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THE STIFFNESS MATRIX OF THE FIXED-END COMPOSITE FRAME ELEMENT

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ABSTRACT. The displacement method of frame analysis involves determination of the element stiffness matrix. In the paper, the stiffness matrix of the fixed-end composite frame element (type “k”) is derived. In general, the composite section consists of concrete part, steel section, reinforcing and prestressing steel. Concrete is considered as an aging, linear viscoelastic material. The relaxation of the prestressing steel is taken into account. The derived expressions can be used with any creep function and, also, for elements with variable cross-sections. The derivation of the element stiffness matrix follows two different approaches. The first approach is based on the integral stress-strain relation for concrete. The linear integral operators are used. In the second approach, the element flexibility matrix and the resisting force vector are determined through the numerical integration of the cross-section deformations. The class of cross-sections which accounts for viscous effects of a composite section is developed. The results of the two approaches are compared in the numerical example.

1. Introduction

In the analysis of structures by the slope deflection method the stiffness matrices are used. In this paper, the stiffness matrix of the fixed-end composite frame element (element type ″k″) is presented. Study the composite member which, in general, consists of concrete (c), prestressing steel (p), steel member (n) and reinforcing steel (m). Concrete is considered as an aging, linear viscoelastic material. The relaxation of the presstresing steel is taken into account. The determination of the stiffness matrix of a composite, type ″k″, frame element is analogous to the direct derivation of the stiffness matrix of a homogeneous elastic element. The stiffness matrix is obtained using the basis stiffness matrix. The flexibility matrix of a composite element is derived firstly, and, by its inversion, the element basic stiffness matrix is found. The procedure is general, and all expressions can be used with any given concrete creep function and for the element with a variable cross section. In the paper, two different methods for derivation of the element stiffness matrix are presented: the mathematical method based on the use of linear integral operators, and the numerical method based on the numerical integration of the cross-section deformations and use of object-oriented classes.

2. The mathematical method

In the mathematical method, the expressions are developed using linear integral operators without any mathematical simplifications introduced, apart from the inevitable approximations referring to the rheological properties of constituent materials. The mathematical method which uses the linear integro-differential operators is given by Mandel in the theory of aging linear viscoelasticity. Since the operators obey the algebra laws of ordinary numbers, the problem is solved symbolically and formally so that the mathematical operations are only noted. Developing the theory of composite structures, Lazic introduced linear integral operators which also obey the algebra laws of ordinary numbers [1]. Using this type of operators, the problem is solved not only symbolically and formally but also the expressions for stresses and displacements are significantly simplified. Their values can be obtained by the least number of mathematical operations. The same procedure is used in the paper.
The basic expressions of the composite cross section [1] are :

$$\eta = \frac{1}{E_u F_i} \widehat{F}'_{11} N + \frac{1}{E_u S_i} \widehat{F}'_{12} M, \quad \kappa = \frac{1}{E_u S_i} \widehat{F}'_{12} N + \frac{1}{E_u J_i} \widehat{F}'_{22} M , \tag{1}$$

where $\eta = \eta(x,t,t_0)$ is the normal strain, $\kappa = \kappa(x,t,t_0)$ is the curvature change of the member axis; E_u is the relative modulus of elasticity, F_i and J_i are transformed cross section area and moment of inertia of this area, $S_i = \sqrt{F_i J_i}$. Operators $\widehat{F}'_{11}$, $\widehat{F}'_{22}$ and $\widehat{F}'_{12}$ are elements of the operator matrix $\left[\widehat{F}'_{hl}\right]_{2,2}$ .Starting from the basic expressions of the composite cross section, Eq.(1), and using the principle of virtual forces, the relationship between the basic statically independent forces (S_{ik}, M_{ik}, M_{ki}) and deformation independent quantities ($\Delta l_{ik}, \tau_{ik}, \tau_{ki}$) for the composite fixed-end member, is established [2],[3]. This relationship, due to viscoelasticity of concrete and relaxation of prestressing steel, is integral and symbolically is presented by operators. The principle of superposition, which is valid in the theory of viscoelasticity, is used. This relationship is given in the following form:

$$\begin{aligned} \Delta l_{ik} &= \widehat{\delta}'_{ik,s} S_{ik} - \widehat{\alpha}'_{ik,s} M_{ik} + \widehat{\alpha}'_{ki,s} M_{ki} \\ \tau_{ik} &= -\widehat{\alpha}'_{ik,s} S_{ik} + \widehat{\alpha}'_{ik} M_{ik} - \widehat{\beta}'_{ik} M_{ki} \\ \tau_{ki} &= \widehat{\alpha}'_{ki,s} S_{ik} - \widehat{\beta}'_{ik} M_{ik} + \widehat{\alpha}'_{ki} M_{ki} \end{aligned} \tag{2}$$

The symmetric operator flexibility matrix is introduced:

$$\left[\widehat{f}'\right] = \begin{bmatrix} \widehat{\delta}'_{ik,s} & -\widehat{\alpha}'_{ik,s} & \widehat{\alpha}'_{ki,s} \\ -\widehat{\alpha}'_{ik,s} & \widehat{\alpha}'_{ik} & -\widehat{\beta}'_{ik} \\ \widehat{\alpha}'_{ki,s} & -\widehat{\beta}'_{ik} & \widehat{\alpha}'_{ki} \end{bmatrix} \tag{3}$$

The operators $\widehat{\delta}'_{ik,s}, \widehat{\alpha}'_{ik,s}, \widehat{\alpha}'_{ki,s}, \widehat{\alpha}'_{ik}, \widehat{\beta}'_{ik}, \widehat{\alpha}'_{ki}$ are associated with the corresponding functions which represent the change of length of the member and deformation angles due to separately acting unity forces S_{ik}, M_{ik} and M_{ki}. These operators are linear integral operators and obey the algebra laws of ordinary numbers, including the commutative law.

The inverse matrix of the operator flexibility matrix (3) is the symmetric operator matrix, which is called the basic operator stiffness matrix:

$$\left[\widehat{K}'_o\right]=\begin{bmatrix} \widehat{N}'_{ik} & \widehat{S}'_{ik} & \widehat{S}'_{ki} \\ \widehat{S}'_{ik} & \widehat{A}'_{ik} & \widehat{B}'_{ik} \\ \widehat{S}'_{ki} & \widehat{B}'_{ik} & \widehat{A}'_{ki} \end{bmatrix} \tag{4}$$

The operators $\widehat{N}'_{ik}, \widehat{S}'_{ik}, \widehat{S}'_{ki}, \widehat{A}'_{ik}, \widehat{B}'_{ik}$ are associated with the corresponding functions which represent the generalized forces at *i* and *k* element ends.

The solution procedure for the stiffness matrix of the composite frame element type ″k″ is analogous to the direct procedure for determination of the stiffness matrix of the homogeneous elastic element. The operator stiffness matrix $\left[\widehat{K}'\right]$is derived using the basis stiffness matrix $\left[\widehat{K}'_o\right]$ and the equilibrium matrix of the element $\left[C\right]$:

$$\left[\widehat{K}'\right]=\left[C\right]\left[\widehat{K}'_o\right]\left[C\right]^T \tag{5}$$

where:

The operator stiffness matrix $\left[\widehat{K}'\right]$ has the following form:

$$\left[\widehat{K}'\right]=\begin{bmatrix} \widehat{N}'_{ik} & -\frac{1}{l}(\widehat{S}'_{ik}+\widehat{S}'_{ki}) & -\widehat{S}'_{ik} & -\widehat{N}'_{ik} & \frac{1}{l}(\widehat{S}'_{ik}+\widehat{S}'_{ki}) & -\widehat{S}'_{ki} \\ & \frac{1}{l^2}(\widehat{C}'_{ik}+\widehat{C}'_{ki}) & \frac{1}{l}\widehat{C}'_{ik} & \frac{1}{l}(\widehat{S}'_{ik}+\widehat{S}'_{ki}) & -\frac{1}{l^2}(\widehat{C}'_{ik}+\widehat{C}'_{ki}) & \frac{1}{l}\widehat{C}'_{ki} \\ & & \widehat{A}'_{ik} & \widehat{S}'_{ik} & -\frac{1}{l}\widehat{C}'_{ik} & \widehat{B}'_{ik} \\ & & & \widehat{N}'_{ik} & -\frac{1}{l}(\widehat{S}'_{ik}+\widehat{S}'_{ki}) & \widehat{S}'_{ki} \\ & symmetrical & & & \frac{1}{l^2}(\widehat{C}'_{ik}+\widehat{C}'_{ki}) & -\frac{1}{l}\widehat{C}'_{ki} \\ & & & & & \widehat{A}'_{ki} \end{bmatrix} \tag{6}$$

where:

$$\widehat{C}'_{ik}=\widehat{A}'_{ik}+\widehat{B}'_{ik}\,,\quad \widehat{C}'_{ki}=\widehat{A}'_{ki}+\widehat{B}'_{ik}\,,\quad l=l_{ik} \tag{7}$$

2. The numerical method

In the second method, the element flexibility matrix and the resisting force vector are determined through the numerical integration of the cross-section deformations. Object oriented approach is applied. The class of cross-sections suitable for this type of nonlinearities is developed. The cross-sections of this class can evaluate cross-sections flexibility and stiffness and also generate deformation over time.

The objects of cross-section class may have multiple sections made of different materials. Viscous deformations are generated by the material class depending on its stress history. The class of materials can generate these nonlinearities. Detailed explanation of the developed cross section class is out of scope of this paper and can be found in [5].

The cross-section class establishes relations between the generalised cross sectional deformations $\{\varepsilon^R\}$ and the cross sectional forces $\{F^R\}$ reduced to the reference point R.

$$\begin{aligned}\{\varepsilon^R\} &= \{\varepsilon_x^R, \gamma_{xy}^R, \gamma_{xz}^R, \theta_x, \kappa_y, \kappa_z\}^T \\ \{F^R\} &= \{N, T_y, T_z, M_t^R, M_y^R, M_z^R\}^T\end{aligned} \tag{8}$$

$$\{\varepsilon^R\} = [C^R]\{F^R\} \Leftrightarrow \varepsilon_i^R = C_{ij}^R F_j^R \tag{9}$$

The elastic properties of a composite cross-section change over time, so the relations have to be established reduced to the reference point, instead of the centroid of the cross-section. In the case of plane frame, the previous expression reduces to:

$$\begin{aligned}\{\varepsilon^R\} &= \{\varepsilon_x^R, \gamma_{xz}^R, \kappa_y\}^T \\ \{F^R\} &= \{N, T_z, M_y^R\}^T\end{aligned} \tag{10}$$

In the simplest case, i.e. for the ideally elastic eccentric beam that has z coordinate of centroid z_{CR}, the cross-section flexibility matrix reduced to the reference point is:

$$\left[C^R\right] = \begin{bmatrix} \frac{1}{EA} + \frac{z_{CR}^2}{EIy} & 0 & -\frac{z_{CR}}{EIy} \\ 0 & \frac{1}{GAz} & 0 \\ -\frac{z_{CR}}{EIy} & 0 & \frac{1}{EIy} \end{bmatrix} \tag{11}$$

Also, in this case, the cross section rigidity matrix is:

$$K^R = \left[C^R\right]^{-1} = \begin{bmatrix} EA & 0 & EAz_{CR} \\ 0 & GAz & 0 \\ EAz_{CR} & 0 & EAz_{CR}^2 + EIy \end{bmatrix} \tag{12}$$

Integration of deformation, in the case of element with constant cross-sections over the length, leads to the following expressions for the flexibility matrix coefficients:

$$[f]=\begin{bmatrix} l\cdot C_{11}^{R} & C_{12}^{R}-\dfrac{l\cdot C_{13}^{R}}{2} & C_{12}^{R}+\dfrac{l\cdot C_{13}^{R}}{2} \\ C_{12}^{R}-\dfrac{l\cdot C_{13}^{R}}{2} & \dfrac{C_{22}^{R}}{l}+C_{23}^{R}-l\dfrac{C_{33}^{R}}{3} & \dfrac{C_{22}^{R}}{l}+l\dfrac{C_{33}^{R}}{6} \\ C_{12}^{R}+\dfrac{l\cdot C_{13}^{R}}{2} & \dfrac{C_{22}^{R}}{l}+l\dfrac{C_{33}^{R}}{6} & \dfrac{C_{22}^{R}}{l}+C_{23}^{R}-l\dfrac{C_{33}^{R}}{3} \end{bmatrix} \qquad (13)$$

Inverting this matrix the basic stiffness matrix can be obtained, and transformation of Eq. (5) leads to the element stiffness matrix.

The compatibility condition of the composite cross-section imposes the requirement of the same cross-sectional deformation for all of its parts. Therefore, the flexibility matrix for the composite cross-section can be calculated by inverting the sum of the rigidity matrix for all of its parts. Once the flexibility matrix of the composite section is obtained, we can calculate the flexibility and stiffness matrices of the element. In the case of element with constant cross-section, the element flexibility matrix is given with Eq. 13.

The increase of viscous deformation is calculated by the material class. Depending on the stress history, viscous properties and time increments, material generates increments in deformations. These increments result in changes of internal forces for previously committed values of sectional deformations. Nonlinear numerical procedures need to be applied. In the nonlinear procedure, frame element generates the change of its residual forces. These changes affect equilibrium in nodes and the system needs to be balanced in iterations.

The results of the two explained methods are compared in the following numerical example.

3. Numerical example

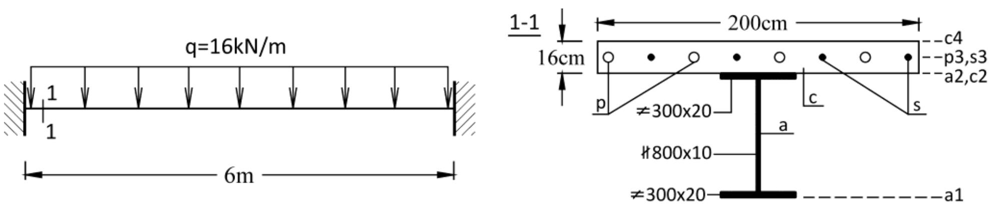

Figure 1. Composite girder and its cross-section

The composite girder shown in Fig. 1 with constant cross-section 1-1 is analyzed. The stresses at characteristic points of the cross-section at fixed end are calculated in time t_o and time t. Stresses are determined due to given uniform loading and due to concrete shrinkage in accordance with the two previously explained methods (exact method and the finite element method with object oriented approach), as well as, with the approximate EM method [4]. The results of the analysis are given in Table 1.

Example data are: *Concrete (c)* $E_{co}=E_{cm}=30$GPa, $\varphi_r=3.5$, $\varepsilon_s = -30\cdot10^{-5}$*; Prestressing steel* (p)*:* $E_p=210$GPa, $A_p=100\text{cm}^2$, $\zeta_p=8\%$*; Structural steel (a):* $E_a=200\text{GPa}=E_u$*; Reinforcing steel (s):* $E_s=200$GPa, $A_s=80\text{cm}^2$.

Using the creep function in accordance with the ageing theory in the exact method and the creep function in accordance with the hereditary theory in the EM method, the upper and lower bounds are determined and results obtained by other theories should take place between them.

Table 1. Results of analysis

H	MPa	t_o	EM		OOFEM	Exact
q	$\sigma_{c4}=$	0.220	0.085		0.060	0.049
	$\sigma_{c2}=$	0.040	0.038		0.033	0.035
	$\sigma_{s3}=$	0.868	1.840		1.904	2.099
	$\sigma_{p3}=$	0.912	1.778		1.853	2.046
	$\sigma_{a2}=$	0.267	1.129		1.205	1.358
	$\sigma_{a1}=$	-6.042	-6.341		-6.180	-6.420
shrinkage	$\sigma_{c4}=$		2.676		2.726	3.063
	$\sigma_{c2}=$		2.545		2.611	2.952
	$\sigma_{s3}=$		18.320		21.852	23.303
	$\sigma_{p3}=$		17.697		21.080	22.459
	$\sigma_{a2}=$		16.357		19.563	20.906
	$\sigma_{a1}=$		-4.252		-5.073	-5.415

4. Conclusion

Two different methods for determination of the stiffness matrix of a composite frame element are presented in the paper. Both of them produce similar results when used with the same material parameters and viscous functions, as is confirmed in the given numerical example.

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