# Fast spatial behavior in higher order in time equations and systems 

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#### Abstract

In this work, we consider the spatial decay for high-order parabolic (and combined with a hyperbolic) equation in a semi-infinite cylinder. We prove a Phragmén-Lindelöf alternative function and, by means of some appropriate inequalities, we show that the decay is of the type of the square of the distance to the bounded end face of the cylinder. The thermoelastic case is also considered when the heat conduction is modeled using a high-order parabolic equation. Though the arguments are similar to others usually applied, we obtain new relevant results by selecting appropriate functions never considered before.


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## 1. Introduction

Parabolic high-order (in time) equations arise in the study of viscoelasticity, fluid mechanics or heat conduction. We can cite the work of Lebedev and Gladwell [16], where the authors propose high order in time viscoelastic solids. We can also consider the generalized Burgers fluids [33], which correspond to a parabolic third order in time equation (anti-plane shear). Moreover, we can recall the recent theories concerning dual-phase-lag [34] and three-phase lag [4] for the heat conduction. In short, we can say that parabolic high-order equations model a big quantity of thermomechanical problems.

The knowledge of the spatial behavior of the solutions for equations and systems is an important topic in mechanics and mathematics. From a mechanical point of view, it is related to the Saint-Venant principle and, from a mathematical point of view, with the Phragmén-Lindelöf principle. Mathematical studies about the spatial behavior have been proposed for elliptic, hyperbolic and parabolic equations [2,5-8,11-14, 19, 20, 23-25]. The list of contributions in this theory is huge, but we want to focus our attention to the parabolic case. Perhaps, the first contribution in this line was done by Knowles [15], where the exponential decay for the solutions was obtained. However, it is worth recalling the work of Horgan et al [9] and extended by Horgan and Quintanilla [10] for functionally graded materials. These contributions provide spatial decay estimates of the kind of the exponential of the "square" of the distance to the boundary where the perturbations hold. They represent an improvement in the sense that the spatial decay for the transient classical heat equation is faster than the spatial decay for the static heat equation. Later, some extensions to these contributions were proposed [28,29]. Furthermore, the combination with the elastic equation has been also considered [17,31]. However, the first contribution concerning the spatial behavior for high order ( $n$-order) of a partial differential equation was given in [32]. In this last contribution, the parabolic (and hyperbolic) transient problem was studied with the help of a weighted Poincaré inequality. In the parabolic case, an exponential decay (linear in the distance to the bounded boundary) was obtained.

In this paper, we want to improve this last result. We are going to obtain a Phragmén-Lindelöf alternative for a function defined on the cross-section and we will prove that the decay is of the type obtained in $[9,10]$. We also study the thermoelastic problem when the heat conduction is determined by a high-order parabolic equation. It is worth recalling that in a recent paper [30] the author showed that the decay would be faster than any exponential of a linear expression of the distance. Here, we give a new precise decay improving the ones presented previously. Although the arguments proposed have been considered in many other contributions by different authors, in this work we introduce new functionals which allow us to obtain the improvement in the knowledge of the decay.

In the next section, we propose the parabolic high-order problem that we will study later. To this end, we need to recall several inequalities which are summarized in the third section. In the fourth section, we obtain a Phragmén-Lindelöf alternative for a cross-sectional measure. In the fifth section, we prove a faster decay estimate. In the sixth section, we consider the thermoelastic problem and we prove a decay estimate of the type of the exponential of a second order polynomial. Finally, we give some examples where the results obtained can be applied.

## 2. The problem

In this paper, we study the spatial behavior of the problem determined by the equation:

$$
\begin{equation*}
\dot{\hat{u}}=\Delta \tilde{u} \tag{2.1}
\end{equation*}
$$

in a semi-infinite cylinder (or strip) $B=[0, \infty) \times D$, with the boundary conditions:

$$
\begin{align*}
& u(\boldsymbol{x}, t)=0 \text { for a.e. } \boldsymbol{x} \in[0, \infty) \times \partial D \\
& u\left(0, x_{2}, x_{3}, t\right)=f\left(x_{2}, x_{3}, t\right) \text { for a.e. }\left(x_{2}, x_{3}\right) \in D \tag{2.2}
\end{align*}
$$

and the initial conditions:

$$
\begin{equation*}
u(\boldsymbol{x}, 0)=\dot{u}(\boldsymbol{x}, 0)=\ldots=u^{(n)}(\boldsymbol{x}, 0)=0 \quad \text { for a.e. } \boldsymbol{x} \in B \tag{2.3}
\end{equation*}
$$

In Eq. (2.1) we have used the notations:

$$
\begin{equation*}
\tilde{u}=b_{0} u+b_{1} \dot{u}+\cdots+b_{n} u^{(n)}, \quad \hat{u}=a_{1} u+a_{2} \dot{u}+\cdots+a_{n+1} u^{(n)} . \tag{2.4}
\end{equation*}
$$

We refer the reader to Sect. 7 for some specific examples of these higher-order equations.
As usual, we need to impose that

$$
f\left(x_{2}, x_{3}, 0\right)=0 \quad \text { for a.e. }\left(x_{2}, x_{3}\right) \in \partial D
$$

to guarantee the compatibility of the conditions.
In this paper, we assume that $a_{n+1}>0$ and $b_{n}>0$. Of course, the case $a_{n+1}<0$ and $b_{n}<0$ can be considered in a similar way. We note that the existence of the solutions to problem (2.1)-(2.3) as well as their regularity can be obtained in view of the results in [21], once we combine these ideas with the ones presented in the appendix of [22].

## 3. Some useful inequalities

To obtain our results it will be useful to recall (and to deduce) several inequalities. First, we recall the weighted Poincaré inequality which states that

$$
\begin{equation*}
\int_{0}^{t} e^{-2 \omega s} f^{2}(s) \mathrm{d} s \leq \omega^{-2} \int_{0}^{t} e^{-2 \omega s}\left|f^{\prime}(s)\right|^{2} \mathrm{~d} s \tag{3.1}
\end{equation*}
$$

where $\omega>0$ and $f(0)=0$.
From the above inequality we can deduce several inequalities which will be useful in our approach.
In view of the inequality (3.1), the systematic use of the Hölder and A-G inequalities allows us to obtain

$$
\begin{align*}
\int_{0}^{t} \int_{D} e^{-2 \omega s}|\tilde{u}(s)|^{2} \mathrm{~d} a \mathrm{~d} s= & b_{n}^{2} \int_{0}^{t} \int_{D} e^{-2 \omega s}|\tilde{u}(s)|^{2} \mathrm{~d} a \mathrm{~d} s \\
& +2 b_{n} \int_{0}^{t} \int_{D} e^{-2 \omega s} u^{(n)}\left(b_{0} u+\cdots+b_{n-1} u^{(n-1)}\right) \mathrm{d} a \mathrm{~d} s \\
& +\int_{0}^{t} \int_{D} e^{-2 \omega s}\left(b_{0} u+\cdots+b_{n-1} u^{(n-1)}\right)^{2} \mathrm{~d} a \mathrm{~d} s \\
\leq & b_{n}^{2} \int_{0}^{t} \int_{D} e^{-2 \omega s}\left|u^{(n)}\right|^{2} \mathrm{~d} a \mathrm{~d} s+2 b_{n}\left(\int_{0}^{t} \int_{D} e^{-2 \omega s}\left(u^{(n)}\right)^{2} \mathrm{~d} a \mathrm{~d} s\right)^{1 / 2} \\
& \times\left[\left|b_{0}\right|\left(\int_{0}^{t} \int_{D} e^{-2 \omega s}|u|^{2} \mathrm{~d} a \mathrm{~d} s\right)^{1 / 2}\right. \\
& \left.+\cdots+\left|b_{n-1}\right|\left(\int_{0}^{t} \int_{D} e^{-2 \omega s}\left|u^{(n-1)}\right|^{2} \mathrm{~d} a \mathrm{~d} s\right)^{1 / 2}\right]+\cdots \\
\leq & b_{n}^{2} \int_{0}^{t} \int_{D} e^{-2 \omega s}\left|u^{(n)}\right|^{2} \mathrm{~d} a \mathrm{~d} s+A \omega^{-1} \int_{0}^{t} \int_{D} e^{-2 \omega s}\left|u^{(n)}\right|^{2} \mathrm{~d} a \mathrm{~d} s \\
& +B \omega^{-2} \int_{0}^{t} \int_{D} e^{-2 \omega s}\left|u^{(n)}\right|^{2} \mathrm{~d} a \mathrm{~d} s \\
\leq & C_{1} \omega^{-1} \int_{0}^{t} \int_{D} e^{-2 \omega s}\left(a_{n+1} b_{n}\right)\left|u^{(n)}\right|^{2} \mathrm{~d} a \mathrm{~d} s \\
& +b_{n}^{2} \int_{0}^{t} \int_{D} e^{-2 \omega s}\left|u^{(n)}\right|^{2} \mathrm{~d} a \mathrm{~d} s \tag{3.2}
\end{align*}
$$

where $C_{1}$ is a positive calculable constant.
The next inequality we consider is the following one:

$$
\begin{aligned}
& \left|\int_{0}^{t} \int_{D} e^{-2 \omega s}\left(b_{0} u+b_{1} \dot{u}+\cdots+b_{n-1} u^{(n-1)}\right)\left(a_{1} u+\cdots+a_{n+1} u^{(n)}\right) \mathrm{d} a \mathrm{~d} s\right| \\
& \quad \leq \frac{1}{16} \int_{0}^{t} \int_{D} e^{-2 \omega s} b_{n} a_{n+1}\left|u^{(n)}\right|^{2} \mathrm{~d} a \mathrm{~d} s
\end{aligned}
$$

where $\omega$ is large enough, which can be obtained in a similar way.

We will also need the following inequalities:

$$
\begin{gathered}
\left|\int_{0}^{t} \int_{0}^{s} \int_{D} e^{-2 \omega \tau} \hat{u}\left(b_{0} \dot{u}+b_{1} \ddot{u}+\cdots+b_{n-1} u^{(n)}\right) \mathrm{d} a \mathrm{~d} \tau \mathrm{~d} s\right| \\
\leq \frac{\omega}{8} \int_{0}^{t} \int_{0}^{s} \int_{D} e^{-2 \omega \tau} b_{n} a_{n+1}\left|u^{(n)}\right|^{2} \mathrm{~d} a \mathrm{~d} \tau \mathrm{~d} s \\
\left|\int_{0}^{t} \int_{0}^{s} \int_{D} e^{-2 \omega \tau} b_{n} u^{(n)}\left(a_{1} \dot{u}+\cdots+a_{n} u^{(n)}\right) \mathrm{d} a \mathrm{~d} \tau \mathrm{~d} s\right| \\
\leq \frac{\omega}{16} \int_{0}^{t} \int_{0}^{s} \int_{D} e^{-2 \omega \tau} b_{n} a_{n+1}\left|u^{(n)}\right|^{2} \mathrm{~d} a \mathrm{~d} \tau \mathrm{~d} s
\end{gathered}
$$

where $\omega$ is again large enough.

## 4. Phragmén-Lindelöf alternative

In this section, we obtain a Phragmén-Lindelöf alternative for the solutions to our problem (2.1)-(2.3) for a measure defined in the cross section of the cylinder. We first define the function

$$
\begin{equation*}
G(z, t)=\frac{1}{2} \int_{0}^{t} \int_{0}^{s} \int_{D} e^{-2 \omega \tau}|\tilde{u}(\tau)|^{2} \mathrm{~d} a \mathrm{~d} \tau \mathrm{~d} s \tag{4.1}
\end{equation*}
$$

We have

$$
\frac{\partial G}{\partial z}=\int_{0}^{t} \int_{0}^{s} \int_{D} e^{-2 \omega \tau} \tilde{u}(\tau) \tilde{u}_{, 1}(\tau) \mathrm{d} a \mathrm{~d} \tau \mathrm{~d} s
$$

and

$$
\begin{aligned}
\frac{\partial^{2} G}{\partial z^{2}} & =\int_{0}^{t} \int_{0}^{s} \int_{D} e^{-2 \omega \tau}\left(\tilde{u}_{, 1} \tilde{u}_{, 1}+\tilde{u} \tilde{u}_{, 11}\right) \mathrm{d} a \mathrm{~d} \tau \mathrm{~d} s \\
& =\int_{0}^{t} \int_{0}^{s} \int_{D} e^{-2 \omega \tau}\left(|\nabla \tilde{u}|^{2}+\tilde{u} \dot{\hat{u}}\right) \mathrm{d} a \mathrm{~d} \tau \mathrm{~d} s .
\end{aligned}
$$

But we find that

$$
\begin{aligned}
\tilde{u} \dot{\hat{u}}= & \left(b_{0} u+\cdots+b_{n-1} u^{(n-1)}\right)\left(a_{1} \dot{u}+\cdots+a_{n+1} u^{(n+1)}\right)+b_{n} a_{n+1} u^{(n)} u^{(n+1)} \\
& +b_{n} u^{(n)}\left(a_{1} \dot{u}+\cdots+a_{n} u^{(n)}\right) \\
=\frac{d}{d t} & {\left.\left[b_{0} u+\cdots+b_{n-1} u^{(n-1)}\right) \hat{u}\right]-\left(b_{0} \dot{u}+\cdots+b_{n-1} u^{(n)}\right) \hat{u}+\frac{1}{2} b_{n} a_{n+1} \frac{d}{d t}\left|u^{(n)}\right|^{2} } \\
& +b_{n} u^{(n)}\left(a_{1} \dot{u}+\cdots+a_{n} u^{(n)}\right),
\end{aligned}
$$

and so, we obtain

$$
\begin{aligned}
\frac{\partial^{2} G}{\partial z^{2}} & =\int_{0}^{t} \int_{D} e^{-2 \omega s}\left(\frac{1}{2} b_{n} a_{n+1}\left|u^{(n)}\right|^{2}+\left(b_{0} u+\cdots+b_{n-1} u^{(n-1)}\right) \hat{u}\right) \mathrm{d} a \mathrm{~d} s \\
& +\int_{0}^{t} \int_{0}^{s} \int_{D} e^{-2 \omega \tau}\left[|\nabla \tilde{u}|^{2}+\omega b_{n} a_{n+1}\left|u^{(n)}\right|^{2}-\left(b_{0} \dot{u}+\cdots+b_{n-1} u^{(n)}\right) \hat{u}\right. \\
& \left.+2 \omega\left(b_{0} u+\cdots+b_{n-1} u^{(n-1)}\right) \hat{u}+b_{n} u^{(n)}\left(a_{1} \dot{u}+\cdots+a_{n} u^{(n)}\right)\right] \mathrm{d} a \mathrm{~d} \tau \mathrm{~d} s .
\end{aligned}
$$

Therefore, in view of the inequalities provided in the previous section, it follows that

$$
\begin{align*}
\frac{\partial^{2} G}{\partial z^{2}} & \geq \frac{1}{4} \int_{0}^{t} \int_{D} e^{-2 \omega s} b_{n} a_{n+1}\left|u^{(n)}\right|^{2} \mathrm{~d} a \mathrm{~d} s \\
& +\int_{0}^{t} \int_{0}^{s} \int_{D} e^{-2 \omega \tau}\left[|\nabla \tilde{u}|^{2}+\frac{\omega}{2} b_{n} a_{n+1}\left|u^{(n)}\right|^{2}\right] \mathrm{d} a \mathrm{~d} \tau \mathrm{~d} s \tag{4.2}
\end{align*}
$$

whenever $\omega$ is large enough. Thus, it leads

$$
\begin{equation*}
\frac{\partial^{2} G}{\partial z^{2}}(z, t) \geq 2 \lambda^{2} G(z, t) \tag{4.3}
\end{equation*}
$$

where $\lambda^{2}$ is the known Poincaré constant for the cross section $D$. Inequality (4.3) has been previously studied in the context of spatial estimates (see [18]).

From here we can obtain that either

$$
\begin{equation*}
G(z, t) \geq G\left(z_{0}, t\right) e^{\sqrt{2} \lambda\left(z-z_{0}\right)} \tag{4.4}
\end{equation*}
$$

for every $z \geq z_{0}$ and where $G\left(z_{0}, t\right)>0$, or the exponential decay

$$
\begin{equation*}
G(z, t) \leq G(0, t) e^{-\sqrt{2} \lambda z} \tag{4.5}
\end{equation*}
$$

is satisfied.
So we can deduce the following.
Theorem 4.1. The function $G(z, t)$ defined in (4.1) for the solutions to problem (2.1)-(2.3) either it satisfies the increasing estimate (4.4) or the decay estimate (4.5).

## 5. Fast decay

In this section, we prove a decay estimate for the solutions to problem (2.1)-(2.3) of the type of the exponential of the distance to the part of the boundary where the perturbations are imposed. First, we note that

$$
\frac{\partial G}{\partial t}=\frac{1}{2} \int_{0}^{t} \int_{D} e^{-2 \omega s}|\tilde{u}|^{2} \mathrm{~d} a \mathrm{~d} s
$$

From (3.2) and (4.2) we also find that

$$
\begin{equation*}
G \frac{\partial^{2} G}{\partial z^{2}}-\left(\frac{\partial G}{\partial z}\right)^{2} \geq 2 \lambda^{2} G^{2}(z)+C^{-1} \frac{\partial G}{\partial t} G, \tag{5.1}
\end{equation*}
$$

where $C$ is a computable positive constant depending on the constitutive coefficients and $\omega$ of the form:

$$
C=2\left(C_{1} \omega^{-1}+\frac{b_{n}}{a_{n+1}}\right)
$$

Inequality (5.1) is well-known (see Equation (3.16) in [10]).
If we denote by $P(z, t)=G(z, t)^{1 / 2}$ we can write

$$
\frac{\partial^{2} P}{\partial z^{2}} \geq \lambda^{2} P+C^{-1} \frac{\partial P}{\partial t}
$$

We also note that $P(z, 0)=0$ for $z \geq 0$ and

$$
P(0, t)=\left(\int_{0}^{t} \int_{0}^{s} \int_{D(0)} e^{-2 \omega \tau}|\tilde{f}(\tau)|^{2} \mathrm{~d} a \mathrm{~d} \tau \mathrm{~d} s\right)^{1 / 2}=g(t)
$$

where $g(0)=0$.
Let $P(z, t)=\exp \left(-\lambda^{2} t C\right) \Phi(z, t)$. It then follows that

$$
\begin{aligned}
& \Phi_{z z}-C^{-1} \Phi_{t} \geq 0 \text { for } z \geq 0 \text { and } t \geq 0, \\
& \Phi(z, 0)=0 \text { for } z \geq 0, \\
& \Phi(0, t)=\exp \left(\lambda^{2} t C\right) g(t) \text { for } t>0 .
\end{aligned}
$$

An upper bound for $\Phi(z, t)$ follows from the maximum principle by using the solution to the problem

$$
\eta_{t}=C \eta_{z z}
$$

with the initial condition $\eta(z, 0)=0$, when $z \geq 0$, and the boundary condition:

$$
\eta(0, t)=\exp \left(\lambda^{2} t C\right) g(t) \quad \text { for } t>0
$$

We know that

$$
P(z, t) \leq \exp \left(-\lambda^{2} t C\right) \eta(z, t)
$$

The function $\eta(z, t)$ is well known (see Carslaw and Jaeger [3, p. 64]) and so, we have

$$
\eta(z, t)=\exp \left(\lambda^{2} t C\right) g(t) F(z, t)
$$

where

$$
\begin{aligned}
F(z, t)= & \frac{1}{2} \exp (-\lambda z) \operatorname{erfc}\left\{\frac{C^{-1 / 2} z}{2 t^{1 / 2}}-\left(\lambda^{2} C t\right)^{1 / 2}\right\} \\
& +\frac{1}{2} \exp (\lambda z) \operatorname{erfc}\left\{\frac{C^{-1 / 2} z}{2 t^{1 / 2}}+\left(\lambda^{2} C t\right)^{1 / 2}\right\},
\end{aligned}
$$

and

$$
\operatorname{erfc}(x)=\frac{2}{\pi^{1 / 2}} \int_{x}^{\infty} e^{-s^{2}} \mathrm{~d} s
$$

Therefore,

$$
P(z, t) \leq g(t) F(z, t)
$$

Since we know that

$$
\pi^{1 / 2} \operatorname{erfc} c(x) \leq \frac{1}{x} e^{-x^{2}}
$$

we can conclude that (see Abramowitz and Stegun [1, p. 98]), for $z>2 \lambda t C$, the estimate

$$
\begin{equation*}
P(z, t) \leq g(t)\left[\frac{2 C^{-3 / 2} z(t / \pi)^{1 / 2} \exp \left(-\lambda^{2} t C\right)}{C^{-2} z^{2}-4 \lambda^{2} t^{2}} \exp \left(-\frac{C^{-1} z^{2}}{4 t}\right)\right] \tag{5.2}
\end{equation*}
$$

holds.
Theorem 5.1. The solutions to problem (2.1)-(2.3) decaying to zero when the distance to the bounded boundary increases satisfy estimate (5.2).

We remark that we can choose $\omega$ large enough to guarantee that the decay is "almost" of the type $\exp \left(-\frac{a_{n+1} z^{2}}{8 b_{n} t}\right)$.

## 6. Thermoelastic system

In this section, we extend the estimates obtained in the previous section to the thermoelastic case, that is, a fast decay of the decaying solutions. Thus, we consider the system:

$$
\begin{align*}
& \rho \ddot{u}_{i}=\mu u_{i, j j}+(\lambda+\mu) u_{j, j i}-\beta \theta, i \\
& c\left(a_{1} \dot{\theta}+a_{2} \dot{\theta}+\cdots+a_{n+1} \theta^{(n+1)}\right)=b_{0} \Delta \theta+b_{1} \Delta \dot{\theta}+\cdots+b_{n} \Delta \theta^{(n)}  \tag{6.1}\\
& \quad-\beta\left(a_{1} \dot{u}_{i, i}+a_{2} \ddot{u}_{i, i}+\cdots+a_{n+1} u_{i, i}^{(n+1)}\right),
\end{align*}
$$

with the boundary conditions:

$$
\begin{align*}
& u_{i}(\boldsymbol{x}, t)=\theta(\boldsymbol{x}, t)=0 \text { for a.e. } \boldsymbol{x} \in[0, \infty) \times \partial D, \\
& u_{i}\left(0, x_{2}, x_{3}, t\right)=f_{i}\left(x_{2}, x_{3}, t\right) \text { for a.e. }\left(x_{2}, x_{3}\right) \in D,  \tag{6.2}\\
& \theta\left(0, x_{2}, x_{3}, t\right)=f\left(x_{2}, x_{3}, t\right) \text { for a.e. }\left(x_{2}, x_{3}\right) \in D,
\end{align*}
$$

and the initial conditions:

$$
\begin{equation*}
u_{i}(\boldsymbol{x}, 0)=\dot{u}_{i}(\boldsymbol{x}, 0)=\theta(\boldsymbol{x}, 0)=0 \quad \text { for a.e. } \boldsymbol{x} \in B . \tag{6.3}
\end{equation*}
$$

Again, we assume that $f_{i}\left(x_{2}, x_{3}, 0\right)=f\left(x_{2}, x_{3}, 0\right)=0$ for a.e. $\left(x_{2}, x_{3}\right) \in \partial D^{1}$.
It is worth noting that here $u_{i}$ is the displacement vector, $\theta$ is the temperature, $\lambda$ and $\mu$ are the Lamé constants, $\rho$ is the mass density, $c$ is the heat capacity and $\beta$ is the coupling coefficient.

To make the calculations easier we assume that

$$
\rho>0, \quad \mu>0, \quad \lambda+\mu \geq 0, \quad c>0 .
$$

We also assume that $a_{n+1}>0$ and $b_{n}>0$.
In order to study the problem it is worth writing the displacement equation as

$$
\rho \ddot{\tilde{u}}_{i}=\mu \tilde{u}_{i, j j}+(\lambda+\mu) \tilde{u}_{j, j i}-\beta \tilde{\theta}_{, i} .
$$

We can define the function

$$
G_{0}(z, t)=-\int_{0}^{t} \int_{0}^{s} \int_{D} e^{-2 \omega \tau}\left[\mu \tilde{u}_{i, 1} \dot{\hat{u}}_{i}+(\lambda+\mu) \tilde{u}_{k, k} \dot{\hat{u}}_{1}+\beta \tilde{\theta} \dot{\hat{u}}_{1}+\tilde{\theta}_{, 1} \tilde{\theta}\right] \mathrm{d} a \mathrm{~d} \tau \mathrm{~d} s
$$

In this section, we want to obtain a new spatial decay estimate. Therefore, we assume that

$$
\begin{equation*}
\lim _{z \rightarrow \infty} G_{0}(z, t)=0 \quad \text { uniformly as } \quad O\left(z^{-3}\right) \tag{6.4}
\end{equation*}
$$

[^0]We obtain that

$$
\begin{aligned}
& G_{0}(z, t)=\int_{0}^{t} \int_{0}^{s} \int_{B(z)} e^{-2 \omega \tau}\left[\rho \ddot{\tilde{u}}_{i} \dot{\hat{u}}_{i}+\mu \tilde{u}_{i, j} \dot{\hat{u}}_{i, j}+(\lambda+\mu) \tilde{u}_{i, i} \dot{\hat{u}}_{j, j}\right. \\
& \left.+|\nabla \tilde{\theta}|^{2}+c \tilde{\theta} \dot{\theta}\right] d v d \tau \mathrm{~d} s,
\end{aligned}
$$

where $B(z)=\left\{\boldsymbol{x} \in B ; x_{1} \geq z\right\}$. We note that

$$
\begin{aligned}
& \ddot{\tilde{u}}_{i} \dot{\hat{u}}_{i}=\left(b_{0} \ddot{u}_{i}+b_{1} \dddot{u}_{i}+\cdots+b_{n} u_{i}^{(n+2)}\right)\left(a_{1} \dot{u}_{i}+a_{2} \ddot{u}_{i}+\cdots+a_{n+1} u_{i}^{(n+1)}\right) \\
&= b_{n} a_{n+1} u_{i}^{(n+2)} u_{i}^{(n+1)}+b_{n} u_{i}^{(n+2)}\left(a_{1} \dot{u}_{i}+a_{2} \ddot{u}_{i}+\cdots+a_{n} u_{i}^{(n)}\right) \\
& \quad+\left(b_{0} \ddot{u}_{i}+b_{1} \dddot{u}_{i}+\cdots+b_{n-1} u^{(n+1)}\right)\left(a_{1} \dot{u}_{i}+a_{2} \ddot{u}_{i}+\cdots+a_{n+1} u_{i}^{(n+1)}\right) \\
&= \frac{d}{d t}\left[\frac{1}{2} b_{n} a_{n+1} u_{i}^{(n+1)} u_{i}^{(n+1)}+b_{n} u_{i}^{(n+1)}\left(a_{1} \dot{u}_{i}+a_{2} \ddot{u}_{i}+\cdots+a_{n} u_{i}^{(n)}\right)\right] \\
&+\left(b_{0} \ddot{u}_{i}+b_{1} \dddot{u}_{i}+\cdots+b_{n-1} u^{(n+1)}\right)\left(a_{1} \dot{u}_{i}+a_{2} \ddot{u}_{i}+\cdots+a_{n+1} u_{i}^{(n+1)}\right) \\
& \quad-b_{n} u_{i}^{(n+1)}\left(a_{1} \ddot{u}_{i}+\cdots+a_{n} u_{i}^{(n+1)}\right) \\
& \tilde{u}_{i, j} \dot{\hat{u}}_{i, j}=\left(b_{0} u_{i, j}+b_{1} \dot{u}_{i, j}+\cdots+b_{n} u_{i, j}^{(n)}\right)\left(a_{1} \dot{u}_{i, j}+a_{2} \ddot{u}_{i, j}+\cdots+a_{n+1} u_{i, j}^{(n+1)}\right) \\
&=b_{n} a_{n+1} u_{i, j}^{(n)} u_{i, j}^{(n+1)}+b_{n} u_{i, j}^{(n)}\left(a_{1} \dot{u}_{i, j}+a_{2} \ddot{u}_{i, j}+\cdots+a_{n} u_{i, j}^{(n)}\right) \\
& \quad+\left(b_{0} u_{i, j}+b_{1} \dot{u}_{i, j}+\cdots+b_{n-1} u_{i, j}^{(n-1)}\right)\left(a_{1} \dot{u}_{i, j}+a_{2} \ddot{u}_{i, j}+\cdots+a_{n+1} u_{i, j}^{(n+1)}\right) \\
&=\frac{d}{d t}\left[\frac{1}{2} b_{n} a_{n+1} u_{i, j}^{(n)} u_{i, j}^{(n)}+\left(b_{0} u_{i, j}+b_{1} \dot{u}_{i, j}+\cdots+b_{n-1} u_{i, j}^{(n-1)}\right) \hat{u}_{i, j}\right] \\
& \quad+b_{n} u_{i, j}^{(n)}\left(a_{1} \dot{u}_{i, j}+a_{2} \ddot{u}_{i, j}+\cdots+a_{n} u_{i, j}^{(n+1)}\right) \\
& \quad-\left(b_{0} \dot{u}_{i, j}+b_{1} \ddot{u}_{i, j}+\cdots+b_{n-1} u_{i, j}^{(n)}\right) \hat{u}_{i, j} .
\end{aligned}
$$

In a similar way, we also have

$$
\begin{aligned}
& \tilde{u}_{i, i} \hat{u}_{j, j}=\frac{\mathrm{d}}{\mathrm{~d} t}\left[\frac{1}{2} b_{n} a_{n+1} u_{i, i}^{(n)} u_{j, j}^{(n)}+\left(b_{0} u_{i, i}+b_{1} \dot{u}_{i, j}+\cdots+b_{n-1} u_{i, i}^{(n-1)}\right) \hat{u}_{j, j}\right] \\
& \quad+b_{n} u_{i, i}^{(n)}\left(a_{1} \dot{u}_{j, j}+a_{2} \ddot{u}_{j, j}+\cdots+a_{n} u_{i, j}^{(n)}\right) \\
& \quad-\left(b_{0} \dot{u}_{i, i}+b_{1} \ddot{u}_{i, i}+\cdots+b_{n-1} u_{i, i}^{(n)}\right) \hat{u}_{j, j} .
\end{aligned}
$$

We then obtain

$$
\begin{aligned}
& G_{0}(z, t)=\int_{0}^{t} \int_{B(z)} e^{-2 \omega s}\left[\frac { 1 } { 2 } b _ { n } a _ { n + 1 } \left(\rho u_{i}^{(n+1)} u_{i}^{(n+1)}+\mu u_{i, j}^{(n)} u_{i, j}^{(n)}\right.\right. \\
& \left.\left.\quad+(\lambda+\mu) u_{i, i}^{(n)} u_{j, j}^{(n)}+c\left|\theta^{(n)}\right|^{2}\right)+F_{1}\right] d v d s+\int_{0}^{t} \int_{0}^{s} \int_{B(z)} e^{-2 \omega \tau}\left[\frac { \omega } { 2 } b _ { n } a _ { n + 1 } \left(c\left|\theta^{(n)}\right|^{2}\right.\right. \\
& \left.\left.\quad+\rho u_{i}^{(n+1)} u_{i}^{(n+1)}+\mu u_{i, j}^{(n)} u_{i, j}^{(n)}+(\lambda+\mu) u_{i, i}^{(n)} u_{j, j}^{(n)}\right)+\omega F_{1}+F_{2}+|\nabla \tilde{\theta}|^{2}\right] d v d \tau \mathrm{~d} s,
\end{aligned}
$$

where

$$
\begin{aligned}
F_{1}= & \rho b_{n} u_{i}^{(n+1)}\left(a_{1} \dot{u}_{i}+\cdots+a_{n} u_{i}^{(n)}\right)+\mu\left(b_{0} u_{i, j}+\cdots+b_{n-1} u_{i, j}^{(n-1)}\right) \hat{u}_{i, j} \\
& +(\lambda+\mu)\left(b_{0} u_{i, i}+\cdots+b_{n-1} u_{i, i}^{(n-1)}\right) \hat{u}_{j, j}+c\left(b_{0} \theta+\cdots+b_{n-1} \theta^{(n-1)}\right) \hat{\theta}, \\
F_{2}= & \left(b_{0} \ddot{u}_{i}+\cdots+b_{n-1} u_{i}^{(n+1)}\right)\left(a_{1} \dot{u}_{i}+a_{2} \ddot{u}_{i}+\cdots+a_{n+1} u_{i}^{(n+1)}\right) \\
& -b_{n} u_{i}^{(n+1)}\left(a_{1} \ddot{u}_{i}+\cdots+a_{n} u_{i}^{(n+1)}\right)+b_{n} u_{i, j}^{(n)}\left(a_{1} \dot{u}_{i, j}+\cdots+a_{n} u_{i, j}^{(n)}\right) \\
& -\left(b_{0} \dot{u}_{i, j}+b_{1} \ddot{u}_{i, j}+\cdots+b_{n-1} u_{i, j}^{(n)}\right) \hat{u}_{i, j}+b_{n} u_{i, i}^{(n)}\left(a_{1} \dot{u}_{j, j}+\cdots+a_{n} u_{j, j}^{(n)}\right) \\
& -\left(b_{0} \dot{u}_{i, i}+\cdots+b_{n-1} u_{i, i}^{(n)}\right) \hat{u}_{j, j}+c b_{n} \theta^{(n)}\left(a_{1} \hat{\theta}+\cdots+a_{n} \theta^{(n)}\right) \\
& -c\left(b_{0} \dot{\theta}+\cdots+b_{n-1} \theta^{(n)}\right) \hat{\theta} .
\end{aligned}
$$

By choosing $\omega$ large enough we can obtain

$$
\begin{aligned}
& G_{0}(z, t) \geq \frac{1}{4} \int_{0}^{t} \int_{B(z)} e^{-2 \omega s} b_{n} a_{n+1}\left[\rho u_{i}^{(n+1)} u_{i}^{(n+1)}+\mu u_{i, j}^{(n)} u_{i, j}^{(n)}+(\lambda+\mu) u_{i, i}^{(n)} u_{j, j}^{(n)}\right. \\
& \left.\quad+c\left|\theta^{(n)}\right|^{2}\right] \mathrm{d} v \mathrm{~d} s+\int_{0}^{t} \int_{0}^{s} \int_{B(z)} e^{-2 \omega \tau}\left[|\nabla \theta|^{2}+\frac{\omega}{2} b_{n} a_{n+1}\left(\rho u_{i}^{(n+1)} u_{i}^{(n+1)}\right.\right. \\
& \left.\left.\quad+\mu u_{i, j}^{(n)} u_{i, j}^{(n)}+(\lambda+\mu) u_{i, i}^{(n)} u_{j, j}^{(n)}+c\left|\theta^{(n)}\right|^{2}\right)\right] \mathrm{d} v \mathrm{~d} \tau \mathrm{~d} s .
\end{aligned}
$$

We consider now

$$
G_{1}(z, t)=\int_{z}^{\infty} G_{0}(\xi, t) \mathrm{d} \xi
$$

and so, we have

$$
\begin{aligned}
\frac{\partial G_{1}}{\partial t}= & -\int_{0}^{t} \int_{B(z)} e^{-2 \omega s}\left[\mu \tilde{u}_{i, 1} \dot{\hat{u}}_{i}+(\lambda+\mu) \tilde{u}_{1, i} \dot{\hat{u}}_{i}+\beta \tilde{\theta} \dot{\hat{u}}_{i}\right] d v d s \\
& +\frac{1}{2} \int_{0}^{t} \int_{D(z)}|\tilde{\theta}|^{2} \mathrm{~d} a \mathrm{~d} s, \\
\frac{\partial G_{1}}{\partial z}(z, t)= & -G_{0}(z, t), \\
\frac{\partial^{2} G_{1}(z, t)}{\partial z^{2}} \geq & \frac{1}{4} \int_{0}^{t} \int_{D(z)} e^{-2 \omega s} b_{n} a_{n+1}\left[\rho u_{i}^{(n+1)} u_{i}^{(n+1)}+\mu u_{i, j}^{(n)} u_{i, j}^{(n)}+c\left|\theta^{(n)}\right|^{2}\right. \\
& \left.+(\lambda+\mu) u_{i, i}^{(n)} u_{j, j}^{(n)}\right] \mathrm{d} a \mathrm{~d} s+\int_{0}^{t} \int_{0}^{s} \int_{D(z)} e^{-2 \omega \tau}\left[|\nabla \theta|^{2}+\frac{\omega}{2} b_{n} a_{n+1}\left(\rho u_{i}^{(n+1)} u_{i}^{(n+1)}\right.\right. \\
& \left.\left.+\mu u_{i, j}^{(n)} u_{i, j}^{(n)}+(\lambda+\mu) u_{i, i}^{(n)} u_{j, j}^{(n)}+c\left|\theta^{(n)}\right|^{2}\right)\right] \mathrm{d} a \mathrm{~d} \tau \mathrm{~d} s .
\end{aligned}
$$

From here, the argument is again standard (see, for instance, [26, 27]).
We can obtain the existence of two positive constants $\beta_{1}$ and $R$ such that

$$
\frac{\partial G_{1}}{\partial t} \leq-\beta_{1} \frac{\partial G}{\partial z}+R \frac{\partial^{2} G_{1}}{\partial z^{2}}
$$

If we denote

$$
\mathcal{H}(z, t)=e^{a^{2} R t} e^{-a z} G_{1}(z, t), \quad a=\frac{\beta_{1}}{2 R},
$$

we obtain

$$
R \frac{\partial \mathcal{H}}{\partial z^{2}} \geq \frac{\partial \mathcal{H}}{\partial t}
$$

An argument similar to the one proposed in the previous section shows that

$$
G_{1}(z, t) \leq e^{a(z-a R t)} \sup _{0 \leq s \leq t}\left[e^{2 R s} G_{1}(0, s)\right] N(z, t),
$$

where $N(z, t)=\frac{z}{(4 \pi R)^{1 / 2}} \int_{0}^{t} s^{-3 / 2} \exp \left(-\frac{z^{2}}{4 \pi s}\right)$, and a change of variable implies that $N(z, t)$ $=\operatorname{erfc} \frac{z}{(4 C t)^{1 / 2}}$.

Therefore, we conclude that

$$
\begin{equation*}
G_{1}(z, t) \leq \frac{A(t)}{z} \exp \left(a z-\frac{z^{2}}{4 R t}\right) \tag{6.5}
\end{equation*}
$$

where

$$
A(t)=(4 R t)^{1 / 2} e^{-a^{2} R t} \sup _{0 \leq s \leq t} e^{a^{2} R s} G_{1}(0, s)
$$

We remark that we can obtain upper bounds for this function $A(t)$ in terms of the boundary conditions following the arguments already used in [17,27].

It is clear that these estimates imply that the decay at the infinite is of the type of $\exp \left(\frac{-z^{2}}{4 R t}\right)$ which we summarize as follows.

Theorem 6.1. The solutions to problem (6.1)-(6.3) that satisfy condition (6.4) decay in the form (6.5).
We note that, for $\omega$ large enough, we can choose $R$ as near as we want to the value $2 \frac{b_{n}}{c a_{n+1}}$. Therefore, asymptotically the rate of decay that we have obtained for the function $G_{1}$ approaches to $\exp \left(-\frac{c a_{n+1} z^{2}}{8 b_{n} t}\right)$.

## 7. A few examples

In this section we give several elementary examples where the results obtained in this paper can be applied.

### 7.1. Parabolic equation

We give here several examples of parabolic equations of higher order. The first example corresponds to the linearized form of generalized Burgers' fluid. From [33] we know that the system determining the evolution of this fluid is given by

$$
\rho\left(\dot{\boldsymbol{v}}+\lambda_{1} \ddot{\boldsymbol{v}}+\lambda_{2} \dddot{\boldsymbol{v}}\right)=-\nabla q+\eta_{1} \Delta \boldsymbol{v}+\eta_{2} \Delta \dot{\boldsymbol{v}}+\eta_{3} \Delta \ddot{\boldsymbol{v}}
$$

where $\rho, \lambda_{1}, \lambda_{2}, \eta_{1}, \eta_{2}$ and $\eta_{3}$ are positive constants.
In the case that we consider anti-plane shear deformations:

$$
v_{1}=v\left(x_{2}, x_{3}\right), \quad v_{2}=v_{3}=0,
$$

we obtain the equation:

$$
\rho\left(\dot{v}+\lambda_{1} \ddot{v}+\lambda_{2} \dddot{v}\right)=\eta_{1} \Delta^{*} v+\eta_{2} \Delta^{*} \dot{v}+\eta_{3} \Delta^{*} \ddot{v},
$$

where $\Delta^{*}=\frac{\partial^{2}}{\partial x_{2}^{2}}+\frac{\partial^{2}}{\partial x_{3}^{2}}$.
A second example could correspond to the dual-phase-lag heat equation [34] written as follows:

$$
\dot{T}+\tau_{q} \ddot{T}+\frac{\tau_{q}^{2}}{2} \dddot{T}+\cdots+\frac{\tau_{q}^{n}}{n!} T^{(n+1)}=k\left(\Delta T+\tau_{T} \Delta \dot{T}+\cdots+\frac{\tau_{T}^{n}}{n!} \Delta T^{(n)}\right)
$$

where $k, \tau_{q}$ and $\tau_{T}$ are three positive constants. Perhaps the most known case corresponds to the equation:

$$
\dot{T}+\tau_{q} \ddot{T}+\frac{\tau_{q}^{2}}{2} \dddot{T}=k\left(\Delta T+\tau_{T} \Delta \dot{T}+\frac{\tau_{T}^{2}}{2} \Delta \ddot{T}\right)
$$

In 2007, Roy Chouduri [4] proposed to generalize the Tzou proposition to include the type II and III Green-Naghdi theories, being the most well-known example the one written as

$$
\ddot{T}+\tau_{q} \dddot{T}=k^{*} \Delta T+\tau_{\nu}^{*} \Delta \dot{T}+k \tau_{T} \Delta \ddot{T}
$$

where $k, k^{*}$ are two positive constants, $\tau_{\nu}^{*}=k^{*} \tau_{\nu}+k$ and $\tau_{q}, \tau_{\nu}$ and $\tau_{T}$ are three positive relaxation parameters.

In the reference [16] the authors proposed a constitutive equation in viscoelasticity of the form:

$$
C(\partial / \partial t) \sigma_{i j}=A(\partial / \partial t) \varepsilon_{k k} \delta_{i j}+2 B(\partial / \partial t) \varepsilon_{i j}
$$

where $A, B, C$ are polynomials, $\sigma_{i j}$ is the stress tensor, $\varepsilon_{i j}$ is the strain tensor and $\delta_{i j}$ is the Kronecker symbol.

The anti-plane shear deformations can be obtained now from the equation:

$$
u^{(n+1)}+a_{n} u^{(n)}+\cdots+a_{2} u^{(2)}=\mu\left(b_{0} \Delta u+\cdots+b_{n} \Delta u^{(n-1)}+\Delta u^{(n)}\right),
$$

where $\mu$ is a positive parameter and whenever degree $(C)=\operatorname{degree}(B)-1$.

### 7.2. Thermoelasticity

Now, we give a couple of examples where we can apply the results of Sect. 6.
We note that the dual-phase-lag thermoelasticity has been studied by many authors. We can consider the case:

$$
\begin{aligned}
& \rho \ddot{u}_{i}=\mu u_{i, j j}+(\lambda+\mu) u_{j, j i}-\beta \theta_{, i}, \\
& c\left(\dot{\theta}+\tau_{q} \ddot{\theta}+\frac{\tau_{q}^{2}}{2} \dddot{\theta}\right)=k\left(\Delta \theta+\tau_{T} \Delta \dot{\theta}+\frac{\tau_{T}^{2}}{2} \Delta \ddot{\theta}\right)-\beta\left(\dot{u}_{i, i}+\tau_{q} \ddot{u}_{i, i}+\frac{\tau_{q}^{2}}{2} \dddot{u}_{i, i}\right) .
\end{aligned}
$$

A thermoelastic system corresponding to the three-phase-lag thermoelasticity could be (see [4]):

$$
\begin{aligned}
& \rho \ddot{u}_{i}=\mu u_{i, j j}+(\lambda+\mu) u_{j, j i}-\beta \theta_{, i}, \\
& c\left(\ddot{\theta}+\tau_{q} \dddot{\theta}\right)=k^{*} \Delta \theta+\tau_{\nu} \Delta \dot{\theta}+k \tau_{T} \Delta \ddot{\theta}-\beta\left(\ddot{u}_{i, i}+\tau_{q} \dddot{u}_{i, i}\right) .
\end{aligned}
$$

It is worth noting that we could also study systems where the derivation of the temperature could correspond to higher order. For instance, we could consider systems of the form:

$$
\begin{aligned}
& \rho \ddot{u}_{i}=\mu u_{i, j j}+(\lambda+\mu) u_{j, j i}-\beta \theta_{, i}, \\
& c\left(\dot{\theta}+\tau_{q} \ddot{\theta}+\cdots+\frac{\tau_{q}^{n}}{n!} \theta^{(n+1)}\right)=k\left(\Delta \theta+\tau_{T} \Delta \dot{\theta}+\cdots+\frac{\tau_{T}^{n}}{n!} \Delta \theta^{(n)}\right) \\
& \quad-\beta\left(\dot{u}_{i, i}+\tau_{q} \ddot{u}_{i, i}+\cdots+\frac{\tau_{q}^{n}}{n!} u_{i, i}^{(n+1)}\right) .
\end{aligned}
$$

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[^0]:    ${ }^{1}$ Also the existence and regularity of solutions to problem (6.1)-(6.3) can be obtained as we already mentioned in Sect. 2.

