UNDERSTANDING OF DIFFERENTIAL EQUATIONS IN A HIGHLY HETEROGENEOUS STUDENT GROUP

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ABSTRACT

Differential equations (DEs) are an important mathematical concept for a wide variety of disciplines in engineering. Hence, students need to develop a good understanding of the basic concepts of DEs. However, they encounter many difficulties when studying DEs and often exclusively focus on procedural knowledge. This study therefore investigates the difficulties concerning DEs encountered by engineering students at a university of applied sciences in Germany.

In contrast to previous studies on this topic our investigation differs in two aspects. First, the group of first-year engineering students at this university is highly heterogeneous; e.g. while some begin their studies immediately after secondary school, others have completed vocational training and joined the workforce for some time. Second, the engineering study programs considered here provide for only two semesters of mathematics and do not include specific courses on (ordinary) differential equations. The subject of DEs is dealt with in a three- to four-week period at the end of the second semester.

We conducted think-aloud interviews lasting about 45 min with 9 students after completion of the relevant course. We found that the main difficulties students

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experience are connected to: substantial lack of prior knowledge, attempting (sometimes unsuccessfully) to apply memorized procedures, and a failure to understand both the difference between a DE and a function and what a solution to a DE is.

The results shall be used to design three to four collaborative-group worksheets that build on students' ways of thinking and aim at improving students' conceptual understanding.

1 INTRODUCTION

1.1 Importance of Differential Equations

Differential equations (DEs) are an important mathematical concept for a wide variety of disciplines in engineering (e.g. mechanical, electrical, chemical and green engineering, applied chemistry, and applied computer science at our university). Students need to develop a deep understanding of the basic concepts of differential equations so that they can assign meaning to DEs from their field later in their study programs and apply them correctly. Literature and our experiences show, that students encounter many difficulties when studying DEs [1-3] and often exclusively focus on procedural knowledge.

1.2 Distinction of Common Difficulties with DEs

The difficulties with DEs can roughly be divided into those with the equations themselves and those with respect to the solutions of DEs [2]. Distinctions of difficulties in literature are first according to different comprehension levels, e.g. action, process or object level understanding in APOS theory [5], and second according to different representations like graphical, verbal, numerical, symbolic or physical [6]. We refrain from a detailed listing of common difficulties and misunderstandings connected to DEs here, but include them in the results section.

1.3 Students' Background

Previous studies on students' understanding of DEs focused mainly on students enrolled at universities in mathematics, physics, teaching mathematics and teaching physics programmes. The students attended several mathematics modules before entering a (full) course on ordinary differential equations. In some cases, students had to fulfil certain requirements (e.g., minimum grade in other math courses, passed algebraic tests) to be considered within the studies.

In contrast, (1) the composition of the students at our university of applied sciences (Hochschule) is very heterogeneous and (2) hardly any prior knowledge can be assumed. Nevertheless, a basic understanding of differential equations should be imparted to all students.

(1) The group of first-year engineering students at our university is highly heterogeneous since more people are allowed to study here compared to regular universities. Not only "Abitur" (high school degree) but also work experience, a good

vocational degree, and others qualify for a study programme. Some students have (very) little knowledge of physics and mathematics. Difficulties to work / learn self-dependent, to organize themselves and to study continuously are also common. In addition to the large differences in prior knowledge and abilities among students, there are large differences in personal qualifications as well that can affect the studying itself: children, difficulties with the German language, financial problems and / or work which is not related to their studies.

(2) The engineering study programs considered here provide for only two semesters of mathematics (5 ECTS each) and do not include specific courses on (ordinary) differential equations. The subject of DEs is dealt with in a three- to four-week period at the end of the second semester where three weeks is the standard and four weeks is the exception.

1.4 Teaching Differential Equations

There are various approaches to increasing students' deep understanding. All have in common that activating methods are used. The methods range from guided task sheets [4] to computer-based, numerical approaches [7] to project work on topics chosen by students themselves [8]. Various approaches attempt to achieve at least a basic conceptual understanding and usually focus on specific representations, especially graphical and / or numerical understanding.

With this study, we want to find out what essential difficulties prevent (our) students from understanding what makes differential equations special, how to "read" or get an overview of them, and what a solution to a differential equation is. The step-by-step analysis of the students’ thinking process is necessary to design collaborative-group worksheets (like the McDermott Tutorials in Introductory Physics [9]). The worksheets will be designed to build on students’ ways of thinking allowing students to arrive at a (deeper) conceptual understanding of DEs.

2 METHODOLOGY

2.1 Semistructured Interviews

We conducted think-aloud interviews lasting about 45 min with 9 students after completion of the relevant course. The purpose of the interview was communicated to the interviewee, and it was thereby made clear that the research objects were their thinking processes and their difficulties, and not their performance in terms of a correct response to the questions asked. Students were asked to solve two tasks and to communicate their thoughts. The interviewer did not intervene to tell students whether a thought or an answer was correct or wrong. All she did was asking questions about the specific reasoning, how students gained certain answers or what they would do next, e.g. “How did you determine that graph xy is not a solution?”, “How did you realize that …”, “How would you proceed with that?”, or “Restate the task with your own words.”

The interviews were recorded and transcribed. To find out students' difficulties, the correctness of answers was assessed as well as their reasoning and the relevance and strengths of the argument's given.

2.2 Tasks of the Interviews

The interviews consisted of two tasks:

1.) Which graph could be a solution to the differential equation $y'(x) = -a \cdot x \cdot y(x)$? (Fig. 1)

2.) The number of fish[2] in a lake increases the faster the more fish there are in that lake. A certain number of fish is taken from the lake each year to sell.

Can this problem be described by a differential equation?

Set up the differential equation for the fish problem.

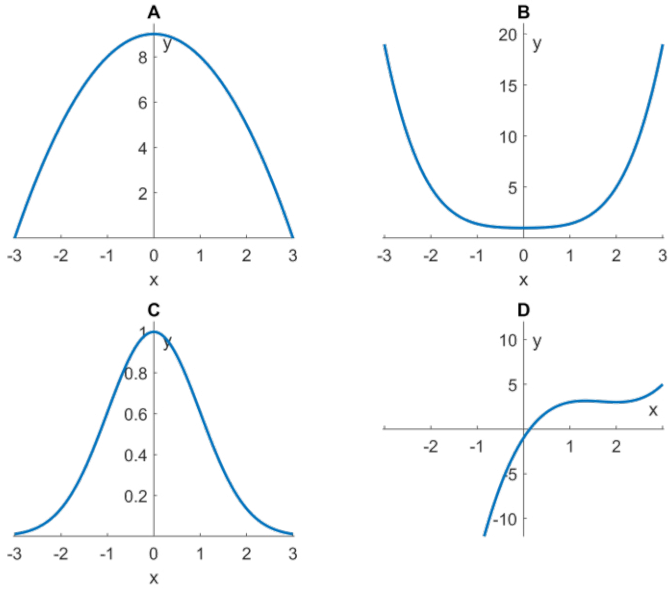

Fig. 1. Which graph could be a solution to the differential equation $y'(x) = -a \cdot x \cdot y(x)$? [10]

There are multiple ways to answer the first task. Some of them are:

First, the differential equation could be solved ($y(x) = a \cdot e^{-\frac{1}{2}x^2}$) by separation of variables. The graph that fits this function can only be C.
Second, y' is the slope, so "values" (the signs, the magnitudes and its courses) for x and y could be read from the graphs and put into the formula.
Or third, the algebraic functions for A to D could be set up and then be inserted into the DE. If a true statement is obtained, the graph is a solution.

The fish problem in the second task can be modelled by a DE since the growth rate of fish depends on the number of fish in the lake. The DE is $y'(x) = ay(x) - c$ with the constants a and c. y describes the number of fish, x the time passed and $y'(x)$ the growth rate of fish.

3 RESULTS

3.1 Categorization of Student Difficulties

We found that there are several substantial difficulties that students experience when dealing with the topic of differential equations. Those difficulties are connected to

i. substantial lack of basic knowledge from algebra and calculus,

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[2] Initially, "fish population" was used instead of "number of fish", which caused language difficulties for some students.

ii. applying memorized procedures,
iii. understanding the difference between a differential equation (DE) and a function, and
iv. the understanding what a solution to a DE is.

Difficulties connected to other areas also exist e.g. with regard to homogeneous / inhomogeneous DEs, initial conditions or obtaining solutions of DEs.
We think, that many difficulties are associated with key concepts. Key concepts are concepts that have to be understood in order to be able to proceed within the field. To overcome difficulties related to DEs a good understanding of those key concepts, especially from the categories iii. and iv., would be necessary. Once those key concepts are understood and internalized, difficulties might not appear at all or can easily be overcome.

3.2 Subdivision of student difficulties

Students can understand key concepts as well as face difficulties on different levels and within different representations. We want to call those levels basic, advanced and comprehensive. They are inspired by, but not identical to the action, process and object levels in APOS theory [5]. Typical representations are graphical, verbal, numerical, symbolic or physical [6].

For this paper we use the levels in the following way:

Basic level understanding of a concept in a certain representation is more or less procedural knowledge that does not necessarily require a deeper understanding. In extreme cases mathematical procedures can be applied without any mathematical understanding at all. Students can perform the action e.g. on a sheet of paper (external) but need not to be able to perform it purely mentally. Students do not need to know why the procedures are applied nor need to make general statements nor switch representations.

Advanced level understanding of a concept in contrast is more comprehensive. Students can make generalisations within the same representation, work with different inputs (e.g. different numbers, variables, graphs, or physical contexts) and manipulate the concept. They recognize when and how the concept can be (properly) applied and make first connections to different representations. Students can perform the actions internally. For a particular concept and representation understanding can be complete at the advanced level.

Comprehensive level understanding means that students can deduce more comprehensive, general statements. They do this by encapsulating multiple (key) concepts and / or by combining different representations. Students switch appropriately between different representations, as needed. Comprehensive level understanding is very abstract and examples are not needed anymore for explanations.

In the next sections, the levels basic, advanced and comprehensive were assigned to the difficulties. The assignment is not yet final. It occurs that a difficulty is assigned to two levels. For example, an assignment to basic and advanced level may occur when certain aspects of these difficulties could also be worked through procedurally but in order to really use the key concepts behind that difficulty an advanced level understanding is necessary. If difficulties are assigned to the advanced as well as the comprehensive level, the actual classification of this difficulty depends on whether it is related to only one representation or to the interaction of several ones.

3.3 Basic knowledge from algebra and calculus

i. Difficulty rearranging equations (*basic level*)
Some students make serious mistakes when rearranging equations. For instance, one student rearranged $y'(x) = -xy(x)$ to $y'(x) - y(x) = -x$ and another one did not know how to transform $-\ln(10)$ $(= \ln\frac{1}{10})$.

ii. Difficulty remembering and applying differentiation rules (*basic level*)
Some students struggle to remember differentiation rules. Some could not remember how to differentiate x^n, one student wrote $(a^{-x})' = -xa^{-x-1}$ (correct would be $a^x \cdot \ln(a)$ for positive a), and one student did not remember how to differentiate e^{-ax}. When students realise that they cannot remember the rules, they feel very uncertain, get stuck with the whole problem and cannot concentrate on continuing with the problem anymore.

iii. Difficulties in dealing with diagrams (*basic level, advanced level*)
Some students do not know what exactly is shown in a diagram or how to get the relevant data like x, y, or y' from it. For instance, one student said in task 1 where graphs A to D were given (Fig. 1) "… because I don't know, A to D, are they $y(x)$ or $y'(x)$?" and another one said "… but I don't know, what $y(x)$ looks like; it could have any sign." A third student put $x = 2$ into the equation from task 1, got $y'(2) = -2 \cdot y(2)$ but then did not know how to get information about $y(2)$ and $y'(2)$ from the graphs.

iv. Difficulties with infinitesimal quantities (*advanced level, comprehensive level*)
The difficulties relate to the understanding of infinitesimals and can occur in different representations (e.g. symbolic, graphical or verbal). Some students do not seem to know the difference between a variable y and an infinitesimal small change of that variable dy, as well as the difference between $\frac{dy}{dt}$ and $\frac{y}{t}$. dy is sometimes considered to be the same as y and sometimes to be $d \cdot y$. E.g. one student put the wrong equation $y'(x) = \frac{y(x)}{dx} + c$ (correct is $y'(x) = \frac{dy}{dx}$) into $y'(x) = -a \cdot x \cdot y(x)$, (omitted c) and obtained $\frac{y(x)}{dx \cdot y(x)} = -x$. He rearranged this to $\frac{1}{dx} = -x$ and to $\frac{1}{d} = -x^2$. Another one said for the fish problem: "The number of fish is growing per time, so it is $\frac{y}{t}$." Then he compared it to the velocity of a car which he said would be $v = \frac{s}{t}$ (correct is $v = \frac{ds}{dt}$).

v. Difficulties in dealing with mathematical problems
Many students think, a given problem must be solvable in an easy and quick way

with one correct solution. They cannot imagine that there are multiple very different correct solutions or that a problem requires quite some effort. If they do not find an answer in one way, most cannot stop at that point and try to find a different way of solving the initial problem. It seems, that many students do not have a basic confidence in their mathematical abilities.

3.4 What is a differential equation?

vi. Difficulty in characterising a differential equation correctly (*basic level*)
Many students forget very quickly after the mathematics course how to characterise a DE that is given in an algebraic form, e.g. linear – nonlinear, homogeneous – inhomogeneous (and correctly naming the disturbing function), and constant – non-constant coefficients.

vii. Difficulties with text-based problems (*basic level, advanced level*)
Many students do not recognize whether a given text-based problem describes a DE or not or think it is a DE for the wrong reasons. They cannot correctly assign variables (e.g. independent variable, dependent variable, derivative) to physical quantities nor assign the correct meaning to terms and variables in the DE.
E.g. for the fish problem (task 2) at least one student thought it was a linear problem because the growth in the number of fish was not recognized as the first derivative of the number of fish. ("[When I say the number of fish doubles every year] then I would already have [writes y=2x+50].") One student thought he would need specific values and only then it would be a DE (which would correspond to the initial value problem).

viii. Difficulty seeing how functions and DEs are related to each other (*comprehensive level*)
DEs are a (for students new) way to describe functions compared to the explicit form they are familiar with from school or preceding mathematics courses. For some students, however, DEs and functions are fundamentally different. If something can be described as or be graphically displayed as a function known to them (e.g. a parabola or straight line in a diagram) it is no longer associated with a DE. For instance, they think that there is "no need" for a DE if something can be described by a "regular" function. Students said "With a parabola, the slope is not dependent on the function itself, so a normal parabola does not form a DE.", or "The slope is always the same for a straight line. I would not need to work with a differential equation."

ix. Difficulties regarding the dependencies of variables (*comprehensive level*)
Many students, intentionally or unintentionally, consider independent or dependent variables as (positive) constants.
E.g. in the DE $y'(x) = -a \cdot x \cdot y(x)$ either x or $y(x)$ are considered to be constant (one student said "y must be negative so that the sign on the left is correct." and thereby kept x and y' as positive constants). Students are not clear about the serious implications of the dependence of y' on y. Sometimes the function is associated with symmetry solely because y' is dependent on y (which is a wrong

argument).
Some students also do not understand, that the dependence on the independent variable, e.g. on time, is implemented into a DE already by $y(x)$ and need not appear explicitly. Therefore, students often explicitly took the time into their DEs (not realizing what this actually would mean).

x. Difficulty setting up the differential equation for a certain problem and checking its validity (*comprehensive level*)
Students do not explicitly think about variables and assign them consciously before setting up a differential equation. They set up wrong equations that, in the fish problem, e.g. explicitly include the time ("Of course I need the time in my equation because I want to know, how many fish there are at a certain time.").
Students e.g. wrote $y'(x) = e^{y(x)}$, $y' = y^x$ or $y' = e^x$ since they "knew" the number of fish is growing exponentially with time. One student thought, a DE must be differentiated in some way.

3.5 What is a solution to a differential equation?

xi. Difficulty using memorized procedures to solve a DE (*basic level*)
Some students cannot remember the rules to solve a DE. This is because they did not understand them but only learned the procedures by heart. Soon after the exam this knowledge is forgotten. E.g. no student remembered how to solve the DE in task 1.

xii. Difficulty verifying that a given (or found) algebraic function is a solution to a DE. (*basic level, advanced level*)
Some students do not know the goal of putting $y(x)$ into the DE (results in true or false statement). One student got the (correct) result $-2x = x^3 - 9x$ in task 1 when putting the squared function $y(x) = -x^2 + 9$ into the DE and said "Now I have a cubic function. That's a new context and I would need to interpret that." He did not realize that the equation is a false statement. Another one said at that point "I can rearrange this any way I like. So how would I know if this is a solution?"

xiii. Difficulty understanding that functions are the solutions to differential equations. (*advanced level*)
Many students make a fundamental distinction between a solution of a DE and a function as they know them. Some students cannot imagine that the solution of a DE is represented in a diagram and looks like a "normal" function. Some students do not know at all, what a solution to a DE is – neither in an algebraical, graphical nor physical representation. The difficulties regarding this concept overlap with those in item viii.

Note that we concentrated on the most basic concepts and most fundamental difficulties that could realistically be covered in an introductory mathematics course in the second semester. There are many more difficulties connected to solutions of differential equations and difficulties in all representations. Examples are the concept that an algebraic function can be solution to multiple differential equations, that one

DE will have multiple solutions (and therefore multiple graphs in a diagram), and the concepts behind different characterisations of DEs.

4 OUTLOOK

Through interviews we revealed major difficulties students face when dealing with differential equations and solutions of differential equations. Those insights shall be used to design three to four collaborative-group worksheets (tutorials) that build on students' ways of thinking. The tutorials will aim at improving students' conceptual understanding. The learning goals for the tutorials are that students understand

- what the difference between a function and a differential equation is,

- what the terms and variables in a differential equation mean and do (mathematically),

- how to set up a differential equation from a given problem, and

- what a solution to a differential equation is and how it differs from the DE.

The tutorials shall use as little recourse to (memorized) prior basic knowledge from calculus and algebra, as possible, but students shall learn this necessary knowledge along the way. Classroom observations as students work through the tutorials will indicate whether difficulties are addressed and lead to fruitful discussions. In a future publication we hope to go more into the details about creating and presenting the tutorials.

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