# Edge Multiway Cut and Node Multiway Cut are NP-complete on subcubic graphs 

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#### Abstract

We show that Edge Multiway Cut (also called Multiterminal Cut) and Node Multiway Cut are NP-complete on graphs of maximum degree 3 (also known as subcubic graphs). This improves on a previous degree bound of 11 . Our NP-completeness result holds even for subcubic graphs that are planar.


2012 ACM Subject Classification Mathematics of computing $\rightarrow$ Graph theory; Theory of computation $\rightarrow$ Graph algorithms analysis; Theory of computation $\rightarrow$ Problems, reductions and completeness

Keywords and phrases multiway cut; subcubic graphs; planar graphs

## 1 Introduction

In the Edge Multiway Cut problem, also known as the Multiterminal Cut problem, we are given an input graph $G=(V, E)$, a subset $T$ of its vertices, and an integer $k$. The goal is to decide if there exists a set $S \subseteq E$ such that $|S| \leq k$ and for any pair of vertices $\{u, v\} \in T, G \backslash S$ does not contain a path between $u$ and $v$. In the Node Multiway Cut problem, the objective is the same, except we ask for a subset of vertices instead of edges to achieve it. Here we restrict the vertices to not belong to $T$; that is, we ask for a set $S \subseteq V \backslash T$ such that $|S| \leq k$ and for any pair of vertices $\{u, v\} \in T, G \backslash S$ does not contain a path between $u$ and $v$. Both problems have been studied extensively in various contexts [1, 2, 3, 4, 13, 14, 11, 6, 18, and can be thought of as natural duals to the famous STEINER Tree problem. Moreover, if $|T|=2$, they reduce to the Minimum Cut problem, which is well known to be solvable in polynomial time [8].

In one of the first studies of Edge Multiway Cut, Dahlhaus et al. [7] showed that the problem is NP-complete for all fixed $|T| \geq 3$. In addition, and more important to this study, they proved that Edge Multiway Cut is NP-complete on planar graphs of maximum degree 11 [7, Theorem 2b]. The authors claimed that with a variant of their construction
and more complicated arguments they believe they could reduce the maximum degree of their instance to 6 , but no further arguments were given. In this paper, we seek a more significant improvement of the degree bound.

We note that there have been many works that prove that graph problems are NPcomplete on graphs of bounded degree. In particular, many problems are NP-complete already on graphs of maximum degree 3, even if the graph is planar. Well-known examples include Vertex Cover and Independent Set [16], List Colouring [10], Dominating Set [9], Independent Dominating Set [5], Edge Dominating Set [21], Max-Cut [20], Disjoint Paths [15], and Path-width [17]. Intriguingly, Edge Multiway Cut and Node Multiway Cut are not currently part of this list, even though they are extensively studied.

## Our Results

We prove that Edge Multiway Cut and Node Multiway Cut are NP-complete on planar subcubic graphs. That is, graphs that are both planar and have maximum degree 3 .

In spirit, our construction for Edge Multiway Cut is similar to the one by Dahlhaus et al. [7]. A main issue with their construction is that terminals can have degree up to 6 , for which a local replacement strategy seems difficult. Hence, in order to upper-bound the maximum degree of our constructed graph by 3 , we needed to build different gadgets and leverage several structural properties of the edge multiway cut in the resulting instance. This makes for a significantly more involved and technical proof. Crucially, we first prove NP-completeness for a weighted version of the problem on graphs of maximum degree 5 , in which the terminals have degree 3 . Then we replace weighted edges and high-degree vertices with appropriate gadgets. Finally, the NP-completeness for Node Multiway Cut follows from the hardness of Edge Multiway Cut in a standard manner.

For sake of completeness, we note that Edge Multiway Cut and Node Multiway Cut can be solved in linear time on graphs of maximum degree 2 by a simple greedy algorithm. Hence, we obtain a dichotomy result based on the degree of the input graph.

Another implication of our work is that Edge Multiway Cut and Node Multiway Cut are so-called C123-problems [12]. Hence, one can obtain a complete characterization of their computational complexity on $\mathcal{H}$-subgraph-free graphs, which are graphs that exclude a finite set $\mathcal{H}$ of graphs as a subgraph. We refer to [12] for details and a full proof.

## 2 Preliminaries

The line graph of an undirected graph $G=(V, E)$ is the graph $L(G)$ containing a vertex for every edge in $E$. Two vertices of $L(G)$ are connected by an edge if and only if their corresponding edges in $E$ have a common end-point.

When we subdivide an edge $(u, v)$, we create a new vertex $w$, add the edges $(u, w)$ and $(w, v)$ and delete the edge $(u, v)$. When we $k$-subdivide an edge $(u, v)$, we delete the edge $(u, v)$, add to the graph a $P_{k}$ (a path ok $k$ vertices) and add an edge between $u$ and one endpoint of the $P_{k}$ and $v$ and the other endpoint of the $P_{k}$. By $k$-subdivision of a graph, we refer to the graph formed by $k$-subdividing each of its edges. We denote the $k$-subdivision of a graph $G$ by $k e(G)$.

In Weighted Edge Multiway Cut, we are given as input a graph $G$, a set $T \subseteq V(G)$, a function $\omega: E(G) \longrightarrow \mathbb{Q}^{+}$, and an integer $k$. The goal is to decide if there exists an edge multiway cut of total weight at most $k$. If the image of $\omega$ is the set $X$, we denote the corresponding Weighted Edge Multiway Cut problem as $X$-Edge Multiway Cut.

## 3 Edge Multiway Cut

In this section, we show that Edge Multiway Cut is NP-complete on subcubic graphs. We reduce the problem from Planar 2P1N-3SAT, which is a restricted version of 3-SAT. Given a CNF-formula $\Phi$ with the set of variables $X$ and the set of clauses $C$, the incidence graph of the formula is the graph $G_{X, C}$ which is a bipartite graph with one of the partitions containing a vertex for each variable and the other partition containing a vertex for each clause of $\Phi$. There exists in $G_{X, C}$ an edge between a variable-vertex and a clause-vertex if and only if the variable appears in the clause. We now define Planar 2P1N-3SAT as follows.

Planar 2P1N-3SAT
Input: A set $X=\left\{x_{1}, \ldots, x_{n}\right\}$ of variables and a CNF formula $\Phi$ with each clause containing at most three literals and each variable occurring twice positively and once negatively in $\Phi$ such that $G_{X, C}$ is planar.
Question: Is there an assignment $\mathcal{A}: X \rightarrow\{0,1\}$ that satisfies $\Phi$ ?
The above problem was shown to be NP-complete in [7]. By their construction, each variable occurs in at least two clauses having size two. This property becomes important later in our NP-completeness proof.

We show the reduction in two steps. In the first step, we reduce from Planar 2P1N3 SAT to $\{1,2,3,6\}$-Edge Multiway Cut restricted to planar graphs. In the second step, we show how to make the instance unweighted while keeping it planar and its maximum degree bounded above by 3 .


Figure 1 This figure shows the gadgets for the variables (top) as well as those for the clauses (bottom). The bottom-left gadget corresponds to a clause with three literals whereas the bottom-right one corresponds to a clause with two literals. The terminals are depicted as red squares.

- Theorem 1. Edge Multiway Cut is NP-complete on the class of planar subcubic graphs.

Proof. Clearly, Edge Multiway Cut is in NP. We reduce Edge Multiway Cut from Planar 2P1N-3SAT. Let $\Phi$ be a given CNF formula with at most three literals in each clause and each variable occurring twice positively and once negatively. Without loss of generality we assume that each clause has size at least 2. By the reduction in [7], every


Figure 2 The figure shows a link structure formed by the connector edges of a clause-triangle and its corresponding variable-triangle.
variable occurs in at least two clauses of size 2 . Let $\left\{x_{i} \mid 1 \leq i \leq n\right\}$ be the set of variables in $\Phi$ and $\left\{c_{j} \mid 1 \leq j \leq m\right\}$ be the set of clauses. The incidence graph $G_{X, C}$ is planar. For each vertex corresponding to a clause $c_{j}$ in $C$ and each vertex corresponding to a variable $x_{i} \in X$, we create a clause-gadget (depending on the size of the clause) and a variable-gadget as in Figure 1 In a variable gadget, the positive literal is attached to the diamond and the negative one to the hat by edges of weight 3 . Each degree- 2 vertex in the gadget acts as a connector. If $x_{i} \in c_{j}$ and $x_{i} \in c_{k}$, then we connect the degree- 2 vertices of the diamond to some degree- 2 vertex of the gadgets for $c_{j}$ and $c_{k}$, each by an edge of weight 6 . We shall refer to the connecting edge as the link. The edges of the triangles adjacent to the link are called connector-edges and the one not adjacent to the link is called the base of the triangle. If $\overline{x_{i}} \in c_{l}$, then we connect the degree- 2 vertex of the hat and some degree- 2 vertex on the gadget for $c_{l}$. The graph thus created is denoted by $G$ and it is planar because $G_{X, C}$ is planar.

Since $\Phi$ is an instance of Planar 2P1N-3SAT, each degree-2 vertex in the variablegadget is incident on exactly one link and corresponds to one occurrence of the variable. Similarly, each degree-2 vertex of a clause gadget is incident on exactly one link and each of its triangles corresponds to a literal in the clause. The variable and clause connections are depicted in Figure 3. The structure formed by the two pairs of connector edges and the link is called the link structure. See Figure 2. The graph $G$ has a total of $2 n+2 m$ terminals and since each variable occurs twice positively and once negatively in $\Phi$, it has $3 n$ link structures. Let $T$ be the set of its terminals.

We replace all the edges in $G$ of weights greater than 1 by as many parallel edges between their end-vertices as the weight of the edge. Each of these parallel edges has weight 1. We refer to this graph as $G^{\prime}$. Next, for all vertices in $G^{\prime}$ of degree grater than 3, we replace each of the vertices by a large honeycomb (hexagonal grid), say of size $1000 \times 1000$, as depicted in Figure 4 Note that none of the terminals have degree greater than 3. The neighbours of the high degree vertex, of which there are at most eight, are now attached to distinct degree-2 vertices on the boundary of the honeycomb such that the distance along the boundary between any pair of them is 100 cells of the honeycomb. These degree- 2 vertices on the boundary are called the attachment points of the honeycomb. The edges, not belonging to the honeycomb, that are incident on these attachment points are called attaching-edges. Let the resultant graph be $\tilde{G}$. Note that the maximum degree of any vertex in $\tilde{G}$ is 3 , and
all the edge weights are equal to 1 . Also, none of the transformations introduce any edge crossings in the graph and hence, $\tilde{G}$ is planar. $\tilde{G}$ has size bounded by a polynomial in $n+m$. We set $k=7 n+2 m$.

For the sake of simplicity, we shall first argue that $\Phi$ is a Yes instance of Planar $2 \mathrm{P} 1 \mathrm{~N}-3 \mathrm{SAT}$ if and only if $(G, T, k)$ is a yes instance of $\{1,2,3,6\}$ - Edge Multiway Cut. Later, we show that the same holds for $\tilde{G}$ by proving that no edge of any of the honeycombs is ever present in any minimum edge multiway cut in $\tilde{G}$. We defer the proof of this claim for now.

Suppose that $\mathcal{A}$ is a truth assignment satisfying $\Phi$. Then, we create a set of edges $S \subseteq E(G)$, as follows:

- If a variable is true, add to $S$ all the three edges of the hat in the corresponding gadget. If a variable is false, add to $S$ all the five edges of the diamond.
- For each clause, pick a true literal in it and add to $S$ all the three edges of the clausetriangle corresponding to this literal.
- Finally, for each link structure with none of its edges in $S$ yet, add the two connector-edges of its clause-triangle to $S$.
$\triangleright$ Claim 2. $\quad S$ is an edge multiway cut of $(G, T)$ of weight at most $7 n+2 m$.
Proof. For each variable, either the positive literal is true, or the negative one. Hence, either all the three edges of its hat are in $S$ or all the five edges of the diamond. Therefore, all the paths between terminal pairs of the form $x_{i}-\overline{x_{i}}$, for all $1 \leq i \leq n$, are disconnected in $G \backslash S$. Consider the link structure in Figure 2. By our choice of $S$, at least one endpoint of each link in $G \backslash S$ is a vertex of degree 1, hence a dead-end. Therefore, no path connecting any terminal pair in $G \backslash S$ passes through any link. As all the paths in $G$ between a variable-terminal and a clause-terminal must pass through some link, we know that all terminal pairs of this type are disconnected in $G \backslash S$. Since $\mathcal{A}$ is a satisfying truth assignment of $\Phi$, all the edges of one triangle from every clause-gadget are in $S$. Hence, all the paths between terminal pairs of the form $c_{j}^{+}-c_{j}^{-}$, for all $1 \leq j \leq m$, are disconnected in $G \backslash S$. Hence $S$ is an edge multiway cut.

It remains to show that its weight is at most $7 n+2 m$. Since $\mathcal{A}$ satisfies each clause of $\Phi$ at least once, there are exactly $m$ triangle-bases of weight 2 from the clause-gadgets in $S$. Similarly, the variable-gadgets contribute exactly $n$ bases to $S$. Finally, for each of the $3 n$ link structures, either the two connector-edges of the variable-triangle are in $S$ or the two connector-edges of the clause-triangle. Together, they contribute a weight of $6 n$ to the total weight of $S$. Therefore, $S$ is an edge multiway cut in $G$ of weight at most $7 n+2 m$.

Conversely, assume that $(G, T, k)$ is a yes instance of $\{1,2,3,6\}$-Edge Multiway Cut. Hence, there exists an edge multiway cut of $(G, T)$ of weight at most $7 n+2 m$. We shall demonstrate an assignment that satisfies $\Phi$. Before that, we shall discuss some structural properties of a minimum-weight multiway cut. In the following arguments, we assume that the clauses under consideration have size three, unless otherwise specified. While making the same arguments for clauses of size two is easier, we prefer to argue about clauses of size three for generality.
$\triangleright$ Claim 3 (adapted from [7]). If $e$ is an edge in $G$ incident on a vertex $v$ of degree $\geq 2$ such that $e$ has weight greater than or equal to the sum of the other edges incident on $v$, then there exists a minimum-weight multiway cut in $G$ that does not contain $e$.
$\triangleright$ Claim 4 ([7]). If a minimum-weight edge multiway cut contains an edge of a cycle, then it contains at least two edges from that cycle.

It follows from Claim 3 that there exists a minimum-weight multiway cut that neither contains the edges incident on the terminals nor does it contain the links. Among the minimum-weight multiway cuts that satisfy Claim 3, we shall select the one that contains the maximum number of connector-edges and from the ones that satisfy both the aforementioned properties, we shall pick the one that contains the maximum number of triangle-bases from clause-gadgets of size two. Let $S$ be a minimum multiway cut that fulfills all these requirements.

We say that a terminal $t$ outside a gadget is reachable from one of the terminals on a gadget if any path from the gadget-terminal to $t$ is cut by $S$ only by edges of the gadget. A link of a gadget reaches a terminal $t$ if it lies on a path from some terminal on the gadget to $t$ and any edge on this path that does not belong to the gadget, is not contained in $S$.


Figure 3 Shown in the figure is the variable interface of $x_{i}$. The positive literal $x_{i}$ occurs in the clauses $c_{j}$ and $c_{g}$, whereas $\overline{x_{i}}$ occurs in $c_{h}$. The black rectangles on the clause gadgets depict that no terminal is reachable through that path from any gadget in the figure.
$\triangleright$ Claim 5. $\quad S$ contains exactly one base of a triangle from each variable gadget.
Proof. Suppose that there exists a minimum-weight multiway cut containing two bases of some variable gadget, say that of $x_{i}$. By Claim 4 it must also contain at least three connector edges from the variable gadget: at least two connector edges (of the two triangles) of the diamond and at least one connector-edge of the hat. We claim that at least all the outer connector edges must be in $S$. If for some triangle the outer connector-edge is not in the cut, then any terminal outside the gadget must not be reachable from it. If some terminal were to be reachable, then a path from one of the variable-terminals to it would have existed through the outer connector-edge of the link structure formed by this triangle and its variable counterpart. This contradicts the feasibility of $S$. Given that no terminal outside the gadget is reachable, we can replace the inner connector edges by their adjacent outer ones.

Henceforth, we shall assume that all the outer connector edges of the $x_{i}$-gadget are in $S$. We now distinguish several cases.

1. Assume that no terminal outside the $x_{i}$-gadget is reachable. In that case, we can remove one of the two bases from the multiway cut without connecting any terminal pairs. This is so because in order to disconnect $x_{i}$ from $\overline{x_{i}}$, it suffices for $S$ to contain either the base of the diamond along with the two outer connector edges or the base and outer connector-edge of the hat. No other terminal pairs are connected via the gadget. This contradicts the minimality of $S$.
2. Next, suppose that exactly one link of the $x_{i}$-gadget reaches some terminal $t$. Then, we remove from $S$ the base of a triangle that is not attached to the link and add the remaining connector-edge of the triangle that is attached to the link. Consequently, $t$ is not reachable from the gadget. Since no other link reached any terminals and $x_{i}$ remains disconnected from $\overline{x_{i}}$, we get a multiway cut satisfying Claim 3 that has strictly more connector-edges than $S$. This is a contradiction to our choice of $S$.
3. Suppose that exactly two links of the $x_{i}$-gadget reach two distinct terminals $t$ and $t^{\prime}$, respectively. Then at least four connector edges of the gadget must be in $S$, or else $t$ would be connected to $t^{\prime}$ via this gadget. In particular, both the connector-edges of one of the two triangles attached to the links that reach $t$ and $t^{\prime}$, must be in $S$. We can remove from $S$ one of the two bases and add instead the remaining connector-edge of the other triangle. Now, neither $x_{i}$ nor $\overline{x_{i}}$ are connected to $t$ or $t^{\prime}$, nor are they connected to each other as one base and its corresponding outer connector(s) are still in $S$. The transformations result in a minimum-weight multiway cut satisfying Claim 3 and having strictly more connector-edges than $S$, a contradiction!
4. Suppose that all the three links of the $x_{i}$-gadget reach distinct terminals $t, t^{\prime}, t^{\prime \prime}$, respectively. Then, at most one connector-edge of the $x_{i}$-gadget is not in $S$ or else at least one pair of terminals among $\left\{\left(t, t^{\prime}\right),\left(t^{\prime}, t^{\prime \prime}\right),\left(t^{\prime \prime}, t\right)\right\}$ would remain connected via the gadget. We replace one of the bases in $S$ with this connector-edge. The resulting multiway cut is no heavier. To see that it is also feasible, note that both the terminals on the $x_{i}$-gadget are disconnected from $\left\{t, t^{\prime}, t^{\prime \prime}\right\}$ because all the connector-edges of this gadget are in the multiway cut. The terminals $x_{i}$ and $\overline{x_{i}}$ are disconnected from each other because one triangle-base and its connector(s) are still in the multiway cut. Hence, we obtain a minimum-weight multiway cut with strictly more number of connector-edges than $S$, a contradiction!
5. Finally, we assume that at least two links of the $x_{i}$-gadget reach exactly one terminal $t$ outside the gadget. Recall that every variable occurs in at least two clauses of size two and $S$ is a minimum-weight multiway cut containing the maximum number of bases from clauses of size two.
Suppose that there exists a size-two clause-gadget $c$, connected to the $x_{i}$-gadget, that does not contain $t$. However, $t$ can be reached by at least two links of the $x_{i}$-gadget. Then, $S$ must contain two base-connector pairs from $c$. In this case, we may remove the base of one of the two triangles of $c$ and add the remaining two connector-edges of $c$. This transformation does not increase the weight of the multiway cut as the base of the clause-triangle has weight 2 while the connectors have weight 1 each. The only potential terminal pair that could get connected by the transformation is the pair of terminals on $c$ itself. However, a base and connector-edge of one of its triangles is still in the cut, and hence no new connections are made. This leads to a contradiction to our choice of $S$ as the transformed cut has strictly more connector-edges than $S$.
Suppose that $t$ appears in one of the size-two clause-gadgets, $c^{\prime}$, connected to the $x_{i}$-gadget. Since no other terminal is reachable from the $x_{i}$-gadget, the base and one connector-edge of the triangle of $c^{\prime}$ that $t$ is not attached to must be in $S$. We consider the path through the link that is not attached to $c^{\prime}$ but reaches $t$. This path must pass through some other clause-gadget $c^{\prime \prime}$ connected to the $x_{i}$-gadget. If it is a size-two clause-gadget, by arguments in the preceding paragraph, we run into a contradiction. Therefore, $c^{\prime \prime}$ must be a clause-gadget corresponding to a size-three clause. Regardless of which triangle is attached to the link from the $x_{i}$-gadget, the base and one connector-edge of both the outer triangles of $c^{\prime \prime}$ must be in $S$. If not, then at least one terminal on this clause-gadget
would have a path to $t$ via the $x_{i}$-gadget, contradicting the feasibility of $S$. We can remove from $S$ the base and connector-edge of the outer-triangle attached to a link that reaches $t$. Instead, we add the base and outer connector-edge of the triangle in $c^{\prime}$ that $t$ is attached to. The terminal attached to this triangle becomes exposed after the transformation. However, we claim that no terminal other than $t$ was reachable from this clause-gadget prior to the replacement. If there existed such a terminal, it would have reached $t$ via the path through the $c^{\prime \prime}$ and subsequently the $x_{i}$-gadget. This would contradict the feasibility of $S$. Since the multiway cut we obtain has the same weight, satisfies Claim 5, has no less connectors than $S$ but contains at least one more base of a clause-gadget of size two, we contradict our choice of $S$.
$\triangleright$ Claim 6. There cannot exist a link structure in $G$ that contributes less than two edges to $S$, such that the clause-triangle of the link structure contributes no connector-edges to $S$.

Proof. Towards a contradiction, suppose that such a link structure exists. Let the clausegadget containing the link structure be $c$ and the variable-gadget containing it be $x_{i}$. By Claim5, we know that there exists a triangle on each variable-gadget that does not contribute its base to $S$. Therefore, at least one variable-terminal is reachable from the clause-gadgets linked to it. This implies that the clause triangle of the link structure is the middle triangle of $c$, or else there would exist a path between a variable-terminal on the $x_{i}$-gadget and the closest clause-terminal on $c$. Since $S$ is feasible, it must contain the base and at least one connector-edge of each of the two outer triangles of $c$. Else, at least one of the clause-terminals would be reachable from the exposed variable-terminal of the $x_{i}$-gadget. It must also be the case that no other terminal is reachable from the other links of $c$ or else the variable-terminal would be connected to them. Now, we can remove one of the two bases from $S$, and add the two connector-edges of the middle triangle without compromising the feasibility of the multiway cut. Thus, there exists a multiway cut of no greater weight than $S$, satisfying Claim 3. and containing two more connector-edges (those of the clause-triangle of the link structure). This is a contradiction to our choice of $S$.
$\triangleright$ Claim 7. $\quad S$ contains at least two edges from each link structure.
Proof. Suppose that there exists a link structure $\ell$ that contributes less than two edges to $S$. Suppose that $\ell$ connects the clause-gadget $c$ and the variable-gadget $x_{i}$. By Claim 6 , we know that the clause-triangle of $\ell$ must contribute an edge to $S$. Therefore, none of the connectors of the variable-triangle attached to $\ell$ are in $S$. As a result, a variable-terminal of $x_{i}$ is reachable from some terminal on $c$ via $\ell$.

Suppose that the left connector-edge of the clause-triangle of $\ell$ is in $S$. We claim that no terminal is reachable via the other link structures attached to $c$ and neither of the two clause-terminals can be reached via the other connector-edge. If some terminal $t$ were to be reachable, then there would exist a path between $t$ and one variable-terminal via $\ell$, thereby contradicting the feasibility of $S$. This implies that at least two pairs of a base and a connector-edge, one from each outer triangle of $c$, must be in $S$. So, we can remove one of the bases of some outer triangle of $c$, and add the two connector-edges of the variable-triangle of $\ell$. We thereby obtain a valid multiway cut, because no terminal path is created by the replacement. This cut is no heavier than $S$ and satisfies Claim 3 However, it contains strictly more number of connector-edges than $S$, which contradicts our choice of $S$. We can argue symmetrically for the case if the right connector-edge were to be in $S$.


Figure 4 Construction of $\tilde{G}$ from $G$ by replacing every edge of weight greater than 1 by as many parallel edges as its weight and then replacing the vertices of degree greater than 3 by a honeycomb of size $1000 \times 1000$.
$\triangleright$ Claim 8. If there exists a multiway cut of weight at most $7 n+2 m$ for $(G, T)$, then there exists a satisfying truth assignment for $\Phi$.

Proof. Let $S$ be the multiway cut defined before. The immediate consequence of Claims 5 and 7 is that the weight of $S$ is at least $n+2 \cdot(3 n)=7 n . S$ must also contain at least one base per clause gadget lest the two terminals on a clause-gadget remain connected. Therefore, its weight is at least $7 n+2 m$. Since it is a multiway cut of weight at most $7 n+2 m$, it has exactly one base per clause gadget.

We also claim that for each link structure, if one of the triangles attached to it has its base in $S$, then the other one cannot: note that if both the triangles had their bases in $S$, then each of them would also have a connector-edge in $S$ by Claim 4 . By Claim 7 and the assumption that the weight of $S$ is at most $7 n+2 m$, the other two connector-edges of the link structure are not in $S$. Since at most one base per variable/clause-gadget can be in $S$, there would be a path between one of the variable-terminals and one of the clause-terminals in the linked gadgets through the link structure, a contradiction to $S$ being a multiway cut for $(G, T)$ ! Figure 5 shows one such case.

We now define the truth assignment $\mathcal{A}$. For each variable-terminal, if the diamond has its base in $S$, we make it "false", otherwise if the hat has its base in $S$ we make it "true". Each clause-gadget has exactly one triangle contributing its base to $S$. From the above argument, we know that the variable-triangle linked to this clause-triangle must not contribute its base to $S$. Hence, every clause-gadget is attached to one literal triangle such that its base is not in $S$, and is therefore "true". Hence, every clause is satisfied by the truth assignment $\mathcal{A}$ and $\Phi$ is a Yes instance of Planar $2 \mathrm{P} 1 \mathrm{~N}-3 \mathrm{SAT}$.

Having proven that $\{1,2,3,6\}$-Edge Multiway Cut is NP-complete on planar subcubic graphs, we now proceed to prove that (unweighted) Edge Multiway Cut is NP-complete on planar subcubic graphs. The proof follows from the observation that the honeycombs of $\tilde{G}$ (defined before) do not contribute any edge to any minimum multiway cut for $(\tilde{G}, T)$.


Figure 5 A link structure with the variable-gadget on the left and its clause-gadget on the right. The bold red edges are the ones contained in the multiway cut. The green curve shows the existence of a path between a variable-terminal and a clause-terminal.
$\triangleright$ Claim 9. Any minimum edge multiway cut for $(\tilde{G}, T)$ does not contain any of the honeycomb edges.

Proof. Let $S^{\prime}$ be a minimum multiway cut for $(\tilde{G}, T)$. Recall that $\tilde{G}$ is planar. Note that an s-t cut in a planar graph corresponds to a simple cycle in the planar dual [19]. Therefore, the dual of a multiway cut comprises several cycles. Let the edges corresponding to $S^{\prime}$ in the planar dual of $\tilde{G}$ be $S^{*}$. In fact, $S^{*}$ induces a planar graph such that exactly one terminal of $T$ is embedded in the interior of each face of this graph. If any face of the $S^{*}$ did not contain a terminal, we could remove the edge in $S^{\prime}$ corresponding to one of the edges of this face. This would not connect any terminal pair, and hence contradicts the minimality of $S^{\prime}$.

Suppose that $S^{\prime}$ contains some of the edges of the honeycomb in $\tilde{G}$ corresponding to the vertex $v \in V\left(G^{\prime}\right)$. We denote the intersection of $S^{\prime}$ with the edges of this honeycomb by $S_{h}^{\prime}$. Let the set of edges corresponding to $S_{h}^{\prime}$ in the planar dual of the honeycomb be $S_{h}^{*}$. By abuse of notation, we also refer to the graph induced by these edges, along with the (outer) vertex formed by contracting all the edges in $S^{*} \backslash S_{h}^{*}$, as $S_{h}^{*}$. Since each face of $S^{*}$ encloses a terminal, each face of $S_{h}^{*}$ must enclose an attachment point of the honeycomb. If not, then we could remove from $S^{\prime}$ an edge in $S_{h}^{\prime}$ corresponding to some edge of the face of $S_{h}^{*}$ not enclosing an attachment point. This does not make any new terminal-terminal connections as the part of the honeycomb enclosed by this face does not contain any path to any of the terminals of $T$. This would be a contradiction to the minimality of $S^{\prime}$.

Next, we observe that no face of $S_{h}^{*}$ can enclose more than one attachment point. Suppose that there exists a face in $S_{h}^{*}$ that encloses two attachment points. Since the two attachment points are separated by 100 cells of the honeycomb, the length of the face boundary must be at least 50 . We could remove all the corresponding 50 edges from $S^{\prime}$ and add all the attachingedges, instead. All the terminal-terminal paths passing through the honeycomb remain disconnected. Since at most 8 attaching-edges can be added, we again get a contradiction to the minimality of $S^{\prime}$.

So, each face of $S_{h}^{*}$ must enclose exactly one attachment point. To enclose the attachment points, each of these faces must cross the boundary of the honeycomb exactly twice. We claim that the faces of $S_{h}^{*}$, enclosing consecutive attachment points on the boundary of the honeycomb, are pairwise edge-disjoint. Suppose that the faces enclosing two consecutive attachment points, $a$ and $a^{\prime}$, share an edge. Then, they must also share an edge that crosses the boundary of the honeycomb. If they do not, then let $e$ be the last edge of the face
enclosing $a$ to cross the boundary and $e^{\prime}$ be the first edge of the face enclosing $a^{\prime}$ to cross the boundary of the honeycomb. The edges $e$ and $e^{\prime}$ along with the other edges not shared between the respective face boundaries bound a region of the plane containing no attachment points, a contradiction!

Therefore, any two faces of $S_{h}^{*}$ enclosing consecutive attachment points share an edge which crosses the boundary of the honeycomb. Without loss of generality, let this edge be closer to $a$. Then, the face enclosing $a^{\prime}$ must contain at least 50 edges as $a$ and $a^{\prime}$ are separated by 100 cells of the honeycomb. This implies that $S_{h}^{\prime}$ contains at least 50 edges. However, we could remove from it all the 50 edges and add all the 8 attaching-edges. This cut is smaller in size and disconnects all the terminal-terminal paths passing through the honeycomb. Once again, we contradict the minimality of $S^{\prime}$.

Since all the faces in $S_{h}^{*}$ enclosing attachment points are edge-disjoint, there are at least $2 \cdot \operatorname{deg}_{G^{\prime}}(v)$ edges in $S_{h}^{\prime}$. We could replace this cut by a smaller cut, namely, the multiway cut formed by removing the edges in $S_{h}^{\prime}$ from $S^{\prime}$ and adding to it all the attaching-edges incident on the attachment points. This cut disconnects all terminal-paths passing through the honeycomb and yet, is smaller in size than $S^{\prime}$, a contradiction to its minimality. Hence, $S^{\prime}$ does not contain any edge of any of the honeycombs.

By the construction of $\tilde{G}$ and Claims 2, 8, and 9, we conclude that Edge Multiway CuT is NP-complete on planar subcubic graphs.

## 4 Node Multiway Cut

We now discuss the node version of the multiway cut problem. We consider the restricted version of the problem where one is not allowed to pick the terminals into the node multiway cut. The problem is defined as follows.

## Node Multiway Cut

Input: Graph $G$, terminals $T \subseteq V(G)$, integer $k$
Question: Does there exist a subset of vertices of $V(G) \backslash T$ of size at most $k$ that pairwise disconnects the terminals of $T$ ?

- Lemma 10. If Edge Multiway Cut is NP-complete on a class $\mathcal{H}$ of graphs, then it is also NP-complete on the class of graphs $\mathcal{H}^{\prime}$ which are built from the graphs of $\mathcal{H}$ by subdividing each edge.

Proof. Let $G^{\prime}$ be the graph $G$ after subdividing each edge. For each edge $e$ in $G$, there exist two edges in $G^{\prime}$. If an edge of $G$ is in the edge multiway cut, then it suffices to pick only one of the two edges created from it in $G^{\prime}$ to disconnect the path $e$ lies on. Hence, $G$ has an edge multiway cut of size at most $k$ if and only if $G^{\prime}$ has an edge multiway cut of size $k$.

- Theorem 11. Node Multiway Cut is NP-complete on the class of planar subcubic graphs.

Proof. Clearly, Node Multiway Cut is in NP. In Theorem 1, we showed that Edge Multiway CuT is NP-complete on the class of planar subcubic graphs. We shall now reduce Node Multiway Cut from Edge Multiway Cut restricted to the class of planar subcubic graphs. Let $G$ be any planar subcubic graph with the set of terminals $T$. We create an instance of Node Multiway Cut by the following operations:

- We 2-subdivide each edge of $G$ and denote the resulting graph by $2 e(G)$.
- Next, we create the line graph of $2 e(G)$, which we denote by $L$. Note that $L$ is planar since the maximum degree of any vertex in $2 e(G)$ is three. It is also subcubic, due to the same reason.
- Finally, we create the terminal set of $L$ as follows: for each terminal $t$ in $2 e(G)$, consider the edges incident on it. In the line graph $L$, these edges must form a clique, $K_{i}$ for $i \in\{1,2,3\}: i=\operatorname{deg}(t)$. In this clique, we pick one vertex and make it a terminal. Let the terminal set in $L$ be denoted by $T_{L}$.
$\triangleright$ Claim 12. There exists an edge multiway cut of $(G, T)$ of size at most $k$ if and only if there exists a node multiway cut of $\left(L, T_{L}\right)$ of size at most $k$.

Proof. We assume that $(G, T)$ has an edge multiway cut $S$ of size at most $k$. By Lemma 10 $2 e(G)$ also has an edge multiway cut of size at most $k$. We claim that there exists an edge multiway cut $S^{\prime}$ of $2 e(G)$ of size at most $k$ which does not contain any edge incident on a terminal. Every edge in $2 e(G)$ is adjacent to some edge with both its ends having degree two. Therefore, if an edge in the edge multiway cut of $2 e(G)$ is incident on a terminal, we can replace it with its adjacent edge, which disconnects all the paths disconnected by the former and does not increase the size of the edge multiway cut. Now, for each edge in $S^{\prime}$ we add its corresponding vertex in $L$ to a set $S_{L}$. Since $S^{\prime}$ pairwise disconnects the terminals in $2 e(G), S_{L}$ disconnects all the terminal cliques from each other. Therefore, $S_{L}$ is a node multiway cut of $L$.

Conversely, let $S_{L}^{\prime} \subseteq V(L) \backslash T_{L}$ be a node multiway cut of ( $L, T_{L}$ ) of size at most $k$. By similar arguments as above, we may assume that $S_{L}^{\prime}$ does not contain any vertex from any terminal-clique. We claim that $G$ has an edge multiway cut of size at most $k$. To that end, we show that $2 e(G)$ has an edge multiway cut of size at most $k$ and appeal to Lemma 10 to prove the same for $G$. We add to the edge multiway cut $S$ the edges of $2 e(G)$ that correspond to the vertices in $S_{L}^{\prime}$. The size of $S$ is clearly at most $k$. To see that it is an edge multiway cut of $2 e(G)$, note that pairwise disconnecting the terminal-cliques of $L$ amounts to pairwise disconnecting the set of edges incident on any terminal in $2 e(G)$ from its counterparts. This, in turn, pairwise disconnects all the terminals in $2 e(G)$.

By our construction and Claim 12, Node Multiway Cut is NP-complete on the class of planar subcubic graphs.

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