

Our paper is also related to the literature on intermarket competition that documents an improvement in market efficiency when competing venues enter a market.[8] Chowdhry and Nanda (1991) extend Kyle (1985) model to accommodate multi-market trading and show that markets with the lowest transaction costs attract liquidity. Closer to our framework, Degryse et al. (2009) analyze the interaction between a dealer market and a crossing network and show that overall welfare is not necessarily enhanced by the introduction of a crossing network. Our setup substantially differs from theirs as we consider a LOB instead of a dealer market and an internalization pool instead of a crossing network. Our model also departs from Buti et al. (2011) who model competition between a PLB and a dark pool by focusing squarely on the tick size. Furthermore, the dark pool that we model has its own discriminatory pricing rule whereas the dark pool in Buti et al. (2011) is based on a derivative pricing rule.

Finally, this paper is related to the literature on broker-dealers' internalization and payment for order flow.[9] Chordia and Subrahmanyam (1995) and Kandel and Marx (1999) show that these practises arise from the existence of the tick size. By contrast, Battalio and Holden (2001) show that when the tick size is set equal to zero, brokers still internalize their clients' orders as they make profits by exploiting their direct relationships with customers. This is consistent with the related empirical

[8] See, for example, Barclay et al. (2003), Bessembinder and Kaufman (1997), Biais et al. (2010), Foucault and Menkveld (2008), Fink et al. (2006), and Goldstein et al. (2008).

[9] Internalization is either the direction of order flows by a broker-dealer to an affiliated specialist, or the execution of order flows by that broker-dealer acting as a market maker.

works.[10]

1.4 Single Market Model

In this Section we introduce the single market framework and in the next one we solve the model and compare the results for two different values of the tick size. In Section 6 we add competition from a dark market where broker-dealers can post quotes at sub-penny increments.

1.4.1 The Market

A market for a security is run over a trading day divided into T periods: $t = 1, ..., T$. At each period t a trader arrives and for simplicity we assume that the size of his order is unitary. Following Parlour (1998), traders have the following linear preferences:

$$U(C_1, C_2; \beta) = C_1 + \beta C_2 \tag{1.1}$$

where C_1 is the cash inflow from selling or buying the security on day 1, while C_2 is the cash inflow from the asset payment on day 2 and is equal to $+v$ $(-v)$ in case of a buy (sell) order. Traders are risk neutral and have a personal trade-off between consumption in the two days equal to β that is a patience indicator drawn from the uniform distribution $U(\underline{\beta}, \overline{\beta})$,

[10] See Chung et al. (2004a and 2004b), Hansch et al. (1999), He et al. (2006), Hendershott and Jones (2005), and Porter and Weaver (1997).

with $0 \leq \underline{\beta} < 1 < \overline{\beta}$. A patient trader has a β close to 1 while an eager one has values of β close either to $\underline{\beta}$ or to $\overline{\beta}$.

Upon arrival at the market in period t, the trader observes the state of the book that is characterized by the number of shares available at each level of the price grid. The latter assembles two prices on the ask ($A_1 < A_2$) and two on the bid side of the market ($B_1 > B_2$), symmetrically distributed around the asset value v. The difference between two adjacent prices, which we name τ, is the tick size. It is equal to the minimum price increment and also corresponds to the minimum inside spread. Thus the possible prices are equal to $A_1 = v + \frac{\tau}{2}$, $A_2 = v + \frac{3\tau}{2}$, $B_1 = v - \frac{\tau}{2}$, and $B_2 = v - \frac{3\tau}{2}$. The state of the book that specifies the number of shares Q_t available at each price level is defined as $S_t = [Q_t^{A_2}, Q_t^{A_1}, Q_t^{B_1}, Q_t^{B_2}]$. As in Seppi (1997) and Parlour (1998), we assume that a trading crowd provides liquidity at the highest levels of the limit order book and prevents traders from quoting prices that are too far away from the top of the book. Besides, traders are allowed to submit limit orders queuing in front of the trading crowd. In this parsimonious way, we can extend Parlour (1998) model to include two price levels where traders can submit orders, and, at the same time, keep the strategy space as small as possible. In addition we can investigate the effects of the tick size reduction on depth at different levels of the book.

The market allows two types of orders: limit orders represented by $+1$ and market orders represented by -1. Traders can submit limit orders to buy (sell) one share at different levels of the bid (ask) prices, or market orders which hit the bid (ask) prices and are executed immediately, or they can decide not to trade. Orders cannot be modified or cancelled after submission, and a trader's strategy at time t is defined by H_t. His strategy space is therefore $H = \{\pm 1^i, 0\}$, where $i = A_2, A_1, B_1$, and B_2. The change in the limit order book induced by the trader's strategy H_t is indicated by h_t and defined as:

$$h_t = [h_t^{A_2}, h_t^{A_1}, h_t^{B_1}, h_t^{B_2}] = \begin{cases} [\pm 1, 0, 0, 0] & \text{if } H_t = \pm 1^{A_2} \\ [0, \pm 1, 0, 0] & \text{if } H_t = \pm 1^{A_1} \\ [0, 0, \pm 1, 0] & \text{if } H_t = \pm 1^{B_1} \\ [0, 0, 0, \pm 1] & \text{if } H_t = \pm 1^{B_2} \\ [0, 0, 0, 0] & \text{if } H_t = 0 \end{cases} \tag{1.2}$$

The state of the book is hence characterized by the following dynamics:

$$S_t = S_{t-1} + h_t \tag{1.3}$$

and the expected state of the book at time t is given by:

$$E[S_{t|t-1}] = S_{t-1} + E[h_t] \tag{1.4}$$

where $E[h_t^i] = \int\limits_{\beta \in \{\beta : H_t(\beta) = \pm 1^i\}} H_t(\beta) d\beta$ for $i = A_2, A_1, B_1, B_2$.

Table 1.2 Order Submission Strategy Space

Strategy	H_t	$U(\cdot)$
Market Sell Order	-1^B	$B-\beta v$
Limit Sell Order	1^{A_k}	$p_t^*(A_k^{N_{-k},N_k}\|S_t)\cdot(A_k-\beta v)$
No Trade	0	0
Limit Buy Order	1^{B_k}	$p_t^*(B_k^{M_{-k},M_k}\|S_t)\cdot(\beta v-B_k)$
Market Buy Order	-1^A	$\beta v-A$

Table 1.2 Order Submission Strategy Space This Table reports in column 3 the payoffs of the order strategies listed in column 1.

1.4.2 Order Submission Decision

To select his order submission strategy, a trader needs to choose an order type and a price. His goal is to maximize his utility, which in this risk neutral setting is equivalent to maximize his payoff, considering all the available strategies. Market orders guarantee immediate executions but higher price opportunity costs, while limit orders enable traders to get better prices at the cost of uncertain execution. Hence in this market traders face the trade-off between execution costs and price opportunity costs.[11] The payoffs of the different strategies available to traders are listed in Table 1.2. Equilibrium strategies are derived in the following Section.

In Table 1.2 we denote by A and B with no subscript the best available quotes, so that for example a market buy order executed at the best

[11] When traders choose a limit rather than a market order, they forgo execution certainty to obtain a better price, and consequently they increase their execution costs. At the same time, however, they reduce their price opportunity cost, which is the cost associated with an execution at a less favourable price.

available price is indicated by -1^A. We indicate by $p_t^*(A_k^{N_{-k},N_k}|S_t)$ (or $p_t^*(B_k^{M_{-k},M_k}|S_t)$) with $k=1,2$ the equilibrium execution probability for a limit sell (or buy) order queuing at the N_k (M_k) position at the price level A_k (B_k), with $N_{-k}=\sum_{d<k} N_d$ ($M_{-k}=\sum_{d<k} M_d$) being the number of shares standing at lower (higher) price levels.[12] This execution probability is conditional on the state of the limit order book, and depends on both the price level at which the order is posted and the depth available on the limit order book. An order posted at A_k and queueing at the N_k position, is executed against the $(N_{-k}+N_k)$-th market order only if $(N_{-k}+N_k-1)$ market orders have already hit both the N_{-k} shares available at lower prices and the N_k-1 shares available at A_k with time priority. If $(N_{-k}+N_k)$ is larger than the number of remaining periods, additional limit orders at that price level will never be executed and $p_t(A_k^{N_{-k},N_k}|S_t)=0$. The execution probability also depends on the state of the other side of the limit order book: a deep book on the bid side increases the incentive for a seller to post limit orders as he knows that incoming buyers will be more inclined to post market orders (due to the long queue on the bid side).[13]

[12] A star superscript indicates equilibrium values.

[13] To simplify the notation, when the best ask is A_k, we indicate the execution probability of a limit order queuing at A_k by $p_t(A_k^{N_k}|S_t)$ instead of using $p_t(A_k^{0,N_k}|S_t)$ and the execution probability of the order standing at the first place of the queue by $p_t(A_k|S_t)$ instead of using $p_t(A_k^{0,1}|S_t)$.

1.4.3 Market Equilibrium

Traders use information from the state of the limit order book to rationally compute different orders' execution probabilities, and then compare the expected payoffs from each order to choose the optimal strategy consistent with their own β.[14] We solve the model by backward induction. At time T, the execution probability for limit orders is zero, and traders submit only market orders or decide not to trade. It can be easily shown that traders' equilibrium strategies are:

$$H_T^*(\beta, S_{T-1}) = \begin{cases} -1^B & \text{if } \beta \in [\underline{\beta}, \frac{B}{v}) \\ 0 & \text{if } \beta \in [\frac{B}{v}, \frac{A}{v}) \\ -1^A & \text{if } \beta \in [\frac{A}{v}, \overline{\beta}] \end{cases} \quad (1.5)$$

where the best ask and bid prices are equal to $A \in \{A_1, A_2\}$ and $B \in \{B_1, B_2\}$ depending on the state of the book. By using these equilibrium strategies together with the distribution of β, we calculate the equilibrium execution probabilities at the best quotes for limit orders submitted at $T-1$:

$$p_{T-1}^*(A\,|S_{T-1}) = \int_{\beta \in \{\beta : H_T^* | S_{T-1} = -1^A\}} \frac{1}{(\overline{\beta} - \underline{\beta})} d\beta = \frac{\overline{\beta} v - A}{(\overline{\beta} - \underline{\beta}) v} \quad (1.6)$$

[14] Differently from Parlour (1998), we do not assume that traders are ex-ante buyers or sellers but we endogenously derive the trader's decision to buy or to sell the asset.

$$p^*_{T-1}(B\,|S_{T-1}) = \int_{\beta\in\{\beta: H^*_T|S_{T-1}=-1^B\}} \frac{1}{(\overline{\beta}-\underline{\beta})} d\beta = \frac{B-\underline{\beta}v}{(\overline{\beta}-\underline{\beta})v} \qquad (1.7)$$

These execution probabilities are the dynamic link between period T and $T-1$. A trader arriving at $T-1$ can choose between a market and a limit order, and his choice is driven by his β value. The following Lemma holds:

Lemma 1 *If at time $t \neq T$ at least one limit order strategy has positive execution probability, there will always exist a β value for which a limit order is optimally selected by the incoming trader.*

After substituting the equilibrium execution probabilities at T given by (1.6) and (1.7) for the case in which a limit order posted at $T-1$ has a positive execution probability on both sides of the market, i.e., $p^*_{T-1}(A_k^{N_{-k},N_k}\,|S_{T-1}) \neq 0$ and $p^*_{T-1}(B_k^{M_{-k},M_k}\,|S_{T-1}) \neq 0$, we obtain the following optimal strategies for $T-1$:[15]

$$H^*_{T-1}(\beta, S_{T-2}) = \begin{cases} -1^B & \text{if } \beta \in [\underline{\beta}, \beta_{1,T-1}) \\ +1^{A_k} & \text{if } \beta \in [\beta_{1,T-1}, \beta_{3,T-1}) \\ +1^{B_k} & \text{if } \beta \in [\beta_{3,T-1}, \beta_{5,T-1}) \\ -1^A & \text{if } \beta \in [\beta_{5,T-1}, \overline{\beta}] \end{cases} \qquad (1.8)$$

[15] The other cases are discussed in the Appendix.

where $\beta_{1,T-1} = \frac{B}{v} - \frac{p^*_{T-1}(A_k|S_{T-1})}{1-p^*_{T-1}(A_k|S_{T-1})} \cdot \frac{A_k - B}{v}$,

$\beta_{3,T-1} = \frac{p^*_{T-1}(A_k|S_{T-1})A_k + p^*_{T-1}(B_k|S_{T-1})B_k}{p^*_{T-1}(A_k|S_{T-1}) + p^*_{T-1}(B_k|S_{T-1})} \cdot \frac{1}{v}$, and $\beta_{5,T-1} = \frac{A}{v} + \frac{p^*_{T-1}(B_k|S_{T-1})}{1-p^*_{T-1}(B_k|S_{T-1})} \cdot \frac{A-B_k}{v}$. Because only one trader can still arrive at the market at T, a limit order posted at $T-1$ has a positive execution probability only if it undercuts all the orders on the book and gains price priority, i.e., $p^*_{T-1}(A_k^{N_{-k},N_k}|S_{T-1}) \neq 0$ only when $N_{-k} = 0$ and $N_k = 1$. Moreover, the greater the limit order execution probability, $p^*_{T-1}(A_k|S_{T-1})$, the smaller the threshold between market sell orders and limit sell orders, $\beta_{1,T-1}$, and the more likely traders submit limit rather than market orders. More generally, if execution probabilities at time t are high enough that execution costs are lower than price opportunity costs, traders will submit limit orders. If instead execution probabilities are low, they will choose market orders.

The optimal price at which a trader will submit a limit order is the result of a trade-off between price opportunity costs and execution costs: a more competitive price implies a higher execution probability due to both the lower risk of being undercut by incoming traders and the fact that the order becomes more attractive for traders on the opposite side of the market. This is, however, obtained at the cost of a lower revenue once the order is executed. This trade-off crucially depends on the relative tick size, $\frac{\tau}{v}$, as shown in the following Lemma:

Lemma 2 *At time $t \neq T$ traders' aggressiveness in the provision of* *liquidity is positively related to the value of* $\frac{\tau}{v}$*.*

From the equilibrium strategies at $T-1$, we can derive the execution probabilities for limit orders submitted in previous periods, and the corresponding equilibrium strategies. The equilibrium is defined as follows:

Definition 1 *Given an initial book* S_0*, a dynamic equilibrium is a set of order submission decisions* $\{H_t^*\}$ *and states of the limit order book* $\{S_t\}$ *such that at each period the trader maximizes his payoff* $U(\cdot)$ *(Table 1.2) according to his Bayesian belief over the execution probabilities* $p^*(\cdot)$*, i.e.,*

$$\{H_t^* := \arg\max\ U(\cdot\,|\,S_{t-1}, p_{t-1}^*)\}$$
$$\{S_t := S_{t-1} + h_t^*\}$$
$$\text{where } h_t^* \text{ is defined by (3.1)}$$

In order to keep the analysis tractable, from here onwards we focus only on the last three periods of the trading game, starting from $T-2$. For our numerical simulation we also assume that the support of the β distribution is $[0,2]$.

1.5 Tick Size and Market Quality

We start with the market (LM) characterized by a large tick size equal to τ that we already presented in Section 1.4, and we compare the

Table 1.3 Price Grid

LM	**Price**	**SM**
A_2	$v + \frac{9}{6}\tau$	a_5
	$v + \frac{7}{6}\tau$	a_4
	$v + \frac{5}{6}\tau$	a_3
A_1	$v + \frac{3}{6}\tau$	a_2
	$v + \frac{1}{6}\tau$	a_1
	$v - \frac{1}{6}\tau$	b_1
B_1	$v - \frac{3}{6}\tau$	b_2
	$v - \frac{5}{6}\tau$	b_3
	$v - \frac{7}{6}\tau$	b_4
B_2	$v - \frac{9}{6}\tau$	b_5

Table 1.3 Price Grid This Table shows the price grid for both the large tick market (LM) and the small tick market (SM), where v indicates the asset value and τ the tick size

resulting equilibrium trading strategies with those obtained when, all else equal, the tick size is set to $\frac{1}{3}\tau$. Both price grids are shown in Table 1.3: on the LM the price grid is $P^{LM} = \{A_2, A_1, B_1, B_2\}$, while on the small tick market (SM) it has five levels on both the ask and the bid side, a_l and b_l, with $l = 1, .., 5$. For the SM the evolution of the state of the book is still characterized by equations (1.3) and (1.4), the main difference being that both S_t^{SM} and h_t^{SM} now consist of ten components instead of four. The trader's strategy space for the SM is much richer thanks to the finer price grid, $H^{SM} = \{\pm 1^j, 0\}$ with $j = \{a_{1:5}, b_{1:5}\}$.

To compare the two markets, we build standard indicators of market quality using traders' equilibrium strategies. For each period t, depth is measured by the number of shares available on the book at different price levels. More precisely, for the LM we define the following depth indicators:

$$DP_t^{i,LM} = E[Q_t^i], \text{ with } i = \{A_{1:2}, B_{1:2}\} \tag{1.9a}$$

$$DPI_t^{LM} = E[Q_t^A + Q_t^B] \tag{1.9b}$$

$$DPT_t^{LM} = \sum_i E[Q_t^i] \tag{1.9c}$$

where DP^i is the expected depth at price i, DPI is the average depth at the best quotes, and DPT is total depth measured by the sum of average depth at all price levels. The average spread is computed as the expected difference between the best ask and bid prices:

$$SP_t^{LM} = E[A - B] \tag{1.10}$$

Finally, volume in period t is measured by the number of orders executed, while liquidity provision is measured by the number of limit orders submitted. Because at each period only one trader arrives, expected volume, VL_t, and liquidity provision, LP_t, are computed as the probability that this trader will submit a market order or a limit order at all price levels:

$$VL_t^{LM} = E[\sum_i \int_{\beta \in \{\beta : H_t = -1^i | S_{t-1}\}} 1 d\beta] \quad (1.11)$$

$$LP_t^{LM} = E[\sum_i \int_{\beta \in \{\beta : H_t = +1^i | S_{t-1}\}} 1 d\beta] \quad (1.12)$$

Indicators of market quality for the SM are computed in a similar way, but using $j = \{a_{1:5}, b_{1:5}\}$. To illustrate the effects of different tick sizes on both liquid and illiquid stocks, we use the initial state of the book as a proxy for liquidity and consider three cases: an empty book for illiquid stocks, and a book with either one or two units on the first (second) level of the LM (SM) price grid for liquid stocks.[16] The following Proposition summarizes the effects of a tick size change on both traders' strategies and market quality:

Proposition 1 *When moving from a large to a small tick market, changes in traders' order submission strategies and market quality depend on the initial state of the book.*

- *For liquid stocks*
  - *liquidity provision increases; spread, total depth and inside depth improve;*

[16] Notice that to compare markets with different tick size we need to start from the same initial state of the book. This implies that in the SM book the price levels not in common with the LM book must be empty.

  - *the effects are the same when the top of the book is very deep, except for inside depth that worsens.*

- *For illiquid stocks the results are the opposite: liquidity provision decreases and spread, total depth and inside depth deteriorate.*

- *All the above effects become stronger for low priced stocks.*

Table 1.4 reports results for market quality under both the large and the small tick size regimes for three different opening states of the limit order book: with 0, 1, or 2 shares on $A_1(a_2)$ and $B_1(b_2)$. By considering books that differ in market depth, we can offer insights on how the effects on market quality of a tick size change can be influenced by the initial level of liquidity.

Consider first a book that opens at $T-2$ with only 1 share on both $A_1(a_2)$ and $B_1(b_2)$. When the tick size is smaller, undercutting is cheaper and competition for the provision of liquidity becomes more intense. This implies that in equilibrium traders switch from market to limit orders that they post at the new best price levels (a_1 and b_1). The enhanced limit order submission increases market depth, both total and at the inside spread, and the increased limit order aggressiveness narrows the spread. These effects become stronger as the stock price decreases. Clearly, when the value of the tick size becomes relatively large compared to the stock price, the benefit of having a finer price grid increases, and the probability that traders switch from market to limit orders posted at

a_1 and b_1 becomes larger.

Consider then a book that opens at $T-2$ with 2 shares at $A_1(a_2)$ and $B_1(b_2)$. Compared to the previous case, a smaller tick size produces effects that are more intense and of the same direction except for depth at the inside spread, which decreases rather than increases. In the large tick size regime there is no room for additional limit orders[17] and traders are forced either to use market orders or to refrain from trading. Hence when the tick size becomes smaller, traders move even more aggressively than before to the top of the book and total depth increases. However, depth at the top of the book decreases as now traders can spread their orders on the additional price levels instead of being clustered at the best prices.

We can therefore conclude that for these two cases a smaller tick size improves liquidity as it narrows the inside spread and increases total depth, but its effect on inside depth depends on the state of the book. If the regime with a deeper book is a good proxy for very liquid stocks, then our results show that for these stocks a smaller tick size can actually decrease depth at the inside spread.

[17] Recall that only one trader arrives at each period and hence between $T-2$ and T at most 2 shares can be executed.

Table 1.4 Tick Size Change ($\tau = 0.1$)**.** This Table compares the large tick market (LM) with the small tick market (SM), and focuses on three cases that represent an illliquid, liquid, and very liquid state of the book respectively: (1) both books open empty at $T-2$, $S_{T-2} = [0000] = [00000\,00000]$ (2) LM opens with one share on the first level of the book (A_1, B_1), and SM opens with one share on the second level (a_2, b_2), $S_{T-2} = [0110] = [00010\,01000]$ (3) LM opens with two shares on the first level (A_1, B_1) and SM with two shares on the second level (a_2, b_2), $S_{T-2} = [0220] = [00020\,02000]$. The Table reports the following statistics for different asset values, $v = \{1, 5, 10, 50\}$: liquidity provision, i.e., probability of limit order submission (LP); trading volume, i.e., probability of market order submission (VL); spread, depth at the BBO and total depth. Columns 11-14 report the difference between the two markets (SM-LM).

Table 1.4 **Tick Size Change** ($\tau = 0.1$).

		LM [$Tick\ Size = \tau$]				SM [$Tick\ Size = \frac{\tau}{3}$]				($\Delta \times 100$)			
S_{T-2}	Asset Price = v	1	5	10	50	1	5	10	50	1	5	10	50
Illiquid	Limit Order = LP	.5860	.1740	.0952	.0206	.5586	.1710	.0942	.0205	-2.74	-.30	-.10	-.01
	Market Order=VL	.4140	.8260	.9048	.9794	.4414	.8290	.9058	.9795	2.74	.30	.10	.01
	Spread	.2751	.3000	.3000	.3000	.2763	.3000	.3000	.3000	.12	0	0	0
	Depth at BBO	.5860	.1740	.0952	.0206	.5586	.1710	.0942	.0205	-2.74	-.30	-.10	-.01
	Total Depth	.5860	.1740	.0952	.0206	.5586	.1710	.0942	.0205	-2.74	-.30	-.10	-.01
Liquid	Limit Order = LP	.1006	.0226	.0114	.0024	.2334	.0492	.0248	.0050	13.28	2.66	1.34	.26
	Market Order=VL	.8994	.9774	.9886	.9976	.7666	.9508	.9752	.9950	-13.28	-2.66	-1.34	-.26
	Spread	.1899	.1977	.1989	.1998	.1699	.1937	.1968	.1994	-2.00	-.40	-.21	-.04
	Depth at BBO	1.1006	1.0226	1.0115	1.0023	1.2334	1.0493	1.0248	1.0050	13.28	2.67	1.33	.27
	Total Depth	1.2012	1.0453	1.0230	1.0047	1.4671	1.0986	1.0496	1.0100	26.59	5.33	2.66	.53
Very Liquid	Limit Order = LP	0	0	0	0	.2334	.0492	.0248	.0050	23.34	4.92	2.48	.50
	Market Order=VL	.9500	.9900	.9950	.9990	.7666	.9508	.9752	.9950	-18.34	-3.92	-1.98	-.40
	Spread	.1000	.1000	.1000	.1000	.0922	.0984	.0992	.0998	-.78	-.16	-.08	-.02
	Depth at BBO	3.0500	3.0100	3.005	3.0010	3.0000	3.0000	3.0000	3.0000	-5.00	-1.00	-.50	-.10
	Total Depth	3.0500	3.0100	3.005	3.0010	3.4668	3.0986	3.0496	3.0100	41.68	8.86	4.46	.90

However, for illiquid stocks, that we proxy by the empty book, the effect of a smaller tick size is to worsen both inside spread and depth. Traders do not have enough incentive to undercut aggressively by posting limit orders at the new top of the book: to avoid being undercut by the next trader, they would have to accept a very low execution price (a_1 and b_1). Therefore, they prefer to submit either more market orders, or, despite the lower execution probability, limit orders at higher levels of the book. Also in this case, when the stock price increases, a smaller tick size tends to produce effects that gradually drop off.

1.6 Dual-Market Model: PLB, IP, and Sub-Penny Trading

In this Section we broaden our comparative analysis of markets with different tick sizes to include intermarket competition. We extend the previous framework by introducing competition between a large tick market that works like a PLB and a special type of dark pool, the IP, that differs from the PLB in that it has a smaller tick size and allows only broker-dealers to supply liquidity. This dual-market model is suitable to investigate sub-penny trading. Such a practice is carried out by those broker-dealers who can access IPs to compete on price with the limit orders posted at the top of the PLB.

Regulators are concerned about the ultimate effects of sub-penny trading on market quality. It is not yet clear whether this practice fosters competition for the provision of liquidity and improves market quality, or whether it only allows highly sophisticated dealers to generate considerable returns from using IPs to step in front of the NBBO. Because the better prices available in the IP intercept market orders sent to the PLB, due to the existence of smart order routers, liquidity demand decreases. This generates two opposite effects on the quality of the PLB. On the one hand, when fewer market orders hit the top of the PLB, depth is preserved, keeping the spread tighter. On the other hand, the reduction of market orders decreases the execution probability of limit orders and hence the incentive for traders to post depth at the top of the PLB, resulting in lower market depth and wider spread. The changing pattern of market orders, i.e., migrating from the PLB to the IP, is affected by the state of the PLB as well as by the tick to price ratio of the stock considered. Agents' choice between market and limit orders hinges on the trade-off between price opportunity costs and execution costs, which are both influenced by stock characteristics.

To discuss this setting, we adapt our previous single market model to embed sub-penny trading. We assume that at each trading period one individual out of two groups of traders arrives at the market: with probability α the incoming trader is a broker-dealer and with the complementary probability he is a regular trader. While a regular trader can only observe and provide liquidity to the PLB, a broker-dealer can observe and use both the PLB and the IP. The IP is characterized by a smaller tick size so that the broker-dealer can undercut orders posted by other traders at the top of the PLB. Furthermore, in order to approximate the nature of real IPs, we assume that the IP does not have a trading crowd sitting at a_5 and b_5. Finally, while only broker-dealers can post limit orders at the IP, all traders can take advantage of the liquidity offered by both trading platforms. This assumption is consistent with the existence of a smart order routing technology (SOR)[18] that allows all investors to simultaneously access multiple sources of liquidity (Butler, 2010).

SORs allow traders to search the best quotes on the consolidated limit order book (PLB&IP) but they do not necessarily reveal the state of the IP. In reality, the more sophisticated the SOR technology, the better traders' inference of the state of the IP. Hence, to consider different regimes of IP pre-trade transparency, we will first assume perfect inference on the state of the IP, and then extend the model to include partial inference and Bayesian learning.

To sum up, we consider two protocols: the benchmark (PLB), where only one trading platform is available to all traders, and the PLB&IP framework, where an IP competes with the PLB. The latter case further differentiates into a transparent and an opaque setting, where the lack of transparency refers to the IP market. To introduce a certain degree of

[18] Examples are ITG Dark Aggregator and Smartrade Liquidity Aggregator.

uncertainty on the state of the IP, we assume that at $T-2$ the IP opens either empty or with one unit on the first level of the book with equal probability. The following Proposition presents the results.

Proposition 2 *When an IP is added to a PLB, trade migrates to the IP, and traders' order submission strategies and market quality change as follows:*

- *for illiquid stocks, PLB market quality, measured by depth and inside spread, deteriorates because liquidity provision decreases. The effects are stronger both for low priced stocks and when the IP market is opaque;*
- *for liquid stocks, the effect of sub-penny trading is to foster price competition so that in the PLB spread and depth improve, yet liquidity provision worsens. The effects are weaker both for low priced stocks and when the IP market is opaque.*
- *The IP is used more intensively by broker-dealers when the stock is both liquid and low priced and when their proportion (α) increases.*
- *When considering markets with a smaller tick size, all the effects diminish.*

Table 1.5 Sub-penny Trading - Illiquid stock ($\tau = 0.1$**,** $\alpha = 10\%$)**.** This Table focuses on the comparison between the single market model, in which only the public limit order book (PLB) with tick size τ is active, and the dual market model, in which the PLB and the internalization pool (IP) with tick size $\frac{\tau}{3}$ compete. The broker-dealer's arrival rate is $\alpha = 10\%$. At $T-2$ the PLB opens empty, $S_{T-2} = [0000]$, while the IP opens with equal probability either empty, [00000 00000], or with one unit on the first level of the book, [00001 10000]. We consider two asset values, $v = \{1, 10\}$, and differentiate results depending on whether the IP is transparent or opaque. The following statistics are reported for both the PLB and the IP: liquidity provision (LP), i.e., probability of limit order submission; trading volume (VL), i.e., probability of market order submission; and limit order submission probabilities at different levels of the book. The Table also reports the probability of "no trading" and, for the PLB, spread, depth at the BBO and total depth.

Table 1.5 Sub-penny Trading - Illiquid stock

S_{T-2} = [0000]&[00000 00000] or [0000]&[00001 10000] with equal probabilities											
$\tau = 0.1$		$v = 1$					$v = 10$				
		PLB	PLB&IP: Transp. IP		PLB&IP: Opaque IP		PLB	PLB&IP: Transp. IP		PLB&IP: Opaque IP	
		$T-2$	$T-2$	$(\Delta \times 100)$	$T-2$	$(\Delta \times 100)$	$T-2$	$T-2$	$(\Delta \times 100)$	$T-2$	$(\Delta \times 100)$
LP (PLB)		.5860	.3176	(-26.84)	.2604	(-32.56)	.0952	.0512	(-4.40)	.0448	(-5.04)
LP (IP)			0	(0)	0	(0)		0	(0)	0	(0)
VL (PLB)		.4140	.2128	(-20.12)	.3544	(-5.96)	.9048	.4524	(-45.24)	.4753	(-42.95)
VL (IP)			.4696	(46.96)	.3852	(38.52)		.4964	(49.64)	.4799	(47.99)
No Trading		0	0	(0)	0	(0)	0	0	(0)	0	(0)
Spread (PLB)		.2751	.2887	(1.36)	.3000	(2.49)	.3000	.3000	(0)	.3000	(0)
Depth BBO (PLB)		.5860	.3176	(-26.84)	.2605	(-32.55)	.0952	.0512	(-4.40)	.0448	(-5.04)
Total Depth (PLB)		.5860	.3176	(-26.84)	.2605	(-32.55)	.0952	.0512	(-4.40)	.0448	(-5.04)
			PLB	IP	PLB	IP		PLB	IP	PLB	IP
Tick Size		τ	τ	$\frac{\tau}{3}$	τ	$\frac{\tau}{3}$	τ	τ	$\frac{\tau}{3}$	τ	$\frac{\tau}{3}$
	$A_2 = a_5$	.1684	.1025	0	.1302	0	.0476	.0256	0	.0224	0
	a_4	-	-	0	-	0	-	-	0	-	0
	a_3	-	-	0	-	0	-	-	0	-	0
Limit	$A_1 = a_2$	.1246	.0563	0	0	0	0	0	0	0	0
Order	a_1	-	-	0	-	0	-	-	0	-	0
Sub	b_1	-	-	0	-	0	-	-	0	-	0
Prob.	$B_1 = b_2$	.1246	.0563	0	0	0	0	0	0	0	0
	b_3	-	-	0	-	0	-	-	0	-	0
	b_4	-	-	0	-	0	-	-	0	-	0
	$B_2 = b_5$	.1684	.1025	0	.1302	0	.0476	.0256	0	.0224	0

Table 1.6 Sub-penny Trading - Liquid stock ($\tau = 0.1$, $\alpha = 10\%$)**.** This Table focuses on the comparison between the single market model, in which only the public limit order book (PLB) with tick size τ is active, and the dual market model, in which the PLB and the internalization pool (IP) with tick size $\frac{\tau}{3}$ compete. The broker-dealer's arrival rate is $\alpha = 10\%$. At $T-2$ the PLB opens with one share on the first level of the book, $S_{T-2} = [0110]$, while the IP opens with equal probability either empty, [00000 00000], or with one unit on the first level of the book, [00001 10000]. We consider two asset values, $v = \{1, 10\}$, and differentiate results depending on whether the IP is transparent or opaque. The following statistics are reported for both the PLB and the IP: liquidity provision (LP), i.e., probability of limit order submission; trading volume (VL), i.e., probability of market order submission; and limit order submission probabilities at different levels of the book. The Table also reports the probability of "no trading" and, for the PLB, spread, depth at the BBO and total depth.

Table 1.6 **Sub-penny Trading - Liquid stock** ($\tau = 0.1$, $\alpha = 10\%$).

S_{T-2} = [0110]&[00000 00000] or [0110]&[00001 10000] with equal probabilities											
		$v = 1$					$v = 10$				
		PLB	PLB&IP: Transp. IP		PLB&IP: Opaque IP		PLB	PLB&IP: Transp. IP		PLB&IP: Opaque IP	
		$T-2$	$T-2$	$(\Delta \times 100)$	$T-2$	$(\Delta \times 100)$	$T-2$	$T-2$	$(\Delta \times 100)$	$T-2$	$(\Delta \times 100)$
LP (PLB)		.1006	.0462	(-5.44)	.0499	(-5.07)	.0114	.0054	(-.60)	.0056	(-.58)
LP (IP)			.0122	(1.22)	.0120	(1.20)		.0011	(0.11)	.0011	(0.11)
VL (PLB)		.8994	.4429	(-45.65)	.4638	(-43.56)	.9886	.4936	(-49.50)	.4961	(-49.25)
VL (IP)			.4911	(49.11)	.4743	(47.43)		.4991	(49.91)	.4972	(49.72)
No Trading		0	.0076	(.76)	0	(0)	0	.0008	(.08)	0	(0)
Spread (PLB)		.1899	.1443	(-4.56)	.1464	(-4.35)	.1989	.1494	(-4.95)	.1496	(-4.93)
Depth BBO (PLB)		1.1006	1.5571	(45.65)	1.5362	(43.56)	1.0115	1.5064	(49.49)	1.5039	(49.24)
Total Depth (PLB)		1.2012	1.6034	(40.22)	1.5617	(36.05)	1.0230	1.5118	(48.88)	1.5040	(48.10)
			PLB	IP	PLB	IP		PLB	IP	PLB	IP
Tick Size		τ	τ	$\frac{\tau}{3}$	τ	$\frac{\tau}{3}$	τ	τ	$\frac{\tau}{3}$	τ	$\frac{\tau}{3}$
	$A_2 = a_5$	.0503	.0231	0	.0249	0	.0057	.0027	0	.0028	0
	a_4	-	-	0	-	0	-	-	0	-	0
	a_3	-	-	0	-	0	-	-	0	-	0
Limit	$A_1 = a_2$	0	0	.0005	0	.0006	0	0	.0000	0	.0000
Order	a_1	-	-	.0056	-	.0054	-	-	.0006	-	.0006
Sub	b_1	-	-	.0056	-	.0054	-	-	.0006	-	.0006
Prob.	$B_1 = b_2$	0	0	.0005	0	.0006	0	0	.0000	0	.0000
	b_3	-	-	0	-	0	-	-	0	-	0
	b_4	-	-	0	-	0	-	-	0	-	0
	$B_2 = b_5$	.0503	.0231	0	.0249	0	.0057	.0027	0	.0028	0

Table 1.5 focuses on the case in which the PLB opens empty at $T-2$, that we use to proxy illiquid stocks, while Table 1.6 shows results for a book that opens with one share on A_1 and B_1 which should offer intuitions for more liquid stocks. Table 1.5 shows that on the PLB both depth and limit orders decrease when the IP is added. When traders perceive the potential competition from broker-dealers, they react by supplying less liquidity to the PLB. Furthermore, this effect generally outweighs the reduction in market orders resulting from their interception by the IP, so that overall the inside spread worsens. These effects become weaker as the stock price increases: when the tick to price ratio becomes very small, the profitability of liquidity provision, and hence of undercutting, declines so that the IP competition becomes less relevant. When instead the IP is opaque, these effects become stronger: the uncertainty on the IP depth and on the actual level of competition makes traders even more reluctant to post limit orders at the PLB.

For more liquid stocks (Table 1.6) the main effect of sub-penny trading is to foster price competition. When the IP platform is introduced, broker-dealers submit limit orders to the IP at a_1 to undercut the existing depth at A_1, thus intercepting incoming market orders away from the PLB. The resulting reduction of liquidity demand improves spread and depth on the PLB. However, the switch of market orders from the PLB to the IP as well as of broker-dealers' limit orders implies that volume and liquidity provision worsen on the public venue.

In this case, when the stock price increases, the effects of the IP competition become stronger due to the reaction of both market orders and limit orders. When the tick to price ratio decreases, market orders become central to the choice of traders' optimal submission strategies and therefore their increased reduction has a greater effect on the PLB quality. This positive effect is reinforced by the fact that the change in limit orders becomes almost irrelevant, and hence, compared to the $v = 1$ case, their reduction almost disappears. When instead the IP market becomes opaque, uncertainty increases for regular traders and therefore the effects on both limit and market orders lessen. As a result, the IP positive effect on PLB market quality decreases.

By comparing traders' equilibrium strategies for illiquid and liquid stocks, we observe that broker-dealers post orders at the IP more intensively for liquid stocks in which competition for liquidity provision in the PLB is higher. This effect is stronger for low priced stocks in which providing liquidity is more convenient than taking liquidity, due to the larger tick to price ratio. Finally, when the proportion α of broker-dealers is increased from 10% to 20% (Table 1.7), the sub-penny activity builds up in the IP and hence the effects on the PLB spread and depth are magnified.

Table 1.7 Sub-penny Trading - Liquid stock ($\tau = 0.1$**,** $\alpha = 20\%$)**.** This Table focuses on the comparison between the single market model, in which only the public limit order book (PLB) with tick size τ is active, and the dual market model, in which the PLB and the internalization pool (IP) with tick size $\frac{\tau}{3}$ compete. The broker-dealer's arrival rate is $\alpha = 20\%$. At $T-2$ the PLB opens with one share on the first level of the book, $S_{T-2} = [0110]$, while the IP opens with equal probability either empty, [00000 00000], or with one unit on the first level of the book, [00001 10000]. We consider two asset values, $v = \{1, 10\}$, and differentiate results depending on whether the IP is transparent or opaque. The following statistics are reported for both the PLB and the IP: liquidity provision (LP), i.e., probability of limit order submission; trading volume (VL), i.e., probability of market order submission; and limit order submission probabilities at different levels of the book. The Table also reports the probability of "no trading" and, for the PLB, spread, depth at the BBO and total depth.

Table 1.7 **Sub-penny Trading - Liquid stock** ($\tau = 0.1$, $\alpha = 20\%$).

S_{T-2} = [0110]&[00000 00000] or [0110]&[00001 10000] with equal probabilities											
		$v = 1$					$v = 10$				
		PLB	PLB&IP: Transp. IP		PLB&IP: Opaque IP		PLB	PLB&IP: Transp. IP		PLB&IP: Opaque IP	
		$T-2$	$T-2$	$(\Delta \times 100)$	$T-2$	$(\Delta \times 100)$	$T-2$	$T-2$	$(\Delta \times 100)$	$T-2$	$(\Delta \times 100)$
LP (PLB)		.1006	.0422	(-5.84)	.0454	(-5.52)	.0114	.0050	(-.64)	.0052	(-.62)
LP (IP)			.0244	(2.44)	.0240	.0240		.0022	(0.22)	.0022	(0.22)
VL (PLB)		.8994	.4362	(-46.32)	.4548	(-44.46)	.9886	.4932	(-49.54)	.4952	(-49.34)
VL (IP)			.4904	(49.04)	.4758	.4758		.4990	(49.90)	.4974	(49.74)
No Trading		0	.0068	(.68)	0	(0)	0	.0006	(0.06)	.0	(0)
Spread (PLB)		.1899	.1436	(-4.63)	.1455	(-4.44)	.1989	.1493	(-4.96)	.1495	(-4.94)
Depth BBO (PLB)		1.1006	1.5639	(46.33)	1.5452	(44.46)	1.0115	1.5071	(49.56)	1.5048	(49.33)
Total Depth (PLB)		1.2012	1.6061	(40.49)	1.5690	(36.78)	1.0230	1.5121	(48.91)	1.5077	(48.47)
			PLB	IP	PLB	IP		PLB	IP	PLB	IP
Tick Size		τ	τ	$\frac{\tau}{3}$	τ	$\frac{\tau}{3}$	τ	τ	$\frac{\tau}{3}$	τ	$\frac{\tau}{3}$
	$A_2 = a_5$	.0503	.0211	0	.0227	0	.0057	.0025	0	.0026	0
	a_4	-	-	0	-	0	-	-	0	-	0
	a_3	-	-	0	-	0	-	-	0	-	0
Limit	$A_1 = a_2$	0	0	.0009	0	.0011	0	0	.0000	0	.0000
Order	a_1	-	-	.0113	-	.0109	-	-	.0011	-	.0011
Sub	b_1	-	-	.0113	-	.0109	-	-	.0011	-	.0011
Prob.	$B_1 = b_2$	0	0	.0009	0	.0011	0	0	.0000	0	.0000
	b_3	-	-	0	-	0	-	-	0	-	0
	b_4	-	-	0	-	0	-	-	0	-	0
	$B_2 = b_5$	.0503	.0211	0	.0227	0	.0057	.0025	0	.0026	0

Table 1.8 Sub-penny Trading - Illiquid and Liquid stocks - Small Tick Size ($\tau = \frac{0.1}{3}$**,** $\alpha = 10\%$)**.** This Table focuses on the comparison between the single market model, in which only the public limit order book (PLB) with tick size τ is active, and the dual market model, in which the PLB and the internalization pool (IP) with tick size $\frac{\tau}{3}$ compete. The broker-dealer's arrival rate is $\alpha = 10\%$. At $T-2$ the PLB opens either empty (Panel A), $S_{T-2} = [0000]$, or with one share on the first level of the book (Panel B), $S_{T-2} = [0110]$, while the IP opens with equal probability either empty, [0000][00000 00000], or with one unit on the first level of the book, [00001 10000]. We consider two asset values, $v = \{1, 10\}$, and differentiate results depending on whether the IP is transparent or opaque. For both the PLB and the IP liquidity provision (LP), i.e., probability of limit order submission, and trading volume (VL), i.e., probability of market order submission, are reported. The Table also reports the probability of "no trading" and, for the PLB, spread, depth at the BBO and total depth.

Table 1.8 **Sub-penny Trading - Illiquid and Liquid stocks - Small Tick Size** ($\tau = \frac{0.1}{3}$, $\alpha = 10\%$).

Panel A: $S_{T-2} = [0000]\&[00000\,00000]$ or $[0000]\&[00001\,10000]$ with equal probabilities										
	$v = 1$					$v = 10$				
	PLB	PLB&IP: Transp. IP		PLB&IP: Opaque IP		PLB	PLB&IP: Transp. IP		PLB&IP: Opaque IP	
	$T-2$	$T-2$	$(\Delta \times 100)$	$T-2$	$(\Delta \times 100)$	$T-2$	$T-2$	$(\Delta \times 100)$	$T-2$	$(\Delta \times 100)$
LP (PLB)	.5860	.3176	(-26.84)	.2604	(-32.56)	.0952	.0512	(-4.40)	.0448	(-5.04)
LP (IP)		0	(0)	0	(0)		0	(0)	0	(0)
VL (PLB)	.4140	.2128	(-20.12)	.3544	(-5.96)	.9048	.4524	(-45.24)	.4753	(-42.95)
VL (IP)		.4696	(46.96)	.3852	(38.52)		.4964	(49.64)	.4799	(47.99)
No Trading	0	0	(0)	0	(0)	0	0	(0)	0	(0)
Spread (PLB)	.2751	.2887	(1.36)	.3000	(2.49)	.3000	.3000	(0)	.3000	(0)
Depth BBO (PLB)	.5860	.3176	(-26.84)	.2604	(-32.56)	.0952	.0512	(-4.40)	.0448	(-5.04)
Total Depth (PLB)	.5860	.3176	(-26.84)	.2604	(-32.56)	.0952	.0512	(-4.40)	.0448	(-5.04)
Panel B: $S_{T-2} = [0110]\&[00000\,00000]$ or $[0110]\&[00001\,10000]$ with equal probabilities										
	$v = 1$					$v = 10$				
	PLB	PLB&IP: Transp. IP		PLB&IP: Opaque IP		PLB	PLB&IP: Transp. IP		PLB&IP: Opaque IP	
	$T-2$	$T-2$	$(\Delta \times 100)$	$T-2$	$(\Delta \times 100)$	$T-2$	$T-2$	$(\Delta \times 100)$	$T-2$	$(\Delta \times 100)$
LP (PLB)	.1006	.0462	(-5.44)	.0499	(-5.07)	.0114	.0054	(-.60)	.0056	(-.58)
LP (IP)		.0122	(1.22)	.0120	(1.20)		.0012	(.12)	.0012	(.12)
VL (PLB)	.8994	.4429	(-45.65)	.4638	(-43.56)	.9886	.4936	(-49.50)	.4961	(-49.25)
VL (IP)		.4911	(49.11)	.4743	(47.43)		.4990	(49.90)	.4972	(49.72)
No Trading	0	.0076	(.76)	0	(0)	0	.0008	(.08)	0	(0)
Spread (PLB)	.1899	.1443	(-4.56)	.1464	(-4.35)	.1989	.1494	(-4.95)	.1496	(-4.93)
Depth BBO (PLB)	1.1006	1.5571	(45.65)	1.5362	(43.56)	1.0114	1.5064	(49.50)	1.5039	(49.25)
Total Depth (PLB)	1.2012	1.6033	(40.21)	1.5861	(38.49)	1.0228	1.5118	(48.90)	1.5040	(48.67)

Finally, our model allows us to evaluate how different tick size values affect competition for the provision of liquidity between regular and dark markets. Table 1.8 shows that when the tick size is $\tau/3$ rather than τ, all the effects of sub-penny trading decrease. A smaller tick size, in fact, reduces the profits for broker-dealers from posting orders in the IP and consequently their activity on this trading platform.

1.7 Welfare

We now turn to investigate how different tick size values can affect the welfare of market participants. Following Degryse et al. (2009) and Goettler et al. (2005), we measure total welfare (W) as the sum of the expected welfare in each period $E(W_t)$, i.e., $W = \sum_t E(W_t)$, where $E(W_t)$ is the sum of agents' expected gains from both market (first term) and limit orders (second term):

$$E(W_t) = \sum_i \int_{\beta \in \{\beta : H_t = -1^i | S_{t-1}\}} |i - \beta v| \cdot f(\beta) d\beta \\ + \sum_i \int_{\beta \in \{\beta : H_t = +1^i | S_{t-1}\}} |i - \beta v| \cdot p_t^*(i\,|S_t) \cdot f(\beta) d\beta \tag{1.13}$$

As this measure only allows us to calculate the absolute change in welfare, to obtain a relative indicator, we compute the expected gains from trade accruing to market participants in a market without frictions, $\overline{W}$. In a frictionless market a first-best allocation of resources is achieved and all orders are executed at the fundamental value of the asset:

$$\overline{W} = \sum_t E(\overline{W}_t) = \int_{\underline{\beta}}^{\overline{\beta}} |v - \beta v| \ f(\beta) \, d\beta \tag{1.14}$$

The results obtained are summarized in the following Proposition.

Proposition 3 .

- *In the single market model a smaller tick size makes traders better off when the stock is liquid and worse off when it is illiquid. Changes in traders' welfare increase with the tick to price ratio.*
- *Competition from an IP makes all traders better off, and the improvement decreases with the tick to price ratio.*

The effects on welfare of a change in the tick size depend on the state of the book (Table 1.9). As we have previously shown, when the book opens shallow, a smaller tick size worsens liquidity with the result that traders' welfare decreases. When instead the market opens deep, a smaller tick size improves the spread and makes the market deeper, thus increasing traders' gains from trade. Finally, consistently with the results obtained on market quality, the effects of a tick size reduction are stronger for low priced stocks, whereas they tend to vanish for high priced stocks.

Table 1.9 Welfare - Single Market Model ($\tau = 0.1$, $\alpha = 20\%$)**.** This Table reports the expected gains from trade over the three trading periods for both the single market model (PLB) and the market without frictions, i.e. the first best (FB). Results are reported for both the large tick market (LM), and the small tick market (SM); columns 10 to 13 report the difference in welfare between the two trading protocols (SM-LM). In addition, results are reported for four different values of the asset (v), and for three different initial states of the book, respectively with 0, 1, and 2 shares on the first level of LM (A_1, B_1) and the second level of SM (a_2, b_2).

Table 1.9 **Welfare - Single Market Model** ($\tau = 0.1$, $\alpha = 20\%$).

	LM [$Tick\ Size = \tau$]				SM [$Tick\ Size = \frac{\tau}{3}$]				($\Delta \times 100$)			
v	1	5	10	50	1	5	10	50	1	5	10	50
FB	1.5	7.5	15	75	1.5	7.5	15	75				
$S_{T-2} = [0000] = [00000\,00000]$												
PLB	1.1761	7.0841	14.5684	74.5539	1.1723	7.0837	14.5683	74.5539	-.38	-.04	-.01	-.00
$\frac{PLB-FB}{FB}$	-.2159	-.0555	-.0288	-.0059	-.2185	-.0555	-.0288	-.0059	-.38	-.04	-.01	-.00
$S_{T-2} = [0110] = [00010\,01000]$												
PLB	1.2616	7.2337	14.7295	74.7259	1.2881	7.2395	14.7324	74.7265	2.65	.58	.29	.06
$\frac{PLB-FB}{FB}$	-.1589	-.0355	-.0180	-.0037	-.1413	-.0347	-.0178	-.0036	2.65	.58	.29	.06
$S_{T-2} = [0220] = [00020\,02000]$												
PLB	1.3335	7.3268	14.8258	74.8251	1.3535	7.3313	14.8283	74.8257	2.00	.45	.25	.06
$\frac{PLB-FB}{FB}$	-.1110	-.0231	-.0116	-.0023	-.0977	-.0225	-.0114	-.0023	2.00	.45	.25	.06

Table 1.10 Welfare - PLB&IP Market ($\tau = \{0.1, \frac{0.1}{3}\}$, $\alpha = 10\%$)**.** This Table reports the expected gains from trade over the three trading periods for both the single market model (PLB) and the model in which the PLB competes with the internalization pool (PLB&IP). At $T-2$ the PLB opens either empty (Panel A), $S_{T-2} = [0000]$, or with one share on the first level of the book (Panel B), $S_{T-2} = [0110]$, while the IP opens with equal probability either empty, [0000][00000 00000], or with one unit on the first level of the book, [00001 10000]. Welfare in the PLB&IP framework is computed for the two categories of agents - broker-dealers (BD) and regular traders (RT) -, and for the two protocols in which the IP is either transparent (PLB&IP_tra) or opaque (PLB&IP_op) to regular traders. The Table also reports in square brackets the percentage difference with the welfare computed for the PLB in the single market model. In addition, results are reported for two different values of the asset, $v = \{1, 10\}$.

Table 1.10 **Welfare - PLB&IP Market** ($\tau = \{0.1, \frac{0.1}{3}\}$, $\alpha = 10\%$).

Panel A: $S_{T-2} = [0000]\&[00000\,00000]$ or $[0000]\&[00001\,10000]$ with equal probabilities $[\frac{\Delta}{PLB} \times 100]$								
v	τ	PLB	PLB&IP_tra	BD	RT	PLB&IP_op	BD	RT
1	0.1	1.1761	1.3555 [15.2538]	1.3566 [15.3473]	1.3554 [15.2453]	1.2222 [3.9197]	1.2570 [6.8787]	1.2183 [3.5881]
10	0.1	14.5684	15.9143 [9.2385]	15.9143 [9.2385]	15.9143 [9.2385]	14.6165 [.3302]	14.6797 [.7640]	14.6095 [.2821]
1	$\frac{0.1}{3}$	1.3673	1.5199 [11.1607]	1.520 [11.1680]	1.5199 [11.1607]	1.3791 [.8630]	1.4021 [2.5452]	1.3766 [.6802]
10	$\frac{0.1}{3}$	14.8522	16.1358 [8.6425]	16.1358 [8.6425]	16.1358 [8.6425]	14.8700 [.1198]	14.8904 [.2572]	14.8677 [.1044]
Panel B: $S_{T-2} = [0110]\&[00000\,00000]$ or $[0110]\&[00001\,10000]$ with equal probabilities $[\frac{\Delta}{PLB} \times 100]$								
v	τ	PLB	PLB&IP_tra	BD	RT	PLB&IP_op	BD	RT
1	0.1	1.2616	1.3276 [5.2315]	1.3315 [5.5406]	1.3272 [5.1997]	1.3365 [5.9369]	1.3305 [5.4613]	1.3372 [5.9924]
10	0.1	14.7295	14.8071 [.5268]	14.8075 [.5295]	14.8070 [.5262]	14.8195 [.6110]	14.8072 [.5275]	14.8209 [.6205]
1	$\frac{0.1}{3}$	1.4130	1.4376 [1.7410]	1.4381 [1.7764]	1.4376 [1.7410]	1.4415 [2.0170]	1.4379 [1.7622]	1.4419 [2.0453]
10	$\frac{0.1}{3}$	14.9088	14.9351 [.1764]	14.9351 [.1764]	14.9351 [.1764]	14.9393 [.2046]	14.9351 [.1764]	14.9398 [.2079]

To evaluate the effects on welfare of the opportunity to trade in an IP, we consider broker-dealers and regular traders separately. As reported in Table 1.10, competition from the IP increases total welfare for both liquid and illiquid stocks, and for both broker-dealers and regular traders. In fact, even though only broker-dealers can post limit orders to the IP, due to the existence of smart order routers, all traders can take advantage of the liquidity offered. This effect decreases with the tick size because broker-dealers have a lower incentive to provide liquidity on the IP.

1.8 Empirical implications

The equilibrium strategies derived from our model generate several empirically testable predictions. We find that the behaviour of traders crucially depends on the liquidity of the limit order book and on the price of the stock considered. Starting with the single market model, we show that a tick size reduction can improve or worsen the indicators of market quality, depending on the initial state of the book:

> **Prediction 1** When the tick size is smaller, depth worsens if the book is shallow and improves if it is deep. For the inside spread the effect is not monotonic with respect to book liquidity: this market indicator worsens for shallow books, improves for deep ones but worsens again when the book is extremely deep.

Most of the existing empirical evidence does not distinguish between liquid and illiquid stocks, and shows that when the tick size is reduced, the inside spread decreases but depth does not necessarily improve.[19] On the other hand, when stocks are classified according to their degree of liquidity, as in Bourghelle and Declerck (2004), results show that as liquidity decreases, the percentage spread increases and the quoted depth decreases. Similarly, our prediction is in line with the empirical findings of Golstein and Kavajecz (2000), who sort NYSE stocks according to their levels of liquidity and price. They show that as the tick size is reduced, there is an overall improvement of the quoted spread and depth for liquid stocks, whereas market quality deteriorates for illiquid ones. Golstein and Kavajecz also show that both effects become stronger as the stock price decreases, and that for low priced illiquid stocks the negative effect on the quoted spread and depth is the greatest. The latter results are consistent with our second prediction:

Prediction 2 Both in deep and in shallow markets, the effect of a smaller tick size gradually increases with the tick to price ratio.

Both predictions can be directly tested by sorting stocks by their price and the level of liquidity and applying an event study of tick size

[19] See Ahn et al. (1996), Bacidore (1997), Bessembinder (2003), Harris (1994), and Porter and Weaver (1997).

change. The key is to select a sample of stocks with enough independent variation in liquidity and in price.

When we extend the analysis to a framework which includes inter-market competition between a primary market and an IP, we obtain additional testable implications. The endogenous strategic interaction between regular traders and more sophisticated broker-dealers having access to dark platforms generates distinctive empirical patterns for liquidity supply and liquidity demand. More precisely, competition from an IP affects traders' desire to supply liquidity which generates empirical predictions about the dynamics of order flows. These again depend on the stocks' characteristics.

> **Prediction 3** When the primary market is deep, the IP attracts market orders, thus preserving liquidity and ultimately enhancing depth and spread on the PLB. When the primary market is shallow, competition from the IP deters liquidity provision on the PLB where depth and spread worsen. All these effects decrease as the tick size gets smaller.

> **Prediction 4** For illiquid stocks, a high tick to price ratio amplifies the effects on market quality of competition from IPs, while the opposite holds for liquid stocks.

These theoretical findings are directly testable either cross-sectionally, by selecting stocks with different degrees of liquidity and price levels; or by using time series of order data. A final empirical implication of our model is that:

Prediction 5 Broker-dealers' activity on IPs is more intensive on liquid and low priced stocks.

This is an interesting topic for future research, when data on dark markets becomes available.

1.9 Policy Discussion and Conclusions

Today's financial markets are characterized by extensive intermarket competition, both between fully transparent venues and between transparent and dark pools of liquidity. In such a world, the regulation of the minimum price increment is extremely important, as it defines how markets compete for liquidity. In this paper, we analyze the impact of the tick size for market quality and traders' welfare in a public limit order market. We also analyze how the entry of a dark venue with a finer price grid (trading in sub-pennies) affects traders' strategies, and therefore results in changes in market quality and traders' welfare.

First of all, we discuss the effects of a smaller tick size within the context of a limit order book and show that these effects depend on the liquidity of the stock and on the underlying value of the asset. In this respect, the model's results show that for illiquid stocks market quality worsens with a smaller tick size, whereas for liquid stocks it improves. Such effects are relevant for low priced stocks, whereas they vanish as the price of the security increases. These results suggest that the objective of tick size regulation should be the definition of a minimum price change that is consistent with the stock's main attributes and should be related to its liquidity and price.

The model is then extended to include competition from an IP. This dual-market model accounts for the changing nature of liquidity provision around the world, which is now dominated by competition among regular and dark markets, and in which a growing number of dark pools allows broker-dealers to post limit orders at fractions of the tick size. This extension is well suited to investigate the issue of sub-penny trading, which is one of the main concerns addressed by the SEC in the April 2010 concept release on Equity Market Structure. Our model suggests that sub-penny trading undertaken by broker-dealers on IPs can have negative effects on the market quality for illiquid stocks, while it is beneficial to the quality of the market for liquid stocks.

By drawing on our model we can also comment the conclusions of the SEC Report on Decimalization (2012), which was required by the JOBS Act (2012) and which states that "the Commission should not proceed with the specific rulemaking to increase the tick size." The JOBS Act advised the SEC to consider the possibility of increasing the tick size for low priced stocks and of eventually undertaking a pilot program, under which some small and mid-cap stocks would trade at increments greater than a penny. The objective of this proposal was to restore the economic incentive to support liquidity for these stocks, that over time had experienced a reduction of the inside spread. Our model reveals that a greater tick size for illiquid low-price stocks could on the one hand stimulate competition for the provision of liquidity and result in smaller rather than wider spreads, while on the other it could lead to an increase in broker-dealers profits from sub-penny trading. According to our results, it would therefore appear that an increase of the tick size would not necessarily restore the interest of liquidity providers for these stocks.

Appendix A

A.1 Proof of Lemma 1

Proof. At any period $t \neq T$, a trader selects his optimal strategy H_t^* by comparing the payoffs of all the strategies described in Table 1.2, $H_t = \{-1^B, +1^{A_k}, +1^{B_k}, -1^A, 0\}$. Assume that $p_t^*(A_k^{N_{-k}, N_k}) > 0$.[20] We compute the threshold $\beta_{-1^B,0}$ between $H_t = -1^B$ and $H_t = 0$ by equating the profits from the two strategies $(B - \beta v = 0)$ and we obtain $\beta_{-1^B,0} = \frac{B}{v}$. Similarly, we compute the threshold between $H_t = -1^B$

[20] To simplify the notation, we omit to write that all the execution probabilities are conditional on S_t.

and $H_t = +1^{A_k}$ and obtain $\beta_{-1^B,1^{A_k}} = \frac{B}{v} - \frac{p_t^*(A_k^{N_{-k},N_k})}{1-p_t^*(A_k^{N_{-k},N_k})} \cdot \frac{A_k - B}{v}$. Under the assumption that $p_t^*(A_k^{N_{-k},N_k}) > 0$, then $\beta_{-1^B,1^{A_k}} < \beta_{-1^B,0}$ and hence there always exists a value for $\beta \in (\beta_{-1^B,1^{A_k}}, \beta_{-1^B,0})$ such that $H_t^*(\beta, S_{t-1}) = +1^{A_k}$. A similar result holds for the bid side. Clearly, traders' equilibrium β-thresholds, and hence strategies, crucially depend on the state of the book which affects the execution probability of limit orders. There are four possible scenarios. When there is room for limit orders on both sides of the market, equilibrium traders' strategies for $t \neq T$ are:

$$H_t^*(\beta, S_{t-1}) = \begin{cases} -1^B & \text{if } \beta \in [\underline{\beta}, \beta_1) \\ +1^{A_k} & \text{if } \beta \in [\beta_1, \beta_3) \\ +1^{B_k} & \text{if } \beta \in [\beta_3, \beta_5) \\ -1^A & \text{if } \beta \in [\beta_5, \overline{\beta}] \end{cases} \quad \text{if} \quad \begin{matrix} p_t^*(A_k^{N_{-k},N_k}) \neq 0 \\ \& \; p_t^*(B_k^{M_{-k},M_k}) \neq 0 \end{matrix} \tag{1.15a}$$

where $\beta_1 = \frac{B}{v} - \frac{p_t^*(A_k^{N_{-k},N_k})}{1-p_t^*(A_k^{N_{-k},N_k})} \cdot \frac{A_k - B}{v}$, $\beta_3 = \frac{p_t^*(A_k^{N_{-k},N_k})A_k + p_t(B_k^{M_{-k},M_k})B_k}{p_t^*(A_k^{N_{-k},N_k}) + p_t(B_k^{M_{-k},M_k})} \cdot \frac{1}{v}$ and $\beta_5 = \frac{A}{v} + \frac{p_t^*(B_k^{M_{-k},M_k})}{1-p_t^*(B_k^{M_{-k},M_k})} \cdot \frac{A - B_k}{v}$. When instead the book opens full either on the ask or on the bid side, the equilibrium strategies are respectively:

$$H_t^*(\beta, S_{t-1}) = \begin{cases} -1^B & \text{if } \beta \in [\underline{\beta}, \beta_1) \\ 0 & \text{if } \beta \in [\beta_1, \beta_4) \\ +1^{B_k} & \text{if } \beta \in [\beta_4, \beta_5) \\ -1^A & \text{if } \beta \in [\beta_5, \overline{\beta}] \end{cases} \quad \text{if} \quad \begin{array}{c} p_t^*(A_k^{N_{-k}, N_k}) = 0 \\ \& \ p_t^*(B_k^{M_{-k}, M_k}) \neq 0 \end{array} \tag{1.15b}$$

$$H_t^*(\beta, S_{t-1}) = \begin{cases} -1^B & \text{if } \beta \in [\underline{\beta}, \beta_1) \\ +1^{A_k} & \text{if } \beta \in [\beta_1, \beta_2) \\ 0 & \text{if } \beta \in [\beta_2, \beta_5) \\ -1^A & \text{if } \beta \in [\beta_5, \overline{\beta}] \end{cases} \quad \text{if} \quad \begin{array}{c} p_t^*(A_k^{N_{-k}, N_k}) \neq 0 \\ \& \ p_t^*(B_k^{M_{-k}, M_k}) = 0 \end{array} \tag{1.15c}$$

where $\beta_2 = \frac{A_k}{v}$ and $\beta_4 = \frac{B_k}{v}$. Finally, when the book is full on both sides, equilibrium strategies are:

$$H_t^*(\beta, S_{t-1}) = \begin{cases} -1^B & \text{if } \beta \in [\underline{\beta}, \beta_1) \\ 0 & \text{if } \beta \in [\beta_1, \beta_5) \\ -1^A & \text{if } \beta \in [\beta_5, \overline{\beta}] \end{cases} \quad \text{if} \quad \begin{array}{c} p_t^*(A_k^{N_{-k}, N_k}) = 0 \\ \& \ p_t^*(B_k^{M_{-k}, M_k}) = 0 \end{array} \tag{1.15d}$$

If $p_t^*(A_k^{N_{-k}, N_k}) = p_t^*(B_k^{M_{-k}, M_k}) = 0$, then $H_t^*(\beta, S_{t-1}) = H_T^*(\beta, S_{t-1})$. ■

A.2 Proof of Lemma 2

Proof. As an example, we consider the ask side in period $T-1$; the other cases can be derived in a similar way. At $T-1$ limit orders have a positive execution probability on both A_1 and A_2 only when the book opens empty on both sides, $S_{T-1} = [0, 0, 0, 0]$.[21] In this case traders can optimally select their level of price aggressiveness. Profits from the two available limit order strategies are:

$$H_{T-1} = +1^{A_2} \quad : \quad (A_2 - \beta v) \cdot p^*_{T-1}(A_2 \,|[1000]\,) = (A_2 - \beta v) \cdot \frac{\overline{\beta} v - A_2}{(\overline{\beta} - \underline{\beta})v}$$
$$H_{T-1} = +1^{A_1} \quad : \quad (A_1 - \beta v) \cdot p^*_{T-1}(A_1 \,|[0100]\,) = (A_1 - \beta v) \cdot \frac{\overline{\beta} v - A_1}{(\overline{\beta} - \underline{\beta})v}$$

A limit order at A_1 is optimal if $\exists \beta$ such that $(A_1 - \beta v) \cdot p^*_{T-1}(A_1 \,|[0100]\,) > \max\{B_2 - \beta v,\ (A_2 - \beta v) \cdot p^*_{T-1}(A_2 \,|[1000]\,)\}$; in this case the threshold between $H_{T-1} = -1^B$ and $H_{T-1} = +1^{A_1}$ is smaller than the threshold between $H_{T-1} = -1^B$ and $H_{T-1} = +1^{A_2}$. Specifically, as $\beta_{-1^B, 1^{A_k}} = \beta_1 \,|_{B=B_2, A_k} = \frac{B_2}{v} - \frac{p^*_{T-1}(A_k|S_t\,)}{1 - p^*_{T-1}(A_k|S_t\,)} \cdot \frac{A_k - B_2}{v}$, in order for $\beta_{-1^B, 1^{A_1}} < \beta_{-1^B, 1^{A_2}}$ the lower selling price (by one tick, τ) must be compensated by a higher execution probability. As $p^*_{T-1}(A_1 \,|[0100]\,) - p^*_{T-1}(A_2 \,|[1000]\,) = \frac{\overline{\beta} v - A_1}{(\overline{\beta} - \underline{\beta})v} - \frac{\overline{\beta} v - A_2}{(\overline{\beta} - \underline{\beta})v} = \frac{\tau}{(\overline{\beta} - \underline{\beta})v}$ is an increasing function of the relative tick size, for $H_{T-1} = +1^{A_1}$ to be an optimal strategy, $\frac{\tau}{v}$ must be larger than $\widehat{\frac{\tau}{v}}$, where $\widehat{\frac{\tau}{v}}$ solves $(A_1 - \beta v) \cdot \frac{\overline{\beta} v - A_1}{(\overline{\beta} - \underline{\beta})v} - (A_2 - \beta v) \cdot \frac{\overline{\beta} v - A_2}{(\overline{\beta} - \underline{\beta})v} = 0$. ■

A.3 Proof of Proposition 1

Proof. We consider the illiquid stocks for which the book opens at $T-2$ as [0000]. We analyze first a LM where at $T-2$ traders' strat-

[21] From here onwards, to simplify the notation, we omit commas, i.e., [0000] represents an empty limit order book.

egy space is $\{-1^B, +1^{A_2}, +1^{A_1}, +1^{B_1}, +1^{B_2}, -1^A, 0\}$; each strategy corresponds to an opening book at $T-1$ equal to [0000], [1000], [0100], [0010], [0001], and [0000] respectively. Consider the book that opens at $T-1$ with one share on A_2, i.e., [1000]. Traders' strategy space is $\{-1^B, +1^{A_1}, +1^{B_1}, +1^{B_2}, -1^A, 0\}$ and the corresponding payoffs are:

$H_{T-1} = -1^B$	:	$B_2 - \beta v$
$H_{T-1} = +1^{A_1}$	:	$(A_1 - \beta v) \cdot p^*_{T-1}(A_1 \,\vert [1100])$
$H_{T-1} = +1^{B_1}$	:	$(\beta v - B_1) \cdot p^*_{T-1}(B_1 \,\vert [1010])$
$H_{T-1} = +1^{B_2}$	:	$(\beta v - B_2) \cdot p^*_{T-1}(B_2 \,\vert [1001])$
$H_{T-1} = -1^A$	:	$\beta v - A_2$
$H_{T-1} = 0$	:	0

where the execution probabilities are given by (1.6) and (1.7). After comparing these payoffs, we obtain the equilibrium strategies at $T-1$ that depend on the relative tick size $\frac{\tau}{v}$, as shown in Lemma 2. If the relative tick size is small such that traders are not aggressive and post limit orders at higher levels of the book, the equilibrium strategies at $T-1$ are given by Eq. (1.8):

$$H^{*LM}_{T-1}(\beta,[1000]) = \begin{cases} -1^{B} & \text{if } \beta \in [\underline{\beta}, \beta_{1,T-1}\,|_{A_1,B=B_2}) \\ +1^{A_1} & \text{if } \beta \in [\beta_{1,T-1}\,|_{A_1,B=B_2}, \beta_{3,T-1}\,|_{A_1,B_2}) \\ +1^{B_2} & \text{if } \beta \in [\beta_{3,T-1}\,|_{A_1,B_2}, \beta_{5,T-1}\,|_{A=A_2,B_2}) \\ -1^{A} & \text{if } \beta \in [\beta_{5,T-1}\,|_{A=A_2,B_2}, \overline{\beta}] \end{cases}$$

where $\beta_{1,T-1}\,|_{A_1,B=B_2} = \frac{B_2}{v} - \frac{p^*_{T-1}(A_1|[1100])}{1-p^*_{T-1}(A_1|[1100])} \cdot \frac{A_1-B_2}{v}$, $\beta_{3,T-1}\,|_{A_1,B_2} = \frac{p^*_{T-1}(A_1|[1100])A_1+p^*_{T-1}(B_2|[1001])B_2}{p^*_{T-1}(A_1|[1100])+p^*_{T-1}(B_2|[1001])} \cdot \frac{1}{v}$

and $\beta_{5,T-1}\,|_{A=A_2,B_2} = \frac{A_2}{v} + \frac{p^*_{T-1}(B_2|[1001])}{1-p^*_{T-1}(B_2|[1001])} \cdot \frac{A_2-B_2}{v}$. This allows us to compute the execution probability of the strategy $H_{T-2} = +1^{A_2}$ that turns the book into [1000] at $T-1$:

$$\begin{aligned} p^*_{T-2}(A_2\,|[1000]) \;=\; & \frac{\overline{\beta}-\beta_{5,T-1}\big|_{A=A_2,B_2}}{\overline{\beta}-\underline{\beta}} \\ & + \frac{\beta_{5,T-1}\big|_{A=A_2,B_2}-\beta_{3,T-1}\big|_{A_1,B_2}}{\overline{\beta}-\underline{\beta}} \cdot p^*_{T-1}(A_2\,|[1001]) \\ & + \frac{\beta_{1,T-1}\big|_{A_1,B=B_2}-\underline{\beta}}{\overline{\beta}-\underline{\beta}} \cdot p^*_{T-1}(A_2\,|[1000]) \end{aligned}$$

The first term is the execution probability at $T-1$, whereas the other two represent the execution probability at T. Similarly, we compute the execution probabilities for the other order types to compare all possible trader's payoffs at $T-2$:

$H_{T-2} = -1^B$	:	$B_2 - \beta v$
$H_{T-2} = +1^{A_2}$	:	$(A_2 - \beta v) \cdot p^*_{T-2}(A_2 \,\vert [1000]\,)$
$H_{T-2} = +1^{A_1}$	:	$(A_1 - \beta v) \cdot p^*_{T-2}(A_1 \,\vert [0100]\,)$
$H_{T-2} = +1^{B_1}$	:	$(\beta v - B_1) \cdot p^*_{T-2}(B_1 \,\vert [0010]\,)$
$H_{T-2} = +1^{B_2}$	:	$(\beta v - B_2) \cdot p^*_{T-2}(B_2 \,\vert [0001]\,)$
$H_{T-2} = -1^A$	:	$\beta v - A_2$
$H_{T-2} = 0$	:	0

Hence equilibrium strategies at $T-2$ are:

$$H^{*LM}_{T-2}(\beta, [0000]) = \begin{cases} -1^B & \text{if } \beta \in [\underline{\beta}, \beta_{1,T-2}\,|_{A_2,B=B_2}\,) \\ +1^{A_2} & \text{if } \beta \in [\beta_{1,T-2}\,|_{A_2,B=B_2}\,, \beta_{3,T-2}\,|_{A_2,B_2}\,) \\ +1^{B_2} & \text{if } \beta \in [\beta_{3,T-2}\,|_{A_2,B_2}\,, \beta_{5,T-2}\,|_{A=A_2,B_2}\,) \\ -1^A & \text{if } \beta \in [\beta_{5,T-2}\,|_{A=A_2,B_2}\,, \overline{\beta}] \end{cases}$$

where $\beta_{1,T-2}\,|_{A_2,B=B_2} = \frac{B}{v} - \frac{p^*_{T-2}(A_2|[1000]\,)}{1-p^*_{T-2}(A_2|[1000]\,)} \cdot \frac{A_2-B}{v}$, $\beta_{3,T-2}\,|_{A_2,B_2} = \frac{p^*_{T-2}(A_2|[1000]\,)A_2+p^*_{T-2}(B_2|[0001]\,)B_2}{p^*_{T-2}(A_2|[1000]\,)+p^*_{T-2}(B_2|[0001]\,)} \cdot \frac{1}{v}$, $\beta_{5,T-2}\,|_{A=A_2,B_2} = \frac{A}{v} + \frac{p^*_{T-2}(B_2|[0001]\,)}{1-p^*_{T-2}(B_2|[0001]\,)} \cdot \frac{A-B_2}{v}$.

When instead $\frac{\tau}{v}$ is large, submitting limit orders on the first level of the book becomes an optimal strategy, and the equilibrium strategies at

$T-2$ are derived from a modified version of Eq. (1.15a):

$$H^{*LM}_{T-2}(\beta,[0000]) = \begin{cases} -1^{B} & \text{if } \beta \in [\underline{\beta}, \beta_{1,T-2}\,|_{A_1,B=B_2}) \\ +1^{A_1} & \text{if } \beta \in [\beta_{1,T-2}\,|_{A_1,B=B_2}, \beta_{6,T-2}\,|_{A_1,A_2}) \\ +1^{A_2} & \text{if } \beta \in [\beta_{6,T-2}\,|_{A_1,A_2}, \beta_{3,T-2}\,|_{A_2,B_2}) \\ +1^{B_2} & \text{if } \beta \in [\beta_{3,T-2}\,|_{A_2,B_2}, \beta_{7,T-2}\,|_{B_1,B_2}) \\ +1^{B_1} & \text{if } \beta \in [\beta_{7,T-2}\,|_{B_1,B_2}, \beta_{5,T-2}\,|_{A=A_2,B_1}) \\ -1^{A} & \text{if } \beta \in [\beta_{5,T-2}\,|_{A=A_2,B_1}, \overline{\beta}] \end{cases}$$

where $\beta_{6,T-2}\,|_{A_1,A_2} = \frac{p^*_{T-2}(A_1|[0100])A_1 - p^*_{T-2}(A_2|[1000])A_2}{p^*_{T-2}(A_1|[0100]) - p^*_{T-2}(A_2|[1000])} \cdot \frac{1}{v}$, $\beta_{7,T-2}\,|_{B_1,B_2} = \frac{p^*_{T-2}(B_1|[0010])B_1 - p^*_{T-2}(B_2|[0001])B_2}{p^*_{T-2}(B_1|[0010]) - p^*_{T-2}(B_2|[0001])} \cdot \frac{1}{v}$.

We solve the same problem for the SM and directly present the equilibrium strategies at $T-2$ for the case with an empty book, that we indicate with $S_t = [0]$, and a small relative tick size value, $\frac{\tau}{v}$:

$$H^{*SM}_{T-2}(\beta,[0]) = \begin{cases} -1^{b} & \text{if } \beta \in [\underline{\beta}, \beta_{1,T-2}\,|_{a_5,b=b_5}) \\ +1^{a_5} & \text{if } \beta \in [\beta_{1,T-2}\,|_{a_5,b=b_5}, \beta_{3,T-2}\,|_{a_5,b_5}) \\ +1^{b_5} & \text{if } \beta \in [\beta_{3,T-2}\,|_{a_5,b_5}, \beta_{5,T-2}\,|_{a=a_5,b_5}) \\ -1^{a} & \text{if } \beta \in [\beta_{5,T-2}\,|_{a=a_5,b_5}, \overline{\beta}] \end{cases}$$

When instead $\frac{\tau}{v}$ is large, the equilibrium strategies at $T-2$ are modified as follows:

$$H^{*SM}_{T-2}(\beta,[0]) = \begin{cases} -1^{b} & \text{if } \beta \in [\underline{\beta}, \beta_{1,T-2}\,|_{a_1,b=b_5}) \\ +1^{a_1} & \text{if } \beta \in [\beta_{1,T-2}\,|_{a_1,b=b_5}, \beta_{6,T-2}\,|_{a_1,a_5}) \\ +1^{a_5} & \text{if } \beta \in [\beta_{6,T-2}\,|_{a_1,a_5}, \beta_{3,T-2}\,|_{a_5,b_5}) \\ +1^{b_5} & \text{if } \beta \in [\beta_{3,T-2}\,|_{a_5,b_5}, \beta_{7,T-2}\,|_{b_1,b_5}) \\ +1^{b_1} & \text{if } \beta \in [\beta_{7,T-2}\,|_{b_1,b_5}, \beta_{5,T-2}\,|_{a=a_5,b_1}) \\ -1^{a} & \text{if } \beta \in [\beta_{5,T-2}\,|_{a=a_5,b_1}, \overline{\beta}] \end{cases}$$

To compare the LM with the SM in terms of liquidity provision and executed volume, it is sufficient to compare the thresholds that make the trader indifferent between submitting a market order and the most attractive among the available limit order strategies:

$$\widehat{\beta}^{LM}_{5,T-2}\,|_{A=A_2,B_k} = \max_{B_k} \beta_{5,T-2}\,|_{A=A_2,B_k} = \max_{B_k}\{\frac{A_2}{v} + \frac{p^*_{T-1}(B_k|S_{T-1})}{1-p^*_{T-1}(B_k|S_{T-1})} \cdot \frac{A_2-B_k}{v}\}$$

$$\widehat{\beta}^{SM}_{5,T-2}\,|_{a=a_5,b_l} = \max_{b_l} \beta_{5,T-2}\,|_{a=a_5,b_l} = \max_{b_l}\{\frac{a_5}{v} + \frac{p^*_{T-1}(b_l|S_{T-1})}{1-p^*_{T-1}(b_l|S_{T-1})} \cdot \frac{a_5-b_l}{v}\}$$

After substituting the equilibrium execution probabilities, we find that for a market buy order $\widehat{\beta}^{SM}_{5,T-2}\,|_{a=a_5,b_l} < \widehat{\beta}^{LM}_{5,T-2}\,|_{A=A_2,B_k}$, and for a market sell order $\widehat{\beta}^{SM}_{1,T-2}\,|_{a_l,b=b_5} > \widehat{\beta}^{LM}_{1,T-2}\,|_{A_k,B=B_2}$. Consequently, volume is higher in the SM:

$$
\begin{aligned}
VL_{T-2}^{SM} &= \frac{\overline{\beta}-\widehat{\beta}_{5,T-2}^{SM}\big|_{a=a_5,b_l}}{\overline{\beta}-\underline{\beta}} + \frac{\widehat{\beta}_{1,T-2}^{SM}\big|_{a_l,b=b_5}-\underline{\beta}}{\overline{\beta}-\underline{\beta}} \\
&> \frac{\overline{\beta}-\widehat{\beta}_{5,T-2}^{LM}\big|_{A=A_2,B_k}}{\overline{\beta}-\underline{\beta}} + \frac{\widehat{\beta}_{1,T-2}^{LM}\big|_{A_k,B=B_2}-\underline{\beta}}{\overline{\beta}-\underline{\beta}} = VL_{T-2}^{LM}
\end{aligned}
$$

When the book opens empty at $T-2$, no trading ($H_{T-2}=0$) is never optimal and hence in this single market model the submission probabilities of market and limit orders are complements. Thus, as $VL_{T-2}^{LM} < VL_{T-2}^{SM}$, we obtain that $LP_{T-2}^{LM} > LP_{T-2}^{SM}$. Furthermore, inside depth is equal to both total depth and liquidity provision, $DPI_{T-2} = DPT_{T-2} = LP_{T-2}$, where:

$$
LP_{T-2} = \frac{\widehat{\beta}_{5,T-2}-\widehat{\beta}_{1,T-2}}{\overline{\beta}-\underline{\beta}}
$$

Consequently, also total and inside depth are lower in the SM: $DPI_{T-2}^{LM} = DPT_{T-2}^{LM} > DPI_{T-2}^{SM} = DPT_{T-2}^{SM}$. Finally, in order to compute the spread, we consider two cases. When $\frac{\tau}{v}$ is large, we obtain:

$$
\begin{aligned}
SP_{T-2}^{LM} &= E[A-B] \\
&= 3\tau\text{-}\left(\frac{\beta_{6,T-2}\big|_{A_1,A_2}-\beta_{1,T-2}\big|_{A_2,B=B_2}}{\overline{\beta}-\underline{\beta}} + \frac{\beta_{5,T-2}\big|_{A=A_2,B_2}-\beta_{7,T-2}\big|_{B_1,B_2}}{\overline{\beta}-\underline{\beta}}\right)\cdot\tau \\
&< 3\tau\text{-}\left(\frac{\beta_{6,T-2}\big|_{a_1,a_5}-\beta_{1.T-2}\big|_{a_1,b=b_5}}{\overline{\beta}-\underline{\beta}} + \frac{\beta_{5,T-2}\big|_{a=a_5,b_1}-\beta_{7,T-2}\big|_{b_1,b_5}}{\overline{\beta}-\underline{\beta}}\right)\cdot\frac{4\tau}{3} \\
&= SP_{T-2}^{SM}
\end{aligned}
$$

where, for example, $\frac{\beta_{6,T-2}\big|_{A_1,A_2}-\beta_{1,T-2}\big|_{A_2,B=B_2}}{\overline{\beta}-\underline{\beta}}$ is the probability of a limit sell order posted at A_1 and $\frac{\beta_{5,T-2}|_{A=A_2,B_2} - \beta_{7,T-2}|_{B_1,B_2}}{\overline{\beta}-\underline{\beta}}$ is the probability of a limit buy order posted at B_1.When instead $\frac{\tau}{v}$ is small, so that traders in equilibrium post limit orders only at $A_2 = a_5$ and $B_2 = b_5$, we obtain that $SP^{LM}_{T-2} = SP^{SM}_{T-2} = 3\tau$.

The proofs for liquid, [0110], and very liquid stocks, [0220], are obtained following the same procedure and are available from the authors upon request. ■

A.4 Proof of Proposition 2

Proof. We provide a proof for liquid stocks, i.e., a PLB that opens at $T-2$ as [0110]. The proof for illiquid stocks follows a similar procedure and is available from the authors upon request.

1) Liquid stocks, transparent IP

When the IP is transparent, regular traders' (RT) equilibrium strategies depend on the initial state of the IP. We consider two cases: an empty IP, [0110]&[0], and an IP with one share on the first level of the book, [0110]&[1]. To obtain market quality indicators that are comparable with the opaque IP case, we take the average of the values obtained for the two cases.

(1.1) When the opening book is [0110]&[0], at $T-2$ the overall trader's strategy space, considering both broker-dealers (BD) and RT, is

$\{-1^B, +1^i, +1^j, -1^A, 0\}$ with $i = A_{1:2}$ and $B_{1:2}$, $j = a_{1:5}$ and $b_{1:5}$. Therefore, at the beginning of $T-1$ there are 17 possible states of the books: one share added to the i-th level of the PLB and no shares added to the IP (4 cases), one share added to the j-th level of the IP and no shares added to the PLB (10 cases), one share taken from the PLB (2 cases) or no trading (1 case). We compute the optimal equilibrium strategy for different types of traders conditional on each case. Because at T all traders can observe the best available price, the equilibrium strategies of RT and BD are the same (market orders only). As a result the orders' execution probabilities at $T-1$ do not depend on the type of trader arriving at T, for example: $p^{*RT}_{T-1}(A_k|S^{PLB}_{T-1}, S^{IP}_{T-1}) = p^{*BD}_{T-1}(A_k|S^{PLB}_{T-1}, S^{IP}_{T-1}) = p^{*}_{T-1}(A_k|S^{PLB}_{T-1}, S^{IP}_{T-1})$. Suppose $H^*_{T-2} = +1^{A_2}$ so that at $T-1$ the book opens [1110]&[0]. If a RT arrives, his payoffs are:

$H_{T-1} = -1^B$	:	$B_1 - \beta v$
$H_{T-1} = -1^A$	:	$\beta v - A_1$
$H_{T-1} = 0$	:	0

and his equilibrium strategies are:

$$H^{*RT}_{T-1}(\beta, [1110]\&[0]) = \begin{cases} -1^B & \text{if } \beta \in [\underline{\beta}, \beta^{RT}_{1,T-1}|_{B=B_1}) \\ 0 & \text{if } \beta \in [\beta^{RT}_{1,T-1}|_{B=B_1}, \beta^{RT}_{5,T-1}|_{A=A_1}) \\ -1^A & \text{if } \beta \in [\beta^{RT}_{5,T-1}|_{A=A_1}, \overline{\beta}] \end{cases}$$

By using the optimal β thresholds associated with these strategies, we compute the execution probability $H_{T-2} = +1^{A_2}$ conditional on a RT arriving at $T-1$:

$$p^{*RT}_{T-2}(A_2\,|[1110]\&[0]\,) = \frac{\overline{\beta}-\beta^{RT}_{5,T-1}\big|_{A=A_1}}{\overline{\beta}-\underline{\beta}}\cdot p^{*}_{T-1}(A)\,|_{A=A_2}$$

If instead a BD arrives at $T-1$, his payoffs are:

$H_{T-1} = -1^{B}$	:	$B_1 - \beta v$
$H_{T-1} = +1^{a_l}$	:	$(a_l - \beta v)\cdot p^{*}_{T-1}(a_l\vert S^{PLB}_{T-1}, S^{IP}_{T-1})$
$H_{T-1} = +1^{b_l}$	:	$(\beta v - b_l)\cdot p^{*}_{T-1}(b_l\vert S^{PLB}_{T-1}, S^{IP}_{T-1})$
$H_{T-1} = -1^{A}$	:	$\beta v - A_1$
$H_{T-1} = 0$	:	0

and his equilibrium strategies are:

$$H^{*BD}_{T-1}(\beta,[1110]\&[0]) = \begin{cases} -1^{B} & \text{if } \beta\in[\underline{\beta},\beta^{BD}_{1,T-1}\,|_{a_1,B=B_1}\,) \\ +1^{a_1} & \text{if } \beta\in[\beta^{BD}_{1,T-1}\,|_{a_1,B=B_1}\,,\beta^{BD}_{2,T-1}\,|_{a_1,a_2}\,) \\ +1^{a_2} & \text{if } \beta\in[\beta^{BD}_{2,T-1}\,|_{a_1,a_2}\,,\beta^{BD}_{3,T-1}\,|_{a_2,b_2}\,) \\ +1^{b_2} & \text{if } \beta\in[\beta^{BD}_{3,T-1}\,|_{a_2,b_2}\,,\beta^{BD}_{4,T-1}\,|_{b_1,b_2}\,) \\ +1^{b_1} & \text{if } \beta\in[\beta^{BD}_{4,T-1}\,|_{b_1,b_2}\,,\beta^{BD}_{5,T-1}\,|_{A=A_1,b_1}\,) \\ -1^{A} & \text{if } \beta\in[\beta^{BD}_{5,T-1}\,|_{A=A_1,b_1}\,,\overline{\beta}] \end{cases}$$

It follows that, conditional on a BD arriving at $T-1$, the execution probability of the limit order posted at A_2 is:

$$p^{*BD}_{T-2}(A_2\,|[1110]\&[0]\,) = \frac{\overline{\beta}-\beta^{BD}_{5,T-1}\big|_{A=A_1,b_1}}{\overline{\beta}-\underline{\beta}} \cdot p^{*}_{T-1}(A)\,|_{A=A_2}$$

We compute the total execution probability of the limit order posted at A_2 at $T-2$ as the weighted average of the two conditional probabilities:

$$p^{*}_{T-2}(A_2\,|[0110]\&[0]\,) = \alpha p^{*BD}_{T-2}(A_2\,|[1110]\&[0]\,)+(1-\alpha)p^{*RT}_{T-2}(A_2\,|[1110]\&[0]\,)$$

Similarly, we compute the equilibrium strategies for all the other possible states of the book at $T-1$ and obtain the execution probabilities of the different order types available at $T-2$.

If a RT arrives at $T-2$, his strategy space is $\{-1^{B}, +1^{A_2}, +1^{A_1}, +1^{B_1}, +1^{B_2}, -1^{A}, 0\}$. His payoffs are:

$H_{T-2} = -1^{B}$	:	$B_2 - \beta v$
$H_{T-2} = +1^{A_k}$	:	$(A_k - \beta v) \cdot p^{*}_{T-2}(A_k\|[0110]\&[0])$
$H_{T-2} = +1^{B_k}$	:	$(\beta v - B_k) \cdot p^{*}_{T-2}(B_k\|[0110]\&[0])$
$H_{T-2} = -1^{A}$	:	$\beta v - A_2$
$H_{T-2} = 0$	:	0

where, for example, $p^{*}_{T-2}(A_1|[0110]\&[0]) = \alpha p^{*RT}_{T-2}(A_1\,|[0210]\&[0]\,) + (1-\alpha)p^{*BD}_{T-2}(A_1\,|[0210]\&[0]\,)$. By comparing these payoffs, we obtain:

$$H^{*RT}_{T-2}(\beta,[0110]\&[0]) = \begin{cases} -1^{B} & \text{if } \beta \in [\underline{\beta}, \beta^{RT}_{1,T-2}\,|_{A_2,B=B_1}\,) \\ +1^{A_2} & \text{if } \beta \in [\beta^{RT}_{1,T-2}\,|_{A_2,B=B_1}\,, \beta^{RT}_{3,T-2}\,|_{A_2,B_2}\,) \\ +1^{B_2} & \text{if } \beta \in [\beta^{RT}_{3,T-2}\,|_{A_2,B_2}\,, \beta^{RT}_{5,T-2}\,|_{A=A_1,B_2}\,) \\ -1^{A} & \text{if } \beta \in [\beta^{RT}_{5,T-2}\,|_{A=A_1,B_2}\,, \overline{\beta}] \end{cases}$$

If instead a BD arrives at $T-2$, he also selects a trading venue. Thus his strategy space is $\{-1^B, +1^i, +1^j, -1^A, 0\}$ with $i = A_{1:2}$ and $B_{1:2}$, $j = a_{1:5}$ and $b_{1:5}$. His payoffs are:

$H_{T-2} = -1^B$	:	$B_2 - \beta v$
$H_{T-2} = +1^{A_k}$	:	$(A_k - \beta v) \cdot p^*_{T-2}(A_k\|[0110]\&[0])$
$H_{T-2} = +1^{a_l}$	:	$(a_l - \beta v) \cdot p^*_{T-2}(a_l\|[0110]\&[0])$
$H_{T-2} = +1^{b_l}$	:	$(\beta v - b_l) \cdot p^*_{T-2}(b_l\|[0110]\&[0])$
$H_{T-2} = +1^{B_k}$	:	$(\beta v - B_k) \cdot p^*_{T-2}(B_i\|[0110]\&[0])$
$H_{T-2} = -1^A$	:	$\beta v - A_2$
$H_{T-2} = 0$	:	0

where for example $p^*_{T-2}(a_l|[0110]\&[0]) = \alpha p^{*BD}_{T-2}(a_l\,|[0110]\&Q^{a_l} = 1\,) + (1-\alpha)p^{*RT}_{T-2}(a_l\,|[0110]\&Q^{a_l} = 1\,)$. If he submits a limit order to the PLB, his order's execution probability is the same as the one of RT. His equilibrium strategies are:

$$H^{*BD}_{T-2}(\beta,[0110]\&[0]) = \begin{cases} -1^{B} & \text{if } \beta \in [\underline{\beta}, \beta^{BD}_{1,T-2}\,|_{a_1,B=B_1}) \\ +1^{a_1} & \text{if } \beta \in [\beta^{BD}_{1,T-2}\,|_{a_1,B=B_1}\ ,\beta^{BD}_{2,T-2}\,|_{a_1,a_2}) \\ +1^{a_2} & \text{if } \beta \in [\beta^{BD}_{2,T-2}\,|_{a_1,a_2}\ ,\beta^{BD}_{2,T-2}\,|_{a_2,A_2}) \\ +1^{A_2} & \text{if } \beta \in [\beta^{BD}_{2,T-2}\,|_{a_2,A_2}\ ,\beta^{BD}_{3,T-2}\,|_{A_2,B_2}) \\ +1^{B_2} & \text{if } \beta \in [\beta^{BD}_{3,T-2}\,|_{A_2,B_2}\ ,\beta^{BD}_{4,T-2}\,|_{B_2,b_2}) \\ +1^{b_2} & \text{if } \beta \in [\beta^{BD}_{4,T-2}\,|_{B_2,b_2}\ ,\beta^{BD}_{4,T-2}\,|_{b_2,b_1}) \\ +1^{b_1} & \text{if } \beta \in [\beta^{BD}_{4,T-2}\,|_{b_2,b_1}\ ,\beta^{BD}_{5,T-2}\,|_{A=A_1,b_1}) \\ -1^{A} & \text{if } \beta \in [\beta^{BD}_{5,T-2}\,|_{A=A_1,b_1}\ ,\overline{\beta}] \end{cases}$$

At $T-2$ expected volume on the PLB is:

$$\begin{aligned} VL^{PLB}_{T-2}([0110]\&[0]) &= \alpha[\Pr(H^{*BD}_{T-2} = -1^{A_1}) + \Pr(H^{*BD}_{T-2} = -1^{B_1})] \\ &+(1-\alpha)[\Pr(H^{*RT}_{T-2} = -1^{A_1}) + \Pr(H^{*RT}_{T-2} = -1^{B_1})] \\ &= \alpha\Big(\frac{\beta^{BD}_{1,T-2}\big|_{a_1,B=B_1} - \underline{\beta}}{\overline{\beta}-\underline{\beta}} + \frac{\overline{\beta}-\beta^{BD}_{5,T-2}\big|_{A=A_1,b_1}}{\overline{\beta}-\underline{\beta}}\Big) \\ &+(1-\alpha)\Big(\frac{\beta^{RT}_{1,T-2}\big|_{A_2,B=B_1} - \underline{\beta}}{\overline{\beta}-\underline{\beta}} + \frac{\overline{\beta}-\beta^{RT}_{5,T-2}\big|_{A=A_1,B_2}}{\overline{\beta}-\underline{\beta}}\Big) \end{aligned}$$

Because there is no volume executed in the IP, we obtain that:

$$LP^{PLB}_{T-2}([0110]\&[0]) = 1 - VL^{PLB}_{T-2}([0110]\&[0]) - LP^{IP}_{T-2}([0110]\&[0])$$

We also compute the other market indicators for the PLB:

$$\begin{aligned}
DPI^{PLB}_{T-2}([0110]\&[0]) &= 2 - VL^{PLB}_{T-2}([0110]\&[0]) \\
DPT^{PLB}_{T-2}([0110]\&[0]) &= 2 + LP^{PLB}_{T-2}([0110]\&[0]) - VL^{PLB}_{T-2}([0110]\&[0]) \\
SP^{PLB}_{T-2}([0110]\&[0]) &= \tau\{LP^{PLB}_{T-2}([0110]\&[0]) + LP^{IP}_{T-2}([0110]\&[0])\} \\
&\quad +2\tau VL^{PLB}_{T-2}([0110]\&[0])
\end{aligned}$$

(1.2) When the opening book at $T-2$ is $[0110]\&[1]$, we derive the equilibrium strategies for both a RT and a BD in a similar way:

$$H^{*RT}_{T-2}(\beta,[0110]\&[1]) = \begin{cases} -1^b & \text{if } \beta \in [\underline{\beta}, \beta^{RT}_{1,T-2}\mid_{B=b_1}) \\ 0 & \text{if } \beta \in [\beta^{RT}_{1,T-2}\mid_{B=b_1}, \beta^{RT}_{5,T-2}\mid_{A=a_1}) \\ -1^a & \text{if } \beta \in [\beta^{RT}_{5,T-2}\mid_{A=a_1}, \overline{\beta}] \end{cases}$$

$$H^{*BD}_{T-2}(\beta,[0110]\&[1]) = \begin{cases} -1^b & \text{if } \beta \in [\underline{\beta}, \beta^{BD}_{1,T-2}\mid_{a_1,B=b_1}) \\ +1^{a_1} & \text{if } \beta \in [\beta^{BD}_{1,T-2}\mid_{a_1,B=b_1}, \beta^{BD}_{3,T-2}\mid_{a_1,b_1}) \\ +1^{b_1} & \text{if } \beta \in [\beta^{BD}_{3,T-2}\mid_{a_1,b_1}, \beta^{BD}_{5,T-2}\mid_{A=a_1,b_1}) \\ -1^a & \text{if } \beta \in [\beta^{BD}_{5,T-2}\mid_{A=a_1,b_1}, \overline{\beta}] \end{cases}$$

We also compute the corresponding market quality indicators:

$VL^{PLB}_{T-2}([0110]\&[1])$, $DPI^{PLB}_{T-2}([0110]\&[1])$, $DPT^{PLB}_{T-2}([0110]\&[1])$, and $SP^{PLB}_{T-2}([0110]\&[1])$.

(1.3) The market quality indicators for the transparent IP case (T)

are equal to the average of those obtained in (1.1) and (1.2). For example:

$$VL_{T-2}^{PLB,T} = \frac{VL_{T-2}^{PLB}([0110]\&[0])}{2} + \frac{VL_{T-2}^{PLB}([0110]\&[1])}{2}$$

We compare these market indicators with those obtained for the single market model. We first analyze separately the two cases, and start with [0110]&[0]. Without broker-dealers ($\alpha = 0$), the PLB&IP model converges to the single market model. When α is positive, instead, limit orders submitted at $T-2$ have a lower execution probability in the PLB&IP model, as they are more frequently undercut by BD. Therefore both BD and RT submit market orders with a higher probability. From Proposition 1 we know that:

$$VL_{T-2}^{LM} = \frac{\widehat{\beta}_{1,T-2}^{LM}|_{A_k,B=B_1} - \underline{\beta}}{\overline{\beta}-\underline{\beta}} + \frac{\overline{\beta}-\widehat{\beta}_{5,T-2}^{LM}|_{A=A_1,B_k}}{\overline{\beta}-\underline{\beta}}$$

So we obtain that:

$$\begin{array}{lcl}
\beta_{1,T-2}^{RT}|_{A_2,B=B_1} & > & \beta_{1,T-2}^{BD}|_{a_1,B=B_1} > \widehat{\beta}_{1,T-2}^{LM}|_{A_k,B=B_1} \\
\beta_{5,T-2}^{RT}|_{A=A_1,B_2} & < & \beta_{5,T-2}^{BD}|_{A=A_1,b_1} < \widehat{\beta}_{5,T-2}^{LM}|_{A=A_1,B_k}
\end{array}$$

This implies that $VL_{T-2}^{PLB}([0110]\&[0]) > VL_{T-2}^{LM}$. It is straighforward to show that:

$$\begin{aligned} LP^{PLB}_{T-2}([0110]\&[0]) &\leq 1 - VL^{PLB}_{T-2}([0110]\&[0]) < 1 - VL^{LM}_{T-2} = LP^{LM}_{T-2} \\ DPI^{PLB}_{T-2}([0110]\&[0]) &> DPI^{LM}_{T-2} \\ DPT^{PLB}_{T-2}([0110]\&[0]) &> DPT^{LM}_{T-2} \\ SP^{PLB}_{T-2}([0110]\&[0]) &< SP^{LM}_{T-2} \end{aligned}$$

In the case with [0110]&[1] all incoming market orders at $T-2$ are executed in the IP, so that $VL^{PLB}_{T-2}([0110]\&[1]) = 0$. Furthermore, limit orders are only posted to the IP as on the PLB they have a zero execution probability. It follows that $LP^{PLB}_{T-2}([0110]\&[1]) = 0$ and $SP^{PLB}_{T-2}([0110]\&[1]) = \tau$.

Averaging over the two cases, [0110]&[0] and [0110]&[1], we obtain $VL^{PLB,T}_{T-2} < VL^{LM}_{T-2}$, $LP^{PLB,T}_{T-2} < LP^{LM}_{T-2}$, $DPI^{PLB,T}_{T-2} > DPT^{LM}_{T-2}$, $SP^{PLB,T}_{T-2} < SP^{LM}_{T-2}$.

2) Liquid stocks, opaque IP

In this proof we only highlight the differences with the transparent IP framework, therefore we focus only on RT. Consider again the PLB that opens [1110] at $T-1$. If a RT arrives, he will infer the state of the IP from the observed PLB. The RT knows that $H_{T-2} = +1^{A_2}$ is never an equilibrium strategy for a BD if the state of the IP is [1]. So he will update the probabilities associated with $IP = [0]$ and $IP = [1]$ from 1/2 to:

$$\begin{aligned}\Pr\{S^{IP}_{T-2} &= [0]\,\big|S^{PLB}_{T-2} = [1110]\,\} \\ &= \frac{\frac{1}{2}[\alpha \Pr(H^{*RT}_{T-2} = +1^{A_2}) + (1-\alpha)\Pr(H^{*BD}_{T-2} = +1^{A_2})]}{\frac{1}{2}\alpha \Pr(H^{*RT}_{T-2} = +1^{A_2}) + (1-\alpha)\Pr(H^{*BD}_{T-2} = +1^{A_2})} > \frac{1}{2}\end{aligned}$$

$$\begin{aligned}\Pr\{S^{IP}_{T-2} &= [1]\,\big|S^{PLB}_{T-2} = [1110]\,\} \\ &= \frac{\frac{1}{2}(1-\alpha)\Pr(H^{*BD}_{T-2} = +1^{A_2})}{\frac{1}{2}\alpha \Pr(H^{*RT}_{T-2} = +1^{A_2}) + (1-\alpha)\Pr(H^{*BD}_{T-2} = +1^{A_2})} < \frac{1}{2}\end{aligned}$$

To select his optimal trading strategy, he will then compute his expected payoffs using the Bayesian updated probabilities. For example:

$$H_{T-1} = -1^{B} : B_1 \Pr\{S^{IP}_{T-2} = [0]\,|[1110]\,\} + b_1 \Pr\{S^{IP}_{T-2} = [1]\,|[1110]\,\} - \beta v$$

A similar reasoning applies to a RT arriving at T, his payoffs are:

$H_T = -1^B$	:	$E(B\,|S_{T-1}, h_{T-1}, h_{T-2}\,) - \beta v$
$H_T = -1^A$	:	$\beta v - E(A\,|S_{T-1}, h_{T-1}, h_{T-2}\,)$
$H_T = 0$	:	0

where $E(B\,|S_{T-1}, h_{T-1}, h_{T-2}\,)$ is for example the expected execution price of a market sell order given the actual state of the book and the order submissions observed in the previous periods. Differently from the transparent IP framework, here RT and BD strategies are not the same at T. Finally, to analyze the changes in the PLB after the introduction of an IP, we compare the market indicators with those computed both in the proof of Proposition 1 and the case with a transparent IP:

$$
\begin{aligned}
DPI_{T-2}^{PLB,T} &> DPI_{T-2}^{PLB,O} > DPI_{T-2}^{LM} \\
DPT_{T-2}^{PLB,T} &> DPT_{T-2}^{PLB,O} > DPT_{T-2}^{LM} \\
SP_{T-2}^{PLB,T} &< SP_{T-2}^{PLB,O} < SP_{T-2}^{LM}
\end{aligned}
$$

■

A.5 Proof of Proposition 3

Proof. Following Eq. (1.13) and (1.14), we define the change in welfare as:

$$
\begin{aligned}
\sum_t E(W_t - \overline{W}_t) &= \sum_t (-1) \cdot \Big[\sum_i |i - v| \cdot \int_{\beta \in \{\beta : H_t = -1^i | S_{t-1}\}} f(\beta) d\beta \\
&\quad + \sum_i |i - v| \cdot \int_{\beta \in \{\beta : H_t = +1^i | S_{t-1}\}} p_t^*(i \,| S_t) \cdot f(\beta) d\beta \Big]
\end{aligned}
$$

where, for example, in the single market model $|i - v|$ takes values $\{\frac{3\tau}{6}, \frac{9\tau}{6}\}$ for the LM and $\{\frac{\tau}{6}, \frac{3\tau}{6}, \frac{5\tau}{6}, \frac{7\tau}{6}, \frac{9\tau}{6}\}$ for the SM. The results in Tables 9 and 10 come from a comparis comparison of both the equilibrium welfare and the change in welfare with respect to the FB for different values of τ and v. ■

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Chapter 2

Learning to Use Private Information

2.1 Introduction

In finance literature, private information as a source of short-term volatility in stock price has been widely modeled and extensively studied. Informed traders' decisions to buy or sell stocks could reveal their private information. Even if buy or sell orders do not get executed the uninformed traders could extract such private information from the order flow. This is a classic paradigm in the market microstructure literature (Glosten and Milgrom (1985), Kyle (1985), Easley and O'Hara (1987), Glosten (1994), and McLoughlin (2012)), in which informed traders need to make the best use of their information and find a proper trading strategy. To understand how informed traders use private information strategically, this paper distills this situation into a two-player, two-stage asymmetric information game; in this game, only one player is informed of the exact payoff matrix at the beginning while the uninformed one knows two possible payoff matrix, and the actions in the first stage are observed by both players before the second stage starts. As players' actions in the first stage are observed, the uninformed player "infers" which of these two payoff matrix is played in the current round and acts accordingly. In equilibrium, the informed player plays a mixed strategy in the first stage and undertakes the stage-dominant action in the second stage.

Feltovich (1997; 1999; and 2000) (hereafter, F97, F99, and F00 respectively) first studied this game in the laboratory. His results suggest that the informed players slowly learn to use the private information in a strategic way. Also, F97 and F00 compare the relative performance of two competitive learning models which are widely explored in the literature: the reinforcement model (Roth and Erev (1995) and Erev and Roth (1998)) and a two-parameter family of belief-based models adapted from Fudenberg and Levine (1995) and Fudenberg and Levine (1998). In this paper, we modify Feltovich's design, borrow a hybrid learning model, i.e. Experience-Weighted Attraction Model (Camerer and Ho (1999), hereafter CH99) which embraces these two models, and explore the following questions that have not been addressed: 1) Do informed and uninformed players behave differently in the learning process?; 2) Does an increased payoff in the second stage accelerate the learning process? If so, how would these two different informational roles be affected? ; and 3) how do players learn from their experience? Do they evaluate different pieces of information of past plays equally?

F97, F99 and F00 adopt a fixed matching protocol to define a pair of players as the study unit, and assume that the different informational roles share the same sets of behavioral parameters when modeling. We relax this by arguing that the various informational roles would (might?) make players learn differently and the fixed pairing might not be an ideal environment for studying the process of learning due to strategic thinking. On the one hand, the most salient feature of this game, compared to other normal form games studied extensively in the learning literature, is the following asymmetric task for different roles: the informed player needs to do an intertemporal (cross-stage) optimization while the uninformed player does not. The perceived higher expected payoff for the uninformed player intuitively causes passive learning. Therefore, their learning speeds are hypothetically different. On the other hand, when we randomly match players, an informed player's experience will be based on his interactions with a group of uninformed ones, so he is expected to behave less strategically. We believe that this reaction is common in our game of mixed-strategy equilibria and random pairing fits the stock trading situation by which this paper is motivated. Therefore, to accommodate the specific features of this game and to study different learning speeds for two groups of players, we change the matching protocol from a fixed pairing into a random one.

In our new game, the payoffs in the second stage double. Though the theoretical predictions for the informed players remain, our experimental results show that increasing payoffs in the second stage provides stronger incentives and induces faster learning. The behavioral parameters that the EWA model estimates also qualitatively show different impacts of such incentive increment on different informational roles, and confirm our intuition of a passive learning for the uninformed players.

As Camerer (2003) points out, tracking information acquisition is a good way for judging which learning theory is the best due to different minimum information requirements for the updating rules assumed in different models. Adding to Feltovich's research, we introduce the possibility of acquiring the history of past plays into our design. A novelty in our design is that the subjects can freely choose the information he/she would like to look at. We track their information choices and define their updating patterns[1] accordingly. In this game, we find that subjects are inclined to update their beliefs by looking at their partners' past actions and payoffs, which is consistent with a belief-based model. Ideally, the

[1] To distinguish from the updating rule assumed in different learning theories, we use the term "updating pattern" for the one defined by information acquisition.

learning theory should be supported by both information lookup choices and behavioral choices. To characterize individuals' actions in the game, we adopt a modified individual EWA (Ho, Wang and Camerer (2008)). It takes the first ten rounds' individual empirical frequencies and magnifies them by an endogenous multiplier to determine the initial attractions. In this way, we achieve a good compromise that not only considers a common criticism arising from aggregating heterogeneous data for the uniformed learning model estimation, but also avoids the estimation difficulty at the individual level. We find that the dominance of seemingly "fictitious play" matches to the majority of belief-based updating type. In conclusion, the measures of individuals' information acquisition do provide some indirect evidence on how players learn in a game, and tracking information acquisition can be considered as a useful tool to study learning.

In the rest of this paper, we first survey the related literature on the role of information in learning in Section 2.2. In Section 2.3, we briefly discuss the properties of the game and three learning theories, and afterwards we present our experimental design in Section 2.4. Sections 2.5 and 2.6 demonstrate our results on information acquisition and learning behavior. Section 2.7 concludes.

2.2 Experimental Studies of Information in Learning

To characterize whether and how the deviations from equilibrium disappear dynamically, game theorists have studied extensively the learning processes in a variety of games. Different theories have been proposed, and the two leading branches are the reinforcement model (Roth and Erev (1995), Erev and Roth (1998)) and the belief-based model (Fudenberg and Kreps (1992); Fudenberg and Levine (1995); Fudenberg and Levine (1998); Crawford (1995)). The main difference lies in the assumption of player's adjustment rule. Camerer and Ho (1999) propose a sophisticated model that incorporates these two, i.e. Experience-Weighted Attraction Model (hereafter EWA). By parameter restrictions, EWA can be reduced to either a reinforcement model or a belief-based model. It generally fits better than those two reduced ones. One subsequent variation of EWA in Chong, Camerer and Ho (2006) (CCH2006 hereafter) takes into account the players' sophistication, and proposes a model with a mixture population. The strategic sophistication in their incomplete information game differs from the intertemporal optimization here. In their game, the payoff does not have a constant sum, thus the strategic "teachers" sacrifice short-term interests up to some stages would be compensated in later stages. By contrast, our game is a constant sum game, and for each stage either the informed one or his uninformed partner gets a nonpositive payoff. Any "teaching" behavior cannot be compensated by the future plays. Our research focuses on different updating rules rather than the different levels of the strategic sophistication.

Other than the incomplete-information game in CCH2006, most studies on learning concentrate on a variety of normal form games.[2] Camerer (2003) provides a comprehensive review on these studies. Among them, F97, F99, and F00 are the first to study learning within a two-stage game with asymmetrically informed players. Based on his works, we modify the experimental design, track players' information acquisition, understand players' updating patterns and study their learning behaviors in this two-stage asymmetric-information game.[3]

Tracking information search has been adopted to examine strategic thinking in several works (Camerer, Johnson, Rymon, and Sen (1993), Johnson, Camerer, Sen, and Rymon (2002), Costa-Gomes, Crawford, and Broseta (2001), Costa-Gomes and Crawford (2006) and Brocas, Carrillo, Wang, and Camerer (2010)). Subjects are invited to play games with the payoff information shaded. When they intentionally acquire payoff information (either move the mouse into game boxes to reveal payoffs

[2] Such as centipede game by McKelvey and Palfrey (1992); constant-sum game, Mookerjee and Sopher (1994;1997); coordination game, Van Huyck, Battalio, and Beil (1990;1991); a 2×2 game with one mixed-strategy equilibrium Ochs (1995) and Nyarko and Schotter (2002); Cheung and Friedman (1997) focus on individual heterogeneity and analyze four games, i.e. Hawk-Dove, Coordination, Prisoners' Dilemma, Buyer-Seller and Battle of the Sexes.

[3] Following Feltovich's (97, 99 and 00) work we call this game as asymmetric-information game rather than incomplete information game; this also distinguishes itself from the incomplete-information game studied by CCH2006.

or click or click-and-hold the mouse), their search paths are tracked. Such information search behavior potentially provides the experimenters with evidence of the subjects' cognitions. Crawford (2008) summarizes these findings and argues for the methodological value of information acquisition measures.

By contrast, most of works in learning literature take information as experimental treatment (such as Cheung and Friedman (1997), McKelvey and Palfrey (2001) and Nyarko and Schotter (2002)) and compare individuals' learning behaviors under different information conditions. We believe that restricting players' information can potentially change their reasoning for each round, and consequently alter their behaviors under the condition where no restrictions have been imposed. In our design, we allow subjects to freely choose what they would like to know if any. This environment would provide a seemingly low-information condition which helps to discriminate among different theories; meanwhile it offers the possibility of a high-information condition rather than imposing one, so it does not distort players' behaviors. To the best of our knowledge, few works on learning have applied this methodology. Salmon (2004) is the closest one to our work. It examines a 4×4 constant-sum game with mixed strategy equilibria. He resorts to a non-standard experimental design[4] to find rule learning in his game. Knoepfle, Wang and Camerer

[4] Salmon (2004) collects subjects' answers on how they make move decision and how information is related to their decision. The author asks the question every 10 rounds, which allows him to track subjects' dynamic mind. He maps these answers

(2009) and Tang (2001) also have the information of past plays available during plays, but the former paper focuses on comparing the eye-tracked group with the control group and the latter one reports few information acquisition.

2.3 Theory

2.3.1 The Game

In financial market, private information related to the fundamental value of a stock is aggregated into price in the long run, while in the short run stock trading can be roughly considered as a constant-sum game and an informed trader needs to find a proper trading strategy in order to make the best use of the information he holds. The asymmetric-information game, first examined theoretically by Aumann and Maschler (1995) and later by F97, F99 and F00 in the laboratory, can capture this feature vey well. In Aumann and Maschler's analysis, this repeated constant-sum game requires that an informed player fully reveals his private information in the long run. However, in the short run (two-stage case in Feltovich's experiments) it is optimal for him to play a mixed strategy, in which he hides/reveals his private information randomly. This experiment helps to understand whether the informed

into different decision rules, points out that the decision rule is changing dynamically and concludes that the learning rule can be described best as a Win-Stay-Lose-Go learning rule.

trader is able to exploit his private information in a simple and repeated setting, and potentially provides insights on the general belief that the informed traders make profits from private information. A two-stage game is parsimonious to implement our research goal.

This asymmetric-information game has a payoff matrix shown in Table 2.1. The sum of the payoffs in the first stage is one and that in the second stage is $a(\geq 1)$. In an experimental session, such two-stage game is played many rounds. The game structure and the payoff function are common knowledge. At the beginning of each round, Nature chooses the states LEFT and RIGHT with probabilities $(p, 1-p)$; the row player is informed of the true state and hereafter called the informed player, while the column player (hereafter, the uninformed player) knows only the distribution. Two players decide simultaneously their actions. At the beginning of the second stage, both of them are informed of their partners' actions in the first stage but not payoffs. Sum of the payoffs in these two stages is realized at the end for both players. For our research purpose, we take $p = 0.5$ where the two states are symmetric and $a = 1$ and 2.[5] For simplicity, we denote the game as $G(a)$.

The action in this game is to choose A or B, and whether a choice can bring a positive payoff depends on the true state and partner's action. The probability of stage-dominant action in the first stage is denoted as

[5]In Auman and Maschler(1995) analysis, the payoff matrix in the first stage is repeated for the rest, i.e. $a = 1$. Feltovich's experiments have two treatments: $p = 0.5$ and 0.34 with $a = 1$.

Table 2.1 Payoff Matrix for a Two-Stage Asymmetric-Information Game

LEFT Table (p)

1st			Un-	Player
			A	B
Stage	Informed	A	$1,0$	$0,1$
(choice)	Player	B	$0,1$	$0,1$
2nd			Un-	Player
			A	B
Stage	Informed	A	$a,0$	$0,a$
(choice)	Player	B	$0,a$	$0,a$

RIGHT Table ($1-p$)

1st			Un-	Player
			A	B
Stage	Informed	A	$0,1$	$0,1$
(Choice)	Player	B	$0,1$	$1,1$
2nd			Un-	Player
			A	B
Stage	Informed	A	$0,a$	$0,a$
(Choice)	Player	B	$0,a$	$a,0$

$P(sda_1)$, which is the average of $P(sda_1|L)$ and $P(sda_1|R)$ weighted by the frequencies of LEFT and RIGHT respectively. Similarly, $P(sda_2)$ denotes the probability of playing stage-dominant action in the second stage; $P(brcr|A)$ is the uninformed player's probability of best response to completely revealing following an informed player choice of A; Similarly for $P(brcr|B)$. The weighted average of these two is $P(brcr)$. Table 2.2 summarizes these notations and Table 2.3 lists the theoretical predictions of the games we are interested in.

We can easily read from Table 2.3 that G(1) and G(2) have multiple equilibria, respectively, in which the theoretical predictions of the informed player's behaviors are exactly the same. When the states of LEFT and RIGHT occur with equal probability, the optimal strategy for the informed player is to play a mixed strategy (0.5, 0.5), and it is also interpreted as hiding/revealing information with equal probability in the

Table 2.2 Notations

symbol	Meanings: "Probability of ..."
$P(sda_{1,2}\|L)$	stage-dominant action in the first/second stage of the LEFT matrix for Player 1
$P(sda_{1,2}\|R)$	stage-dominant action in the first/second stage of the RIGHT matrix for Player 1
$P(brcr\|A)$	best response to completely revealing following a P1 choice of A for P2
$P(brcr\|B)$	best response to completely revealing following a P1 choice of B for P2
$P(sda_{1,2})$	stage-dominant action in the first/second stage for Player 1
$P(brcr)$	best reponse to completely revealing for Player 2 at the second stage

Table 2.3 Nash Predictions

	Strategy Profiles						Expected Payoff	
	P(sda_1 $\mid L$)	P(sda_1 $\mid R$)	P(brcr$\mid A$)	P(brcr$\mid B$)	P(sda_1)	P(brcr)	Player 1	Player 2
G(1)	.500	.500	β_1^*	1.500-β_1	.500	.750[1]	.75	1.25
G(2)	.500	.500	β_2^{**}	1.250-β_2	.500	.625	1.25	1.75

* $\beta_1 \in [0.5, 1]$

** $\beta_2 \in [0.25, 1]$

[1] F97 derives the theoretical prediction of $P(brcr)$ in G(1) is [.500,1.000], which is not rigorous. Given that the equilibrium strategy profiles are as follows: $P(sda_1\,|L) = P(sda_1\,|R) = \frac{1}{2}$, $P(A_1) = \frac{1}{2}$, $P(brcr\,|A)+P(brcr\,|B) = \frac{3}{2}$ and $P(sda_2\,|L) = P(sda_2\,|R) = 1$, we have the probability of Player 1's choice of A is: $P(A) = pP(sda_1\,|L)+(1-p)(1-P(sda_1\,|R)) = \frac{1}{2}$; Similarly $P(B) = \frac{1}{2}$;Therefore, $P(brcr) = P(brcr\,|A)P(A) + P(brcr\,|B)P(B) = \frac{1}{2} \times \frac{3}{2} = \frac{3}{4}$.

first stage. In the second stage playing the stage-dominant action (revealing the information) does no harm at all. In this way, the informed player optimally maximizes his total payoff of two stages. If the informed player always reveals his private information in the first stage in the hope of getting a positive payoff, it does not help him to maximize his total payoff if his partner would best respond to his actions. Being myopic in this repeated setting cannot fully exploit the information value. This is a crucial difference from other games in learning literature that does not require players to consider intertemporal optimization. Regarding the uninformed player, his optimal strategy is to choose A or B randomly in the first stage following the distribution of true state, and to best-respond to his observations over his partner's action with a certain probability. Our experimental results show shortly how differently the subjects behave in G(1) and G(2) though the theoretical predictions of these two games are similar.

2.3.2 Learning Models and Information Lookups

Adaptive learning model describes how players adjust their decisions over time in response to their experience with analogous games. An adaptive learning model has two main components: players' interaction patterns and players' decision adjustment in response to experience.

Corresponding to different adjustment rules, there are three main classes[6] of adaptive learning models: reinforcement learning, belief-based learning and EWA learning. Each theory has different minimum information requirement.

Reinforcement learning (RE) is a completely unsophisticated model with correspondingly low information requirements. It begins with initial decision propensities, taken as random or uniform over all decisions, and updates them each round in response to the player's realized payoff, raising the propensity of the decision played in proportion to the realized payoff. The propensities then determine the probabilities with which the player plays his pure decisions. Players care about the payoffs those strategies yielded in the past, and they do not generally have beliefs about what others will do. Therefore players' own previous choice and previous payoffs are the minimal information requirement for the RE model.

By contrast, belief-based models (BE) such as best-response dynamics and fictitious play (Fudenberg and Levine (1998)) are more sophisticated, with correspondingly different information requirements. It begins with a set of initial beliefs about others' decisions, usually taken as determined by the structure, i.e. both own and others' payoff functions. They then update beliefs each round in response to others' observed de-

[6] Camerer (2003) provides descriptions on other variants such as sophisticated teaching (Camerer, Ho, and Chong 2002), anticipatory learning (Selten 1991), and rule learning (Stahl 1996).

cisions.[7] If future influences are negligible, beliefs determine decisions via expected current-payoff maximization. Depending on his type, the player may need to know own and others' payoff functions to determine initial beliefs. Current beliefs are a function of the entire history of others' decisions, but all the most recent decisions can be stored in a state vector that includes the previous round's beliefs and, as a measure of precision, the number of observations they are based on. Updating requires at least others' current decisions and possibly their entire histories; and the player also needs to know his own payoff function to determine his optimal decisions.

EWA model is sophisticated as belief-based model but it is conveniently viewed as reinforcement learning with an added parameter that measures the strength of "virtual reinforcement" of unplayed decisions. When the parameter (δ) equals zero, there is no virtual reinforcement, and EWA becomes reinforcement learning. When it equals one, unplayed decisions are reinforced as strongly as the one that was played, in proportion to the payoffs they would have yielded, and EWA becomes

[7] An important issue in belief-based learning is the level of inertia, which reflects subjects' views of the relative value of recent and past observations in predicting current behavior. Inertial models like fictitious play, which adjust beliefs and/or actions only part of the way toward the values suggested by the latest observation taken alone, usually describe behavior better than best-reply dynamics, which give past observations zero weight (Boylan and El-Gamal (1993), Crawford (1995), Mookherjee and Sopher (1994), Van Huyck (1994). However, models with intermediate levels of inertia usually outperform fictitious play and Cournot (Crawford (1995), Mookherjee and Sopher (1994), Van Huyck, Cook, and Battalio (1994). The only work we know of that tries to test heterogeneity and strength of priors against inertia is Camerer and Ho (1998), but they allow only two types of learners and focus on EWA.

belief-based learning. Intermediate values yield blends of both (nonlinear blends, so they can yield quite different dynamics) with information requirements the union of the other two models'. Note that this nested belief-based model requires players' most recent received payoff, forgone payoff and the other players' most recent choice to update beliefs.[8]

Actual information use depends on what information is available and on players' cognitive capacities. In high-information conditions, all the information mentioned above is available to players and all these theories seem plausible to explain the subjects' behaviors. The way of discriminating among different theories is to track which information is necessary and important for a subject to make decisions. This would suggest that the theory that takes such information as a prerequisite for updating is the best one in describing behaviors.

In our game, the game structure and the payoff function are common knowledge for two players, and there are basically four pieces of historical information important for updating: one's own action, one's realized payoff, partner's action and partner's payoff. RE needs at least one's own choice(s) and his own received payoff(s) for updating; BE needs one's partner's choice, and one's own payoff function (it is a weakly richer information set than the set of one's own realized payoff and his foregone

[8]It is possible to know one's received payoff without knowing what other players did. Also once the payoff structure is given like in our experiment, forgone payoff can be calculated by knowing the other's choice. But knowing forgone payoff and received payoff does not mean knowing the whole structure. Here we see the nested belief-based model has fewer minimal information requirements than the general belief-based learning.

payoffs); and EWA needs all these information. If a player acquires, for example, the most recent value of "one's received payoff" in the first place, his behavior is consistent with RE, BE and also EWA. Suppose he continues in searching "one's own choice" and then stops. This just meets the minimum information requirement of RE. However, if he does not acquire his own choice but his partners' and stops, it is not consistent with RE. As there are several paths of information search and several combinations of different information sets, we need to do some information grouping and reduce the search paths to test the theory.

Strictly speaking, the information on partner's payoff is not crucial in discriminating among different theories. Naturally we group partner's actions and partner's payoffs as partner's history. In our experimental design, we have an information menu composed of three pieces of information to identify players' updating patterns: one's own choice, one's received payoff, and the partner's history. A RE learner needs the first two pieces of information; due to the fact that players' payoff function is common knowledge, a BE learner would be satisfied if he knows only the third piece of information; and a EWA-learner wants to know all these information. In this way, based on the players' information acquisition we are able to indirectly observe the updating rule specified by different learning theories.

2.4 Experimental Design

In this paper, we calibrate the availability of historical information with three pieces of information of past plays as described above. Allowing subjects to set up their initial beliefs, we set the first 10 rounds as the normal plays. From the 11th round onwards until the final one, a historical information menu is available on the interface. Figure 2.1 is a sample screen an informed player might face at the beginning of the 12th round. The left half of the screen is a "History" menu. It contains six buttons. Buttons A to button C are a group which lists the records of the most recent 10 rounds; Button D to button F are summaries[9] of the recent plays. Three sets of the past information are as follows: Button A and button D are a player's own past actions; button B and button E are a player's past realized payoffs; button C and button F are the information on his partner's past actions and realized payoffs. These six buttons constitute a salient menu which differs from Tang (2001)'s on-line help technique[10] and such information fragmentation also considers the

[9] A summary is a paragraph that shows empirical frequencies. For example, clicking button F "Summary: True States and His/Her Actions + Payoffs" might have the following paragraph displayed in the below box: "In the past 10 rounds, the occurrence of LEFT is 5 times and of RIGHT is 5 times, that is 50% vs. 50%. In the 1st stage, your partner has chosen A 10 times and B 0 times, that is 100% vs. 0%. In the 2nd stage, your partner has chosen A 0 times and B 10 times, that is 0% vs. 100%. Your partner's accumulated points are 15, average points per round are 1.5".

[10] Tang(2001) allows subjects, before making decisions or during the waiting time after having made their decisions, to acquire a summary of average payoff and choice frequency of each strategy in both groups over a subject-defined length of past rounds by clicking "F1" key. It works like an on-line help. The author reports about half of the subjects never used this function, and only about a fourth of the subjects used this function repeatedly.

potential problem in Salmon (2004)'s design, i.e. a single button may not be informative in a game with mixed-strategy equilibria.[11] The right half of this screen functions as the play of the current round. It contains a box which shows the informed player the current state of Nature, otherwise it is empty for the uninformed player. It also contains an action box for players' entries. At the beginning of the second stage of each round, the action box informs the subjects the previous choices of both players and asks for a new entry.

During all rounds in all sessions, the information choices, the number of information acquisitions, the order of different clicks and the time of each click are recorded in the background. Intuitively, a theory dominates the others if the subjects choose the necessary information required by such theory the most often over time, and the subjects acquire related information in the first place. For information acquisition analysis, we define the following measures: Number of Information Choices, Information Choice Combinations and First Information Choice. Then we group six buttons into three sets: A* (i.e., {A}, {D}, and {A, D}), B* (i.e.,

[11] In Salmon(2004)'s design, the historical information menu is a shaded record list displaying the recent (up to) 15 rounds. One row displays one record, which has 3 cells: subject's own action, realized payoff and partner's action. Clicking one cell reveals one number. Therefore we doubt that for a 4×4 constant-sum game with mixed equilibrium strategy a single click hardly conveys useful information; even for a RE-type individual who acquires information on his own actions and his realized payoffs, the subject has to click on all those 30 cells (15 rows $\times$ 2 columns) to understand better the situation (e.g. approximately how frequent one action has been played). This definitely needs great effort. In this respect, we favor Tang (2001)'s design, i.e. offer a summary for subjects. Therefore, we group into one button. Clicking on, for example, button A would display the subject's own actions in the most recent 10 rounds.

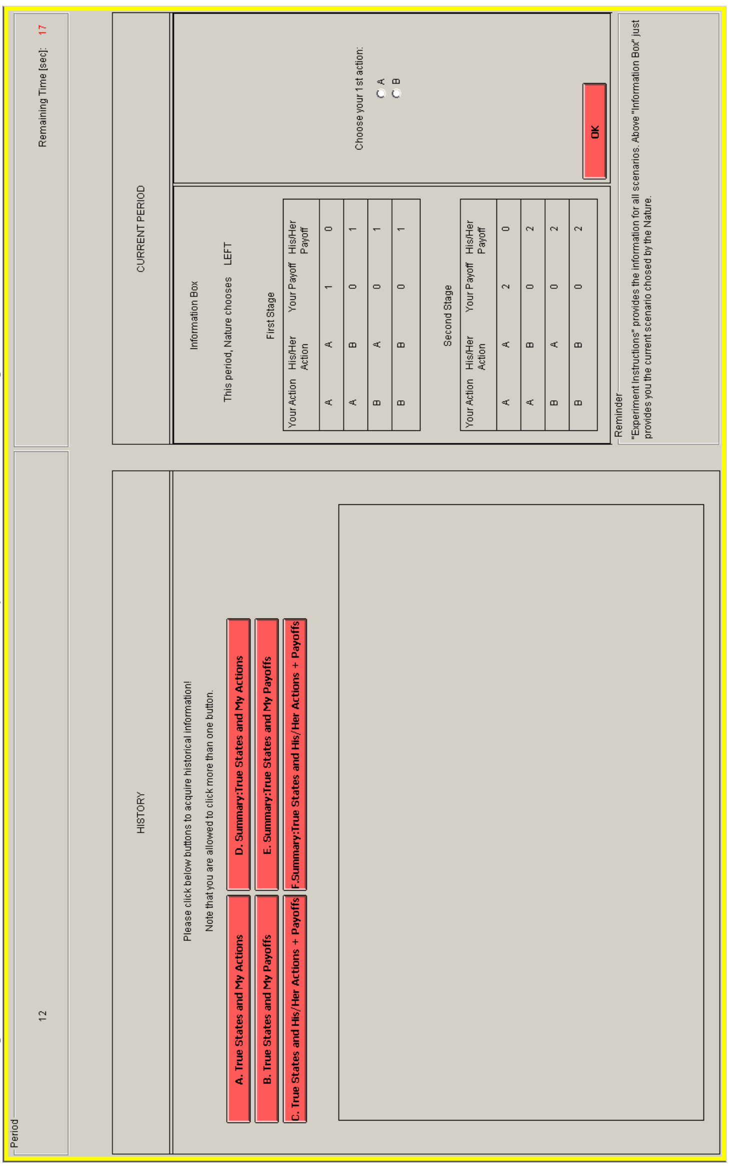


Figure 2.1 Screen for an Informed Player in the First Stage of the Twelfth Round

{B}, {E}, and {B, E}) and C* (i.e., {C}, {F}, and {C, F}). Based on the previous theoretical discussions, these three sets help to discriminate among different theories. RE learners are those who choose information sets A* and B*, BE learners are those who look at C*, and EWA learners are those who acquire the previous two cross-type information combinations. Table 2.4 lists the predictions of different learning models for these information acquisition measures. We also use random re-matching to suppress long-term interaction consideration in this game.[12]

During the fall of 2011, we conduct four sessions with a total of 100 subjects in the Experimental Economics Laboratory at Xiamen University. Subjects are registered students at Xiamen University and are recruited online. The experimental task is coded and run using the Zurich Toolbox for Readymade Economic Experiments (z-Tree) developed by Fischbacher (2007). As shown in Table 2.5, sessions 2 and 3 are similar to that in F99 and F00 except the matching protocol. Sessions 1 and 4 run our new game. These two sessions with increased payoffs in the second stage intend to induce informed players to hide information more often in the first stage by providing them with stronger incentives, though their behaviors have the similar predictions to that in the original game. The comparison between these two and the former ones tells us whether learning can be accelerated when the equilibrium play is more

[12] Feltovich and Oda (2010) and Mckelvey and Palfrey (2001) have documented some significantly experimental results from different matching protocols in a variety of normal form games.

Table 2.4 Measures of Information Acquisition Analysis and Predictions in Three Learning Models[†]

Measures	Reinforcement Learning	Belief-based Learning	EWA
Information Choice(s)	A^*,B^*	C^*	A^*,B^*,C^*
(Min) Num. of Choice(s)	Two	One	Three (maybe two)
Order of Choice(s)	No difference between A^* and B^*.	C^*	-
	The strong form of learning models can have individuals choose the information which is exclusively related to the theory in the first place.		
Explanations	-	-	If a subject is able to infer his own payoffs from his partners' by the fact of constant-sum game, it is possible for him get the needed information without looking at B*.

[†] A^* means either button A or button D or button A and D are clicked during one round for one subject, i.e. $\{\{A\},\{D\},\{A,D\}\}$; Similarly, B^* means $\{\{B\},\{E\},\{B,E\}\}$ and C^* means $\{\{C\},\{F\},\{C,F\}\}$.

Table 2.5 Experimental Sessions

	Date	Time	Subjects	Treatment[†]
1	2011/10/08	14:30-16:30	24	G(2)
2	2011/10/09	09:45-11:45	26	G(1)
3	2011/10/13	09:30-11:30	24	G(1)
4	2011/10/13	14:30-16:30	36	G(2)

[†] The shorthand of treatment is denoted by $G(a)$ where a stands for the sum of the payoffs in the second stage.

attractive. An experimental instruction sample for the treatment G(1) is in Appendix.[13]

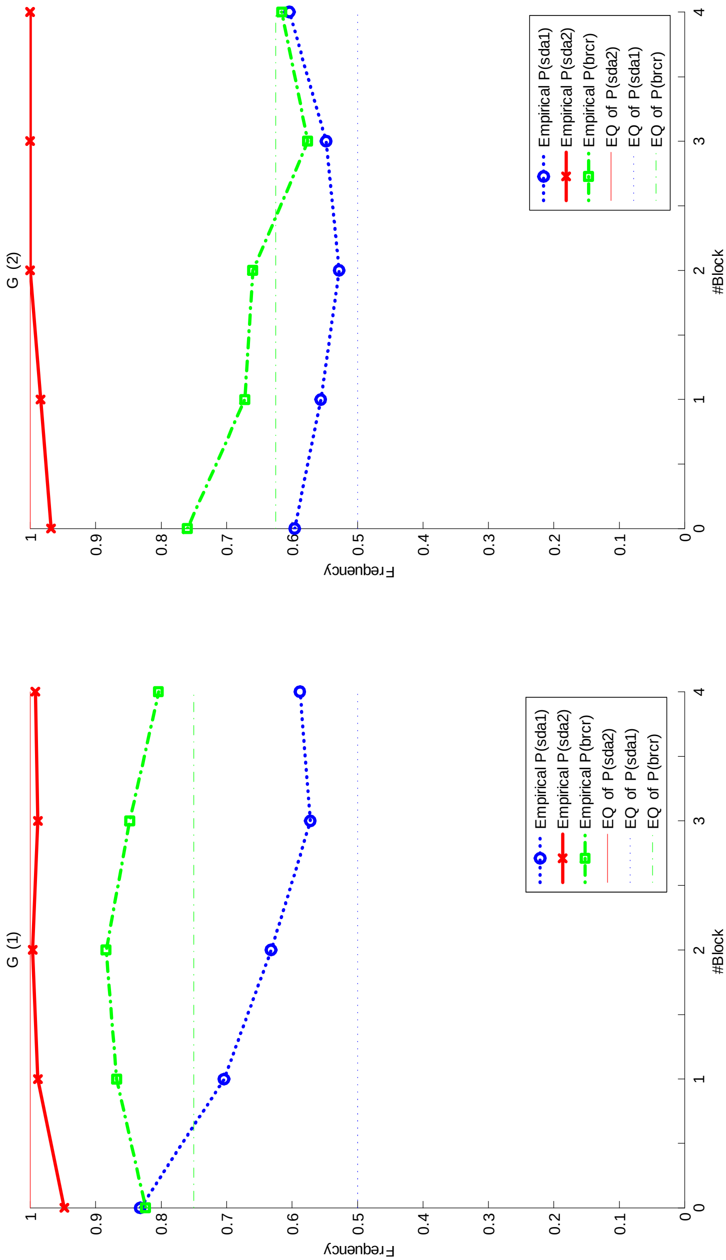
2.5 Aggregate Analysis

2.5.1 Equilibrium Play

For the following aggregate analysis, we pool the data from two sessions for each game and thus have 50 subjects with half informed and half uninformed in both G(1) and G(2). Figure 2.2 takes every 10 rounds as one block and shows the empirical frequencies. In this figure, the left panel is from G(1) while the right one is from G(2). It is clear that: 1) the deviations documented by Feltovich exist in both games. During the first 10 rounds the informed players on average reveal information more often than they should, and most of the uninformed ones believe

[13] We modify the payoff matrix presentation in Feltovich (97, 99 and 00)'s experimental instructions.

Figure 2.2 Experimental Frequencies in Two Games



that their counterparts are revealing the private information and best respond to this belief; 2) the sessions of G(2) have faster learning than those of G(1). This is not predicted by Nash equilibrium but can be explained by learning. As the marginal benefit of hiding in the first stage becomes significant in G(2), the propensity on such strategy in reinforcement learning evolves faster if beginning with the same initial level, while such strategy given the same beliefs over partner in belief-based learning would excel among the other strategies more easily, therefore in both theories the informed player is inclined to behave closer to the equilibrium strategy soon.

It is interesting to note that: during the first 10 rounds, the initial plays of the informed group, especially that of the first stage $P(sda_1)$, deviate from the predictions in both G(1) and G(2) but with significantly different degrees. The average deviation of two sessions in G(1) is (0.832-0.5=) 0.332 but it is (0.596-0.5=) 0.096 in G(2). We can conclude that on average the learning speed of informed players is faster in G(2) during the first 10 rounds. By contrast, this is not true for the uninformed group. The initial deviations of uninformed players in these two games are similar. We conjecture informed players learn in an active way while uninformed players learn passively, and we will show more evidence in the last subsection.

2.5.2 Information Lookups

Previous studies[14] report few information acquisitions of past plays. In our game, the salient information menu as well as the random matching protocol induces a considerable proportion of subjects to acquire information, i.e., 47.75% of information acquisition opportunities (number of subjects $\times$ number of rounds with information menu available) are exploited.

We provide indirect evidence in support of a belief-based theory by examining three measures: Number of Information Choices, First Information Choice, and Information Choice Combinations. We find that subjects in this game are eager to know their partners' actions and payoffs before choosing their own actions. This suggests that the belief-based theory dominates the reinforcement theory.

Figure 2.3 is the number of button-clicks for two games. Each subplot exhibits the contrast between the informed and the uninformed during four blocks[15]. As very few people click more than six times in total during one round, we truncate the number of information choices at six. We calculate the distribution based on all rounds and each group's population. Comparing the left one from G (1) with the right one from G (2), we can see: 1) there are more zero-button-click in the informed group

[14] Tang (2001) reports that about half of the subjects never used it and only about a fourth of the subjects used it repeatedly.

[15] Note that the historical information menu is available from the 11th round onwards, and we have 4 blocks for information analysis.

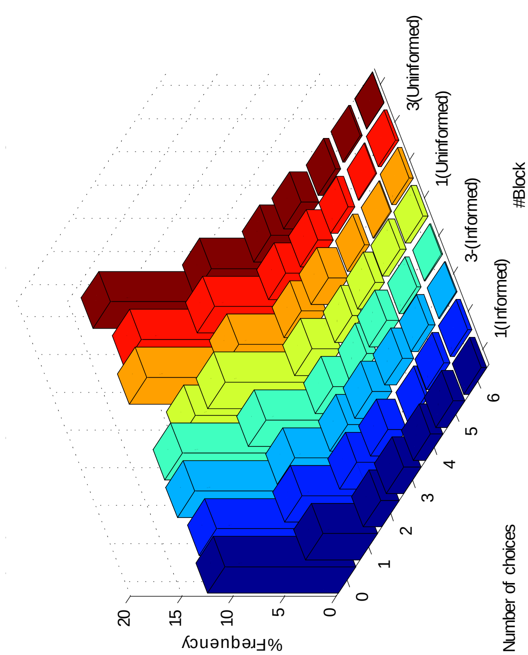
than that in the uninformed counterpart; such asymmetry is more pronounced for G (2); 2) if acquiring information, most subjects click on one button rather than many, and few subjects click on more than three buttons. This would suggest some buttons are considered as key references before moves; 3) the average number of button-clicks does not decrease over time. It is also supported by Figure 2.4 that average lookups gradually decrease as the game repeats in G (1), but increase in G (2).[16]

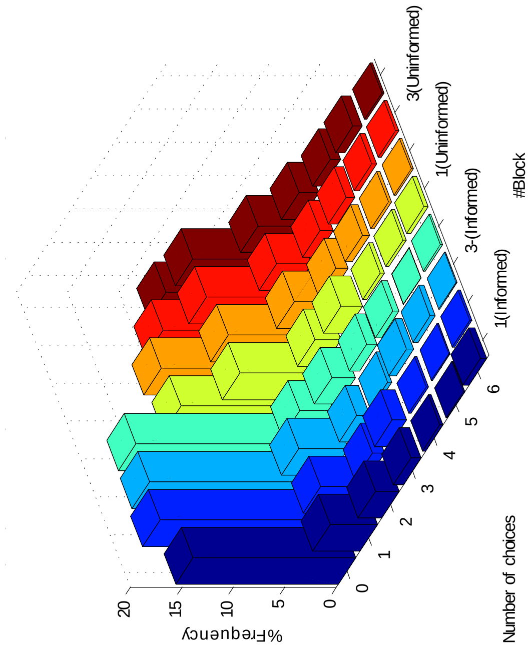
Intuitively, First Information Choice represents how important a piece of information is for a subject. Figure 2.5 shows that both informational roles click on the button C and F in the first place more frequently. In other words, the information on "partner's past actions and payoffs" is the most utilized information for players. Therefore, it seems that the belief-based theory is more plausible than the reinforcement theory. In Figure 2.6, a distribution of information choice combinations further confirms our conjecture.

As mentioned before, we group six buttons into three sets (i.e. A*, B*, and C*) in order to discriminate among different learning models. Figure 2.6 shows the frequencies of different information choice combinations and provides the following observations: 1) consistent with Figure 2.3, informed players acquire information less often; 2) consistent with Figure 2.5, subjects consider C* as the most important informa-

[16] One explanation is the "ending effect". When one session is approaching the end, some of the players start clicking buttons and checking difference pieces of information. Experiments without announcing the total number of rounds may be immune to this effect.

Figure 2.3 Number of Information Choices in Two Games





tion for decision making; and 3) no significant difference in information needs between those two groups across games. Both informational roles examine the same kind of information to make adjustments in these two games.

Figure 2.4 Average Number of Information Choices Over Time in Two Games

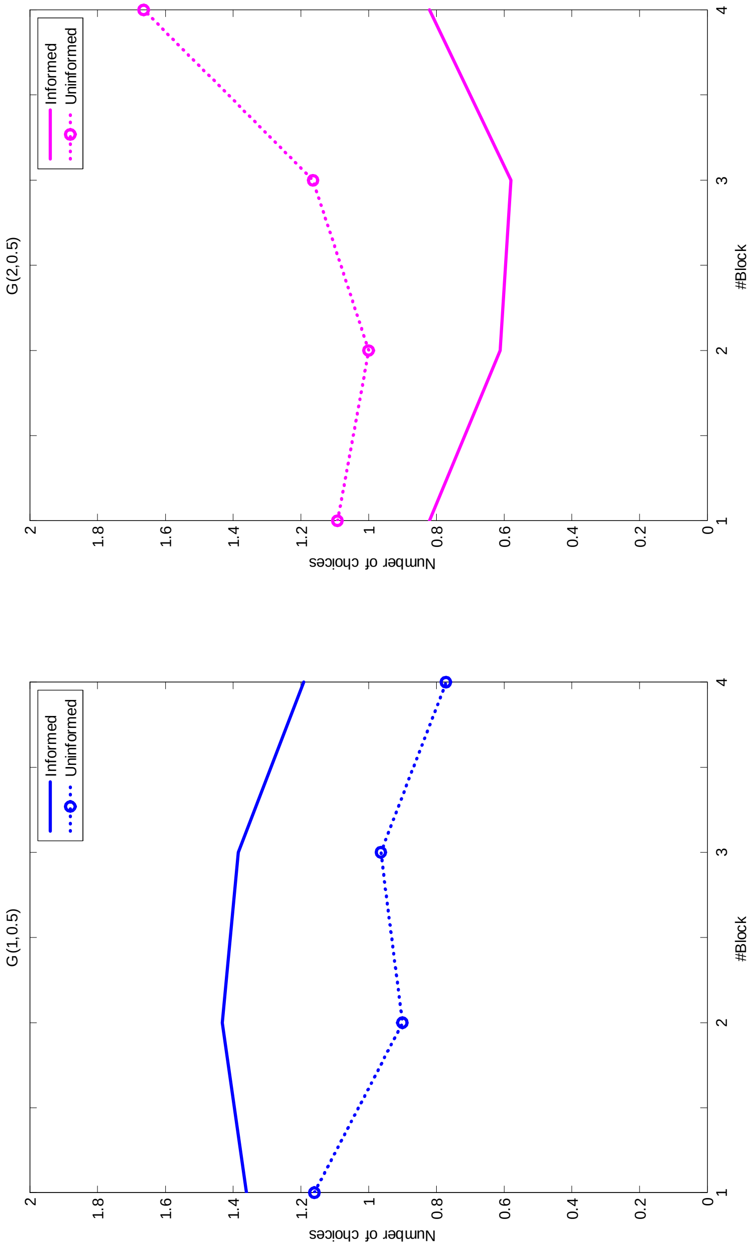


Figure 2.5 First Information Choice in Two Games

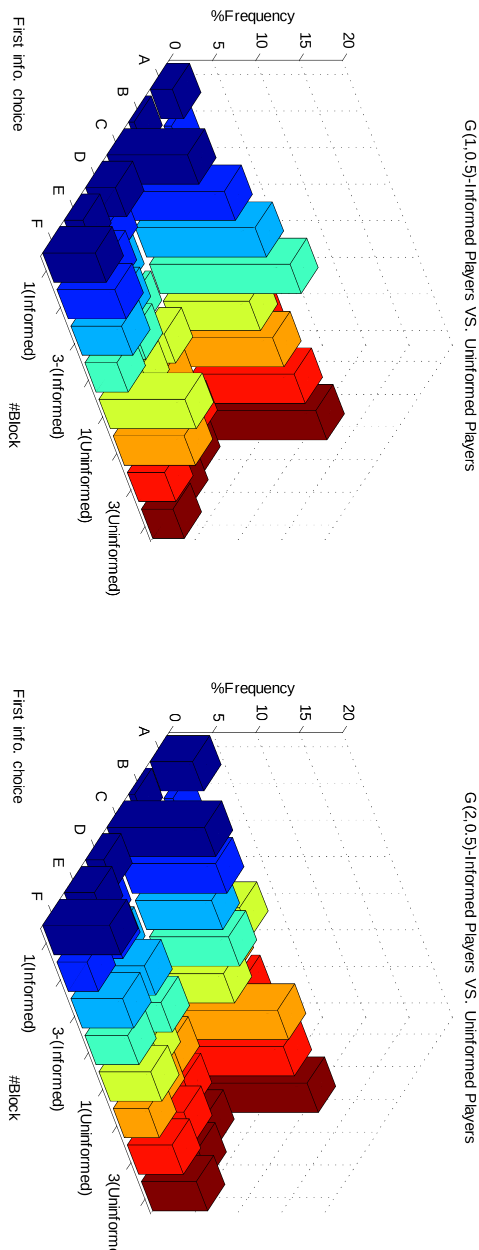
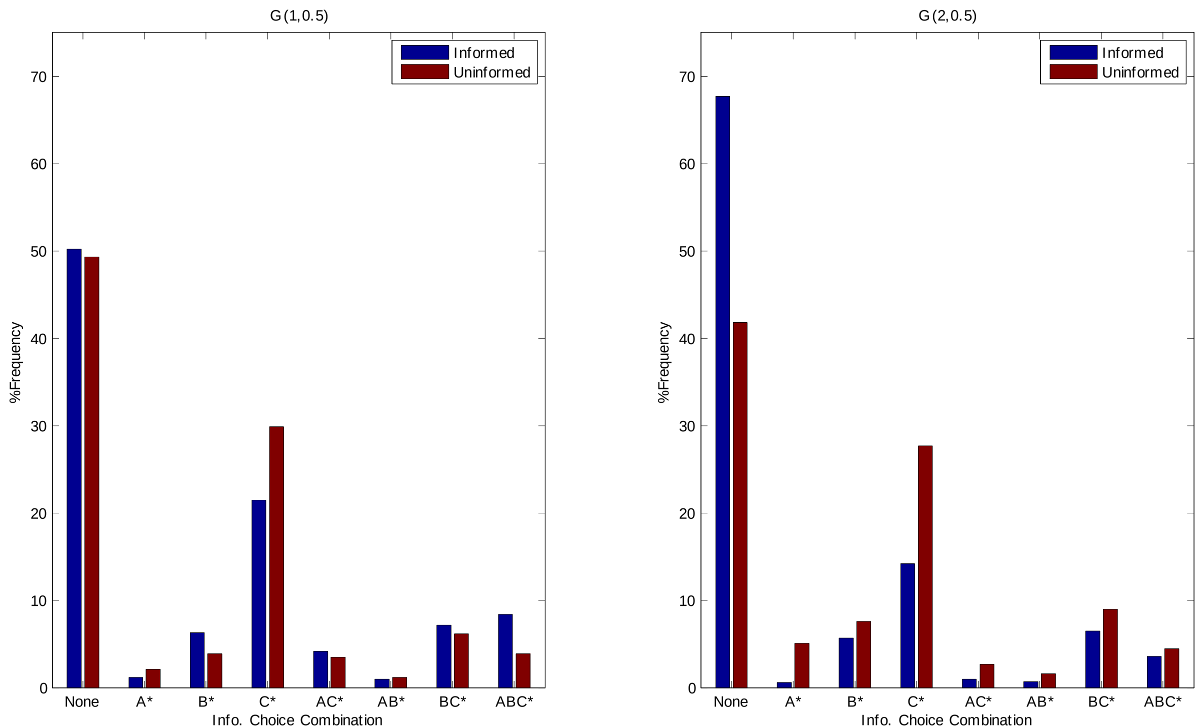


Figure 2.6 Information Choice Combinations[†] within One Round for Two Games



[†] In this figure, A* means either button A or D or both A and D are clicked during one round for one subject, i.e., {{A},{D},{A,D}}; similarly B* means {{B},{E},{B,E}}; C* means {{C},{F},{C,F}}; AC* means the combination of one element in set A* and one element in set C*; Similar definitions for notations AB* and BC* hold; Finally, ABC* means the combination of three elements which come from sets A*, B* and C* respectively.

2.5.3 Structural Estimation and Model Comparisons

To model the behaviors of two informational roles, we consider the structural model EWA which provides measures of learning behaviors. For comparisons, we also consider the reduced RE and the reduced BE models by parameter restrictions. The related notations and the model specifications are introduced below before we present the results.

As we argue in the Introduction, this game has a salient feature that the informed players need to do intertemporal optimization. It is a good reason for believing that informed players take mixed strategies within a round rather than behavioral strategies, thus their strategy space is:{(sda$_1$,sda$_2$), (sda$_1$,non-sda$_2$), (non-sda$_1$,sda$_2$), (non-sda$_1$,non-sda$_2$)}. Besides, taking the Harsanyi Transform, we have eight strategies for informed players in the corresponding normal-form presentation. Similarly for uninformed players, we derive four. Table 2.6 shows the normal-form game. Its theoretical predictions and empirical frequencies in our experiments are found in the last row and column. Players are indexed by i (i=1,... n) and those with odd indices are informed. An individual player's strategy is denoted by s_i. s_{-i} is a strategy combination of player i's partner. The scalar-valued payoff function of player i is $\pi(s_i, s_{-i})$. Denote the actual strategy chosen by player i in round t by $s_i(t)$ and the strategy chosen by his partner by $s_{-i}(t)$, thus $\pi(s_i, s_{-i}(t))$ equals the realized payoff of player i at time t if $s_{i,j} = s_i(t)$; otherwise it equals to

Table 2.6 Normal-Form Presentation for $G(a)$ when p=0.5, Theoretical Predictions and Empirical Frequencies (in square brackets)

	$A, brcr$	$A, nbrcr$	$B, brcr$	$B, nbrcr$	G (1)		G (2)	
L,sda$_1$,sda$_2$	1	$a+1$	0	a	$\frac{1}{4}$	[33.36%]	$\frac{1}{4}$	[29.12%]
L,sda$_1$,nsda$_2$	1	1	0	0	0	[.72%]	0	[.32%]
L,nsda$_1$,sda$_2$	a	0	a	0	$\frac{1}{4}$	[16.32%]	$\frac{1}{4}$	[24.64%]
L,nsda$_1$,nsda$_2$	0	0	0	0	0	[.48%]	0	[.08%]
R,sda$_1$,sda$_2$	0	a	1	$a+1$	$\frac{1}{4}$	[32.16%]	$\frac{1}{4}$	[27.12%]
R,sda$_1$,nsda$_2$	0	0	1	1	0	[.32%]	0	[.08%]
R,nsda$_1$,sda$_2$	a	0	a	0	$\frac{1}{4}$	[16.40%]	$\frac{1}{4}$	[18.16%]
R,nsda$_1$,nsda$_2$	0	0	0	0	0	[.24%]	0	[.48%]
G (1)	$\frac{3}{8}$ [41.28%]	$\frac{1}{8}$ [7.12%]	$\frac{3}{8}$ [43.28%]	$\frac{1}{8}$ [8.32%]				
G (2)	$\frac{5}{16}$ [30.96%]	$\frac{3}{16}$ [17.68%]	$\frac{5}{16}$ [34.72%]	$\frac{3}{16}$ [16.64%]				

his simulated/foregone payoffs given the other player is playing $s_{-i}(t)$.

EWA model has two key variables to characterize the process: the "observation-equivalents" of past experience (the experience, for short) N (t) and the attraction A (t). Each strategy on its space has an attraction which is updated by weighting the player's experience in the game. If player i is informed, i.e., i is an odd number, at time t his j-th strategy has the attraction $A_{i,j}(t)$ defined as:

$$A_{i,j}(t) = \frac{\varphi^{in} N^{in}(t-1) A_{i,j}(t-1) + [\delta^{in} + (1-\delta^{in}) I(s_{i,j}, s_{-i}(t))] \pi(s_{i.j}, s_{-i}(t))}{N^{in}(t)}$$

where $N^{in}(t) = \rho^{in} N^{in}(t-1) + 1$; φ^{in},δ^{in} and ρ^{in} are behavioral parameters restricted within [0,1]; the indicator function $I(s_{i,j}, s_{-i}(t)) = 1$ if $s_{i,j} = s_i(t)$, and 0 otherwise.

The attraction $A_{i,j}(t)$ evolves with endogenous initial conditions $A_{i,j}(0)$ and $N^{in}(0)$. The attraction further determines player i's probability of playing j-th strategy in the next round in a logit way:

$$\widehat{P}_{i.j}(t+1) = \frac{e^{\lambda^{in} A_{i,j}(t)}}{\sum_{j'} e^{\lambda^{in} A_{i,j'}(t)}}$$

If player i is uninformed, the attractions and the predicted probabilities are defined in a similar way with a set of behavioral parameters φ^{un}, δ^{un}, ρ^{un} and two initial conditions $A_{i,j}(0)$ and $N^{un}(0)$. By assuming different sets of behavioral parameters, we investigate the learning behaviors of two groups. Hereafter to simplify the illustrations, we use notations without superscripts to denote both unless specified.

In this model, both ρ and φ are depreciation rates. Their different values allow the attractions and the experiences to evolve at different rates. Other things being equal, the smaller φ is, the less weight is placed on previous attractions, and the faster a player learns. Though theoretically attractions can evolve faster than experience or the other way round, following Ho, Camerer and Chong (2007), a restriction, $\rho \leq \varphi$ or $\rho = \varphi(1-\kappa)$ where $0 \leq \kappa \leq 1$, restricts experiences to evolve faster than attractions.

This dynamic process starts with some endogenous values of attractions and experiences, i.e. $A_{i,j}(0)$ and $N(0)$. To guarantee that the weights $N(t)$ rise over time, following CH99, we impose $0 \leq N(0) \leq \frac{1}{1-\rho}$. In order to make the value of $N(0)$ interpretable as a weight on initial attractions relative to reinforcing payoffs, we also restrict the range of $A_{i,j}(0)$ to be less than or equal to the difference between the minimum and maximum payoffs in the entire game. To avoid chewing up many degrees of freedom, we define the initial attractions of different strategies for one player as his own empirical frequencies proportionally magnified by a multiplier shared by the same informational role. This is different from CH99's treatment. CH99 assumes different players have the same initial attractions of their strategies and simplify $A_{i,j}(0)$ into $A_j(0)$. Our treatment, in comparison, focuses on two informational roles, accommodates individual heterogeneity and avoids the identification problem. The multiplier shared by each informational role is called initial attraction factor (AF) and we reduce several $A_{i,j}(0)$'s into two: $AF^{in}(0)$ for the informed group and $AF^{un}(0)$ for the uninformed group. The value of the initial attraction factors can be interpreted as the strengths on their past plays. Admittedly, this modification is not costless. But, as we run ten more rounds than Feltovich's experiments, taking aside the first 10 rounds seems affordable.

Furthermore, we consider the reduced RE and BE models for comparisons. Several additional restrictions are imposed:

$$\begin{aligned}
\text{Reduced RE model } (RRE) &: \quad \delta = 0, \rho = 0, N(0) = 1 \\
\text{Reduced BE model } (RBE) &: \quad \delta = 1, \rho = \varphi, N(0) = \frac{1}{1-\rho}
\end{aligned}$$

To gather an effective sample for estimation, we exclude those who rarely learn, i.e. those who either do not adjust their strategies or choose information only a few times. We set the threshold at 5, i.e., if an informed player (an uninformed player) changes his first-stage (second-stage) strategy less than 5 times; and/or he chooses to look at the information less than 5 times out of 40. The final sample for estimation is the action data from 61 subjects during 40 rounds. Denote the total number of subjects in this pool by N and the number of repetitions of the games as T rounds. We take 70% of the overall sample size for calibration and the rest for validation. The log-likelihood function is:

$$LL(AF(0), N(0), \varphi, \rho, \delta, \lambda) = \sum_{t=1}^{0.7T} \sum_{i=1}^{N} \ln(\sum_{j=1}^{m_i} I(s_{i.j}, s(t)) \cdot \widehat{P}_{i,j}(t))$$

The maximum likelihood method is applied. Three criteria are used to evaluate it in the calibration phase. They are log-likehood values (LL), Akaike information criteria (AIC) and Bayesian information criteria. For the validation phase, LL and mean-standard deviation (MSD) are calculated.

Table 2.7 shows the results of EWA, RRE and RBE. Since EWA is more general than the other two models, it necessarily fits the data better. It has the largest log-likelihood value, AIC and BIC among these three models. To check the overfit problem, it performs well in prediction too. Chi-squares are calculated for the likelihood ratio test. In G (1), RRE has 160.7241 and RBE 100.2704 while in G (2) we have 177.9877 and 125.5092 for RRE and RBE respectively. Compared to the critical value of relevant degrees of freedom, all of them suggest that EWA performs significantly better. Besides, RBE fits the data better than RRE but predicts slightly worse (MSD criterion). This is consistent with F00's findings[17].

[17] F00, Table IV on page 621 and Table V on page 626.

The interesting results are the value comparisons of these behavioral parameters. First, both informational roles have smaller values of in G(2) than in G(1), and such value deduction is stronger for the informed group. Following the interpretations of φ in Kocher and Sutter (2005), we see that both learn faster in G(2) than in G(1); and that the deduction in φ is greater for the informed group than the other one means, when the equilibrium strategy becomes attractive, the informed group enhances the learning process more than its counterpart. It supports our intuition that the uninformed player is more passive in learning due to a higher perceived (expected) payoff. Second, the informed players always have larger sensitivities of the predicted probabilities to the attractions. In G (1), λ of the informed is 3.8204, while that of the uninformed is 1.3171. In G (2), they are 1.4957 and 0.3427 respectively. This reflects the feature of asymmetric payoffs for these two groups. In the payoff matrix in Table 1, only one among four combinations of two players' actions brings the informed player a positive payoff, and consequently in dynamic process the changes of their strategies relative to the attractions are more sensitive than their partners. Third, our initial attraction factor AF of the informed group is smaller than that of the uninformed group, and it is larger in G (2) than that in G (1). Such comparisons can match with the following observations from Figure 2. The behaviors of the informed group change more from the first to the second block than that of the informed group, and they deviate from the equilibria less at the beginning

Table 2.7 EWA Estimation and Model Comparisons

Model		EWA		Reduced RE Model (RRE)		Reduced BE Model (RBE)	
Informational Role		Informed	Un-	Informed	Un-	Informed	Un-
G (1)	No. of Sbj.	17	9	17	9	17	9
	φ	0.9895	0.9508	0.9622	0.9859	0.9573	0.9557
	ρ	0.9319	0.8384	0	0	0.9573	0.9557
	δ	0.5065	0.7238	0	0	1	1
	λ	3.8204	1.3171	0.5149	0.1546	9.0337	3.0127
	N_0	5.3261	2.8671	1	1	23.4391	22.5484
	AF	0.3841	0.6520	1	0.5	0.1294	0.2046
	Calibration (70%T)		*Validation*	*Calibration*	*Validation*	*Calibration*	*Validation*
	LL	-1037.86	LL -424.26	-1118.22	-453.55	-1087.99	-441.76
	AIC	-1049.86	MSD 0.0783	-1124.22	0.0773	-1093.99	0.0798
	BIC	-1057.61	χ^2 -	-1128.10	160.7241	-1097.87	100.2704
G (2)	No. of Sbj.	14	21	14	21	14	21
	φ	0.8946	0.9181	0.9622	0.9149	0.9619	0.9311
	ρ	0.751	0.6442	0	0	0.9619	0.9311
	δ	0.7036	0.5768	0	0	1	1
	λ	1.4957	0.3427	0.2951	0.0987	5.8142	0.9756
	N_0	2.441	2.8106	1	1	26.2724	14.5113
	AF	0.6259	1	1	0.5	0.1217	0.3432
	Calibration (70%T)		*Validation*	*Calibration*	*Validation*	*Calibration*	*Validation*
	LL	-1352.12	LL -561.97	-1441.11	-612.11	-1414.87	-597.69
	AIC	-1364.12	MSD 0.093	-1447.11	0.0961	-1420.87	0.0968
	BIC	-1371.87	χ^2 -	-1450.99	177.9877	-1424.75	125.5092

of G (2) than in the game of G (1). This is because the initial attraction factor is a multiplier over the individual frequencies of the first 10 rounds' plays for forming initial attractions. It explains how the past plays are linked to players' evaluation of initial attractions over different strategies. The larger it is, the closer players' evaluation of initial attractions is to what they actually play during the first 10 rounds.

Lastly, the following two observations are driven by the nature of constant-sum game as described in CH99: 1) In both games and for both players, the value of ρ is very close to φ. 2) The values of δ for two informational roles are neither close to one nor zero but within [0.4, 0.7]. Decoded from the model setup, it means both informational roles, especially the informed players in G (1) and the uninformed ones in G (2), reinforce the realized payoffs more than the foregone payoffs.

To sum up, we find indirect evidence to support a belief-based model from information acquisition analysis, i.e. players choose to look at their partners' history in order to make their own decisions; meanwhile we bypass calibrations with two models separately and consider both within a uniformed EWA framework. The reduced belief-based model outperforms the reduced reinforcement model for calibration, but it predicts slightly worse than the reduced reinforcement model. Furthermore, EWA model helps us to capture different behaviors (as well as the learning speeds) of two informational roles in these two similar games.

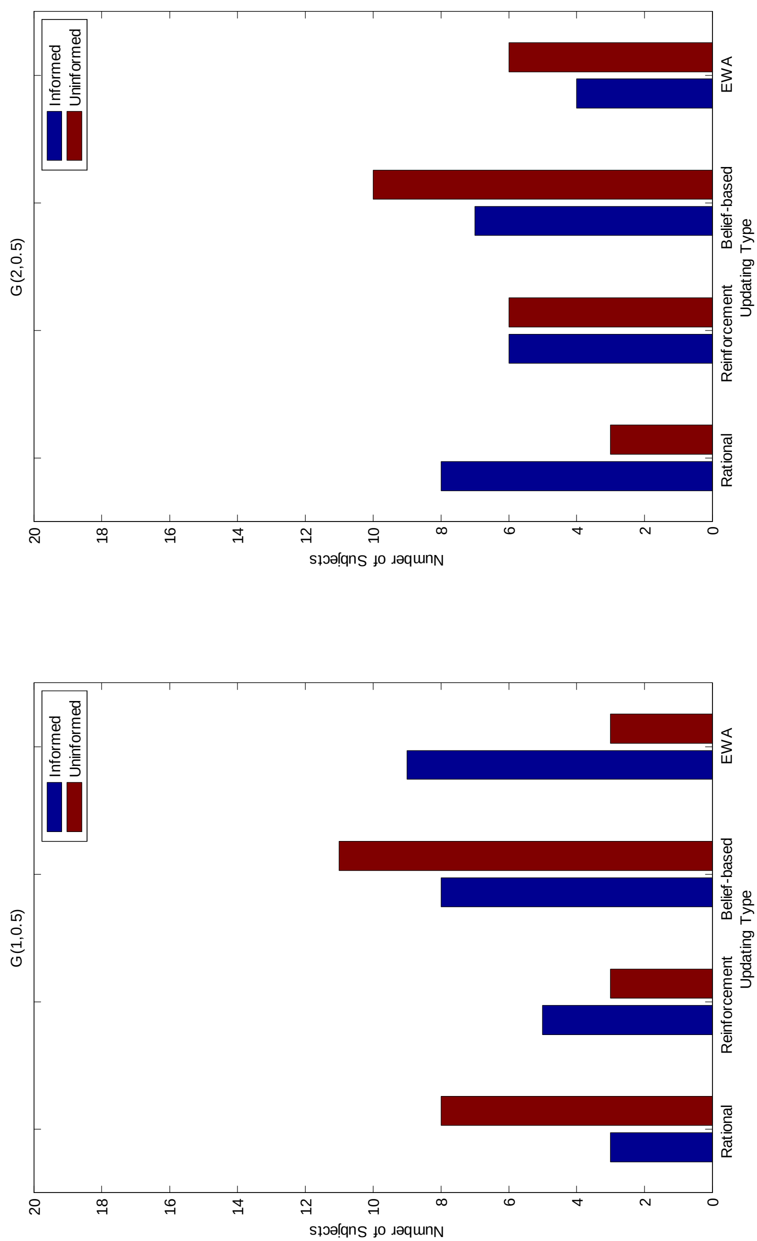
2.6 Updating Patterns and Cluster Estimations

As Schotter (2009) points out, "various models are posited as describing learning behavior of subjects on the individual level yet are tested by aggregating data across subjects and time". In this section we look at subjects' heterogeneity in learning. Moreover, we investigate whether a player's information acquisition behavior provides clues on how he learns. The best fitting model would be the one supported by both information lookup choices and action decision.

As the first step, we define subjects' updating patterns based on their information choices. These updating patterns correspond to different learning theories. More precisely, we map one's information choice combination to his updating pattern. Players of RE learning are those who acquire information sets A*, B*, and/or AB* most often over time; players of BE learning are those who acquire information set C*; and EWA type are those who choose the cross-type information combinations denoted by AC*, BC*, and ABC*. Finally if a subject does not acquire information (here we take the threshold as 35 non-acquisitions out of 40 opportunities), he is classified as rational. Taking each subject's 40 rounds' information choices, we define his updating type by the mode of his information choices. Figure 2.7 is the distribution of different learning patterns in two games. It is clear that BE type has the largest total population among three learning types. The other two learning types have slightly smaller populations. For different informational roles, BE types dominate RE types in both games. If these reveal the “real” updating patterns, it interestingly cautions us that the model would be mis-specified when we use either one type of models to do estimation based on the aggregated action data.

Figure 2.7 Distribution of Updating Type Updating Type is defined as follows: if during 40 rounds a subject does not acquire information for more than 35 rounds, he is considered to have zero-acquisition (allowing the case of curiosity) and he is of “Rational” type. If a subject acquires information, his type is determined by the information choice combination he chooses the most often which corresponds to a specific learning theory. For example, if a subject acquires A* the most times among all his choices during 40 rounds, he is of BE type. The three learning types embrace different information choice combinations: RE has A* and AB*; Belief-based has C*; and EWA has AC*, BC* and ABC*.

Figure 2.7 Distribution of Updating Type



In order to see how each type of players behaves, we have the following cluster analysis. Other than the two informational groups in previous aggregate analysis, here we have six clusters defined by both informational roles and updating patterns. Each cluster has a set of behavioral parameters and initial conditions. The initial attractions are defined to be the multiplication of individual empirical frequencies in the first ten rounds and the initial attraction factor of his cluster. As mentioned before, we have a restriction $\rho = \varphi(1-\kappa)$ where $0 \leq \kappa \leq 1$, and we could draw a cube of EWA parameters. The cube takes , and as three dimensions. To facilitate the model comparisons, we also label some other typical learning models in Figure 2.8: Fictitious Play and Cournot Play from the group of BE learning, and Average Reinforcement and Cumulative Reinforcement from the group of RE learning.

Previously Figure 2.7 shows that two informational roles update their beliefs in a similar way. Are the estimations on their actions consistent? Let's look at Figure 2.8. The solid round point, the solid diamond and the solid square are the behavioral parameters of the informed groups while the unfilled ones are that of the uninformed ones. Shapes represent different updating types. We can observe that there are roughly longer distances between the solid and unfilled parameter points than the distances between every two different-shape points. It suggests that the behavior of the informed group is different from the uninformed group, and within the same group there are insignificant differences among different belief-updating types. In other words, the informational role has a larger impact on players' learning than different updating patterns.

Finally, in order to have a closer look at all these individuals, we offer analysis at the individual level. Ho, Wang and Camerer (2008) carry out individual analysis by allowing different parameters for different subjects in the EWA framework. Here we assign each subject an initial attraction factor which further determines the initial attractions by multiplying his empirical frequencies in the first 10 rounds. It saves some degrees of freedom without loss of individual heterogeneity. The technique is similar as that in the previous section.

The behaviors shown in Figure 2.9 are close to the fictitious plays. The dominance of the green round points coincides with the majority of those seemingly fictitious plays. This clearly suggests that the evidence from information acquisition matches with the evidence from actions. In other words, which kind of information players look at indirectly suggest how they learn. More detailed descriptions of this figure are: almost all green round points concentrate in the corner of "Fictitious play", especially for the informed players; for the uninformed group in G (2), we can find great difference in the BE learners' and RE learners' behaviors; in both games the EWA-type sometimes behaves the same as the BE-type (i.e., G (1)-Uninformed) but sometimes behaves differently (i.e., G (2)-Uninformed).

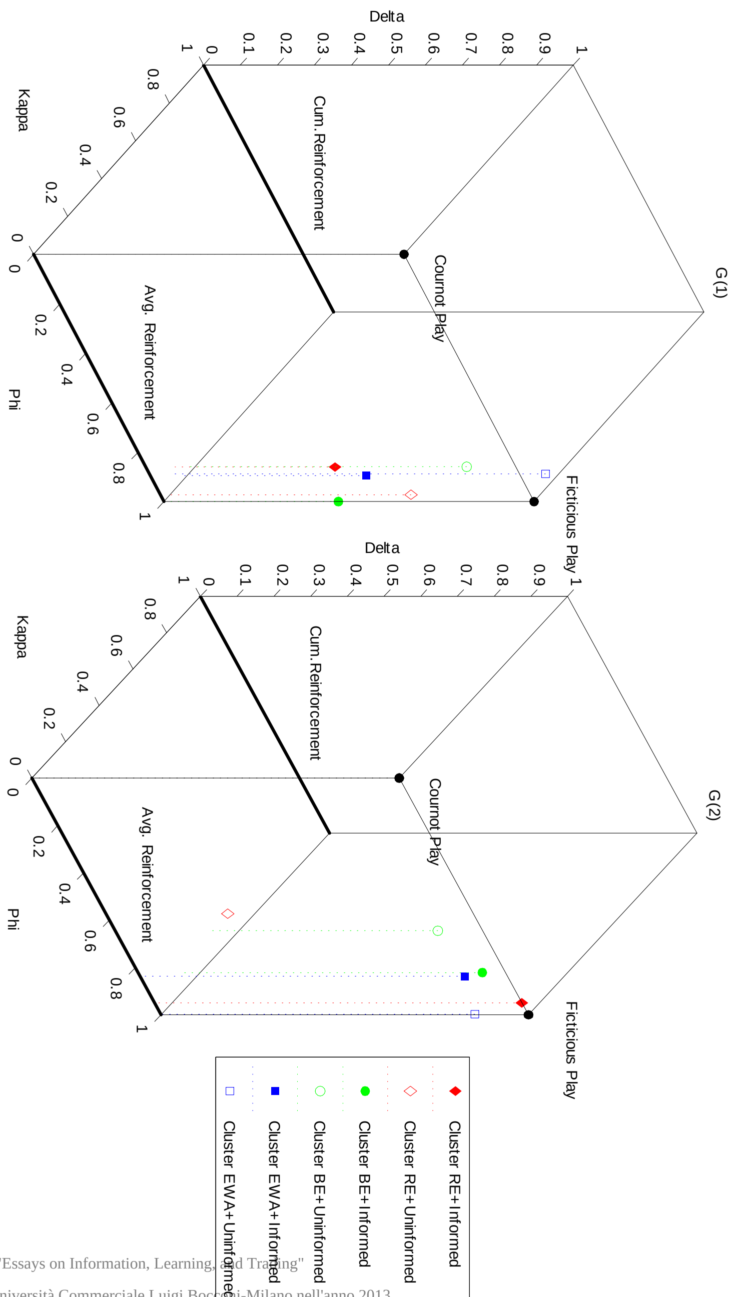
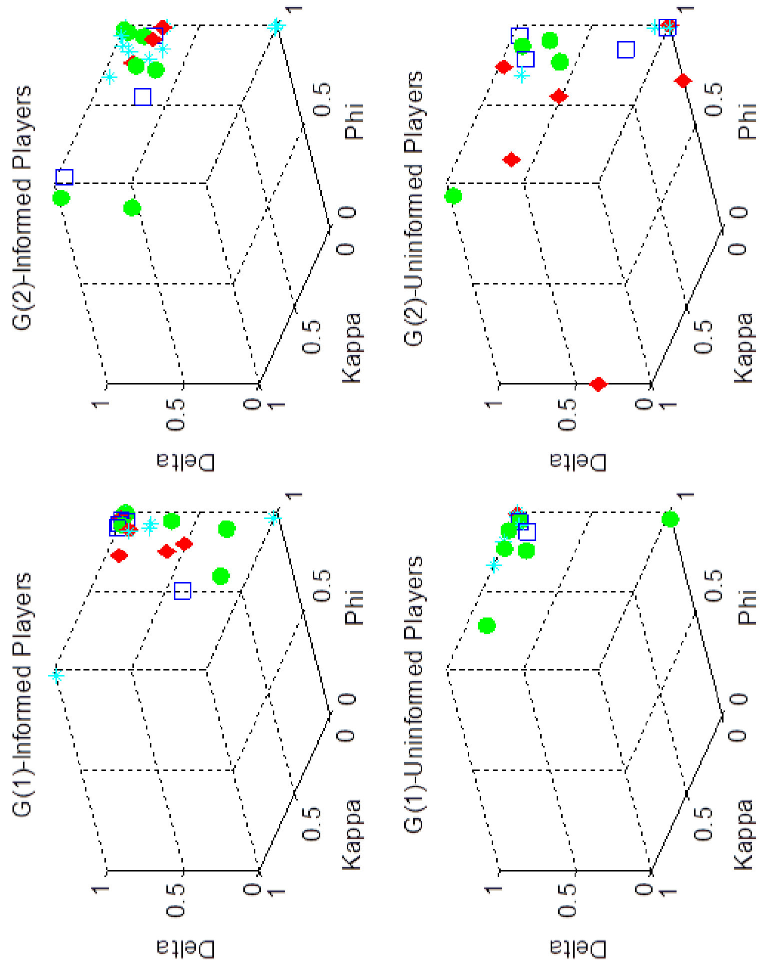


Figure 2.8 Cluster Estimations

Figure 2.9 Individual Learning Behaviors and Updating Type



2.7 Conclusions

In this paper, we study players' learning behaviors in a modified two-stage asymmetric-information game. We investigate whether and how two informational roles learn differently in this game; whether stronger incentives would accelerate the learning process and how we capture learning speed in this game; and whether information-lookup choices provide clues on the way players learn.

We capture different learning speeds for two informational roles by adapting the hybrid EWA model in CH99. By comparing the behavioral parameters in two similar games, i.e., Feltovich's game and our increased-payoff game, we show that: 1) stronger incentives accelerate learning; 2) a relatively high expected payoff for the uninformed drives his passive learning. In the meantime, we find the informed player has a larger sensitivity of the predicted probabilities to the attractions of strategies and a smaller initial attraction factor than the uninformed one.

Second, our experimental design adds some thoughts to Feltovich's research on the relative performance of reinforcement and belief-based models. In this research, by tracking players' information choices of past plays, we provide indirect evidence in support of a belief-based model, i.e., players choose to look at their partners' history in order to make their own decisions. Within the uniformed EWA framework, we have shown that the reduced belief-based model fits the data better than the reduced reinforcement one even though it does not significantly outperform in validation phase. This suggests the calibrations with only action data would make us study learning in the dark, and tracking information acquisition together with exploring the hybrid feature of EWA would shed light on this issue.

Lastly, our cluster estimation shows that informational role has a greater impact on learning than information acquisition pattern. In other words, whether an individual is privately informed is more important than his analogous experience in helping him to learn to play an optimal strategy. In addition, our individual estimation shows that the majority of belief-based type estimated by actions matches with the dominance of belief-based updating type defined by information acquisition measures. This confirms that tracking information acquisition could be a complementary tool to study learning.

Appendix B

(Original instructions are in Chinese. This is one translation for treatment G (1).)

Experiment Instructions

Summer 2011

Welcome to the FEEL laboratory! This is a decision making experiment. You will play a game repeated 50 rounds (the latter 40 rounds are slightly different from the first 10 rounds). Your goal is to earn as many points as possible. Each point is 0.75RMB. At the end of this session, you will get paid in cash plus show-up fee 5RMB. The payment is calculated as follows:

$$\text{Total Payment} = 5 \text{ (Show-up Fee)}$$
$$+\text{total points you have earned} \times \text{convert rate } (0.75)$$

If you have any questions at any time, feel free to ask the experimenter.

1. The Players

There are two types of players, informed player and uninformed player. Your type is chosen randomly at the beginning of this session and will stay the same throughout the whole session. This session consists of many rounds, each divided into two stages. (You will get more information on how to play this game in next section.) Each round you will be randomly assigned a partner, who is of the opposite type. You two are a fixed pair for just one round. You will not be told the identity of your partner—not even at the end of this experiment.

2. Sequence of Play in Each Round

The session is made up of many rounds. In one round, the game will be as follows:

The round begins. One of the two payoff tables on next page is chosen by the computer, and the informed player is told which table was chosen. There is a 50-50 chance of either table being chosen.

Stage 1 begins. You will be assigned a partner. You and your partner each choose an action (either A or B). Notice that if you are an informed player, your action and your partner's action determine your payoff. However, if you are uninformed player, not only your partner's action and yours matter but also the table matters.

Stage 1 ends. Each player is told what action his/her partner chose. Naturally the informed player gets to know her payoff for the stage. But the uninformed one knows only the actions.

Stage 2 begins. You and your partner each choose an action (either A or B).

Stage 2 ends (This round ends). Both players are told the true state, their actions in two stages of this round and their payoffs in this round.

Note: Points are accumulated across rounds and you will know your total points at the end of this session.

3. The Payoff Tables

There are two payoff tables, the Left table and the Right table. In each round you and your partner will be assigned one of these tables. The informed player is told at the beginning of the round which table was chosen; the uninformed player is not told until the end of the round. Each player has two possible moves, A and B. In each stage, either you or your partner will get a point. The point is awarded based on the table you have been assigned, your move, and your partner's move.

If you are an informed player, your payoff is as follows:

LEFT Table (50%)

	Actions		Payoffs	
1st Stage	Yours	His/Hers	Yours	His/Hers
	A	A	1	0
	A	B	0	1
	B	A	0	1
	B	B	0	1
2nd Stage	Actions		Payoffs	
	Yours	His/Hers	Yours	His/Hers
	A	A	1	0
	A	B	0	1
	B	A	0	1
	B	B	0	1

RIGHT Table (50%)

	Actions		Payoffs	
1st Stage	Yours	His/Hers	Yours	His/Hers
	A	A	0	1
	A	B	0	1
	B	A	0	1
	B	B	1	0
2nd Stage	Actions		Payoffs	
	Yours	His/Hers	Yours	His/Hers
	A	A	1	0
	A	B	0	1
	B	A	0	1
	B	B	0	1

If you are an uninformed player, your payoff is as follows:

LEFT Table (50%)

	Actions		Payoffs	
1st	Yours	His/Hers	Yours	His/Hers
Stage	A	A	0	1
	A	B	1	0
	B	A	1	0
	B	B	1	0
	Actions		Payoffs	
2nd	Yours	His/Hers	Yours	His/Hers
Stage	A	A	0	1
	A	B	1	0
	B	A	1	0
	B	B	1	0

RIGHT Table (50%)

	Actions		Payoffs	
1st	Yours	His/Hers	Yours	His/Hers
Stage	A	A	1	0
	A	B	1	0
	B	A	1	0
	B	B	0	1
	Actions		Payoffs	
2nd	Yours	His/Hers	Yours	His/Hers
Stage	A	A	1	0
	A	B	1	0
	B	A	1	0
	B	B	0	1

Either you or your partner gets one point for each stage.

Example 1: Suppose that the Left table was chosen, and that both players choose B in the first stage. Then the informed player gets to know that she gets 0 point while her partner 1 point at this stage. The uninformed player does not know this until the end of the round. And at this moment (the end of Stage 1), the uninformed player does know that 1) his partner has information and chose B, and 2) if the Left table was chosen he gets 1 point, and if the Right table was chosen, he gets 0 point.

Example 2: Suppose instead that the Right table was chosen, and both players choose B in the first stage. Then the informed player gets to know that she gets 1 point. The uninformed player does not know this until the end of the round. And at this moment (the end of Stage 1), the uninformed player does know that 1) his partner has information and chose B, and 2) if the Left table was chosen he gets 1 point, and if the Right table was chosen, he gets 0 point.

Example 3: Suppose that the Left table was chosen, the informed player chooses A, and the uninformed player chooses B in the first stage. Then the informed player gets to know that she gets 0 point. The uninformed player knows that he gets 1 point—in this case his payoff is the same no matter which payoff table had been chosen.

Summary: If the Left table is chosen, only both players choose A makes the informed player get 1 point and the uninformed player get 0 point; while if the Right table is chosen only both players choose B makes the informed player get 1 point and the uninformed player get 0 point. In the rest cases, the informed player gets less than the uninformed player.

4. Playing the game

Once the game begins, your computer screen will be divided into 2 parts (Please refer to Sample Screen on Pages 4 and 5). The top part shows the round number and remaining time for decision-making. The rest is the main screen. Left part shows the true state if you are an informed player otherwise it is empty. Right part needs your entry. If you are an uninformed player, you do not know what the true state is and you can always check the payoff table by looking at previous page of this instruction (you can even tear down the previous page if you feel more comfortable).

In each stage, the computer will ask you to type in a move (either {A} or {B}). After you have done this, click "ok" to continue. After the system has received you and your partner's choices, you will move onto next step.

At the end of each round, uninformed players are told which table had been chosen, and all players are told their total payoff for the round. After observing these results, click "OK" to go on to the next round. Please refer to Figure 3 on Page 5.

5. Historical Information Acquisition

From the 11th round onwards (that is Round 11-50), we offer you an information menu which contains the most recent 10 rounds' history. Sample screen is Figure 4 on Page 5. This information menu has 6 buttons: A. True States and My Actions; B. True States and My Payoffs C. True States and His/Her Actions+ Payoffs; D. Summary: True States and and My Actions; E. Summary: True States and My Payoffs F. Summary: True States and His/Her Actions + Payoffs; If clicking these buttons, you will be able to get the corresponding information. You can click on more than one button, but the information of this click will disappear when you click on the next button.

This information menu is free for you to choose. Please think carefully and make your choice(s).

6. End of This Session

At the end of this experiment, we would like to get your answers to two questions on your screen. Please write down your answers and suggestions on a piece of paper (which has placed on your table). In the meantime, you will also read your accumulated points and total payment on your screen. You are recommended to write it down on your answer sheet as well. When you leave, please bring your belongings and submit the answer sheet. The experimenter will pay you in cash.

Thanks for your attention. Wish you earn as much as possible!

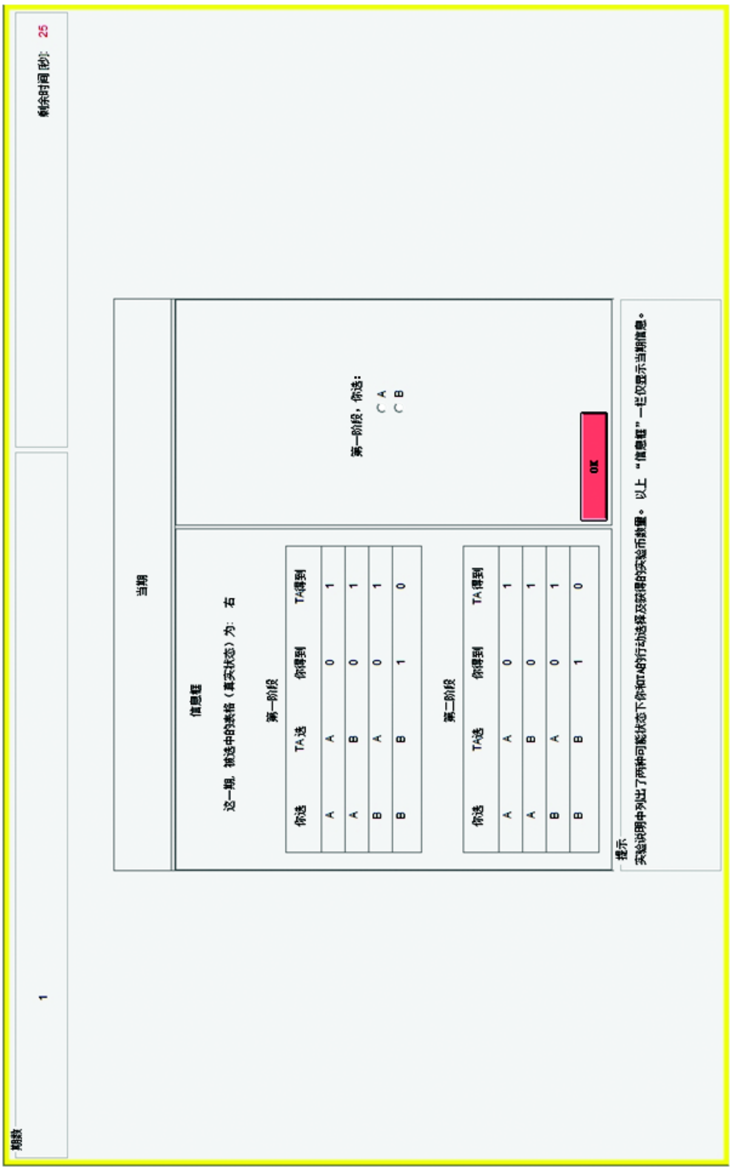


Figure 1 Sample Screen for an Informed Player in the First Stage

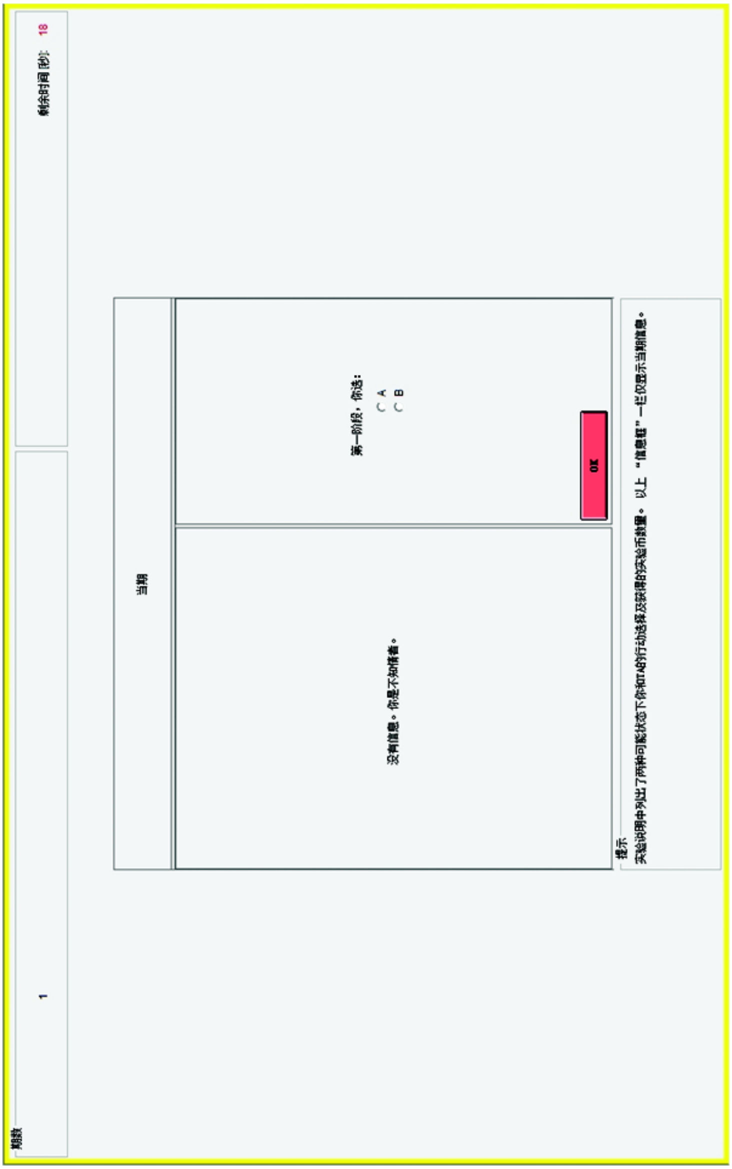


Figure 2 Sample Screen for an Uninformed Player in the First Stage

Figure 3 Sample Screen at the End of One Round

Figure 4 Sample Screen for an Informed Player with Historical Information Menu Available

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Chapter 3

Information Asymmetry and Order Aggressiveness

3.1 Introduction

Non-intermediated, electronic order-driven markets are widely run for stock trading over the world. Two basic types of orders play roles in functioning: limit orders supply liquidity at some specified prices while market orders take liquidity from the best quotes. As such limit order markets can implement "competition-proof" price schedules and guarantee information efficiency, they are "inevitable" (Glosten 1994). However, it is theoretically unclear how informed and uninformed traders interact via limit order books by placing different orders to affect the liquidity as well as market efficiency. Understanding this process helps evaluating information efficiency of limit order markets. This paper proposes an equilibrium model of information-based trade in which both informed and uninformed traders can choose to buy or sell and to submit limit or market orders. We focus on how the information asymmetry is related to order aggressiveness and aim to address the following questions: 1) which of market orders and limit orders do informed traders submit?; 2) how does such informed trade affect market quality?

Early market microstructure literature (e.g., Glosten 1994; Seppi 1997; Handa, Schwartz, and Tiwari 2003) assume that informed traders use market orders. An ongoing effort in analyzing the validity of this assumption has been made in relevant empirical (Keim and Madhavan 1995), theoretical (Chakravarty and Holden 1995; Kaniel and Liu 2006, hereafter KL) and experimental (Bloomfield, O'Hara, and Saar 2005, hereafter BOS) works. We now understand that limit orders can be a vehicle for informed trading. Inspired by Glosten (1994) and BOS, in this paper, we consider two determinants in informed traders' decisions of liquidity provision: fundamental volatility and competition among informed traders.

Glosten (1994) makes the assumption that informed traders use market orders, and he argues for it by two reasons: discounted information value due to immediate disclosure and competition among informed traders.[1] The former one is also emphasized by KL which finds that limit orders convey more information than market orders in some equilibrium with long-lived private information. The latter determinant, i.e., competition among informed traders, has been agreed by several empirical works (Biais Hillion and Spatt 1995; Ranaldo 2004; Beber and Caglio 2005) and a very recent experiment in Stockl (2012). The idea is that only when information is long-lived and the competition among informed traders is not fierce enough, informed traders would prefer to use limit orders.

Additionally, the interesting phenomenon of liquidity provision shift documented by BOS can not be easily explained by these two determinants. In the lab, BOS finds that, as trading progresses, the informed traders shift from market orders to limit orders while the uninformed traders make the reverse shift. This is also empirically confirmed by Anand, Chakravarty, and Martell (2005) and Menkhoff, Osler, and Schmeling (2010) (hereafter MOS). The intuition is that, at the beginning, informed traders take liquidity with market orders to make profits from their private information; in doing so, they push prices closer to the true values, and at some point they can be better off by submitting limit orders and earning the bid-ask spread. In other words, how far the stock is traded away from the true value, or precisely the difference between

[1] This intuition coincides with what Holden and Subrahmanyam (1992) conveys, though the latter paper has a batch market as that in Kyle (1985).

the current mid-quote and the true value that informed traders know, plays a crucial role in their liquidity provision decision.

In our model setup, to capture this crucial factor, we assume that a limit order book is spread symmetrically around the asset value in the first period and informed traders know the future realization of it (either a high value or a low value with equal probabilities). The price change, that is also named as fundamental volatility, corresponds to the distance between the current mid-quote and the true value in BOS's 2005 experiment. At the same time, we consider the arrival rate of an informed trader in each period, a proxy for informed population, to analyze the competition effect that Glosten (1994) suggests. These two determinants shape our main results: conditional on the favorable realization of the liquidation value, informed buyers and sellers are more aggressive than their uninformed counterparts respectively; While an uninf may switch his trade direction to capture the bid-ask spread, and the probability of limit order submissions is frustrated by a marginal increment in volatility; when volatility increases, an informed trader exploits MOs more often to implement his trading goal and he gains from trade and such aggressiveness can match with the experimental findings; in addition, both the volatility and the proportion of informed traders frustrate uninformed traders' willingness to participate in the market and their willingness to provide liquidity if they desire.

Based on the optimal strategies of the informed and the uninformed traders, we also find that the total welfare can be improved by the informed trading. This is derived from the market efficiency brought by the informed traders who gain from the private information and behave rationally in the limit order markets. However, the welfare of the uninformed traders would be deprived partially by a larger proportion of informed participation. Note that we do not examine the effect of information life that KL emphasizes, but take their findings to justify our model setup.

The basic model we present in this paper assumes a discrete price grid for trading, and risk-neutral agents with private evaluations over the asset. This is similar to that in Parlour (1998). Differing from her model, we also assume that an innovation of the asset value that is realized at the end of the trading game and only informed traders know the realization of it. Therefore, the informed trader who might arrive in the first period needs to take into account the impact of his order on the future trader's order decision, and behaves strategically; while the uninformed trader who might arrive in the second period tries to infer the fundamental value from the states of the limit order book and update his beliefs over the fundamental value endogenously. For simplicity, in our basic model the market is open for two periods. An extension of this basic model into a three-period one would share some features with Buti, Rindi, Wen, and Werner (2011)[2] that reveals how order submission strategy is affected by tick size. In general, compared to the close line

[2] It is presented as the first chapter of this thesis.

of research (Buti and Rindi 2011; Buti, Rindi, Wen, and Werner 2011; Buti, Rindi, and Werner 2011; and their predecessor Parlour 1998), information asymmetry added here enriches the literature on limit order markets and improves our understanding on how information plays a role in trading.

It is worthwhile to note that our paper is not the first theoretical one that considers asymmetric information in limit order markets. Though relatively nascent, there are two important works. Goettler, Parlour, and Rajan (2009) numerically solves a dynamic model of limit order markets with asymmetric information. It considers an endogenous information acquisition choice that is not investigated by our model. Rosu (2010) adds asymmetric information into his previous continuous model Rosu (2009), but the strong assumption of cancelling and revising orders instantaneously does not help to capture the feature of the environment with asymmetric-information. More discussions are postponed in the next section.

Section 3.2, that follows, reviews the related literature. Section 3.3 outlines our basic model and Section 3.4 characterizes the features of equilibrium and derive our main results. At the end, Section 3.5 concludes with several remarks. All proofs are collected in the Appendix.

3.2 Literature Review

To understand how new information is incorporated into prices, market microstructure literature has extensively examined the role of asymmetric information in the market. As the order-driven market emerged in 1990s, the field's focus has shifted from the traditional batch auction (e.g., Glosten and Milgrom 1985; Kyle 1985; and Easley and O'Hara 1987) to the limit order market. Theoretical interests shift from the question of setting an optimal bid-ask spread to how to capture discriminatory execution property of limit order books, i.e., orders can be executed at several prices. In line with Glosten (1994)'s seminal paper, models of Biais, Martimort, and Rochet (2000) and Back and Baruch (2012) regard the trading procedure as a common-value auction. Both informational and risk-sharing motivations drive the trading. By contrast, the other larger group of works modeling limit order markets (surveyed by Parlour and Seppi 2008) assumes different private evaluations among traders (and thus the existence of asymmetric information is not a must) and focuses on their order decisions in the markets. The trading procedure is regarded as a private-value auction. This is also the side our paper stands by. We survey the most relevant theoretical works as well as some recent empirical findings.

First, a limited number of theoretical works has embedded asymmetric information into models of limit order markets. Extending Foucault

(1999), Handa, Schwartz, and Tiwari (2003) finds that the bid-ask spread is a function of both the adverse selection cost and differences in valuation. As they assume that the private information is publicly revealed after one trade, they restrict informed traders to trade only by means of market orders. This simplification is justified by what KL asserts: long-lived information is crucial for informed traders to choose to supply rather than demand liquidity. Another extension work is Rosu (2010) which adds asymmetric information into Rosu (2009). It finds that the decision of liquidity provision depends on whether informational advantage is below a cutoff; surprisingly, the price impact of a market order does not depend on the fraction of informed traders; and a higher fraction of informed traders generates a smaller bid-ask spread. Rosu's model has exogenous arrival rates of the patient and impatient sellers/buyers and thus exogenous decisions of limit orders and market orders to buy and sell. The vital problem lies in the strong assumption of cancelling and revising orders instantaneously. Limit orders do not face any pick-off risk, and thus the key feature of asymmetric information environment is missing. Goettler, Parlour, and Rajan (2009) builds a stochastic dynamic model of limit order markets in which traders choose to buy or sell at different prices. Their model also endogenizes traders' decisions of information acquisition. Though the model steps forward in terms of realism, it loses analytic tractability.

Second, as uninformed traders in our model update their beliefs over the true value based on the state of the visible limit order book, this paper closely compares to a growing body of literature on hidden orders and dark pools. One of the possible rationale to use hidden orders is that informed traders hide their private information and disguise themselves as uninformed ones by displaying the same order size. In the two-period model of Moinas (2010), liquidity suppliers are assumed to be asymmetrically informed of the fundamental value in the first period and liquidity takers are assumed to update their beliefs based on the depths of limit order book in the second period. However, since her model does not allow traders to choose to be liquidity supplier or demander, it can not address the informed traders' liquidity provision decisions. Alternative rationale to use hidden orders, what Buti, Rindi, and Werner (2011) proposes, is to reduce the price impact and compete in liquidity provision. The information asymmetry that Buti, Rindi, and Werner (2011) considers does not indicate different information endowment of fundamental values among traders, but it lies in different accesses to trading venues, i.e., some traders can view one more limit order book than the rest when two trading venues are open at the same time.

Lastly, on the empirical side, as the informativeness of limit order book heavily relies on the order submission of the informed, it is important to understand how aggressively, and when, informed traders supply limit orders. Biais, Hillion and Spatt (1995) categorizes different order types of different order sizes into seven groups of aggressiveness. It analyzes how order aggressiveness relates to the market conditions as well as information content of the limit order book in French stock market. Ranaldo (2004) offers similar analysis on Swiss stock market. BC pursues these research by comparing periods when trading is likely to be information-based from other periods. They find that before a positive earning surprise during which informed buyers most likely arrive, buyers are less aggressive than expected. Differing from their methodology, MOS utilizes a comprehensive transaction data from Moscow Interbank Currency Exchange, identifies individual participants and characterize them as informed or uninformed ones, and shows that informed traders use aggressive limit orders more often than uninformed ones when volatility increases. In answer to the narrower question of whether limit orders or market orders informed traders submit, Harris and Hasbrouck (1996), Cooney and Sias (2004), and Hall and Hautsch (2007) contribute.

3.3 The Basic Model

3.3.1 Market Structure

In this market, one asset is traded for T periods and time is denoted by $t = 0, 1, \ldots, T$. In the basic model, $T = 2$. The fundamental value of this asset at the beginning of this trading game is assumed as v_0 and the liquidation value at the end is $\widetilde{v}$. It is realized as $v = v_0 + \widetilde{\Delta}$ where the innovation $\widetilde{\Delta}$ takes the values $+\Delta$ and $-\Delta$ ($\Delta < v_0$) with equal probabilities. Traders arrive in the market sequentially. They can be either informed or uninformed with probabilities α and $(1-\alpha)$ respectively. An informed trader knows perfectly the current and future values of the asset while an uninformed one does not have such information. Both of them have access to the limit order book, which means that the states of the limit order book are observable to both groups.[3] The trading procedure follows the usual rules of price and time priority.[4]

All risk-neutral traders know their own patience type β upon arrival and need to decide whether to submit a limit or a market order to buy or sell. Their utilities are the differences between the private evaluations which depend on their degrees of patience and the cash outflow/inflow upon executions of their orders. A trader arriving in period t and having a market order executed immediately or a limit order executed in period $t^{'}$, where $1 \leq t < t^{'} \leq T$, has the following utility:

$$U(\widetilde{v}, P_{t'}) = \begin{cases} \beta\widetilde{v} - P_t & \text{if he submits a market order to buy} \\ (\beta\widetilde{v} - P_{t'})\,\mathrm{Pr}(\cdot) & \text{if he submits a limit order to buy} \\ (P_{t'} - \beta\widetilde{v})\,\mathrm{Pr}(\cdot) & \text{if he submits a limit order to sell} \\ P_t - \beta\widetilde{v} & \text{if he submits a market order to sell} \end{cases}$$

The transaction price P_t represents the best quote available in pe-

[3]Most of market microstructure literature implicitly take this assumption except the growing body of literature on dark pools and hidden (iceberg) orders.

[4]Price-time priority defines how orders are prioritised for execution. Orders are first ranked according to their price. Orders of the same price are ranked depending on when they are entered.

riod t and $P_{t'}$ is the price specified by a limit order. Both transaction prices can be either an ask or a bid depending on the direction of trade and the order type. $\Pr(\cdot)$ is the execution probability of a limit order and is determined by the price it specifies, the arrival rate of an informed trader α and the possible strategy of an incoming trader. An informed trader knows the realization of $\widetilde{v}$, while an uninformed one makes an order submission decision based on his expected utility and thus bears adverse selection cost eventually. Since a limit order does not get an execution immediately, the limit order submitted by an uninformed trader in the first period undergoes two sources of uncertainties: one comes from the trader's standard balance between execution costs and price opportunity costs; and the other uncertainty comes from the possibility of an adverse change of prices in this asymmetric-information environment. By assuming that some traders are informed of the realization of $\widetilde{v}$ while others are not, we add the informational advantage (for informed traders) and adverse selection cost (for uninformed traders) into the standard balance captured by β in this limit order market. As in Parlour(1998) and others[5], the patience indicator β models the trade-off between execution costs and price opportunity costs a trader faces in an order-driven market, and endogenizes a trader's order submission decision. When a trader has a patience type β close to one, he evaluates the asset with little discount

[5] Degryse, Van Achter, and Wuyts (2009), Buti and Rindi (2011), Buti, Rindi, Wen, and Werner (2011), and Buti, Rindi, and Werner (2011).

or appreciation and he most probably submits a limit order. A trader who has β far away from one is considered to be impatient as he evaluates the asset with a great appreciation or discount, and thus he submits a market order to buy or sell. To make the following analysis tractable, we assume β to be uniformly distributed around one, *i.e.*, $\beta \sim U(0, 2)$ and its density function $f(\beta)$ is equal to $\frac{1}{2}$ when $\beta \in [0, 2]$ and 0 otherwise.

To sum up, Figure 3.1 shows the timeline of the two-period basic model. The liquidation value at the end of the second period v are known by informed traders. In each period, a trader arrives with a probability α to be informed and its complementary probability $1-\alpha$ to be uninformed. He observes the state of the book and makes his order submission. After two periods, the trading game is over. Game structure is common knowledge for all participants. In the following sections, we focus on the order submission decision of the informed and the uninformed trader in the first period and analyze how private information can be incorporated into the LOB. Order submission as well as the state of the book is shortly explained in the next subsection.

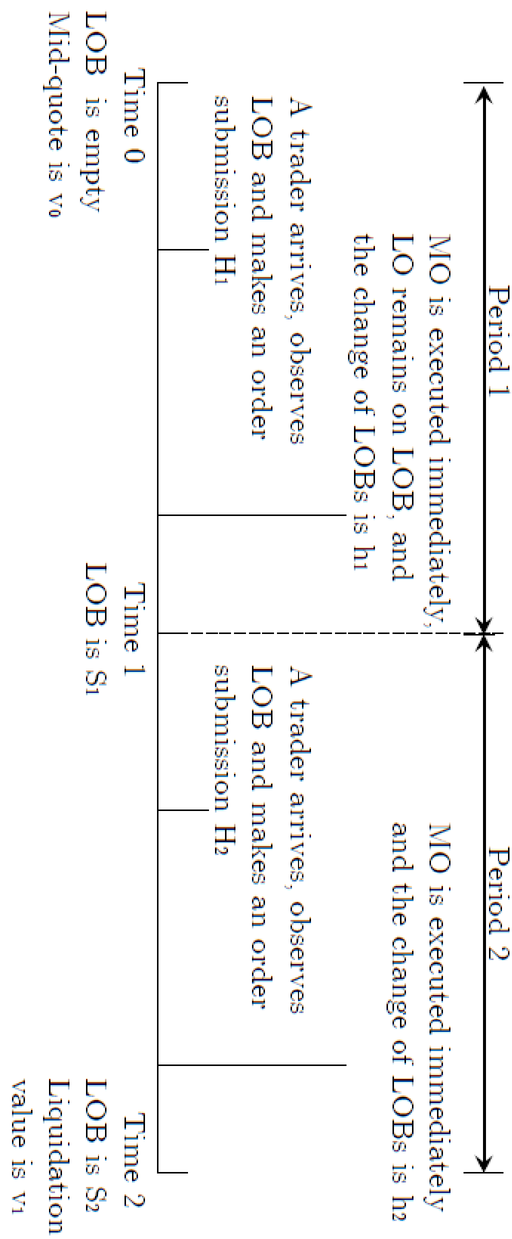

Figure 3.1 Timeline of Two-Period Model

3.3.2 Order Submission Decision

As shown in Figure 3.1, a limit order book (LOB for brevity) opens empty at the beginning. Two prices are assumed to assemble on the price grid, i.e., $A_1 = v_0 + \frac{\tau}{2}$ and $B_1 = v_0 - \frac{\tau}{2}$. The mid-quote of the LOB is v_0. The difference between these two prices is one tick τ which is smaller than the fundamental change Δ. At time t, the state of the LOB that specifies the number of shares Q_t available at these two prices is defined as $S_t = [Q_t^{A_1}, Q_t^{B_1}]$. An empty book is denoted as [00]. In addition, we assume that a trading crowd provides liquidity at $A_2 = v_0 + \frac{3\tau}{2}$ and $B_2 = v_0 - \frac{3\tau}{2}$. The market permits orders of one unit and of two types: limit order (LO) and market order (MO). LOs can be posted at A_1 and B_1. MOs execute against the trading crowd if the book is empty and otherwise against the best quotes available on the LOB.[6]

A buyer (seller) can purchase (sell) one share of the asset by hitting the best ask A_1 or A_2 (the best bid B_1 or B_2), or by submitting a LO at a bid B_1 (at a ask A_1), and otherwise he exits the market with zero profit. Order cancellation is not allowed. We define a trader's strategy at time t as H_t. In some circumstances, superscripts "In" and "Un" are used to specify the strategies of an informed and an uninformed trader respectively, and the notation without any superscript denotes the strategies of

[6] Placing the trading crowd at A_2 and B_2 distinguishs our settings from one-tick market in Parlour(1998) where a trading crowd sits at A_1 and B_1. In her setting, the bid-ask spread is implicitly one tick when the book is empty; and after one order submission the bid-ask spread remains. By contrast, in our model placing a trading crowd at A_2 and B_2 allows us to analyze the changes of bid-ask spread.

both. A LO specifying the price i, a MO executed at the price j, and no trade constitute the strategy space, and they are denoted by $+1^i, -1^j$, and 0 where $i = A_1$ and B_1, and $j = A_{1:2}$ and $B_{1:2}$.[7] A LO provides liquidity while a MO takes liquidity from the LOB. The change of the states of book, h_t, is derived from the order submission and defined as: $\forall t = 1 \ldots T$,

$$S_t - S_{t-1} = h_t \equiv \begin{cases} [\pm 1, 0] & if \ H_t = \pm 1^{A_1} \\ [0, \pm 1] & if \ H_t = \pm 1^{B_1} \\ [0, 0] & otherwise \end{cases} \tag{3.1}$$

Since the execution probability of a LO relates to the possible arrival of a future MO, we need to first analyze a trader's order submission decision in the second period. As the execution probability of a LO in the second period is zero, a trader would either submit a MO or exit the market. Specifically, an informed trader sells if his type β is low, he buys if β is high, and he exits market if the gain-from-trade is zero. More

[7] We do not consider any combination of every two or more single orders in the strategy space. Chakravarty and Holden(1995) finds a combination of a market order to buy and a limit order to sell is more profitable than a market order to buy alone. As their model allows market order walks up and down the limit order book, the limit order acts as a "safety net" for market orders in the opposite direction.

specifically, an informed trader's strategy in the second period H_2^{In} is:

$$H_2^{In} = \begin{cases} -1^B & if\ \beta \in [0, \frac{B}{v}) \\ 0 & if\ \beta \in [\frac{B}{v}, \frac{A}{v}) \\ -1^A & if\ \beta \in [\frac{A}{v}, 2] \end{cases} \tag{3.2}$$

Two threshold values $\frac{B}{v}$ and $\frac{A}{v}$ are computed by equating payoffs from MOs to zero. A and B denote the best ask and bid available on the LOB.

An uninformed trader's strategy H_2^{Un} can be similarly solved based on his belief $E(\widetilde{v}|\mathcal{F}_1)$ where the information set at time 1, denoted as $\mathcal{F}_1$, indicates the state of limit order book S_1 here. If $E(\widetilde{v}|\mathcal{F}_1) < v$, the likelihood of a MO to sell (buy) from the uninformed trader is smaller (larger) than that of the same choice from an informed trader.

Backwardly, in the first period a trader needs to decide whether to buy or sell and also balances between a MO executed immediately and a LO executed with some probability in the next period. A trader makes order submission decision based on his utility function. For instance, a LO to sell at A_1 is optimal for an informed trader informed that $v = v_0 - \Delta$, if it has a higher payoff than that from no trade, a MO to sell, a LO to buy, and a MO to buy, i.e.,

$$(A_1 - \beta v)Pa > Max\{0, B_2 - \beta v, (\beta v - B_1)Pb, \beta v - A_2\} \tag{3.3}$$

The execution probability of a LO to sell for an informed trader, denoted by Pa, is equivalent to the probability of a MO hitting A conditional on the updated LOB $S_1 = [10]$ in the second period, and it is further equal to the average probability of an incoming informed and an incoming uninformed trader's MO to buy weighted by their arrival probabilities:

$$\begin{aligned} Pa &= \Pr(A|[10]) = \alpha \int_{\frac{A_1}{v}}^{2} f(\beta)d\beta + (1-\alpha) \int_{\frac{A_1}{E(\widetilde{v}|[01])}}^{2} f(\beta)d\beta \\ &= \alpha(1 - \frac{A_1}{2v}) + (1-\alpha)(1 - \frac{A_1}{2E(\widetilde{v}|[10])}) \end{aligned} \tag{3.4a}$$

Similarly, the execution probability of a LO to buy for an informed trader Pb is calculated as:

$$Pb = \Pr(B|[01]) = \alpha \cdot \frac{B_1}{2v} + (1-\alpha) \cdot \frac{B_1}{2E(\widetilde{v}|[01])} \tag{3.4b}$$

Plugging these two execution probabilities (3.4a) and (3.4b) into the inequality (3.3), we can get the range of β within which it is optimal to submit a LO to sell. Such submission decision is affected directly by the future price v and indirectly by the incoming trader's updated beliefs ($E(\widetilde{v}|[01])$ and $E(\widetilde{v}|[10])$) via execution probabilities.

Similarly, an uninformed trader's utility maximization derives his optimal strategy:

$$H_1^{Un} \in \arg\max\{EU(+1^{A_1}), \beta v - A_2, EU(+1^{B_1}), B_2 - \beta v, 0\} \quad (3.5)$$

$EU(+1^{A_1})$ and $EU(+1^{B_1})$[8] are expected utilities from submitting a LO at A_1 and a LO at B_1 respectively, which can be expressed in forms similar to that in (3.3) as $(A_1 - \beta v_0)Pa^{Un}$ and $(\beta v_0 - B_1)Pb^{Un}$ with the prior belief of the asset v_0. In these formula, two execution probabilities are different from Pa and Pb, and they can be rewritten as $Pa^{Un} = \alpha \cdot (1 - \frac{A_1}{2v_0} - \epsilon_1(\Delta)) + (1-\alpha) \cdot (1 - \frac{A_1}{2E(\tilde{v}|[10])})$ and $Pb^{Un} = \alpha \cdot (\frac{B_1}{2v_0} + \epsilon_2(\Delta)) + (1-\alpha) \cdot \frac{B_1}{2E(\tilde{v}|[01])}$ with two positive variables $\epsilon_1(\Delta)$ and $\epsilon_2(\Delta)$ increasing in Δ, and $\epsilon_1(0) = \epsilon_2(0) = 0$.

3.3.3 Equilibrium

To find the equilibrium of the model, we first fix the uninformed trader's beliefs conditional on different states of the LOB, and then solve for optimal strategies in the resulting trading game. The optimal strategies determine the states of the LOB which sustain our conjectured uninformed trader's beliefs.

The order submission decision is closely related to the fundamental

[8] Denote the execution probabilities conditional on an increasing and a decreasing future price as Pa^+, Pa^-, Pb^+ and Pb^- , e.g., $Pa^+ = Pa|_{v=v_0+\Delta}$ and $Pa^- = Pa|_{v=v_0-\Delta}$. The expected utilities gained from a LO to sell and a LO to buy are: $EU(+1^{A_1}) = \frac{1}{2}(A_1 - \beta(v_0+\Delta))Pa^+ + \frac{1}{2}(A_1 - \beta(v_0 - \Delta))Pa^-$ and $EU(+1^{B_1}) = \frac{1}{2}(\beta(v_0+\Delta) - B_1)Pb^+ + \frac{1}{2}(\beta(v_0-\Delta) - B_1)Pb^-$

value and the uninformed trader's beliefs. How an uninformed trader updates his beliefs determines what order he submits, and since his order submission changes the state of the LOB, it should have been embedded into the previous trader's order decision in equilibrium. Both an informed and an uninformed trader in the first period are subject to the belief-updating process via execution probabilities due to the uncertainty of an incoming trader's informational role. Therefore, it is necessary for us to set out the uninformed trader's belief-updating process. In the first period, an uninformed trader has a prior belief $E(\widetilde{v}) = v_0$. In the second period, he observes the state of LOB and knows that in the first period an informed trader arrives with the probability α. Hence, he does the following Bayesian updating, provided he observes $S_1 = [01]$ and $[10]$ respectively:

$$\begin{aligned}
E(\widetilde{v}|\,[01]) &= \alpha \cdot [\frac{\Pr(\,H_1^{In} = +1^{B_1}\,|\,v_0 + \Delta)}{2\Pr(S_1 = [01])}(v_0 + \Delta) \\
&\quad + \frac{\Pr(\,H_1^{In} = +1^{B_1}\,|\,v_0 - \Delta)}{2\Pr(S_1 = [01])}(v_0 - \Delta)] \\
&\quad + (1-\alpha) \cdot \frac{\Pr(H_1^{Un} = +1^{B_1})}{\Pr(S_1 = [01])} v_0 \\
E(\widetilde{v}|\,[10]) &= \alpha \cdot [\frac{\Pr(\,H_1^{In} = +1^{A_1}\,|\,v_0 + \Delta)}{2\Pr(S_1 = [10])}(v_0 + \Delta) \\
&\quad + \frac{\Pr(\,H_1^{In} = +1^{A_1}\,|\,v_0 - \Delta)}{2\Pr(S_1 = [10])}(v_0 - \Delta)] \\
&\quad + (1-\alpha) \cdot \frac{\Pr(H_1^{Un} = +1^{A_1})}{\Pr(S_1 = [10])} v_0
\end{aligned}$$

where $\Pr(S_1 = [01]) = \alpha \frac{\Pr(H_1^{In} = +1^{B_1} \big| v_0 + \Delta) + \Pr(H_1^{In} = +1^{B_1} \big| v_0 - \Delta)}{2} + (1 - \alpha) \Pr(H_1^{Un} = +1^{B_1})$ and

$\Pr(S_1 = [10]) = \alpha \frac{\Pr(H_1^{In} = +1^{A_1} \big| v_0 + \Delta) + \Pr(H_1^{In} = +1^{A_1} \big| v_0 - \Delta)}{2} + (1 - \alpha) \Pr(H_1^{Un} = +1^{A_1})$.

When $\alpha = 0$, no informed trader arrives in the first period and the state of the LOB is not informative, and both $E(\widetilde{v}|\,[01])$ and $E(\widetilde{v}|\,[10])$ degenerate to the prior belief v_0. As the arrival rate of an informed trader α increases, the LOB becomes informative and these two expectation values start deviating from the prior belief. Intuitively, a LO to sell more likely appears when the price shifts downward than when the price shifts upward; while a LO to buy more likely appears in the presence of an upward shift than of a downward shift; these two expectation values are adjusted in opposite directions and furthermore both distances are positively linked to α and Δ. After some calculations, they are rewritten as, for $\forall \alpha \in [0, 1)$:

$$E(\widetilde{v}|\,[01]) = v_0 + e_2(\Delta, \alpha) \tag{3.6a}$$

$$E(\widetilde{v}|\,[10]) = v_0 - e_1(\Delta, \alpha) \tag{3.6b}$$

where $e_1(\Delta, \alpha), e_2(\Delta, \alpha) \in [0, \Delta)$ increase in Δ and α, and $e_1(\Delta, 0) = e_2(\Delta, 0) = e_1(0, \alpha) = e_2(0, \alpha) = 0$.

These beliefs, (3.6a), (3.6b), and $E(\widetilde{v}|\,[00])$ that can be similarly calculated, help an uninformed trader make his order submission in the second period and further affects a trader's decision in the first period.

An equilibrium of this model is a fixed point at which traders' behaviors are optimal and consistent with their beliefs and their information. It is defined formally below.

Definition 2 *Given the initial book S_0 and the arrival rate of an informed trader α where $\alpha \in [0,1]$, an equilibrium in this two-period model is a set of order submission decisions $\{H_1, H_2\}$ which best responds to traders' information and Bayesian beliefs, and states of the limit order book based on which the beliefs are formed. An equilibrium includes:*

$$\{H_{1,2} := \arg\max U(\cdot \,|\, E(\widetilde{v}), E(\widetilde{v} | \mathcal{F}_1))\} \text{ and } S_1 \in \mathcal{F}_1$$
$$\{S_t := S_{t-1} + h_t\} \; \forall t = 1,2 \text{ and } h_t \text{ is defined by (3.1)}$$

3.4 Volatility and Informed Trading

In this section, we provide results based on the equilibrium of the basic model, especially the equilibrium strategies of an informed and an uninformed trader in the first period. First, we are interested in what orders informed traders submit. An informed trader's choice of a LO/MO determines what order is informative and how the shape of the LOB is affected. Consequently, we are also interested in: if such informed population increases, does the market become more efficient? We show that informed trading can enhance efficiency in the sense that the midquote of the LOB becomes closer to the fundamental value. This positive finding is consistent with the very recent experiment by Stockl

(2012).

In our model, the payoff of a LO as well as the comparison between a MO and a LO directly depends on the fundamental volatility Δ[9], and indirectly depends on the arrival rate of an informed trader α, i.e., our proxy for an informed population. When we look into an informed and an uninformed trader's utility maximization problem, these two determinants sketch the picture of our main results. The first subsection looks at how volatility influences a trader's order submission in equilibrium while the subsections 3.4.2 and 3.4.3 analyze the indirect influence from the arrival rate of an informed trader on trading behaviors and market welfare.

3.4.1 What Orders Do Informed Traders Submit?

In equilibrium, since an informed trader knows the realization of liquidation value, he can always gain from trade. He optimally submits a MO to buy or sell, or a LO to buy or sell depending on his patience type.[10] The standard balance to decide an order type between the price

[9] It is different from price volatility that can be defined by computing the percentage change of transaction prices.

[10] In fact, when the volatility is very large, a MO to buy can be non-optimal. To restrict the following analysis into the case where all four strategies are optimal for different informed traders, we impose the following assumption: Δ is not larger than $\widehat{\Delta}$ such that (in Figure 2) $\beta_6 = \frac{A_2 - B_1 Pb}{1 - Pb} \cdot \frac{1}{v_0 - \widehat{\Delta}} \geq 2$. In our simulation exercise, $\widehat{\Delta} = 0.26$ given $v_0 = 1$; $\widehat{\Delta} = 2.1$ given $v_0 = 5$.

This assumption does not conflict with the foundation based on which our model has been distilled. .....according to...Daily stock return is rarely high as 30% or even 40% of the stock value. Any potentially "big" surprise would be usually disseminated into the market gradually day after another around the earning-annoucement period.

opportunity costs and the execution costs plays a role. Exiting from the market is non-optimal. Lemma 1 states this formally.

Lemma 3 *It is non-optimal for an informed trader to exit the market.*

Figure 3.2 plots the optimal strategy of informed traders of all patience types $\beta \in [0, 2]$. The left subfigure shows the case with a positive price change while the right subfigure the case with a negative price change. In each one, four dashed lines represent the payoffs of different types of orders for all patience types $\beta \in [0, 2]$. The black solid line is the maximum utility that a trader can achieve by optimally choosing his order. In both subfigures, four strategies, i.e., a MO to sell hitting the price B_2, a LO to buy at B_1, and a MO to buy hitting the price A_2, are possibly used by informed traders depending on their patience level. These two subfigures illuminate how the discrepancy between the liquidation value and the mid-quote of the LOB affects the current buy and sell activities. Suppose there is no surprise at the end, i.e., liquidation value is equal to v_0, the sell and buy activities should be symmetric: those of types $\beta \leq 1$ are sellers and the rest are buyers. When there is a positive surprise, for example, the informed patient trader who is indifferent between participating buy or sell activities in the no-surprise case, i.e., the one with $\beta = 1$, would now submit a LO to buy. His gain-from-trade is Δ discounted by the execution uncertainty. Symmetrically, when there is a negative surprice, this trader would submit a LO to sell to capture his

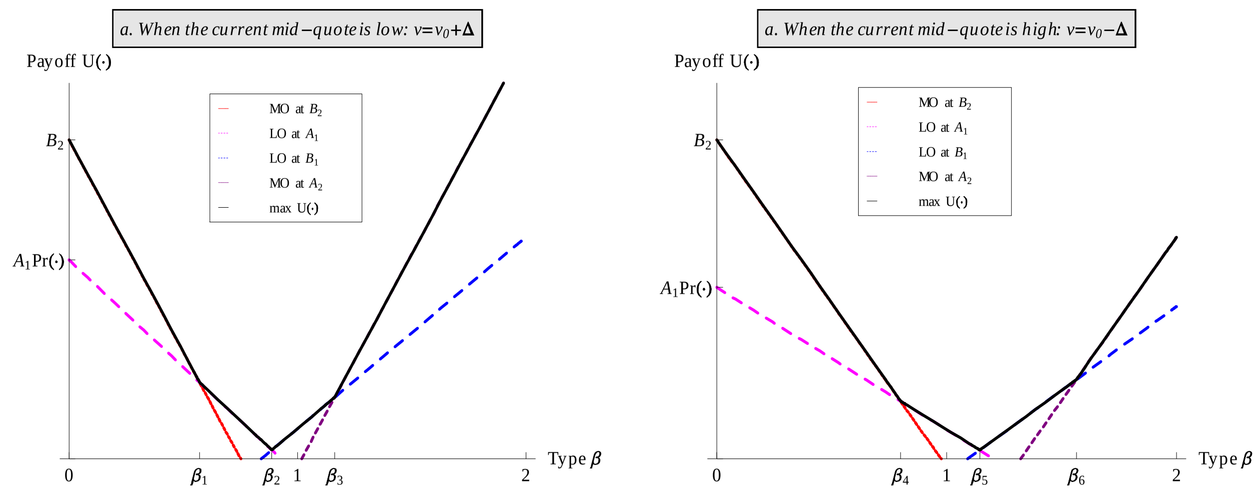

Figure 3.2 Optimal Strategy of An Informed Trader in the First Period

Figure 3.2 Optimal Strategy of An Informed Trader in the First Period This figure plots the possible strategies and the optimal ones for an informed trader of type $\beta \sim U[0,2]$ conditional on two realizations of the liquidation value. Four possible strategies are: a market order at B_2, a limit order at A_1, a limit order at B_1, and a market order at A_2. Their payoffs are depicted respectively by red line, pink dashed line, blue dashed line and violet dotted line. The optimal strategy is the one that yields the highest payoff among these four and depicted by a thick dark line.

gain-from-trade. In general, there are less sell than buy activities when the current mid-quote is low: the cross point of the lines of "LO at A_1" and "LO at B_1" in the left subfigure (i.e., a "threshold" trader who is indifferent between being a patient seller and being a patient buyer) is smaller than 1, while it is larger than 1 in the right subfigure when the mid-quote is already high.

If the current mid-quote v_0 is low and the liquidation value is higher, the informed trader would exploit this trading opportunity to buy; while if the current mid-quote is already high, he would sell the asset. This is quite straightforward. Now, as a further step, the informed trader needs to choose between a MO and a LO. Suppose that the liquidation value is the same as the current mid-quote, or say there is no surprise in liquidation value, whether to submit a LO or a MO depends on the comparison between a price improvement of one tick and a lowered execution probability. When the price improvement outweighs the increased execution uncertainty, submitting a LO is optimal. Now when the price is realized favorably different from v_0 (and such difference is larger than the tick τ as we assume), a MO that guarantees immediate execution can guarantee the gain-from-trade; as volatility increases such advantage makes a MO more attractive than a LO. As volatility increases, therefore, an informed trader would switch from a LO to a MO conditional on good news and switch from a MO to a LO conditional on bad news in the first period. The feature of an endogenous trade direction and an endogenous choice of order type in our model captures this intuition.

Proposition 4 *Given that the future price goes up (down), an informed trader is more likely to buy (sell); furthermore, as the volatility increases, this informed buyer (seller) is more likely to use a MO rather than a LO.*

Table 3.1 provides comparative statistics on equilibrium strategies of both an informed and an uninformed trader, given asset value $v_0 = 1$, tick size $\tau = 0.1$,[11] and small volatilities $\Delta = 0.11$ $(0.11v_0)$ and 0.25 $(0.25v_0)$. The choice of 0.11 is based on the assumption $\Delta > \tau$ while the choice of 0.25 is related to the equilibrium strategy; when $\Delta \geq 0.26$, a MO to buy becomes non-optimal for the informed. We restrict our analysis within the regime where four strategies are all optimal for the informed. As Lemma 1 suggests, informed traders always participate in the market. The strategy of exiting the market can only be optimal for an uninformed trader under some conditions, which we show shortly in Lemma 2. When the current mid-quote is low, i.e., $v = v_0 + \Delta$, the probability of the informed's sell activitiy (Row 3) is lower than that of their buy activitity (Row 6). The opposite holds when the currrent mid-quote is high, i.e., numbers on Row 9 are larger than those on Row 12. Furthermore, if we compare the compositions of LOs and MOs that make up the buy and sell activity seperately, we can find that: conditional on

[11] Simulations for an asset with a high price ($v_0 = 5$) are provided at the end of this section as Tables 3.4- 3.6 (App.1-3) for interested readers. Findings are qualitatively the same as presented here for the case $v_0 = 1$ and thus additional explanations are omitted.

a positive surprise, the informed buyer exploits a MO with the probaility 73.15% and a LO with the complementary probability in the parentheses of Column 3; as the volatility rises, given all else equal, the probability for an informed buyer to exploit the MO becomes larger, i.e., 78.27% in the parentheses of Column 6.

Table 3.1 Optimal Order Submissions of An Informed and An Uninformed Trader This table provides equilibrium strategies of an informed and an uninformed trader in the first period given $v_0 = 1$ and $\tau = 0.1$. Results are reported for different arrival rates of an informed trader $\alpha = \{10\%, 50\%, 90\%\}$ and two volatility scenarios $\Delta = 0.11$ and 0.25: sell activity (rows 3, 9, 15, and 17) and buy activity (rows 6, 12, 16, and 20) with the detailed decomposition by order type (i.e., MOs to sell hitting B_2, LOs to sell at A_1 , LOs to buy at B_1, and MOs to buy hitting A_2). The last row reports the probability of no trade (NT) for an uninformed trader. In the parentheses of Columns 3-8, the proportions of LOs and MOs that make up the sell/buy activity are reported in percentage.

Table 3.1 **Optimal Order Submissions of An Informed and An Uninformed Trader** (Parameter Values: v_0=1, τ=0.1)

	Volatility $\Delta =$	0.11 ($0.11v_0$)			0.25 ($0.25v_0$)*		
	Informed Arrival Rate $\alpha =$	10%	50%	90%	10%	50%	90%
(1) An Informed	Sell Activity	0.4506 (100%)	0.4515 (100%)	0.4525 (100%)	0.4003 (100%)	0.4017 (100%)	0.4036 (100%)
Trader's Order	- MO at B_2	0.2998 (66.53)	0.2926 (64.81)	0.2845 (62.87)	0.2653 (66.28)	0.2517 (62.66)	0.2339 (57.95)
Submissions	- LO at A_1	0.1508 (33.47)	0.1589 (35.19)	0.1680 (37.13)	0.1350 (33.72)	0.1500 (37.34)	0.1697 (42.05)
Conditional on	Buy Activity	0.5494 (100%)	0.5485 (100%)	0.5475 (100%)	0.5997 (100%)	0.5983 (100%)	0.5964 (100%)
$v = v_0+\Delta$	- MO at A_2	0.4019 (73.15)	0.4072 (74.24)	0.4130 (75.43)	0.4694 (78.27)	0.4768 (79.69)	0.4876 (81.76)
	- LO at B_1	0.1475 (26.85)	0.1413 (25.76)	0.1345 (24.57)	0.1303 (21.73)	0.1215 (20.31)	0.1088 (18.24)
(2) An Informed	Sell Activity	0.5614 (100%)	0.5599 (100%)	0.5585 (100%)	0.6653 (100%)	0.6602 (100%)	0.6558 (100%)
Trader's Order	- MO at B_2	0.3787 (67.46)	0.3885 (69.39)	0.3973 (71.14)	0.4555 (68.47)	0.4833 (73.21)	0.5044 (76.91)
Submissions	- LO at A_1	0.1827 (32.54)	0.1714 (30.61)	0.1612 (28.86)	0.2098 (31.53)	0.1769 (26.79)	0.1514 (23.09)
Conditional on	Buy Activity	0.4386 (100%)	0.4401 (100%)	0.4415 (100%)	0.3347 (100%)	0.3398 (100%)	0.3442 (100%)
$v = v_0-\Delta$	- MO at A_2	0.2496 (56.91)	0.2385 (54.19)	0.2277 (51.57)	0.1031 (30.80)	0.0579 (17.04)	0.0122 (3.54)
	- LO at B_1	0.1890 (43.09)	0.2016 (45.81)	0.2138 (48.43)	0.2316 (69.20)	0.2819 (82.96)	0.3320 (96.46)
(3) An	Sell Activity	0.4999 (100%)	0.4996 (100%)	0.4994 (100%)	0.4995 (100%)	0.4979 (100%)	0.4902 (100%)
Uninformed	- MO at B_2	0.3353 (67.07)	0.3379 (67.63)	0.3403 (68.14)	0.3390 (67.87)	0.3530 (70.90)	0.3658 (74.62)
Trader's Order	- LO at A_1	0.1646 (32.93)	0.1617 (32.37)	0.1591 (31.86)	0.1605 (32.13)	0.1449 (29.10)	0.1244 (25.38)
Submissions	Buy Activity	0.5001 (100%)	0.5004 (100%)	0.5006 (100%)	0.5005 (100%)	0.5021 (100%)	0.4969 (100%)
Based on	- MO at A_2	0.3348 (66.95)	0.3363 (67.21)	0.3389 (67.70)	0.3362 (67.17)	0.3439 (68.49)	0.3583 (72.11)
His Expected	- LO at B_1	0.1653 (33.05)	0.1641 (32.79)	0.1617 (32.30)	0.1643 (32.83)	0.1582 (31.51)	0.1386 (27.89)
Payoff	No Trade	0	0	0	0	0	0.0129

* When Δ is in the domain of 0.11 (recall one of our assumptions is that $\Delta > \tau$) and 0.25, a MO to buy and sell, and a LO to buy and sell are optimal for both the informed and the uninformed trader. When Δ is equal to or larger than 0.26, given $v_0 = 1$ and $\tau = 0.1$, a MO to buy becomes non-optimal for some informed traders. More simulation results are provided upon request.

Interestingly, this finding is consistent with the experimental findings of BOS. In BOS, as trade progresses, the distance between the midquote and the true value becomes smaller, i.e., Δ decreases in our model, informed traders submit more LOs and less MOs. Note that in the experimental sessions informed traders and uninformed traders interact at the same time, while informed traders switch the role of liquidity demanders to liquidity providers, uninformed traders change the role of liquidity providers to liquidity demanders. In our model, the traders arrive sequentially with random information-endownment. We can not find the "mirror effect" on the uninformed traders in our model. To complete our characterization of equilibrium, we articulate the effect of volatility Δ on the uninformed trader's LO submission as below.

Lemma 4 *As the volatility Δ becomes larger, other things being equal, an uninformed trader less probably submits a LO and more probably exits the market with no order submission.*

Table 3.1 also provides the optimal order submission of an uninformed trader in the first period. Since the uninformed trader bears a higher execution uncertainty due to a lower unconditional expected probability of MO submissions in the second period, he submits a LO less often. For both buyers and sellers, the probabilities of LO submissions decrease as volatility increases, as shown by Rows 17 and 20. It is also remarkable to read that the adverse selection cost can be so large that the uninformed trader is discouraged and he exits the market. The probability of "No Trade" becomes non-zero when volatility is 0.25 and the informed proportion is 90% in this table. How the high informed proportion contributes into the frustration of the uninformed's trade is discussed in the next subsection.

From this table, it is also noted that even for an uninformed trader, the exploitations of a LO and of a MO are not equal in equilibrium. In order to better describe how aggressively an informed trader behaves in this market, we define an indicator AGG by the percentage difference of the exploitation of the aggressive strategies by an informed buyer (seller) and by an uninformed buyer (seller). Aggressive strategy in this two-period model stands for MOs only and, therefore, the measure of order aggressiveness for an informed seller conditional on a positive surprise in liquidation value $AGG^{sell,+}$ is defined as:

$$\begin{aligned}
AGG^{sell,+} &= \frac{\text{nominator}}{P(H_1^{Un} = -1^{B_2} \,|\text{uninformed sell,} v_0 + \Delta\,)} \\
[\text{nominator} &= P(H_1^{In} = -1^{B_2} \,|\text{informed sell,} v_0 + \Delta\,) \\
-P(H_1^{Un} &= -1^{B_2} \,|\text{uninformed sell,} v_0 + \Delta\,)] \\
&= \frac{P(H_1^{In} = -1^{B_2} \,|\text{informed sell,} v_0 + \Delta\,)}{P(H_1^{Un} = -1^{B_2} \,|\text{uninformed sell,} v_0 + \Delta\,)} - 1
\end{aligned}$$

When the informed trader exploits a MO to implement his trading goal more than his uninformed conterpart, he is considered to be aggressive. This indicator is positive. When the opposite occurs, this informed trader is considered to be passive and the indicator is negative. The larger it is, generally speaking, the more aggressively an informed trader behaves. Similarly, we define the indicator for an informed seller conditional on a negative surprise as well as an informed buyer conditional on differenet realizations of liquidation value. As the positive (negative) surprise is a piece of good news for a buyer (seller), we finally define the aggressiveness indicator conditional good news and bad news separately. They are:

$$AGG^{g} = \frac{AGG^{\text{buy},+} + AGG^{\text{sell},-}}{2} \tag{3.7a}$$

$$AGG^{b} = \frac{AGG^{\text{sell},+} + AGG^{\text{buy},-}}{2} \tag{3.7b}$$

Based on Table 3.1, Table 3.2 derives and reports our measure of informed's order aggressiveness. When the volatility rises, all else being equal, the informed sellers exploit MO less often than their uninformed counterpart if he is informed of a positive surprise in liquidation value, and thus the values of the indicator of informed aggressiveness on row 3 are negative; and if the informed buyer is informed of the same news he would behave more aggressively, i.e., values on row 4 are positive. Symmetrically, when the liquidation value go downward and becomes lower than the current mid-quote, the informed sellers and buyers behaves

Table 3.2 Order Aggressiveness in Two-Period Model This table provides the measure of order aggressiveness given $v_0 = 1$ and $\tau = 0.1$. Results are reported for different arrival rates of an informed trader $\alpha = \{10\%, 50\%, 90\%\}$ and two volatility scenarios $\Delta = 0.11$ and 0.25: sell-side aggressiveness (rows 3 and 5) and buy-side aggressiveness (rows 4 and 6) as well as the average of them conditional on the different realizations of liquidity value (rows 7 and 8).

* More simulation results are provided upon request.

Table 3.2 **Order Aggressiveness in Two-Period Model** (Parameter Values: v_0=1, τ=0.1)

	Volatility $\Delta =$	0.11 ($0.11v_0$)			0.25 ($0.25v_0$)*		
	Informed Arrival Rate $\alpha =$	10%	50%	90%	10%	50%	90%
Conditional on	(1) Sell-side ($AGG^{\text{sell},+}$ ×100)	-0.80	-4.18	-7.73	-2.35	-11.62	-22.34
$v = v_0 + \Delta$	(2) Buy-side ($AGG^{\text{buy},+}$ ×100)	9.27	10.46	11.43	16.52	16.35	13.38
Conditional on	(3) Sell-side ($AGG^{\text{sell},-}$ ×100)	0.76	3.25	5.08	1.92	6.88	6.67
$v = v_0 - \Delta$	(4) Buy-side ($AGG^{\text{buy},-}$ ×100)	-14.99	-19.36	-23.82	-54.14	-75.12	-95.08
Average	Good News: $0.5 \times [(2) + (3)]$	5.02	6.85	8.25	9.22	11.62	10.02
Aggressiveness	Bad News: $0.5 \times [(1) + (4)]$	-7.90	-11.77	-15.78	-28.24	-43.37	-58.71

in the opposite way. In general, an informed trader who is informed of good news exploits a MO more often than an uninformed one while he behaves less aggressively if informed of bad news.

3.4.2 How Is Market Affected by Informed Trading?

This subsection examines how the arrival rate of an informed trader affects the informed trader's aggressiveness in order submission decision. In equilibrium, an informed and an uninformed trader have different submission strategies in the first period, and the arrival rate of an informed trader directly affects the informativeness of the LOB as well as the updated beliefs in the second period based on which a future uninformed trader submits his MO. The arrival rate α affects the liquidity provision decisions of the informed and the uninformed trader by changing the execution probabilities of LOs. Suppose that the informed trader submits a favorable LO conditional on the direction of the price change, i.e., a LO to buy conditional on $v_0 + \Delta$ or a LO to sell conditional on $v_0 + \Delta$. A higher α frustrates the informed traders' LO submissions by decreasing execution probabilities. For instance, when the liquidation value is higher than the current mid-quote, the belief over the fundamental value conditional on the book [01], i.e., (3.6a), is smaller than $v = v_0 + \Delta$, and the probability of a MO submission from the future uninformed seller $\frac{B_1}{2E(\widetilde{v}|[01])}$ is consequently larger than that of a future informed one $\frac{B_1}{2v}$ in (3.2). Thus when calculating the execution probability of a LO to buy, i.e., (3.4b), a higher α results in a smaller value of the execution probability and thus a lower incentive for an informed trader to submit a LO to buy in the first period. Similarly, due to the updated belief $E(\widetilde{v}|\,[10])$ is lower than $v_0 + \Delta$, the future uninformed MO to buy is lower than informed MO to buy and a smaler α results in a higher execution probability of a LO to sell submitted in the first period. To sum up, being informed of $v_0 + \Delta$, informed traders more probably submits LOs to buy and less probably submits LOs to sell. If the informed trader is informed of a negative surprise, the opposite pattern can be argued.

Since these execution probabilities affect both an informed and an uninformed trader in the first period, the above reasoning is also true for an uninformed trader. In addition, an uninformed trader relies on the prior belief v_0 rather than any precise information, and his optimal strategy does not depend on the realized states of v. Hence, when an uninformed trader maximizes his expected utility across different states of $\widetilde{v}$, the effect of α on the future's execution probabilities intra-play with the execution probability. Specifically, a higher α makes an uninformed trader less probably submits a LO and it is likely to have some uninformed traders exit the market.

Proposition 5 *As the arrival rate of an informed trader α becomes larger, other things being equal:*

(1) an informed trader less probably submits a LO conditional on good news but more probably submits a LO conditional on bad news;

(2) an uninformed trader less probably submits a LO and more probably exits the market with no order submission.

Table 3.1 report the simulation results obtained for different arrival rates α. First, we can read that an informed trader less probably submits a LO when α rises. Specifically, When an informed trader is optimal to buy conditional on a positive surprise and to sell conditional on a negative surprise, i.e., rows 6-8 and 9-11 in Table 3.1, the marginal effect of α is positive on the probability of such favorable LO submissions. While such marginal effect of α is negative on the probability of adverse LO submissions, rows 3-5 and 12-14 show that informed traders more probably submits LOs as α rises. As far as an uninformed trader is concerned, the probabilities of his LO submission on both sides (i.e., rows 17 and 20) decrease in α. It is noted that when the voaltility is very high and the informed proportion is very high, the uninformed trader might exit the market, i.e., with a probability 0.0129 when $\Delta = 0.25$ and $\alpha = 90\%$.

3.4.3 Price Efficiency and Welfare

The origin of the interest on informed trader's order submission is derived from the concern of price efficiency in financial market. Intuitively, when informed trader more probably submits a MO, the spread of the limit order book is inevitably widened. The book is pulled downward/upward to the liquidation value that informed traders know. The more probably an informed trader arrives, the faster the book evolves towards the liqudation value. Given all else equal, the larger volatility is, the more significant his contribution in price revelation is. At the same time, when informed traders trade on their information and bring price efficiency into the market, they also financially benefit from the information they know. It brings up the concern on the welfare of the uninformed traders. Financial market is an information-sensitive arena. Information and the ability of processing trade-related information are important for traders. From the regulator's point of view, frostering a healthy and efficient market needs to balance between both efficiency and welfare.

In this subsection, we analyze how welfare is affected by informed trading. Total welfare is defined as the sum of the expected welfare in each period of two groups of traders, i.e., $W = \alpha \sum_t E(W_t^{In}) + (1 - \alpha) \sum_t E(W_t^{Un})$. The expected welfare of informed traders at time t is further defined by summing up the gains from both market and limit orders, i.e., $E(W_t^{In}) = \int_{\beta \in \{\beta : H_t^{In} = -1^{P_t}\}} |P_t - \beta v| \cdot f(\beta) d\beta + \int_{\beta \in \{\beta : H_t^{In} = +1^{P_{t'}}\}} |P_{t'} - \beta v| \cdot \Pr(\cdot) \cdot f(\beta) d\beta$; and the expected welfare of uninformed ttraders at time t, $E(W_t^{In})$, can be similarly defined by substituting H_t^{In} by H_t^{Un}.

To examine how the information asymmetry affects the welfare in the market, we consider an environment free of asymmetric information. It is an alternative setting with no informed traders, i.e., $\alpha = 0$. By comparing the changes of welfare, we can qualify how uninformed traders' welfare is affected by information asymmetry as well as the market conditions.

Proposition 6 *Compared to the no-information environment, informed traders benefit from trading at the cost of uninformed traders' welfare, despite the fact that they bring welfare improvement in the total amount of all market participants.*

Table 3.3 reports that how the total welfare as well as those of different informational roles' are affected by the arrival rate of an informed trader. Firstly, when α becomes larger, the informed traders' welfare improves. Translated in to the numbers in this table, the percentage change of the informed traders' welfare compared to the benchmark case are positive for all arrival rates of an informed trader. Secondly, the uninformed traders' welfare are worsened as the percentage changes are negative. However, the total welfare is improved by the exsitence of informed trading. These observations are true for different volatilities when we compare the upper panel and the lower panel of this table.

At the end of this section, we provides simulations for an asset with a high price ($v_0 = 5$) for interested readers. They are Table 3.4, Table 3.5, and Table 3.6. Findings are qualitatively the same as presented here for the case $v_0 = 1$ and thus additional explanations are omitted.

Table 3.3 Welfare in Two-Period Model This table provides the measure of welfare given $v_0 = 1$ and $\tau = 0.1$. Results are reported for different arrival rates of an informed trader $\alpha = \{10\%, 30\%, 50\%, 70\%, 90\%\}$ and two volatility scenarios $\Delta = 0.11$ and 0.25: informed traders' welfare (rows 3 and 7), uninformed traders' welfare (rows 4 and 8), and the total welfare (rows 5 and 9). The last five columns report the difference in percentage of columns 3-7 and column 2 (the benchmark: no informed traders) .

* More simulation results are provided upon request.

Table 3.3 **Welfare in Two-Period Model** (Parameter Values: v_0=1, τ=0.1)

Volatility $\Delta =$		0.11 ($0.11v_0$)*			Percentage Change ($\times$100)		
Informed Arrival Rate $\alpha =$	0%	10%	50%	90%	10%	50%	90%
(1) Informed Traders' Welfare		0.7676	0.7678	0.7683	1.6823	1.7088	1.7684
(2) Un- Traders' Welfare	0.7549	0.7547	0.7540	0.7537	-0.0265	-0.1192	-0.1590
(3) Total=α (1)+(1-α)$\times$ (2)	0.7549	0.7560	0.7609	0.7668	0.1444	0.7948	1.5757
Volatility $\Delta =$		0.25 ($0.25v_0$)			Percentage Change ($\times$100)		
Informed Arrival Rate $\alpha =$	0%	10%	50%	90%	10%	50%	90%
(4) Informed Traders' Welfare		0.8240	0.8256	0.8289	9.1535	9.3655	9.7960
(5) Un- Traders' Welfare	0.7549	0.7537	0.7509	0.7513	-0.1590	-0.5299	-0.4769
(6) Total=α (4)+(1-α)$\times$ (5)	0.7549	0.7607	0.7883	0.8211	0.7723	4.4178	8.7687

Table 3.4 (App. 1) Optimal Order Submissions of An Informed and An Uninformed Trader This table provides equilibrium strategies of an informed and an uninformed trader in the first period given $v_0 = 5$ and $\tau = 0.1$. Results are reported for different arrival rates of an informed trader $\alpha = \{10\%, 50\%, 90\%\}$ and two volatility scenarios $\Delta = 0.11$ and 0.25: sell activity (rows 3, 9, 15, and 17) and buy activity (rows 6, 12, 16, and 20) with the detailed decomposition by order type (i.e., MOs to sell hitting B_2, LOs to sell at A_1 , LOs to buy at B_1, and MOs to buy hitting A_2). The last row reports the probability of no trade (NT) for an uninformed trader. In the parentheses of Columns 3-8, the proportions of LOs and MOs that make up the sell/buy activity are reported in percentage.

Table 3.4 **Optimal Order Submissions of An Informed and An Uninformed Trader** (Parameter Values: v_0=5, τ=0.1)

	Volatility $\Delta =$	0.11 (1.1τ)			0.25 (2.5τ)*		
	Informed Arrival Rate $\alpha =$	10%	50%	90%	10%	50%	90%
(1) An Informed	Sell Activity	0.4893 (100%)	0.4893 (100%)	0.4893 (100%)	0.4763 (100%)	0.4763 (100%)	0.4764 (100%)
Trader's Order	- MO at B_2	0.4553 (93.05)	0.4550 (92.99)	0.4546 (92.91)	0.4431 (93.03)	0.4423 (92.86)	0.4415 (92.67)
Submissions	- LO at A_1	0.0340 (6.95)	0.0343 (7.01)	0.0347 (7.09)	0.0332 (6.97)	0.0340 (7.14)	0.0349 (7.33)
Conditional on	Buy Activity	0.5107 (100%)	0.5107 (100%)	0.5107 (100%)	0.5237 (100%)	0.5237 (100%)	0.5236 (100%)
$v = v_0+\Delta$	- MO at A_2	0.4770 (93.40)	0.4773 (93.46)	0.4777 (93.54)	0.4910 (93.76)	0.4917 (93.89)	0.4924 (94.04)
	- LO at B_1	0.0337 (6.60)	0.0334 (6.54)	0.0330 (6.46)	0.0327 (6.24)	0.0320 (6.11)	0.0312 (5.96)
(2) An Informed	Sell Activity	0.5113 (100%)	0.5112 (100%)	0.5112 (100%)	0.5263 (100%)	0.5262 (100%)	0.5261 (100%)
Trader's Order	- MO at B_2	0.4760 (93.10)	0.4763 (93.17)	0.4767 (93.25)	0.4901 (93.12)	0.4910 (93.31)	0.4918 (93.48)
Submissions	- LO at A_1	0.0353 (6.90)	0.0349 (6.83)	0.0345 (6.75)	0.0362 (6.88)	0.0352 (6.69)	0.0343 (6.52)
Conditional on	Buy Activity	0.4887 (100%)	0.4888 (100%)	0.4888 (100%)	0.4737 (100%)	0.4738 (100%)	0.4739 (100%)
$v = v_0-\Delta$	- MO at A_2	0.4533 (92.76)	0.4529 (92.66)	0.4525 (92.57)	0.4370 (92.25)	0.4361 (92.04)	0.4352 (91.83)
	- LO at B_1	0.0354 (7.24)	0.0359 (7.34)	0.0363 (7.43)	0.0367 (7.75)	0.0377 (7.96)	0.0387 (8.17)
(3) An	Sell Activity	0.5000 (100%)	0.5000 (100%)	0.5000 (100%)	0.5000 (100%)	0.4999 (100%)	0.4999 (100%)
Uninformed	- MO at B_2	0.4654 (93.08)	0.4655 (93.10)	0.4656 (93.12)	0.4655 (93.10)	0.4660 (93.22)	0.4665 (93.32)
Trader's Order	- LO at A_1	0.0346 (6.92)	0.0345 (6.90)	0.0344 (6.88)	0.0345 (6.90)	0.0339 (6.78)	0.0334 (6.68)
Submissions	Buy Activity	0.5000 (100%)	0.5000 (100%)	0.5000 (100%)	0.5000 (100%)	0.5001 (100%)	0.5001 (100%)
Based on	- MO at A_2	0.4654 (93.08)	0.4655 (93.10)	0.4656 (93.12)	0.4655 (93.10)	0.4660 (93.18)	0.4665 (93.28)
His Expected	- LO at B_1	0.0346 (6.92)	0.0345 (6.90)	0.0344 (6.88)	0.0345 (6.90)	0.0341 (6.82)	0.0336 (6.72)
Payoff	No Trade	0	0	0	0	0	0

* The choice of Δ, 0.11 and 0.25, is determined to facilitate the comparison with the case with $v_0 = 1$. More simulation results are provided upon request.

Table 3.5 (App. 2) Order Aggressiveness in Two-Period Model This table provides the measure of order aggressiveness given $v_0 = 5$ and $\tau = 0.1$. Results are reported for different arrival rates of an informed trader $\alpha = \{10\%, 50\%, 90\%\}$ and two volatility scenarios $\Delta = 0.11$ and 0.25: sell-side aggressiveness (rows 3 and 5) and buy-side aggressiveness (rows 4 and 6) as well as the average of them conditional on the different realizations of liquidity value (rows 7 and 8).

* More simulation results are provided upon request.

Table 3.5 **Order Aggressiveness in Two-Period Model** (Parameter Values: v_0=5, τ=0.1)

	Volatility $\Delta =$	0.11 (0.11τ)			0.25 (0.25τ)*		
	Informed Arrival Rate $\alpha =$	10%	50%	90%	10%	50%	90%
Conditional on $v = v_0 + \Delta$	(1) Sell-side ($AGG^{sell,+} \times 100$)	-0.03	-0.12	-0.23	-0.08	-0.38	-0.69
	(2) Buy-side ($AGG^{buy,+} \times 100$)	0.35	0.39	0.45	0.70	0.76	0.81
Conditional on $v = v_0 - \Delta$	(3) Sell-side ($AGG^{sell,-} \times 100$)	0.02	0.08	0.14	0.02	0.14	0.21
	(4) Buy-side ($AGG^{buy,-} \times 100$)	-0.35	-0.48	-0.59	-0.91	-1.22	-1.55
Average Aggressiveness	Good News: $0.5 \times [(2) + (3)]$	0.18	0.23	0.30	0.36	0.45	0.51
	Bad News: $0.5 \times [(1) + (4)]$	-0.19	-0.30	-0.41	-0.49	-0.80	-1.12

Table 3. 6 (App. 3) Welfare in Two-Period Model This table provides the measure of welfare given $v_0 = 5$ and $\tau = 0.1$. Results are reported for different arrival rates of an informed trader $\alpha = \{10\%, 30\%, 50\%, 70\%, 90\%\}$ and two volatility scenarios $\Delta = 0.11$ and 0.25: informed traders' welfare (rows 3 and 7), uninformed traders' welfare (rows 4 and 8), and the total welfare (rows 5 and 9). The last five columns report the difference in percentage of columns 3-7 and column 2 (the benchmark: no informed traders) .

* More simulation results are provided upon request.

Table 3.6 **Welfare in Two-Period Model** (Parameter Values: v_0=5, τ=0.1)

Volatility $\Delta =$		0.11 $(0.11\tau)^*$			Percentage Change ($\times 100$)		
Informed Arrival Rate $\alpha =$	0%	10%	50%	90%	10%	50%	90%
(1) Informed Traders' Welfare		4.71411	4.71411	4.71412	0.05144	0.05141	0.05156
(2) Un- Traders' Welfare	4.71169	4.71168	4.71164	4.71162	-0.00021	-0.00096	-0.00153
(3) Total=α (1)+(1-α)$\times$ (2)	4.71169	4.71192	4.71288	4.71387	0.00495	0.02523	0.04625
Volatility $\Delta =$		0.25 (0.25τ)			Percentage Change ($\times 100$)		
Informed Arrival Rate $\alpha =$	0%	10%	50%	90%	10%	50%	90%
(4) Informed Traders' Welfare		4.7242	4.7242	4.7243	0.2659	0.2660	0.2668
(5) Un- Traders' Welfare	4.7117	4.7116	4.7115	4.7113	-0.0013	-0.0049	-0.0079
(6) Total=α (4)+(1-α)$\times$ (5)	4.7117	4.7129	4.7178	4.7230	0.0254	0.1306	0.2393

3.5 Conclusions

Our static model embeds asymmetric information into the model of Parlour (1998). It has a closed-form solution and, furthermore, it provides insights on the classical question of what orders informed traders submit and on the recent attention of whether the informed traders use orders more aggressively.

Our equilibrium shows that volatility and competition among informed traders affect the choice of liquidity supply and demand. When the price is known to evolve favorable, the informed buyers and sellers are more aggressive than their uninformed counterparts. When the volatility increases, they exploits MOs more. Such aggressiveness, at the same time, is also positively correlated with the competition among informed traders. In general, we also show that both the volatility and the proportion of informed traders frustrate uninformed traders' willingness to participate in the market and their willingness to provide liquidity if they desire. Total welfare of market participants is improved by informed trading but the informed traders gain at the cost of their uninformed counterpart's welfare.

Admittedly, volatility and competition among informed traders are two independent determinants of order submission decision in our analysis, but they are not exclusive to each other in stock trading. For example, fundamental volatility (the information value defined in BOS) decreases dramatically if the competition among informed traders is fierce. Whether, and how, these two determinants play with each other in affecting the equilibrium is beyond the scope of this paper.

This paper takes a simple framework of two-period to look at how information asymmetry relates to traders' liquidity provision decision. If we extend the model into a dynamic way (with more than three periods), this discrete model becomes far more complicated than being tractable even if some assumptions are made. For example, one possibility is to take Foucault's assumption that a LO has one-period life, i.e., after one period the LO either executes or expires; it makes an individual's decision of order submission directly relates to the next trader's action but not the rest of traders' for execution. Still, such dynamic discrete system hardly works more than three periods. This is because the beliefs in this multiple-stage incomplete-information game would expand exponentially and makes the model computationally challenged. Alternatively, if we follow what Handa, Schwartz, and Tiwari (2003) assumes, i.e., the private information has one-period life and it becomes public after one period, the dynamic structure would collapse into a series of simple two-period ones as we study here.

Our model fairly characterizes some basic features of limit order markets such as price-time priority imposed on trading procedure and long-lived LOs. However, for simplicity, we assume the order size to be one. Considering orders of the size more than one unit and allowing it to walk up/down the LOB, i.e., marketable LO, can provide insights on how information advantage relates to the decision of order-splitting. Intuitively, when the volatility is very large, the favorable informed traders split orders to mimic what the uninformed would do in order to hide their information; while the adverse side of informed traders submit large orders to avoid the incoming price change. When the volatility is very small, the decision of submitting a large order or splitting it into small orders depends on the comparison between expected payoffs from both strategies: a large order likely reveals the private information while several small orders increase trading costs. One potential challenge of this extension is that, as the strategy space is expanded more than being doubled, the complexity of solving the equilibrium rises dramatically.

Appendix C

C.1 Proof of Lemma 3

Proof. An informed trader solves his utimility maximization problem by comparing the payoffs from LOs and MOs. Since $A_1 > B_2$ and $A_2 > B_1$, we have $\frac{B_2}{v} < \frac{A_1}{v}$ (which means the line of "LO at A_1" intercepts the x-axis at $\frac{A_1}{v}$ on the right hand side of $\frac{B_2}{v}$ where the line of "MO at B_2" intercepts the x-axis, as showed in Figure 3.2) and $\frac{B_1}{v} < \frac{A_2}{v}$(which means the line of "LO at B_1" intercepts the x-axis at $\frac{B_1}{v}$ on the left hand side of $\frac{A_2}{v}$ where the line of "MO at A_2" intercepts the x-axis, as showed in Figure 3.2); furthermore, given that $\frac{A_1}{v} > \frac{B_1}{v}$, there is no regime so that the optimal one among these four strategies generates zero profits. ■

C.2 Proof of Proposition 4

Proof. Below we prove the case when the future price goes up. The case with a decreasing future price ($v = v_0 - \Delta$) holds by symmetric arguments.

Given that four strategies are optimal for informed traders, to prove that more buy than sell conditional on a high realization of the liquidation value is equivalent to show that the cross point of "a LO at A_1" and "a LO at B_1" is smaller than 1. By equating payoffs of LOs at A_1 and B_1, i.e., $(A_1 - \beta_2 v)Pa = (\beta_2 v - B_1)Pb$, we have $\beta_2 = \frac{A_1 Pa + B_1 Pb}{(Pa+Pb)\cdot v}$. It is trivial to have $\beta_2 < 1$ as long as $\Delta > 0$.

Now in order to show the second half of this Proposition, we need to prove that $\frac{2-\beta_3}{2-\beta_2}$ is positively correlated with Δ. β_3 is the cross point of the payoffs from "a LO at B_1" and "a MO at A_2" and it is, therefore, equal to $\beta_3 = \frac{A_2 - B_1 Pb}{(1-Pb)\cdot v}$. After subsituting back into $\frac{2-\beta_3}{2-\beta_2}$, we have the MO submission proportion becomes a function of Pa and Pb denoted by $h(Pa, Pb)$. From (3.4a) and (3.4b), we can derive that $Pa'(\Delta)\Big|_{v_0+\Delta} > 0$ and $Pb'(\Delta)\Big|_{v_0+\Delta} < 0$. Taking the first derivatives of $h(Pa, Pb)$ with respect to Δ, we yield the result. ■

C.3 Proof of Lemma 4

Proof. The utility maximization problem of an uninformed trader (3.5) is replicated here: $H_1^{Un} \in \arg\max\{EU(+1^{A_1}), \beta v_0 - A_2, EU(+1^{B_1}), B_2 - \beta v_0, 0\}$ with $EU(+1^{A_1}) = \frac{1}{2}(A_1 - \beta(v_0+\Delta))Pa^+ + \frac{1}{2}(A_1 - \beta(v_0-\Delta))Pa^-$ and $EU(+1^{B_1}) = \frac{1}{2}(\beta(v_0+\Delta) - B_1)Pb^+ + \frac{1}{2}(\beta(v_0-\Delta) - B_1)Pb^-$. Pa^+ denotes $Pa|_{v_0+\Delta}$ for short, Pa^- is $Pa|_{v_0-\Delta}$, and Pb^+ and Pb^- are similarly denoted. The strategy of submitting a LO to sell at A_1 is better than "no trade" if $\frac{1}{2}(A_1 - \beta(v_0+\Delta))Pa^+ + \frac{1}{2}(A_1 - \beta(v_0-\Delta))Pa^- > 0$. We get that the threshold value β_{21}^{Un} is $\frac{A_1(Pa^+ + Pa^-)}{v_0(Pa^+ + Pa^-) + \Delta(Pa^+ - Pa^-)}$(which is smaller than $\frac{A_1}{v_0}$); similarly, we get that the threshold type β_{22}^{Un} between submitting a LO to buy and "no trade" is $\frac{B_1(Pb^+ + Pb^-)}{v_0(Pb^+ + Pb^-) + \Delta(Pb^+ - Pb^-)}$(which is larger than $\frac{B_1}{v_0}$ because $(Pb^+ - Pb^-) < 0$). If $\Delta = 0$, $\beta_{21}^{Un} = \frac{A_1}{v_0} > \frac{B_1}{v_0} = \beta_{22}^{Un}$, thus those traders with their patience types within the rage of $(\beta_{22}^{Un}, \beta_{21}^{Un})$ can be better off by trading. When Δ rises, β_{21}^{Un} decreases and β_{22}^{Un} increases. When Δ is large enough so that $\beta_{21}^{Un} < \beta_{22}^{Un}$, there exists some traders who are better off by exiting the market without order submission.

Now we need to discuss two cases separately in which the LO submissions of uninformed traders decrease in Δ (Lemma 2) and decrease in α (the second part of Proposition 5).

Case 1: when Δ is small and exiting the market is non-optimal

Taking an uninformed trader's utility maximization problem, we define two threshold types β_1^{Un} and β_2^{Un} in the same way as that in the proof of Lemma 1: $\beta_1^{Un} = \frac{B_2 - A_1 Pa^{Un}}{(1-Pa^{Un})\cdot v_0}$ and $\beta_2^{Un} = \frac{A_1 Pa^{Un} + B_1 Pb^{Un}}{[Pa^{Un}+Pb^{Un}]\cdot v_0}$. Similarly, for a buyer, we can find the threshold type β_3 between a LO to buy and a MO to buy and it is $\beta_3 = \frac{A_2 - B_1 Pb^{Un}}{(1-Pb^{Un})\cdot v_0}$. The total probability of LO submissions is $\beta_3^{Un} - \beta_1^{Un}$ and it is a function of Pa^{Un} and Pb^{Un}. Keep in mind that, the first part of Pa^{Un} (Pb^{Un}), i.e., $\alpha \cdot [1 - \frac{A_1}{2v_0} - \epsilon_1(\Delta)]$ ($\alpha \cdot [\frac{B_1}{2v_0} + \epsilon_2(\Delta)]$), embeds the prior belief over the fundamental value instead of the true value v_1 and it approaches to $1 - \frac{A_1}{2v_0}$ ($\frac{B_1}{2v_0}$) with a distance $\epsilon_1(\Delta)$ ($\epsilon_2(\Delta)$) increasing in Δ. Substituting (3.6a) and (3.6b) into Pa^{Un} and Pb^{Un}, we have $Pa^{Un} = \alpha \cdot [1 - \frac{A_1}{2v_0} - \epsilon_1(\Delta)] + (1-\alpha)\cdot(1 - \frac{A_1}{2(v_0 - e_1(\Delta,\alpha))})$ and $Pb^{Un} = \alpha \cdot (\frac{B_1}{2v_0} + \epsilon_2(\Delta)) + (1-\alpha)\frac{B_1}{2(v_0+e_2(\Delta,\alpha))}$. Further substituting them into $\beta_3^{Un} - \beta_1^{Un}$ and taking the first-order derivative with respect to Δ, we get a negative value.

When we take the first-order derivative with respect to α, we have $\frac{d(\beta_3^{Un} - \beta_1^{Un})}{d\alpha} = \frac{\partial \beta_3^{Un}}{\partial Pb^{Un}} \frac{\partial Pb^{Un}}{\partial \alpha} - \frac{\partial \beta_3^{Un}}{\partial Pa^{Un}} \frac{\partial Pa^{Un}}{\partial \alpha}$. Since $\frac{\partial \beta_3^{Un}}{\partial Pb^{Un}}$ and $\frac{\partial \beta_3^{Un}}{\partial Pa^{Un}}$ are negative, we need to compare $\frac{\partial Pa^{Un}}{\partial \alpha}$ with $\frac{\partial Pb^{Un}}{\partial \alpha}$. They are $(1-\frac{A_1}{2v_0}-\epsilon_1(\Delta))-(1-\frac{A_1}{2(v_0-e_1(\Delta,\alpha))})+(1-\alpha)(-\frac{A_1}{2(v_0-e_1(\Delta,\alpha))})'$ and $\frac{B_1}{2v_0}+\epsilon_2(\Delta)-\frac{B_1}{2(v_0+e_2(\Delta,\alpha))}+(1-\alpha)(\frac{B_1}{2(v_0+e_2(\Delta,\alpha))})'$ respectively. $\frac{\partial Pb^{Un}}{\partial \alpha}$ is generally positive and $\frac{\partial Pa^{Un}}{\partial \alpha}$ is positive except when α is large. Thus $\frac{d(\beta_3^{Un}-\beta_1^{Un})}{d\alpha}$ is negative except when α is large.

Case 2: when Δ is large and exiting the market can be optimal

In this case, the total probability of LO submission becomes: $(\beta_{21}^{Un}-\beta_1^{Un})+(\beta_3^{Un}-\beta_{22}^{Un})$. Similar calculations yield the results. ■

C.4 Proof of Proposition 5

Proof. The proof of the second part is contained in Appendix C.3. Below I show only the first part of this Proposition.

From (3.6a) and (3.6b), we know that when $\alpha=0$, $E(v_1|\,[01])=E(v_1|\,[10])=v_0$, thus (3.4a) and (3.4b) are equal to $1-\frac{A_1}{2v_0}$and $\frac{B_1}{2v_0}$respectively. So, $Pa^{In}=Pb^{In}$ when $\alpha=0$.To focus on the impact of α, we use $Pa(\alpha)$ and $Pb(\alpha)$ to denote Pa and Pb respectively. And we have $Pa(0)=Pb(0)=\frac{B_1}{v_0}\equiv m<0.5$.

Now let's consider two derivatives at the origin and simplify the notations by removing the superscripts "In": $Pa^{'}(0)$ and $Pb^{'}(0)$.

$$Pa^{'}(0)=\lim_{\alpha\to 0^+}\frac{Pa^{In}(\alpha)-Pa^{In}(0)}{\alpha}$$
$$=\lim_{\alpha\to 0^+}\frac{\alpha(1-\frac{A_1}{2v_1})+(1-\alpha)(1-\frac{A_1}{2(v_0-e_1)})-(1-\frac{A_1}{2v_0})}{\alpha}$$
$$=(\frac{A_1}{2v_0}-\frac{A_1}{2v_1})+\lim_{\alpha\to 0^+}\frac{\frac{A_1}{2v_0}-\frac{A_1}{2(v_0-e_1)}}{\alpha}(1-\alpha)$$
$$=(\frac{A_1}{2v_0}-\frac{A_1}{2v_1})+\lim_{\alpha\to 0^+}\frac{\frac{A_1}{2v_0}-\frac{A_1}{2(v_0-e_1)}}{\alpha}-\lim_{\alpha\to 0^+}(\frac{A_1}{2v_0}-\frac{A_1}{2(v_0-e_1)})$$

$$= (\frac{A_1}{2v_0} - \frac{A_1}{2v_1}) + \frac{A_1}{2v_0}\frac{e_1^{'}\big|_{\alpha=0}}{v_0} = \frac{A_1}{2}(\frac{1}{v_0} - \frac{1}{v_1} - \frac{e_1^{'}\big|_{\alpha=0}}{v_0^2})$$

$$Pb^{'}(0) = \lim_{\alpha\to 0^+} \frac{Pb^{In}(\alpha) - Pb(0)}{\alpha}$$

$$= \lim_{\alpha\to 0^+} \frac{\alpha\frac{B_1}{2v_1} + (1-\alpha)\cdot\frac{B_1}{2(v_0+e_2)} - \frac{B_1}{2v_0}}{\alpha}$$

$$= (\frac{B_1}{2v_1} - \frac{B_1}{2v_0}) + \lim_{\alpha\to 0^+} \frac{\frac{B_1}{2(v_0+e_2)} - \frac{B_1}{2v_0}}{\alpha}(1-\alpha)$$

$$= (\frac{B_1}{2v_1} - \frac{B_1}{2v_0}) + \lim_{\alpha\to 0^+} \frac{\frac{B_1}{2(v_0+e_2)} - \frac{B_1}{2v_0}}{\alpha} - \lim_{\alpha\to 0^+}(\frac{B_1}{2(v_0+e_2)} - \frac{B_1}{2v_0})$$

$$= (\frac{B_1}{2v_1} - \frac{B_1}{2v_0}) - \frac{B_1}{2v_0}\frac{e_2^{'}\big|_{\alpha=0}}{v_0} = \frac{-B_1}{2}(\frac{1}{v_0} - \frac{1}{v_1} + \frac{e_2^{'}\big|_{\alpha=0}}{v_0^2})$$

Given what Lemma 1 suggests, we need to discuss two cases: when both LO and MO are optimal for sellers and buyers, and when the change of prices is so large that LO at $A_1(B_1)$ is not optimal for any seller (buyer) when adverse information arrives. Below we show that this lemma holds when the future price goes up, and omit the proof of the case when the future price goes down as it can be obtained by symmetric arguments.

Define β_3 is a trader type who evaluates a limit order to buy (the strategy $+1^{B_1}$) and a market order to buy (the strategy -1^{A_2}) equal, i.e., $(\beta_3 v_1 - B_1)Pb = \beta_3 v_1 - A_2 \Rightarrow \beta_3 = \frac{A_2 - B_1 Pb}{v_0 - v_1 Pb}$; In (**??**), $\beta_1(= \frac{B_2 - A_1 Pa}{v_0 - v_1 Pa})$ is the threshold for submitting a LO and a MO to sell. The distance between these two thresholds is a range of traders who submit LOs. We need to prove that the distance $(\beta_3 - \beta_1 = g(\alpha))$ decreases in the arrival rate of an informed trader α.We have:

$$g(0) = \frac{A_2 - B_1 Pb(0)}{v_0 - v_1 Pb(0)} - \frac{B_2 - A_1 Pa^{In}(0)}{v_0 - v_1 Pa^{In}(0)} = \frac{(3+m)\tau}{v_0 - v_1 m}$$

$$g'(0) = [(\frac{A_2 - B_1 Pb}{v_0 - v_1 Pb})^{'}_{Pb} \cdot Pb^{'} - (\frac{B_2 - A_1 Pa}{v_0 - v_1 Pa})^{'}_{Pa} \cdot Pa^{'}]\Big|_{\alpha=0}$$

$$= \frac{A_2 v_1 - B_1 v_0}{(v_0 - v_1 m)^2} \cdot (-\frac{B_1}{2})(\frac{1}{v_0} - \frac{1}{v_1} + \frac{e_2^{'}\big|_{\alpha=0}}{v_0^2}) - \frac{A_1 v_0 - B_2 v_1}{(v_0 - v_1 m)^2} \cdot \frac{A_1}{2}(\frac{1}{v_0} - \frac{1}{v_1} -$$

$\frac{e_1^{'}\big|_{\alpha=0}}{v_0^2})$

Given $(\frac{1}{v_0} - \frac{1}{v_1} - \frac{e_1^{'}\big|_{\alpha=0}}{v_0^2})$ and $(\frac{1}{v_0} - \frac{1}{v_1} + \frac{e_2^{'}\big|_{\alpha=0}}{v_0^2})$ are positive, $g'(0)$ results in a negative value.

$\because g(\alpha) \approx g(0) + g'(0) \cdot \alpha$ for $\alpha > 0 \therefore \exists 0 < \alpha^{'} < \alpha^{''}, g(\alpha^{'}) > g(\alpha^{''})$.

■

C.5 Proof of Proposition 6

Proof. Given that Proposition 5 holds, following the definition of welfare, we yield the result. More details are available upon request. ■

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