AN INTRODUCTIONAL LECTURE ON CHAOTIC SYSTEMS THROUGH LORENZ ATTRACTOR AND FORCED LOTKA VOLTERRA EQUATION FOR INTERDISCIPLINARY EDUCATION

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Conference Key Areas: *Mathematics at the heart of Engineering, Co-creation with students*
Keywords: *Interdisciplinary engineering education, theory-based engineering course, practice-based learning concepts*

ABSTRACT

Is it possible to predict the future? How accurate is the prediction for the future? These questions are fascinating and intriguing ones in particular for young generations who look at their future with curiosity. For a long time, many have tried to quantitatively predict future behavior of systems more accurately with techniques such as time series analysis and derived dynamical models based on observed data. The paper proposes a lecture structure in which elements of chaos, which greatly impacts the predictive capabilities of dynamical models, are introduced through two classical examples of nonlinear dynamical systems, namely Lorenz attractor and Lotka-Volterra equations. In a possible lecture, these two structures are introduced in a basic and intuitive way, followed by equilibria analyses and Lyapunov control approaches. The paper intends to give a possible structure of an interdisciplinary lecture in chaotic systems, for all students in general and non-engineering students in particular, to kindle students' interest in challenging ideas and models. By presenting an intuitive learning-based approach and the results of the implementation, the paper contributes to the discourse on interdisciplinary education. The lecture is a part of a course within a Complementary Study at Leuphana Unversity of Lüneburg. The material which inspired the proposed lecture structure is taken from the scripts of the Master Complementary Course titled Modelling and Control of Dynamical Systems using Linear and Nonlinear Differential Equations held at Leuphana University of Lüneburg.

1 INTRODUCTION

The presented lecture can be used in different contexts: in Bachelors courses for engineers as well as for Masters courses dedicated to non-engineers (non-engineering minor programs,

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doi:10.5821/conference-9788412322262.1457

complementary study). In this way, the paper provides important insights into how to teach the first and only control course in non-control engineering programs. In fact, an important point of the complementary study at Leuphana University in Lüneburg, Germany, is a holistic vision of the culture in which also non-engineers can profit from knowledge which typically is just for engineers. Therefore, such liberal education demands and encourages the intellectual and personal development of each and every specific student profile, in which new ideas to teach and to connect naturally or technically with human sciences represent a prerequisite.

In this paper, a modified Lotka-Volterra model with fixed initial condition has been employed to examine whether there is a pattern in chaos situations and where equilibrium points are that do not change over time, see [1]. It is important that the course starts recalling basic knowledge of differential equations and, before starting with a lecture dedicated to sliding mode control, the course should introduce elements of nonlinear differential equations considering the fundamental direct method of Lyapunov [2] related to the stability of a solution of a differential equation.

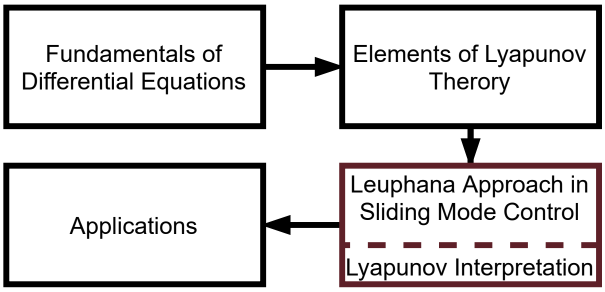

Fig. 1. Structure of the proposed course

Figure 1 shows the structure of the proposed course of this lecture in which the control of chaotic systems using Lyapunov control is proposed as an application. There are theoretical backgrounds about Lorenz's chaos theory and Lotka-Volterra model along with the explanation of forced Lotka-Volterra equations in Section 2. In Section 3, differential equations of Lorenz attractor of chaos theory as well as forced Lotka-Volterra equations in chaos scenarios with Jacobian matrix are computed manually in order to find equilibrium points and then they are followed by manual computation of Lyapunov equation for controllers. In Section 4, the simulations in Matlab/Simulink regarding Section 3 has been described. Later, conclusion and some remarks are devoted in Section 5.

2 MODELLING

Several kinds of dynamical systems from nature, and also synthetic ones, can be chaotic in their solutions. Two important ones are Lorenz attractors and forced Lotka-Volterra models.

2.1 Lorenz attractor

In the context of meteorology and fluid dynamics, Edward Lorenz came up with the simplest equation to explain the observed chaotic phenomena in the Earth atmosphere [3]. The result are three nonlinear differential equations $\dot{x}$, $\dot{y}$ and $\dot{z}$,

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \sigma\left(y(t) - x(t)\right), \tag{1}$$

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = px(t) - y(t) - x(t)z(t), \tag{2}$$

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = x(t)y(t) - \beta z(t), \tag{3}$$

where x is the rate of convective motion, for instance how fast the rolls are rotating, y is the temperature difference between the ascending and descending currents, and z is the distortion (from linearity) of the vertical temperature profile, see [3] for details on the practical interpretation of the parameters and variables of this example system. Parameters σ, p, and β depend on condition of the fluid, the heat input, etc., but they are assumed to be constant throughout experiments. So, in the simulations from Lorenz, $\sigma = 10$, $p = 28$, and $\beta = \frac{8}{3}$, see [4]. When the aspect of movement is transferred to the coordinate planes of the x, y, and z axes, a 'strange butterfly' shape can be seen. It is also called Lorenz's Butterfly Attractor because its shape resembles a butterfly. Here, the characteristic of chaos shows that it is sensitive to initial conditions but at the same time it is still possible to find the underlying pattern. Another, structurally simpler class of dynamical systems that can exhibit chaotic behavior under certain conditions is the dynamics of predator and prey populations in nature.

2.2 Lotka-Volterra models and their control

When there are preys, there are predators. It can also be applied to the other way around. When there are predators, there are preys. From this phenomenon, one can see that there must be a pattern or trend of prey and predator population over time. This pattern has been formed into equations by Alfred J. Lotka in 1910, analyzing prey-predator interactions as known as Lotka-Volterra model and predicting the population rate of change over time, see [5]. So the original Lotka-Volterra model is described by the following pair of equations, see [1]:

$$\frac{\mathrm{d}N_1(t)}{\mathrm{d}t} = \alpha_1 N_1(t) - \beta_1 N_1(t) N_2(t), \tag{4}$$

$$\frac{\mathrm{d}N_2(t)}{\mathrm{d}t} = -\alpha_2 N_2(t) + \beta_2 N_1(t) N_2(t), \tag{5}$$

where $N_1(t)$ and $N_2(t)$ are the population of prey and predator respectively, α_1 is the growth rate of prey and α_2 is the die out rate of predator. The term - $\beta_1 N_1(t) N_2(t)$ represents the loss rate of prey due to collisions with predator and $\beta_2 N_1(t) N_2(t)$ represents the growth rate of the population of predator due to collisions with prey. The equations have periodic solutions under the assumptions that none of parameters α_1, α_2, β_1, β_2 are negative, see [1]. From the model, the population of predator seemingly follow the pattern of the population of prey putting both populations into loop of increase and decrease. In this paper, the modified Lotka-Volterra (forced Lotka-Volterra) model which has been introduced by Gause, see [6], has been employed in order to obtain chaotic behaviour

$$\frac{\mathrm{d}N_1(t)}{\mathrm{d}t} = \alpha_1 N_1(t) - \beta_1 \sqrt{N_1(t)} N_2(t) - \gamma_1 (N_1(t))^2, \tag{6}$$

$$\frac{\mathrm{d}N_2(t)}{\mathrm{d}t} = -\alpha_2 N_2(t) + \beta_2 \sqrt{N_1(t)} N_2(t), \tag{7}$$

where $\gamma_1 (N_1(t))^2$ is a logistic term and also a penalty due to lack of room for prey as its number grows and the terms - $\beta_1 \sqrt{N_1(t)} N_2(t)$ and $\beta_2 \sqrt{N_1(t)} N_2(t)$ are now not proportional to $N_1(t)$

but to $\sqrt{N_1(t)}$. In other words, the original Lotka-Volterras collision terms are proportional to $N_1(t)$ while Gauses collision is obtained when it is proportional to $\sqrt{N_1(t)}$ of the prey community which contributes to collision and provides saturation effect for the collision. The saturation effect causes a population explosion of prey when the logistic term is disregarded, see [6]. In this paper, time-variance will be considered since recovery of grass causes a periodic oscillation of the grass-eating prey's population. Then the growth rate of prey turns out to be as following, which according to [1] yields chaotic behavior:

$$\alpha_1(t) = 2\left[a - b\cos(\omega t)\right] \tag{8}$$

where a is the positive constant and b and ω are the amplitude and angular frequency of the oscillating part of the growth rate respectively. Furthermore, these two forced Lotka-Volterra equation will be extended again using control inputs which are $\mu_1(t)$ and $\mu_2(t)$. It is possible to control the population of both species by intervening in their interaction. In other words, population dynamics can be affected by controlling amount of food provided, or controlling the population of species by introducing or withdrawing individuals from the habitat.

$$\frac{\mathrm{d}N_1(t)}{\mathrm{d}t} = \alpha_1 N_1(t) - \beta_1\sqrt{N_1(t)}N_2(t) - \gamma_1 N_1(t)^2 + \mu_1(t), \tag{9}$$
$$\frac{\mathrm{d}N_2(t)}{\mathrm{d}t} = -\alpha_2 N_2(t) + \beta_2\sqrt{N_1(t)}N_2(t) + \mu_2(t). \tag{10}$$

Here, $\mu_{1,2}(t)$ will be regarded as controlling the population of two species by introducing or withdrawing individuals since time-variance for the amount of food has been already considered in form of $\alpha_1(t)$. Even in the presence of chaotic behavior in the populations, a controller can help to stabilize the populations from the outside at an arbitrary level.

3 SYSTEM ANALYSIS AND CONTROL

3.1 Equilibrium points of Lorenz attractor

With Lorenz equations (1-3), equilibrium points fulfil the conditions $\dot{x} = 0$, $\dot{y} = 0$, $\dot{z} = 0$

$$0 = \sigma\left(y(t) - x(t)\right), \tag{11}$$
$$0 = px(t) - y(t) - x(t)z(t), \tag{12}$$
$$0 = x(t)y(t) - \beta z(t), \tag{13}$$

Since parameters σ, p, β are given as constants, the solutions are $x = 0$ or $x = \pm\sqrt{\beta(p-1)}$, $y = x$, and $z = p - 1$. Therefore, the equilibrium points of x, y, z including the origin are [4]

$$C^{\pm} = (\pm\sqrt{\beta(p-1)}, \pm\sqrt{\beta(p-1)}, p-1). \tag{14}$$

Using parameters $\sigma = 10$, $p = 28$, and $\beta = \frac{8}{3}$ as fixed by Lorenz, computing equilibrium points by hand results in $C^{+} = (8.5, 8.5, 27)$, $C^{-} = (-8.5, -8.5, 27)$, which corresponds with Fig. 2.

3.2 Limited equilibrium of Forced Lotka Volterra using Jacobian Matrix

Using the forced Lotka-Volterra model in Section 2.2, limited equilibrium points can be reached Solving these two conditions,

$$0 = \alpha_1 N_1(t) - \beta_1\sqrt{N_1(t)}N_2(t) - \gamma_1(N_1(t))^2, \tag{15}$$
$$0 = -\alpha_2 N_2(t) + \beta_2\sqrt{N_1(t)}N_2(t), \tag{16}$$

apart from the trivial solution $N_{1,2}(t) = 0$ yields the following two equilibria:

$$N_1(t) = \frac{\alpha_1}{\gamma_1},\ N_2(t) = 0 \text{ or } N_1(t) = \frac{\alpha_2^2}{\beta_2^2},\ N_2(t) = \frac{\alpha_1\alpha_2\beta_2^2 - \alpha_2^3\gamma_1}{\beta_1\beta_2^3}. \tag{17}$$

For sake of brevity, we will consider just the first two equilibrium points in terms of local stability using the Lyaponov criterium which concerns the local stability. We can see that these two equilibrium points are locally unstable. Point 1 gives trivial solution because when there is no prey ($N_1(t) = 0$), there will be no predator as well ($N_2(t) = 0$). In other words, when one species will go extinct, the existence of the other species will also be endangered. When it comes to determining whether equilibrium points in both Point 1 and 2 are stable, the eigenvalues of the Jacobian Matrix J should be examined. Assuming that $f_1(t) = \frac{\mathrm{d}N_1(t)}{\mathrm{d}t}$ and $f_2(t) = \frac{\mathrm{d}N_2(t)}{\mathrm{d}t}$, Jacobian matrix of forced Lotka-Volterra can be described as following.

$$J = \begin{bmatrix} \frac{\partial f_1(t)}{\partial N_1(t)} & \frac{\partial f_1(t)}{\partial N_2(t)}, \\ \frac{\partial f_2(t)}{\partial N_1(t)} & \frac{\partial f_2(t)}{\partial N_2(t)} \end{bmatrix} = \begin{bmatrix} \alpha_1 - 2\gamma_1 N_1(t) - \frac{\beta_1 N_2(t)}{2\sqrt{N_1(t)}} & -\beta_1\sqrt{N_1(t)} \\ \frac{\beta_2 N_2(t)}{2\sqrt{N_1(t)}} & -\alpha_2 + \beta_2\sqrt{N_1(t)} \end{bmatrix}. \tag{18}$$

In order to analyze the equilibria stability, Lyapunov stability theory for time-variant systems states that the eigenvalues of $J_p^T + J_p$ must have a negative real part [7]. Two points are investigated. First, when point 1 is used,

$$J_{p_1} = \begin{bmatrix} \alpha_1 & 0 \\ 0 & -\alpha_2 \end{bmatrix}, \tag{19}$$

where we have two eigenvalues λ_1: $2\alpha_1$ and $-2\alpha_2$.

$$\det(\lambda_1 I - J_{p_1}^T - J_{p_1}) = 0 \Rightarrow \det \begin{bmatrix} \lambda_1 - 2\alpha_1 & 0 \\ 0 & \lambda_1 + 2\alpha_2 \end{bmatrix} = 0. \tag{20}$$

The determinant formula shows that λ_1 equals $2\alpha_1$ which has a positive value, thus making the system locally unstable around this equilibrium point. On the other hand when using point 2, the result is as follows

$$J_{p_2} = \begin{bmatrix} -\alpha_1 & -\beta_1\sqrt{\frac{\alpha_1}{\gamma_1}} \\ 0 & -\alpha_2 + \beta_2\sqrt{\frac{\alpha_1}{\gamma_1}} \end{bmatrix}, \tag{21}$$

and so

$$J_{p_2}^T + J_{p_2} = \begin{bmatrix} -2\alpha_1 & -\beta_1\sqrt{\frac{\alpha_1}{\gamma_1}} \\ -\beta_1\sqrt{\frac{\alpha_1}{\gamma_1}} & -2\alpha_2 + 2\beta_2\sqrt{\frac{\alpha_1}{\gamma_1}} \end{bmatrix}, \tag{22}$$

where we have two values for λ_2:

$$\det(\lambda_2 I - J_{p_2}^T - J_{p_2}) = 0 \Rightarrow \det \begin{bmatrix} \lambda_2 + 2\alpha_1 & \beta_1\sqrt{\frac{\alpha_1}{\gamma_1}} \\ \beta_1\sqrt{\frac{\alpha_1}{\gamma_1}} & \lambda_2 + 2\alpha_2 - 2\beta_2\sqrt{\frac{\alpha_1}{\gamma_1}} \end{bmatrix} = 0. \tag{23}$$

The two solutions for λ_2 are complicated, with the same real (and non-zero imaginary) part

$$\mathrm{Re}(\lambda_2) = -\alpha_1 - \alpha_2 + \beta_2\sqrt{\frac{\alpha_1}{\gamma_1}}, \tag{24}$$

so in order to investigate stability, it is enough to consider the real part of the two solutions of λ_2 and the stability condition (depending on the system model parameters) becomes:

$$-\alpha_1 - \alpha_2 + \beta_2\sqrt{\frac{\alpha_1}{\gamma_1}} < 0. \tag{25}$$

3.3 Controllers using Lyapunov equation

With controllers $\mu_{1,2}(t)$ which allows for convergence to an arbitrary level, a Lyapunov equation is used to make sure that the arbitrarily desired population size combination is stable. According to the Lyapunov theory, it can be described as

$$v(N_1(t), N_2(t)) = \frac{1}{2}[(N_1(t) - N_{1d}(t))^2 + (N_2(t) - N_{2d}(t))^2], \tag{26}$$

where $N_{1d}(t)$ is desired population of prey and $N_{2d}(t)$ is desired population of predator. In this paper, the desired population of prey and predator have been set for 200 and 300 respectively. Assuming that $(N_1(t) - N_{1d}(t))$ is $S_1(t)$ and $(N_2(t) - N_{2d}(t))$ is $S_2(t)$, the result of computing $\dot{v}(N_1(t), N_2(t))$ is as following.

$$\dot{v}(N_1(t), N_2(t)) = (N_1(t) - N_{1d}(t))\frac{\mathrm{d}N_1(t)}{\mathrm{d}t} + (N_2(t) - N_{2d}(t))\frac{\mathrm{d}N_2(t)}{\mathrm{d}t} \tag{27}$$

$$= S_1(t)\frac{\mathrm{d}N_1(t)}{\mathrm{d}t} + S_2(t)\frac{\mathrm{d}N_2(t)}{\mathrm{d}t}. \tag{28}$$

Substituting equations $\frac{\mathrm{d}N_1(t)}{\mathrm{d}t}$ and $\frac{\mathrm{d}N_2(t)}{\mathrm{d}t}$ with forced Lotka-Volterra equations, $\dot{v}(N_1(t), N_2(t))$ can also be described as

$$\dot{v}(N_1(t), N_2(t)) = S_1(t)[\alpha_1 N_1(t) - \beta_1\sqrt{N_1(t)}N_2(t) - \gamma_1(N_1(t))^2 + \mu_1(t)] + S_2(t)[-\alpha_2 N_2(t) + \beta_2\sqrt{N_1(t)}N_2(t) + \mu_2(t)] \tag{29}$$

The input terms $\mu_1(t)$ and $\mu_2(t)$ which both consist of an equivalent and a corrective control term, can then be derived considering requirement $\dot{v} \leq 0$ of the sliding mode control approach

$$\mu_1(t) = \mu_{1eq}(t) + \mu_{1corr}(t), \tag{30}$$

$$\mu_{1eq}(t) = -\alpha_1 N_1(t) + \beta_1\sqrt{N_1(t)}N_2(t) + \gamma_1(N_1(t))^2, \tag{31}$$

$$\mu_{1corr}(t) = -\eta_1 \tanh(S_1(t)), \tag{32}$$

$$\mu_2(t) = \mu_{2eq}(t) + \mu_{2corr}(t), \tag{33}$$

$$\mu_{2eq}(t) = \alpha_2 N_2(t) - \beta_2\sqrt{N_1(t)}N_2(t), \tag{34}$$

$$\mu_{2corr}(t) = -\eta_2 \tanh(S_2(t)). \tag{35}$$

Here, $\eta_{1,2}$ determines how strong the input term will affect the equation to which it is applied, which means in this case, how strong it controls the population of prey and predator in order to stabilize a population combination, where it naturally would not be possible. In simulation for the convergence under the control, η_1 and η_2 have been set to 1 so that the desired population of prey and predator can be reached. Furthermore, the corrective term contains the hyperbolic tangent function as a substition of sign function $\mathrm{sign(S_{1,2})}$ for leading to a smoother graph with less oscillations in the simulink model.

4 SIMULINK MODEL IN MATLAB

The simulation of Lorenz attractor has been implemented in Matlab/Simulink to check whether it is possible to find pattern in chaotic situations. Instead plots of the considered Lorenz attractor have been visualized to describe the state evolutions of variables x, y, and z with Fig. 2. The Simulink implementation to see how the population dynamics of prey and predator converges using chaotic methods and controllers shows the following. Without connecting controller parts, both of the population of prey and predator show chaotic, unpredictable patterns in the respective scope blocks individually.

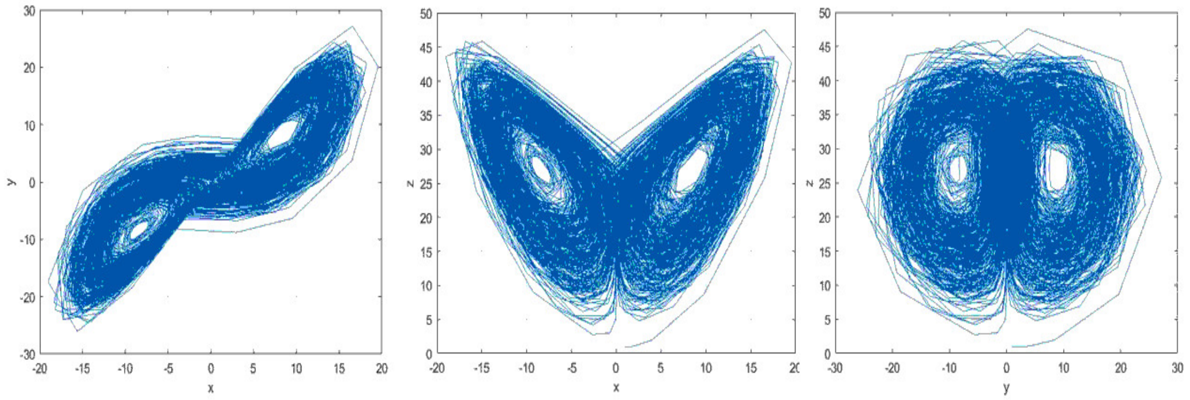

Fig. 2. 2D visualization of the three states of the Lorenz attractor

(1).png

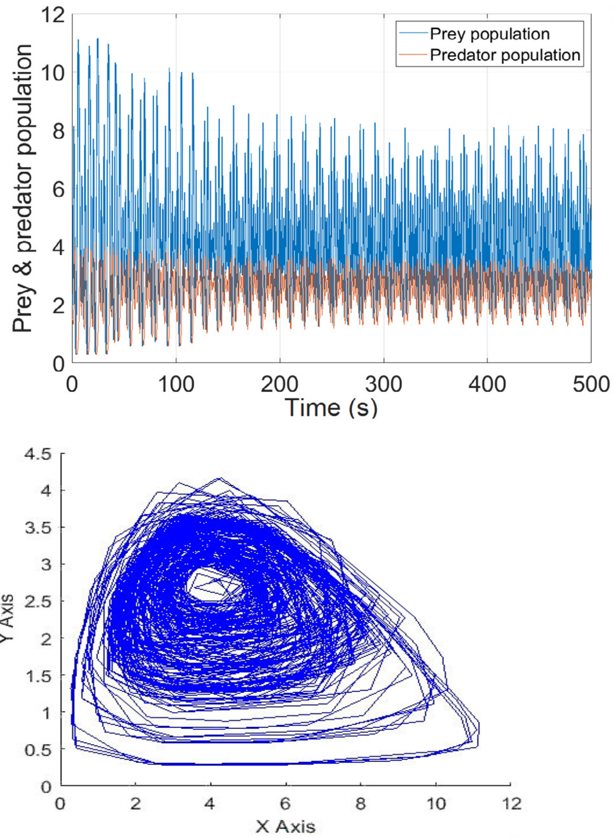

Fig. 3. Population evolution (left) of prey and predator without controllers and phase plot (right)

In Fig. 3, it can be observed that the population of prey ($N_1(t)$) fluctuates between 0 and 12 over time and the population of predator ($N_2(t)$) fluctuates between 0 and 4.5. Nevertheless, the XY graph shows that the population of prey and predator still converges into the center of the circle-like figure predicting that there will be approx. 4 preys (X axis) and 3 predators (Y axis) as time goes. With the controller part, the results show that the population of prey (X axis) and predator(Y axis) eventually reaches the desired population size which is 200 and 300, respectively, see Fig. 4.

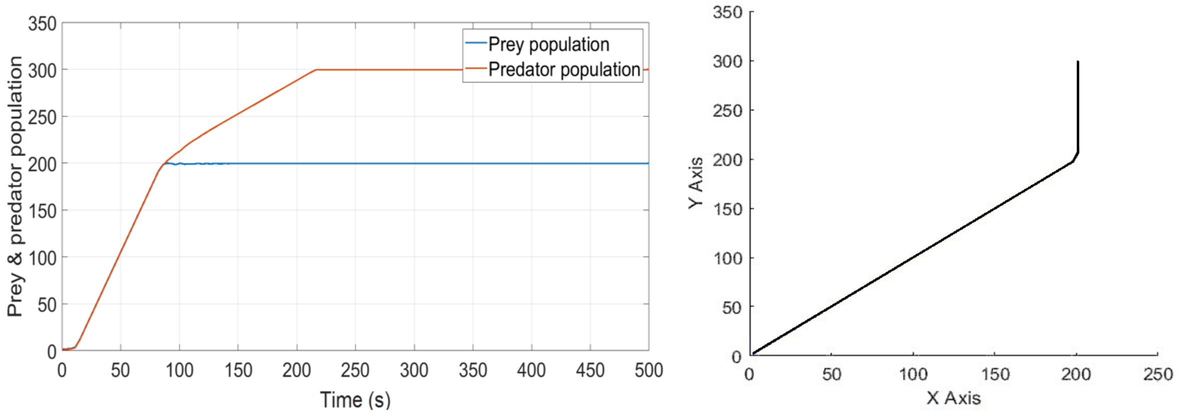

Fig. 4. Population evolution (left) of prey and predator with controllers and phase plot (right)

5 CONCLUSION

The real world is full of uncertain and chaotic phenomena. Therefore, employing simple linear prediction models with some assumptions is often not enough to enhance the accuracy of model-based predictions. Lorenz attractor from chaos theory and the forced Lotka-Volterra model are introduced as examples for systems that impede such predictions, due to extreme sensitivity w.r.t. the initial conditions. In case the initial conditions (and the model) are known with certainty, the chaotic behavior is deterministic and, hence, also predictable, with or without controller, yet more difficult. The presence of multiple equilibrium points implies that it is hard to predict real-life phenomena but thanks to simulation studies, it is still possible to make analyze convergence of the states. Furthermore, it is possible to deal with chaotic phenomena of fields other than meteorology using Lorenz attractor model and forced Lotka-Volterra equations. Promising applications can be found in economics (market shares of complementary and substitute goods), immunology of infectious diseases (populations of infected COVID-19 patients and susceptibles, etc.) or electronic devices (Chua's circuit). In execution of the course, the students' progress of learning was extensive.

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