On Linear Equations in Modules

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Resumo: Nesta nota obtemos uma condição necessária e suficiente para que uma equação linear proveniente de uma aplicação linear entre dois módulos sobre um anel principal admita uma solução.

Abstract: A necessary and sufficient condition for a linear equation arising from a linear mapping between two modules over a principal ring to admit a solution is established.

palavras-chave: anéis principais; módulos; equações lineares.

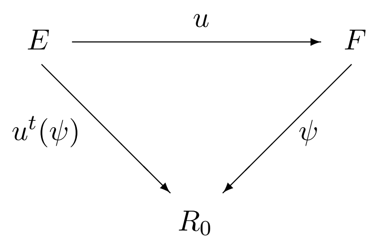
keywords: principal rings; modules; linear equations.

1 Introduction

It is known (p. 162 of [1]) that for a linear equation $u(x) = y_0$ arising from a linear mapping u between two vector spaces over a field and an element y_0 of the codomain of u to admit a solution, it is necessary and sufficient that y_0 be an element of the orthogonal of the kernel of the transpose of u. Nevertheless that fact cannot be extended to the context of modules. As a matter of fact, in Exercise 10, p. 265 of [1], the construction of a linear mapping u which is neither injective nor surjective and whose transpose is bijective is indicated. Therefore any element y_0 of the codomain of u which does not belong to the image of u belongs to the orthogonal of the kernel of the transpose of u. The main purpose of this note is to obtain an extension of the above-mentioned result, valid in the context of modules over a principal ring, in whose statement the concept of dual of a module is understood in a known sense.

2 Linear equations in modules over a principal ring

Let R be an arbitrary principal ring, K the field of fractions of R and R_0 the R-module K/R. Then R_0 is an injective R-module [2, A X.18], a fact that will play a central role in our work (see the proof of Proposition 2.1). For each R-module E the dual of E is the R-module E' of all R-linear mappings from E into R_0 [4; 7, p. 116]. For two arbitrary R-modules E, F and an arbitrary R-linear mapping u from E into F, u^t will denote the R-linear mapping from F' into E' defined by $u^t(\psi) = \psi \circ u$ for $\psi \in F'$.



The next result will be important for our purposes.

Proposition 2.1 *Let E be an R-module and $x \in E \backslash \{0\}$. Then there is a $\varphi \in E'$ such that $\varphi(x) \neq 0$.*

Proof: Since the result is well known when R is a field, we shall assume that R is not a field. Let $\pi : K \to R_0$ be the canonical surjection, $M = [x]$ and let $\theta \in K \backslash R$ be fixed. Since R_0 in an injective R-module, the R-linear mapping

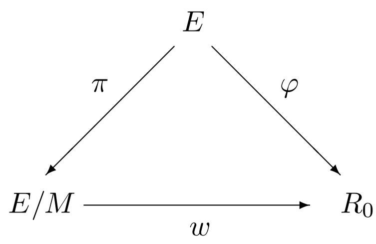
$$v : \lambda x \in M \mapsto \pi(\lambda\theta) \in R_0$$

can be extended to an R-linear mapping $\varphi \in E'$. Moreover, $\varphi(x) = v(x) = \pi(\theta) \neq 0$, which concludes the proof.

Definition 2.2 *Let E be an R-module, $A \subset E$ and $B \subset E'$. The orthogonal of A (resp. B) is the submodule $A^\perp = \{\varphi \in E'; \varphi(x) = 0$ for all $x \in A\}$ of E' (resp. $B^\perp = \{x \in E; \varphi(x) = 0$ for all $\varphi \in B\}$ of E).*

Proposition 2.3 *Let M be a submodule of an R-module E and $x \in E \backslash M$. Then there exists a $\varphi \in E'$ such that $\varphi \in M^\perp$ and $\varphi(x) \neq 0$.*

Proof: Let $\pi : E \to E/M$ be the canonical surjection; $\pi(x) \neq 0$ because $x \notin M$. By Proposition 2.1, there is a $w \in (E/M)'$ so that $w(\pi(x)) \neq 0$. Then $\varphi := w \circ \pi \in E'$, $\varphi \in M^{\perp}$ and $\varphi(x) = w(\pi(x)) \neq 0$.



Corollary 2.4 *If M is a submodule of an R-module E, then $M = M^{\perp\perp}$, where $M^{\perp\perp} := \left(M^{\perp}\right)^{\perp}$.*

Proof: Obviously, $M \subset M^{\perp\perp}$. On the other hand, if $x \in E \backslash M$, Proposition 2.3 ensures the existence of a $\varphi \in M^{\perp}$ such that $\varphi(x) \neq 0$; consequently, $x \in E \backslash M^{\perp\perp}$.

Proposition 2.5 *If u is an R-linear mapping from an R-module E into an R-module F and A is a subset of E, then $(u(A))^{\perp} = (u^t)^{-1}(A^{\perp})$. In particular, $(Im(u))^{\perp} = Ker(u^t)$.*

Proof: For $\psi \in F'$, $\psi \in (u(A))^{\perp}$ if and only if $(u^t(\psi))(x) = 0$ for all $x \in A$, which is the same as $u^t(\psi) \in A^{\perp}$, which finally means that $\psi \in (u^t)^{-1}(A^{\perp})$.

Corollary 2.6 *For u as in Proposition 2.5, one has $Im(u) = (Ker(u^t))^{\perp}$.*

Proof: By Corollary 2.4 and Proposition 2.5,

$$Im(u) = (Im(u))^{\perp\perp} = ((Im(u))^{\perp})^{\perp} = (Ker(u^t))^{\perp}.$$

Theorem 2.7 *Let u be an R-linear mapping from an R-module E into an R-module F and $y_0 \in F$. In order that the equation $u(x) = y_0$ admits a solution $x \in E$, it is necessary and sufficient that $y_0 \in (Ker(u^t))^{\perp}$.*

Proof: Follows immediately from Corollary 2.6.

In the special case where R is a discrete valuation ring, Theorem 2.7 was proved in [6] by means of topological arguments.

Finally we would like to mention that topological analogues of results obtained in the present note may be found, for example, in [3] and [5].

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