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Logarithmic convexity for third-order in time partial differential equations

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Abstract

In this short note, we want to describe the logarithmic convexity argument for third-order in time partial differential equations. As a consequence, we first prove a uniqueness result whenever certain conditions on the parameters are satisfied. Later, we show the instability of the solutions if the initial energy is less or equal than zero.

Keywords Third order in time partial differential equations, Logarithmic convexity, Uniqueness, Instability.

1 Introduction

With regard to differential equations, both ordinary and partial, one is interested in knowing if solutions exist, and if solutions do exist, if they are

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unique, and whether the solutions depend continuously on data. One is also interested in knowing whether the solution is stable or otherwise. A problem is said to be well-posed if the solution to the equations exists, is unique, and depends continuously on the data, a notion that can be traced back to the work of Hadamard [1]. Equations that are not well-posed are referred to as ill-posed. There has been considerable work on ill-posed problems and of special relevance to this work is the use of logarithmic convexity arguments to study issues of uniqueness, continuous dependence on data and instability for solutions of these problems (see Knops [2]).

Logarithmic convexity arguments have been used to establish instability results for ill-posed partial differential equations, particularly with those concerned with the response of continua such as the partial differential equations for the motion of elastic bodies, the heat equation, etc. Ill-posed problems occur frequently when one studies inverse problems and problems backward in time. Such problems usually present the possibility of non-unique solutions. Based on the values of the parameters that appear in the partial differential equation, the problem may be well-posed or ill-posed in that a solution might not exist, or the solution might not be unique, etc. Invariably, the conditions under which the equations are ill-posed occur, that is, the assumptions that lead to such ill-posedness, stem from an error in describing the physics of the problem, when one is concerned with the equations governing a physical problem.

At times, one may not be able to prove the existence of solution to the partial differential equation of interest, but one might yet be able to establish the uniqueness of the solution and its continuous dependence on data. Also, one might be able to answer questions concerning stability or instability of the solution if a solution exists. With regard to stability analysis, one usually defines a functional based on the solution, which when the parameters that appear in it take on appropriate values becomes a positive definite quantity and can be associated with the energy of the system, and one can study the stability of the solution to the partial differential equation. However, if the parameters take on values which imply that the functional is not positive definite, then using logarithmic convexity arguments one can show the blow up of the functional. That is, one can show that an erroneous physical assumption leads to unacceptable physical response. An interesting analysis of the same is the situation when the Elasticity Tensor is not positive definite (see [3–6]). In such a situation one can use logarithmic convexity arguments to prove instability of the solution.

Logarithmic convexity arguments have been used frequently in partial differential equations that are first and second order in time. The method has been used in the study of the Moore-Gibson Thompson heat equation (third order in time) [6], and for the high order backward in time parabolic equations [7]. In the last paper, the uniqueness of solutions was established. Given the paucity of studies of problems where partial differential equations that have derivatives of order higher than two, we investigate equations that have third order in time using the notion of logarithmic convexity. Uniqueness and instability of solutions will be a direct consequence of our approach, but it is worth noting that many other qualitative results could be developed.

The paper is structured as follows. In the next section, we describe briefly the problem for which the uniqueness and the instability of solutions are proved in the third section. The main idea is to use the logarithmic convexity arguments. As an example of application, in the fourth section the specific case corresponding to the three-phase-lag model is presented. An extension to the analysis in a Hilbert space is finally discussed.

2 Basic Equations

Let us denote by B a three-dimensional bounded domain whose boundary ∂B is assumed smooth enough so that the divergence theorem can be applied.

We will study several qualitative properties of the solutions to the problem determined by the equation:

$$\lambda_0 \dot{u} + \lambda_1 \ddot{u} + \lambda_2 \dddot{u} = \kappa_1 \Delta u + \kappa_2 \Delta \dot{u} + \kappa_3 \Delta \ddot{u} \quad \text{in} \quad B. \tag{1}$$

To define a well-posed problem we need to impose the boundary condition:

$$u(\boldsymbol{x}, t) = 0 \quad \text{for a.e.} \quad \boldsymbol{x} \in \partial B, \quad t \geq 0, \tag{2}$$

and the initial conditions, for a.e. $\boldsymbol{x} \in B$,

$$u(\boldsymbol{x}, 0) = u^0(\boldsymbol{x}), \quad \dot{u}(\boldsymbol{x}, 0) = u^1(\boldsymbol{x}), \quad \ddot{u}(\boldsymbol{x}, 0) = u^2(\boldsymbol{x}). \tag{3}$$

It is worth noting that the following equality

$$E(t) + 2 \int_0^t D(s)\, ds = E(0) \tag{4}$$

holds for the solutions to problem (1)-(3), where

$$E(t) = \int_B \Big[(\lambda_1 \dot{u} + \lambda_2 \ddot{u} - \kappa_3 \Delta \dot{u})^2 + \kappa_1 \lambda_1 (\nabla (u + \xi \dot{u}))^2 \\ + \xi (\kappa_2 \lambda_1 - \kappa_1 \lambda_2) |\nabla \dot{u}|^2 + \kappa_1 \kappa_3 |\Delta u|^2 + \lambda_0 \lambda_2 |\dot{u}|^2 \Big] \, dv, \tag{5}$$

and

$$D(t) = \int_B \left[(\lambda_1\kappa_2 - \lambda_2\kappa_1)|\nabla \dot{u}|^2 + \lambda_0\lambda_1|\dot{u}|^2 + \kappa_2\kappa_3|\Delta\dot{u}|^2 + \lambda_0\kappa_3|\nabla\dot{u}|^2 \right] dv. \quad (6)$$

Here, $\xi = \lambda_2\lambda_1^{-1}$. We note that this equality is satisfied for any sign for the parameters.

A direct consequence of (4)-(6) is that, if we assume that $\lambda_i > 0$ and $\kappa_i > 0$ for $i = 1, 2, 3$, and $\lambda_1\kappa_2 - \lambda_2\kappa_1 > 0$, we obtain stability of solutions.

3 Logarithmic convexity

In this section, we consider logarithmic convexity argument for the problem determined by equation (1) with boundary conditions (2) and initial conditions (3). As usual, the basic idea is to work with a "good" function satisfying an appropriate inequality (see (11)). Thus, we define the function

$$\begin{aligned} G_{\omega,t_0}(t) &= \int_B (\lambda_1 u + \lambda_2\dot{u} - \kappa_3\Delta u)^2 dv + \omega(t+t_0)^2 \\ &+ \int_0^t \int_B \left[(\lambda_1\kappa_2 - \lambda_2\kappa_1 + \lambda_0\kappa_3)|\nabla u|^2 + \lambda_0\lambda_1|u|^2 + \kappa_2\kappa_3|\Delta u|^2 \right] dvds, \quad (7) \end{aligned}$$

where ω and t_0 are two positive constants to be chosen later.

Since we desire that this function defines a measure on the solutions, we would need to assume that

$$\lambda_0\lambda_1 \geq 0, \quad \kappa_2\kappa_3 \geq 0 \quad \text{and} \quad \lambda_1\kappa_2 + \kappa_3\lambda_0 \geq \lambda_2\kappa_1. \quad (8)$$

From now onwards, we will assume that inequalities (8) hold. In fact, we could also assume that at least one of the inequalities is strictly positive.

A direct differentiation of (7) with respect to time gives

$$\begin{aligned} \dot{G}_{\omega,t_0}(t) &= 2\int_B (\lambda_1 u + \lambda_2\dot{u} - \kappa_3\Delta u)(\lambda_1\dot{u} + \lambda_2\ddot{u} - \kappa_3\Delta\dot{u})\, dv \\ &+ \int_B \left[(\lambda_1\kappa_2 - \lambda_2\kappa_1 + \lambda_0\kappa_3)|\nabla u^0|^2 + \lambda_0\lambda_1|u^0|^2 + \kappa_2\kappa_3|\Delta u^0|^2 \right] dv \\ &+ 2\int_0^t \int_B \left[(\lambda_1\kappa_2 - \lambda_2\kappa_1 + \lambda_0\kappa_3)\nabla u\nabla\dot{u} + \lambda_0\lambda_1 u\dot{u} + \kappa_2\kappa_3\Delta u\Delta\dot{u} \right] dvds \\ &+ 2\omega(t+t_0). \end{aligned} \quad (9)$$

It is worth noting that the following equality holds:

$$\begin{aligned}\dot{G}_{\omega,t_0}(t) &= 2\int_B (\lambda_1 u + \lambda_2 \dot{u} - \kappa_3 \Delta u)(\lambda_1 \dot{u} + \lambda_2 \ddot{u} - \kappa_3 \Delta \dot{u})\, dv \\ &+ \int_B \Big[(\lambda_1\kappa_2 - \lambda_2\kappa_1 + \lambda_0\kappa_3)|\nabla u|^2 + \lambda_0\lambda_1 |u|^2 + \kappa_2\kappa_3|\Delta u|^2\Big]\, dvds \\ &+ 2\omega(t+t_0).\end{aligned}$$

We also have

$$\begin{aligned}\ddot{G}_{\omega,t_0}(t) = {} & 2\int_B (\lambda_1 \dot{u} + \lambda_2 \ddot{u} - \kappa_3 \Delta \dot{u})^2\, dv \\ & + 2\int_B \Big[(\lambda_1 u + \lambda_2 \dot{u} - \kappa_3 \Delta u)(\lambda_1 \ddot{u} + \lambda_2 \dddot{u} - \kappa_3 \Delta \ddot{u})\Big]\, dv \\ & + 2\int_B \Big[(\lambda_1\kappa_2 - \lambda_2\kappa_1 + \lambda_0\kappa_3)\nabla u \nabla \dot{u} + \lambda_0\lambda_1 u\dot{u} + \kappa_2\kappa_3 \Delta u \Delta \dot{u}\Big]\, dv + 2\omega.\end{aligned}$$

If we define the function

$$\begin{aligned}I = {} & (\lambda_1 u + \lambda_2 \dot{u} - \kappa_3 \Delta u)(\lambda_1 \ddot{u} + \lambda_2 \dddot{u} - \kappa_3 \Delta \ddot{u}) \\ & + (\lambda_1\kappa_2 - \lambda_2\kappa_1 + \lambda_0\kappa_3)\nabla u \nabla \dot{u} + \lambda_0\lambda_1 u\dot{u} + \kappa_2\kappa_3 \Delta u \Delta \dot{u},\end{aligned}$$

we have

$$\begin{aligned}I = {} & (\lambda_1 u + \lambda_2 \dot{u} - \kappa_3 \Delta u)(\kappa_1 \Delta u + \kappa_2 \Delta \dot{u} - \lambda_0 \dot{u}) \\ & + (\lambda_1\kappa_2 - \lambda_2\kappa_1 + \lambda_0\kappa_3)\nabla u \nabla \dot{u} + \lambda_0\lambda_1 u\dot{u} + \kappa_2\kappa_3 \Delta u \Delta \dot{u}.\end{aligned}$$

Therefore, we find that

$$\int_B I\, dv = -\int_B \Big[\lambda_1\kappa_1(\nabla u + \xi \nabla \dot{u})^2 + \xi(\kappa_2\lambda_1 - \kappa_1\lambda_2)|\nabla \dot{u}|^2 + \kappa_1\kappa_3|\Delta u|^2 + \lambda_0\lambda_2|\dot{u}|^2\Big]\, dv.$$

This then leads to

$$\begin{aligned}\ddot{G}_{\omega,t_0}(t) = {} & 4\int_B (\lambda_1 \dot{u} + \lambda_2 \ddot{u} - \kappa_3 \Delta \dot{u})^2\, dv + 2(\omega - E(0)) \\ & + 4\int_0^t \int_B \Big[(\lambda_1\kappa_2 - \kappa_1\lambda_2 + \lambda_0\kappa_3)|\nabla \dot{u}|^2 + \kappa_2\kappa_3|\Delta u|^2 + \lambda_0\lambda_1|\dot{u}|^2\Big]\, dvds. \quad (10)\end{aligned}$$

In view of equalities (7)-(10) and after a systematic use of the Hölder inequality one obtains that

$$\ddot{G}_{\omega,t_0}(t) G_{\omega,t_0}(t) - \left(\dot{G}_{\omega,t_0} - \frac{\eta}{2}\right)^2 \geq -2(\omega + E(0)) G_{\omega,t_0}(t), \qquad (11)$$

where

$$\eta = 2\int_B \left[(\kappa_2\lambda_1 - \kappa_1\lambda_2 + \lambda_0\kappa_3)|\nabla u^0|^2 + \lambda_0\lambda_1|u^0|^2 + \kappa_2\kappa_3|\Delta u^0|^2 \right] dv.$$

We will use the inequality (11) to establish the main result.

3.1 Uniqueness

We will deduce from inequality (11) the uniqueness of the solutions whenever conditions (8) hold.

In fact, to prove the uniqueness of the solutions it is enough to see that the unique solution corresponding to the null initial conditions is the null solution. We note that, in the case that we impose null initial conditions, inequality (11) implies that

$$\ddot{G}G - (\dot{G})^2 \geq 0, \tag{12}$$

where we denote $G(t) = G_{0,0}(t)$.

From inequality (12) we obtain that

$$\frac{d^2}{dt^2}\left[\ln G(t)\right] \geq 0 \quad \text{for} \quad t \in [0, T],$$

and so, it follows that $\ln G(t)$ is a convex function. Hence, we can deduce that

$$G(t) \leq G(0)^{1-t/T} G(T)^{t/T} \quad \text{for} \quad 0 \leq t \leq T.$$

Therefore, we find that $G(t)$ vanishes in the interval $[0, T]$ and we can conclude that

$$u(t) = 0 \quad \text{for} \quad 0 \leq t \leq T.$$

It leads to the uniqueness of solutions that we state as follows.

Theorem 1 *Let us assume that conditions (8) hold. Therefore, the problem determined by (1)-(3) has at most one solution.*

It is worth noting that, in the case that λ_2 and κ_3 are different from zero, the uniqueness of solutions is a consequence of the results obtained by [8] and [7]. However, in the case that $\kappa_3 = 0$, we obtain a new uniqueness result for the equation

$$\lambda_0\dot{u} + \lambda_1\ddot{u} + \lambda_2\dddot{u} = \kappa_1\Delta u + \kappa_2\Delta\dot{u},$$

with the initial and boundary conditions (2) and (3), respectively.

We note that the uniqueness is guaranteed whenever

$$\lambda_0\lambda_1 > 0 \quad \text{and} \quad \kappa_2\lambda_1 > \lambda_2\kappa_1. \tag{13}$$

In virtue of the results obtained in [8], we are aware of an existence and uniqueness result for this problem when $\kappa_2\lambda_2 > 0$. It is clear that our result applies for a new class of assumptions. At the same time, this result extends the one obtained in [6] for the Moore-Gibson-Thompson equation.

We can use the same analysis for the equation:

$$\lambda_0(\boldsymbol{x})\dot{u} + \lambda_1\ddot{u} + \lambda_2\dddot{u} = (b_{ij}(\boldsymbol{x})u_{,i})_{,j} + (k_{ij}(\boldsymbol{x})\dot{u}_{,i})_{,j}. \tag{14}$$

In this case, we need to assume that $\lambda_1\lambda_0(\boldsymbol{x})$ and that $\lambda_1 k_{ij}(\boldsymbol{x}) - \lambda_2 b_{ij}(\boldsymbol{x})$ is positive definite.

It is worth noting that, for this equation, we must use the new functions:

$$\begin{aligned}
E(t) &= \int_B \Big[(\lambda_1\dot{u} + \lambda_2\ddot{u})^2 + \lambda_1 b_{ij}(\boldsymbol{x})(u_{,i} + \xi\dot{u}_{,i})(u_{,j} + \xi\dot{u}_{,j}) \\
&\quad + \xi(\lambda_1 k_{ij}(\boldsymbol{x}) - \lambda_2 b_{ij}(\boldsymbol{x}))\dot{u}_{,i}\dot{u}_{,j} + \lambda_2\lambda_0(\boldsymbol{x})|\dot{u}|^2\Big]\, dv, \\
D(t) &= \int_B \Big[(\lambda_1 k_{ij}(\boldsymbol{x}) - \lambda_2 b_{ij}(\boldsymbol{x}))\dot{u}_{,i}\dot{u}_{,j} + \lambda_1\lambda_0(\boldsymbol{x})|\dot{u}|^2\Big]\, dv, \\
G_{\omega,t_0}(t) &= \int_B (\lambda_1 u + \lambda_2\dot{u})^2 + 2\omega(t+t_0)^2 \\
&\quad + \int_0^t \int_B \Big[(\lambda_1 k_{ij}(\boldsymbol{x}) - \lambda_2 b_{ij}(\boldsymbol{x}))u_{,i}u_{,j} + \lambda_1\lambda_0(\boldsymbol{x})u^2\Big]\, dvds.
\end{aligned}$$

3.2 *Instability of solutions*

In this subsection we will prove the instability of solutions whenever we assume that $E(0) \leq 0$. As $E(t)$ denotes the energy associated with the equation (1), then assuming it to be negative is not a physically meaningful assumption. Thus, it is not surprising that a consequence of such an assumption is instability of the solution. In fact, we will show that the solution grows in an exponential way.

Under the assumption $E(0) < 0$, we can choose $\omega = -E(0)$. Then, inequality (11) implies that

$$\ddot{G}_{\omega,t_0}(t)G_{\omega,t_0}(t) \geq \dot{G}_{\omega,t_0}(t)(\dot{G}_{\omega,t_0}(t) - \eta),$$

and we can write

$$\frac{\ddot{G}_{\omega,t_0}(t)}{\dot{G}_{\omega,t_0}(t)} \geq \frac{\dot{G}_{\omega,t_0}(t)}{G_{\omega,t_0}(t)}. \tag{15}$$

Now, we select t_0 large enough to guarantee that $\dot{G}_{\omega,t_0}(0) - \eta > 0$. From estimate (15) it follows that

$$\ln\left[\frac{\dot{G}_{\omega,t_0}(t)-\eta}{G_{\omega,t_0}(t)}\right] \geq \ln\left[\frac{\dot{G}_{\omega,t_0}(0)-\eta}{G_{\omega,t_0}(0)}\right]. \tag{16}$$

Estimate (16) implies that

$$\dot{G}_{\omega,t_0}(t) \geq \frac{\dot{G}_{\omega,t_0}(0)-\eta}{G_{\omega,t_0}(0)} G_{\omega,t_0}(t)+\eta.$$

After an integration we obtain

$$G_{\omega,t_0}(t) \geq \frac{G_{\omega,t_0}(0)\dot{G}_{\omega,t_0}(0)}{\dot{G}_{\omega,t_0}(0)-\eta} \exp\left[\frac{\dot{G}_{\omega,t_0}(0)-\eta}{G_{\omega,t_0}(0)} t\right] - \frac{\eta G_{\omega,t_0}(0)}{\dot{G}_{\omega,t_0}(0)-\eta}. \tag{17}$$

It then follows that

$$G(t) \geq \frac{G_{\omega,t_0}(0)\dot{G}_{\omega,t_0}(0)}{\dot{G}_{\omega,t_0}(0)-\eta} \exp\left[\frac{\dot{G}_{\omega,t_0}(0)-\eta}{G_{\omega,t_0}(0)} t\right] - \frac{\eta G_{\omega,t_0}(0)}{\dot{G}_{\omega,t_0}(0)-\eta} - \omega(t+t_0)^2,$$

which gives an exponential type growth of the solution.

In the case that $E(0)=0$ and $\dot{G}_{0,0}(0)>\eta$, we can also obtain the previous inequality when $\omega=t_0=0$.

Therefore, we have proved the following.

Theorem 2 *Let us assume that conditions (8) hold and let us consider the solution such that $E(0)<0$ or $E(0)=0$ and $\dot{G}_{0,0}(0)>\eta$. Then, this solution grows in an exponential way.*

The previous theorem applies to the general case but it is possible to show that, if we consider $\kappa_1<0$ and $u^1=u^2=0$, $u^0\neq 0$, then we can obtain that $E(0)<0$. The result also applies when $\kappa_3=0$.

In particular, the analysis also applies to the equation (14).

4 Three-phase-lag model

Let us consider the three-phase-lag equation. The parabolic version is given by

$$\ddot{T}+\tau_1\dddot{T}=\kappa^*\Delta T+(\kappa+\kappa^*\tau_3)\Delta\dot{T}+\kappa\tau_2\Delta\ddot{T}. \tag{18}$$

Here, τ_1, τ_2 and τ_3 are three positive constants which represent the relaxation parameters, κ is the thermal conductivity (assume to be positive) and κ^* is the rate thermal conductivity. It is worth noting that the axioms of thermomechanics do not imply that κ^* must be positive. Therefore, we could assume that it is positive or negative. We note that, in this case, we have

$$\lambda_0 = 0, \quad \lambda_1 = 1, \quad \lambda_2 = \tau_1, \quad \kappa_1 = \kappa^*, \quad \kappa_2 = \kappa + \kappa^* \tau_3,$$
$$\kappa_3 = \kappa \tau_2, \quad \xi = \tau_1.$$

Conditions (8) are satisfied whenever

$$\kappa + \kappa^* \tau_3 > 0, \quad \kappa + \kappa^* \tau_3 > \tau_1 \kappa^*.$$

We note that in the case

$$\tau_3 \kappa^* > -\kappa \quad \text{and} \quad \kappa^* < 0$$

the assumptions are satisfied.

Finally, we remark that, in this case, $E(0) < 0$ if we assume that $u^0 \neq 0$ but $u^1 = u^2 = 0$. Therefore, the instability of solutions can be obtained.

The backward in time version of equation (18) is

$$-\ddot{T} + \tau_1 \dddot{T} = -\kappa^* \Delta T + (\kappa + \kappa^* \tau_3) \Delta \dot{T} - \kappa \tau_2 \Delta \ddot{T}. \tag{19}$$

In this case, we have

$$\lambda_0 = 0, \quad \lambda_1 = 1, \quad \lambda_2 = \tau_1, \quad \kappa_1 = -\kappa^*, \quad \kappa_2 = \kappa + \kappa^* \tau_3,$$
$$\kappa_3 = -\kappa \tau_2.$$

If we assume that $\tau_1 \kappa^* < \kappa + \kappa^* \tau_3 < 0$, then conditions (8) hold. This assumption would be satisfied for suitable choices of the parameter when $\kappa^* < 0$. This gives a new uniqueness result as well as an instability result for equation (19).

5 Further comments

The proposed analysis concerning the logarithmic convexity argument can be adapted "word by word" to the equations proposed in a Hilbert space:

$$\lambda_0 \dot{u} + \lambda_1 \ddot{u} + \lambda_2 \dddot{u} = \kappa_1 A u + \kappa_2 A \dot{u} + \kappa_3 A \ddot{u},$$

when A is a symmetric positive definite operator. In this case, the energy of the system is

$$E(t) = \|\lambda_1\dot{u} + \lambda_2\ddot{u} - \kappa_3 A\dot{u}\|^2 + \lambda_1\kappa_1\|A^{1/2}(u+\xi\dot{u})\|^2$$
$$+\xi(\kappa_2\lambda_1 - \lambda_2\kappa_1)\|A^{1/2}\dot{u}\|^2 + \lambda_0\lambda_1\|\dot{u}\|^2 + \kappa_1\kappa_3\|Au\|^2,$$

and the dissipation will be

$$D(t) = (\kappa_2\lambda_1 - \lambda_2\kappa_1)\|A^{1/2}\dot{u}\|^2 + \lambda_0\lambda_1\|\dot{u}\|^2 + \kappa_2\kappa_3\|A\dot{u}\|^2 + \lambda_0\kappa_3\|A^{1/2}\dot{u}\|^2.$$

In this case, we can define the function

$$G_{\omega,t_0}(t) = \|\lambda_1 u + \lambda_2\dot{u} - \kappa_3 Au\|^2 + 2\omega(t+t_0)^2$$
$$+\int_0^t \left[(\kappa_2\lambda_1 - \lambda_2\kappa_1 + \lambda_0\kappa_3)\|A^{1/2}u\|^2 + \lambda_0\lambda_1\|u\|^2 + \kappa_2\kappa_3\|Au\|^2\right] ds.$$

We can consider the viscoelastic proposition introduced by Lebedev and Gladwell [9] to consider the system of equations

$$\lambda_1\ddot{u}_i + \lambda_2\dddot{u}_i = \left(\kappa_1 C_{ijkl}u_{k,l} + \kappa_2 C_{ijkl}\dot{u}_{k,l} + \kappa_3 C_{ijkl}\ddot{u}_{k,l}\right)_{,j},$$

where λ_i and κ_i are given constants, and C_{ijkl} is the elasticity tensor satisfying

$$C_{ijkl} = C_{klij},$$

and there exists a positive constant C such that

$$\int_B C_{ijkl}u_{i,j}u_{k,l}\,dv \geq C\int_B u_{i,j}u_{i,j}\,dv$$

for every vector function (u_i) vanishing at the boundary.

We can consider the Hilbert space $\mathcal{H} = [L^2(B)]^3$ and so, we can define the operator

$$A_i\boldsymbol{u} = (C_{ijkl}u_{k,l})_{,j}, \quad \boldsymbol{A} = (A_i).$$

Our arguments can be used in this situation.

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