



## STUDENTS' UNDERSTANDING OF DOUBLE INTEGRALS – IMPLICATIONS FOR THE ENGINEERING CURRICULUM

**T Khemane**<sup>1</sup>

University of Cape Town  
Cape Town, South Africa

**P Padayachee**

University of Cape Town  
Cape Town, South Africa

**C Shaw**

University of Cape Town  
Cape Town, South Africa

**Conference Key Areas:** *Mathematics at the heart of Engineering,*

**Keywords:** *Engineering curriculum, mathematics, double integrals, misconceptions*

### ABSTRACT

Mathematics plays a significant role in engineering students' education. To undergraduate engineering students, calculus concepts are foundational to their engineering courses. One such concept is the double integral. It is thus important to ensure that students not only learn this concept but also engage to understand it and are able to apply this knowledge in relevant engineering courses. This research paper focuses on the following two components: Firstly, the relevance of double integrals to the engineering curriculum. And secondly, students' understanding of the double integral concept.

We present the relevance of double integrals in the engineering curriculum by looking at the use of this concept in different engineering fields. We explored students' understanding of double integrals and administered a test to 35 second year engineering students enrolled in an undergraduate Calculus III course. In a qualitative study, the performance of students was used to analyse the type of misconceptions they have in double integration. The findings reveal that the students encounter difficulties with graphical representation of surfaces and region of integration. In addition, students struggle with changing the order of integration and performing the integration process. While some of these errors are conceptual, others are really due to carelessness in the procedure.

---

<sup>1</sup> *Corresponding Author*

*T Khemane*

*Khmta029@myuct.ac.za*



Further analysis indicate that some misconceptions are a result of misunderstandings in prerequisite courses. The results will be useful to Mathematics educators who are keen in finding sources of misconceptions. The research may be used in designing functional teaching and learning instruments to rectify misconceptions in double integrals.



## 1 INTRODUCTION

### 1.1 Background

Undergraduate engineering students taking mathematics courses in the first and second years of their programmes are introduced to calculus concepts as a foundation for other engineering courses. Thus, for their success in professional engineering programmes and their future practice, the understanding of such mathematical concepts and the ability to interpret and apply them in future courses is essential for engineering students.

In South Africa, engineering students enrolled in a four-year programme encounter mathematics courses, of which calculus is a large part, in their first two years of study. Students are introduced to differential and integral calculus and their related applications during their first-year calculus courses [1] and this concept is further expanded on in the second-year vector calculus course. At the University of Cape Town, it is mandatory for engineering students to complete three semesters of calculus.

Integral calculus consists of the study of fundamental concepts useful in engineering. It also has various applications in fields such as physics and engineering [2] and is used widely in courses such as thermodynamics, mechanics, and hydraulics. For example, in thermodynamics, energy is a function of pressure, temperature, volume, and entropy; in engineering, density may be a function of  $x, y$ , and  $z$  [3]. The significance of calculus in engineering is highlighted in a study [4] where the correlation between students' understanding of derivative and integral calculus with thermodynamics is researched. The aforementioned study reports that results of the thermodynamics subject were affected by students' comprehension of derivatives and integrals and makes a convincing argument to improve calculus learning in order to increase the quality of achievements in thermodynamics and moreover other topics that involve integrals. It stresses the importance of monitoring engineering students' understanding and experience of calculus concepts.

The body of research on calculus topics show that students have difficulties in limits, derivatives, and integrals. Their difficulties are a result of various errors and certain misconceptions in the calculus concepts [5]. Common errors, persistent misconceptions and misunderstandings should be identified to inform and improve the teaching and learning of mathematics to facilitate student success. While researchers have documented students' understanding of various concepts in single variable calculus such as on difficulties and errors made by students in the topic of limits and integration [5,6], fewer studies exist about their understanding of multivariable calculus concepts making a case for this research study[3].

Double integration is a fundamental concept in many engineering applications and widely used in courses across the engineering disciplines. Integral calculus is a prerequisite for further coursework as it consists of important concepts such as the fundamental theorem of calculus and integral-area relationships. The focus of this

research is to investigate the misconceptions in double integration in vector calculus among undergraduate engineering students.

## 1.2 Relevance of double integrals to Engineering

In the Mechanical engineering curriculum, vector calculus is a prerequisite course for the two third year courses namely control systems, and stress analysis and materials. In the latter, the course content covers topics such as theories of failure due to different loading conditions and material processes.

In theories of failure, finding beam deflections by double integration is one of the basic calculations students should know. They have to use boundary conditions and integrate twice; firstly, to find the slope and secondly the deflection. At this point, students apply double integration knowledge from their vector calculus course to calculate beam deflections.

For a cantilever beam in Figure 1 subjected to a combination of loading, the method of double integration is used to determine the deflection at the free end by integrating this deflection equation  $EI \frac{d^2y}{dx^2} = 5 - \frac{20x^2}{2}$  twice.

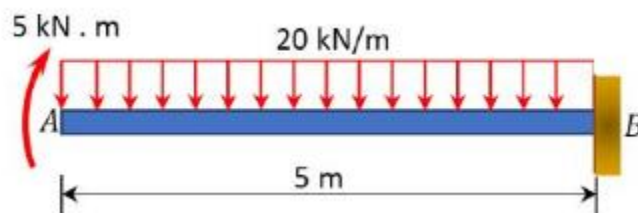


Fig. 1. Beam deflection calculation

With this example, it is clear that students have to understand the concept of double integration to minimise carrying misconceptions to future courses like the stress analysis and materials. Civil engineering students also apply the same concept in calculating beam and column deflections.

## 2 METHODOLOGY

### 2.1 Introduction

A mixed method approach was used drawing on quantitative and qualitative research data. Statistical data analysis was used to describe students' performance in the test on double integrals. The qualitative data was in the form of document analysis of the test, including their reflective notes and conducting interviews with students.

### 2.2 Participants

The research participants were second-year undergraduate engineering students at the University of Cape Town taking vector calculus. All participants had completed two first year calculus courses. Students in the class were invited to participate in the study,

with all students agreeing that their class test could be used. Seven students agreed to be interviewed. They had given written consent and allowed the researcher to use their responses in the study.

### 2.3 Research method

The researcher designed the double integrals test to investigate students' understanding of and their reflections on the concept. Five questions were prepared for the test, discussed with the second researcher, and refined thereafter. The test included a reflection question at the end of each of the five questions, added to allow students to make sense of the material or experiences and the conditions that shaped those experiences. In the research study, the reflection questions help identify how the student perceived their response to the question.

Students wrote the test in 75 minutes. After the test marking was completed, an interview schedule was developed to learn about students' perspectives, experiences, and mathematics knowledge that shaped their understanding of double integrals. Amidst the circumstances presented by the pandemic, and students' unavailability for a physical meeting, the interviews were conducted and recorded using video conferencing technology. The qualitative data collected were thematically analysed through comparing and contrasting students' responses and interview results. These themes were students' misconceptions.

## 3 RESULTS

### 3.1 Performance statistics

The quantitative data analysis is presented as performance statistics of the test. This is followed by the analysis of the test responses, students' reflections and interviews culminating in the identification of the double integral misconceptions students displayed.

Overall statistical analysis on the test is presented below. Figure 2 shows the distribution of marks. The minimum mark obtained in the test was 2 and the maximum marks obtained were 28. The mean and the standard deviation were  $16.06 \pm 6.86$  marks. Eighteen students (51%) achieved a passing grade in the test. In as much as a fair performance is observed, the overall test performance, looking at misconceptions students have and the errors they made, was not satisfactory.

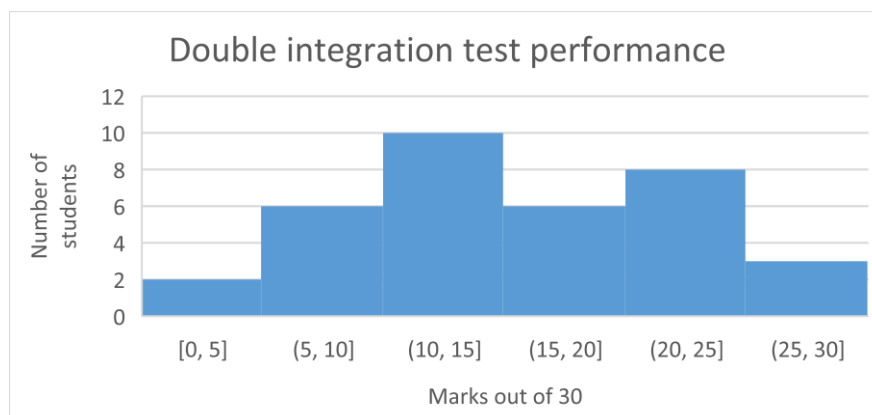


Fig. 2. Test performance

## 3.2 Misconceptions

Misconceptions are recognised as conceptual difficulties that students have. These may delay the learning process and hinder mathematical conceptual understanding. In this research, the following seven misconceptions were identified: Graphical representation of surfaces and the region of integration, difficulty with changing the order of integration, difficulty in setting up the integral, algebraic difficulties, treating a double integral as a triple integral, and changing the coordinate system from cartesian to polar coordinates. The first two prominent misconceptions will be discussed in the following sections.

### 3.2.1 Misconception 1: Graphical representation of surfaces and the region of integration

Graphical representation of the region of integration and of the three-dimensional surfaces was the primary misconception identified from the test analysis. This misconception is prevalent in all test questions. Sketching the region of integration is fundamental to setting up an integral to be evaluated. Despite students' knowledge of integration and integration techniques, incorrect double integrals setups will result in incorrect results. To this end, students' prior knowledge of curve sketching is necessary.

Curve sketching is a large focus in first-year calculus and with students sketching planes and intersecting planes. This prior knowledge would have been extended to include an introduction to 3D surfaces in second year vector calculus where students are required to recognise the equations and names of the typical surfaces (cylinders and quadric surfaces) and be able to sketch them. Sketching the region of integration requires the students' ability to draw curves and lines.

As an example, in question 1, students were required to sketch a plane  $z = 5$  above the rectangle  $R = \{x, y: 0 \leq x \leq 2, 1 \leq y \leq 4\}$ . It is common practice to sketch the region of integration before setting up the Integral. Figure 3 shows Mary's correct response

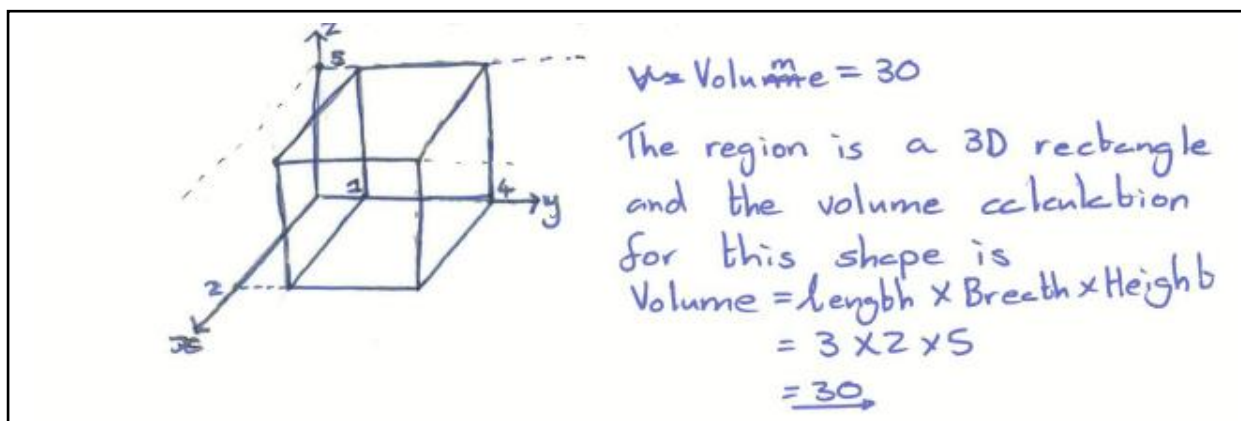


Fig. 3. Mary's correct response to sketching  $z=5$  above the given region

In figure 3, part of the plane  $z = 5$  above the region of integration is drawn. The region of integration lies in the  $xy$  plane and is defined by the rectangle. Five students were not able to draw the plane and the region of integration at all. When interviewed, some students mentioned that they knew that they had to draw a plane but were not able to draw it in three dimensions. Jessica drew a line to represent a plane in three dimensions and later during an interview said that although she knew that  $z = 5$  represented a plane, it was difficult to represent a plane in three dimensions on paper.

Figure 4a shows Jessica's region of integration. The region shows that the limits in the  $x$  and  $y$  axes were swapped and emphasises a further issue of plotting intervals incorrectly. When interviewed, the student demonstrated that they understood the required region. Figure 4b shows the plane  $z = 5$ . The student could tell that this particular plane lies flat at a value of  $z = 5$  but could not represent it in three dimensions. Instead of a plane, a line was drawn above an unknown region of integration. Since this misconception is fundamental, it is important to ensure that first year calculus equips students with necessary curve sketching skills and revision of this should be done at the beginning of vector calculus.

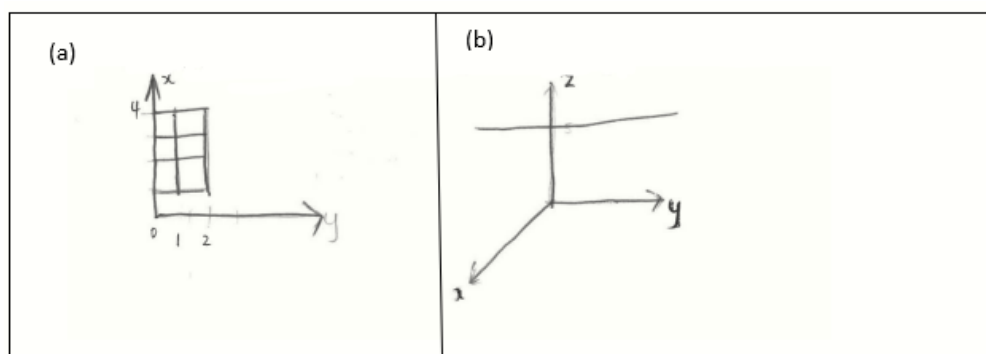


Fig. 4. Jessica's region of integration and plane sketches

Some students could not interpret the plane  $z = 5$  as a function  $f(x, y)$  with a value of 5 for all values of  $x$  and  $y$ . They were used to the plane with the following equation  $ax + by + cz = k$ , yet they could not tell in the case of  $z = 5$  that  $a, b = 0, c = 1$  and  $k = 5$ . The understanding of what a double integral represents was also crucial in interpreting the sketch. Students with ability to interpret a double integral as being used to calculate volume under a surface could have drawn the cuboid in figure 3.

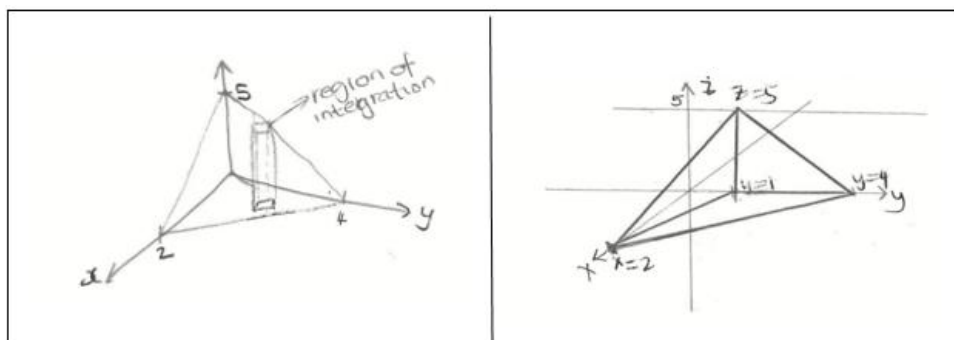


Fig. 5. Incorrect sketches from two students

One way of understanding double integrals is by relating them to the Riemann sum. Thus, students can set up the integral  $\iint_R 5dA$  by thinking about  $\sum \sum f(x_i y_j) \Delta A$  as an area (determined by the region of integration in the  $xy$  plane) multiplied by the height function  $z=5$ . Interpreting double integrals this way can be helpful in sketching the three-dimensional surfaces.

In some cases, students' interpretation and understanding of the limits of integration can be a determining factor in sketching the region of integration. In question 1 for example, students were given that  $0 \leq x \leq 2$  and  $1 \leq y \leq 4$ . To draw the rectangular region in the  $xy$  plane, the limits can be interpreted as the area enclosed by the following straight lines  $x = 0, x = 2, y = 1$  and  $y = 4$ . This rectangle can be extended upwards to have a cuboid representative of the plane  $z = 5$  above the region of integration. Figure 5 reveals, that the two students could not interpret the limits of integration and hence their sketches were not representative of a rectangular region of integration even though this was indicated in the question.

In addition, it is important for students to distinguish a rectangular region from non-rectangular regions. Students with non-rectangular regions ended up with limits of integration representative of a rectangular region. John's region of integration shown in the double integral in figure 6 represents a rectangle not a circle as in the picture on the left. When interviewed, students expressed that they often struggle identifying the upper and lower limits of integration even with the correct sketch. This, they say it is because they want simpler limits to work with when integrating. Students should understand that the limits of integration describe an area and they should express bounds of this area. Taken in the  $x$  direction first the bounds are  $f_1(y) \leq x \leq f_2(y)$  and the limits in the  $y$  direction will range from  $a \leq y \leq b$ ,  $a, b$  are elements of real numbers.

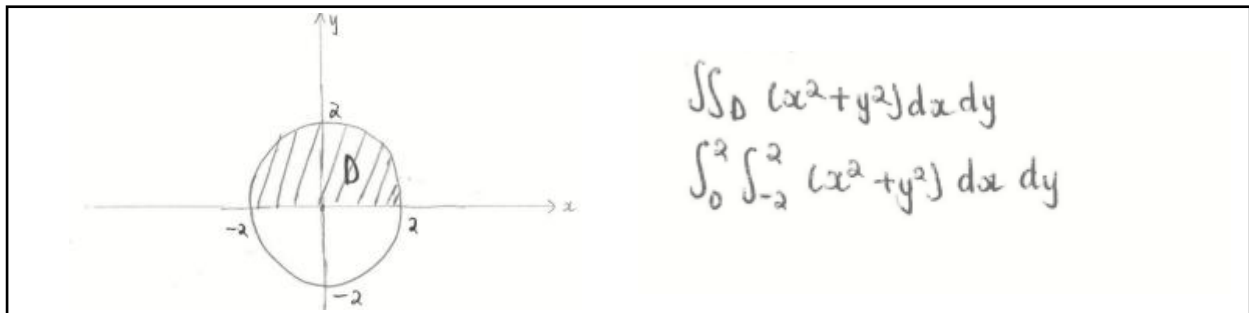


Fig. 6. John's limits of integration representative of a rectangle

### 3.2.2 Misconception 2 : Difficulty in Changing the Order of Integration

The change of the order of integration is important in cases where the given integrand is complicated and results in the integration being difficult or not possible to compute.

The integral  $\int_c^d \int_a^b dx dy$  is written in a way that the integration with respect to  $x$  is carried out before the integration with respect to  $y$ . Assuming that the limits of integration were constants, changing the order of integration will result in simply  $\int_a^b \int_c^d dy dx$ . The difficulty is experienced when the region of integration is not



rectangular. To achieve changing the order of integration in this context, students are required to sketch the region of integration before determining new limits of integration. This misconception arose in response to question 4 of the test. Students were required to change the order of integration of the following double integral  $\int_0^2 \int_{y^2}^4 ye^{x^2} dx dy$ . They were to begin by drawing the region of integration with the resultant expected integral of this form ;  $\int_0^4 \int_0^{\sqrt{x}} ye^{x^2} dy dx$ . When asked if they were able to evaluate this integral, twenty-six students responded with “No”, mostly citing that the integrand  $ye^{x^2}$  is too complex to integrate with respect to  $x$ . Nine students however commented that it was possible to evaluate the integral since the limits of integration were given. Two of the nine students highlighted incorrectly that knowledge of integration by parts could be employed to solve the question.

Confirming misconception 1, 30% of the class presented incorrect sketches and therefore were not successful in changing the order of integration. Even though this cohort of students had difficulties with sketching the region of integration, they acknowledged its importance in identifying limits of integration. Some students' responses revealed that they did not understand the limits of integration and what these represented. With the given integral  $\int_0^2 \int_{y^2}^4 ye^{x^2} dx dy$ , it is expected that students interpret the inner integral limits as  $y^2 \leq x \leq 4$ . The outer limits will therefore be  $0 \leq y \leq 2$ . The four inequalities culminate to produce the region of integration on the  $xy$ -plane. Jeffry's region of integration and the integral setup revealed this misconception. In figure 7, Jeffry represented the parabola  $x = y^2$  as a straight line. Furthermore, the student did not understand that, when setting up the integral, changing the order of integration requires the limits to also change to first in the  $y$  direction and followed by the  $x$  direction. With the current limits, Jeffry suggests that the boundaries are  $4 \leq y \leq y^2$  and  $0 \leq x \leq 4$ . The equation  $y = y^2$  is the case for the two points  $y = 0$  and  $y = 1$ .

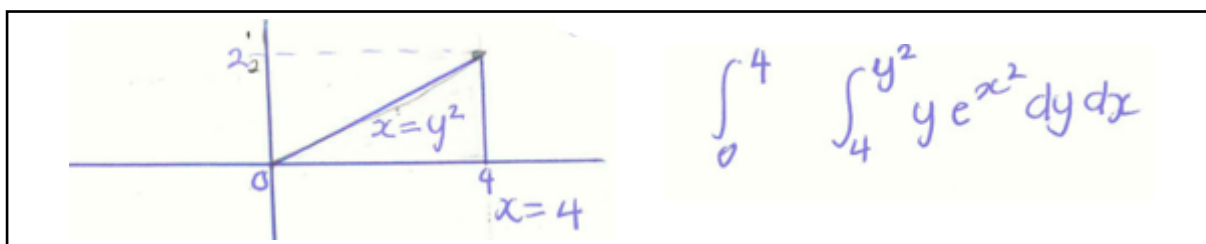


Fig. 7. Jeffry's change of order of integration

A few students did not understand that changing the order of integration involves changing the limits of integration accordingly. They simply changed the  $dx dy$  to  $dy dx$  without altering the corresponding limits of integration. This is a serious misconception and, it was impossible for such students to succeed in effectively changing the order of integration.

In the question, three students not only denoted the incorrect region of integration but also an incorrect integrand. This is demonstrated through Jordan's response in figure 8. The integrand in question is the function  $f(x, y) = ye^{x^2}$ . In changing the integrand to  $y\ln(x^2)$ , Jordan effectively changed the question. It is not clear how the student arrived at this integrand. Because Jordan did not change the order and the limits, it is concluded that the student did not understand what it entailed to perform the change of order of integration.


$$\int_0^2 \int_{y^2}^4 y \ln x^2 dx dy$$

Fig. 8. Jordan's change of order of integration

#### 4 SUMMARY

This study highlights the relevance of double integrals to the engineering curriculum. Therefore, students' understanding of the double integral concept is non-negotiable if they are to avoid difficulties in engineering courses. To identify and address misconception in double integration, difficulties experienced by vector calculus students in double integrals were studied. The prominent misconceptions included difficulty with graphical representation of surfaces and regions of integration, setting up of the double integral and changing the order of integration. It is recommended that the software application for the visualization of multivariate functions should be explored further in the teaching of this course to better students' achievements in determining the domain of integration and the integration bounds in order to solve double integrals.

Misconceptions in double integration also affect students' performance in other sections in vector calculus such as line integrals, surface integrals and Stokes' Theorem, making it equally important for teaching to focus on these misconceptions to improve performance in vector calculus

Researching students' difficulties help educators design appropriate teaching and learning activities. It is suggested that vector calculus educators liaise with first year calculus educators to discuss misconceptions resulting from prerequisite knowledge.



## REFERENCES

- [1] Maharaj, A (2014), An APOS analysis of natural science students' understanding of integration, *Journal of Research in Mathematics Education*, Vol. 3, No. 1, pp. 54-73.
- [2] Jones, SR (2015), Areas, anti-derivatives, and adding up pieces: Definite integrals in pure mathematics and applied science contexts, *Journal of Mathematical Behavior*, Vol. 38, pp. 9-28.
- [3] Dorko, A, Weber, E (2014), Generalising calculus ideas from two dimensions to three: how multivariable calculus students think about domain and range, *Research in Mathematics education*, Vol. 16, No. 3, pp. 269-287.
- [4] Saepuzaman, D (2016), Correlation between students' Understanding on Derivative and Integral Calculus with Thermodynamics, *1st International Conference of Mathematics and Science Education*, Atlantis press, Indonesia, Vol 57, pp. 218-220
- [5] Li Li, V, Hazizah Julaihi, N, Howe Eng, T (2017), Misconceptions and errors in learning integral calculus, *Asian Journal of University Education*, Vol. 3, No. 1, pp. 17-39
- [6] Martinez-Planell, R, Trigueros, M (2020), Students' understanding of Riemann sums for integrals of functions of two variables, *Journal of Mathematical Behavior*, Vol. 59, pp. 1-26