Generalized Nash Equilibrium Seeking in Population Games under the Brown-von Neumann-Nash Dynamics

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Abstract— This paper investigates the problem of generalized Nash equilibrium (GNE) seeking in population games under the Brown-von Neumann-Nash dynamics and subject to general affine equality constraints. In particular, we consider that the payoffs perceived by the decision-making agents are provided by a so-called payoff dynamics model (PDM), and we show that an appropriate PDM effectively steers the agents to a GNE. More formally, using Lyapunov stability theory, we provide sufficient conditions to guarantee the asymptotic stability of the set of generalized Nash equilibria of the game, for the case when the game is a so-called stable game (also known as contractive game). Furthermore, we illustrate the application of the considered framework to an energy market game considering coupled equality constraints over the players decisions.

I. INTRODUCTION

Population games provide an evolutionary game theoretical framework to model and analyze the non-cooperative strategic interaction of a large society of agents [1], [2]. Under such a framework, each agent selects exactly one strategy at a time but, often in time, each agent is granted a revision opportunity to revise (and possibly update) her selected strategy. To revise their strategies, agents are equipped with a so-called revision protocol, which provides the conditional switch rates between strategies according to their associated payoffs [2, Chapter 4]. Under the assumption of a large number of agents, the aforementioned evolutionary process can be arbitrarily well described by an ordinary differential equation, here referred to as the evolutionary dynamics model (EDM), which models the strategic distribution of the society over time. Hence, whether the agents converge to a Nash equilibrium can be determined by analyzing the corresponding EDM.

Depending on the form of the revision protocol, several EDMs might emerge. In this paper, we focus on a particular (yet popular) EDM known as the Brown-von Neumann-Nash (BNN) dynamics [3]. Note that such an EDM is not only one of the six fundamental EDMs studied in the seminal work of [2], but has also been recently exploited in control applications including water distribution [4] and

real-time demand response [5], among others. Thus, the BNN dynamics are relevant both from the theoretical and practical perspectives.

In this paper, we regard the problem of generalized Nash equilibrium (GNE) seeking in population games under the BNN dynamics. In the field of game theory, the problem of GNE seeking refers to the task of reaching a Nash equilibrium subject to coupled constraints over the players' decisions. Namely, a GNE is a self-enforceable state where no player can benefit by unilaterally deviating from her selected strategy, and where certain coupled constraints over the players' decisions are satisfied. In the context of population games, on the other hand, a GNE is a (self-enforceable) strategic distribution of the society of agents where no agent can benefit by unilaterally deviating from her selected strategy, and certain (coupled) constraints on the strategic distribution of the society are satisfied. As such, the problem of GNE seeking is relevant for multi-agent control applications involving coupled constraints.

Contributions: Inspired by the ideas on dynamic payoff mechanisms [6], in this paper we consider that the payoffs perceived by the society agents are provided by a so-called payoff dynamics model (PDM), and we show that an appropriate PDM can be used for GNE seeking under the BNN dynamics regarding general affine equality constraints. As the main technical contribution, by analyzing the feedback interconnection between the PDM and the BNN dynamics, we provide sufficient conditions to guarantee the asymptotic stability of the set of generalized Nash equilibria for the class of stable population games [7] (also known as contractive games [6]). As illustration, we apply the considered framework to a multiplayer game comprising an energy market scenario subject to coupled affine equality constraints. To the best of our knowledge, this is the first paper that formally studies the problem of GNE seeking under the BNN dynamics.

Related work: Whilst the problem of GNE seeking in multiplayer games has been addressed from different perspectives [8], [9], [10], [11], such a problem has received limited attention from the context of population games and EDMs. Some exceptions include [12], [13] and [14]. For instance, the approach in [12] considers the task of GNE seeking under the so-called replicator dynamics and under affine constraints; the approach in [13] regards the problem of GNE seeking under various (mixtures of) EDMs and under box constraints; and the approach in [14] studies the problem of GNE seeking under the so-called Smith dynamics [15] and

under affine equality constraints.

In contrast with the aforementioned previous works, the approach considered in this paper has the following novelties. First, our approach guarantees the coincidence between the equilibrium states of the society of agents and the corresponding set of generalized Nash equilibria of the game (such a property does not hold under the approach in [12]). Second, our approach is able to handle general affine equality constraints and not only box constraints as the approach in [13]. Third, we provide complete sufficient conditions for asymptotic stability under the family of stable (equivalently, contractive) games, and not only for potential games as in [14]. In order to achieve this goal, we exploit the connection between GNE seeking problems and variational inequalities [16]. Moreover, different from [14], we analyze the more general case where the society of agents is comprised of multiple populations (in [14] only the single population case is studied). Thus, the framework considered in this paper generalizes the one in [14].

Structure of the paper: The rest of this paper is organized as follows. In Section II, we introduce some preliminary concepts on population games and formally state the considered EDM regarding the BNN dynamics. Then, in Section III, we describe our considered approach for GNE seeking under affine equality constraints, and we provide the corresponding theoretical analyses. Later, in Section IV, we illustrate the application of the considered framework to an energy market game including multiple players (populations). Finally, in Section V, we provide some concluding remarks and future directions of research. Due to space limitations, the complete proofs of the theoretical results can be found online in the extended version of the paper[1].

Notations: The notation $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, while $\mathbb{R}^n_{\geq 0}$ and $\mathbb{R}^n_{>0}$ denote the nonnegative and positive orthants of $\mathbb{R}^n$, respectively. The notations $\mathbb{Z}_{\geq 1}$ and $\mathbb{Z}_{\geq 2}$ refer to the sets of integers greater than or equal to 1 and 2, respectively. Given a collection of column vectors $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_N$, the notation $\mathrm{col}\,(\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_N)$ denotes the stacked column vector, and $\mathrm{diag}\,(\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_N)$ denotes the diagonal matrix whose main diagonal is given by the vector $\mathrm{col}\,(\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_N)$. In contrast, given a collection of square matrices $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_N$, the notation $\mathrm{diag}\,(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_N)$ denotes the block diagonal matrix with the matrices $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_N$ in its main diagonal. Additionally, $\mathbf{1}_M\,(\mathbf{0}_M)$ denotes the M-dimensional column vector with all its elements equal to 1 (0), and we omit the sub-index whenever the dimensions are clear from context. Similarly, $\mathbf{1}_{M \times M}$ denotes the $M \times M$ matrix with all its elements equal to 1. On the other hand, given a finite set of indices $\mathcal{I} = \{1, 2, \ldots, n\}$, the notation $|\mathcal{I}|$ corresponds to the cardinality of $\mathcal{I}$, i.e., $|\mathcal{I}| = n$. Finally, given a vector $\mathbf{z} \in \mathbb{R}^n$, the notation $\mathrm{supp}\,(\mathbf{z})$ refers to the support of $\mathbf{z}$.

II. POPULATION GAMES AND EVOLUTIONARY DYNAMICS MODEL

Consider a society of agents divided into $N \in \mathbb{Z}_{\geq 1}$ disjoint populations indexed by $\mathcal{P} = \{1, 2, \ldots, N\}$. Each population $k \in \mathcal{P}$ is comprised of a large number of decision-making agents whose available set of strategies is $\mathcal{S}^k = \{1, 2, \ldots, n^k\}$, with $n^k \in \mathbb{Z}_{\geq 2}$. Throughout, for all $k \in \mathcal{P}$, the amount of agents in population k are modeled as a (constant) mass $m^k \in \mathbb{R}_{>0}$, and the portion of agents selecting strategy $i \in \mathcal{S}^k$ at population k is denoted as $x_i^k \in \mathbb{R}_{\geq 0}$. Hence, by letting $\mathbf{x}^k = \mathrm{col}\,\left(x_1^k, x_2^k, \ldots, x_{n^k}^k\right)$ be the strategic distribution of population k, it follows that the set of possible strategic distributions of population k is

$$\Delta^k = \left\{\mathbf{x}^k \in \mathbb{R}^{n^k}_{\geq 0} : \mathbf{1}_{n^k}^\top \mathbf{x}^k = m^k\right\}, \quad \forall k \in \mathcal{P}.$$

Moreover, by letting $\mathbf{x} = \mathrm{col}\,\left(\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^N\right)$ be the strategic distribution of the entire society, it follows that the set of possible strategic distributions of the society is

$$\Delta = \{\mathbf{x} \in \mathbb{R}^n_{\geq 0} : \mathbf{1}_{n^k}^\top \mathbf{x}^k = m^k, \forall k \in \mathcal{P}\},$$

where $n = \sum_{k \in \mathcal{P}} n^k$.

Remark 1: In this paper, especially in Section III, we let $\mathbf{x} = \mathrm{col}\,\left(\mathbf{x}^1, \ldots, \mathbf{x}^N\right) \in \mathbb{R}^n$ be equivalently written as $\left(\mathbf{x}^k, \mathbf{x}^{-k}\right) \in \mathbb{R}^n$, for all $k \in \mathcal{P}$, where $\mathbf{x}^{-k} = \mathrm{col}\,\left(\mathbf{x}^1, \ldots, \mathbf{x}^{k-1}, \mathbf{x}^{k+1}, \ldots, \mathbf{x}^N\right) \in \mathbb{R}^{n-n^k}$ is the strategic distribution of all populations except k. Namely, regardless of k, it always holds that $\left(\mathbf{x}^k, \mathbf{x}^{-k}\right) = \mathrm{col}\,\left(\mathbf{x}^1, \ldots, \mathbf{x}^N\right) = \mathbf{x}$. That is, the ordering is preserved regardless of k.

To model the temporal evolution of the strategic distribution of the society, let $t \in \mathbb{R}_{\geq 0}$ denote the continuous-time index, and let $\mathbf{x}(t)$ be the value of $\mathbf{x}$ at time t. Moreover, let $p_i^k(t) \in \mathbb{R}^n$ be the payoff received by the agents selecting strategy $i \in \mathcal{S}^k$ at population $k \in \mathcal{P}$ at time t, and let $\mathbf{p}(t) = \mathrm{col}\,\left(\mathbf{p}^1(t), \mathbf{p}^2(t), \ldots, \mathbf{p}^N(t)\right) \in \mathbb{R}^n$ be the payoff vector at time t, where $\mathbf{p}^k(t) = \mathrm{col}\,\left(p_1^k(t), p_2^k(t), \ldots, p_{n^k}^k(t)\right) \in \mathbb{R}^{n^k}$, for all $k \in \mathcal{P}$. Following the framework in [2, Chapter 4], the decision-making agents are assumed to be equipped with a revision protocol to update their strategies. More precisely, a revision protocol is a map of the form $\rho_{ij}^k : \Delta^k \times \mathbb{R}^{n^k} \to \mathbb{R}_{\geq 0}$, i.e., $\rho_{ij}^k\left(\mathbf{x}^k(t), \mathbf{p}^k(t)\right) \in \mathbb{R}_{\geq 0}$, which provides the conditional switch rate from strategy $i \in \mathcal{S}^k$ to strategy $j \in \mathcal{S}^k$ at population $k \in \mathcal{P}$, given the strategic distribution $\mathbf{x}^k(t)$ and the payoff vector $\mathbf{p}^k(t)$ (throughout, we let $\rho_{ij}^k(t) \triangleq \rho_{ij}^k\left(\mathbf{x}^k(t), \mathbf{p}^k(t)\right)$ for simplicity). Depending on the form of $\rho_{ij}^k(t)$, different evolutionary dynamics might be considered [2, Chapter 5]. In this paper, we assume that

$$\rho_{ij}^k(t) = \left[p_j^k(t) - \frac{1}{m^k} \sum_{\ell \in \mathcal{S}^k} x_\ell^k(t) p_\ell^k(t)\right]_+, \quad (1)$$

for all $i, j \in \mathcal{S}^k$ and all $k \in \mathcal{P}$, and where $[\cdot]_+ \triangleq \max(\cdot, 0)$. Consequently, the evolutionary dynamics model (EDM) that

describes the evolution of $\mathbf{x}(t)$ over time is

$$\dot{x}_i^k(t) = m^k \left[\hat{p}_i^k(t)\right]_+ - x_i^k(t) \sum_{j \in \mathcal{S}^k} \left[\hat{p}_j^k(t)\right]_+, \quad (2a)$$

$$\hat{p}_i^k(t) = p_i^k(t) - \frac{1}{m^k} \sum_{\ell \in \mathcal{S}^k} x_\ell^k(t) p_\ell^k(t), \quad (2b)$$

for all $i \in \mathcal{S}^k$ and all $k \in \mathcal{P}$. For the sake of compactness, it is convenient to rewrite the EDM in (2) in matrix form as

$$\dot{\mathbf{x}}(t) = \mathbf{M}\left[\hat{\mathbf{p}}(t)\right]_+ - \mathrm{diag}\left(\mathbf{x}(t)\right)\mathbf{T}\left[\hat{\mathbf{p}}(t)\right]_+, \quad (3a)$$

$$\hat{\mathbf{p}}(t) = \mathbf{p}(t) - \mathbf{M}^{-1}\mathbf{T}\left(\mathbf{x}(t) \odot \mathbf{p}(t)\right), \quad (3b)$$

where $[\cdot]_+$ is applied element-wise; $\odot$ denotes the Hadamard product; and

$$\mathbf{M} = \mathrm{diag}\left(m^1 \mathbf{1}_{n^1}, m^2 \mathbf{1}_{n^2}, \ldots, m^N \mathbf{1}_{n^N}\right) \in \mathbb{R}^{n \times n},$$
$$\mathbf{T} = \mathrm{diag}\left(\mathbf{1}_{n^1 \times n^1}, \mathbf{1}_{n^2 \times n^2}, \ldots, \mathbf{1}_{n^N \times n^N}\right) \in \mathbb{R}^{n \times n}.$$

The EDM in (2), or equivalently in (3), is well-known in the literature as the Brown-von Neumann-Nash (BNN) dynamics [3], and their deduction from the aforementioned revision protocol $\rho_{ij}^k(t)$ in (1) can be found in [7, Section 4]. Notice that, as shown in Fig. 1, such an EDM can be considered as a continuous-time dynamical system with input $\mathbf{p}(t)$ and state vector $\mathbf{x}(t)$. Clearly, to interpret $\mathbf{x}(t)$ as the strategic distribution of the society, it must hold that $\mathbf{x}(t) \in \Delta$, for all $t \geq 0$. Such a property holds under the following assumption in conjunction with Lemma 1 (to be introduced in short).

Standing Assumption 1: $\mathbf{x}(0) \in \Delta$.

Furthermore, the BNN dynamics have some (well-known [17]) important properties that play a crucial role in our forthcoming analyses. For the sake of completeness, we formally present such properties in Lemma 1.

Lemma 1: Consider the BNN dynamics characterized by the EDM in (2). The following three properties hold.

1) $\mathbf{x}(t) \in \Delta$, for all $t \geq 0$.
2) For any $k \in \mathcal{P}$, if $\dot{\mathbf{x}}^k(t) \neq \mathbf{0}$, then $\left(\dot{\mathbf{x}}^k(t)\right)^\top \mathbf{p}^k(t) > 0$.
3) $\dot{\mathbf{x}}(t) = \mathbf{0}$ if and only if it holds that

$$x_i^k(t) > 0 \Rightarrow p_i^k(t) = \max_{j \in \mathcal{S}^k} p_j^k(t), \quad \forall i \in \mathcal{S}^k, \forall k \in \mathcal{P}.$$
(4)

In Lemma 1, the first statement verifies the forward time invariance of Δ under the EDM in (2); the second statement is usually termed as the positive correlation property and plays an important role in our forthcoming stability analysis; and the third statement is known as the Nash stationarity property and characterizes the equilibria set of the EDM in (2) in terms of the payoff vector $\mathbf{p}(t)$.

Having introduced the population games framework and the considered EDM, we now proceed to formally state the GNE seeking problem that is studied in this paper.

III. GNE Seeking under Affine Equality Constraints

In this section, we formally state the GNE seeking task under affine equality constraints and present our proposed approach to solve such a problem.

$$\mathbf{x}(0) \in \Delta$$

$$\begin{array}{c} \mathbf{p}(t) \rightarrow \boxed{\begin{array}{l} \dot{\mathbf{x}}(t) = \mathbf{M}[\hat{\mathbf{p}}(t)]_+ - \mathrm{diag}\left(\mathbf{x}(t)\right)\mathbf{T}[\hat{\mathbf{p}}(t)]_+ \\ \hat{\mathbf{p}}(t) = \mathbf{p}(t) - \mathbf{M}^{-1}\mathbf{T}\left(\mathbf{x}(t) \odot \mathbf{p}(t)\right) \end{array}} \rightarrow \mathbf{x}(t) \end{array}$$
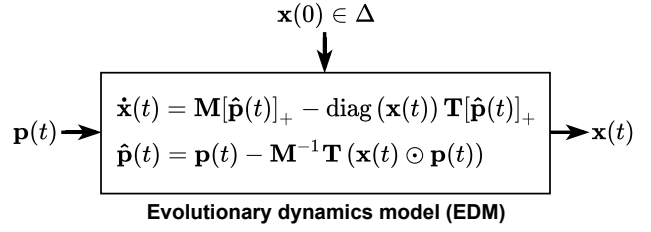
Evolutionary dynamics model (EDM)

Fig. 1. Considered EDM regarding the BNN dynamics.

Consider the scenario where each population $k \in \mathcal{P}$ seeks to solve the optimization problem (in variables $\mathbf{x}^k$) given by

$$\max_{\mathbf{x}^k \in \mathbb{R}^{n^k}} \psi^k\left(\mathbf{x}^k, \mathbf{x}^{-k}\right) \quad \text{s.t.} \ \left(\mathbf{x}^k, \mathbf{x}^{-k}\right) \in \Omega^k\left(\mathbf{x}^{-k}\right), \quad (5)$$

where $\left(\mathbf{x}^k, \mathbf{x}^{-k}\right) = \mathrm{col}\left(\mathbf{x}^1, \ldots, \mathbf{x}^N\right)$ (recall Remark 1); $\psi^k : \mathbb{R}_{\geq 0}^{n^k} \times \mathbb{R}_{\geq 0}^{n-n^k} \to \mathbb{R}$ is the objective function of population k (whose domain is assumed to be $\mathbb{R}_{\geq 0}^n \supset \Delta$); and

$$\Omega^k\left(\mathbf{x}^{-k}\right) = \left\{\mathbf{x}^k \in \Delta^k : \left(\mathbf{x}^k, \mathbf{x}^{-k}\right) \in \mathcal{X}\right\},$$
$$\mathcal{X} = \left\{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = \mathbf{b}\right\}.$$

Here, $\mathbf{A} \in \mathbb{R}^{C_= \times n}$, $\mathbf{b} \in \mathbb{R}^{C_=}$ characterize the $C_= \in \mathbb{Z}_{\geq 1}$ equality constraints to be considered. In particular, note that $\Omega^k\left(\mathbf{x}^{-k}\right)$ is the set of feasible strategic distributions of population k with respect to $\mathbf{x}^{-k}$ and $\mathcal{X}$. Hence, viewed from the society level, the populations of agents are engaged in the game $\mathcal{G} = \left(\mathcal{P}, \{\Omega^k(\cdot)\}_{k \in \mathcal{P}}, \{\psi^k(\cdot)\}_{k \in \mathcal{P}}\right)$. Throughout, we impose the following assumptions on the game $\mathcal{G}$.

Standing Assumption 2: For all $k \in \mathcal{P}$, $\psi^k\left(\mathbf{x}^k, \mathbf{x}^{-k}\right)$ is concave and continuously differentiable with respect to $\mathbf{x}^k$ for every $\mathbf{x}^{-k} \in \mathbb{R}_{\geq 0}^{n-n^k}$.

Standing Assumption 3: The set $\Delta \cap \mathcal{X} \cap \mathbb{R}_{>0}^n$ is non-empty, and the matrix $\hat{\mathbf{A}} = \left[\mathbf{A}^\top, \mathbf{A}_\Delta^\top\right]^\top \in \mathbb{R}^{(C_=+N) \times n}$ is full row rank (thus, $C_= \leq n - N$). Here, $\mathbf{A}_\Delta \in \mathbb{R}^{N \times n}$ is given by $\mathbf{A}_\Delta = \left[\mathbf{A}_\Delta^1, \mathbf{A}_\Delta^2, \ldots, \mathbf{A}_\Delta^N\right]$, where $\mathbf{A}_\Delta^k \in \mathbb{R}^{N \times n^k}$ has its k-th row equal to $\mathbf{1}_{n^k}^\top$ and all its other rows equal to $\mathbf{0}_{n^k}^\top$, for all $k \in \mathcal{P}$ (namely, note that by setting $\mathbf{m} = \mathrm{col}\left(m^1, m^2, \ldots, m^N\right)$, the set Δ can be equivalently written as $\Delta = \left\{\mathbf{x} \in \mathbb{R}_{\geq 0}^n : \mathbf{A}_\Delta \mathbf{x} = \mathbf{m}\right\}$).

Furthermore, based on the considered framework, we define the set of generalized Nash equilibria of the game $\mathcal{G}$ as follows.

Definition 1: Consider the pseudo-gradient $\mathbf{f} : \mathbb{R}_{\geq 0}^n \to \mathbb{R}^n$ defined as

$$\mathbf{f}(\mathbf{x}) = \mathrm{col}\left(\nabla_{\mathbf{x}^1}\psi^1(\mathbf{x}), \nabla_{\mathbf{x}^2}\psi^2(\mathbf{x}), \ldots, \nabla_{\mathbf{x}^N}\psi^N(\mathbf{x})\right),$$

where, for all $k \in \mathcal{P}$, $\nabla_{\mathbf{x}^k}\psi^k(\mathbf{x}) \in \mathbb{R}^{n^k}$ denotes the gradient of $\psi^k(\mathbf{x}^k, \mathbf{x}^{-k})$ with respect to $\mathbf{x}^k$ at $\mathbf{x} = \left(\mathbf{x}^k, \mathbf{x}^{-k}\right)$. The set of generalized Nash equilibria of the game $\mathcal{G}$ is

$$\mathrm{GNE}(\mathbf{f}) = \left\{\mathbf{x} \in \Delta \cap \mathcal{X} : \mathbf{x} \in \arg\max_{\mathbf{y} \in \Delta \cap \mathcal{X}} \mathbf{y}^\top \mathbf{f}(\mathbf{x})\right\}.$$

According to Definition 1, we highlight that the set $\mathrm{GNE}(\mathbf{f})$ coincides with the set of solutions of the variational

inequality $\mathrm{VI}\,(\Delta \cap \mathcal{X}, -\mathbf{f})$ (see [18]), which is given by

$$\mathrm{SOL}\,(\Delta \cap \mathcal{X}, -\mathbf{f})$$
$$= \left\{ \mathbf{x} \in \Delta \cap \mathcal{X} : (\mathbf{y} - \mathbf{x})^\top \left(-\mathbf{f}(\mathbf{x})\right) \geq 0, \, \forall \mathbf{y} \in \Delta \cap \mathcal{X} \right\}.$$

Hence, using this observation in conjunction with [16, Theorem 2.1], it is straightforward to obtain the following result.

Lemma 2: If $\mathbf{x} \in \mathrm{GNE}\,(\mathbf{f})$, then, for all $k \in \mathcal{P}$,

$$\mathbf{x}^k \in \arg\max_{\mathbf{y}^k} \psi^k \left(\mathbf{y}^k, \mathbf{x}^{-k}\right) \quad \text{s.t. } \mathbf{y}^k \in \Omega^k \left(\mathbf{x}^{-k}\right).$$

Therefore, according to Definition 1 and Lemma 2, if $\mathbf{x}(t) = \mathrm{col}\left(\mathbf{x}^1(t), \mathbf{x}^2(t), \ldots, \mathbf{x}^N(t)\right) \in \mathrm{GNE}(\mathbf{f})$, then $\mathbf{x}^k(t)$ comprises a solution of the optimization problem in (5), for all $k \in \mathcal{P}$. Consequently, the goal is to design a mechanism that steers the society of agents to the set $\mathrm{GNE}(\mathbf{f})$, i.e., to a GNE of the game $\mathcal{G}$ (c.f., Definition 1).

In Section III-A, we introduce our proposed approach for GNE seeking under the population games framework of Section II. For the forthcoming analyses, we impose the following assumption on the pseudo-gradient $\mathbf{f}(\cdot)$.

Standing Assumption 4: The pseudo-gradient $\mathbf{f}(\mathbf{x})$ is continuously differentiable with respect to $\mathbf{x}$. Moreover, $\mathbf{f}(\cdot)$ is contractive in the sense that $(\mathbf{x} - \mathbf{y})^\top \left(\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\right) \leq 0$, for all $\mathbf{x}, \mathbf{y} \in \Delta$.

Remark 2: Standing Assumption 4 implies two important properties of the set $\mathrm{GNE}(\mathbf{f})$. First, recall that, according to Definition 1, the set $\mathrm{GNE}(\mathbf{f})$ coincides with the set of solutions of the variational inequality $\mathrm{VI}\,(\Delta \cap \mathcal{X}, -\mathbf{f})$. In consequence, since $\Delta \cap \mathcal{X}$ is nonempty, convex, and compact (c.f., Standing Assumption 3), and $\mathbf{f}(\cdot)$ is continuous, it follows from [18, Corollary 2.2.5] that $\mathrm{GNE}(\mathbf{f})$ is nonempty and compact. Thus, under the considered assumptions, the existence of a GNE of the game $\mathcal{G}$ is guaranteed. Second, the contractivity condition on $\mathbf{f}(\cdot)$ implies that $\mathbf{f}(\cdot)$ corresponds to a so-called stable game [7, Section 2.2] (equivalently, contractive game [6, Definition 6]). Therefore, it follows from [7, Theorem 2.1] that $(\dot{\mathbf{x}}(t))^\top \mathbf{D_x f}\,(\mathbf{x}(t))\,\dot{\mathbf{x}}(t) \leq 0$, for all $t \geq 0$. Here, $\mathbf{D_x f}(\cdot) \in \mathbb{R}^{n \times n}$ is the Jacobian matrix of $\mathbf{f}(\mathbf{x})$ with respect to $\mathbf{x}$; and $\dot{\mathbf{x}}(t)$ is given by (3).

A. A Payoff Dynamics Model Approach

According to the framework in Section II (c.f., Fig. 1), observe that the only viable mechanism to steer the strategic distribution of the society is through the payoff signal $\mathbf{p}(t)$. Hence, in this section we propose a payoff dynamics model (PDM) to guide the society of agents to a GNE of the game $\mathcal{G}$. That is, to a strategic distribution $\mathbf{x} \in \mathrm{GNE}(\mathbf{f})$.

Following the approach in [14], in this paper we consider the PDM given by

$$\dot{\boldsymbol{\mu}}(t) = \mathbf{Ax}(t) - \mathbf{b}, \tag{6a}$$
$$\mathbf{p}(t) = \mathbf{f}\,(\mathbf{x}(t)) - \mathbf{A}^\top \boldsymbol{\mu}(t), \tag{6b}$$

where $\boldsymbol{\mu}(t) = \mathrm{col}\left(\mu_1(t), \mu_2(t), \ldots, \mu_{C_=}(t)\right) \in \mathbb{R}^{C_=}$. Consequently, observe that the EDM in (2) and the PDM in (6) are interconnected in a closed-loop configuration as shown in Fig. 2. Namely, the PDM in (6) plays the role of a feedback controller that takes as input $\mathbf{x}(t)$ and outputs $\mathbf{p}(t)$.
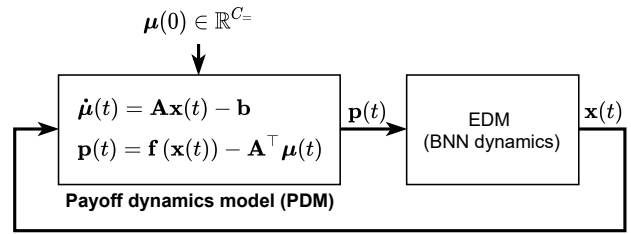

Fig. 2. Feedback connection of the considered EDM-PDM system.

Remark 3: Notice that although the PDM in (6) has the same form of the one in [14, (1)], the theoretical analyses in [14] do not contemplate the BNN dynamics in (2). Hence, in this paper we must provide the corresponding theoretical analyses. Furthermore, the results presented in [14] are limited to the scope of potential games. That is, to games where the objective functions of all populations are aligned to a global potential function. Namely, the results in [14] are limited to potential games because the potential nature of the game is employed to prove the non-emptiness and compactness of the equilibria set of the interconnected EDM-PDM system (see [14, Theorem 2 and Proposition 3]). In contrast, in this paper we overcome such a difficulty by exploiting the connection between GNE seeking problems and variational inequalities [16], [18]. This allows us to provide complete sufficient conditions for asymptotic stability for the more general family of continuously differentiable stable/contractive games (which contemplates the potential games of [14] as a particular case).

B. Analysis of the Considered EDM-PDM System

In this section, we provide the formal analysis of the feedback system comprised of the EDM in (2) and the PDM in (6) (c.f., Fig. 2). Throughout, we refer to such a system simply as the EDM-PDM system.

We start our discussion by characterizing the equilibria set of the considered EDM-PDM system.

Theorem 1: Consider (3), (6), and Definition 1. It holds that $\mathrm{col}\,(\dot{\mathbf{x}}(t), \dot{\boldsymbol{\mu}}(t)) = \mathbf{0}$, if and only if $\mathbf{x}(t) \in \mathrm{GNE}\,(\mathbf{f})$.

Namely, Theorem 1 shows the coincidence between the equilibria set of the EDM-PDM system and the set of generalized Nash equilibria of the game $\mathcal{G}$. We now state our main result on the asymptotic stability of $\mathrm{GNE}\,(\mathbf{f})$.

Theorem 2: Consider (3), (6), and Definition 1. The set $\mathrm{GNE}\,(\mathbf{f})$ is asymptotically stable under the considered EDM-PDM system.

Theorem 2 shows that the considered PDM in (6) effectively steers the strategic distribution of the society to a GNE of the game $\mathcal{G}$. Hence, the proposed framework can indeed be applied to GNE seeking tasks under affine equality constraints. We now proceed to illustrate the application of the framework to a non-cooperative energy market game.

IV. AN ENERGY MARKET GAME

Consider an energy market game[2] where $N \in \mathbb{Z}_{\geq 1}$ players (energy management systems) compete to purchase energy over a time horizon of $T \in \mathbb{Z}_{\geq 1}$ time slots. Let $\mathcal{P} = \{1, 2, \dots, N\}$ denote the set of players (populations), let $\mathcal{T} = \{1, 2, \dots, T\}$ be the aforementioned time horizon, and let $\mathcal{S}^k \subseteq \mathcal{T}$ be the set of time slots (strategies) where player $k \in \mathcal{P}$ competes. Throughout, it is assumed that $|\mathcal{S}^k| = n^k \geq 2$, for all $k \in \mathcal{P}$, and that $\mathcal{T} = \cup_{k \in \mathcal{P}} \mathcal{S}^k$ (which implies that $n \geq T$).

Based on the considered problem, we interpret $x_i^k \in \mathbb{R}_{\geq 0}$ as the energy to be purchased by player $k \in \mathcal{P}$ for time slot $i \in \mathcal{S}^k$, and we interpret $m^k \in \mathbb{R}_{>0}$ as the total energy requirement of player $k \in \mathcal{P}$. That is, the purchased energy profile of each player k must satisfy that $\sum_{i \in \mathcal{S}^k} x_i^k = m^k$. Furthermore, there are also some system-level constraints that couple the players decisions. Namely, let $\mathcal{T}_d \subset \mathcal{T}$ be a subset of time slots subject to certain energy demand constraints (to be defined in short), and let $\mathcal{P}_j \subseteq \mathcal{P}$ denote the set of players that compete in time slot j, for all $j \in \mathcal{T}_d$. The system-level energy demand constraints require that $\sum_{k \in \mathcal{P}_j} x_j^k = d_j$, for all $j \in \mathcal{T}_d$, where $d_j \in \mathbb{R}_{\geq 0}$ represents an energy demand to be satisfied at time slot $j \in \mathcal{T}_d$.

Assumption on the parameters: Following Standing Assumption 3, it is assumed that, for all $k \in \mathcal{P}$ and all $j \in \mathcal{T}_d$, the parameters m^k and d_j are such that there exists some $\hat{\mathbf{x}} \in \mathbb{R}_{>0}^n$ such that $\sum_{i \in \mathcal{S}^k} \hat{x}_i^k = m^k$ and $\sum_{k \in \mathcal{P}_j} \hat{x}_j^k = d_j$. Moreover, it is assumed that $|\mathcal{T}_d| \leq n - N$ (which implies that $\mathcal{T}_d \subset \mathcal{T}$).

Remark 4: Note that under the studied framework, one may also consider inequality constraints of the form $\sum_{k \in \mathcal{P}_j} x_j^k \leq d_j$ simply by introducing a fictitious player (population) whose energy profile represents the surplus of energy in the market. Namely, if $x_j^\ell \in \mathbb{R}_{\geq 0}$ denotes the energy to be purchased by a fictitious player $\ell \in \mathcal{P}$ for time slot $j \in \mathcal{T}$, then the equality constraint $x_j^\ell + \sum_{k \in \mathcal{P}_j} x_j^k = d_j$ implies that $\sum_{k \in \mathcal{P}_j} x_j^k \leq d_j$ (assuming by convention that $\ell \notin \mathcal{P}_j$). To ease the exposition, however, we only consider equality constraints in our numerical example.

Now, to define the optimization objective of each player $k \in \mathcal{P}$, let $\mathbf{C}^k \in \mathbb{R}^{T \times n^k}$ be a matrix such that each column of $\mathbf{C}^k$ has exactly one element equal to 1 and the rest equal to 0; each row of $\mathbf{C}^k$ has at most one element equal to 1; and the j-th element of the i-th column of $\mathbf{C}^k$ is 1 if and only if player k competes in time slot $j \in \mathcal{T}$. As an example, if $T = 10$ and player $\ell \in \mathcal{P}$ competes in time slots $\mathcal{S}^\ell = \{2, 5, 7\}$, then $\mathbf{C}^\ell = [\mathbf{e}_2, \mathbf{e}_5, \mathbf{e}_7]$, where $\mathbf{e}_i$ denotes the i-th column of the 10×10 identity matrix. Furthermore, let $\mathbf{C} = [\mathbf{C}^1, \mathbf{C}^2, \dots, \mathbf{C}^N] \in \mathbb{R}^{T \times n}$ be the concatenation of the $\mathbf{C}^k$ matrices of all players. Namely, $\mathbf{C}\mathbf{x}$ corresponds to the collective energy demand for all time slots.

Based on the previous formulations, we let $\mathbf{J} : \mathbb{R}^n \to \mathbb{R}^T$ be the pricing function for the energy market, which

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is given by $\mathbf{J}(\mathbf{x}) = \mathbf{D}\mathbf{C}\mathbf{x} + \bar{\mathbf{J}}$, where $\mathbf{D} \in \mathbb{R}_{\geq 0}^{T \times T}$ is a diagonal matrix and $\bar{\mathbf{J}} \in \mathbb{R}_{\geq 0}^T$ (namely, the price of energy increases with the total demand); and we let $Q^k : \mathbb{R}_{\geq 0}^{n^k} \to \mathbb{R}$ be the individual cost of player $k \in \mathcal{P}$, which is given by $Q^k(\mathbf{x}^k) = \sum_{i \in \mathcal{S}^k} \left((\alpha_i^k/2)(x_i^k)^2 + \beta_i^k x_i^k \right)$, where $\alpha_i^k \in \mathbb{R}_{\geq 0}$ and $\beta_i^k \in \mathbb{R}$. Consequently, each player $k \in \mathcal{P}$ seeks to solve the optimization problem (in variables $\mathbf{x}^k$) given by

$$\max_{\mathbf{x}^k \in \Omega^k(\mathbf{x}^{-k})} \psi^k(\mathbf{x}^k, \mathbf{x}^{-k}) := -(\mathbf{J}(\mathbf{x}))^\top \mathbf{C}^k \mathbf{x}^k - Q^k(\mathbf{x}^k),$$

where $\Omega^k(\mathbf{x}^{-k}) = \{\mathbf{x}^k \in \Delta^k : (\mathbf{x}^k, \mathbf{x}^{-k}) \in \mathcal{X}\}$, with $\mathcal{X} = \{\mathbf{x} \in \mathbb{R}^n : \sum_{k \in \mathcal{P}_j} x_j^k = d_j, \ \forall j \in \mathcal{T}_d\}$. Thus,

$$\psi^k(\mathbf{x}^k, \mathbf{x}^{-k}) = -\sum_{\ell \in \mathcal{P}} \mathbf{x}^{\ell\top} \mathbf{C}^{\ell\top} \mathbf{D}\mathbf{C}^k \mathbf{x}^k - \bar{\mathbf{J}}^\top \mathbf{C}^k \mathbf{x}^k$$
$$- Q^k(\mathbf{x}^k), \quad \left[\text{using } \mathbf{C}\mathbf{x} = \sum_{\ell \in \mathcal{P}} \mathbf{C}^\ell \mathbf{x}^\ell \right],$$

and, therefore,

$$\nabla_{\mathbf{x}^k} \psi^k(\mathbf{x}) = -2\mathbf{C}^{k\top} \mathbf{D}\mathbf{C}^k \mathbf{x}^k - \sum_{\ell \in \mathcal{P} \setminus \{k\}} \mathbf{C}^{k\top} \mathbf{D}\mathbf{C}^\ell \mathbf{x}^\ell$$
$$- \mathbf{C}^{k\top} \bar{\mathbf{J}} - \boldsymbol{\alpha}^k \odot \mathbf{x}^k - \boldsymbol{\beta}^k,$$

with $\boldsymbol{\alpha}^k = \text{col}(\alpha_1^k, \dots, \alpha_{n^k}^k)$ and $\boldsymbol{\beta}^k = \text{col}(\beta_1^k, \dots, \beta_{n^k}^k)$. In consequence, the pseudo-gradient $\mathbf{f}(\cdot)$ for the energy market game is $\mathbf{f}(\mathbf{x}) = -\mathbf{S}\mathbf{x} - \mathbf{C}^\top \bar{\mathbf{J}} - \boldsymbol{\alpha} \odot \mathbf{x} - \boldsymbol{\beta}$, with

$$\mathbf{S} = \text{diag}(\mathbf{C}^{1\top} \mathbf{D}\mathbf{C}^1, \dots, \mathbf{C}^{N\top} \mathbf{D}\mathbf{C}^N) + \mathbf{R}^\top \mathbf{R},$$
$$\mathbf{R} = \left[\sqrt{\mathbf{D}}\mathbf{C}^1, \sqrt{\mathbf{D}}\mathbf{C}^2, \dots, \sqrt{\mathbf{D}}\mathbf{C}^N \right]^\top,$$
$$\boldsymbol{\alpha} = \text{col}(\boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2, \dots, \boldsymbol{\alpha}^N),$$
$$\boldsymbol{\beta} = \text{col}(\boldsymbol{\beta}^1, \boldsymbol{\beta}^2, \dots, \boldsymbol{\beta}^N).$$

Here, $\mathbf{D} = \sqrt{\mathbf{D}}\sqrt{\mathbf{D}}$. Since $\mathbf{D}$ is diagonal with non-negative diagonal elements, it immediately follows that $\mathbf{S}$ is a positive semidefinite matrix, and, therefore, the pseudo-gradient $\mathbf{f}(\cdot)$ satisfies Standing Assumption 4 (c.f., Remark 2 and [7, Theorem 2.1]). Hence, for the considered energy market game, the set $\text{GNE}(\mathbf{f})$ is asymptotically stable under the studied EDM-PDM system (c.f., Theorem 2).

Without loss of generality, we let $N = 10$, $T = 20$, and, for all $k \in \mathcal{P}$, we randomly sample $\mathcal{S}^k$ such that for each time slot $i \in \mathcal{T}$ there is a 0.5 probability that $i \in \mathcal{S}^k$. Moreover, we ensure that $|\mathcal{S}^k| \geq 2$, for all $k \in \mathcal{P}$, and we set $\mathcal{S}^{10}$ as the complement of $\mathcal{S}^1$ to ensure that $\mathcal{T} = \cup_{k \in \mathcal{P}} \mathcal{S}^k$ (for reference, after sampling we have $n = 99$). On the other hand, we randomly sample the (nonzero) elements of $\mathbf{D}$, $\bar{\mathbf{J}}$, $\boldsymbol{\alpha}$, and $\boldsymbol{\beta}$, from $[0, 1]$, $[2, 4]$, $[1, 10]$, and $[0, 1]$, respectively. Finally, we randomly set $\mathcal{T}_d = \{1, 2, 4, 5, 6, 7, 8, 10, 12, 14, 15, 16, 19, 20\}$ (hence, $C_= = 14$), and, for all $k \in \mathcal{P}$ and all $j \in \mathcal{T}_d$, we randomly sample m^k and d_j from $[3, 4]$ and $[1, 1.5]$, respectively (and we numerically verify that $\Delta \cap \mathcal{X} \cap \mathbb{R}_{>0}^n \neq \emptyset$).

In Fig. 3 we present the trajectory of the selected performance index, i.e., $\|\mathbf{x}(t) - \mathbf{x}^*\|_2 / \|\mathbf{x}(0) - \mathbf{x}^*\|_2$, where $\mathbf{x}^* \in \text{GNE}(\mathbf{f})$. Moreover, in Fig. 4 we depict the trajectories

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[2]An energy market game can be regarded as a form of Cournot competition, and various control engineering applications can be considered under such an abstraction [9], [19], [20].
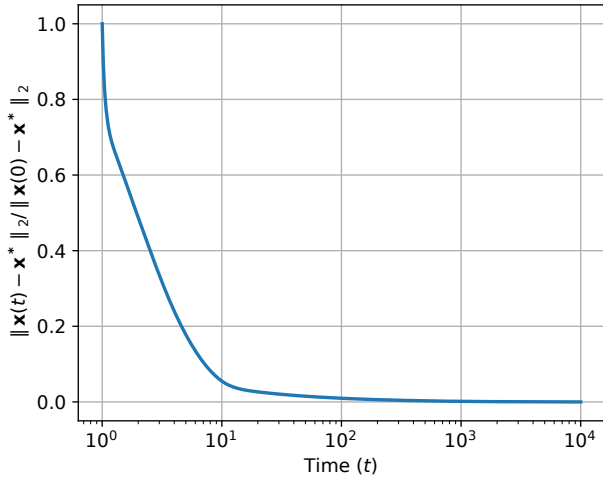
Fig. 3. Trajectory of the selected performance index for the considered energy market game under the EDM-PDM system given by (2) and (6). Here, without loss of generality, $x_i^k(0) = m^k/n^k$, for all $i \in \mathcal{S}^k$ and all $k \in \mathcal{P}$, and $\boldsymbol{\mu}(0) = \mathbf{0}$.
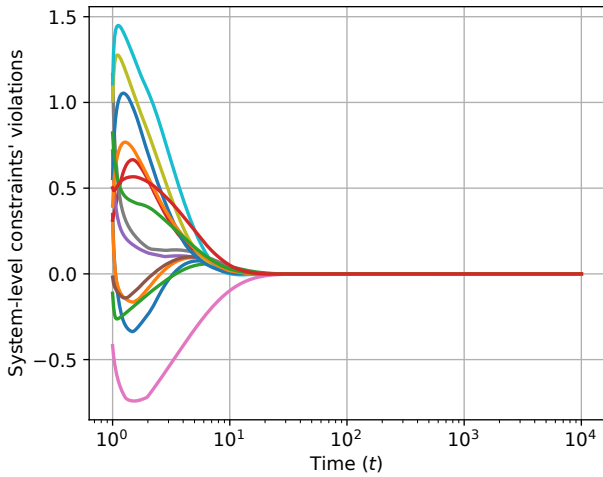

Fig. 4. Trajectories of the (coupled) system-level constraints' violations. Here, a value of 0 implies that the corresponding constraint is satisfied. The key observation in this plot is the fact that all the constraint's violations eventually converge to 0.

of $\sum_{k \in \mathcal{P}_j} x_j^k(t) - d_j$, for all $j \in \mathcal{T}_d$. As shown in Fig. 3, it is verified that the considered EDM-PDM system reaches a GNE of the underlying energy market game. Furthermore, as shown in Fig. 4, it is verified that the reached GNE indeed satisfies the coupled constraints over the players decisions.

V. CONCLUDING REMARKS

This paper has studied the problem of generalized Nash equilibrium seeking under affine equality constraints in (multi) population games under the Brown-von Neumann-Nash dynamics. We have provided sufficient conditions to guarantee the asymptotic stability of the corresponding equilibria set, and we have illustrated the application of the framework to an energy market game with coupled constraints over the players' decisions.

Future work should focus on the extension of the framework to other families of evolutionary dynamics, e.g., the replicator dynamics, as well as on the characterization of the convergence rate of the resulting dynamical systems.

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