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On the numerical approximation of a problem involving a mixture of a MGT viscous material and an elastic solid

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Abstract In this work, we analyze, from the numerical point of view, a problem including a mixture made of a MGT viscoelastic solid and an elastic solid. The corresponding variational problem is a linear system composed of two coupled hyperbolic equations written in terms of the acceleration of the first constituent and the velocity of the second one. Then, fully discrete approximations are introduced by using the finite element method and the implicit Euler scheme. A discrete stability property and a priori error estimates are proved. Finally, some one-dimensional numerical simulations are shown to demonstrate the accuracy of the proposed approximations and the behaviour of the solution.

Keywords Mixture · Moore-Gibson-Thompson material · Viscoelasticity · Finite elements · Error estimates · Numerical simulations

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1 Introduction

In the sixties of the previous century a big deal was developed to model generalized fluids and solids. A relevant example corresponds to mixtures of interacting materials. It was trying to consider new materials in such a way that they satisfied certain properties. While we can remember [4, 10, 7–9, 11, 17, 20–23, 27, 31] as pioneering works, we can also say that the amount of contributions in this theory is immense. One of the main applications of these models involving mixtures is the theory of the well-known composites, with an increasing use in the automotive industry.

The so-called Moore-Gibson-Thompson (MGT) equation has deserved a big interest recently. This equation came from the study of the mechanics of fluids; however, it has also been proposed as a heat equation in a way similar to the proposition of the Maxwell-Cattaneo heat equation. Therefore, we can also consider a thermoelastic theory based on the Moore-Gibson-Thompson heat equation. In fact, many contributions have been published on this theory over the last two years (see [1–3, 5, 6, 15, 16, 18, 24, 25, 30, 33] among others). At the same time, it has also been shown how to obtain a viscous effect for different materials [14, 15, 29] following the Moore-Gibson-Thompson proposition.

It is well known that the thermal waves obtained by the use of the Fourier law (also for the type III Green-Naghdi heat conduction) propagate instantaneously. As a consequence, the causality principle is violated. For this reason, new constitutive equations for the heat flux vector have been proposed as the Maxwell-Cattaneo law (see [26]), the type II Green-Naghdi theory or some types of the phase-lag theories. They save the drawback and propose heat conduction theories free of the instantaneous propagation effect of the thermal waves. Certainly, this aspect has been reflected very often in the literature. However, to our knowledge there is not a similar criticism with respect to the instantaneous propagation of the mechanical waves. In fact, the mechanical waves for the Kelvin-Voigt (KV) viscoelasticity are also affected by the instantaneous propagation effect (see [32, p. 39]), and therefore they also violate the causality principle. It would be suitable to propose alternative laws for the viscoelasticity saving this drawback as we know that it has been done for the heat conduction. A natural possibility is to consider the viscoelastic theory based on the MGT equation which eliminates this phenomenon [28]. In short, we can say that the MGT-viscoelasticity is more realistic than the KV viscoelasticity in order to describe viscoelastic effects by using partial differential equations (in a similar way as the alternative theories for the heat conduction). As mixtures of viscoelastic and elastic materials have been considered (very often) in the literature, we think that it is needed to propose such theory in the case that the viscous material is proposed from the MGT theory. Nevertheless, we recognize that this theory is (until now) theoretical and academical and it has not been applied yet in the context of the industrial applications. In this sense, it is suitable to recall that the analytical study for this theory has been developed recently [19]. In fact, the authors showed there how to model a mixture of a MGT viscoelastic solid and an elastic solid. They gave the suitable framework to obtain the existence and uniqueness of solutions as well as the exponential decay of the solutions.

In this work, we also consider this kind of materials but we now analyze the problem from a numerical point of view, proving a discrete stability property, providing an a priori error analysis and performing some numerical simulations. The plan of this work is the following. In the next section we describe the basic equations describing these solids, as well as the required assumptions, and we recall an existence and uniqueness result recently proved in [19]. In Section 3 we introduce a fully discrete algorithm by using the finite element method and the implicit Euler scheme, we prove a discrete stability property and we obtain a main a priori error estimates result by using a discrete version of Gronwall's inequality and some well-known results on the approximation by the finite elements. Finally, in Section 4 we present some one-dimensional numerical simulations to demonstrate the accuracy of the approximations and the behaviour of the solution.

2 The variational formulation of the problem

In this section, we provide the variational formulation for a mixture of a Moore-Gibson-Thompson viscous solid with an elastic solid, from the theory of mixture of materials with memory, and we recall an existence and uniqueness result (see [19] for further details).

We denote by B a multi-dimensional region in $\mathbb{R}^d$, $d = 1, 2, 3$, such that its boundary ∂B is smooth enough to apply the divergence theorem for $d = 2, 3$. Moreover, let $[0, T]$, with $T > 0$, be the time interval of interest.

Let $\boldsymbol{u} = (u_i)_{i=1}^d$ and $\boldsymbol{w} = (w_i)_{i=1}^d$ be the displacement of the first and second constituents, respectively.

Our system becomes (see [19]):

$$\begin{aligned}\rho_1(\tau \dddot{u}_i + \ddot{u}_i) &= \Big(A^*_{ijrs} u_{r,s} + A_{ijrs}\dot{u}_{r,s} + B^*_{ijrs} w_{r,s}\Big)_{,j} \\ &\quad - a^*_{ij}(u_j + \tau \dot{u}_j - w_j), \qquad (1)\\ \rho_2 \ddot{w}_i &= \Big(B^*_{rsij}(u_{r,s} + \tau \dot{u}_{r,s}) + C^*_{ijrs} w_{r,s}\Big)_{,j} \\ &\quad + a^*_{ij}(u_j + \tau \dot{u}_j - w_j). \qquad (2)\end{aligned}$$

Here, ρ_1 and ρ_2 are the mass density of each component, A^*_{ijrs}, A_{ijrs}, B^*_{ijrs}, C^*_{ijrs} and a^*_{ij} are constitutive tensors whose properties will be described later, and τ is a positive constant.

To define a problem based on this system we need to impose the boundary conditions:

$$u_i(\boldsymbol{x}, t) = w_i(\boldsymbol{x}, t) = 0 \quad \forall \boldsymbol{x} \in \partial B, \quad i = 1, \dots, d, \qquad (3)$$

and the initial conditions, for all $\boldsymbol{x} \in B$ and $i = 1, \dots, d$,

$$\begin{aligned}&u_i(\boldsymbol{x}, 0) = u_i^0(\boldsymbol{x}), \quad \dot{u}_i(\boldsymbol{x}, 0) = v_i^0(\boldsymbol{x}), \quad \ddot{u}_i(\boldsymbol{x}, 0) = \xi_i^0(\boldsymbol{x}), \\ &w_i(\boldsymbol{x}, 0) = w_i^0(\boldsymbol{x}), \quad \dot{w}_i(\boldsymbol{x}, 0) = e_i^0(\boldsymbol{x}),\end{aligned} \qquad (4)$$

where u_i^0, v_i^0, ξ_i^0, w_i^0 and e_i^0 are prescribed functions.

In this paper we will assume that

(i) $\rho_1(\boldsymbol{x})$ and $\rho_2(\boldsymbol{x})$ are strictly positive.

(ii) There exists a positive constant C such that

$$A^*_{ijrs}\xi_{ij}\xi_{rs} + 2B^*_{ijrs}\xi_{ij}\eta_{rs} + C^*_{ijrs}\eta_{ij}\eta_{rs} \geq C(\xi_{ij}\xi_{ij} + \eta_{ij}\eta_{ij})$$

for every tensors ξ_{ij} and η_{ij}.

(iii) There exists a positive constant $\tilde{C}$ such that

$$a^*_{ij}\xi_i\xi_j \geq \tilde{C}\xi_i\xi_j$$

for every vector ξ_i.

(iv) There exists a constant greater than one C^{**} such that

$$A_{ijrs}\xi_{ij}\xi_{rs} \geq C^{**}\tau A^*_{ijrs}\xi_{ij}\xi_{rs}$$

for every tensor ξ_{ij}.

In order to provide the numerical approximation of problem (1)-(4) in the next section, we will obtain its variational formulation. Thus, let $H = [L^2(B)]^d$ and denote by $(\cdot,\cdot)_H$ the scalar product in this space, with corresponding norm $\|\cdot\|_H$. Moreover, let us define the variational space $V = [H_0^1(B)]^d$ and, in order to simplify the notation, let the following linear operators be given:

$$\begin{array}{ll}
A^*(\boldsymbol{u},\boldsymbol{v}) = (A^*_{ijrs}u_{r,s}, v_{i,j})_{L^2(B)} & \forall \boldsymbol{u} = (u_i)_{i=1}^d, \boldsymbol{v} = (v_i)_{i=1}^d \in V,\\
A(\boldsymbol{u},\boldsymbol{v}) = (A_{ijrs}u_{r,s}, v_{i,j})_{L^2(B)} & \forall \boldsymbol{u} = (u_i)_{i=1}^d, \boldsymbol{v} = (v_i)_{i=1}^d \in V,\\
B^*(\boldsymbol{u},\boldsymbol{v}) = (B^*_{ijrs}u_{r,s}, v_{i,j})_{L^2(B)} & \forall \boldsymbol{u} = (u_i)_{i=1}^d, \boldsymbol{v} = (v_i)_{i=1}^d \in V,\\
C^*(\boldsymbol{u},\boldsymbol{v}) = (C_{ijrs}u_{r,s}, v_{i,j})_{L^2(B)} & \forall \boldsymbol{u} = (u_i)_{i=1}^d, \boldsymbol{v} = (v_i)_{i=1}^d \in V,\\
a^*(\boldsymbol{u},\boldsymbol{v}) = (a^*_{ij}u_j, v_i)_{L^2(B)} & \forall \boldsymbol{u} = (u_i)_{i=1}^d, \boldsymbol{v} = (v_i)_{i=1}^d \in V.
\end{array}$$

Then, applying Green's formula to equations (1)-(2) and using boundary conditions (3) we have the following weak problem.

Problem VP. *Find the acceleration of the first constituent* $\boldsymbol{\xi} : [0,T] \to V$, *and the velocity of the second constituent* $\boldsymbol{e} : [0,T] \to V$ *such that* $\boldsymbol{\xi}(0) = \boldsymbol{\xi}^0 = (\xi_i^0)_{i=1}^d$, $\boldsymbol{e}(0) = \boldsymbol{e}^0 = (e_i^0)_{i=1}^d$, *and, for a.e.* $t \in (0,T)$,

$$\begin{aligned}
&\rho_1(\tau\dot{\boldsymbol{\xi}}(t) + \boldsymbol{\xi}(t), \boldsymbol{r})_H + A^*(\boldsymbol{u}(t),\boldsymbol{r}) + A(\boldsymbol{v}(t),\boldsymbol{r}) + B^*(\boldsymbol{w}(t),\boldsymbol{r})\\
&\qquad + a^*(\boldsymbol{u}(t) + \tau\boldsymbol{v}(t) - \boldsymbol{w}(t), \boldsymbol{r}) = 0 \quad \forall \boldsymbol{r} \in V, \qquad (5)\\
&\rho_2(\dot{\boldsymbol{e}}(t), \boldsymbol{z})_H + C^*(\boldsymbol{w}(t),\boldsymbol{z}) + B^*(\boldsymbol{u}(t) + \tau\dot{\boldsymbol{v}}(t), \boldsymbol{z})\\
&\qquad - a^*(\boldsymbol{u}(t) + \tau\boldsymbol{v}(t) - \boldsymbol{w}(t), \boldsymbol{z}) = 0 \quad \forall \boldsymbol{z} \in V, \qquad (6)
\end{aligned}$$

where we recall that the displacement of the first constituent $\boldsymbol{u}$, the velocity of the first constituent $\boldsymbol{v}$ and the displacement of the second constituent $\boldsymbol{w}$ are then recovered from the relations:

$$\boldsymbol{v}(t) = \int_0^t \boldsymbol{\xi}(s)\,ds + \boldsymbol{v}^0, \quad \boldsymbol{u}(t) = \int_0^t \boldsymbol{v}(s)\,ds + \boldsymbol{u}^0, \qquad (7)$$

$$\boldsymbol{w}(t) = \int_0^t \boldsymbol{e}(s)\,ds + \boldsymbol{w}^0. \qquad (8)$$

Here, the initial values are given by $\boldsymbol{u}^0 = (u_i^0)_{i=1}^d$, $\boldsymbol{v}^0 = (v_i^0)_{i=1}^d$ and $\boldsymbol{w}^0 = (w_i^0)_{i=1}^d$.

The following result which states the existence of a unique solution to Problem VP has been recently proved in [19].

Theorem 1 *Let assumptions (i)-(iv) hold. If the initial conditions have the following regularity:*

$$\begin{aligned}
&\boldsymbol{u}^0 \in [H_0^2(B)]^d, \quad \boldsymbol{v}^0 \in [H^1(B)]^d, \quad \boldsymbol{\xi}^0 \in [L^2(B)]^d,\\
&\boldsymbol{w}^0 \in [H_0^1(B)]^d, \quad \boldsymbol{e}^0 \in [L^2(B)]^d,
\end{aligned}$$

there exists a unique solution $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\xi}, \boldsymbol{w}, \boldsymbol{e})$ *to Problem* VP *such that*

$$\begin{aligned}
&\boldsymbol{u} \in C^3([0,T]; H) \cap C^2([0,T]; V),\\
&\boldsymbol{w} \in C^2([0,T]; H) \cap C^1([0,T]; V).
\end{aligned}$$

3 Numerical analysis of a fully discrete scheme

In this section, we will numerically analyze the variational problem VP. We proceed in two steps. First, in order to provide the spatial approximation, the domain $\overline{B}$ is assumed polyhedral and so, let us denote by $\mathcal{T}^h$ its regular triangulation in the sense of [13]. We define the finite dimensional space $V^h \subset V$ as follows,

$$V^h = \{\boldsymbol{z}^h \in [C(\overline{B})]^d \cap V \; ; \; \boldsymbol{z}^h_{|Tr} \in [P_1(Tr)]^d \quad \forall Tr \in \mathcal{T}^h\}, \tag{9}$$

where $P_1(Tr)$ represents the space of polynomials of degree less or equal to one in the element Tr, i.e. the finite element space V^h is composed of continuous and piecewise affine functions. Here, $h > 0$ denotes the spatial discretization parameter. Moreover, the discrete initial conditions, denoted by $\boldsymbol{u}^{0h}$, $\boldsymbol{v}^{0h}$, $\boldsymbol{\xi}^{0h}$, $\boldsymbol{w}^{0h}$ and $\boldsymbol{e}^{0h}$, are given by

$$\begin{aligned}
&\boldsymbol{u}^{0h} = \mathcal{P}^h\boldsymbol{u}^0, \quad \boldsymbol{v}^{0h} = \mathcal{P}^h\boldsymbol{v}^0, \quad \boldsymbol{\xi}^{0h} = \mathcal{P}^h\boldsymbol{\xi}^0,\\
&\boldsymbol{w}^{0h} = \mathcal{P}^h\boldsymbol{w}^0, \quad \boldsymbol{e}^{0h} = \mathcal{P}^h\boldsymbol{e}^0,
\end{aligned} \tag{10}$$

where $\mathcal{P}^h$ is the usual finite element interpolation operator over V^h (see, for instance, [13]).

In order to consider the discretization of the time derivatives, we define a uniform partition of the time interval $[0,T]$, denoted by $0 = t_0 < t_1 < \cdots < t_N = T$, with step size $k = T/N$ and nodes $t_n = n\,k$ for $n = 0, 1, \ldots, N$. Moreover, for a continuous function $z(t)$, we use the notation $z_n = z(t_n)$ and, for the sequence $\{z_n\}_{n=0}^N$, we denote by $\delta z_n = (z_n - z_{n-1})/k$ its corresponding divided differences.

Using the well-known implicit Euler scheme, we can fully approximate the variational problem VP in the following form.

Problem VP^{hk}. *Find the discrete acceleration of the first constituent* $\boldsymbol{\xi}^{hk} = \{\boldsymbol{\xi}_n^{hk}\}_{n=0}^N \subset V^h$ *and the discrete velocity of the second constituent* $\boldsymbol{e}^{hk} = \{\boldsymbol{e}_n^{hk}\}_{n=0}^N \subset V^h$ *such that* $\boldsymbol{\xi}_0^{hk} = \boldsymbol{\xi}^{0h}$, $\boldsymbol{e}_0^{hk} = \boldsymbol{e}^{0h}$, *and, for* $n = 1, \ldots, N$,

$$\rho_1(\tau\delta\boldsymbol{\xi}_n^{hk} + \boldsymbol{\xi}_n^{hk}, \boldsymbol{r}^h)_H + A^*(\boldsymbol{u}_n^{hk}, \boldsymbol{r}^h) + A(\boldsymbol{v}_n^{hk}, \boldsymbol{r}^h) + B^*(\boldsymbol{w}_n^{hk}, \boldsymbol{r}^h)$$
$$+a^*(\boldsymbol{u}_n^{hk} + \tau\boldsymbol{v}_n^{hk} - \boldsymbol{w}_n^{hk}, \boldsymbol{r}^h) = 0 \quad \forall \boldsymbol{r}^h \in V^h, \tag{11}$$
$$\rho_2(\delta\boldsymbol{e}_n^{hk}, \boldsymbol{z}^h)_H + C^*(\boldsymbol{w}_n^{hk}, \boldsymbol{z}^h) + B^*(\boldsymbol{u}_n^{hk} + \tau\boldsymbol{v}_n^{hk}, \boldsymbol{z}^h)$$
$$-a^*(\boldsymbol{u}_n^{hk} + \tau\boldsymbol{v}_n^{hk} - \boldsymbol{w}_n^{hk}, \boldsymbol{z}^h) = 0 \quad \forall \boldsymbol{z}^h \in V^h, \tag{12}$$

where the discrete displacement of the first constituent $\boldsymbol{u}^{hk} = \{\boldsymbol{u}_n^{hk}\}_{n=0}^N$, the discrete velocity of the first constituent $\boldsymbol{v}^{hk} = \{\boldsymbol{v}_n^{hk}\}_{n=0}^N$ and the discrete displacement of the second constituent $\boldsymbol{w}^{hk} = \{\boldsymbol{w}_n^{hk}\}_{n=0}^N$ are then recovered from the next relations:

$$\boldsymbol{v}_n^{hk} = k\sum_{j=1}^n \boldsymbol{\xi}_j^{hk} + \boldsymbol{v}^{0h}, \quad \boldsymbol{u}_n^{hk} = k\sum_{j=1}^n \boldsymbol{v}_j^{hk} + \boldsymbol{u}^{0h}, \tag{13}$$
$$\boldsymbol{w}_n^{hk} = k\sum_{j=1}^n \boldsymbol{e}_j^{hk} + \boldsymbol{w}^{0h}. \tag{14}$$

It is obvious that the existence of a unique solution to Problem VP^{hk} follows from the Lax-Milgram lemma taking into account assumptions (i)-(iv).

The aim of this section is to provide the numerical analysis of Problem VP. First, we have the following discrete stability result.

Lemma 1 *Under the assumptions of Theorem 1, it follows that the sequences* $\{\boldsymbol{u}^{hk}, \boldsymbol{v}^{hk}, \boldsymbol{\xi}^{hk}, \boldsymbol{w}^{hk}, \boldsymbol{e}^{hk}\}$, *generated by Problem* VP^{hk}, *satisfy the stability estimate:*

$$\|\boldsymbol{\xi}_n^{hk}\|_H^2 + \|\boldsymbol{v}_n^{hk}\|_V^2 + \|\boldsymbol{u}_n^{hk}\|_V^2 + \|\boldsymbol{e}_n^{hk}\|_H^2 + \|\boldsymbol{w}_n^{hk}\|_V^2 \le C,$$

where C *is a positive constant assumed to be independent of the discretization parameters* h *and* k.

Proof In order to simplify the writing, we remove the superscripts h and k in all the variables. Moreover, for the sake of simplicity in the calculations we assume that $\tau = 1$.

First, we take as a test function $\boldsymbol{r}^h = \boldsymbol{\xi}_n^{hk}$ in discrete equation (11) to obtain

$$\rho_1(\delta\boldsymbol{\xi}_n + \boldsymbol{\xi}_n, \boldsymbol{\xi}_n)_H + A^*(\boldsymbol{u}_n, \boldsymbol{\xi}_n) + A(\boldsymbol{v}_n, \boldsymbol{\xi}_n) + B^*(\boldsymbol{w}_n, \boldsymbol{\xi}_n)$$
$$+a^*(\boldsymbol{u}_n + \boldsymbol{v}_n - \boldsymbol{w}_n, \boldsymbol{\xi}_n) = 0.$$

Keeping in mind that

$$(\delta\boldsymbol{\xi}_n, \boldsymbol{\xi}_n)_H \ge \frac{1}{2k}\Big[\|\boldsymbol{\xi}_n\|_H^2 - \|\boldsymbol{\xi}_{n-1}\|_H^2\Big],$$
$$a^*(\boldsymbol{v}_n, \boldsymbol{\xi}_n) \ge \frac{\tilde{C}}{2k}\Big[\|\boldsymbol{v}_n\|_H^2 - \|\boldsymbol{v}_{n-1}\|_H^2\Big],$$

where we have used assumption (iii), it follows that

$$\frac{\rho_1}{2k}\Big[\|\boldsymbol{\xi}_n\|_H^2-\|\boldsymbol{\xi}_{n-1}\|_H^2\Big]+A^*(\boldsymbol{u}_n,\boldsymbol{\xi}_n)+A(\boldsymbol{v}_n,\boldsymbol{\xi}_n)+B^*(\boldsymbol{w}_n,\boldsymbol{\xi}_n)$$
$$+\frac{\tilde{C}}{2k}\Big[\|\boldsymbol{v}_n\|_H^2-\|\boldsymbol{v}_{n-1}\|_H^2\Big]\leq C\Big(\|\boldsymbol{u}_n\|_H^2+\|\boldsymbol{w}_n\|_H^2+\|\boldsymbol{\xi}_n\|_H^2\Big). \quad (15)$$

Secondly, taking as a test function $\boldsymbol{z}^h=\boldsymbol{e}_n^{hk}$ in discrete equation (12) we have

$$\rho_2(\delta\boldsymbol{e}_n,\boldsymbol{e}_n)_H+C^*(\boldsymbol{w}_n,\boldsymbol{e}_n)+B^*(\boldsymbol{u}_n+\boldsymbol{v}_n,\boldsymbol{e}_n)$$
$$-a^*(\boldsymbol{u}_n+\boldsymbol{v}_n-\boldsymbol{w}_n,\boldsymbol{e}_n)=0,$$

and using the estimates (obtained applying again assumption (iii)):

$$(\delta\boldsymbol{e}_n,\boldsymbol{e}_n)_H\geq\frac{1}{2k}\Big[\|\boldsymbol{e}_n\|_H^2-\|\boldsymbol{e}_{n-1}\|_H^2\Big],$$
$$a^*(\boldsymbol{w}_n,\boldsymbol{e}_n)\geq\frac{\tilde{C}}{2k}\Big[\|\boldsymbol{w}_n\|_H^2-\|\boldsymbol{w}_{n-1}\|_H^2\Big],$$

we find that

$$\frac{\rho_2}{2k}\Big[\|\boldsymbol{e}_n\|_H^2-\|\boldsymbol{e}_{n-1}\|_H^2\Big]+C^*(\boldsymbol{w}_n,\boldsymbol{e}_n)+B^*(\boldsymbol{u}_n+\boldsymbol{v}_n,\boldsymbol{e}_n)$$
$$+\frac{\tilde{C}}{2k}\Big[\|\boldsymbol{w}_n\|_H^2-\|\boldsymbol{w}_{n-1}\|_H^2\Big]\leq C\Big(\|\boldsymbol{u}_n\|_H^2+\|\boldsymbol{v}_n\|_H^2+\|\boldsymbol{e}_n\|_H^2\Big). \quad (16)$$

Combining the previous estimates (15) and (16) and observing that

$$A(\boldsymbol{v}_n,\boldsymbol{\xi}_n)=\frac{1}{2k}\Big[A(\boldsymbol{v}_n,\boldsymbol{v}_n)-A(\boldsymbol{v}_{n-1},\boldsymbol{v}_{n-1})+A(\boldsymbol{v}_n-\boldsymbol{v}_{n-1},\boldsymbol{v}_n-\boldsymbol{v}_{n-1})\Big],$$
$$C^*(\boldsymbol{w}_n,\boldsymbol{e}_n)=\frac{1}{2k}\Big[C^*(\boldsymbol{w}_n,\boldsymbol{w}_n)-C^*(\boldsymbol{w}_{n-1},\boldsymbol{w}_{n-1})+C^*(\boldsymbol{w}_n-\boldsymbol{w}_{n-1},\boldsymbol{w}_n-\boldsymbol{w}_{n-1})\Big],$$
$$B^*(\boldsymbol{v}_n,\boldsymbol{e}_n)+B^*(\boldsymbol{w}_n,\boldsymbol{\xi}_n)=\frac{1}{k}\Big[B^*(\boldsymbol{v}_n,\boldsymbol{w}_n)-B^*(\boldsymbol{v}_{n-1},\boldsymbol{w}_{n-1})$$
$$+B^*(\boldsymbol{v}_n-\boldsymbol{v}_{n-1},\boldsymbol{w}_n-\boldsymbol{w}_{n-1})\Big],$$
$$A(\boldsymbol{v}_n-\boldsymbol{v}_{n-1},\boldsymbol{v}_n-\boldsymbol{v}_{n-1})+C^*(\boldsymbol{w}_n-\boldsymbol{w}_{n-1},\boldsymbol{w}_n-\boldsymbol{w}_{n-1})$$
$$+2B^*(\boldsymbol{v}_n-\boldsymbol{v}_{n-1},\boldsymbol{w}_n-\boldsymbol{w}_{n-1})\geq 0,$$

where we have used assumptions (ii) and (iv), we obtain the estimates:

$$\frac{\rho_1}{2k}\Big[\|\boldsymbol{\xi}_n\|_H^2-\|\boldsymbol{\xi}_{n-1}\|_H^2\Big]+A^*(\boldsymbol{u}_n,\boldsymbol{\xi}_n)+\frac{1}{2k}\Big[A(\boldsymbol{v}_n,\boldsymbol{v}_n)-A^*(\boldsymbol{v}_{n-1},\boldsymbol{v}_{n-1})\Big]$$
$$+\frac{\rho_2}{2k}\Big[\|\boldsymbol{e}_n\|_H^2-\|\boldsymbol{e}_{n-1}\|_H^2\Big]+\frac{1}{2k}\Big[C^*(\boldsymbol{w}_n,\boldsymbol{w}_n)-C^*(\boldsymbol{w}_{n-1},\boldsymbol{w}_{n-1})\Big]$$
$$+\frac{1}{k}\Big[B^*(\boldsymbol{v}_n,\boldsymbol{w}_n)-B^*(\boldsymbol{v}_{n-1},\boldsymbol{w}_{n-1})\Big]+\frac{\tilde{C}}{2k}\Big[\|\boldsymbol{v}_n\|_H^2-\|\boldsymbol{v}_{n-1}\|_H^2\Big]$$
$$+\frac{\tilde{C}}{2k}\Big[\|\boldsymbol{w}_n\|_H^2-\|\boldsymbol{w}_{n-1}\|_H^2\Big]+B^*(\boldsymbol{u}_n,\boldsymbol{e}_n)$$
$$\leq C\Big(\|\boldsymbol{u}_n\|_H^2+\|\boldsymbol{v}_n\|_H^2+\|\boldsymbol{w}_n\|_H^2+\|\boldsymbol{\xi}_n\|_H^2\Big).$$

Now, summing up to n the above estimates, we have

$$\rho_1\|\boldsymbol{\xi}_n\|_H^2 + k\sum_{j=1}^n A^*(\boldsymbol{u}_j,\boldsymbol{\xi}_j) + A(\boldsymbol{v}_n,\boldsymbol{v}_n) + \rho_2\|\boldsymbol{e}_n\|_H^2 + C^*(\boldsymbol{w}_n,\boldsymbol{w}_n)$$
$$+2B^*(\boldsymbol{v}_n,\boldsymbol{w}_n) + \tilde{C}\|\boldsymbol{v}_n\|_H^2 + \tilde{C}\|\boldsymbol{w}_n\|_H^2 + k\sum_{j=1}^n B^*(\boldsymbol{u}_j,\boldsymbol{e}_j)$$
$$\leq Ck\sum_{j=1}^n \Big(\|\boldsymbol{u}_j\|_H^2 + \|\boldsymbol{v}_j\|_H^2 + \|\boldsymbol{w}_j\|_H^2 + \|\boldsymbol{\xi}_j\|_H^2\Big) + C\Big(\|\boldsymbol{\xi}^0\|_H^2 + \|\boldsymbol{v}^0\|_V^2$$
$$+\|\boldsymbol{u}^0\|_V^2 + \|\boldsymbol{e}^0\|_H^2 + \|\boldsymbol{w}^0\|_V^2\Big).$$

Observing that, thanks again to properties (ii) and (iv),

$$A(\boldsymbol{v}_n,\boldsymbol{v}_n) + C^*(\boldsymbol{w}_n,\boldsymbol{w}_n) + 2B^*(\boldsymbol{v}_n,\boldsymbol{w}_n) \geq C(\|\boldsymbol{v}_n\|_V^2 + \|\boldsymbol{w}_n\|_V^2),$$

it follows that

$$\|\boldsymbol{\xi}_n\|_H^2 + k\sum_{j=1}^n A^*(\boldsymbol{u}_j,\boldsymbol{\xi}_j) + \|\boldsymbol{v}_n\|_V^2 + \|\boldsymbol{w}_n\|_V^2 + \|\boldsymbol{e}_n\|_H^2 + k\sum_{j=1}^n B^*(\boldsymbol{u}_j,\boldsymbol{e}_j)$$
$$\leq Ck\sum_{j=1}^n \Big(\|\boldsymbol{u}_j\|_H^2 + \|\boldsymbol{v}_j\|_H^2 + \|\boldsymbol{w}_j\|_H^2 + \|\boldsymbol{\xi}_j\|_H^2\Big) + C\Big(\|\boldsymbol{\xi}^0\|_H^2 + \|\boldsymbol{v}^0\|_V^2$$
$$+\|\boldsymbol{u}^0\|_V^2 + \|\boldsymbol{e}^0\|_H^2 + \|\boldsymbol{w}^0\|_V^2\Big).$$

Finally, keeping in mind that

$$k\sum_{j=1}^n A^*(\boldsymbol{u}_j,\boldsymbol{\xi}_j) = A^*(\boldsymbol{u}_n,\boldsymbol{v}_n) - k\sum_{j=1}^{n-1} A^*(\boldsymbol{v}_j,\boldsymbol{v}_{j-1}) - A^*(\boldsymbol{u}^0,\boldsymbol{v}^0),$$
$$k\sum_{j=1}^n B^*(\boldsymbol{u}_j,\boldsymbol{e}_j) = B^*(\boldsymbol{u}_n,\boldsymbol{w}_n) - k\sum_{j=1}^{n-1} B^*(\boldsymbol{v}_j,\boldsymbol{w}_{j-1}) - B^*(\boldsymbol{u}^0,\boldsymbol{w}^0),$$
$$\|\boldsymbol{u}_n\|_V^2 \leq Ck\sum_{j=1}^n \|\boldsymbol{v}_j\|_V^2 + \|\boldsymbol{u}^0\|_V^2,$$

using a discrete version of Gronwall's inequality (see [12]) we obtain the desired stability property.

Now, our aim is to show an a priori error analysis. We have the following.

Theorem 2 *Let the assumptions of Theorem 1 still hold. If we denote by* $(\boldsymbol{u},\boldsymbol{v},\boldsymbol{\xi},\boldsymbol{w},\boldsymbol{e})$ *and* $(\boldsymbol{u}^{hk},\boldsymbol{v}^{hk},\boldsymbol{\xi}^{hk},\boldsymbol{w}^{hk},\boldsymbol{e}^{hk})$ *the respective solutions to problems*

VP and VP^{hk}, then we have the following a priori error estimates for all $\boldsymbol{r}^h=\{\boldsymbol{r}_n^h\}_{n=0}^N$, $\boldsymbol{z}^h=\{\boldsymbol{z}_n^h\}_{n=0}^N\subset V^h$,

$$\begin{aligned}
&\max_{0\le n\le N}\Big\{\|\boldsymbol{\xi}_n-\boldsymbol{\xi}_n^{hk}\|_H^2+\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_V^2+\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_V^2+\|\boldsymbol{e}_n-\boldsymbol{e}_n^{hk}\|_H^2+\|\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}\|_V^2\Big\}\\
&\quad\le Ck\sum_{j=1}^N\Big(\|\boldsymbol{\xi}_j-\boldsymbol{r}_j^h\|_V^2+\|\dot{\boldsymbol{\xi}}_j-\delta\boldsymbol{\xi}_j\|_H^2+\|\dot{\boldsymbol{v}}_j-\delta\boldsymbol{v}_j\|_V^2+\|\dot{\boldsymbol{e}}_j-\delta\boldsymbol{e}_j\|_H^2\\
&\qquad+\|\boldsymbol{e}_j-\boldsymbol{z}_j^h\|_V^2+\|\dot{\boldsymbol{w}}_j-\delta\boldsymbol{w}_j\|_V^2\Big)+C\max_{0\le n\le N}\Big[I_n+\|\boldsymbol{e}_n-\boldsymbol{z}_n^h\|_H^2+\|\boldsymbol{\xi}_n-\boldsymbol{r}_n^h\|_H^2\Big]\\
&\qquad+\frac{C}{k}\sum_{j=1}^{N-1}\Big[\|\boldsymbol{\xi}_j-\boldsymbol{r}_j^h-(\boldsymbol{\xi}_{j+1}-\boldsymbol{r}_{j+1}^h)\|_H^2+\|\boldsymbol{e}_j-\boldsymbol{z}_j^h-(\boldsymbol{e}_{j+1}-\boldsymbol{z}_{j+1}^h)\|_H^2\Big]\\
&\qquad+C\Big(\|\boldsymbol{\xi}^0-\boldsymbol{\xi}^{0h}\|_H^2+\|\boldsymbol{v}^0-\boldsymbol{v}^{0h}\|_V^2+\|\boldsymbol{u}^0-\boldsymbol{u}^{0h}\|_V^2+\|\boldsymbol{e}^0-\boldsymbol{e}^{0h}\|_H^2\\
&\qquad+\|\boldsymbol{w}^0-\boldsymbol{w}^{0h}\|_V^2\Big),
\end{aligned}\tag{17}$$

where $C>0$ is again a positive constant assumed to be independent of the discretization parameters, but depending on the continuous solution, and I_n is the integration error defined as

$$I_n=\Big\|\int_0^{t_n}\boldsymbol{v}(s)\,ds-k\sum_{j=1}^n\boldsymbol{v}_j\Big\|_V^2.\tag{18}$$

Proof Again, in order to simplify the calculations we assume that $\tau=1$.

First, we will derive the estimates for the acceleration of the first constituent. Thus, subtracting variational equation (5) at time t_n for a test function $\boldsymbol{r}=\boldsymbol{r}^h\in V^h$ and discrete variational equation (11) it follows that, for all $\boldsymbol{r}^h\in V^h$,

$$\begin{aligned}
&\rho_1(\dot{\boldsymbol{\xi}}_n-\delta\boldsymbol{\xi}_n^{hk}+\boldsymbol{\xi}_n-\boldsymbol{\xi}_n^{hk},\boldsymbol{r}^h)_H+A^*(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk},\boldsymbol{r}^h)+A(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk},\boldsymbol{r}^h)\\
&\quad+B^*(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk},\boldsymbol{r}^h)+a^*(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}+(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk})-(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}),\boldsymbol{r}^h)=0.
\end{aligned}$$

Then, we have, for all $\boldsymbol{r}^h\in V^h$,

$$\begin{aligned}
&\rho_1(\dot{\boldsymbol{\xi}}_n-\delta\boldsymbol{\xi}_n^{hk}+\boldsymbol{\xi}_n-\boldsymbol{\xi}_n^{hk},\boldsymbol{\xi}_n-\boldsymbol{\xi}_n^{hk})_H+A^*(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk},\boldsymbol{\xi}_n-\boldsymbol{\xi}_n^{hk})\\
&\quad+A(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk},\boldsymbol{\xi}_n-\boldsymbol{\xi}_n^{hk})+B^*(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk},\boldsymbol{\xi}_n-\boldsymbol{\xi}_n^{hk})\\
&\quad+a^*(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}+\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}-(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}),\boldsymbol{\xi}_n-\boldsymbol{\xi}_n^{hk})\\
&=\rho_1(\dot{\boldsymbol{\xi}}_n-\delta\boldsymbol{\xi}_n^{hk}+\boldsymbol{\xi}_n-\boldsymbol{\xi}_n^{hk},\boldsymbol{\xi}_n-\boldsymbol{r}^h)_H+A^*(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk},\boldsymbol{\xi}_n-\boldsymbol{r}^h)\\
&\quad+A(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk},\boldsymbol{\xi}_n-\boldsymbol{r}^h)+B^*(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk},\boldsymbol{\xi}_n-\boldsymbol{r}^h)\\
&\quad+a^*(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}+\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}-(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}),\boldsymbol{\xi}_n-\boldsymbol{r}^h).
\end{aligned}$$

Taking into account that

$$
\begin{aligned}
&(\dot{\boldsymbol{\xi}}_n-\delta\boldsymbol{\xi}_n^{hk},\boldsymbol{\xi}_n-\boldsymbol{\xi}_n^{hk})_H \geq (\dot{\boldsymbol{\xi}}_n-\delta\boldsymbol{\xi}_n,\boldsymbol{\xi}_n-\boldsymbol{\xi}_n^{hk})_H+\frac{1}{2k}\Big[\|\boldsymbol{\xi}_n-\boldsymbol{\xi}_n^{hk}\|_H^2-\|\boldsymbol{\xi}_{n-1}-\boldsymbol{\xi}_{n-1}^{hk}\|_H^2\Big],\\
&A(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk},\boldsymbol{\xi}_n-\boldsymbol{\xi}_n^{hk})=A(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk},\dot{\boldsymbol{v}}_n-\delta\boldsymbol{v}_n)+A(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk},\delta\boldsymbol{v}_n-\delta\boldsymbol{v}_n^{hk})\\
&\quad=A(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk},\dot{\boldsymbol{v}}_n-\delta\boldsymbol{v}_n)+\frac{1}{2k}\Big[A(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk},\boldsymbol{v}_n-\boldsymbol{v}_n^{hk})-A(\boldsymbol{v}_{n-1}-\boldsymbol{v}_{n-1}^{hk},\boldsymbol{v}_{n-1}-\boldsymbol{v}_{n-1}^{hk})\\
&\qquad\qquad+A(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}-(\boldsymbol{v}_{n-1}-\boldsymbol{v}_{n-1}^{hk}),\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}-(\boldsymbol{v}_{n-1}-\boldsymbol{v}_{n-1}^{hk}))\Big],\\
&a^*(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk},\boldsymbol{\xi}_n-\boldsymbol{\xi}_n^{hk})=a^*(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk},\dot{\boldsymbol{v}}_n-\delta\boldsymbol{v}_n)+a^*(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk},\delta\boldsymbol{v}_n-\delta\boldsymbol{v}_n^{hk})\\
&\qquad\geq a^*(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk},\dot{\boldsymbol{v}}_n-\delta\boldsymbol{v}_n)+\frac{\tilde{C}}{2k}\Big[\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2-\|\boldsymbol{v}_{n-1}-\boldsymbol{v}_{n-1}^{hk}\|_H^2\Big],
\end{aligned}
$$

where we used the notations $\delta\boldsymbol{v}_n=(\boldsymbol{v}_n-\boldsymbol{v}_{n-1})/k$ and $\delta\boldsymbol{\xi}_n=(\boldsymbol{\xi}_n-\boldsymbol{\xi}_{n-1})/k$, we find that, for all $\boldsymbol{r}^h\in V^h$,

$$
\begin{aligned}
&\frac{\rho_1}{2k}\Big[\|\boldsymbol{\xi}_n-\boldsymbol{\xi}_n^{hk}\|_H^2-\|\boldsymbol{\xi}_{n-1}-\boldsymbol{\xi}_{n-1}^{hk}\|_H^2\Big]+\frac{\tilde{C}}{2k}\Big[\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2-\|\boldsymbol{v}_{n-1}-\boldsymbol{v}_{n-1}^{hk}\|_H^2\Big]\\
&\qquad+\frac{1}{2k}\Big[A(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk},\boldsymbol{v}_n-\boldsymbol{v}_n^{hk})-A(\boldsymbol{v}_{n-1}-\boldsymbol{v}_{n-1}^{hk},\boldsymbol{v}_{n-1}-\boldsymbol{v}_{n-1}^{hk})\\
&\qquad+A(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}-(\boldsymbol{v}_{n-1}-\boldsymbol{v}_{n-1}^{hk}),\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}-(\boldsymbol{v}_{n-1}-\boldsymbol{v}_{n-1}^{hk}))\Big]\\
&\qquad+A^*(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk},\boldsymbol{\xi}_n-\boldsymbol{\xi}_n^{hk})+B^*(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk},\delta\boldsymbol{v}_n-\delta\boldsymbol{v}_n^{hk})\\
&\quad\leq C\Big(\|\boldsymbol{\xi}_n-\boldsymbol{r}^h\|_V^2+\|\dot{\boldsymbol{\xi}}_n-\delta\boldsymbol{\xi}_n\|_H^2+\|\dot{\boldsymbol{v}}_n-\delta\boldsymbol{v}_n\|_V^2+\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_V^2+\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_V^2\\
&\qquad+\|\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}\|_V^2+(\delta\boldsymbol{\xi}_n-\delta\boldsymbol{\xi}_n^{hk},\boldsymbol{\xi}_n-\boldsymbol{r}^h)_H\Big).
\end{aligned}
$$

Now, we obtain the error estimates for the velocity of the second constituent and so, we subtract variational equation (6) at time t_n for a test function $\boldsymbol{z}=\boldsymbol{z}^h\in V^h$ and discrete variational equation (12) to get, for all $\boldsymbol{z}^h\in V^h$,

$$
\begin{aligned}
&\rho_2(\dot{\boldsymbol{e}}_n-\delta\boldsymbol{e}_n^{hk},\boldsymbol{z}^h)_H+C^*(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk},\boldsymbol{z}^h)+B^*(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}+\boldsymbol{v}_n-\boldsymbol{v}_n^{hk},\boldsymbol{z}^h)\\
&\qquad-a^*(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}+\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}-(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}),\boldsymbol{z}^h)=0.
\end{aligned}
$$

Therefore, we have, for all $\boldsymbol{z}^h\in V^h$,

$$
\begin{aligned}
&\rho_2(\dot{\boldsymbol{e}}_n-\delta\boldsymbol{e}_n^{hk},\boldsymbol{e}_n-\boldsymbol{e}_n^{hk})_H+B^*(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}+\boldsymbol{v}_n-\boldsymbol{v}_n^{hk},\boldsymbol{e}_n-\boldsymbol{e}_n^{hk})\\
&\qquad+C^*(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk},\boldsymbol{e}_n-\boldsymbol{e}_n^{hk})-a^*(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}+\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}-(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}),\boldsymbol{e}_n-\boldsymbol{e}_n^{hk})\\
&\quad=\rho_2(\dot{\boldsymbol{e}}_n-\delta\boldsymbol{e}_n^{hk},\boldsymbol{e}_n-\boldsymbol{z}^h)_H+B^*(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}+\boldsymbol{v}_n-\boldsymbol{v}_n^{hk},\boldsymbol{e}_n-\boldsymbol{z}^h)\\
&\qquad+C^*(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk},\boldsymbol{e}_n-\boldsymbol{z}^h)-a^*(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}+\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}-(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}),\boldsymbol{e}_n-\boldsymbol{z}^h).
\end{aligned}
$$

Taking now into account that

$$
\begin{aligned}
&(\dot{\boldsymbol{e}}_n-\delta\boldsymbol{e}_n^{hk},\boldsymbol{e}_n-\boldsymbol{e}_n^{hk})_H\geq(\dot{\boldsymbol{e}}_n-\delta\boldsymbol{e}_n,\boldsymbol{e}_n-\boldsymbol{e}_n^{hk})_H+\frac{1}{2k}\Big[\|\boldsymbol{e}_n-\boldsymbol{e}_n^{hk}\|_H^2-\|\boldsymbol{e}_{n-1}-\boldsymbol{e}_{n-1}^{hk}\|_H^2\Big],\\
&C^*(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk},\boldsymbol{e}_n-\boldsymbol{e}_n^{hk})=C^*(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk},\dot{\boldsymbol{w}}_n-\delta\boldsymbol{w}_n)+C^*(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk},\delta\boldsymbol{w}_n-\delta\boldsymbol{w}_n^{hk})\\
&\quad=C^*(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk},\dot{\boldsymbol{w}}_n-\delta\boldsymbol{w}_n)+\frac{1}{2k}\Big[C^*(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk},\boldsymbol{w}_n-\boldsymbol{w}_n^{hk})-C^*(\boldsymbol{w}_{n-1}-\boldsymbol{w}_{n-1}^{hk},\boldsymbol{w}_{n-1}-\boldsymbol{w}_{n-1}^{hk})\\
&\qquad\qquad+C^*(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}-(\boldsymbol{w}_{n-1}-\boldsymbol{w}_{n-1}^{hk}),\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}-(\boldsymbol{w}_{n-1}-\boldsymbol{w}_{n-1}^{hk}))\Big],\\
&a^*(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk},\boldsymbol{e}_n-\boldsymbol{e}_n^{hk})=a^*(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk},\dot{\boldsymbol{w}}_n-\delta\boldsymbol{w}_n)+a^*(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk},\delta\boldsymbol{w}_n-\delta\boldsymbol{w}_n^{hk})\\
&\qquad\geq a^*(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk},\dot{\boldsymbol{w}}_n-\delta\boldsymbol{w}_n)+\frac{\tilde{C}}{2k}\Big[\|\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}\|_H^2-\|\boldsymbol{w}_{n-1}-\boldsymbol{w}_{n-1}^{hk}\|_H^2\Big],
\end{aligned}
$$

where we used now the similar notations $\delta \boldsymbol{e}_n = (\boldsymbol{e}_n - \boldsymbol{e}_{n-1})/k$ and $\delta \boldsymbol{w}_n = (\boldsymbol{w}_n - \boldsymbol{w}_{n-1})/k$, it follows that, for all $\boldsymbol{z}^h \in V^h$,

$$
\begin{aligned}
&\frac{\rho_2}{2k}\Big[\|\boldsymbol{e}_n - \boldsymbol{e}_n^{hk}\|_H^2 - \|\boldsymbol{e}_{n-1} - \boldsymbol{e}_{n-1}^{hk}\|_H^2\Big] + \frac{\tilde{C}}{2k}\Big[\|\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}\|_H^2 - \|\boldsymbol{w}_{n-1} - \boldsymbol{w}_{n-1}^{hk}\|_H^2\Big] \\
&\quad + \frac{1}{2k}\Big[C^*(\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}, \boldsymbol{w}_n - \boldsymbol{w}_n^{hk}) - C^*(\boldsymbol{w}_{n-1} - \boldsymbol{w}_{n-1}^{hk}, \boldsymbol{w}_{n-1} - \boldsymbol{w}_{n-1}^{hk}) \\
&\quad + C^*(\boldsymbol{w}_n - \boldsymbol{w}_n^{hk} - (\boldsymbol{w}_{n-1} - \boldsymbol{w}_{n-1}^{hk}), \boldsymbol{w}_n - \boldsymbol{w}_n^{hk} - (\boldsymbol{w}_{n-1} - \boldsymbol{w}_{n-1}^{hk}))\Big] \\
&\quad + B^*(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}, \delta \boldsymbol{w}_n - \delta \boldsymbol{w}_n^{hk}) + B^*(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}, \delta \boldsymbol{w}_n - \delta \boldsymbol{w}_n^{hk}) \\
&\leq C\Big(\|\boldsymbol{e}_n - \boldsymbol{z}^h\|_V^2 + \|\dot{\boldsymbol{e}}_n - \delta \boldsymbol{e}_n\|_H^2 + \|\dot{\boldsymbol{w}}_n - \delta \boldsymbol{w}_n\|_V^2 + \|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V^2 + \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V^2 \\
&\quad + \|\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}\|_V^2 + (\delta \boldsymbol{e}_n - \delta \boldsymbol{e}_n^{hk}, \boldsymbol{e}_n - \boldsymbol{z}^h)_H\Big).
\end{aligned}
$$

Combining the previous estimates it follows that, for all $\boldsymbol{r}^h, \boldsymbol{z}^h \in V^h$,

$$
\begin{aligned}
&\frac{\rho_1}{2k}\Big[\|\boldsymbol{\xi}_n - \boldsymbol{\xi}_n^{hk}\|_H^2 - \|\boldsymbol{\xi}_{n-1} - \boldsymbol{\xi}_{n-1}^{hk}\|_H^2\Big] + \frac{\tilde{C}}{2k}\Big[\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 - \|\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk}\|_H^2\Big] \\
&\quad + \frac{1}{2k}\Big[A(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{v}_n^{hk}) - A(\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk}, \boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk}) \\
&\quad + A(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk} - (\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk}), \boldsymbol{v}_n - \boldsymbol{v}_n^{hk} - (\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk}))\Big] \\
&\quad + A^*(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}, \boldsymbol{\xi}_n - \boldsymbol{\xi}_n^{hk}) + B^*(\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}, \delta \boldsymbol{v}_n - \delta \boldsymbol{v}_n^{hk}) \\
&\quad + \frac{\rho_2}{2k}\Big[\|\boldsymbol{e}_n - \boldsymbol{e}_n^{hk}\|_H^2 - \|\boldsymbol{e}_{n-1} - \boldsymbol{e}_{n-1}^{hk}\|_H^2\Big] + \frac{\tilde{C}}{2k}\Big[\|\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}\|_H^2 - \|\boldsymbol{w}_{n-1} - \boldsymbol{w}_{n-1}^{hk}\|_H^2\Big] \\
&\quad + \frac{1}{2k}\Big[C^*(\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}, \boldsymbol{w}_n - \boldsymbol{w}_n^{hk}) - C^*(\boldsymbol{w}_{n-1} - \boldsymbol{w}_{n-1}^{hk}, \boldsymbol{w}_{n-1} - \boldsymbol{w}_{n-1}^{hk}) \\
&\quad + C^*(\boldsymbol{w}_n - \boldsymbol{w}_n^{hk} - (\boldsymbol{w}_{n-1} - \boldsymbol{w}_{n-1}^{hk}), \boldsymbol{w}_n - \boldsymbol{w}_n^{hk} - (\boldsymbol{w}_{n-1} - \boldsymbol{w}_{n-1}^{hk}))\Big] \\
&\quad + B^*(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}, \delta \boldsymbol{w}_n - \delta \boldsymbol{w}_n^{hk}) + B^*(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}, \delta \boldsymbol{w}_n - \delta \boldsymbol{w}_n^{hk}) \\
&\leq C\Big(\|\boldsymbol{\xi}_n - \boldsymbol{r}^h\|_V^2 + \|\dot{\boldsymbol{\xi}}_n - \delta \boldsymbol{\xi}_n\|_H^2 + \|\dot{\boldsymbol{v}}_n - \delta \boldsymbol{v}_n\|_V^2 + \|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V^2 + \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V^2 \\
&\quad + \|\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}\|_V^2 + (\delta \boldsymbol{\xi}_n - \delta \boldsymbol{\xi}_n^{hk}, \boldsymbol{\xi}_n - \boldsymbol{r}^h)_H + \|\boldsymbol{e}_n - \boldsymbol{z}^h\|_V^2 + \|\dot{\boldsymbol{e}}_n - \delta \boldsymbol{e}_n\|_H^2 \\
&\quad + \|\dot{\boldsymbol{w}}_n - \delta \boldsymbol{w}_n\|_V^2 + (\delta \boldsymbol{e}_n - \delta \boldsymbol{e}_n^{hk}, \boldsymbol{e}_n - \boldsymbol{z}^h)_H\Big).
\end{aligned}
$$

Keeping in mind that

$$
\begin{aligned}
&B^*(\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}, \delta \boldsymbol{v}_n - \delta \boldsymbol{v}_n^{hk}) + B^*(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}, \delta \boldsymbol{w}_n - \delta \boldsymbol{w}_n^{hk}) \\
&\quad = \frac{1}{k}\Big[B^*(\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{v}_n^{hk}) - B^*(\boldsymbol{w}_{n-1} - \boldsymbol{w}_{n-1}^{hk}, \boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk}) \\
&\quad\quad + B^*(\boldsymbol{w}_n - \boldsymbol{w}_{n-1} - (\boldsymbol{w}_{n-1} - \boldsymbol{w}_{n-1}^{hk}), \boldsymbol{v}_n - \boldsymbol{v}_{n-1} - (\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk}))\Big], \\
&A(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk} - (\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk}), \boldsymbol{v}_n - \boldsymbol{v}_n^{hk} - (\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk})) \\
&\quad\quad + 2B^*(\boldsymbol{w}_n - \boldsymbol{w}_{n-1} - (\boldsymbol{w}_{n-1} - \boldsymbol{w}_{n-1}^{hk}), \boldsymbol{v}_n - \boldsymbol{v}_{n-1} - (\boldsymbol{v}_{n-1} - \boldsymbol{v}_{n-1}^{hk})) \\
&\quad\quad + C^*(\boldsymbol{w}_n - \boldsymbol{w}_n^{hk} - (\boldsymbol{w}_{n-1} - \boldsymbol{w}_{n-1}^{hk}), \boldsymbol{w}_n - \boldsymbol{w}_n^{hk} - (\boldsymbol{w}_{n-1} - \boldsymbol{w}_{n-1}^{hk})) \geq 0,
\end{aligned}
$$

where we have used again assumptions (ii) and (iv), multiplying the above estimates by k and summing up to n we find that

$$\begin{aligned}
&\|\boldsymbol{\xi}_n-\boldsymbol{\xi}_n^{hk}\|_H^2+\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_V^2+A(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk},\boldsymbol{v}_n-\boldsymbol{v}_n^{hk})+\|\boldsymbol{e}_n-\boldsymbol{e}_n^{hk}\|_H^2\\
&\quad+k\sum_{j=1}^n A^*(\boldsymbol{u}_j-\boldsymbol{u}_j^{hk},\delta\boldsymbol{v}_j-\delta\boldsymbol{v}_n^{hk})+\|\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}\|_V^2+C^*(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk},\boldsymbol{w}_n-\boldsymbol{w}_n^{hk})\\
&\quad+2B^*(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk},\boldsymbol{w}_n-\boldsymbol{w}_n^{hk})+k\sum_{j=1}^n B^*(\boldsymbol{u}_j-\boldsymbol{u}_j^{hk},\delta\boldsymbol{w}_j-\delta\boldsymbol{w}_j^{hk})\\
&\leq Ck\sum_{j=1}^n\Big(\|\boldsymbol{\xi}_j-\boldsymbol{r}_j^h\|_V^2+\|\dot{\boldsymbol{\xi}}_j-\delta\boldsymbol{\xi}_j\|_H^2+\|\dot{\boldsymbol{v}}_j-\delta\boldsymbol{v}_j\|_V^2+\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|_V^2+\|\boldsymbol{u}_j-\boldsymbol{u}_j^{hk}\|_V^2\\
&\quad+\|\boldsymbol{w}_j-\boldsymbol{w}_j^{hk}\|_V^2+(\delta\boldsymbol{\xi}_j-\delta\boldsymbol{\xi}_j^{hk},\boldsymbol{\xi}_j-\boldsymbol{r}_j^h)_H+\|\boldsymbol{e}_j-\boldsymbol{z}_j^h\|_V^2+\|\dot{\boldsymbol{e}}_j-\delta\boldsymbol{e}_j\|_H^2\\
&\quad+\|\dot{\boldsymbol{w}}_j-\delta\boldsymbol{w}_j\|_V^2+(\delta\boldsymbol{e}_j-\delta\boldsymbol{e}_j^{hk},\boldsymbol{e}_j-\boldsymbol{z}_j^h)_H\Big)+C\Big(\|\boldsymbol{\xi}^0-\boldsymbol{\xi}^{0h}\|_H^2+\|\boldsymbol{v}^0-\boldsymbol{v}^{0h}\|_V^2\\
&\quad+\|\boldsymbol{e}^0-\boldsymbol{e}^{0h}\|_H^2+\|\boldsymbol{w}^0-\boldsymbol{w}^{0h}\|_V^2\Big)\quad\forall\boldsymbol{r}^h=\{\boldsymbol{r}_j^h\}_{j=0}^n,\ \boldsymbol{z}^h=\{\boldsymbol{z}_j^h\}_{j=0}^n\subset V^h.
\end{aligned}$$

From assumptions (ii) and (iv) it follows that

$$\begin{aligned}
&A(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk},\boldsymbol{v}_n-\boldsymbol{v}_n^{hk})+C^*(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk},\boldsymbol{w}_n-\boldsymbol{w}_n^{hk})+2B^*(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk},\boldsymbol{w}_n-\boldsymbol{w}_n^{hk})\\
&\qquad\geq C\Big(\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_V^2+\|\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}\|_V^2\Big).
\end{aligned}$$

Finally, taking into account that

$$\begin{aligned}
&k\sum_{j=1}^n A^*(\boldsymbol{u}_j-\boldsymbol{u}_j^{hk},\delta\boldsymbol{v}_j-\delta\boldsymbol{v}_n^{hk})=A^*(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk},\boldsymbol{v}_n-\boldsymbol{v}_n^{hk})\\
&\qquad-k\sum_{j=1}^n A^*(\delta\boldsymbol{u}_j-\delta\boldsymbol{u}_j^{hk},\boldsymbol{v}_{j-1}-\boldsymbol{v}_{j-1}^{hk})-A^*(\boldsymbol{u}^0-\boldsymbol{u}^{0h},\boldsymbol{v}^0-\boldsymbol{v}^{0h}),\\
&k\sum_{j=1}^n B^*(\boldsymbol{u}_j-\boldsymbol{u}_j^{hk},\delta\boldsymbol{w}_j-\delta\boldsymbol{w}_n^{hk})=B^*(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk},\boldsymbol{w}_n-\boldsymbol{w}_n^{hk})\\
&\qquad-k\sum_{j=1}^n B^*(\delta\boldsymbol{u}_j-\delta\boldsymbol{u}_j^{hk},\boldsymbol{w}_{j-1}-\boldsymbol{w}_{j-1}^{hk})-B^*(\boldsymbol{u}^0-\boldsymbol{u}^{0h},\boldsymbol{w}^0-\boldsymbol{w}^{0h}),\\
&k\sum_{j=1}^n(\delta\boldsymbol{\xi}_j-\delta\boldsymbol{\xi}_j^{hk},\boldsymbol{\xi}_j-\boldsymbol{r}_j^h)_H=(\boldsymbol{\xi}_n-\boldsymbol{\xi}_n^{hk},\boldsymbol{\xi}_n-\boldsymbol{r}_n^h)_H+(\boldsymbol{\xi}^{0h}-\boldsymbol{\xi}^0,\boldsymbol{\xi}_1-\boldsymbol{r}_1^h)_H\\
&\qquad+\sum_{j=1}^{n-1}(\boldsymbol{\xi}_j-\boldsymbol{\xi}_j^{hk},\boldsymbol{\xi}_j-\boldsymbol{r}_j^h-(\boldsymbol{\xi}_{j+1}-\boldsymbol{r}_{j+1}^h))_H,\\
&k\sum_{j=1}^n(\delta\boldsymbol{e}_j-\delta\boldsymbol{e}_j^{hk},\boldsymbol{e}_j-\boldsymbol{z}_j^h)_H=(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk},\boldsymbol{e}_n-\boldsymbol{z}_n^h)_H+(\boldsymbol{e}^{0h}-\boldsymbol{e}^0,\boldsymbol{e}_1-\boldsymbol{z}_1^h)_H\\
&\qquad+\sum_{j=1}^{n-1}(\boldsymbol{e}_j-\boldsymbol{e}_j^{hk},\boldsymbol{e}_j-\boldsymbol{z}_j^h-(\boldsymbol{e}_{j+1}-\boldsymbol{z}_{j+1}^h))_H,\\
&\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_V^2\leq C\Big(k\sum_{j=1}^n\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|_V^2+I_j+\|\boldsymbol{u}^0-\boldsymbol{u}^{0h}\|_V^2\Big),
\end{aligned}$$

where I_j is the integration error defined in (18), applying a discrete version of Gronwall's inequality (see again [12]), we obtain the a priori error estimates (17).

From the above error estimates result we can deduce the convergence order under suitable regularity conditions. As an example, we have the following corollary which states the linear convergence of the algorithm under some additional regularity conditions.

Corollary 1 *Under the assumptions of Theorem 2, if the solution to Problem VP has the regularity:*

$$\begin{aligned} &\boldsymbol{u} \in C^2([0,T];[H^2(B)]^d) \cap H^4(0,T;H), \\ &\boldsymbol{w} \in C^1([0,T];[H^2(B)]^d) \cap H^3(0,T;H), \end{aligned} \tag{19}$$

and we use the finite element space V^h given in (9), and the discrete initial conditions $\boldsymbol{u}^{0h}$, $\boldsymbol{v}^{0h}$, $\boldsymbol{\xi}^{0h}$, $\boldsymbol{w}^{0h}$ and $\boldsymbol{e}^{0h}$ defined in (10), the linear convergence of the algorithm is deduced; i.e. there exists a positive constant $C > 0$, independent of the discretization parameters h and k, such that

$$\begin{aligned} \max_{0\le n\le N} \Big\{ &\|\boldsymbol{\xi}_n - \boldsymbol{\xi}_n^{hk}\|_H + \|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_V + \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V + \|\boldsymbol{e}_n - \boldsymbol{e}_n^{hk}\|_H \\ &+ \|\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}\|_V \Big\} \le C(h+k). \end{aligned}$$

4 Numerical results in one-dimensional examples

In this final section, we describe the numerical scheme implemented in MATLAB for solving Problem VP^{hk}, and we show some numerical examples to demonstrate the accuracy of the approximations and the behaviour of the solution. For the sake of simplicity, we restrict ourselves to the one-dimensional case (the domain B is assumed to be the interval $(0,1)$) and so, problem (1)-(4) is written as follows:

$$\begin{aligned} &\rho_1(\tau\,\dddot{u} + \ddot{u}) = A^* u_{xx} + A\dot{u}_{xx} + B^* w_{xx} - a^*(u + \tau\dot{u} - w) \quad \text{in} \quad (0,1)\times(0,T), \\ &\rho_2 \ddot{w} = B^*(u_{xx} + \tau\dot{u}_{xx}) + C^* w_{xx} + a^*(u + \tau\dot{u} - w) \quad \text{in} \quad (0,1)\times(0,T), \\ &u(0,t) = u(1,t) = w(0,t) = w(1,t) \quad \text{for a.e. } t \in (0,T), \\ &u(x,0) = u^0(x), \quad \dot{u}(x,0) = v^0(x), \quad \ddot{u}(x,0) = \xi^0(x) \quad \text{for a.e. } x \in (0,1), \\ &w(x,0) = w^0(x), \quad \dot{w}(x,0) = e^0(x) \quad \text{for a.e. } x \in (0,1), \end{aligned}$$

where A^*, A, B^*, C^* and a^* are now given constants representing the corresponding linear operators.

Therefore, proceeding as in Sections 2 and 3, we obtain the following numerical algorithm to solve the one-dimensional version of Problem VP^{hk}.

Given the solution u_{n-1}^{hk}, v_{n-1}^{hk}, ξ_{n-1}^{hk}, w_{n-1}^{hk} and e_{n-1}^{hk} at time t_{n-1}, variables ξ_n^{hk} and e_n^{hk} are obtained by solving the discrete linear system, for all r^h, $z^h \in V^h$,

$$
\begin{aligned}
&\rho_1(\tau\xi_n^{hk} + k\xi_n^{hk}, r^h) + A^*k^3((\xi_n^{hk})_x, r_x^h) + Ak^2((\xi_n^{hk})_x, r_x^h) + a^*k^3(\xi_n^{hk}, r^h)\\
&\quad +k^2B^*((e_n^{hk})_x, r_x^h) + a^*\tau k^2(\xi_n^{hk}, r^h) - a^*k^2(e_n^{hk}, r^h)\\
&\quad = \rho_1\tau(\xi_{n-1}^{hk}, r^h) - A^*k((u_{n-1}^{hk} + kv_{n-1}^{hk})_x, r_x^h) - Ak((v_{n-1}^{hk})_x, r_x^h)\\
&\quad -B^*k((w_{n-1}^{hk})_x, r_x^h) - a^*k(u_{n-1}^{hk} + kv_{n-1}^{hk} + \tau v_{n-1}^{hk} - w_{n-1}^{hk}, r^h),\\
&\rho_2(e_n^{hk}, z^h) + C^*k^2((e_n^{hk})_x, z_x^h) + a^*k^2(e_n^{hk}, z^h) + B^*k((k^2+\tau k)(\xi_n^{hk})_x, z_x^h)\\
&\quad -a^*k((k^2+\tau k)\xi_n^{hk}, z^h)\\
&\quad = \rho_2(e_{n-1}^{hk}, z^h) - B^*k((u_{n-1}^{hk} + kv_{n-1}^{hk})_x, z_x^h) - C^*k((w_{n-1}^{hk})_x, z_x^h)\\
&\quad +a^*k(u_{n-1}^{hk} + kv_{n-1}^{hk} + \tau v_{n-1}^{hk} - w_{n-1}^{hk}, z^h).
\end{aligned}
$$

We note that the numerical scheme leads to a non-symmetrical system which was solved by using LU-method, and it was implemented on a 3.2 Ghz PC using MATLAB. Moreover, a typical run ($h = k = 0.001$) took about 0.37 seconds of CPU time.

4.1 First example: numerical convergence

As an academical example, in order to show the accuracy of the approximations we solve this discrete problem with the following data:

$$
\begin{aligned}
&B = (0,1), \quad T = 1, \quad \rho_1 = 1, \quad \rho_2 = 1, \quad A^* = 3, \quad A = 2,\\
&B^* = 1, \quad a^* = 4, \quad C^* = 2, \quad \tau = \frac{1}{2}.
\end{aligned}
$$

By using the following initial conditions, for all $x \in (0,1)$,

$$u^0(x) = v^0(x) = \xi^0(x) = w^0(x) = e^0(x) = x(x-1),$$

and the (artificial) supply terms, for all $(x,t) \in (0,1)\times(0,1)$,

$$
\begin{aligned}
&F_1(x,t) = e^t(7x(x-1)/2 - 12),\\
&F_2(x,t) = -e^t(x(x-1) + 7),
\end{aligned}
$$

the exact solution to the above one-dimensional problem can be easily calculated and it has the form, for $(x,t) \in [0,1]\times[0,1]$:

$$u(x,t) = w(x,t) = e^t x(x-1).$$

Thus, the approximation errors estimated by

$$
\begin{aligned}
\max_{0\le n\le N} \Big\{ &\|v_n - v_n^{hk}\|_V + \|u_n - u_n^{hk}\|_V + \|\xi_n - \xi_n^{hk}\|_H + \|w_n - w_n^{hk}\|_V\\
&+\|e_n - e_n^{hk}\|_H \Big\}
\end{aligned}
$$

$h \downarrow k \rightarrow$	0.01	0.005	0.002	0.001	0.0005	0.0002	0.0001
$1/2^3$	0.291780	0.287690	0.285235	0.284416	0.284007	0.283761	0.283678
$1/2^4$	0.149900	0.145754	0.143281	0.142457	0.142045	0.141797	0.141715
$1/2^5$	0.079191	0.074944	0.072447	0.071620	0.071208	0.070960	0.070878
$1/2^6$	0.044065	0.039604	0.037055	0.036222	0.035808	0.035560	0.035478
$1/2^7$	0.026901	0.022032	0.019376	0.018528	0.018111	0.017862	0.017780
$1/2^8$	0.018938	0.013426	0.010566	0.009689	0.009264	0.009014	0.008931
$1/2^9$	0.015626	0.009408	0.006217	0.005284	0.004845	0.004590	0.004507
$1/2^{10}$	0.014421	0.007716	0.004138	0.003109	0.002642	0.002380	0.002295
$1/2^{11}$	0.014028	0.007096	0.003221	0.002068	0.001555	0.001277	0.001190
$1/2^{12}$	0.013913	0.006895	0.002867	0.001609	0.001034	0.000730	0.000638
$1/2^{13}$	0.013882	0.006837	0.002747	0.001430	0.000804	0.000464	0.000365

Table 1 Example 1: Numerical errors for some h and k.

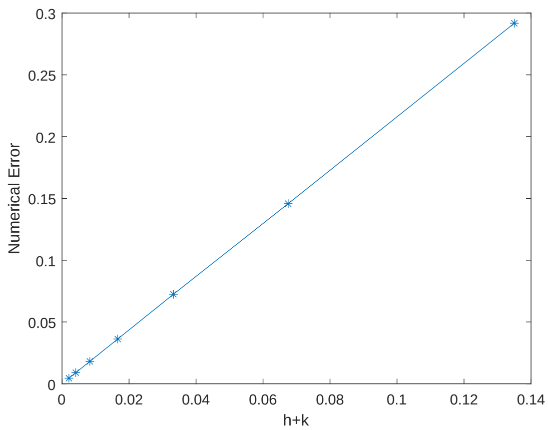

Fig. 1 Example 1: Asymptotic constant error.

are presented in Table 1 for several values of the discretization parameters h and k. Moreover, the evolution of the error depending on the parameter $h+k$ is plotted in Fig. 1. The convergence of the algorithm is clearly observed, and the linear convergence, stated in Corollary 1, is achieved.

If we assume now that there are not supply terms, and we use the final time $T = 20$, the following data:

$$B = (0,1), \quad T = 20, \quad \rho_1 = 1, \quad \rho_2 = 0.1, \quad A^* = 5, \quad A = 2,$$
$$B^* = 1, \quad a^* = 1, \quad C^* = 6, \quad \tau = 0.03,$$

and the initial conditions, for all $x \in (0,1)$,

$$u^0(x) = v^0(x) = \xi^0(x) = x(x-1), \quad w^0(x) = e^0(x) = 0,$$

taking the discretization parameters $h = k = 0.001$, the evolution in time of the discrete energy

$$E_n^{hk} = \rho_1\tau^2||\xi_n^{hk}||_H^2 + \rho_2||e_n^{hk}||_H^2 + (A - \tau A^*)||(u_n^{hk})_x||_H^2 + A^*\tau^2||(v_n^{hk})_x||_H^2$$
$$+a^*||(w_n^{hk})_x||_H^2 + a^*||u_n^{hk}\|_H^2 + a^*\tau^2\|v_n^{hk}\|_H^2 + a^*\|w_n^{hk}\|_H^2$$

is plotted in Fig. 2 (in both natural and semi-log scales). As can be seen, it converges to zero and an exponential decay seems to be achieved.

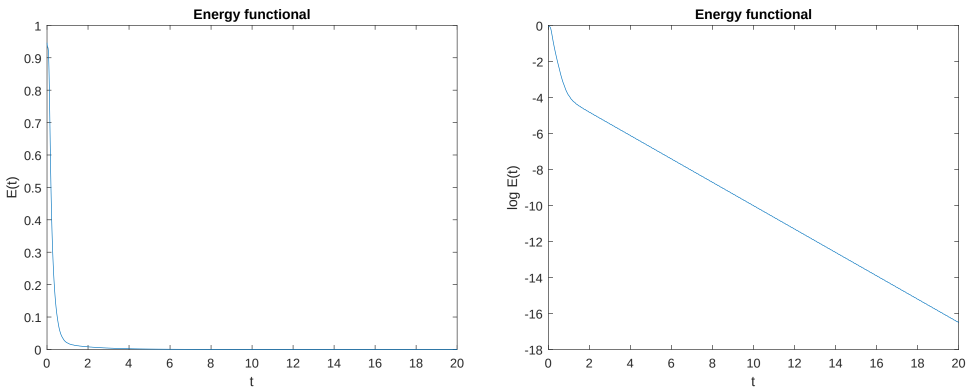

Fig. 2 Example 1: Evolution in time of the discrete energy (natural and semi-log scales).

4.2 Second example: dependence on parameter a^*

In this second example our aim is to study the dependence on the coupling parameter a^* between the two constituents of the mixture.

The following data are used:

$$B = (0, 1), \quad T = 1, \quad \rho_1 = 1, \quad \rho_2 = 0.1, \quad A^* = 5, \quad A = 2,$$
$$B^* = 1, \quad C^* = 6, \quad \tau = 0.03,$$

with the initial conditions, for all $x \in (0, 1)$,

$$u^0(x) = v^0(x) = \xi^0(x) = w^0(x) = e^0(x) = x(x - 1),$$

and again null Dirichlet boundary conditions.

Using the discretization parameters $h = k = 0.001$, in Fig. 3 we show the displacement (upper left), velocity (upper right) and acceleration (lower) of the first constituent of the mixture, for several values of parameter a^*. As can be seen, the domain bends with a quadratic shape but, when the parameter increases, the deformation reduces due to the energy transmission to the second

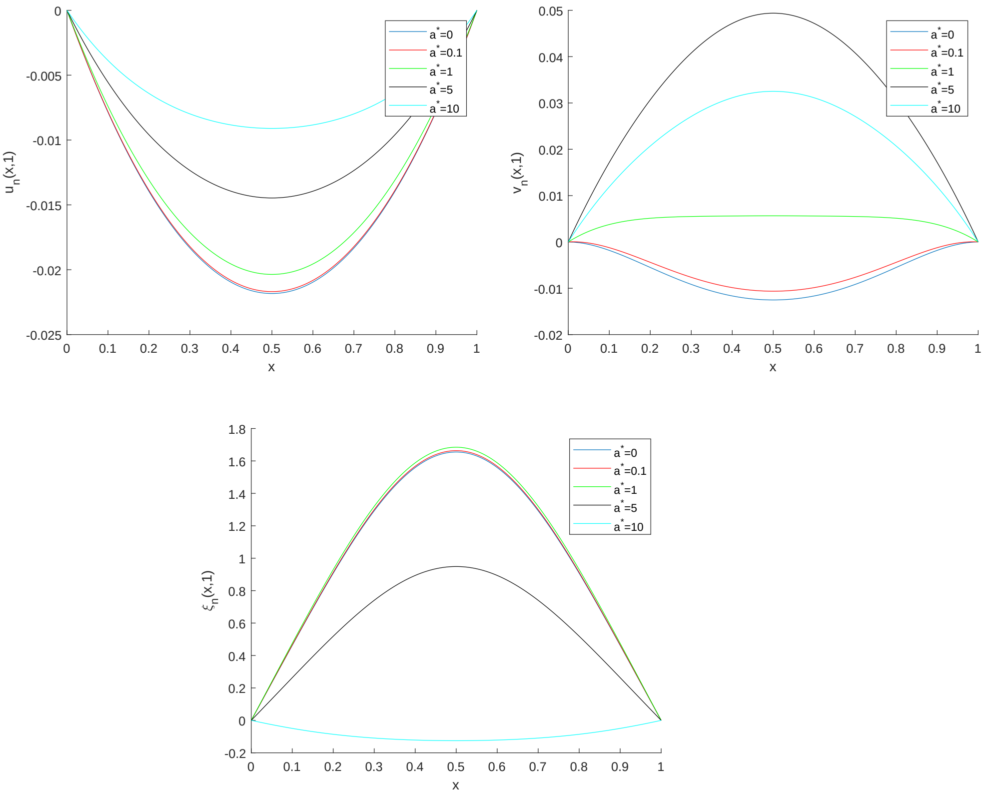

Fig. 3 Example 2: Displacement (upper left), velocity (upper right) and acceleration (lower) of the first constituent for several values of the mixture).

constituent. Even, the shape of the velocity and acceleration changes for the largest values.

The displacement and velocity of the second constituent are plotted in Fig. 4 at final time. Again, we observe a quadratic behaviour for the displacement field but the velocity clearly changes its form for values greater than 5. Moreover, we note that this deformation is produced due to the coupling between the two components of the mixture.

Finally, in Fig. 5 we can see the evolution in time of the discrete energy (defined in the previous example), in both natural and semi-log scales, for some values of the parameter a^*. The exponential energy decay is found as expected, being rather similar in all cases.

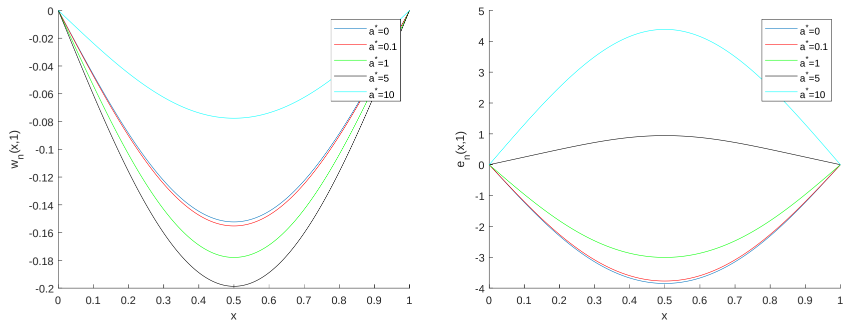

Fig. 4 Example 2: Displacement and velocity of the second constituent at time $T = 1$.

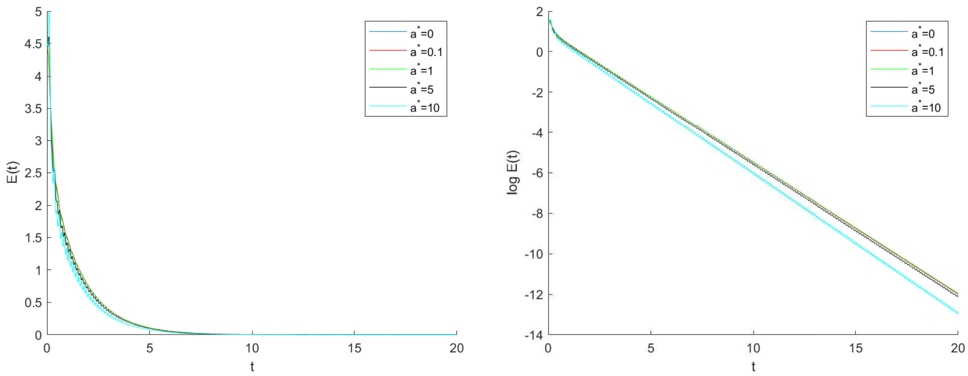

Fig. 5 Example 2: Evolution in time of the discrete energy (natural and semi-log scales) for different values of a^*.

Compliance with Ethical Standards

The authors declare that they have no conflict of interest.

References

1. Abouelregal, A.E., Ahmad, H., Nofal, T.A., Abu-Zinadah, H. Moore-Gibson-Thompson thermoelasticity model with temperature-dependent properties for thermo-viscoelastic orthotropic solid cylinder of infinite length under a temperature pulse, Physica Scripta, 96, 105201 (2021).
2. Abouelregal, A.E., Sedighi, H.M. The effect of variable properties and rotation in a visco-thermoelastic orthotropic annular cylinder under the Moore–Gibson–Thompson heat conduction model, Proceedings of the Institution of Mechanical Engineers, Part L: Journal of Materials: Design and Applications, 235(5), 1004-20 (2021).
3. Abouelregal, A.E., Sedighi, H.M., Shirazi, A.H., Malikan, M., Emereyev, V.A. Computational analysis of an infinite magneto-thermoelastic solid periodically dispersed with

varying heat flow based on non-local Moore–Gibson–Thompson approach, Continuum Mechanics and Thermodynamics, 29, 1-9 (2021).

4. Atkin, R.J., Craine, R.E. Continuum theories of mixtures: basic theory and historical development, Quarterly Journal of Mechanics and Applied Mathematics, 29, 209–244 (1976).
5. Bazarra, N., Fernández, J.R., Quintanilla, R. Analysis of a Moore-Gibson-Thompson thermoelasticity problem, Journal of Computational and Applied Mathematics, 382, 113058 (2021).
6. Bazarra, N., Fernández, J.R., Quintanilla, R. On the decay of the energy for radial solutions in Moore-Gibson-Thompson thermoelasticity, Mathematics and Mechanics of Solids, 26, 1507–1514 (2021).
7. Bedford, A., Drumheller, D.S. Theories of immiscible and structured materials, International Journal of Engineering Science, 21, 863–960 (1983).
8. Bedford, A., Stern, M. A multi-continuum theory of composite elastic materials, Acta Mechanica, 14, 85–102 (1972).
9. Bedford, A., Stern, M. Towards a diffusing continuum theory of composite elastic materials, Journal of Applied Mechanics, 38, 8–14 (1972).
10. Bowen, R.M. Theory of mixtures. In: Continuum Physics III, A.C. Eringen ed., Academic Press, New York, 689–722 (1976).
11. Bowen, R.M., Wiese, J.C. Diffusion in mixtures of elastic materials, International Journal of Engineering Science, 7, 689–722 (1969).
12. Campo, M., Fernández, J.R., Kuttler, K. L., Shillor, M., Viaño, J. M. Numerical analysis and simulations of a dynamic frictionless contact problem with damage, Computer Methods in Applied Mechanics and Engineering, 196(1-3), 476–488 (2006).
13. Ciarlet, P. G. Basic error estimates for elliptic problems. In: Handbook of Numerical Analysis, P.G. Ciarlet and J.L. Lions eds., vol II, 17-351 (1993).
14. Conti, M., Pata, V., Pellicer, M., Quintanilla, R., On the analyticity of the MGT-viscoelastic plate with heat conduction, Journal of Differential Equations, 269, 7862–7880 (2020).
15. Conti, M., Pata, V., Pellicer, M., Quintanilla, R. A new approach to MGT-thermoviscoelasticity, Discrete and Continuous Dynamical Systems, 41, 4645-4666 (2021).
16. Conti, M., Pata, V., Quintanilla, R. Thermoelasticity of Moore-Gibson-Thompson type with history dependence in the temperature, Asymptotic Analysis, 120, 1–21 (2020).
17. Eringen, A.C., Ingram, D.J. A continuum theory of chemically reacting media, International Journal of Engineering Science, 3 197–212 (1965).
18. Fernández, J.R., Quintanilla, R. Moore-Gibson-Thompson theory for thermoelastic dielectrics, Applied Mathematics and Mechanics, 42, 309–316 (2021).
19. Fernández, J.R., Quintanilla, R. On a mixture of a MGT viscous material and an elastic solid, Acta Mechanica, in press (2021), DOI:10.1007/s00707-021-03124-z.
20. Green, A.E., Naghdi, P.M. A dynamical theory of interacting continua,International Journal of Engineering Science, 3, 231–241 (1965).
21. Green, A.E., Naghdi, P.M. A note on mixtures, International Journal of Engineering Science, 6, 631–635 (1968).
22. Ingram, J.D., Eringen, A.C. A continuum theory of chemically reacting media II, International Journal of Engineering Science, 5, 289–322 (1967).
23. Ieşan, D., Quintanilla, R. On the theory of interacting continua with memory, Journal of Thermal Stresses, 25, 1161–1178 (2002).
24. Jangid, K., Mukhopadhyay, S. A domain of influence theorem for a natural stress-heat-flux problem in the Moore-Gibson-Thompson thermoelasticity theory, Acta Mechanica, 232, 1–11 (2021).
25. Jangid, K., Mukhopadhyay, S. A domain of influence theorem under MGT thermoelasticity theory, Mathematics and Mechanics of Solids, 26, 285–295 (2020).
26. Joseph, D.D., Preziosi, L. Heat waves, Reviews of Modern Physics, 61, 41-73 (1989).
27. Kelly, P. A reacting continuum, International Journal of Engineering Science, 2, 129–153 (1964).
28. Ostoja-Starzewski, M., Quintanilla, Q. Spatial behaviour of solutions of the Moore-Gibson-Thompson equation, Journal of Mathematical Fluid Dynamics, 23, 105 (2021).

29. Pellicer, M., Quintanilla, R., On uniqueness and instability for some thermoemechanical problems involving the Moore-Gibson-Thompson equation, Zeitschrift für angewandte Mathematik und Physik, 71 (2020) 84.
30. Quintanilla, R. Moore-Gibson-Thompson thermoelasticity, Mathematics and Mechanics of Solids, 24, 4020–4031 (2019).
31. Rajagopal, K.R., Tao, L. Mechanics of mixtures, In: Series on Advances in Mathematics for Applied Sciences, vol. 35, World Scientific Publishing Co. Inc., River Edge, N. J. (1995).
32. Renardy, M., Hrusa, W.H., Nohel, J.A. Mathematical Problems in Viscoelasticity, Longman Scientific and Techical, London, 1987.
33. Singh, B., Mukhopadhyay, S. Galerkin-type solution for the Moore-Gibson-Thompson thermoelasticity theory, Acta Mechanica, 232, 1273–1283 (2021).