Showing non-realizability of spheres by distilling a tree

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Abstract

In [Zhe20a], Hailun Zheng constructs a combinatorial 3-sphere on 16 vertices whose graph is the complete 4-partite graph $K_{4,4,4,4}$. Such a sphere seems unlikely to be realizable as the boundary complex of a 4-dimensional polytope, but all known techniques for proving this fail because there are just too many possibilities for the $16 \times 4 = 64$ coordinates of its vertices. Known results [PPS12] on polytopal realizability of graphs also do not cover multipartite graphs.

In this paper, we level up the old idea of Grassmann–Plücker relations, and assemble them using integer programming into a new and more powerful structure, called *positive Grassmann–Plücker trees*, that proves the non-realizability of this example and many other previously inaccessible families of simplicial spheres. See [Pfe20] for the full version.

1 Introduction

A *simplicial $(d-1)$-sphere* Σ is a simplicial complex homeomorphic to a $(d-1)$-dimensional sphere. We say that Σ is *non-realizable* if there does not exist a (necessarily simplicial) d-polytope whose boundary complex is isomorphic to Σ.

Example 1 *The following list of* 19 *facets defines a* 3*-sphere* Σ *on* 8 *vertices with* 27 *edges and* 38 *triangles:*

```
+[0123] -[0124] +[0135] -[0146]
+[0157] -[0167] -[0234] +[0345]
-[0456] +[0567] +[1237] -[1246]
-[1267] +[1357] -[2347] +[2456]
-[2457] -[2567] +[3457]
```

The signs define an orientation of Σ.

How to prove that this 3-sphere is non-realizable? In this case, the venerable *Grassmann–Plücker relations* suffice. These are polynomial relations that are satisfied by the determinants of any $d+1$ points of a realization of Σ in d-space. The most basic ones are the *three-term GP relations* $\Gamma(S|ijkl) = 0$ with

$$\Gamma(S|ijkl) = [Sij][Skl] - [Sik][Sjl] + [Sil][Sjk], \qquad (1)$$

which are valid for any subset $S \subset \{1, 2, \dots, n\}$ of size $(d-1)$, and any four indices $i, j, k, l \in \{1, 2, \dots, n\} \setminus S$. A typical 3-term GP relation in our example is

$$\begin{aligned} 0 &= \Gamma(045|1267) \\ &= [04512][04567] - [04516][04527] + [04517][04526]. \end{aligned}$$

By permuting the entries inside these determinants, we can change their signs — even permutations will leave the sign unchanged, while odd permutations will flip it. A particularly advantageous way of changing the signs is

$$\begin{aligned} 0 &= \Gamma(045|1267) \\ &= \quad (-1)[01425](-1)[05674] \\ &\quad - \quad [04651](-1)[24750] \\ &\quad + \quad [01574] \quad [24560] \\ &= [01425][05674] + [04651][24750] + [01574][24560]. \end{aligned}$$

The advantage of rewriting $\Gamma(045|1267)$ in this way is that now *all determinants are positive*! For example, $[01425] > 0$ because $[01425] = -[01245]$, and $[01245]$ is the "signed slack" of the point x_5 with respect to the facet $[0124]$ in the supposed convex realization of Σ; but the orientation of 0124 in Σ is negative by the above list. The other determinants can be similarly checked to be positive.

But this expresses zero as a positive combination of positive numbers, which is impossible; therefore, there is no convex realization of this 3-sphere.

2 The non-realizability of Zheng's 3-sphere

To explain why Zheng's combinatorial 3-sphere Z is important, let's fix definitions. A $(d-1)$-dimensional simplicial complex is *balanced* if its 1-skeleton is *d-colorable* in the graph-theoretic sense, i.e., its vertices can be colored with d colors in such a way that the endpoints of all edges receive different colors. Moreover, a $(d-1)$-dimensional balanced simplicial complex Σ is *balanced k-neighborly* if each k-subset of the vertex set that contains at most one vertex of each color class is actually a face of Σ.

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Now we can say why Zheng's example Z is important — in fact, it is important in at least *two* ways.

First, there has been a lot of work on analogies between combinatorial data in the balanced and the non-balanced settings [JM18, JMNS18, Ven19]. For example, one would like to have a balanced analogue of the celebrated Upper Bound Theorem by McMullen and Stanley. For this, in particular one would like balanced analogues of the extremal examples to even *exist*, i.e., one would like to construct infinite families of balanced k-neighborly simplicial spheres. What Zheng shows in [Zhe20a], however, is that (i) there is no balanced 2-neighborly homology 3-sphere on 12 vertices; (ii) there is no balanced 2-neighborly homology 4-sphere on 15 vertices; (iii) but taking suspensions over Z yields a balanced 2-neighborly homology $(3+m)$-sphere on $16+2m$ vertices for every $m \geq 0$.

The second reason why her example is important lies in the fact that in [PPS12], the authors study which graphs are realizable as the 1-skeleton of polytopes. The case of multipartite graphs was not treated there, and to date the only polytope whose graph is known to be the multipartite graph $K_{4,4,4,4}$ is the 4-dimensional cross polytope.

We can now show for the first time that Z is not realizable as the boundary complex of a convex polytope, and therefore that Z does not yield a new polytope whose graph is $K_{4,4,4,4}$.

Theorem 2 *Zheng's balanced sphere Z is not polytopal.*

Proof. An orientation of the facets of Z is given by the following list:

-[048c]	+[048e]	+[049c]	-[049d]	+[04ad]
-[04ae]	+[059d]	-[059f]	-[05ad]	+[05ae]
-[05be]	+[05bf]	+[068c]	-[068e]	-[069c]
+[069e]	-[079e]	+[079f]	+[07be]	-[07bf]
+[148c]	-[148e]	+[14ae]	-[14af]	-[14bc]
+[14bf]	-[158c]	+[158d]	-[159d]	+[159f]
+[15bc]	-[15bf]	+[168e]	-[168f]	-[16ae]
+[16af]	-[178d]	+[178f]	+[179d]	-[179f]
-[24ad]	+[24af]	+[24bd]	-[24bf]	+[258c]
-[258d]	-[25ac]	+[25ad]	-[268c]	+[268d]
+[269c]	-[269e]	+[26ae]	-[26af]	-[26bd]
+[26bf]	-[279c]	+[279e]	+[27ac]	-[27ae]
-[349c]	+[349d]	+[34bc]	-[34bd]	+[35ac]
-[35ae]	-[35bc]	+[35be]	-[368d]	+[368f]
+[36bd]	-[36bf]	+[378d]	-[378f]	+[379c]
-[379d]	-[37ac]	+[37ae]	-[37be]	+[37bf]

We prove the non-realizability of Z in a similar way as in Example 1, but we allow GP relations in which we do not have full control over the signs. For example, we can express

$$\begin{aligned} 0 &= \Gamma(\texttt{18f|56bd}) \\ &= [\texttt{18f56}][\texttt{18fbd}] - [\texttt{18f5b}][\texttt{18f6d}] + [\texttt{18f5d}][\texttt{18f6b}] \\ &= [\texttt{16f85}][\texttt{18bdf}]^{?} + [\texttt{15fb8}][\texttt{16f8d}] + [\texttt{158df}][\texttt{16f8b}], \end{aligned}$$

where all (black) signs are known to be positive in any realization of Z, but the (red) sign with a question mark can be either positive or negative. For instance, [16f8] is a positively oriented facet because the orientation of [168f] in the given list is negative, and this implies that all determinants of the form [16f8x] must be positive, because all points x lie on the same side of the facet in any convex realization. On the other hand, no four-element subset of $[\texttt{18bdf}]^{?}$ appears in the list of facets of Z, so the sign of that determinant could be positive or negative.

To balance this uncertainty, we look for another GP relation that involves $[\texttt{18bdf}]^{?}$. A candidate is

$$\begin{aligned} 0 &= \Gamma(\texttt{1bf|48de}) \\ &= [\texttt{1bf48}][\texttt{1bfde}] - [\texttt{1bf4d}][\texttt{1bf8e}] + [\texttt{1bf4e}][\texttt{1bf8d}] \\ &= [\texttt{14bf8}][\texttt{1bdef}]^{?} + [\texttt{14bfd}][\texttt{18bef}]^{?} - [\texttt{14bfe}][\texttt{18bdf}]^{?}. \end{aligned}$$

On the one hand, we can eliminate the unknown sign $[\texttt{18bdf}]^{?}$ by forming the polynomial combination

$$[\texttt{14bfe}] \cdot \Gamma(\texttt{18f|56bd}) \; + \; [\texttt{16f85}] \cdot \Gamma(\texttt{1bf|48de}),$$

but on the other hand we now have two additional unknown signs to worry about.

Somewhat surprisingly, we are able to bring this process to a closure by forming the following polynomial combination of GP relations:

$$\begin{aligned} [\texttt{36fb5}]\Bigg([\texttt{36fb4}]\Bigg([\texttt{14bf3}]\Bigg([\texttt{16f85}]\big([\texttt{14bfd}](-\Gamma(\texttt{18f|46be})) \\ +[\texttt{16f84}]\,\Gamma(\texttt{1bf|48de})\big) \\ +[\texttt{16f84}]\,[\texttt{14bfe}]\,\Gamma(\texttt{18f|56bd})\Bigg) \\ +[\texttt{16f84}]\,[\texttt{14bf8}]\,[\texttt{16f85}]\,\Gamma(\texttt{1bf|34de})\Bigg) \\ +[\texttt{16f84}]\,[\texttt{14bf8}]\,[\texttt{14bfe}]\,[\texttt{16f85}](-\Gamma(\texttt{3bf|146d}))\Bigg) \\ +\,[\texttt{16f84}]\,[\texttt{14bf8}]\,[\texttt{14bfd}]\,[\texttt{16f85}]\,[\texttt{36fb4}]\,\Gamma(\texttt{3bf|156e}) \quad . \end{aligned}$$

It is encoded in the *positive Grassmann–Plücker tree* in Figure 1, and multiplying it out as in Figure 2 proves the non-realizability of Z. ☐

One can check that arranging the GP polynomials into a tree, i.e., a graph without cycles, guarantees that the final certificate does not depend on the order in which the certificate is multiplied out.

3 Finding positive Grassmann–Plücker trees

How do we go about finding such certificates? First, we restrict to a useful subclass of GP relations that permit algebraic elimination:

Definition 3 *A three-term GP relation as in* (1) has no adjacent unknown solids *if no two determinants that are multiplied together have unknown sign.*

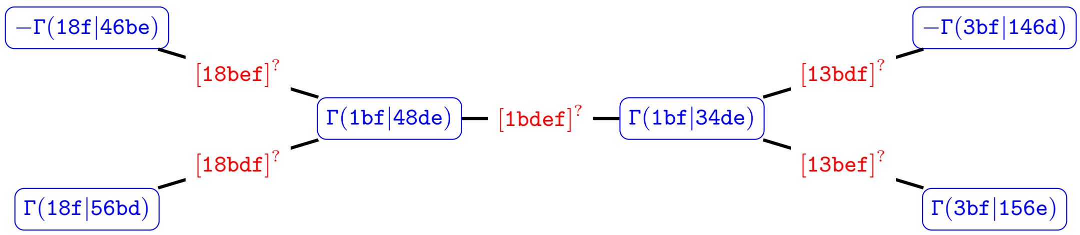

Figure 1: Grassmann–Plücker tree proving the non-realizability of Zheng's 3-sphere

$$
\begin{aligned}
0 = \; & [\texttt{36fb5}]\Big([\texttt{36fb4}]\Big([\texttt{14bf3}]\Big([\texttt{16f85}]\big([\texttt{14bfd}](-[\texttt{16f84}][\texttt{18bef}]^{?} + [\texttt{14bf8}][\texttt{16f8e}] + [\texttt{14e8f}][\texttt{16f8b}]) \\
& \qquad + [\texttt{16f84}]([\texttt{14bf8}][\texttt{1bdef}]^{?} + [\texttt{14bfd}][\texttt{18bef}]^{?} - [\texttt{14bfe}][\texttt{18bdf}]^{?})\big) \\
& \qquad + [\texttt{16f84}]\,[\texttt{14bfe}]\big([\texttt{16f85}][\texttt{18bdf}]^{?} + [\texttt{15fb8}][\texttt{16f8d}] + [\texttt{158df}][\texttt{16f8b}]\big)\Big) \\
& \qquad + [\texttt{16f84}]\,[\texttt{14bf8}]\,[\texttt{16f85}]\big(-[\texttt{14bf3}][\texttt{1bdef}]^{?} + [\texttt{13bdf}]^{?}[\texttt{14bfe}] - [\texttt{13bef}]^{?}[\texttt{14bfd}]\big)\Big) \\
& \qquad + [\texttt{16f84}]\,[\texttt{14bf8}]\,[\texttt{14bfe}]\,[\texttt{16f85}]\big([\texttt{14bf3}][\texttt{36fbd}] + [\texttt{36fb1}][\texttt{34dbf}] - [\texttt{13bdf}]^{?}[\texttt{36fb4}]\big)\Big) \\
& \qquad + [\texttt{16f84}]\,[\texttt{14bf8}]\,[\texttt{14bfd}]\,[\texttt{16f85}]\,[\texttt{36fb4}]\big([\texttt{15fb3}][\texttt{36fbe}] + [\texttt{36fb1}][\texttt{35bef}] + [\texttt{13bef}]^{?}[\texttt{36fb5}]\big) \\
= \; & [\texttt{14bf3}]\,[\texttt{16f84}]\,[\texttt{14bf8}]\,[\texttt{14bfe}]\,[\texttt{16f85}]\,[\texttt{36fb5}]\,[\texttt{36fbd}] + [\texttt{14bf3}]\,[\texttt{16f84}]\,[\texttt{14bfe}]\,[\texttt{15fb8}]\,[\texttt{16f8d}]\,[\texttt{36fb4}]\,[\texttt{36fb5}] \\
& + [\texttt{14bf3}]\,[\texttt{16f84}]\,[\texttt{14bfe}]\,[\texttt{158df}]\,[\texttt{16f8b}]\,[\texttt{36fb4}]\,[\texttt{36fb5}] + [\texttt{14bf3}]\,[\texttt{14bf8}]\,[\texttt{14bfd}]\,[\texttt{16f85}]\,[\texttt{16f8e}]\,[\texttt{36fb4}]\,[\texttt{36fb5}] \\
& + [\texttt{14bf3}]\,[\texttt{14e8f}]\,[\texttt{14bfd}]\,[\texttt{16f85}]\,[\texttt{16f8b}]\,[\texttt{36fb4}]\,[\texttt{36fb5}] + [\texttt{15fb3}]\,[\texttt{16f84}]\,[\texttt{14bf8}]\,[\texttt{14bfd}]\,[\texttt{16f85}]\,[\texttt{36fb4}]\,[\texttt{36fbe}] \\
& + [\texttt{36fb1}]\,[\texttt{16f84}]\,[\texttt{14bf8}]\,[\texttt{14bfd}]\,[\texttt{16f85}]\,[\texttt{36fb4}]\,[\texttt{35bef}] + [\texttt{36fb1}]\,[\texttt{16f84}]\,[\texttt{14bf8}]\,[\texttt{14bfe}]\,[\texttt{16f85}]\,[\texttt{34dbf}]\,[\texttt{36fb5}]\;.
\end{aligned}
$$

Figure 2: The certificate derived from Figure 1. The sign of each determinant with a "?" can be different in different realizations, but the certificate is arranged in such a way that they all cancel. After multiplying out, all surviving determinants are known to be positive in any realization, but the whole certificate must sum to zero. Since this can't happen, the realization cannot exist.

Next, we set up the *GP graph* in which we will search for our certificate. Its nodes are

$$V_\Sigma = \left\{ \begin{array}{r} \pm\Gamma(S|ijkl) : \Gamma(S|ijk\ell) \text{ has no two adjacent} \\ \text{unknown solids} \end{array} \right\},$$

and the edges are labelled with the set $\mathfrak{S}(\Sigma)$ of *normal forms* [Pfe20, Definition 3.2] of solids of Σ. There can be multiple edges between two nodes, but they have different labels: two nodes $\Gamma, \Gamma' \in V_\Sigma$ are joined with an edge $(\overline{\pi}, \{\Gamma, \Gamma'\})$ labelled $\overline{\pi} \in \mathfrak{S}(\Sigma)$ in $E_\Sigma \subseteq \mathfrak{S}(\Sigma) \times \binom{V_\Sigma}{2}$ iff they share a solid π such that $\sigma_i = -\sigma_i'$, where σ_i, σ_i' are the *canonical signs* [Pfe20, Definition 4.4] of the terms containing π in Γ, Γ'.

Said differently, each GP polynomial is connected to many other GP polynomials, and each connecting edge is labelled with an unknown solid common to both polynomials. At each node, there can be at most 3 different labels, but potentially thousands of incident edges with those labels.

Our goal is to find a tree in this graph in such a way that each node has exactly one incident edge labelled with each label occurring at that node. The task is therefore to distill the lucky nodes out of this graph, and for each lucky node the up to three lucky edges out of the thousands of candidates; oh, and we'd like the resulting tree to be as small as possible.

For this, we solve the integer program on the integer indicator variables $\{x_\Gamma : \Gamma \in V_\Sigma\}$ and $\{x_e : e = (\overline{\pi}, \{\Gamma, \Gamma'\}) \in E_\Sigma\}$ defined in Figure 3. The inequalities for these variables $x_\Gamma,\ x_e \in \{0,1\}$ have the following interpretation:

- (2) ensures that both endpoints of an edge present in the solution are present;
- (3) ensures that at most one edge is selected between two selected nodes;
- (4) forces the solution to be a tree with at least one node;
- (5) ensures that if a node Γ with an unknown sign $\overline{\pi}^{?}$ is present in the solution, then there is exactly one edge of that label incident to Γ.

4 More results

We have implemented a search for positive GP trees in the software framework `polymake` [GJ00]. With this

$$\min \sum_{\Gamma \in V_\Sigma} x_\Gamma \quad \text{s.t.} \qquad 2 \sum_{\overline{\pi}:\ e=(\overline{\pi},\{\Gamma,\Gamma'\})\in E_\Sigma} x_e \ \leq\ x_\Gamma + x_{\Gamma'} \qquad \text{for each } \{\Gamma,\Gamma'\} \in \binom{V_\Sigma}{2} \tag{2}$$

$$\sum_{\overline{\pi}:\ e=(\overline{\pi},\{\Gamma,\Gamma'\})\in E_\Sigma} x_e \ \leq\ 1 \qquad \text{for each } \{\Gamma,\Gamma'\} \in \binom{V_\Sigma}{2}, \tag{3}$$

$$1 + \sum_{e\in E_\Sigma} x_e \ =\ \sum_{\Gamma\in V_\Sigma} x_\Gamma \tag{4}$$

$$\sum_{\Gamma':\ e=(\overline{\pi}^?,\{\Gamma,\Gamma'\})\in E_\Sigma} x_e \ =\ x_\Gamma \qquad \text{for all } \Gamma \in V_\Sigma, \text{for all unknown } \overline{\pi}^? \in \Gamma \tag{5}$$

Figure 3: The integer program for finding positive Grassmann–Plücker trees

implementation, we can prove the non-realizability of several previously inaccessible families.

4.1 Topological Prismatoids

In [CS19], Francisco Criado and Francisco Santos introduced *topological prismatoids*, a combinatorial abstraction of the geometric prismatoids used by Santos [San12] to construct counterexamples to the Hirsch conjecture. Criado and Santos construct four combinatorially distinct non-d-step topological 4-dimensional prismatoids on 14 vertices, referred to as #1039, #1963, #2669 and #3513, which imply the existence of 8-dimensional spheres on 18 vertices whose combinatorial diameter exceeds the Hirsch bound. In [CS19], the question of polytopality of these combinatorial prismatoids was left open.

Theorem 4 *The topological prismatoids #1039, #1963, #2669 and #3513 are not polytopal.* □

4.2 Novik and Zheng's centrally symmetric neighborly d-spheres

In [NZ19], Novik and Zheng give several constructions of centrally symmetric, highly neighborly d-spheres. They are based on a family Δ_n^d of cs-$\lceil \frac{d}{2} \rceil$-neighborly combinatorial d-spheres on $2n \geq 2d+2$ vertices, which arise as the case $i = \lceil \frac{d}{2} \rceil$ of an inductively constructed family $\Delta_n^{d,i}$ of cs-i-neighborly combinatorial d-spheres. Each of *those* contains a certain combinatorial d-ball $B_n^{d,i-1}$, which is the only part that gets deleted in a step of the inductive construction. For $d = 3$, Novik and Zheng's family $\{\Delta_n^3 : n \geq 4\}$ is precisely Jockusch's family from [Joc95].

Theorem 5 *For $n \geq 6$, no member Δ_n^4 of Novik and Zheng's family is realizable.* □

Theorem 6 *(with [Zhe20b]) For $n-2 \geq d \geq 3$, no member Δ_n^d of Novik and Zheng's family is realizable.*

References

[CS19] Francisco Criado and Francisco Santos. Topological Prismatoids and Small Simplicial Spheres of Large Diameter. *Experimental Mathematics*, 2019. `https:doi.org/10.1080/10586458.2019.1641766`.

[GJ00] Ewgenij Gawrilow and Michael Joswig. `polymake`: a framework for analyzing convex polytopes. In *Polytopes—combinatorics and computation (Oberwolfach, 1997)*, volume 29 of *DMV Sem.*, pages 43–73. Birkhäuser, Basel, 2000.

[JM18] Martina Juhnke-Kubitzke and Satoshi Murai. Balanced generalized lower bound inequality for simplicial polytopes. *Sel. Math., New Ser.*, 24(2):1677–1689, 2018.

[JMNS18] Martina Juhnke-Kubitzke, Satoshi Murai, Isabella Novik, and Connor Sawaske. A generalized lower bound theorem for balanced manifolds. *Math. Z.*, 289(3-4):921–942, 2018.

[Joc95] William Jockusch. An infinite family of nearly neighborly centrally symmetric 3-spheres. *J. Comb. Theory, Ser. A*, 72(2):318–321, 1995.

[NZ19] Isabella Novik and Hailun Zheng. Highly neighborly centrally symmetric spheres, 2019.

[Pfe20] Julian Pfeifle. Positive Plücker tree certificates for non-realizability. `https://arxiv.org/abs/2012.11500`, 2020.

[PPS12] Julian Pfeifle, Vincent Pilaud, and Francisco Santos. Polytopality and Cartesian products of graphs. *Isr. J. Math.*, 192:121–141, 2012.

[San12] Francisco Santos. A counterexample to the Hirsch conjecture. *Ann. Math. (2)*, 176(1):383–412, 2012.

[Ven19] Lorenzo Venturello. Balanced triangulations on few vertices and an implementation of cross-flips. *Electron. J. Comb.*, 26(3):research paper p3.61, 25, 2019.

[Zhe20a] Hailun Zheng. Ear decomposition and balanced neighborly simplicial manifolds. *Electron. J. Comb.*, 27(1):research paper p1.10, 17, 2020.

[Zhe20b] Hailun Zheng. personal communication, 2020.