PROMOTING CONCEPTUAL UNDERSTANDING OF DIFFERENTIAL EQUATIONS THROUGH INQUIRY TASKS

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ABSTRACT

Courses in Differential Equations (DEs) have been an important part of engineering education for decades. However, students experience difficulties with the understanding of main concepts including differential equation itself and diverse types of solutions (general, particular, stationary). In this paper, we discuss how the work on non-routine problems on the Existence and Uniqueness Theorems (EUTs) helps students to make sense of DEs and their solutions thus contributing to the development of advanced mathematical thinking.

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1 INTRODUCTION

1.1 Importance of mathematics in engineering curricula

Teaching mathematics to future engineers is challenging; one has to maintain a correct balance between theoretical knowledge and techniques for solving relevant applied problems. On the one hand, "the teaching of 'practical' mathematics is becoming much more focused on the process of modelling of engineering systems - this results in a decrease in the teaching of calculation techniques" [11, p. 9]. Furthermore, due to a rapidly growing use of digital technology in engineering education, "there are significant dangers in losing the teaching of pen-and-paper mathematical techniques to 'button pressing'" [11, p. 10]. On the other hand, Clark [7, p. 149] argued that all structural engineers "should experience a rigorous mathematical education, not necessarily because they will use the mathematics in their future careers, but because of the mode of thinking that such education develops." On a similar note, Devlin stressed that "the main benefit they [software engineers] got from mathematics they learned in academia was the experience of rigorous reasoning with purely abstract objects and structures" [9, p. 22].

Although both educators and students acknowledge the importance of abstract mathematical thinking, educational research often points towards the lack of students' conceptual understanding and the tendency of engineering students to take an instrumental approach to their studies. Many students are surprised by the elevated level of demands set in mathematics courses which are often perceived as obstacles on their path to the engineering degree. One of the difficult but important for engineering curricula courses is that in Differential Equations (DEs).

DEs are used for modelling of a wide spectrum of phenomena including nonlinear oscillations in mechanical systems, complex dynamics of financial markets, currents in electric circuits, and spread of infectious diseases. Students should learn both useful solution techniques and fundamental theoretical results and be prepared for the analysis of applied problems described by DEs using analytical techniques in combination with numerical methods and computer. As Bickley pointed out, "in the end, it is not the number of tricks that the student has learned, but rather the understanding of the concepts and awareness of the relevance of the techniques, which is important – and, finally, his approach to the new learning which an encounter with a new problem may demand" [3, p. 383].

1.2 Research on teaching and learning differential equations

Current research on teaching and learning DEs is rather scarce, there are "fewer than two dozen empirical studies published in top journals" which is surprising "given the centrality of differential equations (DEs) in the undergraduate curriculum" [13, p. 555]. In fact, teaching and learning of DEs at the university level is a quite new area of educational research, and "we need to explore the variety of ways in which content, instruction, and technology can be profitably coordinated to promote student learning" [12, p. 84].

New didactic approaches and modern digital technology have a positive impact on students' understanding of DEs. However, many of them erroneously believe that the success in DEs courses can be achieved by learning only solution routines. Recent empirical research actively explored students' understanding of the concepts of a DE and its solutions [1, 6, 12, 14] revealing many difficulties. Students concentrate attention on specific solution techniques and "often fail to relate them to other concepts or ideas" [6, p. 76]. They experience difficulties with the fundamental concepts including a DE itself, the general and particular solutions [1]. Students "made little or no attempt to place the solution in context, be it a solution to an equation or a DE" [14, p. 48]. Unfortunately, "students were successful in algebraic solutions of DEs, but not in conceptualising DEs and the solution of DEs concepts … algebraic solutions of DEs can be found even without a deep understanding and conceptualization of DEs, which is why students do not feel any need to understand DEs and related concepts" [1, p. 887]. In summary, "research has pointed to the various challenges that students face with this concept" [13, p. 555].

The authors [17] analysed several tasks suggested in the literature for assessing students' conceptual understanding of the general and particular solutions to DEs concluding that only one out of five problems encourages students' inquiry and can contribute to their conceptual understanding of DEs. We argued that the correct formation of the concept as Vygotsky's *scientific concept* can be achieved only through the rigorous explanation of all relevant definitions and theoretical results, including the Existence and Uniqueness Theorems (EUTs), which meaningfully complete the definition of a solution to a DE and link all important notions. We believe that conceptual understanding of DEs can be fostered by the use of inquiry-based pedagogy [12] and non-routine problems [2]; first steps in this direction were recently made by the authors [17-19].

The research question we address in this paper is: *How does the work on non-routine problems impact the conceptual understanding of the notion of a DE and its solutions by senior engineering students?*

2 METHODOLOGY

2.1 Inquiry-based mathematic education

Contemporary trends shift mathematics teaching from instructor-centred to student-centred, and the terms inquiry and inquiry-based mathematics education (IBME) appear increasingly often in research literature and in educational policy documents. Simply put, inquiry-based learning and teaching mean the organisation of students' work similarly to that of professional mathematicians. Rasmussen and Wawro [13] argued that the three important components of inquiry-oriented instruction are (i) student deep engagement in mathematics, (ii) peer-to-peer collaboration, and (iii) instructor inquiry into student thinking. A three-layer inquiry model developed by Jaworski [10] takes this idea further considering the inner layer where students engage into inquiry in a classroom with peers and a lecturer, the middle layer where lecturers engage into professional inquiry aimed at creating new learning opportunities, and the outer layer where education researchers and lecturers extend inquiry further to the developmental research.

The ultimate goal of IBME is to empower students to inquire independently and with confidence. Practical strategies that teach inquiry include rephrasing usual problems as questions, searching for hidden patterns, formulating, and verifying conjectures, designing counterexamples, searching for alternative solutions, etc.

2.2 Conceptual understanding and non-routine problems

Reframing standard textbook tasks into inquiry-oriented ones often turns them into *non-routine problems,* that is, problems "for which students had no algorithm, well-rehearsed procedure or previously demonstrated process to follow" [5, p. 2318]. Such tasks introduce uncertainty and associated risks, their use in teaching is challenging both for lecturers and students, but the benefits are significant – "if more time were spent in classrooms with students engaged in working on cognitively demanding nonroutine tasks, as opposed to exercises in which a known procedure is practised, students' opportunities for thinking and learning would likely be enhanced" [20, p. 92]. By cognitive demands we understand the form and level of thinking needed by students for successful engagement and solution of the given task. Empirical evidence confirms that "the highest gains on a mathematics-performance assessment were related to the extent to which tasks were set up and implemented in ways that engaged students in high levels of cognitive thinking and reasoning. […] Starting with a good task does, however, appear to be a necessary condition, since low-level tasks almost never result in high-level engagement" [21, p. 344]. The use of high-level tasks in teaching encourages student reflections, facilitates generation and exchange of ideas, fosters creativity, stimulates further inquiry, and contributes to the development of advanced mathematical thinking.

2.3 Teaching experiment and data collection

The teaching experiment was organised in a DEs course for senior mechatronics students in their fourth year of studies. A total of thirty-seven students enrolled in this course based on a popular textbook by Boyce and DiPrima [4]. In the final part of the course, students worked for three weeks on an assessed assignment – a set of non-routine problems on EUTs designed by the authors with the focus on the development of conceptual understanding. Our aim was to explore how non-standard questions can challenge students, develop their analytical skills, and contribute to conceptual understanding of important notions and ideas in an ODE course for engineering students. Furthermore, introducing a small group work in the project, we could explore the extent to which individual work and group discussions contributed to students' conceptual understanding of EUTs and influenced their individual solutions submitted for final assessment.

Participation in the teaching experiment was voluntary. We expected that by the time of the assessment students acquired necessary theoretical knowledge and developed required computational skills. In the first week, students worked on the problems individually in the tutorial and submitted a copy of their solutions to the lecturer (script #1). Then they worked on the assignment at home and handed in individual solutions in the next tutorial (script #2). During the second week, students discussed their individual solutions in small groups, audio-recorded the discussions and submitted the audio files to the lecturer. In the last, third week, students presented solutions they agreed upon in small groups to their peers during the tutorial. The lecturer was present in the class but did not participate in the discussions and did not comment on students' solutions. In the end of the third week, students submitted individual solutions (script #3) for grading and received lecturer's feedback by email afterwards. During these three weeks, students had no lecturer's feedback on the scripts but had the possibility to reflect (individually or in groups) about their solutions and modify them, if desired, in the next script. We collected students' written work (three scripts), answers to pre- and post- questionnaires, audio recordings of the small group discussions, and the recording of the class presentation of solutions. The audio records were transcribed, and the data were analysed after the course work was completed and the letter grades were assigned.

3 DATA ANALYSIS

3.1 Sample tasks and expected reasoning

The problems on the EUTs that do not require the use of computer in the course textbook fall into the following four categories: (i) determine (without solving the problem) an interval in which the solution of the given initial value problem is certain to exist – six problems; (ii) state where in the ty-plane the hypotheses of the theorem are satisfied – six problems; (iii) solve the given initial value problem and determine how the interval in which the solution exists depends on the initial value y_0 – four problems; (iv) explain why the existence of two solutions of the given problem does not contradict the uniqueness part of the theorem – *one* problem.

Since the nature of sixteen out of seventeen problems on EUTs in the course textbook [4] is procedural, the problems in our assignment were designed to engage students in a deeper reflection about EUTs and related notions. In this paper, we discuss students' approach to the following two problems.

Problem 1. (a) Verify that $y(x) = C_1 + C_2x^2$ is the general solution of a differential equation $xy'' - y' = 0$. **(b)** Explain why there exists no particular solution of the given equation satisfying initial conditions $y(0) = 0,\ y'(0) = 1$. **(c)** Suggest different initial conditions for this differential equation so that there will exist exactly one particular solution of a new initial value problem. Motivate your choice.

Problem 2. The coefficient $p(x) = \frac{2}{x}$ in a linear differential equation $xy' + 2y = 18x^4$ is discontinuous at $x = 0$. **(a)** According to the EUT will a solution satisfying the initial condition $y(0) = 0$ exist or not? **(b)** How does your answer to part (a) agree with the fact that $y = 3x^4$ is the exact solution of the initial value problem $xy' + 2y = 18x^4$, $y(0) = 0$? Explain.

Both problems are non-routine; no similar examples or problems are discussed in the textbook [4]. The tasks encourage exploration and set cognitive demands at the higher levels of using procedures with connections to concepts and meanings and doing mathematics [21]. In Problem 1, we expected that students (i) verify that a given function is *the general solution* and (ii) show that the first initial condition yields $C_1 = 0$ whereas the second one leads to a meaningless equality $2C_2 \cdot 0 = 1$. The exploration in part (c) includes two options: either choose the initial value $x_0 \neq 0$, in which case the EUT always assures local existence of solution to a system of differential equations $y' = u,\ \ u' - \frac{1}{x}u = 0$, (an easier one) or change the second initial condition to $y'(0) = 0$ obtaining a one-parameter family of solutions $y(x) = C_2x^2$ (a more difficult one). In Problem 2, one has to show first that the discontinuity at $x = 0$ of the coefficient in the differential equation written in the standard form $y' + \frac{2}{x}y = 18x^3$ does not allow applying the EUT, and the theorem is inconclusive. However, the existence of an exact solution emphasises the fact that the EUT provides only sufficient conditions and if these are not met, a unique solution to the given initial value problem may or may not exist.

3.2 Analysis of the students' work

We illustrate students' reasoning using the transcripts of small group discussions, group A with students, A1-A4 and group B with students B1-B5. We selected episodes where the discussions are particularly succinct. Although students are not native speakers of English, we did not edit the original text in transcripts.

Episode 1 – Problem 1 (b).

A2. My approach there was just to put in the initial conditions and then see that $C_1 = 0$, that is okay, and then I tried the derivative $y'(0)$, it's supposed to be equal to 1.

A3. 0 is never equal to 1.

A2. So the equation does not compute.

A3. So it is not possible to determine C2. […]

A2. It is 0=1.

A1. I did the same thing as well, but I tried thinking why is it this way, and my sort of conclusion was that it's in the bottom of a parabola, where the derivative always is 0.
A3. It cannot be anything else in a bottom of a parabola which is a minimum point.
A1. So, therefore if you state that the derivative at that point should be anything other than zero it doesn't make any sense because it's the minimum point.

A2. It has to be a minimum or maximum point.

In Episode 1, students employed analytical reasoning pointing to the inconsistency of the system of algebraic equations for determining coefficients of particular solution (A2 and A3) and combined it with a geometrical argument referring to the particular shape of solution curves (parabolas) and the zero value for the slope of a line tangent to these curves at the origin (A1 and A3).

Episode 2 – Problem 1 (c).

B4. For my part, I just made up some initial conditions, I just tried them, so $y(1) = 2$, $y'(1) = 2$. Then we can get $C_1 = 1$ and $C_2 = 1$. So, this will possibly be a suitable initial condition.

B3. I also made up some initial conditions and tested them. I used $y(0)=1,\ y'(0)=0$, and I get $C_1=1$ and $C_2=\frac{1}{2}$.

B4. Ok, there will possibly be multiple initial conditions.

B1. You can choose any x-value, and arbitrary … anything except $x=0$ will work?

B4. Ok.

B5. I did the same, I used the existence and uniqueness theorem because of discontinuity at x=0, so no guarantee there, but for all other x there is a solution guaranteed.

B2. That might be the correct solution. We must be sure that we have a solution by referring it to the theorem. I think you can show it by solving, too.

The developments in Episode 2 perfectly matched our expectations. We observe that students discussed both possible modifications of initial conditions that ensure the uniqueness of solutions. B1, B4 and B5 opted for a different initial point whereas B3 suggested the only possible value for the derivative, $y'(0)=0$.

Episode 3 – Problem 2 (a).

B4. We have a linear equation, like $xy'+2y=18x^4$, and if we put it in the standard form, we have a coefficient like $p(x)=\frac{2}{x}$ and that will be discontinuous at $x=0$. And according to the existence and uniqueness theorem does a solution satisfying the initial condition $y(0)=0$ exist or not? As a start, I think that is a tricky question […] trick question. So, my suggestion is, due to the discontinuity, the EUT does not apply. The theorem cannot say anything about the existence and uniqueness of that solution.

B2. I agree.

B4. All of us.

B3. Yes.

Episode 3 demonstrates that students understood the meaning of the EUT. They noticed the discontinuity of the coefficient of a DE at $x=0$ and unanimously concluded that in this case the theorem is inconclusive.

Episode 4 – Problem 2 (b).

B5. The existence and uniqueness theorem just says that I cannot say anything, I cannot tell you anything whether there is a solution or not because of this discontinuity, so the solution could exist or not exist. It does not violate the theorem in any way if it exists or does not exist. But it exists.

B4. But should someone improve the theorem to make it better?

B5. Maybe. I think in mathematics a lot of the times you have some theorem it cannot tell you everything. There is no universal answer for everything.

In Episode 4, the student B5 explained that the EUT does not provide necessary but only sufficient conditions for the existence of a unique solution. His concise and correct explanation confirms a good understanding of the issues discussed in Episode 4 where he seemingly was less active.

4 DISCUSSION AND CONCLUSIONS

The teaching experiment reported in this paper was designed to explore how the use of non-routine tasks stimulates student inquiry and contributes to the advancement of the conceptual understanding of the notions of a DE, its solutions, and related theoretical results, including EUTs. Preparing the assignment problems, we employed the *inquiry by design technique* where lecturers "design tasks and projects that stimulate to ask questions, pose problems, and set goals" whereas students "must learn to inquire systematically" and "must actively construct their own knowledge" [16, p. 38]. Working with senior students who had previous experience with other mathematics courses, developed appropriate learning strategies and social skills needed for collaborative work, we deliberately provided no lecturer's support because "students operating at the frontiers of their conceptual knowledge have no reason to build new conceptual structures unless their current knowledge results in obstacles, contradictions or surprises" [8, p. 82]. Substantial academic maturity of students facilitated their engagement in unguided inquiry. Four episodes from small group discussions in Section 3 illustrate an overall success of the teaching experiment and positive impact of the use of non-routine tasks on the development of students' conceptual understanding of the EUTs for DEs.

In line with [1, 6, 14], we also acknowledge students' difficulties with the concept of the general solution of a DE introduced as the expression which contains all its possible solutions [4, p. 11] and defined only for linear DEs. Integration of the first order linear DEs always furnishes the general solution although the fact that no other solutions are available is not emphasised. Furthermore, the concept of the general solution is used in [4] primarily to develop the theory of higher order linear DEs with constant coefficients. For nonlinear DEs, the situation may be much more complex, and the textbook prompts that "the existence of "additional" solutions is not uncommon for nonlinear equations" [4, p. 11]. However, in attempt to facilitate students' learning, many textbooks use the term "general solution" only to discuss linear DEs. This might be one of the main reasons for students' lack of attention to this important concept explaining the difficulties experienced when asked to demonstrate that a given function is the general solution to a DE, especially if the equation is nonlinear. The issue can be resolved if the notion is properly introduced and illustrated with relevant examples where "additional" solutions are produced. We plan to address this important problem in one of our forthcoming papers along with the analysis of changes in the views on teaching of DEs during the last fifty years.

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