Contributions to the theory of near-vector spaces, their geometry, and hyperstructures

by

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Declaration

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Abstract

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This thesis expands on the theory and application of near-vector spaces — in particular, the underlying geometry of near-vector spaces is studied, and the theory of near-vector spaces is applied to hyperstructures.

More specifically, a near-linear space is defined and some properties of these spaces are proved. It is shown that by adding some axioms, the nearaffine space, as defined by André, is obtained. A correspondence is shown between subspaces of nearaffine spaces generated by near-vector spaces, and the cosets of subspaces of the corresponding near-vector space. As a highlight, some of the geometric results are used to prove an open problem in near-vector space theory, namely that a non-empty subset of a near-vector space that is closed under addition and scalar multiplication is a subspace of the near-vector space. The geometric work of this thesis is concluded with a first look into the projections of nearaffine spaces, a branch of the geometry that contains interesting avenues for future research.

Next the theory of hyper near-vector spaces is developed. Hyper near-vector spaces are defined having similar properties to André's near-vector space. Important concepts, including independence, the notion of a basis, regularity, and subhyperspaces are defined, and an analogue of the Decomposition Theorem, an important theorem in the study of near-vector spaces, is proved for these spaces.

Uittreksel

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Hierdie tesis bou op die teorie en toepassing van naby-vektorruimtes — besonderlik word die onderliggende meetkunde van naby-vektorruimtes bestudeer en die teorie van naby-vektorruimtes word toegepas op hiperstrukture.

Spesifiek work 'n naby-lineêre ruimte gedefinieer en sommige eienskappe van hierdie ruimtes word bewys. Dit word bewys dat, deur sekere aksiomas by te las, die naby-affiene ruimte, soos gedefinieer deur André, verkry word. 'n Verwantskap tussen die deelruimtes van naby-affiene ruimtes gegenereer deur naby-vektorruimtes en die resklasse van die deelruimtes van die verwante naby-vektorruimte word bewys. As 'n hoogtepunt word van die meetkundige resultate gebruik om 'n oop probleem op te los in naby-vektorruimteteorie, naamlik dat 'n nie-leë deelversameling van 'n naby-vektorruimte wat geslote is onder optelling en skalaarvermenigvuldiging 'n deelruimte is van die naby-vektorruimte. Die meetkundige werk in dié tesis sluit af met 'n eerste bestudering van projeksies van naby-affiene ruimtes, 'n tak in die meetkunde wat interessante toekomstige navorsingsrigtings bevat.

Volgende word die teorie agter hiper naby-vektorruimtes ontwikkel. Hiper naby-vektorruimtes word gedefinieer soortgelyk aan André se naby-vektorruimte. Belangrike konsepte, insluitent onafhanklikheid, die begrip van 'n basis, regulêriteit en hiper-deelruimtes word gedefinieer en 'n analoog van die Ontbindingstelling, belangrik in die teorie van naby-vektorruimtes, word bewys vir hierdie ruimtes.

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Chapter 1

Introduction

The idea of a vector space was generalised to a structure comprising a bit more non-linearity by various authors, the so-called near-vector space (see [1], [4], [13], and [14]). The weakening of the axioms compared to a vector space, results in a space with only one distributive law holding in general. André's near-vector spaces are the focus of my work. Recently there have been a number of papers investigating their algebraic structure (see for example [9],[11]). For an analysis of the definition André proposed for near-vector spaces, we refer the reader to [8].

Geometry was key to André's motivation for defining his near-vector spaces (see [2] and [3]). These structures give rise to what is known as a nearaffine space, a structure of particular importance in the study of noncommutative geometry (incidence structures in which the line joining two points depends on the order in which you join them). The algebraic and geometric structure of near-vector spaces has been the focus of my work so far.

This thesis adds new insight into the study of near-vector spaces: starting with, a brief study of the behaviour of the quasi-kernel of near-vector spaces. In particular, it is shown that, for a set of compatible elements of the quasi-kernel of a near-vector space, any linear combination of elements of this set will be in the quasi-kernel, This ultimately made the proof of the Decomposition Theorem of near-vector spaces (which states that every near-vector space may be expressed as the direct sum of its maximal regular subspaces) more accessible.

The discovery of two papers by André, *On Finite Non-Commutative Affine Spaces* ([2]) and *Some New Results on Incidence Structures* ([3]) inspired a renewed study in the geometry of near-vector spaces. These papers are in the field of incidence geometry, and lay out the geometric structure of near-vector spaces. The book *Foundations of Incidence Geometry* ([23]) served as reference for the exploration of incidence structures.

In particular, the nearaffine space as defined in [2] and [3] is studied.Similar to affine spaces, these geometries have points, lines, and a parallelism. However, the lines joining two points differ depending on the order in which you connect them. With André's papers as reference, a number of new results are proved, and results from his first paper from the finite case are generalised. This research resulted in a paper, *Geometries with Non-Commutative Joins and their Application to Near-Vector Spaces* ([12]), which has been published online.

In this paper, a near-linear space, a generalisation of a linear space, is defined by taking inspiration from André's definition of a nearaffine space, and some preliminary results are proved. These structures, and the geometries that may arise from them, have produced a new avenue of research in incidence geometry. In particular, it is shown show that, for an arbitrary nearaffine space, a subset of its point set is a subspace if and only if it is a weak subspace — a proof of this result was given in [2] for finite nearaffine spaces, but it contained an omission. Next nearfield spaces (nearaffine spaces over near-vector spaces) are investigated, and it is shown that, as with affine spaces over vector spaces, a subset of the near-vector space is a subspace of the nearfield space if and only if it is the coset of a subspace of a nearaffine space. Finally, the paper ends with a preliminary investigation into projections of nearaffine spaces. These structures are defined similarly to the projective space over a vector space, and initial results show that they are very similar to projective spaces; however, there is still scope for future research.

Subspaces of near-vector spaces were first defined in [9] and it was proved that a subset of a near-vector space is a subspace if and only if it is closed under addition and scalar multiplication; however, the proof of the converse was incomplete, as picked up by Sophie Marques in 2019. In particular, it was unclear how the so-called quasi-kernel of the subset would generate the subset as a group — a requirement for near-vector spaces. This was partially remedied in [10], where Sophie Marques showed that the result holds in the specific case for near-vector spaces over division rings. Subsequently, using the aforementioned geometric results in [12], the result is proved to hold in general. This is of particular interest, as it shows how the understanding of the geometry of near-vector spaces can inform the understanding of their algebraic structure.

In [9], the above result was used to characterise the subspaces of certain near-vector spaces constructed using copies of nearfields. As a result of work mentioned above, these characterisations may now be used. The result also ensures that the kernel of near-linear mappings will necessarily form a subspace of its domain — this fundamentally links near-linear mappings to quotient spaces (first explored by Wessels in [27]). Wessels also defined the subspace inclusion graph of a near-vector space in her PhD thesis and the result will be useful in the further exploration of these graphs of different constructions of near-vector spaces, which now can be identified with the incidence graph of the associated geometry.

Following the chapter on the geometry of near-vector spaces, a new algebraic structure is defined — the hyper near-vector space. This structure is a generalisation of a near-vector space, where instead of an additive group of vectors, one has a canonical hypergroup of vectors (in which the sum of any two vectors gives a non-empty set of vectors). In particular, the hyper near-vector space behaves surprisingly well, allowing for notions of independence, bases, subhyperspaces and even an analogue of André's Decomposition Theorem. The culmination of this work resulted in a second paper, *Hyper near-vector spaces* ([6]), which has been submitted for review.

Throughout this thesis, for any set S, define S^* as $S \setminus \{0\}$.

Chapter 2

Preliminary Material on Near-Vector Spaces

2.1 Definition and some basic results

The purpose of this chapter is to give the reader an overview of the construction of a near-vector space. The idea that André used is to think of the elements of a group as the vectors and a set of endomorphisms of the group as the scalars of the group. This idea gives rise to the following definition.

Definition 2.1. ([1], p. 303)

A pair (V, A) is called a *near-vector space* if the following conditions hold.

1. $(V, +)$ is a group and A is a set of endomorphisms of V.
2. A contains the endomorphisms $0, id_V$, and $-id_V$ (hereafter simply $0, 1, -1$).
3. If A^* is nonempty, then it is a subgroup of $\mathrm{Aut}(V)$.
4. A acts freely on V, that is, for each $x \in V$ and $\alpha, \beta \in A$, if $\alpha x = \beta x$, then $x = 0$ or $\alpha = \beta$.
5. The *quasi-kernel* of V, defined by

   $$Q(V) = \{x \in V \,|\, \forall \alpha, \beta \in A \; \exists \gamma \in A \, [\alpha x + \beta x = \gamma x]\}$$

   generates V as a group, i.e. for every $v \in V$, there exists $u_1, \ldots, u_n \in Q(V)$ such that $v = \sum_{i=1}^{n} u_i$.

It might be strange to add the condition that A^* be non-empty before we assume that it is a subgroup of $\text{Aut}(V)$ in the above definition. In fact, André does not require this condition. However, this is for a very simple reason: when $V = \{0\}$, $1 = -1 = 0$ in A. In fact, any endomorphism of V would be 0, so A^* would be empty. We add this condition, because we want $\{0\}$ to have a near-vector space construction. This is useful, for without it, the trivial vector space is not a near-vector space. However, the case where $V = \{0\}$ is the only case where A^* is empty, so generally results are proven with the assumption that A^* is a group and with the understanding that these results apply to the zero space as well.

Notation. From now on, the quasi-kernel $Q(V)$ of a near-vector space (V, A) will be denoted just as Q, if there is no confusion.

Example 2.2. ([11], p. 89) The smallest example of a proper near-vector space (i.e. a near-vector space that is not also a vector space) is $(V, A) = (\mathbb{Z}_5^2, \mathbb{Z}_5)$, where for each $(x, y) \in V$ and $\alpha \in A$, the corresponding map $\alpha : V \to V$ is defined by

$$\alpha(x, y) = (\alpha x, \alpha^3 y).$$

Here, $Q = \{(x, 0) \mid x \in \mathbb{Z}_5\} \cup \{(0, y) \mid y \in \mathbb{Z}_5\}$.

Some basic results follow.

Lemma 2.3. ([1], p. 297) Let (V, A) be a near-vector space. Then $(V, +)$ is an abelian group.

Proof. Let $a, b, c, d \in V$ such that $a = -c$ and $b = -d$. Then $a+b = (-c)+(-d) = -(d + c)$. But $-(d + c) = (-1)(d + c) = (-1)d + (-1)c = (-d) + (-c)$, since -1 is an endomorphism of V by 2. in Definition 2.1. Therefore $a + b = -(d + c) = (-d) + (-c) = b + a$ and so it follows V is abelian. ■

A natural question to ask would be whether all vector spaces are near-vector spaces. Indeed, this is the case.

Lemma 2.4. Let V be a vector space over a field F. Then (V, F) is a near-vector space. Here, the endomorphisms in F are given by the scalar multiplication of the vector space. In this case, $Q = V$.

The quasi-kernel seems to be a very arbitrary construction. One way to think of it would be as the set of linear axes of the near-vector space. The following results strengthen this idea.

Lemma 2.5. *(Properties of the quasi-kernel, [1], p. 299)* The quasi-kernel Q of a near-vector space (V, A) has the following properties.

a. $0 \in Q$.

b. For each $u \in Q^*$ and $\alpha, \beta \in A$, if $\alpha u + \beta u = \gamma u$, then γ is uniquely determined by α and β.

c. If $u \in Q$ and $\lambda \in A$, $\lambda u \in Q$, i.e. $AQ \subseteq Q$.

d. If $u \in Q$ and $\lambda_i \in A$ for each $i \in \{1, \ldots, n\}$, then $\sum_{i=1}^{n} \lambda_i u = \eta u \in Q$ for some $\eta \in A$.

e. If $u \in Q$ and $\alpha, \beta \in A$, there exists $\gamma \in A$ such that $\alpha u - \beta u = \gamma u$.

2.2 A new addition on A

As with vector spaces, one would generally desire some type of addition for the scalars of near-vector spaces. Unfortunately, $\alpha + \beta$ is not necessarily in A if α and β are. This can be partially remedied by the elements of the quasi-kernel: for each nonzero element of Q, we may define an addition as below.

Definition 2.6. ([1], p.299) Let (V, A) be a near-vector space, with $u \in Q^*$. Define the binary operation $+_u$ on A such that for all $\alpha, \beta \in A$,

$$(\alpha +_u \beta)u = \alpha u + \beta u.$$

The operation $+_u$ is well-defined, by Lemma 2.5b.

Example 2.7. Consider the near-vector space in Example 2.2. As noted before, $Q = \{(x, 0) \,|\, x \in \mathbb{Z}_5\} \cup \{(0, y) \,|\, y \in \mathbb{Z}_5\}$. In this case

$$\alpha +_{(x,0)} \beta = \alpha + \beta$$

for all $\alpha, \beta \in A$ and $x \in \mathbb{Z}_5^*$, and

$$\alpha +_{(0,y)} \beta = \left(\alpha^3 + \beta^3\right)^{\frac{1}{3}}$$

for all $\alpha, \beta \in A$ and $y \in \mathbb{Z}_5^*$.

The next natural question to ask would be whether this addition gives us a group structure on A.

Lemma 2.8. ([1], p.300) Let (V, A) be a near-vector space, with $u \in Q^*$. Then $(A, +_u)$ is an abelian group.

Lemma 2.9. ([1], p.301) Let (V, A) be a near-vector space and suppose that $u, v \in Q^*$ such that $v \notin Au$. Suppose $\alpha u + \beta v = \alpha' u + \beta' v$ for some $\alpha, \alpha', \beta, \beta' \in A$. Then $\alpha = \alpha'$ and $\beta = \beta'$.

Definition 2.10. ([1], p.300) Let $\lambda \in A^*$. Define $\alpha^\lambda = \lambda^{-1}\alpha\lambda$ for all $\alpha \in A$.

Of course, for $u, v \in Q(V)$, $+_u$ and $+_v$ do not in general have to be equal. However, when some elements of the quasi-kernel do have the same addition, we group them together in the following way.

Definition 2.11. ([1], p.301) Let (V, A) be a near-vector space and let $u \in Q^*$. Define the *kernel* $R_u(V) = R_u$ of (V, A) by the set

$$R_u = \{v \in V \mid (\alpha +_u \beta)v = \alpha v + \beta v \text{ for every } \alpha, \beta \in A\}.$$

Lemma 2.12. ([1], p.301) *(Properties of R_u)*

Let (V, A) be a near-vector space and let $u \in Q^*$. Then

a. $u \in R_u$.

b. $R_u \subseteq Q$.

c. $(R_u, +)$ is a subgroup of $(V, +)$.

2.3 Independence and a basis for Q

The next question one might have about André's near-vector space construction would be whether there is a suitable definition of a basis for a near-vector space, and what properties this basis may have that are analogous to that of a vector space. In particular, we would like the definition of a basis for a near-vector space to coincide with that of a vector space when our near-vector space is a vector space.

For this, some compromises have to be made. Because, generally, the elements of a near-vector space outside its quasi-kernel are not well-behaved, we rather build a basis for the quasi-kernel from its elements. The fifth axiom of near-vector spaces then ensures that this basis generates the whole near-vector space. This compromise does not affect our requirement that the definition must coincide with that of a vector space, for the quasi-kernel of a vector space is the whole space, as remarked before.

As near-vector spaces are not the only structure for which we will define independence, it is useful to generalise the concept, by introducing a dependence relation.

Definition 2.13. A *dependence relation* is a relation between a set X and its power set $\mathcal{P}(X)$, denoted by $v \lhd M$ where $v \in X$ and $M \subseteq X$ such that the following conditions hold for $u, v, w \in X$ and $M, N \subseteq X$.

(D1) If $v \in M$, then $v \lhd M$.

(D2) If $w \lhd M$ and $v \lhd N$ for each $v \in M$, then $w \lhd N$.

(D3) If $v \lhd M$ and $v \not\lhd M \setminus \{u\}$, then $u \lhd (M \setminus \{u\}) \cup \{v\}$.

Example 2.14. A trivial (but not very illuminating) example of a dependence relation is the relation $\in$ on any set. A more illuminating example is the relation $\lhd \subseteq V \times \mathcal{P}(V)$ for a vector space V, where $x \lhd U$ if $x \in \text{span}(U)$ for any $x \in V$ and $U \subseteq V$.

The axioms of a dependence relation are given by Van der Waerden in [24] as fundamental theorems of independence, from which all the necessary properties of independence are derived. Later, Pickert defines the above relation in [18].

From a dependence relation, we may define the following concepts.

Definition 2.15. Let X be a set and $\lhd$ be a dependence relation on X. Let $M, N \subseteq X$.

1. If M is finite, then M is *independent* if $x \not\lhd M \setminus \{x\}$ for every $x \in M$.
2. If M is not finite, then M is *independent* if each of its finite subsets is independent.
3. N is said to *depend on* M (or N is *generated by* M) if, for each $x \in N$, there exists a finite subset $M' \subseteq M$ such that $x \lhd M'$.
4. M is a *basis* of X if M is independent and X depends on M.

It is a consequence of Zorn's Lemma that, if X is equipped with a dependence relation, then every independent subset M of X will be contained in a basis of X, a proof of which is presented in [8], p.19. As a direct corollary, we have that X will have a basis, since the empty set vacuously satisfies the condition for being independent. Likewise, every subset of X can be shown to have its own basis (that is, an independent subset that generates it) by restricting the dependence relation to that subset.

The dependence relation is, of course, not the standard generalisation of independence, namely the matroid. It was shortly before Van der Waerden laid out the fundamental theorems of independence, that Whitney's foundational paper, *On the abstract properties of linear dependence* ([28]) was published. However, since matroids are restricted to finite sets, it is useful to rather examine one of its generalisations: the finitary matroid. For this, we give the definition as given by Klee.

Definition 2.16. ([15], p.138) A *finitary matroid* is a system (M, f), where M is a set and $f : \mathcal{P}(M) \to \mathcal{P}(M)$ satisfying the following conditions for all $x, y \in M$ and $X, Y \subseteq M$.

(E) $X \subseteq f(X)$ (f is *enlarging*).

(I) If $X \subseteq Y$, then $f(X) \subseteq f(Y)$ (f is *isotonic*).

(wI) If $x \in f(Y)$, then $f(\{x\} \cup Y) \subseteq f(Y)$ (f is *weakly idempotent*).

(wE) If $y \in f(Y)$ and $y \notin f(Y \setminus \{x\})$, then $x \in f((Y \setminus \{x\}) \cup \{y\})$ (f is *weakly exchanging*).

(F) If $x \in f(Y)$, then there is a finite $U \subseteq Y$ such that $x \in f(U)$.

While, in general, dependence relations and matroids do not seem to be mentioned together, it can be shown that a finite set together with a dependence relation on it serves as yet another equivalent definition of a matroid — a fact that was undoubtably known to the authors who studied independence abstractly. Moreover, more generally, an arbitrary set together with a dependence relation on it form a finitary matroid. A proof of the latter is given below.

Theorem 2.17. *Let X be independent, and suppose $\lhd \subseteq X \times \mathcal{P}(X)$. Then $\lhd$ is a dependence relation if and only if (X, cl) forms a finitary matroid, where $cl : \mathcal{P}(X) \to \mathcal{P}(X)$ such that $cl(M) = \{x \in X \mid x \lhd M\}$.*

Proof. Suppose $\lhd$ is a dependence relation. Let $M, N \subseteq X$ and $x, y \in X$.

(E) Suppose $x \in M$, then $x \lhd M$, so $x \in \mathrm{cl}(M)$.

(I) Suppose $M \subseteq N$, and suppose $x \in \mathrm{cl}(M)$, Then $x \lhd M$. Furthermore, since $M \subseteq N$, for each $u \in M$, $u \in N$, and so $u \lhd N$. It follows $x \lhd N$.

(wI) Suppose $x \in \mathrm{cl}(M)$, then $x \lhd M$. Suppose $y \in \mathrm{cl}(M \cup \{x\})$. Then $y \lhd M \cup \{x\}$. For each $u \in M \cup \{x\}$, $u \lhd M$, so $y \lhd M$, and hence $y \in \mathrm{cl}(M)$.

(wE) Suppose $x \in \mathrm{cl}(M)$ and $x \notin \mathrm{cl}(M \setminus \{y\})$. Then $x \lhd M$ and $x \ntriangleleft M \setminus \{y\}$. It follows $y \lhd (M \setminus \{y\}) \cup \{x\}$, so that $y \in \mathrm{cl}((M \setminus \{y\}) \cup \{x\})$.

(F) Suppose $x \in \mathrm{cl}(M)$, then $x \lhd M$. Let B be a basis for M in terms of $\lhd$. Then there exists some finite subset of B, say M', such that $x \lhd M'$.

Conversely, suppose (X, cl) is a finitary matroid.

(D1) Suppose $x \in M$. Then $x \in \mathrm{cl}(M)$, so that $x \lhd M$.

(D2) Suppose $x \lhd M$ and $u \lhd N$ for each $u \in M$. Then $x \in \mathrm{cl}(M)$, so that $x \in \mathrm{cl}(M')$ for some finite subset $M' = \{u_1, \dots, u_n\}$ of M. Inductively, we show $\mathrm{cl}(M' \cup N) \subseteq \mathrm{cl}(N)$. First, since $u_1 \in \mathrm{cl}(N)$, it follows $\mathrm{cl}(N \cup \{u_1\}) \subseteq \mathrm{cl}(N)$. Suppose then $\mathrm{cl}(N \cup \{u_1, \dots, u_k\}) \subseteq \mathrm{cl}(N)$. Then $u_{k+1} \in \mathrm{cl}(N)$, so that $u_{k+1} \in \mathrm{cl}(N \cup \{u_1, \dots, u_k\})$, and finally $\mathrm{cl}(N \cup \{u_1, \dots, u_{k+1}\}) \subseteq \mathrm{cl}(N \cup \{u_1, \dots, u_k\}) \subseteq \mathrm{cl}(N)$. Hence $\mathrm{cl}(N \cup M') \subseteq \mathrm{cl}(N)$. But $x \in \mathrm{cl}(M') \subseteq \mathrm{cl}(N \cup M') \subseteq \mathrm{cl}(N)$, so $x \lhd N$.

(D3) Suppose $x \lhd M$ and $x \ntriangleleft M \setminus \{y\}$. Then $x \in \mathrm{cl}(M)$ and $x \notin \mathrm{cl}(M \setminus \{y\})$, so that $y \in \mathrm{cl}(M \setminus \{y\} \cup \{x\})$. It follows $y \lhd M \setminus \{y\} \cup \{x\}$.

■

We now return to the near-vector space. André defines a relation $\lhd$ on the quasi-kernel of a near-vector space.

Definition 2.18. ([1], p.302) Let (V, A) be a near-vector space, and define the relation $\lhd \subseteq Q \times \mathcal{P}(Q)$ as follows, for $u \in Q$ and $M \subseteq Q$.

1. $0 \lhd \varnothing$.

2. $u \lhd M$ if there exist $u_1, \dots, u_n \in M$ and $\lambda_1, \dots, \lambda_n \in A$ such that

$$u = \sum_{i=1}^{n} \lambda_i u_i.$$

In the above definition, $\lhd$ is a dependence relation ([1], p. 302). Moreover, if V is a vector space over A, then the independent sets of V defined by $\lhd$ are exactly the independent sets of V obtained from the standard definition of linear independence. A basis for Q is as is defined in Definition 2.15; more precisely,

$B \subseteq Q$ is a basis of Q if B is independent and, for each $u \in Q$, there exists $b_1, \ldots, b_n \in B$ and $\lambda_1, \ldots, \lambda_n \in A$ such that $u = \sum_{i=1}^{n} \lambda_i b_i$. Generally, if B is a basis of Q, then it is also said to be a basis of the corresponding near-vector space V.

Lemma 2.19. ([1], p.302) $M \subseteq Q$ is independent if and only if, for any $n \in \mathbb{N}$ and distinct $u_1, \ldots, u_n \in M$, if $0 = \sum_{i=1}^{n} \lambda_i u_i$ for some $\lambda_1, \ldots, \lambda_n \in A$, then $\lambda_1 = \ldots = \lambda_n = 0$.

Theorem 2.20. *([1], p.303) Let (V, A) be a near-vector space and suppose $S \subseteq Q$ is an independent set. Then there exists a basis B of Q such that $S \subseteq B$.*

As a simple corollary, because $\varnothing$ is independent, the following result holds.

Corollary 2.21. ([1], p.303) The quasi-kernel of every near-vector space has a basis.

Near-vector spaces have a corresponding notion of dimension.

Theorem 2.22. *([1], p.303) Let (V, A) be a near-vector space with B, B' bases of Q. Then $|B| = |B'|$.*

Definition 2.23. ([1], p.303) Let (V, A) be a near-vector space and let B be a basis of Q. Then $|B|$ is called the *dimension* of (V, A). If $|B| = n \in \mathbb{N}$, then (V, A) is called an *n-dimensional* near-vector space, and if B is infinite, then (V, A) is called an *infinite-dimensional* near-vector space.

We end this section with the following result.

Theorem 2.24. *([1], p.304) Let (V, A) be an near-vector space. Let B be a basis for Q. Then for each $v \in V$ there exists $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \in A$, and distinct $u_1, \ldots, u_n \in B$ such that $v = \sum_{i=1}^{n} \lambda_i u_i$. Moreover, this expression is unique.*

2.4 Subspaces and regularity

One might wonder what a suitable definition of a subspace of a near-vector space would be. For a vector space V, a subspace V' of V is a nonempty subset of V that is closed under addition and scalar multiplication. It is tempting to apply this to near-vector spaces. However, in general, it is not immediately clear whether the quasi-kernel of such a subset would necessarily generate it. A more suitable definition would therefore be the following.

Definition 2.25. ([8]) Let (V, A) be a near-vector space and let X be an independent subset of $Q(V)$. Let $V' = \langle AX \rangle$, i.e. V' is generated additively by AX.

Then (V', A) is a *subspace* of (V, A). We also say V' is a subspace of V.

Strictly speaking, we want (V', A') to be a subspace of (V, A), where A' consists of the restrictions of the endomorphisms in A to V'. However, it is useful to remember that we don't really think of these endomorphisms as functions on V, but rather as the scalars of the near-vector space. Therefore, in practice, there is functionally no difference between (V', A) and (V', A'). In fact, when $V' \neq \{0\}$, then A' even has the same cardinality as A by the fixed-point-free property (if $V' = \{0\}$ then all endomorphisms in A must restrict to the identity map of V'). Therefore, in the future, we take (V', A) to mean the same thing as (V', A').

It is routine to show that a subspace (V', A) of a near-vector space (V, A) is a near-vector space itself, and moreover, that every near-vector space (V', A) contained in a near-vector space (V, A) is a subspace of (V, A).

Example 2.26. In the case of Example 2.2, the subspaces of V are as follows.

- $\{(0, 0)\}$ (generated by $X = \varnothing$)
- $\{(x, 0) \,|\, x \in \mathbb{Z}_5\}$ (generated by $X = \{(x, 0)\}$, for any $x \in \mathbb{Z}_5^*$)
- $\{(0, y) \,|\, y \in \mathbb{Z}_5\}$ (generated by $X = \{(0, y)\}$, for any $y \in \mathbb{Z}_5^*$)
- V (generated by $X = \{(x, 0), (0, y)\}$, for any $x, y \in \mathbb{Z}_5^*$)

If X contains two distinct elements from $Q_1 = \{(x, 0) \,|\, x \in \mathbb{Z}_5\}$, then X is dependent, since for any $(x, 0), (x', 0) \in Q_1$ with $x \neq x'$, supposing without loss of generality $x \neq 0$, it follows if $\alpha = x'x^{-1}$ that $\alpha(x, 0) = (x', 0)$, and so $\alpha(x, 0) + (-1)(x', 0) = (0, 0)$. Likewise, by a similar argument, if X contains two elements from $Q_2 = \{(0, y) \,|\, y \in \mathbb{Z}_5\}$, then X is dependent. Furthermore, any subset of Q containing more than two elements will contain more than one element in either Q_1 or Q_2 and can therefore not be independent. Finally, $\{(0, 0)\}$ is not independent by Lemma 2.19, and so the above list of possibilities for X is exhaustive.

Lemma 2.27. ([9], p.2527) Let (V, A) be a near-vector space and let (V', A) be a subspace of (V, A). Then $Q(V') = Q(V) \cap V'$.

Proof. Suppose (V, A) is a near-vector space and suppose V' is a subspace of V. Let $v \in Q(V')$. Then $v \in V'$. Furthermore, since $v \in Q(V')$, it follows for every $\alpha, \beta \in A$ there exists $\gamma \in A$ such that $\alpha v + \beta v = \gamma v$. But then $v \in Q(V)$, so $v \in V' \cap Q(V)$. ■

The above lemma has the following obvious consequence, and is applied in [9], Theorem 5.9, p.2538, while citing the above lemma in its use.

Corollary 2.28. If V' is a subspace of V, then $Q(V') \subseteq Q(V)$.

It is easy to show that any subspace of a near-vector space is a subgroup of V closed under scalar multiplication. A proof was published in [9] that the converse — that any subgroup of V closed under scalar multiplication would also be a subspace — was also true, but an error was picked up in this proof by Sophie Marques in 2019. In particular, it was unclear how the quasi-kernel of such subgroups would generate these sets. In the same year, it was partially proved that was true the converse in the case where $\dim(V) \leq 2$, a proof of which was contained in [20] and which is given below. Independently, the converse was proved to hold in a separate case, for near-vector spaces over division rings by Sophie Marques and was subsequently published in [10] in 2022. The result is finally proved to hold, using a geometric argument, for all near-vector spaces, a proof of which was published in 2022 in [12], and appears in Chapter 3 of this thesis. For the following results, note, in order to prove Corollary 2.28, it is not used that V' is a subspace of V. In fact, one could for any subset V' of V define a quasi-kernel $Q(V')$ in the same way as one would define it for a near-vector space. Then $Q(V') = Q(V) \cap V'$, so $Q(V') \subseteq Q(V)$.

Theorem 2.29. *([20], p.14) Let (V, A) be a near-vector space. Let $V' \subseteq V$ be closed under addition and scalar multiplication. Let B be a basis for V. Suppose that $v \in V'$ such that $v = \sum_{i=1}^{n} \lambda_i u_i$, where $u_2, \ldots, u_n \in B \cap V'$, where $\lambda_1 \neq 0$. Then $u_1 \in V'$.*

Proof. We have that $v = \sum_{i=1}^{n} \lambda_i u_i$, so $\lambda_1 u_1 = v - \sum_{i=2}^{n} \lambda_i u_i$. Then $u_1 = \lambda_1^{-1}(v - \sum_{i=2}^{n} \lambda_i u_i)$. Since V' is closed under addition and scalar multiplication, it follows that $u_1 \in V'$. ■

Corollary 2.30. ([20], p.14) Let (V, A) be a 2-dimensional near-vector space and $V' \subseteq V$. Then (V', A) is a subspace of V if and only if V' is closed under addition and scalar multiplication.

Proof. If V' is a subspace of V, then it is closed under addition and scalar multiplication.

Suppose V' is closed under addition and scalar multiplication. Let B' be a basis for $Q(V')$. We know that $Q(V') \subseteq Q(V)$ by the same argument as in Corollary 2.28.

Therefore B' is an independent subset of $Q(V)$, so it extends to a basis B for $Q(V)$ by Theorem 2.20. Let $v \in V'$ such that $v = \lambda_1 u_1 + \lambda_2 u_2$, where $B = \{u_1, u_2\}$. If u_1 and u_2 are in B', then $B' = B$. Now, if $v \in V$, then $v = \alpha u_1 + \beta u_2$ for some $\alpha, \beta \in A$, since $\{u_1, u_2\}$ is a basis for $Q(V)$. But then $v \in V'$, since $u_1, u_2 \in V'$ and V' is by assumption closed under addition and scalar multiplication. Hence $V = V'$ and so the result follows. Without loss of generality, assume $u_1 \notin B'$. If $\lambda_1 \neq 0$, then by Theorem 2.29 $u_1 \in V'$. But then $u_1 \lhd B' = B' \setminus \{u_1\}$, so that $B' \cup \{u_1\}$ is dependent — a contradiction, since $B' \cup \{u_1\} \subseteq B$ and B is independent. Hence $v = \lambda u_2 \in V'$. But then $u_2 \in V'$ or $\lambda_2 = 0$, since V' is closed under scalar multiplication. If $u_2 \in V'$, then $u_2 \in B'$, so $B' = \{u_2\}$. Then $\langle AB' \rangle = \{\lambda u_2 \,|\, \lambda \in A\} = V'$. If $u_2 \notin V'$, then $B' = \varnothing$, and $\lambda_2 = 0$, so $\langle AB' \rangle = \{0\} = V'$. Therefore, in all cases, V' is a subspace of V. ■

We return to the subspace problem in Chapter 3.

In [1], André defines the notion of compatibility between vectors in the quasi-kernel. The purpose of this is to decompose a near-vector space into well-behaved subspaces, called regular subspaces.

Definition 2.31. ([1], p.305) For a near-vector space (V, A), let $u, v \in Q^*$. u and v are called *compatible* (u cp v) if there exists some $\lambda \in A^*$ such that $u + \lambda v \in Q$.

Lemma 2.32. ([1], p.305) For a near-vector space (V, A), let $u, v \in Q^*$. Then u cp v if and only if there exists some $\lambda \in A \setminus \{0\}$ such that $+_u = +_{\lambda v}$.

Theorem 2.33. *([1], p.306) The relation cp is an equivalence relation on Q^*.*

Definition 2.34. ([1], p.306) A near-vector space is called *regular* if any two nonzero elements in its quasi-kernel are compatible.

Lemma 2.35. ([1], p.306) A near-vector space is regular if and only if it has a basis of mutually pairwise compatible vectors.

We now move on the the Decomposition Theorem for near-vector spaces. For this, we will need the following definition.

Definition 2.36. Let G be an abelian group and let $\{H_i i \in I\}$ be a set of subgroups of G. Then G is called the *direct sum* of the subgroups $H_i, i \in I$ if for every $g \in G$ there exist unique $h_i \in H_i, i \in I$ such that $g = \sum_{i \in I} h_i$.

Theorem 2.37. *(*Decomposition Theorem*, [1], p.306) Every near-vector space V is the direct sum of regular near-vector spaces V_j $(j \in J)$ such that each $u \in Q^*$*

lies in precisely one V_j. The subspaces V_j are maximal regular near-vector spaces.

The proof of the Decomposition Theorem used to be quite involved. In particular, it was tricky to show that each $u \in Q^*$ lies in precisely one maximal regular subspace V_j. I was able to shorten this argument with the addition of the following lemma.

Lemma 2.38. Let (V, A) be a near-vector space, and let $u_1, \ldots, u_n$ be independent elements in Q. Let $u = \sum_{i=1}^n \lambda_i u_i \in Q$ for some $\lambda_1, \ldots, \lambda_n \in A \backslash \{0\}$. Then u cp u_i for all $i \in \{1, \ldots, n\}$.

Proof. Let $\alpha, \beta \in A$. Since $u \in Q$ we know that there exists $\gamma \in A$ such that $\alpha u + \beta u = \gamma u$. We know that, since $\lambda_1, \ldots, \lambda_n$ are nonzero and $u_1, \ldots, u_n$ are independent, u is nonzero, so $\gamma = \alpha +_u \beta$ is uniquely defined by α and β. Now:

$$\begin{aligned}
\alpha u + \beta u &= \gamma u \\
\alpha \sum_{i=1}^n \lambda_i u_i + \beta \sum_{i=1}^n \lambda_i u_i &= \gamma \sum_{i=1}^n \lambda_i u_i \\
\alpha \sum_{i=1}^n \lambda_i u_i + \beta \sum_{i=1}^n \lambda_i u_i - \gamma \sum_{i=1}^n \lambda_i u_i &= 0 \\
\sum_{i=1}^n (\alpha\lambda_i u_i + \beta\lambda_i u_i + (-\gamma)\lambda_i u_i) &= 0 \\
\sum_{i=1}^n (\alpha +_{\lambda_i u_i} \beta +_{\lambda_i u_i} (-\gamma))\lambda_i u_i &= 0.
\end{aligned}$$

Since $u_1, \ldots, u_n$ are independent, it follows that:

$$\alpha +_{\lambda_i u_i} \beta +_{\lambda_i u_i} (-\gamma) = 0,$$

or equivalently

$$\alpha +_{\lambda_i u_i} \beta = \gamma$$

for all $i \in \{1, \ldots, n\}$. Therefore $\alpha +_u \beta = \alpha +_{\lambda_i u_i} \beta$, so that $R_u = R_{\lambda_i u_i}$ for all $i \in \{1, \ldots, n\}$. Therefore, since $u \in R_u$, $\lambda_i u_i \in R_{\lambda_i u_i} = R_u$, and R_u is a subgroup of V, it follows that

$$u + \lambda_i u_i \in R_u$$

for each $i \in \{1, \ldots, n\}$. But $R_u \subseteq Q$, so

$$u + \lambda_i u_i \in Q$$

for each $i \in \{1, \ldots, n\}$. Because λ_i is nonzero for all $i \in \{1, \ldots, n\}$, we may conclude u cp u_i for all $i \in \{1, \ldots, n\}$. ■

We are now ready to prove the Decomposition Theorem.

Proof. Let $B = \{b_i \,|\, i \in I\}$ be a basis for V and $Q \setminus \{0\} /$ cp $= \{Q_j \,|\, j \in J\}$. Define, for each $j \in J$, $B_j = B \cap Q_j = \{b_{ij} \,|\, i \in I_j\}$, and $V_j = \langle AB_j \rangle$. Because B is independent, so is B_j for each $j \in J$, and hence V_j is a subspace of V for each $j \in J$. Now let $v \in V$, and suppose $v = \sum_{j \in J} \sum_{i \in I_j} \lambda_{ij} b_{ij}$, where $\lambda_{ij} \in A$ for each $i \in I$ and $j \in J$, and $\lambda_{ij} = 0$ for all but finitely many $i \in I$ and $j \in J$. Define $v_j = \sum_{i \in I_j} \lambda_{ij} b_{ij}$ for each $j \in J$. Then $v_j \in V_j$ and $v = \sum_{j \in J} v_j$. Moreover, because the decomposition of v in terms of B is unique by Theorem 2.24, each $v_j \in V_j$ is uniquely determined, and so $\sum_{j \in J} v_j$ is the unique decomposition of v in terms of $\{V_j \,|\, j \in J\}$. It follows that V is the direct sum of $V_j, j \in J$.

Next, suppose $u \in Q \setminus \{0\}$. Then, since Q^* is partitioned by Q_j's, $j \in J$, it follows that $u \in Q_i$ for exactly one $i \in J$. We wish to show that $u \in V_i$. Let $u = \sum_{j=1}^{n} \lambda_j b_j$ for some $b_1, \ldots, b_n \in B$ and $\lambda_1, \ldots, \lambda_n \in A^*$. Then u cp b_j for each $j \in \{1, \ldots, n\}$ by Lemma 2.38. It follows that $b_j \in Q_i$ for each $j \in \{1, \ldots, n\}$, so $b_j \in B \cap Q_i = B_i$ for all $j \in \{1, \ldots, n\}$. It follows that $u = \sum_{j=1}^{n} \lambda_j b_j \in \langle AB_i \rangle = V_i$.

Now, suppose $u \in V_k$ for some $k \in J$ such that $i \neq k$. Then, because the unique expression for u in terms of the basis B is $u = \sum_{j=1}^{n} \lambda_j b_j$, $b_1, \ldots, b_n \in B_k$, so $b_1, \ldots, b_n \in Q_k$ — a contradiction, since $b_1, \ldots, b_n \in Q_i$ and $Q_i \cap Q_k = \varnothing$. Hence u lies in exactly one V_j, $j \in J$.

Finally, to show each V_j is maximal, suppose for some $j \in J$ there exists a regular subspace W of V such that $V_j \subseteq W \subsetneq V$. Because W is regular, $Q(W)^*$ contains only compatible vectors. But $Q_j \subseteq Q(V_j) \subseteq Q(W)$. Since Q_j is an equivalence class of cp , it contains a maximal set of compatible elements, and so $Q(W)^* \subseteq Q_j$. Hence $Q_j = Q(W)^*$, and so $Q(W) \subseteq Q(V_j)$, and hence $W \subseteq V_j$. It follows that $V_j = W$, so that W is maximal. ■

Example 2.39. Refer back to Example 2.2. Note that $B = \{(1, 0), (0, 1)\}$ is a basis for V, with $Q/$ cp $= \{Q_1, Q_2\}$, where $Q_1 = \{(x, 0) \,|\, x \in \mathbb{Z}_5^*\}$ and $Q_2 = \{(0, y) \,|\, y \in \mathbb{Z}_5^*\}$, so that $B_1 = B \cap Q_1 = \{(1, 0)\}$ with $V_1 = \{(x, 0) \,|\, x \in \mathbb{Z}_5\}$, and $B_2 = B \cap Q_2 = \{(0, 1)\}$ with $V_2 = \{(0, y) \,|\, y \in \mathbb{Z}_5\}$. It follows that $V = V_1 \oplus V_2$.

2.5 Nearfield theory and Van Der Walt's Theorem

The following section establishes a connection between André's near-vector spaces and nearfields.

Definition 2.40. ([19], p.11) A *(left) nearfield* is a triple $(G, +, \cdot)$ such that the following conditions hold.

- $(G, +)$ is a group.
- $(G^*, \cdot)$ is a group.
- For any $a, b, c \in G$, $a \cdot (b + c) = a \cdot b + a \cdot c$.
- $0 \cdot a = 0$ for all $a \in G$.

While the definition does not require it, it can be shown that the additive group of a nearfield is abelian. The proof of this is omitted.

There is an analogous definition of a right nearfield that has the right distributive law in place of the left one. For more on nearfields, we refer the reader to Pilz ([19]).

Example 2.41. ([**p.257**, 19]) The smallest example of a nearfield that is not a field is the Dickson nearfield $F = (GF(3^2), +, \circ)$, where

$$x \circ y = \begin{cases} x \cdot y \text{ if } x \text{ is a square in } (GF(3^2), +, \cdot); \\ x \cdot y^3 \text{ otherwise.} \end{cases}$$

Nearfields are closely linked to near-vector spaces, in much the same way as fields are linked to vector spaces.

Theorem 2.42. *([1],p.300) Let (V, A) be a near-vector space and $u \in Q^*$. Then $(A, +_u, \circ)$ is a near-field.*

André noted in [1] that one can construct near-vector spaces using copies of a nearfield.

Theorem 2.43. *([1], p.303) Let F be a nearfield and I an index set. Then the* *set*

$$F^{(I)} = \{(\xi_i)_{i\in I} \,|\, \xi_i \in F \text{ and } \xi_i \neq 0 \text{ for only finitely many } i \in I\} \text{ if } I \neq \varnothing,$$
$$F^{\varnothing} = \{0\}$$

is a near-vector space over F, when one defines addition and scalar multiplication componentwise.

A near-vector space constructed as above will always be regular. To show this, let $B = \{b_j \,|\, j \in I\}$, where $b_j = (\delta_{ij})_{i\in I}$ and δ_{ij} is the Kronecker-delta. Then $\alpha b_j + \beta b_j = (\alpha\delta_{ij} + \beta\delta_{ij})_{i\in I} = ((\alpha+\beta)\delta_{ij})_{i\in I} = (\alpha+\beta)b_j$ (because $\delta_{ij} \in \{0,1\}$, the right-distributive law holds). It follows $B \subseteq Q(F^{(I)})^*$, and $+_{b_j} = +$ for all $j \in I$, so that B is a basis of mutually pairwise compatible basis vectors.

Example 2.44. Let $V = F^2$, where F is as defined in as in Example 2.41. Then (V, F) is a regular near-vector space. In this case, $Q \subsetneq V$ ([1], p.312), making it the smallest example of a regular near-vector space where its quasi-kernel is properly contained in it.

André continues on to show that the following holds for any near-vector space.

Theorem 2.45. *([1], p.305) Let (V, A) be a near-vector space with basis $B = \{b_i \,|\, i \in I\}$. Then there exists a bijective map $f : V \to A^{(I)}$ such that $f(\alpha v) = \alpha f(v)$ for each $\alpha \in A$ and $v \in V$.*

It should be noted that the above is not, in general, an additive group isomorphism, as A does not have a unique addition defined on it. In fact, f is an additive group isomorphism if and only if V is a regular near vector space ([1]). To remedy this, Van der Walt gave a way to characterise all finite-dimensional near-vector spaces as the direct sum of nearfields.

Theorem 2.46. *(*Van der Walt's Theorem*, [25], p.301)*

Let $(G, +)$ be a group and $A = D \cup \{0\}$, where D is a fixed-point-free group of automorphisms of G. Then (G, A) is a finite dimensional near-vector space if and only if there exists a finite number of nearfields $F_1, \ldots, F_n$, semigroup isomorphisms $\psi_i : (A, \circ) \to (F_i, \cdot)$ and an additive group isomorphism $\Phi : G \to F_1 \oplus \ldots \oplus F_n$ such that, for any $g \in G$ and $\alpha \in A$ if $\Phi(g) = (x_1, \ldots, x_n)$, then $\Phi(\alpha g) = (\psi_1(\alpha)x_1, \ldots, \psi_n(\alpha)x_n)$.

While Van der Walt's Theorem may only have been stated for finite-dimensional near-vector spaces, the theorem generalises to all near-vector spaces in a natural way – for this, see [10], p.3669.

Chapter 3

Noncommutative Geometry

The work in this chapter appears largely in [12]. The main motivation for André's near-vector space construction was the so-called nearfield space, a generalisation of an incidence structure called an affine space. We start off this chapter by giving a brief overview of incidence structures, and build up to the nearfield space.

3.1 Incidence structures

We use the book *Foundations of Incidence Geometry* by Johannes Ueberberg ([23]) as reference for our definitions for pregeometries and geometries. The concept of a *linear space* is then generalised to that of a *near-linear space*, and projective and affine spaces, are generalised to their suitable near-structures.

Definition 3.1. ([23], p.1) Let I be a non-empty set whose elements are called *types*. A *pregeometry* over the type set I is a triple $\Gamma = (X, *, \text{type})$ fulfilling the following conditions.

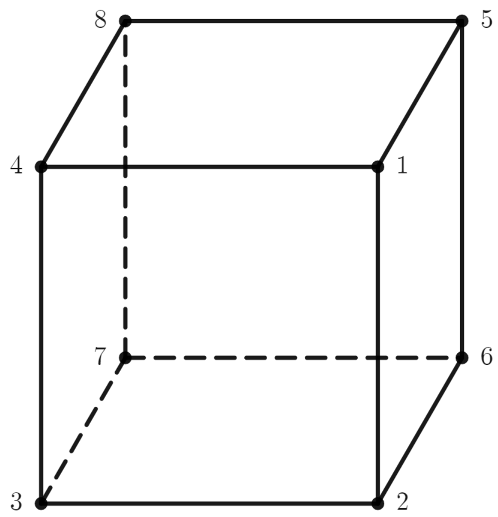
i X is a non-empty set whose elements are called the *elements* of the pregeometry Γ.

ii type is a surjective function from X to I. It is called the *type function* of Γ.

iii $*$ is a reflexive and symmetric relation on X, the so-called *incidence relation*. It fulfills the condition: If x and y are incident elements of the same type, that is $x * y$ and $\text{type}(x) = \text{type}(y)$, we have $x = y$.

The *rank* of a pregeometry over a type set I is $|I|$.

Example 3.2. ([23], p.2) Consider a cube with numbered vertices as in the figure.



Let $X = \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 12,\ 14,\ 15,\ 23,\ 26,\ 34,\ 37,\ 48,\ 56,\ 58,\ 67,\ 78,\ 1234,$ $1256,\ 1458,\ 2367,\ 3478,\ 5678\}$ and $I = \{\text{vertex}, \text{edge}, \text{face}\}$. Let type : $X \to I$ be a function that maps each single digit number to vertex, each two-digit number to edge, and each four digit number to face. Finally, let a vertex and an edge, a vertex and a face, or an edge and a face be incident under $*$ if the digits of the former appear in the latter (in the figure, a vertex-edge, vertex-face, or edge-face pair are incident if the former lies on the latter). Then $\Gamma = (X, *, \text{type})$ is a pregeometry over I.

Definition 3.3. Let $\Gamma = (X, *, \text{type})$ be a pregeometry over type set I. A *flag* F of Γ is a subset of X such that all elements of F are incident with each other. The *rank* of a flag F is $|\text{type}(F)|$. A *chamber* C of Γ is a flag such that $\text{type}(C) = I$. If every flag of Γ is contained in a chamber, then Γ is a *geometry* over the type set I.

Example 3.4. ([23], p.2) Γ of Example 3.2 is a geometry over I. The flags of Γ containing the vertex 1 are $\{1\}, \{1, 12\}, \{1, 14\}, \{1, 15\}, \{1, 1234\}, \{1, 1256\}, \{1, 1458\},$ $\{1, 12, 1234\}, \{1, 12, 1256\}, \{1, 14, 1234\}, \{1, 14, 1458\}, \{1, 15, 1256\}$, and $\{1, 15, 1458\}$.

In essence, what it means for a pregeometry Γ to be a geometry, is that any element of Γ is incident to an element of each type in I. In the next section, we delve into near-linear spaces. These are geometries over the type set $\{\text{point}, \text{line}\}$.

3.2 The near-linear space

In this section we generalise the concept of a linear space to a near-linear space. Recall a *linear space* is a geometry of rank 2 over the type set $\{\text{point}, \text{line}\}$ such that every line is incident to at least two points, there are at least two lines, and, given two distinct points x and y, there is exactly one line incident to both of them ([23], p.4). Elements of type point are called *points*, and elements of type line are called *lines*. We will denote the set of points by $\mathbb{P}$, and the set of lines by $\mathbb{L}$.

From this we can construct a natural surjective map $\sqcup : \mathbb{P}^2 \setminus \triangle_{\mathbb{P}} \to \mathbb{L}$ (here $\triangle_{\mathbb{P}} = \{(x, x) \,|\, x \in \mathbb{P}\}$), and denote $\sqcup(x, y)$ rather as $x \sqcup y$ — this line is called the *join* of x and y. Note that the order of x and y does not matter here: $x \sqcup y$ must be the same as $y \sqcup x$, else there are two distinct lines incident to both x and y. It follows that in a linear space, $\sqcup$ is commutative.

This is the property that is generalised in a near-linear space. The choice of axioms for a near-linear space are derived from the work of André in [2], and each of them serve a purpose in proving some elementary properties of a near-linear space.

Definition 3.5. Let $\mathbf{L}$ be a geometry of rank 2 over the type set $\{\text{point}, \text{line}\}$ with the following incidence structure.

(1) $\mathbb{L} \subseteq \mathcal{P}(\mathbb{P})$, and a point x and line L are incident if and only if $x \in L$.

(2) There is a surjective mapping $\sqcup : \mathbb{P}^2 \setminus \triangle_{\mathbb{P}} \to \mathbb{L}$ such that $(x, y) \mapsto x \sqcup y$.

Then $\mathbf{L}$ is a *near-linear space* if $\mathbf{L}$ satisfies the following conditions.

(L1) $x, y \in x \sqcup y$.

(L2) If x, y, z are distinct points, then $z \in x \sqcup y \setminus \{x\}$ if and only if $x \sqcup y = x \sqcup z$.

(L3) If x, y, z are distinct points with $z \in x \sqcup y = y \sqcup x$, then $x \sqcup z = z \sqcup x$.

(R) There are at least two distinct lines.

(G1) If x, y are points such that $G = x \sqcup y = y \sqcup x$, then for any line $L \neq x \sqcup y$, $|L \cap G| \leq 1$.

(G2) For any two distinct points x and y, there exists a finite sequence of points $p_0 = x, p_1, \ldots, p_n = y$ such that $p_i \sqcup p_{i+1} = p_{i+1} \sqcup p_i$ for all $i \in \{0, \ldots, n-1\}$.

Some consequences of the above, as noted in [2].

- (1) implies that lines are completely determined by the points on them: if L and L' are lines incident to exactly the same points, then $L = L'$.
- It is easy to see in the definition above that, if $\sqcup$ is commutative, then $\mathbf{L}$ is a linear space. Moreover, since lines in a linear space are clearly completely determined by the set of points incident to them, we can choose to consider lines in a linear space as the set of points incident to them. Hence every linear space is a near-linear space.
- (L1) implies that every line contains at least two points.
- If L is a line such that $L = x \sqcup y$, then we call x a *base point* of L.
- A line L is called a *straight line* if there are two points $x, y \in L$ such that $L = x \sqcup y = y \sqcup x$. Lines that are not straight are called *proper lines.*
- (G1) implies that any line can intersect a straight line at most once.
- (G2) implies that there is a finite sequence of straight lines connecting any two points and that any point is incident to a straight line. As a shorthand, two points are said to be *joinable* if this property is satisfied.

As mentioned in [2], the following lemma (there mentioned in the context of near-affine spaces) is a simple consequence of (L3).

Lemma 3.6. ([2], p.68) The following statements are equivalent for a given line L in a near-linear space.

1. L has two base points.
2. Every point of L is a base point of L.
3. If x and y are different points on L then $L = x \sqcup y = y \sqcup x$.

From this lemma, it is clear that every point on a straight line G is a base point of G.

Next, we prove some basic properties of near-linear spaces.

Theorem 3.7. *A near-linear space contains at least two straight lines.*

Proof. Let $\mathbf{L}$ be a near-linear space. By (R), $\mathbf{L}$ has at least two lines. Suppose that one of these lines, say L, is a proper line. Let $L = x \sqcup y$. Then x and y are joinable by (G2). Let $p_0 = x, \ldots, p_n = y$ be a finite sequence as described in (G2).

If $n > 1$, then $\mathbf{L}$ has at least two straight lines. Suppose not. Then $n = 1$, so that $x \sqcup y$ is straight. But $x \sqcup y = L$ is a proper line, so this is a contradiction. ■

Lemma 3.8. In a near-linear space, distinct lines that share a base point intersect exactly once.

Proof. Let L and L' be two distinct lines that share a base point x. Clearly, these two lines intersect at x. Assume now that the lines intersect at a point p different from x. Then p is incident with L and L', hence $L = x \sqcup p = L'$ by (L2) — a contradiction, since L and L' are distinct. Hence L and L' intersect only at x. ■

Next, we introduce the notion of subspaces for near-linear spaces.

Definition 3.9. ([2], p.90) Let $\mathbf{L}$ be a near-linear space, and let U be a set of points of $\mathbf{L}$. U is called a *weak subspace* of $\mathbf{L}$ if for any two points x and y in U, all points of the line $x \sqcup y$ are contained in U. By convention, we say a line L is contained in a weak subspace U whenever $L \subseteq U$.

It is clear that a weak subspace U of a near-linear space satisfies all the conditions of a near-linear space, except possibly (R) and (G2) (where the lines of U are all the lines totally contained in U). We specify further with the following definition.

Definition 3.10. ([2], p.79) Let $\mathbf{L}$ be a near-linear space. A *(strong) subspace* U of $\mathbf{L}$ is a weak subspace such that any pair of points in U are joinable by straight lines completely contained in U.

Some trivial examples of subspaces of near-linear spaces are singletons of points, straight lines, the empty set, and the whole point set $\mathbb{P}$ (in this last case, we say $\mathbf{L}$ is a subspace of itself).

Lemma 3.11. Let $\mathbf{L}$ be a near-linear space. The intersection of an arbitrary family of weak subspaces of $\mathbf{L}$ is a weak subspace of $\mathbf{L}$.

Proof. Let $(U_i)_{i \in I}$ be a family of weak subspaces of $\mathbf{L}$, and let x and y be two points contained in U_i for all $i \in I$. Since U_i is a weak subspace for all $i \in I$, the line $x \sqcup y$ is contained in U_i for all $i \in I$. ■

The above property allows us to create the following definition.

Definition 3.12. Let $\mathbf{L}$ be a near-linear space.

1. A set of points M of $\mathbf{L}$ is called *collinear* if all points of M lie on a common line of $\mathbf{L}$.

2. Let M be a set of points of $\mathbf{L}$, and let

$$\langle M \rangle = \bigcap \{U \,|\, U \text{ is a weak subspace of } \mathbf{L} \text{ containing } M\}\,.$$

   $\langle M \rangle$ is the smallest weak subspace of $\mathbf{L}$ containing M. It is called *the weak subspace generated by M*.

If $M = \{p_i \,|\, i \in I\}$ is a set of points, we may also use the notation $\langle p_i \,|\, i \in I \rangle$ to denote $\langle M \rangle$. Likewise, $\langle p_0, \dots, p_n \rangle = \langle \{p_0, \dots, p_n\} \rangle$.

If U is a weak subspace, then clearly $\langle U \rangle = U$. In particular, $\langle G \rangle = G$ for any straight line G.

The following lemma will be useful later in the chapter.

Lemma 3.13. Let M and M' be sets of points of a near-linear space. Then $\langle M \cup M' \rangle = \langle \langle M \rangle \cup M' \rangle$.

Proof. Since $M \cup M' \subseteq \langle M \rangle \cup M' \subseteq \langle \langle M \rangle \cup M' \rangle$, we have $\langle M \cup M' \rangle \subseteq \langle \langle M \rangle \cup M' \rangle$. Conversely, $\langle M \rangle \subseteq \langle M \cup M' \rangle$, since $M \subseteq M \cup M' \subseteq \langle M \cup M' \rangle$. It follows that $\langle M \rangle \cup M' \subseteq \langle M \cup M' \rangle$, and so $\langle \langle M \rangle \cup M' \rangle \subseteq \langle M \cup M' \rangle$. Hence $\langle M \cup M' \rangle = \langle \langle M \rangle \cup M' \rangle$. ■

Theorem 3.14. *Any near-linear space contains three non-collinear points.*

Proof. Let $\mathbf{L}$ be a linear space. By Theorem 3.7, $\mathbf{L}$ contains at least two distinct straight lines, say L and L'. Let x and y be two distinct points on L, and x' and y' be two distinct points on L'. If L and L' do not intersect, then x, y, y' are three distinct points. If L and L' do intersect, then they intersect at most once by (G1), we may assume at x', so that x, y, y' are distinct points. Now suppose there is a line L'' connecting x, y, and y'. Since L and L'' intersect at two points, they must be the same line, otherwise (G1) would be contradicted. But y' does not lie on L, so this is a contradiction. Hence there is no common line incident to x, y, y', and so the three points are non-collinear. ■

We now introduce hyperplanes.

Definition 3.15. Let $\mathbf{L}$ be a near-linear space. A maximal proper subspace of $\mathbf{L}$ is called a *hyperplane* of $\mathbf{L}$.

Theorem 3.16. *Let* $\mathbf{L}$ *be a near-linear space, and let* H *be a subspace of* $\mathbf{L}$*. Then* H *is a hyperplane of* $\mathbf{L}$ *if every line of* $\mathbf{L}$ *has at least one point in common with* H*.*

Proof. Suppose H is a subspace of a linear space $\mathbf{L}$ such that every line of $\mathbf{L}$ has a point in common with H. Assume there exists a proper subspace U of $\mathbf{L}$ such that H is a proper subspace of U. Let $x \in U \setminus H$ and $y \in \mathbf{L} \setminus U$, the existence of which is ensured by the fact that U is a proper subspace of $\mathbf{L}$. Then the line $x \sqcup y$ meets H in a point z by the assumption. Since z is incident with $x \sqcup y$, it follows $x \sqcup y = x \sqcup z$ by (L2). But x and z are contained in U, hence $x \sqcup z$ is contained in U by definition — a contradiction, since y is incident to $x \sqcup z$, and therefore also contained in U. Hence H must be a hyperplane of $\mathbf{L}$. ■

We close this section by introducing the notion of parallelism and proving some properties.

Definition 3.17. ([2], p.68,73) A *parallelism* on a near-linear space $\mathbf{L}$ is an equivalence relation $\parallel$ on the set of lines of $\mathbf{L}$ such that the following conditions hold:

(P1) For any line L and point x of $\mathbf{L}$, there exists a unique line L' such that x is a base point of L' and $L \parallel L'$. We denote L' by $(x \parallel L)$.

(P2) If L is a straight line and $L \parallel L'$, then L' is a straight line.

(P3) For all points x and y, $x \sqcup y \parallel y \sqcup x$.

Lines L and L' are said to be *parallel* if $L \parallel L'$.

Note that (P2) and (L2) imply (L3): If $x \sqcup y = y \sqcup x$, and z is incident to $x \sqcup y$, then $x \sqcup y = x \sqcup z$, so that $x \sqcup y \parallel x \sqcup z$. Hence by (P2), $x \sqcup z = z \sqcup x$.

Lemma 3.18. In a near-linear space, distinct straight lines that are parallel never intersect.

Proof. Suppose G and G' are two distinct parallel straight lines with intersection point x. Then x is a base point of G and G', so that $G = (x \parallel G) = (x \parallel G') = G'$— a contradiction. Hence G and G' do not intersect, or are not parallel. ■

Theorem 3.19. *(G2') Let* $\mathbf{L}$ *be a near-linear space that admits a parallelism. Then* $\mathbf{L}$ *has at least two non-parallel straight lines.*

Proof. By Theorem 3.7 there are at least two straight lines, say G and G'. If G and G' intersect, then we are done by Lemma 3.18. Suppose not. Let x be a point on G and y be a point on G'. Then by (G2) there exists a finite sequence $x = p_0, \ldots, p_n = y$ such that $p_i \sqcup p_{i+1}$ is straight. Since y is not incident to G, there exists $i \in \{0, \ldots, n-1\}$ such that $p_i \sqcup p_{i+1} \neq G$. Let j be the smallest index for which $p_j \sqcup p_{j+1} \neq G$. Then G intersects $p_j \sqcup p_{j+1}$ at p_j. Hence G and $p_j \sqcup p_{j+1}$ are two straight lines that are not parallel by Lemma 3.18. ∎

3.3 The nearaffine space

By adding some axioms to a near-linear space, we arrive at a nearaffine space, as defined by André in [2].

Definition 3.20. ([1], p.73) A *nearaffine space* **A** is a near-linear space that admits a parallellism $\|$ such that the following condition holds.

(T) If x, y, and z are pairwise distinct points, and x' and y' are different points with $x \sqcup y \parallel x' \sqcup y'$, then

$$(x' \parallel x \sqcup z) \cap (y' \parallel y \sqcup z) \neq \varnothing.$$

If we specialise (T) by putting $x = x'$, then we get the affine version of the axiom of Veblen-Young ([2], p.69).

(V) If x, y, y', and z are pairwise different points and $x \sqcup y = x \sqcup y'$, then

$$(x \sqcup z) \cap (y' \parallel y \sqcup z) \neq \varnothing.$$

Next, we list a few properties of nearaffine spaces.

Theorem 3.21. *([2], p.69) A nearaffine space with a commutative join $\sqcup$ is an affine space.*

Theorem 3.22. *([2], p.73) The following condition holds in a nearaffine space.*

(Pa) (Condition for closed parallelograms) If x, y, and z are pairwise different points such that $x \sqcup y \neq x \sqcup z$, then $(z \parallel x \sqcup y) \cap (y \parallel x \sqcup z) \neq \varnothing$.

The following proof appears in [3], and is expanded upon here with clarifying detail.

Theorem 3.23. *([3], p.208) Let* $\mathbf{A}$ *be a nearaffine space. Then all lines of* $\mathbf{A}$ *have the same cardinality.*

Proof. The proof is given in four steps:

1. *Two intersecting straight lines have the same cardinality:*

   Let G and G' be distinct straight lines, with common point p. Fix $q \in G\setminus\{p\}$ and $q' \in G' \setminus \{p\}$. Let x be a point in $G = p \sqcup q$ different from p. By (V), we know that $G' \cap (x \parallel q \sqcup q') = (p \sqcup q') \cap (x \parallel q \sqcup q') \neq \varnothing$. By (G1), $|G' \cap (x \parallel q \sqcup q')| \leq 1$, so that $|G' \cap (x \parallel q \sqcup q')| = 1$, and so G' and $(x \parallel q \sqcup q')$ have a unique intersection point. Moreover, this unique intersection point cannot be p, else $G = x \sqcup p \parallel q \sqcup q'$, implying that $G = q \sqcup q'$ since they share the common point q. But then $q' \in G$ so that $G = p \sqcup q' = G'$, contradicting that these lines are distinct. We may therefore construct a map $f : G \to G'$ such that $x \mapsto x' \in G' \cap (x \parallel q \sqcup q')$ when $x \neq p$, and $f(p) = p$.
   Suppose that x' is a point on G'. By a similar argument as before, we may construct a map $g : G' \to G$ such that $x' \mapsto x \in G \cap (x' \parallel q' \sqcup q)$ for any $x' \neq p$ and $g(p) = p$. We show f and g are inverses. Let $x \in G \setminus \{p\}$, then $x \sqcup f(x) = (x \parallel q \sqcup q')$. Likewise, $f(x) \sqcup g(f(x)) = (f(x) \parallel q' \sqcup q)$. Hence $x \sqcup f(x) \parallel q \sqcup q' \parallel q' \sqcup q \parallel f(x) \sqcup g(f(x))$ by (P3), hence $x \sqcup f(x) \parallel f(x) \sqcup g(f(x))$. Again by (P3), this implies $f(x) \sqcup x \parallel f(x) \sqcup g(f(x))$ so that $f(x) \sqcup x = f(x) \sqcup g(f(x))$ by (P1). Hence x and $g(f(x))$ are common to G and $f(x) \sqcup x$, so by (G1), $x = g(f(x))$. It follows that f and g are inverses, and so f is a bijection from G to G'. Thus $|G| = |G'|$.

2. *Any two straight lines have the same cardinality:*
   Let G and G' be any two straight lines. Let $x \in G$ and $y \in G'$. By (G2) there is a sequence of points $p_0 = x, p_1, \ldots, p_n = y$ such that $p_i \sqcup p_{i+1}$ is straight. These straight lines intersect at the points p_i, so by (1.) $|G| = |p_0 \sqcup p_1| = \ldots = |p_{n-1} \sqcup p_n| = |G'|$. Hence $|G| = |G'|$.

3. *Any two lines with common base point have the same cardinality:*
   We prove the following by induction: for all $n \in \mathbb{N}$, if L and L' are two lines with a base point p, and $q \in L$, $q' \in L'$ such that q and q' are joinable by a sequence $q = q_0, q_1 \ldots, q_n = q'$, then $|L| = |L'|$.

   Base Case: Suppose $n = 1$, then $q \sqcup q'$ is straight. Let $x \in L = p \sqcup q$ such that $x \neq p$. We know $L' = p \sqcup q'$. Then by (V), we have $L' \cap (x \parallel q \sqcup q') = (p \sqcup q') \cap (x \parallel q \sqcup q') \neq \varnothing$. Since $q \sqcup q'$ is straight, by (P2) we have that $(x \parallel q \sqcup q')$ is straight, so that by (G1), $|L' \cap (x \parallel q \sqcup q')| \leq 1$. Hence $|L \cap (x \parallel q \sqcup q')| = 1$. Moreover, this intersection point is not p, else L intersects the straight line $(x \parallel q \sqcup q')$ at both x and p, contradicting (G1). Hence we may construct a map $f : L \to L'$ such that $f(x) = x' \in L' \cap (x \parallel q \sqcup q')$ if $x \neq p$ and $f(p) = p$. Likewise, we may construct a map $g : L' \to L$ such that $g(x') = x \in L \cap (x' \parallel q \sqcup q')$ when $x' \neq p$ and $g(p) = p$. These functions are inverses: for any point $x \in L$, $x \sqcup f(x) \parallel q \sqcup q'$ is straight, so $x \sqcup f(x) = f(x) \sqcup x$ by (P2). Now $f(x) \sqcup x \parallel q \sqcup q' \parallel f(x) \sqcup g(f(x))$, so that $f(x) \sqcup x \parallel f(x) \sqcup g(f(x))$, and thus $f(x) \sqcup x = f(x) \sqcup g(f(x))$. It follows that $g(f(x))$ and x are common to L and $f(x) \sqcup x$, hence, since $f(x) \sqcup x$ is straight, by (G1) we have that $x = g(f(x))$. It follows g and f are inverses, so that f is a bijection $L \to L'$, hence $|L| = |L'|$.

Inductive Hypothesis: Suppose for all $n \leq k$, if two lines L and L' share the base point p, with two points $q \in L$ and $q' \in L'$ being joinable by a sequence $q_0 = q, q_1, \ldots, q_n = q'$, then $|L_1| = |L_2|$.

Inductive Step: Suppose L and L' share the base point p, with two points $q \in L$ and $q' \in L'$ being joinable by a sequence $q_0 = q, q_1, \ldots, q_{k+1} = q'$. We first consider the case where $k + 1 = 2$ and $q_1 = p$: then $L = p \sqcup q = q \sqcup p$ and $L' = p \sqcup q' = q' \sqcup p$, so that L and L' are straight. By (1.) we are done. In all other cases, if $q_i = p$ for some $i \in \{1, \ldots, k\}$, then by our induction hypothesis, $|L| = |q_{i-1} \sqcup p|$ and $|p \sqcup q_{i+1}| = |L'|$, so that $|L| = |L'|$. Finally, if $q_i \neq p$, for all $i \in \{1, \ldots, k\}$, then, since $q = q_0$ and q_k are joinable by the sequence $q_0, \ldots, q_k$, by the induction hypothesis we have that $|L| = |p \sqcup q_k|$. Furthermore, $q_k \sqcup q' = q_k \sqcup q_{k+1}$ is straight, so by the base case, $|p \sqcup q_k| = |p \sqcup q'| = |L'|$, hence $|L| = |L'|$. By induction we may conclude, if two lines L and L' share the base point p, and two points $q \in L$ and $q' \in L'$ are joinable, then $|L| = |L'|$. Since all points are joinable, any two lines with common base point have the same order.

4. *Any two lines have the same cardinality:*

   Let L be a line with base point p. By (G2) there exists a straight line G such that $p \in G$. Since $p \in G$, p is a base point of G. Hence for all lines L, there exists a straight line G that shares a base point with L. Since all straight lines have the same order by (2.), and all lines with a common base point have the same order by (3.), we may conclude that all lines have the same order.

■

Definition 3.24. ([2], p.74) The cardinality of each line in a nearaffine space **A** is called the *order of* **A**.

By (L1) we always have that the order of a nearaffine space is greater than or equal to 2. If the order of a nearaffine space **A** is 2, then $x \sqcup y = \{x, y\} = y \sqcup x$ for any two points x and y, hence **A** is an affine space ([2], p.69).

3.4 Subspaces of nearaffine spaces

What follows is some general theory of subspaces of a nearaffine space. Note that every nearaffine space is a subspace of itself. We fix a nearaffine space **A**, and assume that the order of **A** is always greater than or equal to 3. We start with the following two results that appear in [2], the first without a proof, which has been added here, and the second with a proof, which has been expanded upon with clarifying detail.

Lemma 3.25. ([2], p.79) Let U be a subspace of **A** and G be a straight line of **A**. Then G is a subspace of U, or $|G \cap U| \leq 1$.

Proof. Suppose G is not a subspace of U. If U contains two points of G, say x and y, then $G = x \sqcup y$ by Lemma 3.6, so U contains at most one point of G. ■

Theorem 3.26. *([2], p.79) Let U be a subspace of* **A**, *and x and L a point and line contained in U. Then $(x \parallel L) \subseteq U$.*

Proof. Let L be a line completely contained in U and let y be a base point of L. For any $x \in U$, if $x = y$ or $x \sqcup y = L$, then $(x \parallel L) = L \subseteq U$, so we consider all $x \in U$ such that $x \neq y$ and $L \neq x \sqcup y$. We show by induction that, for all $n \in \mathbb{N}$, if $y = p_0, p_1, \ldots p_n = x$ is a sequence of points joining a given point x and y in U, then $(p_i \parallel L) \subseteq U$ for all $i \in \{1, \ldots, n\}$.

Base Case: Suppose $x \in U$ such that $n = 1$, i.e. $y \sqcup x$ is straight. Since the order of **A** is greater than or equal to three, we know that there exists a point $z \in x \sqcup y$ different from x and y. Then $z \notin L$, else z and y would be common to L and the straight line $x \sqcup y$, contradicting (G1). Let $x' \in (x \parallel L)$ different from x, then we know $x' \notin x \sqcup y$ by (G1). It follows the points z, x, y and x' are pairwise distinct points. We also have $x \sqcup x' = (x \parallel L) \parallel L$, so that $L = (y \parallel x \sqcup x')$. Furthermore, $z \sqcup x = z \sqcup y = x \sqcup y$, since $z \in x \sqcup y$ and $x \sqcup y$ is straight. Now, it follows from (V) $(z \sqcup x') \cap L = (z \sqcup x') \cap (y \parallel x \sqcup x') \neq \varnothing$. Let $y' \in (z \sqcup x') \cap L$. Since $L \subseteq U$, it follows $y' \in U$. By (L2), it follows $z \sqcup y' = z \sqcup x;'$. Therefore the line $z \sqcup x' = z \sqcup y'$ is contained in U, and so $x' \in U$. It follows $(x \parallel L) = x \sqcup x' \subseteq U$.

Inductive Hypothesis: Suppose that for any point $x \in U$ such that there exists a sequence $y = p_0, \ldots, p_k = x$ joining x and y, we have $(p_i \parallel L) \subseteq U$ for all $i \in \{1, \ldots, k\}$.

Inductive Step: Let $x \in U$ such that there exists a sequence $y = p_0, p_1, \ldots, p_{k+1} = x$ joining y and x. Then $y = p_0, p_1, \ldots, p_k$ is a sequence joining y and $p_k \in U$, hence by the inductive hypothesis, $(p_i \parallel L) \subseteq U$ for all $i \in \{0, \ldots, k\}$. Let $L' = (p_k \parallel L) \subseteq U$. Then by the same argument as in the base case, since $p_k \sqcup x$ is straight, $(x \parallel L') \subseteq U$. But $(x \parallel L') \parallel L' = (p_k \parallel L) \parallel L$, hence $(x \parallel L') = (x \parallel L)$. Hence $(x \parallel L) \subseteq U$, as required.

Hence, if y and x are joinable in U, then $(x \parallel L) \subseteq U$. Since all points in U are joinable in U, it follows for all $x \in U$, $(x \parallel L) \subseteq U$. ■

A consequence of the above theorem is that if U is a subspace of $\mathbf{A}$, then the set of points U, together with the set of lines contained in U satisfy all the conditions to be a nearaffine space with the same parallelism as $\mathbf{A}$, possibly with the exception of (R) ([2], Corollary, p.79).

Theorem 3.27. *([2], p.79) A subspace U of $\mathbf{A}$ that contains two lines is a nearaffine space.*

We now prove some results. The following proof appears in [2], and is expanded upon below with clarifying detail.

Theorem 3.28. *([2], p.80) Let U be a subspace of $\mathbf{A}$ and G a straight line of $\mathbf{A}$. Then*

$$\langle U, G \rangle = \bigcup_{y \in U} (y \parallel G)$$

is a subspace of $\mathbf{A}$.

Proof. Let x and y be two points in $\langle U, G \rangle$. Note that $U \subseteq \langle U, G \rangle$, so if x and y are in U, then $x \sqcup y \subseteq U \subseteq \langle U, G \rangle$. We may assume $x, y \notin U$, then there exist $x', y' \in U$ such that $x \in (x' \parallel G)$ and $y \in (y' \parallel G)$. By (Pa), since $x' \sqcup x \neq x' \sqcup y'$, $(y' \parallel G) \cap (x \parallel x' \sqcup y') = (y' \parallel x' \sqcup x) \cap ((x \parallel x' \sqcup y') \neq \varnothing$. Let $z \in (y' \parallel G) \cap (x \parallel x' \sqcup y')$. Then $x \sqcup z \parallel x \sqcup y'$. Now, if $z' \in x \sqcup z \setminus \{x\}$, then, again by (Pa), $x' \sqcup y' \cap (z' \parallel G) = (x' \parallel x \sqcup z') \cap (z' \parallel x \sqcup x') \neq \varnothing$. Let

$z'' \in x' \sqcup y' \cap (z' \parallel G)$. Then $(z'' \parallel G) = (z' \parallel G)$, and thus $z' \in (z'' \parallel G) \subseteq \langle U, G\rangle$. It follows $x \sqcup z \subseteq \langle U, G\rangle$.

If $x \sqcup y = x \sqcup z$, then we are done, so suppose not, then $y \neq z$ and $y \sqcup z = (y' \parallel G)$. Let $p \in x \sqcup y$ different from x and y. Since $x \sqcup p = x \sqcup y$, by (V), $(x \sqcup z) \cap (p \parallel G) = (x \sqcup z) \cap (p \parallel y \sqcup z) \neq \varnothing$. Let $p' \in (x \sqcup z) \cap (p \parallel G)$, then, since $p' \in x \sqcup z$, $p' \in \langle U, G\rangle$, so that there exists $p'' \in U$ such that $p' \in (p'' \parallel G)$. Then $p' \in (p'' \parallel G)$ and $p' \in (p \parallel G)$, so by (P1), $(p'' \parallel G) = (p \parallel G)$. Hence $p \in (p'' \parallel G) \subseteq \langle U, G\rangle$. Hence $x \sqcup y \subseteq \langle U, G\rangle$.

Next, we show any two points are joinable in U. Let x and y be points in $\langle U, G\rangle$. We may assume $x \in U$ and $y \notin U$, for if x and y are both in U, then they are joinable in U, and therefore also in $\langle U, G\rangle$, and if both x and y are not in U and are both joinable in $\langle U, G\rangle$ with an intermediate point $z \in U$, then they are joinable. Since $y \in \langle U, G\rangle$ there exists a point y' in U such that $y \in (y' \parallel G)$. The points x and y' are joinable in U, say by a sequence $x = p_0, p_1, \ldots, p_n = y' \in U$. But then $x = p_0, \ldots p_n = y', y$ is a sequence joining x and y, since $y' \sqcup y = (y' \parallel G)$ are straight by (P2). Hence all points in $\langle U, G\rangle$ are joinable in $\langle U, G\rangle$. It follows $\langle U, G\rangle$ is a subspace of $\mathbf{A}$. ■

Theorem 3.29. *For a subspace U and straight line G of $\mathbf{A}$, $\langle U, G\rangle$ contains G if and only if $G \cap U \neq \varnothing$.*

Proof. Suppose $G \cap U \neq \varnothing$, then there exists $x \in G \cap U$. Then $G = (x \parallel G) \subseteq \langle U, G\rangle$.

Conversely, suppose $\langle U, G\rangle$ contains G. If $G \subseteq U$, then we are done, so suppose not. Then there exists some $x \in G \setminus U$. Since $G \subseteq \langle U, G\rangle$, it follows $x \in \langle U, G\rangle$, so that there exists some $x' \in U$ such that $x \in (x' \parallel G)$. But $(x' \parallel G) = (x \parallel G) = G$, hence $x' \in G$. It follows $x' \in U \cap G$, so $U \cap G \neq \varnothing$. ■

Theorem 3.30. *Let U be a subspace of $\mathbf{A}$ and G be a straight line with $U \cap G \neq \varnothing$. Then the weak subspace generated by $U \cup G$, namely $\langle U \cup G\rangle$, is a subspace. In particular, $\langle U \cup G\rangle = \langle U, G\rangle$.*

Proof. If $|G \cap U| > 1$, then $G \subseteq U$ by Lemma 3.25 and so $\langle U, G\rangle = U = \langle U\rangle = \langle U \cup G\rangle$. Suppose then G and U intersect in a single point y. Since U and G intersect in y, $\langle U, G\rangle$ is a (weak) subspace containing $U \cup G$, so that $\langle U \cup G\rangle \subseteq \langle U, G\rangle$. We show that $\langle U, G\rangle \subseteq \langle U \cup G\rangle$ by proving the following statement inductively.

For all $n \in \mathbb{N}$, if $x \in \langle U, G\rangle$ is a point outside of $U \cup G$ such that x lies on the line $(x' \parallel G)$ for some point $x' \in U$, and if y and x' are joinable by the sequence $y = p_0, p_1, \ldots, p_n = x'$ in U, then $x \in \langle U \cup G\rangle$.

Base Case: Suppose that $x \in \langle U, G\rangle$ is a point outside of $U \cup G$ such that x lies on the line $(x' \parallel G)$ for some point $x' \in U$ such that $y \sqcup x'$ is straight. Since the order of **A** is greater than or equal to three, there exists a point $z \in y \sqcup x'$ distinct from y and x'. Then $z \sqcup x' = z \sqcup y = y \sqcup x'$, since $y \sqcup x'$ is straight. It follows by (V) that $(z \sqcup x) \cap G = (z \sqcup x) \cap (y \parallel x \sqcup x') \neq \varnothing$. Let $z' \in (z \sqcup x) \cap G$, then $z \sqcup x = z \sqcup z'$. Since $z \in y \sqcup x' \subseteq U$ and $z' \in G$, it follows that $z, z' \in U \cup G$, so that $z \sqcup z' \subseteq \langle U \cup G\rangle$. Hence $x \in \langle U \cup G\rangle$.

Inductive Hypothesis: Suppose that, if $x \in \langle U, G\rangle$ is a point outside of $U \cup G$ such that x lies on the line $(x' \parallel G)$ for some point $x' \in U$, and if y and x' are joinable by the sequence $y = p_0, p_1, \ldots, p_k = x'$ in U, then $x \in \langle U \cup G\rangle$.

Inductive Step: Let $x \in \langle U, G\rangle$ be a point outside of $U \cup G$ such that x lies on the line $(x' \parallel G)$ for some point $x' \in U$, and suppose y and x' are joinable by the sequence $y = p_0, p_1, \ldots, p_{k+1} = x'$ in U. Since $x \notin U$, we know that $x' \sqcup x \neq x' \sqcup p_k$, so that by (Pa), $(p_k \parallel G) \cap (x \parallel x' \sqcup p_k) = (p_k \parallel x' \sqcup x) \cap (x \parallel x' \sqcup p_k) \neq \varnothing$. Let $p'_k \in (p_k \parallel G) \cap (x \parallel x' \sqcup p_k)$. If $p'_k \in U$, then $p_k \sqcup p'_k \subseteq U$, but then, since $x' \sqcup x \parallel p_k \sqcup p'_k$, $x' \sqcup x \subseteq U$ by Theorem 3.26, contradicting that $x \notin U$, hence $p'_k \notin U$. Furthermore, if $p'_k \in G$, since $p'_k \sqcup p_k \parallel G$, it follows that $G = p'_k \sqcup p_k$ so that $p_k = y$ (since U and G intersect in a unique point), which reduces to the base case. Suppose therefore that $p'_k \notin U \cup G$. Then, since $p'_k \in (p_k \parallel G)$ and y and p_k are joinable by the sequence $y = p_0, p_1, \ldots, p_k$ in U, by the inductive hypothesis, $p'_k \in \langle U \cup G\rangle$.

Let $z \in p_k \sqcup x'$ different from p_k and x'. Then $z \sqcup p_k = z \sqcup x' = p_k \sqcup x'$ since $p_k \sqcup x'$ is straight. By (V), $(z \sqcup x) \cap (p_k \sqcup p'_k) \neq \varnothing$. Let $z' \in (z \sqcup x) \cap (p_k \sqcup p'_k)$. Then $z \in U \subseteq \langle U \cup G\rangle$ and $z' \in p_k \sqcup p'_k \subseteq \langle U \cup G\rangle$ so that $z \sqcup z' \subseteq \langle U \cup G\rangle$. It follows that $x \in z \sqcup z' \subseteq \langle U \cup G\rangle$.

It follows by induction, for all $x \in \langle U, G\rangle$ outside of $U \cup G$, $x \in \langle U, G\rangle$, so that $\langle U, G\rangle \subseteq \langle U \cup G\rangle$. Hence $\langle U \cup G\rangle = \langle U, G\rangle$. ∎

The above result can be generalised slightly by noting that, in the above proof, to show that $x \in \langle U \cup G\rangle$, we only needed that $x' \in U \cap (x \parallel G)$ and y are joinable in U. Therefore, using a similar proof as above, we can show the following result.

Corollary 3.31. Let U be a weak subspace and G be a straight line intersecting U in a point x. Let y be a point in U joinable with x in U. Then $(y \parallel G) \subseteq \langle U \cup G \rangle$.

Corollary 3.32. Let $\mathbb{G} = \{G_1, \ldots, G_n\}$ be a set of straight lines intersecting in a common point. Then $\langle \bigcup_{i=1}^{n} G_i \rangle$ is a subspace.

Proof. By induction on n. We know that $\langle \bigcup_{i=1}^{1} G_i \rangle = \langle G_1 \rangle = G_1$ is a subspace, which completes the base case. Suppose then that $\left\langle \bigcup_{i=1}^{k} G_i \right\rangle$ is a subspace for some $k \in \mathbb{N}$. Then

$$\begin{aligned}
\left\langle \bigcup_{i=1}^{k+1} G_i \right\rangle &= \left\langle \left(\bigcup_{i=1}^{k} G_i \right) \cup G_{k+1} \right\rangle \\
&= \left\langle \left\langle \bigcup_{i=1}^{k} G_i \right\rangle \cup G_{k+1} \right\rangle \text{ (by Lemma 3.13)} \\
&= \left\langle \left\langle \bigcup_{i=1}^{k} G_i \right\rangle, G_{k+1} \right\rangle \text{ (by Theorem 3.30)}
\end{aligned}$$

which is a subspace. It follows by induction $\langle \bigcup_{i=1}^{n} G_i \rangle$ is a subspace. ∎

Theorem 3.33. *Let $\mathbb{G} = \{G_i \,|i \in I\}$ be a set of straight lines all intersecting in a point. Then*

$$\langle \mathbb{G} \rangle = \left\langle \bigcup_{i \in I} G_i \right\rangle$$

is a subspace.

Proof. Let x be the intersection point of the lines in $\mathbb{G}$, and let $\mathbb{U}_0 = \{x\}$, and for each $n \in \mathbb{N}$, define $\mathbb{U}_n$ recursively by

$$\mathbb{U}_n = \bigcup_{y \in \mathbb{U}_{n-1}} \bigcup_{G \in \mathbb{G}} (y \parallel G).$$

Note that $\mathbb{U}_n \subseteq \mathbb{U}_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Define $\mathbb{U} = \bigcup_{n \in \mathbb{N}} \mathbb{U}_n$.

1. *For all $n \in \mathbb{N} \cup \{0\}$ and $y \in \mathbb{U}_n \setminus \{x\}$, there exists a sequence $x = p_0, \ldots, p_k = y$ in $\mathbb{U}_n$ such that $k \leq n$ and $(x \parallel p_{i-1} \sqcup p_i) \in \mathbb{G}$ for all $i \in \{1 \ldots, k\}$.*

   We prove this by induction. For the base case, the statement is vacuously satisfied. Suppose then for an arbitrary $n \in \mathbb{N} \cup \{0\}$ that for all points $z \in \mathbb{U}_n$, there exists a sequence $x = p_0, \ldots, p_k = z$ in $\mathbb{U}_n$ such that $k \leq n$ and $(x \parallel p_{i-1} \sqcup p_i) \in \mathbb{G}$ for all $i \in \{1 \ldots, k\}$, and let $y \in \mathbb{U}_{n+1} \setminus \mathbb{U}_n$. Then, by definition, there exists $y' \in \mathbb{U}_n$ and $G \in \mathbb{G}$ such that $y \in (y' \parallel G)$. We know, since $y' \in \mathbb{U}_n$, there exists a sequence $x = p_0, \ldots, p_k = y'$ in $\mathbb{U}_n$ such that $k \leq n$ and $(x \parallel p_{i-1} \sqcup p_i) \in \mathbb{G}$ for all $i \in \{1 \ldots, k\}$. Let $y = p_{k+1}$.

Then $p_0, \ldots, p_k \in \mathbb{U}_n \subseteq \mathbb{U}_{n+1}$ by the inductive hypothesis, and $p_{k+1} \in \mathbb{U}_{n+1}$ by assumption. Furthermore $k+1 \leq n+1$, since $k \leq n$ by the inductive hypothesis. Finally $(x \parallel p_{i-1} \sqcup p_i) \in \mathbb{G}$ for all $i \in \{1 \ldots, k\}$ by the inductive hypothesis, and $p_k \sqcup p_{k+1} = y' \sqcup y \parallel G \in \mathbb{G}$. Hence the statement follows by induction.

2. $\mathbb{U}$ *is a subspace.*

We already know that all points of $\mathbb{U}$ are joinable to x in $\mathbb{U}$ by (1.), and therefore joinable to each other in $\mathbb{U}$ through x. It remains to be shown that for any two points y and z in $\mathbb{U}$, we have $y \sqcup z \subseteq \mathbb{U}$. Let y and z then be two points in $\mathbb{U}$. By (1.) there exist sequences $x = p_0, \ldots, p_n = y$ and $x = p'_0, \ldots, p'_r = z$ such that $G_i = (x \parallel p_{i-1} \sqcup p_i) \in \mathbb{G}$ for all $i \in \{1 \ldots, n\}$ and $G'_j = (x \parallel p'_{j-1} \sqcup p'_j) \in \mathbb{G}$ for all $j \in \{1 \ldots r\}$. Define $\mathbb{G}'$ to be the set $\{G_1, \ldots, G_n, G'_1, \ldots G'_r\}$. By Corollary 3.32, $\langle \bigcup \mathbb{G}' \rangle$ is a subspace.

We show that $\langle \bigcup \mathbb{G}' \rangle \subseteq \mathbb{U}$. Let w be a point of $\langle \bigcup \mathbb{G}' \rangle$, and suppose for contradiction that $w \notin \mathbb{U}$. Then there is a (not necessarily unique) set $\mathbb{G}_w \subseteq \mathbb{G}'$ of minimal cardinality such that $w \in \langle \bigcup \mathbb{G}_w \rangle$ (that is, if $G \in \mathbb{G}_w$, then $w \notin \langle \bigcup (\mathbb{G}_w \setminus \{G\}) \rangle$. By the Well-Ordering Principle, we may assume that w is a point with minimal $|\mathbb{G}_w|$ such that $w \notin \mathbb{U}$. Now, let $G \in \mathbb{G}_w$. Then $w \notin \langle (\mathbb{G}_w \setminus \{G\}) \rangle$. Since $\langle \bigcup (\mathbb{G}_w \setminus \{G\}) \rangle$ is a subspace by Corollary 3.32, we know that $\langle \bigcup \mathbb{G}_w \rangle = \langle \langle \bigcup (\mathbb{G}_w \setminus \{G\}) \rangle \cup G \rangle = \langle \langle \bigcup (\mathbb{G}_w \setminus \{G\}) \rangle, G \rangle$ by Theorem 3.30. Hence there exists $w' \in \langle \bigcup (\mathbb{G}_w \setminus \{G\}) \rangle$ such that $w \in (w' \parallel G)$. But then, since $w' \in \langle \bigcup (\mathbb{G}_w \setminus \{G\}) \rangle$, we know that $|\mathbb{G}_{w'}| \leq |\mathbb{G}_w \setminus \{G\}| < |\mathbb{G}_w|$: hence, by minimality of $|\mathbb{G}_w|$ we have $w' \in \mathbb{U}$. Then $w' \in \mathbb{U}_m$ for some $m \in \mathbb{N} \cup \{0\}$. But $w \in (w' \parallel G)$. Therefore, since $G \in \mathbb{G}_w \subseteq \mathbb{G}$, we know that $w \in \mathbb{U}_{m+1} \subseteq \mathbb{U}$ — a contradiction, since it was assumed $w \notin \mathbb{U}$. Hence we must have that $\langle \bigcup \mathbb{G}' \rangle \subseteq \mathbb{U}$.

Next, we show that $y \in \langle \bigcup \mathbb{G}' \rangle$. We show by induction that $p_i \in \langle \bigcup \mathbb{G}' \rangle$. Obviously, $p_0 = x \in \bigcup \mathbb{G}' \subseteq \langle \bigcup \mathbb{G}' \rangle$, which proves the base case. Suppose then that $p_k \in \langle \bigcup \mathbb{G}' \rangle$ for some $k \in \{0, \ldots, n-1\}$. Then $(p_k \parallel G_{k+1}) \subseteq \langle \bigcup \mathbb{G}' \rangle$ by Theorem 3.26. Since $p_{k+1} \in (p_k \parallel G_{k+1})$, it follows $p_{k+1} \in \langle \bigcup \mathbb{G}' \rangle$. It follows by induction, for all $i \in \{0, \ldots, n\}$, $p_i \in \langle \bigcup \mathbb{G}' \rangle$; in particular $y \in \langle \bigcup \mathbb{G}' \rangle$.

In an analogous way, we also have $z \in \langle \bigcup \mathbb{G}' \rangle$. Now $y, z \in \langle \bigcup \mathbb{G}' \rangle$, so that $y \sqcup z \subseteq \langle \bigcup \mathbb{G}' \rangle$. Since $\langle \bigcup \mathbb{G}' \rangle \subseteq \mathbb{U}$, we may conclude $y \sqcup z \subseteq \mathbb{U}$.

3. $\langle \mathbb{G} \rangle = \mathbb{U}$.

   $(\subseteq)$: This follows from the fact that $\mathbb{U}$ is a (weak) subspace containing $\bigcup \mathbb{G}$.

   $(\supseteq)$: We prove this inclusion by induction: for all $n \in \mathbb{N} \cup \{0\}$, $\mathbb{U}_n \subseteq \langle \mathbb{G} \rangle$. For the base case, note that $\mathbb{U}_0 = \{x\} \subseteq \langle \mathbb{G} \rangle$. Suppose then that $\mathbb{U}_k \subseteq \langle \mathbb{G} \rangle$ for some $k \in \mathbb{N} \cup \{0\}$. Let $y \in \mathbb{U}_{k+1} \setminus \mathbb{U}_k$. Then there exists some $y' \in \mathbb{U}_k$ and $G \in \mathbb{G}$ such that $y \in (y' \parallel G)$. We know $G \subseteq \langle \mathbb{G} \rangle$ and $y' \in \mathbb{U}_k \subseteq \langle \mathbb{G} \rangle$. Furthermore, by (1.) we know that there exists a sequence of straight lines joining x and y' in $\mathbb{U}_k \subseteq \langle \mathbb{G} \rangle$. Now, note that x and y' are joinable in $\langle \mathbb{G} \rangle$. Furthermore, the straight line $G \subseteq \langle \mathbb{G} \rangle$, therefore $\langle \langle \mathbb{G} \rangle \cup G \rangle = \langle \mathbb{G} \rangle$, and so we have by Corollary 3.31 that $(y' \parallel G) \subseteq \langle \langle \mathbb{G} \rangle \cup G \rangle = \langle \mathbb{G} \rangle$. Hence $y \in \langle \mathbb{G} \rangle$. It follows by induction that $\mathbb{U}_n \subseteq \langle \mathbb{G} \rangle$ for all $n \in \mathbb{N} \cup \{0\}$, so that $\mathbb{U} = \bigcup_{n \in \mathbb{N}} \mathbb{U}_n \subseteq \langle \mathbb{G} \rangle$.

Therefore $\langle \mathbb{G} \rangle = \mathbb{U}$ is a subspace. ∎

From now on, for a set $\mathbb{G} = \{G_i \mid i \in I\}$ of straight lines intersecting in a point, we add the notation $\langle G \mid G \in \mathbb{G} \rangle$ and $\langle G_i \mid i \in I \rangle$ to denote the subspace $\langle \mathbb{G} \rangle$.

It is now clear that straight lines generate (at least some) subspaces, so the natural next step is to add a notion of independence to them. For this, we define a matroid on the set of straight lines through a given point. We make use of the following definition.

Definition 3.34. Let x be a point of $\mathbf{A}$. We define $\mathbb{G}_x$ to be the set of straight lines through x.

Theorem 3.35. *Let $cl : \mathcal{P}(\mathbb{G}_x) \to \mathcal{P}(\mathbb{G}_x)$ be the map defined as follows,*

$$cl(\mathbb{G}) = \{G \in \mathbb{G}_x \mid G \subseteq \langle \mathbb{G} \rangle\}.$$

Then $(\mathbb{G}_x, cl)$ is a finitary matroid.

Proof. Let $\mathbb{G}, \mathbb{G}' \subseteq \mathbb{G}_x$ and $G, G' \in \mathbb{G}_x$.

(E) If $G \in \mathbb{G}$, then $G \subseteq \langle \mathbb{G} \rangle$, so $G \in \mathrm{cl}(\mathbb{G})$. Hence $\mathbb{G} \subseteq \mathrm{cl}(\mathbb{G})$.

(I) Suppose $\mathbb{G} \subseteq \mathbb{G}'$. Then $\bigcup \mathbb{G} \subseteq \bigcup \mathbb{G}' \subseteq \langle \mathbb{G}' \rangle$. But $\langle \mathbb{G} \rangle$ is the smallest weak subspace containing $\bigcup \mathbb{G}$, so $\langle \mathbb{G} \rangle = \langle \bigcup \mathbb{G} \rangle \subseteq \langle \mathbb{G}' \rangle$. Hence $\langle \mathbb{G} \rangle \subseteq \langle \mathbb{G}' \rangle$. It follows, if $G \in \mathrm{cl}(\mathbb{G})$, then $G \subseteq \langle \mathbb{G} \rangle \subseteq \langle \mathbb{G}' \rangle$, so that $G \in \mathrm{cl}(\mathbb{G}')$, and thus $\mathrm{cl}(\mathbb{G}) \subseteq \mathrm{cl}(\mathbb{G}')$.

(wI) We prove the stronger condition that cl is idempotent. Let $G \in \mathrm{cl}(\mathrm{cl}(\mathbb{G}))$. Then $G \subseteq \langle \mathrm{cl}(\mathbb{G}) \rangle$. But, since $\bigcup \mathrm{cl}(\mathbb{G}) \subseteq \langle \mathbb{G} \rangle$, it follows that $\langle \mathrm{cl}(\mathbb{G}) \rangle = \langle \bigcup \mathrm{cl}(\mathbb{G}) \rangle \subseteq \langle \mathbb{G} \rangle$. Hence $G \subseteq \langle \mathbb{G} \rangle$, so that $G \in \mathrm{cl}(\mathbb{G})$. Hence $\mathrm{cl}(\mathrm{cl}(\mathbb{G})) \subseteq \mathrm{cl}(\mathbb{G})$, and using (E), $\mathrm{cl}(\mathrm{cl}(\mathbb{G})) = \mathrm{cl}(\mathbb{G})$.

(wE) Suppose $G \in \mathrm{cl}(\mathbb{G} \cup \{G'\}) \setminus \mathrm{cl}(\mathbb{G})$. We know $G' \notin \mathrm{cl}(\mathbb{G})$: otherwise $G' \subseteq \langle \mathbb{G} \rangle$, so that, by Lemma 3.13, $\langle \mathbb{G} \cup \{G'\} \rangle = \langle \langle \mathbb{G} \rangle \cup G' \rangle = \langle \langle \mathbb{G} \rangle \rangle = \langle \mathbb{G} \rangle$. Hence $\mathrm{cl}(\mathbb{G} \cup \{G'\}) \setminus \mathrm{cl}(\mathbb{G}) = \varnothing$, contradicting that $G \in \mathrm{cl}(\mathbb{G} \cup \{G'\}) \setminus \mathrm{cl}(\mathbb{G})$. Therefore $G' \not\subseteq \langle \mathbb{G} \rangle$, so that G' intersects $\langle \mathbb{G} \rangle$ in the point x.

It follows by Theorem 3.30 that $\langle \mathbb{G} \cup \{G'\} \rangle = \langle \langle \mathbb{G} \rangle , G' \rangle$. Now $G \not\subseteq \langle \mathbb{G} \rangle$ and $G \subseteq \langle \langle \mathbb{G} \rangle , G' \rangle$. Let $y \in G \setminus \{x\}$, then $y \in \langle \langle \mathbb{G} \rangle , G' \rangle$ and $y \notin \langle \mathbb{G} \rangle$ (otherwise $x \sqcup y = G \subseteq \langle \mathbb{G} \rangle$). It follows that there exists $y' \in \langle \mathbb{G} \rangle$ such that $y \in (y' \parallel G')$. We know $(y' \parallel G') \parallel G'$, and, since $G \neq G'$ and G and G' intersect in the point x, we know that G and G' are not parallel. Hence $y \sqcup y' = (y' \parallel G')$ is not parallel to G, and therefore not equal to $G = y \sqcup x$. Therefore, by (Pa), it follows $(y' \parallel G) \cap G' = (y' \parallel y \sqcup x) \cap (x \parallel y \sqcup y') \neq \varnothing$. Let $z \in (y' \parallel G) \cap G'$. Since $z \in (y' \parallel G)$ and $y' \in \langle \mathbb{G} \rangle$, it follows $z \in \langle \langle \mathbb{G} \rangle , G \rangle$, so that $G' = x \sqcup z \subseteq \langle \langle \mathbb{G} \rangle , G \rangle$. Furthermore $\langle \langle \mathbb{G} \rangle , G \rangle = \langle \mathbb{G} \cup \{G\} \rangle$ by Theorem 3.30. Hence $G' \subseteq \langle \mathbb{G} \cup \{G\} \rangle$, so that $G' \in \mathrm{cl}(\mathbb{G} \cup \{G\})$. Since $G' \notin \mathrm{cl}(\mathbb{G})$, we may conclude $G' \in \mathrm{cl}(\mathbb{G} \cup \{G\}) \setminus \mathrm{cl}(\mathbb{G})$.

(F) Let $G \in \mathrm{cl}(\mathbb{G})$ and let $y \in G \setminus \{x\}$. Since $G \in \mathrm{cl}(\mathbb{G})$, it follows $G \subseteq \langle \mathbb{G} \rangle$, and in particular, $y \in \langle \mathbb{G} \rangle$. From part 1 of the proof of Theorem 3.33, it follows that there is a sequence $x = p_0, p_1, \ldots, p_n = y$ of points in $\langle \mathbb{G} \rangle$ such that $G_i = (x \parallel p_{i-1} \sqcup p_i) \in \mathbb{G}$. Let $\mathbb{G}' = \{G_i \,|\, 1 \leq i \leq n\}$. We have that $\mathbb{G}' \subseteq \mathbb{G}$ and that $\mathbb{G}'$ is finite. Furthermore, $p_i \in \langle \mathbb{G}' \rangle$ for all $i \in \{0, \ldots, n\}$: if not, there would be a smallest $j \in \{0, \ldots, n\}$ such that $p_j \notin \langle \mathbb{G}' \rangle$. But $x = p_0 \in \langle \mathbb{G}' \rangle$, hence $j > 0$, and thus $p_j \in (p_{j-1} \parallel p_j \sqcup p_{j-1}) = (p_{j-1} \parallel G_j)$. By Theorem 3.26, it follows $(p_{j-1} \parallel G_j) \subseteq \langle \mathbb{G}' \rangle$, and so $p_j \in \langle \mathbb{G}' \rangle$, contradicting that $p_j \notin \langle \mathbb{G}' \rangle$. Hence $p_i \in \langle \mathbb{G} \rangle$ for all $i \in \{0 \ldots, n\}$; in particular $y = p_n \in \langle \mathbb{G}' \rangle$, hence $G = x \sqcup y \subseteq \langle \mathbb{G}' \rangle$, and so $G \in \mathrm{cl}(\mathbb{G}')$.

The construction of a closure operator allows us to define independence, bases, and dimension for the finitary matroid associated with the ground set $\mathbb{G}_x$. It is a well-known result that every independent set of a finitary matroid is contained in a basis, and that every basis of a finitary matroid has the same cardinality (see Section 2.3). Since the empty set is independent, we are assured of the existence of a basis for our matroid.

Theorem 3.36. *Suppose x is a point of $\mathbf{A}$. Let $\mathbb{G}$ be a basis for $\mathbb{G}_x$. Then $\langle \mathbb{G} \rangle = \mathbb{P}$, the point set of $\mathbf{A}$.*

Proof. Suppose not. Then $\langle \mathbb{G} \rangle \subsetneq \mathbb{P}$. It follows there is some point $p \in \mathbb{P}$ such that $p \notin \langle \mathbb{G} \rangle$. By (G2), the points x and p are joinable, say by $x = p_0, \dots p_k = p$. Let $G_i = (x \parallel p_{i-1} \sqcup p_i)$, and let $\mathbb{G}' = \mathbb{G} \cup \{G_i \,|\, 1 \leq i \leq k\}$. Now, $p_i \in \langle \mathbb{G}' \rangle$ for all $i \in \{0 \dots, k\}$. If not, since $p_0 = x \in \mathbb{G}'$, by the Well-Ordering Principle, there is some p_j with minimum index j such that $p_j \notin \langle \mathbb{G}' \rangle$. But $p_j \in p_{j-1} \sqcup p_j = (p_{j-1} \parallel G_j) \subseteq \langle \mathbb{G}' \rangle$ by Theorem 3.26, therefore $p_j \in \langle \mathbb{G}' \rangle$, contradicting our assumption. Therefore, in particular, $p = p_k \in \langle \mathbb{G}' \rangle$. It follows that $\langle \mathbb{G} \rangle \subsetneq \langle \mathbb{G}' \rangle$, and $\mathbb{G} \subsetneq \mathbb{G}'$. Now, suppose $G \in \mathbb{G}' \setminus \mathbb{G}$. Since $\mathbb{G}$ is a maximal independent set, we have that $\mathbb{G} \cup \{G\}$ is not independent. Therefore, $G \in \mathrm{cl}(\mathbb{G})$, and so $G \subseteq \langle \mathbb{G} \rangle$. Hence $\bigcup \mathbb{G}' \subseteq \langle \mathbb{G} \rangle$, so that $\langle \mathbb{G}' \rangle \subseteq \langle \mathbb{G} \rangle$ — a contradiction. Hence $\langle \mathbb{G} \rangle = \mathbb{P}$. ■

Corollary 3.37. Let U be a subspace of $\mathbf{A}$ containing at least one line. Let $x \in U$. Then there exists a maximal independent subset of $\mathbb{G}_x$ of lines contained in U. Moreover, this set generates U.

Proof. If U contains two lines, then U itself is a nearaffine space by Theorem 3.27. Let

$$\mathbb{G}_{x,U} = \{G \in \mathbb{G}_x \,|\, G \subseteq U\}.$$

Let $\mathbb{G}$ be a basis for $\mathbb{G}_{x,U}$. $\mathbb{G}$ is then a basis for the set of straight lines through x contained in U, and so $\langle \mathbb{G} \rangle = U$ by Theorem 3.36.

If U contains only one line, then, since U is a subspace, U is a straight line, and so $\mathbb{G} = \{U\}$ is a set containing a straight line through x such that $\langle \mathbb{G} \rangle = U$. ■

The following result for the finite dimensional case is given in [2] (Proposition 3.2, p.86). Below a generalised proof is given.

Theorem 3.38. *Let U be a subspace of a nearaffine space $\mathbf{A}$ of order greater than or equal to 3. Let $\mathbb{G}_{x,U} = \{G \in \mathbb{G}_x \,|\, G \subseteq U\}$ for any $x \in U$. Let x and y be different points of U, and let $\mathbb{G}$ be a basis for $\mathbb{G}_{x,U}$. Then $\mathbb{G}' = \{(y \parallel G) \,|\, G \in \mathbb{G}\}$ is a basis for $\mathbb{G}_{y,U}$.*

Proof. Firstly, note that $x \in \langle \mathbb{G}' \rangle$ (see part 1. of the proof of Theorem 3.33). Therefore, since for each $G \in \mathbb{G}$, $G = (x \parallel G')$ for some $G' \in \mathbb{G}'$, it follows that $G \subseteq \langle \mathbb{G}' \rangle$, hence $U = \langle \mathbb{G} \rangle \subseteq \langle \mathbb{G}' \rangle$, and therefore $\mathbb{G}'$ generates U. But if $\mathbb{G}'$

generates U, then for each $G' \in \mathbb{G}_{y,U}$, $G' \subseteq U = \langle \mathbb{G}' \rangle$, and so $G' \in \mathrm{cl}(\mathbb{G}')$. Hence $\mathbb{G}_{y,U}$ is generated by $\mathbb{G}'$.

To show $\mathbb{G}'$ is minimal, suppose $\mathbb{G}'' \subsetneq \mathbb{G}'$ such that $\mathbb{G}''$ generates $\mathbb{G}_{y,U}$. Then, if $\mathbb{G}''' = \{(x \parallel G') \mid G' \in \mathbb{G}''\}$, $\mathbb{G}'''$ generates $\mathbb{G}_{x,U}$ by the same argument as in the above paragraph (switching x and y, $\mathbb{G}'$ and $\mathbb{G}'''$, and $\mathbb{G}$ and $\mathbb{G}''$ respectively). But $\mathbb{G}''' \subseteq \mathbb{G}$, contradicting the minimality of $\mathbb{G}$. Hence $\mathbb{G}'$ is a minimal generating set of U. ∎

The above results allows us to define the notion of a basis and dimension for a subspace.

Definition 3.39. Let U be a subspace of $\mathbf{A}$. Let $\mathbb{G}_{x,U} = \{G \in \mathbb{G}_x \mid G \subseteq U\}$ for any $x \in U$. Let $\dim \mathbb{G}_{x,U}$ be the cardinality of a basis for U of straight lines through x for any $x \in U$.

1. $\mathbb{G} \subseteq \mathbb{G}_{x,U}$ is a said to be a basis for U if $\mathbb{G}$ is a basis for $\mathbb{G}_{x,U}$.
2. The dimension of U is defined as follows:

$$\dim U = \begin{cases} \dim \mathbb{G}_{x,U} & \text{for any } x \in U, \text{ if } U \text{ contains a line.} \\ 0 & \text{if } U = \{x\} \text{ for some point } x. \\ -1 & \text{if } U = \varnothing. \end{cases}$$

While André also defines independence, bases, and dimension for subspaces in [2], he does so only for finite nearaffine spaces. It is routine to show that André's definitions coincide exactly with the ones given above when $\mathbf{A}$ is finite.

André proves the result below in the finite case in [2] (Proposition 3.3, p.86). However, the proof does not rely on the fact that the nearaffine space is finite. Therefore, a generalised version of the result is stated, and André's proof is presented without alteration.

Theorem 3.40. *Let U be a subspace of a nearaffine space $\mathbf{A}$ of order greater than or equal to 3 containing at least one line. Let y be a point of $\mathbf{A}$. For some basis $\mathbb{G}$ for U, define*

$$U_y = \langle (y \parallel G) \mid G \in \mathbb{G} \rangle .$$

Then either $U = U_y$ or $U \cap U_y = \varnothing$.

Proof. Suppose $z \in U \cap U_y$. Then by Theorem 3.38, $U = U_z = U_y$. ∎

The following definition is a useful shorthand.

Definition 3.41. ([2], p.90) Let L be a line and U be a subspace of a nearaffine space $\mathbf{A}$ of order greater than or equal to 3. Then we say $L \parallel U$ if $(x \parallel L) \subseteq U$ for every $x \in U$.

Lemma 3.42. ([2], p.88) Let H and H' be hyperplanes such that $H' = H_x$ for some point x of a nearaffine space $\mathbf{A}$. Then if $y \in H_x$ and $L \subseteq H$ for some line L, $(y \parallel L) \subseteq H_x$.

Lemma 3.43. Let U be a subspace, L be a line, and x be a point of a nearaffine space $\mathbf{A}$ of order greater than or equal to 3. If $L \parallel U$, then $L \parallel U_x$.

Proof. If $x \in U$, then we are done by Theorem 3.40, so suppose not. Let $x' \in U$, and let $G = x \sqcup x'$. Suppose first that G is straight. Let $\mathbb{G}$ be a basis for U though x', then clearly $\mathbb{G} \cup \{G\}$ will be a basis for $\langle U, G\rangle$. But $x \in G$, so $x \in \langle U, G\rangle$, hence $\{(x \parallel G') \,|\, G' \in \mathbb{G}\} \cup \{G\}$ is a basis for $\langle U, G\rangle$ as well. Furthermore, $\{(x \parallel G') \,|\, G' \in \mathbb{G}\}$ is a basis for U_x, so it follows $\langle U, G\rangle = \langle U_x, G\rangle$. It follows U and U_x are hyperplanes of $\langle U, G\rangle$. Let $z \in U$, then $(z \parallel L) \subseteq U$, since $L \parallel U$. It follows by Lemma 3.42 that $(x \parallel L) \subseteq U_x$. Hence by Theorem 3.26, $L \parallel U_x$.

If G is not straight, let $x' = p_0, \dots, p_n = x$ be a sequence joining x and x'. Then by inductively applying the above argument,

$$L \parallel U_{p_0} \Rightarrow L \parallel U_{p_1} \Rightarrow \dots \Rightarrow L \parallel U_{p_n}.$$

Since $U = U_{p_0}$ and $U_x = U_{p_n}$, we may conclude $L \parallel U_x$. ∎

Lemma 3.44. ([2], p.88) Let H be a hyperplane and L be a line of a nearaffine space $\mathbf{A}$. Then either $L \cap H \neq \varnothing$ or $L \parallel H$.

To prove the final result in this section, we add the following definition.

Definition 3.45. Let U be a subspace of a nearaffine space $\mathbf{A}$. Define $d_U : U^2 \to \mathbb{N}$ as the length n of the shortest sequence $x = p_0, \dots, p_n = y$ joining x and y in U.

Note that d_U is a metric on U.

Lemma 3.46. Let U be a finite dimensional subspace of a nearaffine space $\mathbf{A}$. If $\dim U = n$, then for any $x, y \in U$, $d_U(x, y) \leq n$.

Proof. By induction on n. If $\dim U = 0$, then $U = \{x\}$ for some point x. Since $d(x, x) = 0 \leq \dim U$, the theorem holds in the base case. Next, suppose that if

$\dim U = k$, then $d(x, y) \leq k$ for any $x, y \in U$. Let W be a $(k+1)$-dimensional subspace, and let $x, y \in W$. Let G be a straight line through x contained in W. Then there exists a basis $G_1, \ldots, G_k, G$ for W of straight lines through x. Let $W' = \langle G_1, \ldots, G_k \rangle$. If $y \in W'$, then we are done by the induction hypothesis. If $y \notin W'$, then, since $W = \langle W', G \rangle$, there exists some $y' \in W'$ such that $y \in (y' \parallel G)$. Hence $y' \sqcup y \parallel G$ and is therefore straight. By the induction hypothesis, $d_{W'}(x, y') \leq k$, hence, since $y' \sqcup y$ is straight, $d_W(x, y) \leq k + 1$, and so the result follows. ■

The next theorem is proven for finite nearaffine spaces.

Theorem 3.47. *([2], p.91) Let* $\mathbf{A}$ *be a finite nearaffine space of order* $n \geq 3$ *and let* U *be a set of points. Then the following are equivalent.*

(A) U *is a weak subspace.*

(B) U *is a subspace.*

(C) U *is a flat, i.e. any line incident with* $x, y \in U$, $x \neq y$, *completely lies in* U.

The argument presented in [2] for $(A) \Rightarrow (B)$ contains a slight error, in that it takes as a given that a weak subspace will contain a straight line (see part 3 of the proof).

We present a new proof for a general (not necessarily finite) nearaffine space. André notes that the proof he presents in [2] is due to Bachmann, and that he was unable to prove the theorem himself without the assumption that all weak subspaces are closed under parallelism. However, as already noted above, even the proof given in [2] contains an implicit assumption.

The proof below shares similarity with that which appears in [2] in that both are proofs by contradiction using a minimality argument — in [2] based on the dimension of the nearaffine space and the cardinality of a weak subspace, and below based on the minimal dimension of a certain subspace U, and $d_U(x, y)$ for any two points in U.

The equivalence to (C) is left as work for a future addition.

Theorem 3.48. *Let* $\mathbf{A}$ *be a nearaffine space of order* $n \geq 3$ *and let* U *be a set of points. Then the following are equivalent.*

(A) U *is a weak subspace.*

(B) U is a subspace.

Proof. $(B) \Rightarrow (A)$ is trivial, so we prove $(A) \Rightarrow (B)$. Suppose there exists a nearaffine space $\mathbf{A}$ with a proper weak subspace W.

1. *There exists a finite-dimensional subspace U containing a proper weak subspace.*

   W is a proper weak subspace of $\mathbf{A}$, so it contains two points x and y which are not joinable in W. It follows $\langle x, y\rangle \subseteq W$ is also a proper weak subspace. Let $x = p_0, \ldots, p_n = y$ be a sequence joining x and y in $\mathbf{A}$ (this is possible by (G2)). Let $G_1, \ldots, G_n$ be such that $G_i = (x \parallel p_{i-1} \sqcup p_i)$. Define $U = \langle G_1, \ldots, G_n\rangle$. Then $p_i \in U$ for all $i \in \{0, \ldots, n\}$: if not, let $j \in \{1, \ldots, n\}$ be the smallest index for which $p_j \notin U$. Clearly $p_0 = x \in U$. Note that $p_j \in p_{j-1} \sqcup p_j = (p_{j-1} \parallel G_j) \subseteq U$ by Theorem 3.26 — a contradiction. Therefore $p_i \in U$ for all $i \in \{0 \ldots, n\}$; in particular $y = p_n \in U$. Hence $x, y \in U$, and so, since U is a subspace containing x and y, $\langle x, y\rangle \subseteq U$. Furthermore, $\dim U \leq n$, since $U = \langle G_1, \ldots, G_n\rangle$.

Assume now that U is a finite-dimensional subspace with smallest dimension n such that U contains a proper weak subspace. Obviously, $n \geq 2$, else there would be a proper weak subspace contained in a straight line, a point, or the empty set. Let $x, y \in U$, $x \neq y$, with minimal $d_U(x, y)$ such that x and y are not joinable in $\langle x, y\rangle$, i.e. if $d_U(p, q) < d_U(x, y)$, then p and q are joinable in $\langle p, q\rangle$.

2. $d_U(x, y) = n$.

   Let $k = d_U(x, y)$ and let $x = p_0, \ldots, p_k = y$ be a sequence joining x and y in U. By Lemma 3.46, $k \leq n$. Now, let $G_i = (x \parallel p_{i-1} \sqcup p_i)$ for all $i \in \{1, \ldots, k\}$. The argument in 1 shows that $U' = \langle G_1, \ldots, G_k\rangle$ is a subspace containing x and y, and therefore also $\langle x, y\rangle \subseteq U'$. By the minimality of $\dim U$, it follows $n = \dim U \leq \dim U' \leq k$. Hence $n = k$.

3. $\langle x, y\rangle \not\subseteq \langle G_1, \ldots, G_{n-1}\rangle$

   This follows, since $\langle G_1, \ldots, G_{n-1}\rangle$ has dimension $n - 1 < n$, and so all its weak subspaces are subspaces.

4. $S = \langle x, y\rangle \cap \langle G_1, \ldots, G_{n-1}\rangle$ *is a strong subspace.*

   Suppose S is a proper weak subspace. We know $S \subseteq \langle G_1, \ldots, G_{n-1}\rangle$. But $\dim \langle G_1, \ldots, G_{n-1}\rangle = n - 1 < n$, contradicting the minimality of $n = \dim U$.

5. *S contains a straight line G through x.*

   First, note $x \in S$. Let $z \in x \sqcup y \setminus y \sqcup x$. Clearly $z \in \langle x, y\rangle$, so $y \sqcup z \subseteq \langle x, y\rangle \subseteq U$. Since $\langle G_1, \ldots, G_{n-1}\rangle$ is a hyperplane of U and $y \sqcup z$ is contained in U, it follows that either $y \sqcup z \cap \langle G_1, \ldots, G_{n-1}\rangle \neq \varnothing$ or $y \sqcup z \parallel \langle G_1, \ldots, G_{n-1}\rangle$ by Lemma 3.44.

   If $y \sqcup z \cap \langle G_1, \ldots, G_{n-1}\rangle \neq \varnothing$, then let $p \in y \sqcup z \cap \langle G_1, \ldots, G_{n-1}\rangle$. If $p = x$, then $y \sqcup z = y \sqcup x$, so that $z \in y \sqcup x$, contradicting our assumption that $z \notin y \sqcup x$. It follows that $p \neq x$ and so S contains two distinct points.

   If $y \sqcup z \parallel \langle G_1, \ldots, G_{n-1}\rangle$, then $(x \parallel y \sqcup z) \subseteq \langle G_1, \ldots, G_{n-1}\rangle$. Let $x' \in (x \parallel y \sqcup z) \setminus \{x\}$. We know then that $x \sqcup x' \parallel y \sqcup z$. It follows by (T) that $(y \parallel x \sqcup z) \cap (z \parallel x' \sqcup z) \neq \varnothing$. Since $x \sqcup z = x \sqcup y \parallel y \sqcup x$, it follows $(y \parallel x \sqcup z) = y \sqcup x$. Furthermore, since $x' \sqcup z \parallel z \sqcup x'$, we have that $(z \parallel x' \sqcup z) = z \sqcup x'$. Hence $(y \sqcup x) \cap (z \sqcup x') \neq \varnothing$. Let $w \in (y \sqcup x) \cap (z \sqcup x')$. Then $w \in y \sqcup x \subseteq \langle x, y\rangle$ and so $z \sqcup w \subseteq \langle x, y\rangle$. But $w \in z \sqcup x'$, and so $z \sqcup x' = z \sqcup w$. Thus $x' \in z \sqcup w \subseteq \langle x, y\rangle$. In addition, since $x' \in x \sqcup x' = (x \parallel y \sqcup z) \subseteq \langle G_1, \ldots, G_{n-1}\rangle$, we have that $x' \in \langle x, y\rangle \cap \langle G_1, \ldots, G_{n-1}\rangle = S$. Thus x and x' are distinct points in S.

   Hence, in all cases, we have that S is not a point. Since S is not a point, but is a subspace, it must contain a straight line G'. Let $G = (x \parallel G')$, then by Theorem 3.26, $G \subseteq S$.

6. *Conclusion of proof*

   We have $G \subseteq S \subseteq \langle x, y\rangle \subseteq U$. Because $G \subseteq U$, we know there exists a basis for U through x containing G, say $G, G'_2, \ldots, G'_n$. Let $V = \langle (y \parallel G'_i) \,|\, i \in \{2, \ldots, n\}\rangle$. We know $x \notin V$, since $d_U(x, y) = n > \dim V$. But $x \in U = U_y$ by Theorem 3.40 and $U_y = \langle V, (y \parallel G)\rangle$, so there exists some $q \in V$ such that $x \in (q \parallel (y \parallel G))$. Hence $q \sqcup x \parallel (y \parallel G) \parallel G$, and so $x \sqcup q = G$ by (P1). Since $q \in G$, it follows $q \in \langle x, y\rangle$, and so $\langle q, y\rangle \subseteq \langle x, y\rangle$. But $d_U(q, y) \leq d_V(q, y) \leq \dim V = n - 1 < n$, so q and y are joinable in $\langle q, y\rangle \subseteq \langle x, y\rangle$. But $x \sqcup q$ is straight, so x and y are joinable in $\langle x, y\rangle$ — a contradiction. Hence all weak subspaces are strong subspaces.

■

3.5 Geometry of near-vector spaces

We now return to near-vector spaces and look at their geometric structure.

Definition 3.49. ([2], p.76) Let (V, A) be a near-vector space with dimension greater than or equal to 2, and let $\mathbb{L}$ consist of all subsets of V of the form $x \sqcup y = A(y - x) + x$ for some x and y in V. We call $\mathbb{L}$ the set of lines and V to be the set of points, with $x \in V$ and $L \in \mathbb{L}$ being incident if $x \in L$. Together, $(V, \mathbb{L})$ form what is known as a *nearfield space*.

Definition 3.50. ([2], p.76) Let $(V, \mathbb{L})$ be a nearfield space and let L and L' be lines of V. Then $L \parallel L'$ if there exists $u, v, v' \in V$ such that $L = Au + v$ and $L' = Au + v'$.

It is worth noting that if $L = Au + v$ and $L' = Au + v'$ for some $u, v, v' \in V$, then $L = L' - v' + v$, so that L' is a translation of L.

The next result characterises straight lines.

Lemma 3.51. ([2], p.75) For all distinct $x, y \in V$, $x \sqcup y = y \sqcup x$ if and only if $x - y \in Q(V)^*$.

The next result shows the connection between nearfield spaces and nearaffine spaces. The result is mentioned in [2] without a full proof, which has been added here.

Theorem 3.52. *([2], p.76) A nearfield space $(V, \mathbb{L})$ is a nearaffine space.*

Proof. (L1) By definition, $x \sqcup y$ contains all elements of the form $\alpha(y - x) + x$ for some $\alpha \in A$. By setting $\alpha = 0$, we obtain x, and by setting $\alpha = 1$, we obtain y.

(L2) Suppose $z \in (x \sqcup y) \setminus \{x\}$. Then there exists some nonzero $\alpha \in A$ such that $z = \alpha(y - x) + x$. Since α is nonzero, it is invertible, and hence we obtain $y = \alpha^{-1}(z - x) + x$. Suppose now that $y' \in x \sqcup y$. Then there exists some $\beta \in A$ such that $y' = \beta(y - x) + x = \beta[(\alpha^{-1}(z - x) + x) - x] + x = \beta\alpha^{-1}(z - x) + x \in x \sqcup z$. Hence $x \sqcup y \subseteq x \sqcup z$. Similarly, if $y' \in x \sqcup z$, then there exists some $\beta \in A$ such that $y' = \beta(z - x) + x = \beta[(\alpha(y - x) + x) + x] + x = \beta\alpha(y - x) + x \in x \sqcup y$. Hence $x \sqcup y = x \sqcup z$.

Conversely, suppose $x \sqcup y = x \sqcup z$. Then $A(y - x) + x = A(z - x) + x$. Therefore for all $\alpha \in A$ there exists $\beta \in A$ such that $\alpha(z - x) + x = \beta(y - x) + x$. In particular, if $\alpha = 1$, then there exists $\beta \in A$ such that $z = \beta(y - x) + x \in x \sqcup y$, as required.

(L3) Suppose $x \sqcup y = y \sqcup x = x \sqcup z$. Then $y - x \in Q(V)$, so that $x \sqcup y \subseteq Q(V) + x$.

Therefore $z \in Q(V) + x$, so $z - x \in Q(V)$. It follows $x \sqcup z = z \sqcup x$.

(P1) Let $L = u \sqcup v$ and $x \in V$. Let $z = x - u$. We show that $x \sqcup (v + z) = L + z$. Note that $A(v - u) + u = A[(v + z) - (u + z)] + u$, so that $L + z = A[(v + z) - (z + u)] + u + z = A[(z + v) - x] + x = x \sqcup (v + z)$. It follows $x \sqcup (v + z)$ is a translation of L, and is therefore parallel to L with base point x.

To prove uniqueness, suppose we have a line L and point x such that $L \parallel x \sqcup y$ and $L \parallel x \sqcup y'$ for some $y, y' \in V$. Then $x \sqcup y \parallel x \sqcup y'$, so there exists some $z \in V$ such that $x \sqcup y = x \sqcup y' + z$. We now have that

$$\begin{aligned} x \sqcup y &= x \sqcup y' + z \\ &= A(y' - x) + x + z \\ &= A[(y' + z) - (x + z)] + (x + z) \\ &= (x + z) \sqcup (y' + z) \end{aligned}$$

Likewise $x \sqcup y' = (x - z) \sqcup (y - z)$. From this, two cases arise: either x is the unique base point of the line $x \sqcup y$ (in which case $x = x + z$, so that $z = 0$ and $x \sqcup y = x \sqcup y'$), or all points on $x \sqcup y$ are base points of the line, with $x + z \in x \sqcup y$. In the latter case, suppose $z \neq 0$, else the result follows immediately. Then $x \sqcup y = x \sqcup (x + z)$ by (L2), so $(-1)(x + z - x) + x = x - z \in x \sqcup y$. Hence $x \sqcup y = x \sqcup (x - z)$, again by (L2). Furthermore, since $z \neq 0$, the base point of the line $x \sqcup y'$ is not unique (since $x \sqcup y' = (x - z) \sqcup (y' - z)$), and $x - z \in x \sqcup y'$. It follows once again by (L2) that $x \sqcup y' = x \sqcup (x - z)$. Hence $x \sqcup y = x \sqcup (x - z) = x \sqcup y'$, as required.

(P2) Suppose $x \sqcup y \parallel x' \sqcup y'$ and $x \sqcup y = y \sqcup x$. Then we know that $x \sqcup y = x' \sqcup y' + z$ for some $z \in V$. It follows $x \sqcup y = x' \sqcup y' + z = (x' + z) \sqcup (y' + z)$ by the same argument as in (P1). By (L1), $y' + z \in (x' + z) \sqcup (y' + z) = x \sqcup y$, so that $x \sqcup y = x \sqcup (y' + z)$ by (L2). By (L3), because $y' + z \in x \sqcup y = y \sqcup x$, we know that $x \sqcup (y' + z) = (y' + z) \sqcup x$, so that $x \sqcup y = (y' + z) \sqcup x$. But $x' + z \in (x' + z) \sqcup (y' + z) = x \sqcup y = (y' + z) \sqcup x$ so that by (L2) $(y' + z) \sqcup x = (y' + z) \sqcup (x' + z)$. Hence $(x' + z) \sqcup (y' + z) = x \sqcup y = (y' + z) \sqcup x = (y' + z) \sqcup (x' + z)$, so that $x' \sqcup y' + z = y' \sqcup x' + z$, and so $x' \sqcup y' = y' \sqcup x'$.

(R) V has at least two basis vectors, say v_1 and v_2. Then $0 \sqcup v_i = Av_i$ for $i \in \{1, 2\}$. Since v_1 and v_2 are independent, it follows $Av_1 \neq Av_2$, so that there are at least two distinct lines.

(T) Suppose $x, y, z \in V$ are pairwise different vectors, and $x', y' \in V$ are different vectors, with $x \sqcup y \parallel x' \sqcup y'$, and let $z' \in V$ such that $x' = x + z'$. We have that

$$\begin{aligned} x' \sqcup (z + z') &= (x + z') \sqcup (z + z') \\ &= A(z - x) + x + z' \\ &= (x \sqcup z) + z'. \end{aligned}$$

Hence $x' \sqcup (z + z') \parallel x \sqcup z$. By (P1) we have $(x' \parallel x \sqcup z) = x' \sqcup (z + z')$. Likewise, $x' \sqcup (y + z') \parallel x' \sqcup y'$, hence $x' \sqcup y' = x' \sqcup (y + z')$, again by (P1). It follows that there is some $\alpha \in A^*$ such that $\alpha(y' - x') + x' = y + z'$. Therefore $y' = \alpha^{-1}(y - x + \alpha x')$, hence:

$$\begin{aligned} y' \sqcup \alpha^{-1}(z - x + \alpha x') &= \alpha^{-1}(y - x + \alpha x') \sqcup \alpha^{-1}(z - x + \alpha x') \\ &= A(\alpha^{-1}(z - y)) + \alpha^{-1}(y - x + \alpha x') \\ &= A(z - y) + y' \\ &= (y \sqcup z) + (y' - y) \end{aligned}$$

It follows $(y' \parallel y \sqcup z) = y' \sqcup \alpha^{-1}(z - x + \alpha x')$. Furthermore, note that, since $y' = \alpha^{-1}(y - x + \alpha x')$, we have that $y' - \alpha^{-1} y = x' - \alpha^{-1} x$, hence we have $\alpha^{-1}(z - y) + y' = \alpha^{-1}(z - x) + x'$.

But

$$\alpha^{-1}(z - y) + y' \in A(z - y) + y' = y' \sqcup \alpha^{-1}(z - x + \alpha x') = (y' \parallel y \sqcup z)$$

and

$$\alpha^{-1}(z - x) + x' \in A(z - x) + x + z' = x' \sqcup (z + z') = (x' \parallel x \sqcup z)$$

Hence $(x' \parallel x \sqcup z) \cap (y' \parallel y \sqcup z) \neq \varnothing$ as required.

(P3) We have $x \sqcup y = A(y - x) + x = A(x - y) + y + (x - y) = y \sqcup x + (x - y)$, hence $x \sqcup y \parallel y \sqcup x$ as required.

(G1) Suppose $G, L \in \mathbb{L}$ such that $G = Au + v$ is straight, $G \neq L = Au' + v'$. Furthermore, suppose G and L have two distinct intercepts, $x = \alpha u + v = \alpha' u' + v'$ and $y = \beta u + v = \beta' u' + v'$. Since G is straight, by Lemma 3.51,

$u \in Q(V)^*$. Furthermore, we may assume L is not straight: if L were straight, then $L = x \sqcup y = G$, since G is straight. It follows $u' \notin Q(V)$. But then $u' \notin Au$, since $Au \subseteq Q(V)$ by Lemma 2.5. Furthermore, from the equalities

$$\alpha u + v = \alpha' u' + v';$$

$$\beta u + v = \beta' u' + v'$$

we may conclude

$$\alpha u - \beta u = \alpha' u' - \beta' u'$$

so that

$$\alpha u - \alpha' u' = \beta u - \beta' u'.$$

Since $u \in Q(V)^*$ and $u' \notin Au$, this allows us to conclude $\alpha = \beta$ and $\alpha' = \beta'$ by Lemma 2.9. But then $x = \alpha u + v = \beta u + v = y$ — a contradiction, since x and y are distinct. Hence G and L can have at most one intersection point.

(G2) To show that for any $x, y \in V$, x and y are joinable is equivalent to showing for all $x \in V$, x and 0 are joinable. So assume now that $x \in V$, we will show that x is joinable to 0. If $x = 0$, then we are done by definition, so assume $x \neq 0$. Let $x = \sum_{i=1}^{n} \lambda_i u_i$, where $u_1, \ldots, u_n \in B$, a basis for V. Let $x_k = \sum_{i=1}^{k} \lambda_i u_i$ for each $k \in \{1, \ldots, n\}$, with $x_0 = 0$. Then, for each $k \in \{1, \ldots, n\}$, we have that: $x_k - x_{k-1} = \lambda_k u_k \in Q(V)$, since $u_k \in B \subseteq Q(V)$ and so $\lambda_k u_k \in Q(V)$ by Lemma 2.5. Hence $x_{k-1} \sqcup x_k$ is straight. It follows that $0 = x_0, x_1, \ldots, x_n = x$ is a finite set of points such that for all $i \in \{0, \ldots, n-1\}$, $x_i \sqcup x_{i+1}$ is straight.

■

Note for the following two results, we assume that (V, A) is a near-vector space with $|A| \geq 3$: if $A = 2$, then V must be a vector space over $\mathbb{Z}_2$, since $\mathbb{Z}_2$ is the only nearfield with two elements (see Van der Walt's Theorem in Section 2.5). Hence, in this case $(V, \mathbb{L})$ would be an affine space over a vector space.

Since $|A| \geq 3$, we know that the order of $(V, \mathbb{L})$ will be greater than or equal to 3 by the fixed-point-free property, and so we may apply all of the results in the previous section.

We now link the notion of a subspace of a nearaffine space to that of a subspace of a near-vector space.

Theorem 3.53. *Let (V, A) be a near-vector space, and let $U \subseteq V$. Then U is a non-empty subspace of the nearaffine space $(V, \mathbb{L})$ if and only if there exists some subspace (W, A) of (V, A) such that U is a coset of W.*

Proof. Suppose $U = x + W$ for some subspace W of (V, A). Then W is a subspace of the nearfield space $(V, \mathbb{L})$ with 0 as a point, since $0 \in W$, and for any $x, y \in W$, $x \sqcup y = A(y - x) + x \subseteq W$. Let $\mathbb{G} = \{G_i \,|\, i \in I\}$ be a basis for W of straight lines through 0, and let $u_i \in G_i^*$. It follows $G_i = A(u_i - 0) + 0 = Au_i$. Therefore $x \sqcup (x + u_i) = Au_i + x \parallel G_i$, hence $(x \parallel G_i) = Au_i + x \subseteq x + W = U$ for each $i \in I$, and thus $W_x = \langle (x \parallel G_i) \,|\, i \in I\rangle \subseteq U$. Next, if $u \in U \setminus \{x\}$, then $w = u - x \in W$, so that $0 \sqcup w \subseteq W$. But then $0 \sqcup w \parallel W$ by Theorem 3.26, and so $0 \sqcup w \parallel W_x$ by Lemma 3.43. Furthermore $0 \sqcup w = Aw = A(u - x) \parallel A(u - x) + x = x \sqcup u$. It follows that $x \sqcup u = (x \parallel 0 \sqcup w) \subseteq W_x$, so that $u \in W_x$, and so $U \subseteq W_x$. Hence $U = W_x$ and is therefore a subspace of the nearfield space $(V, \mathbb{L})$.

Conversely, suppose U be a subspace of the nearaffine space $(V, \mathbb{L})$. Let x be a point of U. If U is a point, then the result is trivial, so suppose $\dim U > 1$. Then by Corollary 3.37, there exists a basis for U of straight lines through x, say $\mathbb{G}' = \{G_i' \,|\, i \in I\}$. Let $\mathbb{G} = \{G_i = (0 \parallel G_i') \,|\, i \in I\}$, and let $u_i \in G_i^*$ for every $i \in I$. Finally, let $X = \{u_i \,|\, i \in I\}$.

For contradiction, assume X is a dependent subset of $Q(V)$. Let $\{u_1, \ldots, u_n\} \subseteq X$, and suppose $\sum_{j=1}^{n} \lambda_j u_j = 0$. We may assume $\lambda_k \neq 0$ for all $k \in \{1, \ldots, n\}$ (otherwise, exclude each u_k such that $\lambda_k = 0$). Consider the partial sums $v_k = \sum_{j=1}^{k} \lambda_j u_j$, where $k \in \{1, \ldots, n\}$, and set $v_0 = 0$. Then $v_{k-1} \sqcup v_k = A(v_k - v_{k-1}) + v_k = A(\lambda_k u_k) + v_{k-1} = Au_k + v_{k-1}$, since $\lambda_k \neq 0$ for all $k \in \{1 \ldots, n\}$. But $G_k = 0 \sqcup u_k = Au_k$, so $G_k \parallel v_{k-1} \sqcup v_k$, therefore $v_{k-1} \sqcup v_k$ is straight for all $k \in \{1, \ldots, n\}$. Furthermore, $v_k \in \langle G_1, \ldots G_k\rangle$ for all $k \in \{1, \ldots, n\}$: if not, there is some v_j with smallest index j such that $v_j \notin \langle G_1, \ldots, G_j\rangle$. But $v_j \in v_j \sqcup v_{j-1} = v_{j-1} \sqcup v_j \parallel G_j$. Hence $v_j \in (v_{j-1} \parallel G_j)$. But $v_{j-1} \in \langle G_1, \ldots G_{j-1}\rangle \subseteq \langle G_1, \ldots G_j\rangle$, so that by Theorem 3.26, $v_j \in \langle G_1, \ldots G_j\rangle$ — a contradiction. In particular, this implies that $v_{n-1} \in \langle G_1, \ldots, G_{n-1}\rangle$. But $v_n = 0 \in \langle G_1, \ldots, G_{n-1}\rangle$, so $v_n \sqcup v_{n-1} = 0 \sqcup v_{n-1} \subseteq \langle G_1, \ldots, G_{n-1}\rangle$. Furthermore $v_n \sqcup v_{n-1} \parallel G_n$, and so $v_n \sqcup v_{n-1} = G_n$, since both lines share the base point 0. Therefore $G_n \subseteq \langle G_1, \ldots, G_{n-1}\rangle$ — a contradiction, since $G_1, \ldots, G_n$ are independent. It follows $\lambda_1 = \ldots = \lambda_n = 0$, and so X is independent.

Define W to be the subgroup generated by AX. Clearly (W, A) is a subspace of the near-vector space (V, A) since X is an independent subset of $Q(V)$. Now, for any $x, y \in W$, $x \sqcup y = A(y - x) + x \subseteq W$, hence W is a weak subspace of $(V, \mathbb{L})$ and therefore a subspace of $(V, \mathbb{L})$. Furthermore, $W = U_0$: for each $i \in I$, $G_i = Au_i \subseteq W$, hence $U_0 = \langle G_i \,|\, i \in I \rangle = \left\langle \bigcup_{i \in I} G_i \right\rangle \subseteq W$. Likewise if $w \in W$, then $w = \sum_{j=1}^{n} \lambda_j u_j$ for some $u_1 \ldots, u_n \in X$ and $\lambda_1, \ldots, \lambda_n \in A$. The above argument then shows that $w = \sum_{j=1}^{n} \lambda_j u_j \in \langle G_1, \ldots, G_n \rangle \subseteq \langle G_i \,|\, i \in I \rangle = U_0$, so $W \subseteq U_0$ and hence $W = U_0$.

Now let $w \in W$. Then $0 \sqcup w = Aw \subseteq W$, and therefore $0 \sqcup w \parallel W = U_0$ by Theorem 3.26, hence $0 \sqcup w \parallel U$ by Lemma 3.43. Furthermore $x \sqcup (x + w) = Aw + x$, and so $0 \sqcup w \parallel x \sqcup (x + w)$, and thus, since $0 \sqcup w \parallel U$, it follows $x \sqcup (x + w) \subseteq U$. It follows $x + w \in U$, and so $x + W \subseteq U$. Conversely, if $u \in U$, then define $w = u - x$. Then $x \sqcup u = A(u - x) + x = Aw + x \parallel Aw = 0 \sqcup w$. Since $x \sqcup u \subseteq U$, we know $x \sqcup u \parallel U$ (again by Theorem 3.26), and so $x \sqcup u \parallel U_0 = W$, again by Lemma 3.43. It follows $0 \sqcup w \subseteq W$, and so $w \in W$, thus $u = x + w \in x + W$. It follows $U \subseteq x + W$, and hence $U = x + W$. ■

A proof of the following result was given in [9]; however, the proof of the converse was incomplete, as picked up by Sophie Marques in 2019, in that it was unclear how the so-called quasi-kernel of the subset would generate the subset as a group — a requirement for near-vector spaces. This was partially remedied in [10] by Sophie Marques, where it was shown that the result holds in the specific case for near-vector spaces over division rings. We now give a the proof of the general result using the geometry we have developed.

Theorem 3.54. *Let $W \subseteq V$ for some near-vector space (V, A). Then (W, A) is a subspace of (V, A) if and only if W is a subgroup of V and $AW \subseteq W$.*

Proof. If $\dim V < 2$, then the only subspaces of V are V itself and $\{0\}$, which trivially satisfy the equivalence. Suppose therefore that $\dim V \geq 2$, then $(V, \mathbb{L})$ is a nearfield space, and therefore a nearaffine space by Theorem 3.52.

The forward direction follows directly from the definition of a subspace.

For the converse, suppose W is a subgroup of V and $AW \subseteq W$. Since W is a subgroup of V, we know W is non-empty. Let $x, y \in W$. Then $y - x \in W$ (since W is a group), so $\alpha(y - x) \in AW \subseteq W$ for all $\alpha \in A$. Therefore $\alpha(y - x) + x \in W$ for all $\alpha \in A$, since W is a group. Thus $x \sqcup y = A(y - x) + x \subseteq W$. It follows W is a weak subspace of $(V, \mathbb{L})$.

It follows by Theorem 3.48, since the order of $(V, \mathbb{L})$ is greater than or equal to 3,

W is a subspace of $(V, \mathbb{L})$. Hence by Theorem 3.53, W is a coset of some subspace V' of (V, A). But since W is a subgroup of V, $W = V'$. Hence (W, A) is a subspace of (V, A). ■

3.6 Projections of nearaffine spaces

In this section, we define projections of nearaffine spaces, in an analogous way to how one would do for affine spaces. Inspiration is drawn from Ueberberg in [23] (Chapter 3). We will only present an elementary introduction to this topic, and leave further exploration for future research.

Definition 3.55. Let $\mathbf{A}$ be a nearaffine space of dimension at least 3, and let q be an arbitrary fixed point of $\mathbf{A}$. Then the projection of $\mathbf{A}$, $\mathbf{P} = P(A)$, is a geometry of rank 2 over the type set $\{\text{point}, \text{line}\}$ where points and lines are defined as follows.

(a) Points of $\mathbf{P}$ are the lines $[x] = q \sqcup x$ for all points $x \neq q$ of $\mathbf{A}$.

(b) Lines of $\mathbf{P}$ are defined as follows: for any two distinct points $[x]$ and $[y]$

$$[x][y] = \bigcup \{(z \parallel q \sqcup x) \,|\, z \in q \sqcup y\}.$$

(c) A point $[z]$ and line $[x][y]$ of $\mathbf{P}$ are incident if $[z] \subseteq [x][y]$.

The following two results are useful in this section.

Theorem 3.56. *Let* $\mathbf{P}$ *be the projection of nearaffine space* $\mathbf{A}$*. For any distinct points* $[x]$ *and* $[y]$ *of* $\mathbf{P}$*, we have* $[x][y] = [y][x]$*.*

Proof. Let $w \in [x][y]$. Then there exists some $z \in q \sqcup y$ such that $w \in (z \parallel q \sqcup x)$, i.e $(z \parallel q \sqcup x) = z \sqcup w$. By (L2), $q \sqcup y = q \sqcup z$. Since $[x] \neq [y]$, we know $q \sqcup x \neq q \sqcup y$. Since $q \sqcup x$ and $q \sqcup y$ share a base point and are not equal, it implies that they are not parallel. Therefore, since by (P3) $z \sqcup q \parallel q \sqcup z = q \sqcup y$, and $z \sqcup w = (z \parallel q \sqcup x) \parallel q \sqcup x$, we have that $z \sqcup q$ and $z \sqcup w$ are not parallel, and therefore not equal. Therefore, by (Pa), $(q \parallel z \sqcup w) \cap (w \parallel z \sqcup q) \neq \varnothing$. But $(q \parallel z \sqcup w) = (q \parallel q \sqcup x) = q \sqcup x$ by (P1), and $(w \parallel z \sqcup q) = (w \parallel q \sqcup z) = (w \parallel q \sqcup y)$ by (P3). Hence $(q \sqcup x) \cap (w \parallel q \sqcup y) \neq \varnothing$. Now, let $z' \in (q \sqcup x) \cap (w \parallel q \sqcup y)$. Since $z' \in (w \parallel q \sqcup y)$, we have that $w \sqcup z' = (w \parallel q \sqcup y)$. Therefore $z' \sqcup w \parallel w \sqcup z' = (w \parallel q \sqcup y) \parallel q \sqcup y$. Hence $z' \sqcup w = (z' \parallel q \sqcup y)$, and thus $w \in (z' \parallel q \sqcup y)$, with $z' \in q \sqcup x$. It follows that $w \in [y][x]$ so that $[x][y] = [y][x]$. ■

Theorem 3.57. *Let* $\mathbf{P}$ *be the projection of an affine space* $\mathbf{A}$*. Then* $z \in [x][y]$ *if and only if* $[z] \subseteq [x][y]$*.*

Proof. If $[z] \subseteq [x][y]$, then clearly $z \in [x][y]$. Suppose then that $z \in [x][y]$. Then there exists $w \in q \sqcup x$ such that $z \in (w \parallel q \sqcup y)$. Let $z' \in [z] = q \sqcup z$. If $z' = q$ or $z' = z$, then we are done, so suppose not. By (L2), $q \sqcup z = q \sqcup z'$. Therefore by (V), we know that $(q \sqcup w) \cap (z' \parallel z \sqcup w) \neq \varnothing$. Let $w' \in (q \sqcup w) \cap (z' \parallel z \sqcup w)$. Then $w' \sqcup z' \parallel z' \sqcup w' = (z' \parallel z \sqcup w) \parallel z \sqcup w \parallel w \sqcup z = (w \parallel q \sqcup y) \parallel q \sqcup y$. Furthermore, $w' \in q \sqcup w = q \sqcup x$. Hence $z' \in (w' \parallel q \sqcup y)$, with $w' \in q \sqcup x$, so that $z' \in [x][y]$. Thus $[z] \subseteq [x][y]$. ■

The next three results aim to show which axioms of a projective space are satisfied by the projection of a nearaffine space. See [23], p.10 for more details.

Theorem 3.58. *The projection* $\mathbf{P}$ *of a nearaffine space* $\mathbf{A}$ *satisfies the axiom of Veblen-Young: If* $[p], [x], [y], [a]$*, and* $[b]$ *are five points of* $\mathbf{P}$ *such that the lines* $[x][y]$ *and* $[a][b]$ *meet in a point* $[p]$*, the lines* $[x][a]$ *and* $[y][b]$ *also meet in a point.*

Proof. Since $[p] \subseteq [x][y] = [y][x]$ and $[p] \subseteq [a][b]$, we know that there exits $w \in q \sqcup y$ and $w' \in q \sqcup a$ such that $p \in (w \parallel q \sqcup x) \cap (w' \parallel q \sqcup b)$. By (L2) and (P1): $p \sqcup w \parallel w \sqcup p = (w \parallel q \sqcup x) \parallel q \sqcup x$ and $p \sqcup w' \parallel w' \sqcup p = (w' \parallel q \sqcup b) \parallel q \sqcup b$. Since $p \sqcup w$ and $p \sqcup w'$ are not parallel, they are not equal, and so by (Pa) we have $(w' \parallel p \sqcup w) \cap (w \parallel p \sqcup w') \neq \varnothing$. Let $z \in (w' \parallel p \sqcup w) \cap (w \parallel p \sqcup w')$. But $p \sqcup w \parallel w \sqcup p = (w \parallel q \sqcup x) \parallel q \sqcup x$, and $p \sqcup w' \parallel w' \sqcup p = (w' \parallel q \sqcup b) \parallel q \sqcup b$. Hence $z \in (w' \parallel q \sqcup x) \cap (w \parallel q \sqcup b)$. Since $w \in q \sqcup y$ and $w' \in q \sqcup a$, it follows $z \in [a][x] = [x][a]$ and $z \in [y][b]$. Hence $[z] \subseteq [x][a]$ and $[z] \subseteq [y][b]$ by Theorem 3.57, so that $[x][a]$ and $[y][b]$ intersect at $[z]$. ■

Theorem 3.59. *Let* $\mathbf{P}$ *be the projection of a nearaffine space* $\mathbf{A}$*. Then every line of* $\mathbf{P}$ *has at least three points.*

Proof. Let $[x][y]$ be an arbitrary line of $\mathbf{P}$. Obviously $[x] \subseteq [x][y]$ and $[y] \subseteq [x][y]$. Since $q \sqcup x = [x] \neq [y] = q \sqcup y$, it follows by (Pa) that $(x \parallel q \sqcup y) \cap (y \parallel q \sqcup x) \neq \varnothing$. Let $z \in (x \parallel q \sqcup y) \cap (y \parallel q \sqcup x)$. Then, in particular, $z \in (x \parallel q \sqcup y)$ with $x \in q \sqcup x$, hence $z \in [x][y]$ so that $[z] \subseteq [x][y]$ by Theorem 3.57. Thus $[x], [y]$, and $[z]$ are three different points on the line $[x][y]$, and so $[x][y]$ contains three points. ■

Theorem 3.60. *Let* $\mathbf{P}$ *be the projection of a nearaffine space* $\mathbf{A}$*. Then there are at least two lines in* $\mathbf{P}$*.*

Proof. Suppose $\mathbf{P}$ has only one line. Since the dimension of $\mathbf{A}$ is at least 3, there are at least three independent straight lines through q in $\mathbf{A}$, say $q \sqcup x, q \sqcup y, q \sqcup z$. Since $\mathbf{P}$ has only one line, we may conclude $[x][y] = [x][z]$. Note the following.

$$\begin{aligned}[x][y] &= \bigcup \{(w \parallel q \sqcup x) \mid w \in q \sqcup y\} \\ &= \bigcup_{w \in q \sqcup y} (w \parallel q \sqcup x) \\ &= \langle q \sqcup y, q \sqcup x \rangle .\end{aligned}$$

Since $[x][y] = [x][z]$, we know $z \in [x][y]$, and so $[z] \subseteq [x][y]$ by Theorem 3.57. It follows $q \sqcup z \subseteq \langle q \sqcup y, q \sqcup x \rangle$, so that $q \sqcup z \in \mathrm{cl}(\{q \sqcup x, q \sqcup y\}) = \mathrm{cl}(\{q \sqcup x, q \sqcup y, q \sqcup z\} \setminus \{q \sqcup z\})$ — a contradiction, since $q \sqcup x, q \sqcup y$, and $q \sqcup z$ are independent. Hence $\mathbf{P}$ has at least two lines. ■

While the projections of nearaffine spaces come close to satisfying the requirements of a projective space, they do not satisfy all of them. For example, if $z \in [x][y]$, then $[x][y] \neq [x][z]$ in general, as will be illustrated in an example at the end of the chapter, and hence $[x]$ and $[z]$ are incident to two distinct lines. However, we have partial satisfaction of this requirement, as illustrated by the following results.

Theorem 3.61. *Let $\mathbf{P}$ be the projection of a nearaffine space $\mathbf{A}$, and suppose x is a point of $\mathbf{A}$ such that $q \sqcup x$ is straight. Then for any points $y, z \notin [x]$, if $z \in [x][y]$, then $[x][y] = [x][z]$.*

Proof. Suppose $z \in [x][y]$ and that $q \sqcup x$ is straight. We show $[x][y] = [x][z]$.

$(\subseteq)$: Let $w \in [x][y]$. Then there exists some $y' \in q \sqcup y$ such that $w \in (y' \parallel q \sqcup x)$. Therefore $y' \sqcup w \parallel q \sqcup x$. Likewise, $z \in [x][y]$, so there exists some $y'' \in q \sqcup y$ such that $z \in (y'' \parallel q \sqcup x)$. Therefore $y'' \sqcup z \parallel q \sqcup x$. It follows $y'' \sqcup z \parallel y' \sqcup w$.

Now, if $y' = y''$, then $y'' \sqcup z = y' \sqcup w$ by (P1), and so $w \in y'' \sqcup z = z \sqcup y''$, since by (P2), $y'' \sqcup z$ is straight. Therefore $w \in (z \parallel q \sqcup x)$, and so $w \in [x][z]$. If $y' \neq y''$, then by (V), $(q \sqcup z) \cap (y' \sqcup w) = (q \sqcup z) \cap (y' \parallel y'' \sqcup z) \neq \varnothing$. Let $z' \in (q \sqcup z) \cap (y' \sqcup w)$. We know $y' \sqcup w$ is straight by (P2), and so, since $z', w \in y' \sqcup w$, We have $z' \sqcup w = y' \sqcup w \parallel q \sqcup x$. Hence $w \in (z' \parallel q \sqcup x)$, and so $w \in [x][y]$.

$(\supseteq)$: Suppose $w \in [x][z]$. Then there exists $z' \in q \sqcup z$ such that $w \in (z' \parallel q \sqcup x)$, hence $z' \sqcup w \parallel q \sqcup x$. Since $z \in [x][y]$, $[z] \subseteq [x][y]$, and so $z' \in [x][y]$. It follows that there exists $y' \in q \sqcup y$ such that $z' \in (y' \parallel q \sqcup x)$, hence $y' \sqcup z' \parallel q \sqcup x$. But $y' \sqcup z' = z' \sqcup y'$ by (P2), hence $z' \sqcup w = z' \sqcup y'$ by (P1). Therefore $w, y' \in y' \sqcup z'$, and so, since $z' \sqcup y'$ is straight, it follows $y' \sqcup w = y' \sqcup z' \parallel q \sqcup x$. Hence $w \in (y' \parallel q \sqcup x)$, and so $w \in [x][y]$. ∎

Corollary 3.62. Let **P** be the projection of a nearaffine space **A**, and suppose x and y are points of **A** such that $q \sqcup x$ and $q \sqcup y$ are straight. Then, for all $z \in [x][y]$, $[x][z] = [x][y] = [y][z]$.

Proof. We know $[x][y] = [y][x]$ by Theorem 3.56, and so $z \in [y][x]$. Since $q \sqcup x$ is straight, $[x][y] = [x][z]$, and since $q \sqcup y$ is straight, $[y][x] = [y][z]$. Hence $[x][z] = [x][y] = [y][x] = [y][z]$. ∎

The next natural step is to specify further, in the case where the nearaffine space is a nearfield space.

Definition 3.63. Let (V, A) be a near-vector space of dimension greater than or equal to 3. Then the *projection of* V, $\mathbf{P} = P(V)$, is the projection of the nearfield space induced by V, where $q = 0$.

Lemma 3.64. Let **P** be the projection of a near-vector space (V, A). Let $x \in V^*$. Then the point $[x]$ of **P** is the set Ax.

Proof. $[x] = 0 \sqcup x = A(x - 0) + 0 = Ax$. ∎

Lemma 3.65. Let **P** be the projection of a near-vector space (V, A). Let $[x]$ and $[y]$ be points of **P**. Then

$$[x][y] = Ax + Ay.$$

Proof. Let $z \in [x][y]$. Then there exists $w \in 0 \sqcup y = Ay$ such that $z \in (w \parallel 0 \sqcup x) = Ax + w$. Hence $z \in Ax + Ay$.

Conversely, suppose $z \in Ax + Ay$. Then there exists $\alpha \in A$ such that $z \in Ax + \alpha y$. We know $w = \alpha y \in Ay = 0 \sqcup y$. Hence $z \in Ax + w = (w \parallel 0 \sqcup x)$, where $w \in Ay = 0 \sqcup y$. It follows $z \in [x][y]$. ∎

In general, we do not have that that if $[z]$ is incident to $[x][y]$, then $[x][y] = [x][z] = [y][z]$. This is illustrated with the following example.

Example 3.66. Let $(V, A) = (\mathbb{R}^3, \mathbb{R})$, where $\alpha(x, y, z) = (\alpha x, \alpha^3 y, \alpha^3 z)$. This is a near-vector space by Van der Walt's Theorem (see Section 2.5). Let $x = (1, 1, 0)$

and $y = (1, 0, 1)$. Then the line $[x][y]$ contains $z = 1(1, 1, 0) + 1(1, 0, 1) = (2, 1, 1)$. By Theorem 3.57, we have that $[z]$ is incident to $[x][y]$.

Furthermore, we have $w = 1(1, 1, 0) + 2(1, 0, 1) = (1, 1, 0) + (2, 0, 8) = (3, 1, 8) \in [x][y]$.

Suppose then that $w \in [x][z]$. Then $(3, 1, 8) = \alpha(1, 1, 0) + \beta(2, 1, 1)$ for some $\alpha, \beta \in \mathbb{R}$. Then we have the following.

$$\begin{aligned} 3 &= \alpha + 2\beta \\ 1 &= \alpha^3 + \beta^3 \\ 8 &= \beta^3 \end{aligned}$$

The third equation clearly gives us that $\beta = 2$, hence $3 = \alpha + 2(2)$ so that $\alpha = -1$. However, $(-1)^3 + (2)^3 = 7 \neq 1$, contradicting that $\alpha^3 + \beta^3 = 1$. Therefore there are no $\alpha, \beta \in \mathbb{R}$ such that $(3, 1, 1) = \alpha(1, 1, 0) + \beta(2, 1, 1)$, and thus $(3, 1, 1) \in [x][y]$ but $(3, 1, 1) \notin [x][z]$.

Chapter 4

Hyper Near-Vector Spaces

Hyper nearrings and hyper vector spaces have been defined and studied (see [5] and [22], for example). Thus it is natural to progress to the notion of a hyper near-vector space.

In this chapter we define and study hyper near-vector spaces that have similar properties to André's near-vector spaces. Important concepts including independence, the notion of a basis, regularity and subhyperspaces are defined. We give some interesting first examples of hyper near-vector spaces. Most notably, we prove that there is a Decomposition Theorem for these spaces into maximal regular subhyperspaces.

4.1 Preliminary material on hyperstructures

Below we give the preliminary material we will need on hypergroups and hyper vector spaces. For further reference, we refer the reader to [7].

Definition 4.1. Let J be a nonempty set. A mapping $\circ : J \times J \to \mathcal{P}^*(J)$, where $\mathcal{P}^*(J)$ is the set of all nonempty subsets of J, is called a *hyperoperation* on J.

From the above definition, if A and B are two nonempty subsets of J and $x \in J$, then $A \circ B = \cup_{a\in A b\in B} a \circ b$, $x \circ A = \{x\} \circ A$, $A \circ x = A \circ \{x\}$. From now on we will write $\{x\}$ and x interchangeably, when there is no room for confusion.

Definition 4.2. A *quasicanonical hypergroup* is a pair $(H, +)$, where $+$ is a hyperoperation on H satisfying the following.

1. $(H, +)$ is a *hypergroup,* i.e.

    (a) $a + (b + c) = (a + b) + c$ for all $a, b, c \in H$ ($(H, +)$ is a *semihypergroup*)

(b) $a + H = H + a = H$ for all $a \in H$ (($H, +$) is a *quasihypergroup*)

2. H has a *scalar identity*, i.e. there exists $0 \in H$ such that, for all $x \in H$, $x + 0 = \{x\}$.

3. Every element has a *unique inverse*, i.e. for all $x \in H$, there exists a unique $-x \in H$ such that $0 \in x + (-x)$.

4. H is *reversible*, i.e. if $x \in y + z$, then $z \in (-y) + x$.

If H is commutative, (i.e. $a + b = b + a$ for all $a, b \in H$), then H is called a *canonical hypergroup.*

Definition 4.3. A non-empty subset K of a canonical hypergroup H is a *canonical subhypergroup* if K is also a quasi-canonical hypergroup.

We note that it is well-known that canonical subhypergroups are closed under intersection.

Definition 4.4. Let $(H, +)$ and $(K, \circ)$ be canonical hypergroups with scalar identities 0 and e respectively. Let $f : H \to K$.

- f is a *homomorphism* if for all $x, y \in H$, $f(x+y) \subseteq f(x) \circ f(y)$ and $f(0) = e$.
- f is a *good homomorphism* if for all $x, y \in H$, we have $f(x+y) = f(x) \circ f(y)$ and $f(0) = e$.
- f is an *isomorphism* if it is a homomorphism and its inverse f^{-1} is a homomorphism.
- f is an *endomorphism* if $(K, \circ) = (H, +)$ and f is a homomorphism.
- f is an *automorphism* if it is an isomorphism and an endomorphism.

As with any algebraic structure, the automorphisms of a canonical hypergroup form a group, which, for a hypergroup H we will denote $\text{Aut}(H)$.

A proof in [7], p.44 is presented that shows a homomorphism is an isomorphism if and only if it is bijective and good.

In 1990 [5], Dašić introduced the concept of hypernear-rings.

Definition 4.5. ([5], p.75) A triple $(R, +, \cdot)$ is called a *hypernear-ring* if the following axioms hold.

- $(R, +)$ is a quasicanonical hypergroup.

- $(R, \cdot)$ is a semigroup having 0 as a left absorbing element, i.e. $x \cdot 0 = 0$ for all $x \in R$.
- The multiplication $\cdot$ is distributive with respect to the hyperoperation $+$ on the left side, i.e. $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in R$.

If in addition, $(R^*, \cdot)$ is a quasicanonical hypergroup, $(R, +, \cdot)$ is called a *hyper-nearfield*.

In 1990, Tallini [22] introduced the notion of a hyper vector space over a field, while Vougiouklis [26] introduced weak hyper vector spaces. Recently in 2020, Al Tahan and Davvaz [21] introduced a hyper vector space over a Krasner hyperfield. This is the definition of a hyper vector space we focus on. We will see that it is most fitting since we will show that every hyper vector space is a hyper near-vector space.

Definition 4.6. ([16], p.307-308) A *Krasner hyperring* is an algebraic structure $(R, +, \cdot)$ which satisfies the following axioms for all $x, y, z \in R$.

1. $(R, +)$ is a canonical hypergroup.
2. $(R, \cdot)$ is a semigroup having zero as bilaterally absorbing element, i.e. $x \cdot 0 = 0 \cdot x = 0$.
3. $x \cdot (y + z) = x \cdot y + x \cdot z$
4. $(x + y) \cdot z = x \cdot y + x \cdot z$

A commutative Krasner hyperring $(R, +, \cdot)$ with identity 1 is a *Krasner hyperfield* if $(R^*, \cdot)$ is a group.

It is clear that a $(R, +, \cdot)$ is a commutative Krasner hyperring if it is a hypernear-ring with commutative $+$ and $\cdot$.

Definition 4.7. ([21], p.62) Let F be a Krasner hyperfield. A canonical hypergroup $(V, +)$ together with a map $\cdot : F \times V \to V$ is called a *hyper vector space over F* if for all $a, b \in F$ and $x, y \in V$ the following conditions hold.

1. $a \cdot (x + y) = a \cdot x + a \cdot y$
2. $(a + b) \cdot x = a \cdot x + b \cdot x$
3. $a \cdot (b \cdot x) = (ab) \cdot x$
4. $a \cdot (-x) = (-a) \cdot x = -(a \cdot x)$

5. $x = 1 \cdot x$

We will also need the definition of a weak hyper vector space.

Definition 4.8. ([21], p.62) Let F be a Krasner hyperfield. A canonical hypergroup $(V, +)$ together with a map $\cdot : F \times V \to V$ is called a *weak hyper vector space over* F if for all $a, b \in F$ and $x, y \in V$ the following conditions hold.

1. $a \cdot (x + y) \subseteq a \cdot x + a \cdot y$
2. $(a + b) \cdot x \subseteq a \cdot x + b \cdot x$
3. $a \cdot (b \cdot x) = (ab) \cdot x$
4. $a \cdot (-x) = (-a) \cdot x = -(a \cdot x)$
5. $x = 1 \cdot x$

Note that, given a weak hyper vector space V over a Krasner hyperfield F, one may construct for each $a \in F$ a map $a : V \to V$ such that $a(v) = a \cdot v$, which by the first property of a weak hyper vector space is a homomorphism of V. Now, if $a \neq 0$, then it has an inverse a^{-1}, since

$$a(a^{-1}(v)) = a(a^{-1} \cdot v) = a \cdot (a^{-1} \cdot v) = (aa^{-1}) \cdot v = 1 \cdot v = v$$

for all $v \in V$. It follows that each nonzero a is an isomorphism, and is therefore a good homomorphism. Hence $a \cdot (x + y) = a(x + y) = a(x) + a(y) = a \cdot x + a \cdot y$. Furthermore, $0 \cdot (x + y) = 0 = 0 + 0 = 0 \cdot x + 0 \cdot y$. It follows that, for all $a \in F$ and all $x, y \in V$, $a \cdot (x + y) = a \cdot x + a \cdot y$.

4.2 Hyper near-vector spaces definition and preliminary results

In this section we define our hyper near-vector space, give some examples and prove our main results. In order to do this we begin by replacing the additive group of vectors in [1] with a canonical hypergroup and define the scalar multiplication as a group of endomorphisms as before. This is similar to how [21] generalises a vector space. The notion of the quasi-kernel is generalised in a suitable way so that its elements maintain the structural properties of André's near-vector space.

Definition 4.9. A *hyper near-vector space* is a pair (V, A) which satisfies the following conditions.

1. $(V, +)$ is a canonical hypergroup and A is a set of endomorphisms of V.
2. A contains the endomorphisms 0, 1 and -1.
3. A^* is a subgroup of the group $\mathrm{Aut}(V)$.
4. If $\alpha x = \beta x$ with $x \in V$ and $\alpha, \beta \in A$, then $\alpha = \beta$ or $x = 0$, i.e. A *acts fixed point free on* V.
5. $V = \langle Q(V) \rangle$, i.e. V is generated additively by the *quasi-kernel,* $Q(V)$, where

$$Q(V) = \{x \in V \,|\, \forall \alpha, \beta \in A,\, \alpha x + \beta x \subseteq Ax\}\,.$$

Because A^* is a set of isomorphisms, and $0 \in A$ is itself good, it follows each endomorphism in A is good. We view A as the set of scalars of V.

In order to compare hyper near-vector spaces, we need the following definition.

Definition 4.10. Let (V, A_1) and (W, A_2) be hyper near-vector spaces over A. Then maps $\phi : V \to W$ and $\eta : A_1 \to A_2$ form a *homomorphism* if η is a semigroup isomorphism, $\phi(0) = 0$ and for any $x, y \in V$ and $\alpha \in A_1$ we have $\phi(x + y) \subseteq \phi(x) + \phi(y)$ and $\phi(\alpha x) = \eta(\alpha)\phi(x)$, and a *good homomorphism* if in addition $\phi(x) + \phi(y) \subseteq \phi(x + y)$.

We say that two hyper near-vector spaces (V, A) and (W, A) are *isomorphic* (written $(V, A) \cong (W, A)$) if there is a bijective good homomorphism $\phi : V \to W$.

If, in the definition above, we have that $A_1 = A_2$, often η is implicitly taken to be the identity map on A, unless expressly otherwise stated.

It is known that every vector space is a near-vector space with the quasi-kernel the entire space. We now prove the analogous result for hyper vector spaces.

Lemma 4.11. Every hyper vector space is a hyper near-vector space.

Proof. Let V be a hyper vector space over F.

1. By definition $(V, +)$ is a canonical hypergroup and F is a set of endomorphisms of V.
2. F contains the endomorphisms 0, 1 and -1 by definition.
3. F^* is a subgroup of the group $\mathrm{Aut}(V)$ since for any $\alpha, \beta^{-1} \in F^*, \alpha\beta^{-1} \in F^*$ and it is not difficult to check that every $\alpha \in F^*$ is a bijection of $(V, +)$.

4. Suppose that $\alpha x = \beta x$ with $x \neq 0$. Then $0 \in \alpha x - \beta x = (\alpha - \beta)x$ so that $0 \in \alpha - \beta$. Thus $-\beta = -\alpha$, so by the uniqueness of inverses, $\alpha = \beta$. Thus F acts fixed point free on V, as explained under Definition 4.8.

5. $V = \langle Q(V) \rangle$, where $Q(V) = V$ by Definition 4.7 (2).

■

Below we give a first example of a hyper near-vector space.

Example 4.12. Let $V = \{0, a, b, c\}$ be a set with the hyperoperation $\oplus$ defined as follows:

$\oplus$	0	a	b	c
0	0	a	b	c
a	a	$\{0, a\}$	c	$\{b, c\}$
b	b	c	$\{0, b\}$	$\{a, c\}$
c	c	$\{b, c\}$	$\{a, c\}$	V

Then $(V, \oplus)$ is a canonical hypergroup ([17], p.549). If we now take $A = \{0, 1\}$, then since $-1 = 1$, we have that (V, A) is a hyper near-vector space. A quick check shows that $Q(V) = \{0, a, b\}$. We note that (V, A) is also a weak hyper vector space.

We now prove some useful properties of the quasi-kernel, as done in Section 2.1.

Lemma 4.13. Let (V, A) be a hyper near-vector space. The quasi-kernel Q has the following properties.

(a) $0 \in Q$.

(b) For $u \in Q^*$, if $\alpha u + \beta u = A'u \subseteq Au$, then A' is uniquely determined by α and β.

(c) If $u \in Q$ and $\lambda \in A$, then $\lambda u \in Q$, i.e. $Au \subseteq Q$.

(d) If $u \in Q$ and $\lambda_i \in A$, $i = 1, 2, \ldots, n$, then $\sum_{i=1}^{n} \lambda_i u = A'u \subseteq Q$ for some $A' \subseteq A$.

Proof.

(a) Let $\alpha, \beta \in A$. Then $\alpha 0 + \beta 0 = 0 + 0 = \{0\} = A0$. Thus $0 \in Q$.

(b) Suppose that for all $\alpha, \beta \in A$ we have that $\alpha u + \beta u = A'u$ and $\alpha u + \beta u = A''u$, where $A', A'' \subseteq A$. If $\alpha \in A'$, then $\alpha u \in A'u$ and $\alpha u \in A''u$. Thus $\alpha u = \alpha' u$

for some $\alpha' \in A''$. Since $u \neq 0$, by the fixed point free property, we have that $\alpha = \alpha'$. Thus $A' \subseteq A''$. Similarly, it can be shown that $A'' \subseteq A'$, so that $A' = A''$.

(c) Suppose $u \in Q$ and $\lambda \in A$. There are two cases to consider:
Case 1: $\lambda = 0$
Then $\lambda u = 0u = 0 \in Q$ by (a).
Case 2: $\lambda \neq 0$
Let α, β be elements of A. Then

$$\begin{aligned} \alpha(\lambda u) + \beta(\lambda u) &= (\alpha\lambda)u + (\beta\lambda)u \\ &= \lambda' u \text{ for some } \lambda' \in A \text{ since } u \in Q. \end{aligned}$$

Since $\lambda \neq 0$, $\lambda' u = (\lambda'\lambda^{-1})\lambda u$. Thus $\lambda u \in Q$, so $Au \subseteq Q$.

(d) We prove the result using induction on n. From (c), if $u \in Q, \lambda u \in Q$ for $\lambda \in A$. Now suppose that $\sum_{i=1}^{k} \lambda_i u \subseteq Au$, say $\sum_{i=1}^{k} \lambda_i u = A' u$. Then

$$\begin{aligned} \sum_{i=1}^{k} \lambda_i u &= A' u + \lambda_{k+1} u \\ &= \bigcup_{\lambda \in A'} (\lambda u + \lambda_{k+1} u) \subseteq Au. \end{aligned}$$

■

Lemma 4.14. Let (V, A) and (W, A) be hyper near-vector spaces over A and $\phi : V \to W$ be a good homomorphism. Then $\phi(Q(V)) \subseteq Q(W)$.

Proof. Let $u \in Q(V)$ and $\alpha, \beta \in A$. Then $\alpha u + \beta u \subseteq Au$, so that $\alpha\phi(u) + \beta\phi(u) = \phi(\alpha u + \beta u) \subseteq \phi(Au) = A\phi(u)$. It follows $\phi(u) \in Q(W)$. ■

4.3 An addition on A

As mentioned in Section 2.2, André introduced a special addition on the group of scalars. We do the same below.

Definition 4.15. Let (V, A) be a hyper near-vector space. For $u \in Q^*$, we define an operation $+_u$ on A as follows. For all $\alpha, \beta \in A$,

$$\alpha +_u \beta = A',$$

where $\alpha u + \beta u = A'u$.

Example 4.16. Returning to Example 4.12 we have that for all $\alpha, \beta \in A$,

$$\alpha +_a \beta = \alpha +_b \beta.$$

We now prove that the addition on A results in it having the structure of a canonical hypergroup.

Lemma 4.17. Let (V, A) be a hyper near-vector space. Then $(A, +_u)$ is a canonical hypergroup for each $u \in Q^*$.

Proof. By the uniqueness of A' in Lemma 4.13, we have that $+_u$ is well-defined. Let $\alpha, \beta, \gamma \in A$. We verify each of the axioms for a canonical hypergroup.

1. (a) Let $u \in Q^*$. Then

$$\begin{aligned}(\alpha +_u (\beta +_u \gamma))u &= \alpha u + (\beta +_u \gamma)u \\ &= \alpha u + (\beta u + \gamma u) \\ &= (\alpha u + \beta u) + \gamma u \text{ (since V is a semihypergroup)} \\ &= ((\alpha +_u \beta) +_u \gamma)u.\end{aligned}$$

Since $u \neq 0$, by the fixed point free property, we have that

$$\alpha +_u (\beta +_u \gamma) = (\alpha +_u \beta) +_u \gamma.$$

(b)

$$\begin{aligned}(\alpha +_u A)u &= \alpha u + Au \\ &= \bigcup_{\beta \in A} (\alpha u + \beta u).\end{aligned}$$

We want to show that $\alpha +_u A = A$. Since for all $\beta \in A$, we have that $\alpha u + \beta u \subseteq Au$, we have that $(\alpha +_u A)u = \bigcup_{\beta \in A}(\alpha u + \beta u) \subseteq Au$. Thus by the fixed point free property, $\alpha +_u A \subseteq A$. Now let $\lambda \in A$. Then

$$\begin{aligned}\lambda u \in \alpha u - \alpha u + \lambda u &= \alpha u + (-\alpha +_u \lambda)u \\ &= (\alpha +_u A')u \text{ where } A' = -\alpha +_u \lambda.\end{aligned}$$

Now we have that $\lambda u \in (\alpha +_u A')u$ where $\alpha +_u A' \subseteq \alpha +_u A$. Thus $\lambda u \in (\alpha +_u A)u$ and by using the fixed point free property we have that $\lambda \in \alpha +_u A$.

2.

$$\begin{aligned}(\alpha +_u \beta)u &= \alpha u + \beta u \\ &= \beta u + \alpha u \text{ (since } (V,+) \text{ is commutative)} \\ &= (\beta +_u \alpha)u.\end{aligned}$$

Hence, by the fixed point free property, $\alpha +_u \beta = \beta +_u \alpha$.

3. We claim 0 is the scalar identity of $(A, +_u)$.

$$\begin{aligned}(\alpha +_u 0)u &= \alpha u + 0u \\ &= \alpha u + 0 \\ &= \{\alpha u\} \\ &= \{\alpha\}\, u\end{aligned}$$

Hence, by the fixed point free property, $\alpha +_u 0 = \{\alpha\}$.

4. We claim $-\alpha = (-1) \circ \alpha$ is the unique inverse of α in $(A, +_u)$.

$$\begin{aligned}(\alpha +_u (-\alpha))u &= \alpha u + (-1)(\alpha u) \\ &= \alpha u - \alpha u\end{aligned}$$

Now, since $-\alpha u$ is the unique inverse of αu in $(V, +)$, we have that $0u = 0 \in \alpha u - \alpha u = (\alpha +_u (-\alpha))u$. It follows that $0 \in \alpha +_u (-\alpha)$ by the fixed point free property. For uniqueness, suppose $0 \in \alpha +_u \lambda$ for some $\lambda \in A$. Then $0 = 0u \in (\alpha +_u \lambda)u = \alpha u + \lambda u$, hence, from the uniqueness of the inverse of αu in $(V, +)$, it follows that $\lambda u = -\alpha u$. By the fixed point free property, it follows that $\lambda = -\alpha$.

5. Suppose $\alpha \in \beta +_u \gamma$. Then $\alpha u \in (\beta +_u \gamma)u = \beta u + \gamma u$. Since $(V, +)$ is

reversible, it follows $\gamma u \in -\beta u + \alpha u = (-\beta +_u \alpha)u$, and so $\gamma \in -\beta +_u \alpha$ by the fixed point free property.

■

In fact, we can show more, i.e. we have a hyper-nearfield.

Lemma 4.18. Let $u \in Q^*$. Then $(A, +_u, \circ)$ is a hyper-nearfield.

Proof. Since $(A, +_u)$ is a canonical hypergroup, and $(A^*, \circ)$ is a group by definition, it remains to be shown that the left distributive law holds and that $0 \in A$ is bilaterally absorbing. Let $\alpha, \beta, \gamma \in A$, then

$$\begin{aligned} \alpha \circ (\beta +_u \gamma)u &= \alpha(\beta u + \gamma u) \\ &= (\alpha \circ \beta)u + (\alpha \circ \gamma)u \\ &= (\alpha \circ \beta +_u \alpha \circ \gamma)u. \end{aligned}$$

By the fixed point free property it follows that $\alpha \circ (\beta +_u \gamma) = \alpha \circ \beta +_u \alpha \circ \gamma$. Furthermore $(0 \circ \alpha)u = 0(\alpha u) = 0 = 0u$ and $(\alpha \circ 0)u = \alpha(0u) = \alpha 0 = 0 = 0u$, hence by the fixed point free property, $0\alpha = \alpha 0 = 0$ for all $\alpha \in A$. ■

Next we show that for any nonzero element of the quasi-kernel, the hyper near-field from the previous lemma is isomorphic to all of those where the addition is defined in terms of scalar multiples of it.

Lemma 4.19. For every $u \in Q^*$ and $\lambda \in A^*$, $(A, +_u, \circ) \cong (A, +_{\lambda u}, \circ)$.

Proof. Let $u \in Q^*$ and $\lambda \in A^*$. Define $\theta : (A, +_{\lambda u}, \circ) \to (A, +_u, \circ)$ so that $\theta(\alpha) = \lambda^{-1}\alpha\lambda$. Let $\alpha, \beta \in A$, then

$$\begin{aligned} \theta(\alpha +_{\lambda u} \beta)u &= \lambda^{-1}(\alpha +_{\lambda u} \beta)\lambda u \\ &= \lambda^{-1}(\alpha\lambda u + \beta\lambda u) \\ &= \lambda^{-1}\alpha\lambda u + \lambda^{-1}\beta\lambda u \\ &= \theta(\alpha)u + \theta(\beta)u \\ &= (\theta(\alpha) +_u \theta(\beta))u. \end{aligned}$$

Therefore $\theta(\alpha +_{\lambda u} \beta) = \theta(\alpha) +_u \theta(\beta)$ by the fixed point free property. Furthermore,

$$\begin{aligned}\theta(\alpha\beta) &= \lambda^{-1}\alpha\beta\lambda \\ &= \lambda^{-1}\alpha(\lambda\lambda^{-1})\beta\lambda \\ &= \theta(\alpha)\theta(\beta).\end{aligned}$$

Hence θ is a homomorphism. But $\lambda(\theta(\alpha))\lambda^{-1} = \lambda\lambda^{-1}\alpha\lambda\lambda^{-1} = \alpha$, and $\theta(\lambda\alpha\lambda^{-1}) = \lambda^{-1}(\lambda\alpha\lambda^{-1})\lambda = \alpha$, so that θ is invertible, with $\theta^{-1} : \alpha \mapsto \lambda\alpha\lambda^{-1}$. Hence θ is an isomorphism. ■

4.4 Independence and a basis for $Q(V)$

As noted in Section 2.3, André defined independence in terms of a dependence relation in [1]. We follow the same route to defining independence.

Definition 4.20. Let (V, A) be a hyper near-vector space. We define a relation between Q and $\mathcal{P}(Q)$ as follows:
$v \lhd M \subseteq Q$ if there exists $n \in \mathbb{N}$, $u_i \in M$ for $i \in \{1, \ldots, n\}$, and $\lambda_1, \ldots, \lambda_n \in A$ such that

$$v \in \sum_{i=1}^{n} \lambda_i u_i.$$

Theorem 4.21. *The relation defined in Definition 4.20 is a dependence relation.*

Proof. (D1) Let $v \in M$. Then since $v \in \{v\} = v + 0v$, we have that $v \lhd M$.

(D2) Suppose that $w \lhd M$ and $v \lhd N$ for all $v \in M$, where M and N are subsets of Q. Then $w \in \sum_{i=1}^{n_i} \lambda_i v_i$ for some $v_i \in M$ and $\lambda_i \in A$, $i \in \{1, \ldots, n\}$, and so, for each $i \in \{1, \ldots, n\}$, $v_i \in \sum_{j=1}^{n} \eta_{ji} u_{ji}$, where $u_{ij} \in N$ and $\eta_{ij} \in A$ for all $j \in \{1, \ldots, n_i\}$. Now $\lambda_i v_i \in \lambda_i \sum_{j=1}^{n_i} \eta_{ji} u_{ji}$ and thus $\sum_{i=1}^{n_i} \lambda_i v_i \subseteq \sum_{i=1}^{n} \lambda_i (\sum_{j=1}^{n_i} \eta_{ji} u_{ji})$.
Thus $w \in \sum_{i=1}^{n} \lambda_i (\sum_{j=1}^{n_i} \eta_{ji} u_{ji}) = \sum_{i=1}^{n} \sum_{j=1}^{n_i} \lambda_i \eta_{ji} u_{ji}$, so that $w \lhd N$.

(D3) Let $v \lhd M$ and $v \not\lhd M \setminus \{u\}$. Then $v \in \sum_{i=1}^{n} \lambda_i u_i$ where $u_i \in M$ for $i \in \{1, \ldots, n\}$. Since $v \not\lhd M \setminus \{u\}$, we must have that u is equal to one of the u_i. To see this, suppose it is not the case, then $\{u_1, \ldots, u_n\} \subseteq M \setminus \{u\}$ and $v \in \sum_{i=1}^{n} \lambda_i u_i$, so $v \lhd M \setminus \{u\}$, a contradiction. Suppose then, without loss of generality, that $u = u_1$. Then $v \in \lambda_1 u + \sum_{i=2}^{n} \lambda_i u_i$. So there exists $x \in \sum_{i=2}^{n} \lambda_i u_i$ such that $v \in \lambda_1 u + x = x + \lambda_1 u$. Thus $\lambda_1 u \in (-x) + v$ by the reversibility property. This implies that $u \in \lambda_1^{-1}(-x + v) \subseteq$

$\lambda^{-1}(-\sum_{i=2}^{n}\lambda_i u_i+v)=\sum_{i=2}^{n}-\lambda_1^{-1}\lambda_i u_i+\lambda_1^{-1}v$. Thus $\{u_2,\ldots,u_n\}\subseteq M\setminus\{u\}$, so that $\{u_2,\ldots,u_n\}\cup\{v\}\subseteq(M\setminus\{u\})\cup\{v\}$.

■

As in Section 2.3, we say that a subset M of Q is *independent* if and only if for all $x\in M$ we have that $x\ntriangleleft M\setminus\{x\}$.

The next result will be useful.

Lemma 4.22. A subset M of Q is independent if and only if for all $n\in\mathbb{N}$ and $u_1,\ldots,u_n\in M$ with $u_i\neq u_j$ when $i\neq j$ and $\lambda_i\in A$ for $i\in\{1,\ldots,n\}$ if

$$0\in\sum_{i=1}^{n}\lambda_i u_i,$$

then $\lambda_i=0$ for $i\in\{1,\ldots,n\}$.

Proof. Suppose that $M\subseteq Q$ is independent and that $0\in\sum_{i=1}^{n}\lambda_i u_i$ where $u_i\in M$ and $\lambda_i\in A$ for $i\in\{1,\ldots,n\}$. Assume, without loss of generality, that $\lambda_1\neq 0$. Then $0\in\lambda_1u_1+\sum_{i=2}^{n}\lambda_iu_i$. Thus $-\lambda_1u_1\in\sum_{i=2}^{n}\lambda_iu_i$, otherwise $0\notin\sum_{i=1}^{n}\lambda_iu_i$. Then we have that $u_1\in(-\lambda_1)^{-1}\sum_{i=2}^{n}\lambda_iu_i$. Thus $u_1\in\sum_{i=2}^{n}(-\lambda_1)^{-1}\lambda_iu_i$, so that $u_1\triangleleft M\setminus\{u_1\}$, a contradiction.
Conversely, suppose that $M\subseteq Q$ such that for any $u_1,\ldots,u_j\in M$ with $u_i\neq u_j$ when $i\neq j$ we have that $0\in\sum_{i=1}^{n}\lambda_iu_i$ implies that $\lambda_i=0$ for $i\in\{1,\ldots,n\}$. Let $x\in M$ and suppose that $x\triangleleft M\setminus\{x\}$, then there exist $u_1,\ldots,u_n\in M\setminus\{x\}$ and $\lambda_i\in A$ such that $x\in\sum_{i=1}^{n}\lambda_iu_i$. Then $0\in\sum_{i=1}^{n}\lambda_iu_i-x=\sum_{i=1}^{n}\lambda_iu_i+(-1)x$, a contradiction since $-1\neq 0$. ■

We define a *basis* for a hyper near-vector space in the standard way, as in Section 2.3, from the above dependence relation, i.e. it is an independent generating set for the quasi-kernel. As for near-vector spaces, we show that every vector in the hyper near-vector space has a unique representation in terms of the basis elements.

Lemma 4.23. Let (V,A) be a hyper near-vector space, and let $B=\{u_i\,|\,i\in I\}$ be a basis of Q. Then each $u\in V$ is an element of a unique linear combination of elements of B, i.e. there exists $\lambda_i\in F$, with $\lambda_i\neq 0$ for at most a finite number of $i\in I$, which are uniquely determined by u and B, such that

$$u\in\sum_{i\in I}\lambda_iu_i.$$

Proof. Since $\langle Q(V)\rangle = V$, we know that there exists $x_1, \ldots, x_n \in Q(V)$ such that $u \in \sum_{j=1}^{n} x_j$. Since B is a basis for $Q(V)$, it follows for all $j \in \{1, \ldots, n\}$, $x_j \lhd B$, so that $x_j \in \sum_{i\in I} \lambda_{ij} u_i$, where $\lambda_{ij} \in A$ for all $i \in I$. It follows

$$u \in \sum_{j=1}^{n} x_j$$
$$u \in \sum_{j=1}^{n} \sum_{i\in I} \lambda_{ij} u_i$$
$$u \in \sum_{i\in I} (\lambda_{i1} u_i + \ldots + \lambda_{in} u_i)$$
$$u \in \sum_{i\in I} (\lambda_{i1} +_{u_i} \ldots +_{u_i} \lambda_{in}) u_i$$

It follows there exists $\eta_i \in \lambda_{i1} +_{u_i} \ldots +_{u_i} \lambda_{in}$ such that $u \in \sum_{i\in I} \eta_i u_i$.

For uniqueness, suppose that $u \in \sum_{i\in I} \lambda_i u_i = \sum_{i\in I} \lambda'_i u_i$ for the index set I. Then $0 \in u + (-u) \subseteq \sum_{i\in I} \lambda_i u_i - \sum_{i\in I} \lambda'_i u_i = \sum_{i\in I} A_i u_i$ where $A_i = \lambda_i +_{u_i} (-\lambda'_i) \subseteq A$.

Thus there exists $\eta_i \in A_i$ such that $0 \in \sum_{i\in I} \eta_i u_i$. This implies that $\eta_i = 0$ for all $i \in I$. It follows for each $i \in I$, $0 \in \lambda_i +_{u_i} (-\lambda'_i)$, i.e. $-\lambda'_i$ is the unique inverse of λ_i. Thus for each $i \in I$, $\lambda_i = \lambda'_i$. ■

The unique linear combination above is referred to as the *decomposition* of u in terms of a basis B of Q. A basis B of Q is referred to as a basis of V, as B generates V. The following result is an analogue of Lemma 3.2 in [1].

Lemma 4.24. Let (V, A) be a hyper near-vector space with basis $B = \{b_i \mid i \in I\}$, and let $\lambda_i \in A^*$ for all $i \in I$. Then $B' = \{\lambda_i b_i \mid i \in I\}$ is a basis of V.

Proof. Suppose there exists $\eta_i \in A$ such that $0 \in \sum_{i\in I} \eta_i(\lambda_i b_i) = \sum_{i\in I} (\eta_i\lambda_i) b_i$. Then, since B is independent, $\eta_i\lambda_i = 0$ for all $i \in I$, and thus $\eta_i = \eta_i\lambda_i\lambda_i^{-1} = 0\lambda_i^{-1} = 0$ for all $i \in I$. Hence B' is independent. Furthermore, if $x \in Q$ has decomposition $x \in \sum_{i\in I} \alpha_i b_i$, then $x \in \sum_{i\in I} \alpha_i(\lambda_i^{-1}\lambda_i) b_i = \sum_{i\in I} (\alpha_i\lambda_i^{-1})\lambda_i b_i$, so that $x \lhd B'$, hence B' generates Q (and therefore V). It follows B' is a basis for Q. ■

In Section 2.5, it is mentioned that André showed that every near-vector space

(V, A) with basis $B = \{b_i \,|\, i \in I\}$ is isomorphic to the set of families $(x_i)_{i\in I}$ where x_i is in A for each $i \in I$ and $x_i = 0$ for some cofinite subset of I. We prove the analogue in the next result.

Theorem 4.25. *Let (V, A) be a hyper–near-vector space with basis $B = \{b_i \,|\, i \in I\}$. Let*

$$A^{(I)} = \{(\lambda_i)_{i\in I} \,|0 \neq \lambda_i \in A \text{ for at most finitely many } i \in I\}.$$

For $(\alpha_i)_{i\in I}, (\beta_i)_{i\in I} \in A^{(I)}$, define $(\alpha_i)_{i\in I} + (\beta_i)_{i\in I} = \{(\gamma_i)_{i\in I} \,|\, \gamma_i \in \alpha_i +_{b_i} \beta_i\}$ and $\lambda(\alpha_i)_{i\in I} = (\lambda\alpha_i)_{i\in I}$. Then $V \cong A^{(I)}$.

Proof. Define $\phi : V \to A^{(I)}$ so that, if $v \in \sum_{i\in I} \lambda_i b_i$, with at most finitely many λ_i's nonzero, then $\phi(v) = (\lambda_i)_{i\in I}$. Take $v, w \in V$ with decompositions $v \in \sum_{i\in I} \lambda_i b_i$ and $w \in \sum_{i\in I} \eta_i b_i$. Let $u \in v + w$. Then $u \in \sum_{i\in I} \lambda_i b_i + \sum_{i\in I} \eta_i b_i = \sum_{i\in I}(\lambda_i +_{b_i} \eta_i)b_i$, so that $u \in \sum_{i\in I} \gamma_i b_i$, for some $\gamma_i \in \lambda_i +_{b_i} \eta_i$. It follows $\phi(u) = (\gamma_i)_{i\in I} \in (\lambda_i)_{i\in I} + (\eta_i)_{i\in I} = \phi(v) + \phi(w)$, so that $\phi(v + w) \subseteq \phi(v) + \phi(w)$. Conversely, suppose $(\gamma_i)_{i\in I} \in (\lambda_i)_{i\in I} + (\eta_i)_{i\in I} = \phi(v) + \phi(w)$. Then $\gamma_i \in \lambda_i +_{b_i} \eta_i$, so that $\sum_{i\in I} \gamma_i b_i \subseteq \sum_{i\in I}(\lambda_i +_{b_i} \eta_i)b_i = v + w$. Let $u \in \sum_{i\in I} \gamma_i b_i$. Then $u \in v + w$ and $\phi(u) = (\gamma_i)_{i\in I}$. It follows $(\gamma_i)_{i\in I} = \phi(u) \in \phi(v+w)$, so that $\phi(v)+\phi(w) = \phi(v+w)$.

Next, note that $\lambda w \in \lambda \sum_{i\in I} \eta_i b_i = \sum_{i\in I} \lambda\eta_i b_i$, so that $\phi(\lambda w) = (\lambda\eta_i)_{i\in I} = \lambda(\eta_i)_{i\in I} = \lambda\phi(w)$.

Finally, to show ϕ is surjective, for any $(\alpha_i)_{i\in I} \in A^{(I)}$, let $u \in \sum_{i\in I} \alpha_i b_i$, then $\phi(u) = (\alpha_i)_{i\in I}$. For injectivity, suppose $\phi(v) = \phi(w)$. Then $(0)_{i\in I} \in \phi(v) - \phi(w) = \phi(v - w)$. Let $x \in V \setminus \{0\}$, then $x = \sum_{i\in I} \lambda_i b_i$ such that $\lambda_j \neq 0$ for some $j \in I$. It follows $\phi(x) = (\lambda_i)_{i\in I} \neq (0)_{i\in I}$, since $\lambda_j \neq 0$. Therefore $\ker\phi = \{0\}$, so that $0 \in v - w$, hence $v = w$. ■

Corollary 4.26. Let (V, A) be a hyper–near-vector space with basis $B = \{b_i \,|\, i \in I\}$. Suppose $x, y \in V$ such that x and y have the same decomposition in terms of B, i.e. $x, y \in \sum_{i\in I} \lambda_i b_i$ for some $\lambda_i \in A$ for each $i \in I$. Then $x = y$.

Proof. Take $\phi : V \to A^{(I)}$ from the previous theorem. Then $\phi(x) = \phi(y) = (\lambda_i)_{i\in I}$. Since ϕ is injective, this implies $x = y$. ■

The above result reveals more: suppose $U = \{u_1, \ldots, u_n\}$ is independent, and consider the sum $\sum_{i=1}^{n} \lambda_i u_i$. Since U is independent, it is contained in a basis B of Q, and hence $\sum_{i=1}^{n} \lambda_i u_i$ is the decomposition of some unique element by the corollary above, i.e. $\sum_{i=1}^{n} \lambda_i u_i = \{x\}$. It therefore is clear that any *independent* *sum* (a linear combination of independent elements of $Q(V)$) contains exactly one element.

4.5 Compatibility and regularity

Regularity and compatibility are central to the study of near-vector spaces. We define these below and develop the theory as André does in [1], and as explored in Section 2.4.

Definition 4.27. Let (V, A) be a hyper near-vector space. The elements u, v of Q^* are called *compatible* (u cp v) if there exists a $\lambda \in A^*$ such that $+_u = +_{\lambda v}$.

We note that for a near-vector space (V, A), two vectors $u, v \in Q^*$ are said to be compatible if there exists a $\lambda \in A^*$ such that $u + \lambda v \in Q$ and it is shown that this is equivalent to there existing a $\lambda \in A^*$ such that $+_u = +_{\lambda v}$. This is not the case for hyper near-vector spaces. Referring back to Example 4.12, a cp b, but $a \oplus b = c \notin Q^*$, so we do not have the second statement. We will motivate our choice of definition a bit later in the chapter.

Next we show that compatibility induces an equivalence relation on Q^*, a fact that becomes central to the proof of the Decomposition Theorem, as we will see.

Lemma 4.28. The compatibility relation cp is an equivalence relation on Q^*.

Proof. (i) Reflexivity

It is clear that for all $u \in Q^*$, we have that $+_u = +_{1u}$.

(ii) Symmetry

Suppose that $+_u = +_{\lambda v}$ for $\lambda \in A^*$. Now let $\alpha, \beta \in A$, then

$$\begin{aligned}(\alpha +_{\lambda^{-1}u} \beta)\lambda^{-1}u &= \alpha\lambda^{-1}u + \beta\lambda^{-1}u \\ &= (\alpha\lambda^{-1} +_u \beta\lambda^{-1})u \\ &= (\alpha\lambda^{-1} +_{\lambda v} \beta\lambda^{-1})u.\end{aligned}$$

Thus, since $u \neq 0$, we have that $(\alpha +_{\lambda^{-1}u} \beta)\lambda^{-1} = \alpha\lambda^{-1} +_{\lambda v} \beta\lambda^{-1}$. Next we have,

$$\begin{aligned}(\alpha\lambda^{-1} +_{\lambda v} \beta\lambda^{-1})\lambda v &= \alpha\lambda^{-1}\lambda v + \beta\lambda^{-1}\lambda v \\ &= \alpha v + \beta v \\ &= (\alpha +_v \beta)v.\end{aligned}$$

Thus by the fixed point free property, $(\alpha\lambda^{-1} +_{\lambda v} \beta\lambda^{-1})\lambda = \alpha +_v \beta$. Hence we finally have that $\alpha +_{\lambda^{-1}u} \beta = [(\alpha +_{\lambda^{-1}u} \beta)\lambda^{-1}]\lambda = (\alpha\lambda^{-1} +_{\lambda v} \beta\lambda^{-1})\lambda = \alpha +_v \beta$.

(iii) Transitivity
Suppose that $+_v = +_{\lambda u}$ and $+_u = +_{\lambda' w}$ for $\lambda, \lambda' \in A^*$. Then since $+_v = +_{\lambda u}$, we have that $+_{\lambda^{-1}v} = +_u = +_{\lambda' w}$, so that $+_v = +_{\lambda\lambda' w}$.

■

We give a second example of a hyper near-vector space.

Example 4.29. Let $X = \{0, 1\}$ with the hyperoperation $+_X$ defined as follows:

$+_X$	0	1
0	0	1
1	1	X

It is not difficult to verify that $(X, +_X)$ is a canonical hypergroup.
Take $V = X \times \mathbb{Z}_2$, with $\oplus$ defined for all $(a, b), (a', b') \in V$, by

$$(a, b) \oplus (a', b') = \{(x, y) | x \in a +_X a' \text{ and } y \in b +_{\mathbb{Z}_2} b'\}.$$

We then have the following table for $(V, \oplus)$:

$\oplus$	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$(0,0)$	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$(0,1)$	$(0,1)$	$(0,0)$	$(1,1)$	$(1,0)$
$(1,0)$	$(1,0)$	$(1,1)$	$\{(0,0),(1,0)\}$	$\{(0,1),(1,1)\}$
$(1,1)$	$(1,1)$	$(1,0)$	$\{(0,1),(1,1)\}$	$\{(1,0),(0,0)\}$

$(V, \oplus)$ is a canonical hypergroup and if we take $A = \{0, 1\}$, then since $-1 = 1$, we have that (V, A) is a hyper near-vector space. A quick check shows that $Q(V) = \{(0, 0), (0, 1), (1, 0)\}$. In addition, $+_{(0,1)} \neq +_{(1,0)}$, since $+_{(0,1)} = +_{\mathbb{Z}_2}$, while $+_{(1,0)} = +_X$. Thus $(0, 1)$ is not compatible with $(1, 0)$. We note that (V, A) is not a weak hyper vector space, and therefore also not a hyper vector space.

Lemma 4.30. Let (V, A), (W, A) be hyper near-vector spaces and $\phi : V \to W$ a good homomorphism. Let $u, v \in Q(V)^*$. Then the following properties hold.

1. If $W = V$, then $\phi(u)$ cp u if and only if $\phi(u) \neq 0$.
2. $\phi(u)$ cp $\phi(v)$ if and only if $\phi(u) \neq 0 \neq \phi(v)$ and u cp v.

Proof. By Lemma 4.14 we know $\phi(u), \phi(v) \in Q(W)$.

1. Suppose $V = W$, and suppose u cp $\phi(u)$. Then $\phi(u) \neq 0$, since cp is an equivalence relation on $Q(V)^*$. Conversely, suppose $\phi(u) \neq 0$ and let $\alpha, \beta \in A$. Then the following holds.

$$\begin{aligned} \alpha u + \beta u &= (\alpha +_u \beta)u \\ \phi(\alpha u + \beta u) &= \phi((\alpha +_u \beta)u) \\ \alpha\phi(u) + \beta\phi(u) &= (\alpha +_u \beta)\phi(u) \\ (\alpha +_{\phi(u)} \beta)\phi(u) &= (\alpha +_u \beta)\phi(u) \end{aligned}$$

   Since $\phi(u) \neq 0$, it follows by the fixed-point-free property that $\alpha +_{\phi(u)} \beta = \alpha +_u \beta$. Hence $+_{\phi(u)} = +_u$, so that $\phi(u)$ cp u.

2. Suppose $\phi(u)$ cp $\phi(v)$. Then $\phi(u) \neq 0 \neq \phi(v)$, since cp is an equivalence relation on $Q(W)^*$. Let $\lambda \in A^*$ such that $+_{\phi(u)} = +_{\lambda\phi(v)}$. Then by the same argument as above we have $+_u = +_{\phi(u)}$ and $+_{\lambda v} = +_{\phi(\lambda v)} = +_{\lambda\phi(v)}$, hence $+_u = +_{\lambda v}$ so that u cp v.

   Conversely, if u cp v and $\phi(u) \neq 0 \neq \phi(v)$, let $\lambda \in A^*$ such that $+_u = +_{\lambda v}$. Then once again we have $+_{\phi(u)} = +_u$ and $+_{\lambda\phi(v)} = +_{\phi(\lambda v)} = +_{\lambda v}$, hence $+_{\phi(u)} = +_{\lambda\phi(v)}$. It follows that $\phi(u)$ cp $\phi(v)$.

■

The next result shows that each vector in the quasi-kernel is compatible with each basis vector in its decomposition.

Lemma 4.31. Let (V, A) be a hyper near-vector space and let $u_1, \ldots, u_n$ be independent elements in Q. Let $u \in \sum_{i=1}^n \lambda_i u_i$ such that $u \in Q$ for some $\lambda_1, \ldots, \lambda_n \in A^*$. Then u cp u_i for all $i \in \{1, \ldots, n\}$.

Proof. Let $\alpha, \beta \in A$. Since $u \in Q$ we know that there exists $A' \subseteq A$ such that $\alpha u + \beta u = A'u$. We know that, since $\lambda_1, \ldots, \lambda_n$ are nonzero and $u_1, \ldots, u_n$ are independent, u is nonzero, so $A' = \alpha +_u \beta$ is uniquely defined by α and β. Now, let $\gamma \in A'$, then:

$$\begin{aligned}
\gamma u &\in \alpha u + \beta u \\
0 &\in \alpha u + \beta u - \gamma u \\
0 &\in \alpha \sum_{i=1}^{n} \lambda_i u_i + \beta \sum_{i=1}^{n} \lambda_i u_i - \gamma \sum_{i=1}^{n} \lambda_i u_i \\
0 &\in \sum_{i=1}^{n} (\alpha \lambda_i u_i + \beta \lambda_i u_i - \gamma \lambda_i u_i) \\
0 &\in \sum_{i=1}^{n} (\alpha +_{\lambda_i u_i} \beta +_{\lambda_i u_i} (-\gamma)) \lambda_i u_i
\end{aligned}$$

It follows that there exists $\eta_1, \ldots, \eta_n \in A$ such that $\eta_i \in (\alpha +_{\lambda_i u_i} \beta +_{\lambda_i u_i} (-\gamma))$ and $0 \in \sum_{i=1}^{n} \eta_i \lambda_i u_i$. But then $\eta_i \lambda_i = 0$ for each $i \in \{1, \ldots, n\}$ by Lemma 4.22, and hence $\eta_i = 0$ for each $i \in \{1, \ldots, n\}$. Now, since $0 \in (\alpha +_{\lambda_i u_i} \beta +_{\lambda_i u_i} (-\gamma))$, it follows that $\gamma \in \alpha +_{\lambda_i u_i} \beta$ for each i. Hence $A' \subseteq \alpha +_{\lambda_i u_i} \beta$.

Conversely, suppose without loss of generality $\alpha +_{\lambda_1 u_1} \beta \nsubseteq A'$. We know that $u \in \sum_{i=1}^{n} \lambda_i u_i$, so $\lambda_1 u_1 \in \sum_{i=2}^{n} \lambda_i u_i - u$. If $\{u_2, \ldots, u_n, u\}$ is independent, it follows that $\alpha +_{\lambda_1 u_1} \beta \subseteq \alpha +_u \beta = A'$, contradicting the assumption. Let $\eta, \eta_2 \ldots, \eta_n \in A$ such that $0 \in \eta u + \sum_{i=2}^{n} \eta_i u_i$. Then

$$\begin{aligned}
0 &\in \eta u + \sum_{i=2}^{n} \eta_i u_i \\
0 &\in \eta \sum_{i=1}^{n} \lambda_i u_i + \sum_{i=2}^{n} \eta_i u_i \\
0 &\in \eta \lambda_1 u_1 + \sum_{i=2}^{n} (\eta \lambda_i u_i + \eta_i u_i) \\
0 &\in \eta \lambda_1 u_1 + \sum_{i=2}^{n} (\eta \lambda_i +_{u_i} \eta_i) u_i
\end{aligned}$$

It follows there exist $\xi_2, \dots \xi_n \in A$ such that $\xi_i \in \eta\lambda_i +_{u_i} \eta_i$ for all $i \in \{2, \dots n\}$ and $0 \in \eta\lambda_1 u_1 + \sum_{i=2}^{n} \xi_i u_i$. But since $u_1 \dots, u_n$ are independent, it follows $\eta\lambda_1 = \xi_i = 0$ for all $i \in \{2, \dots, n\}$, so that $\eta = 0$ and $0 \in \eta\lambda_i +_{u_i} \eta_i = 0 +_{u_i} \eta_i = \{\eta_i\}$, hence $\eta_i = 0$ for all $i \in \{2, \dots, n\}$. It follows $u, u_2, \dots, u_n$ are independent, so $\alpha +_{\lambda_1 u_1} \beta \subseteq \alpha +_u \beta = A'$, a contradiction. Hence $\alpha +_u \beta = \alpha +_{\lambda_i u_i} \beta$ for all $\alpha, \beta \in A$, and so $+_u = +_{\lambda_i u_i}$ for all $i \in \{1, \dots, n\}$, hence u cp u_i for all $i \in \{1 \dots, n\}$. ■

We now define regularity.

Definition 4.32. Let (V, A) be a hyper near-vector space. V is said to be *regular* if every pair of nonzero elements of the quasi-kernel are compatible.

Example 4.33. Referring back to Example 4.12 and 4.29, Example 4.12 is regular and Example 4.29 is non-regular.

As with near-vector spaces, we can prove that regularity is determined by the regularity of the basis elements.

Theorem 4.34. *A near vector space V is regular if and only if there exists a basis which consists of mutually pairwise compatible vectors.*

Proof. Suppose V is regular. Then, by definition, any two vectors of Q^* are compatible. Therefore, every basis of Q consists of mutually pairwise compatible vectors.
Conversely, suppose there exists a basis $B = \{u_i | i \in I\}$ of mutually pairwise compatible vectors. Let $u, v \in Q^*$, then $u \in \sum_{i=1}^{n} \lambda_i u_i$ with $u_i \in B$ for some $\lambda_1, \dots, \lambda_n \in A$ and $v \in \sum_{i=1}^{n} \eta_i u_i$ with $u_i \in B$ for some $\eta_1, \dots, \eta_n \in A$. Since the u_i for $i \in \{1, \dots, n\}$ are independent, we can apply Lemma 4.31. Thus u is compatible to each u_i for $i \in \{1, \dots, n\}$. Similarly, v is compatible to each u_i for $i \in \{1, \dots, n\}$. Thus by the transitivity of the compatibility relation, we have that u is compatible to v. ■

The following result is an analogue of Theorem 4.2 in [1].

Theorem 4.35. *Let (V, A) be a near-vector space and $u \in Q^*$. Let F be the nearfield defined by $(F, +, \cdot) = (A, +_u, \circ)$. Then V is regular if and only if $V \cong F^{(I)}$, as defined in Theorem 4.25, with $(\alpha_i)_{i\in I} + (\beta_i)_{i\in I} = \{(\gamma_i)_{i\in I} \,|\, \gamma_i \in \alpha_i +_u \beta_i\}$ and $\lambda(\alpha_i)_{i\in I} = (\lambda\alpha_i)_{i\in I}$.*

Proof. Suppose V is regular, then there exists a basis $B = \{b_i \,|\, i \in I\}$ of V such that $u \in B$. Since V is regular, u cp b_i for all $i \in I$, therefore there exists

$\lambda_i \in A^*$ such that $+_u = +_{\lambda_i b_i}$ for all $i \in I$. Therefore by Theorem 4.25, $V \cong A^{(I)}$ with $(\alpha_i)_{i\in I} + (\beta_i)_{i\in I} = \{(\gamma_i)_{i\in I} \,|\, \gamma_i \in \alpha_i +_{\lambda_i b_i} \beta_i\}$ for all $(\alpha_i)_{i\in I} + (\beta_i)_{i\in I} \in A^{(I)}$. But since $+_u = +_{\lambda_i b_i}$, it follows $(\alpha_i)_{i\in I} + (\beta_i)_{i\in I} = \{(\gamma_i)_{i\in I} \,|\, \gamma_i \in \alpha_i +_u \beta_i\}$ for all $(\alpha_i)_{i\in I} + (\beta_i)_{i\in I} \in A^{(I)}$, so that $A^{(I)} = F^{(I)}$, and so $V \cong F^{(I)}$.

Conversely, suppose $V \cong F^{(I)}$. Let $\phi : V \mapsto F^{(I)}$ be an isomorphism and let $b_j = \phi^{-1}((\delta_{ij})_{i\in I})$ for all $j \in I$, where δ_{ij} is the Kronecker delta symbol. Then $b_j \in Q(V)$ and $+_{b_j} = +_u$ for all $j \in I$:

$$\begin{aligned}
\alpha b_j + \beta b_j &= \alpha\phi^{-1}((\delta_{ij})_{i\in I}) + \beta\phi^{-1}((\delta_{ij})_{i\in I})\\
&= \phi^{-1}(\alpha(\delta_{ij})_{i\in I}) + \phi^{-1}(\beta(\delta_{ij})_{i\in I})\\
&= \phi^{-1}((\alpha\delta_{ij})_{i\in I} + (\beta\delta_{ij})_{i\in I})\\
&= \phi^{-1}(\alpha\delta_{ij} +_u \beta\delta_{ij})_{i\in I})\\
&= \phi^{-1}(((\alpha +_u \beta)\delta_{ij})_{i\in I}) \text{ (Since } \delta_{ij} \in \{0,1\}\text{, it satisfies the right distributive law.)}\\
&= \phi^{-1}((\alpha +_u \beta)(\delta_{ij})_{i\in I})\\
&= (\alpha +_u \beta)\phi^{-1}((\delta_{ij})_{i\in I})\\
&= (\alpha +_u \beta)b_j.
\end{aligned}$$

Moreover, $B = \{b_i \,|\, i \in I\}$ is a basis for V. To see this, if $0 \in \sum_{j\in I} \lambda_j b_j$, then $(0)_{j\in I} = \phi(0) \in \phi(\sum_{j\in I} \lambda_j b_j) = \sum_{j\in I} \lambda_j \phi(b_j) = \sum_{j\in I} \lambda_j (\delta_{ij})_{i\in I}) = \{(\lambda_i)_{i\in I}\}$, hence $\lambda_j = 0$ for all $j \in I$ and B is independent. Furthermore, if $x \in Q$ and $\phi(x) = (\eta_i)_{i\in I}$, then $\phi(x) \in \sum_{j\in I} \eta_j (\delta_{ij})_{i\in I} = \phi(\sum_{i\in I} \eta_i b_i)$, and so $x \in \sum_{i\in I} \eta_i b_i$, since ϕ is injective. It follows $x \lhd B$ and thus B generates Q (and therefore V). Hence B is a basis consisting of mutually pairwise compatible vectors, so that V is regular by Theorem 4.34. ■

The above result motivates our choice of definition for compatibility. Referring back to Example 4.12, we have a basis of mutually compatible vectors, namely $B = \{a, b\}$. Should we have chosen the alternative definition, these two would not be compatible and so the hyper near-vector space would not be regular. This would not correspond to the above result, as $V \cong X^2$ where X is defined as in Example 4.29.

4.6 Subhyperspaces of V and the Decomposition Theorem

Next we define the notion of a subhyperspace, the final missing requirement to prove an analogue of the Decomposition Theorem.

Definition 4.36. If (V, A) is a hyper near-vector space and $\emptyset \neq V' \subseteq V$ is such that V' is the canonical subhypergroup of $(V, +)$ generated additively by $AX = \{ax \,|\, x \in X, a \in A\}$, where X is an independent subset of $Q(V)$, then we say that (V', A) is a *subhyperspace* of (V, A), or simply V' is a *subhyperspace* of V if A is clear from the context.

If $(V, +)$ is generated additively by AX, we will write $V = \langle AX \rangle$.

Lemma 4.37. Let (V, A) be a hyper near-vector space and V' be a subhyperspace of V. Then $Q(V') = V' \cap Q(V)$.

Proof. Suppose $v \in V' \cap Q(V)$, then $v \in V'$ and $v \in Q(V)$, so that for all $\alpha, \beta \in Q(V)$, $\alpha v + \beta v \subseteq Av$. It follows $v \in Q(V')$.

Conversely, suppose $v \in Q(V')$. Then $v \in V'$ and for all $\alpha, \beta \in A$, $\alpha v + \beta v \subseteq Av$. It follows $v \in Q(V)$ and so $v \in Q(V) \cap V'$. ∎

Corollary 4.38. Let (V, A) be a hyper near-vector space, and suppose U and W are subhyperspaces of V. Then $U \subseteq W$ if and only if $Q(U) \subseteq Q(W)$.

Proof. Suppose $U \subseteq W$, then $Q(V) \cap U \subseteq Q(V) \cap W$, hence $Q(U) \subseteq Q(W)$. Conversely, if $Q(U) \subseteq Q(W)$, let $X \subseteq Q(V)$ such that $U = \langle AX \rangle$. Then $X \subseteq Q(V) \cap U = Q(U) \subseteq Q(W)$, so that X is an independent subset of $Q(W)$. It follows there exists a basis X' for $Q(W)$ such that $X \subseteq X'$. Therefor $AX \subseteq AX'$, and hence $U = \langle AX \rangle \subseteq \langle AX' \rangle = W$. ∎

In the next proposition we prove when the union of two subhyperspaces will be a subhyperspace. The proof is by Howell and appearr as follows in [6].

Proposition 4.39. *Let (V, A) be a hyper near-vector space and W_1, W_2 subhyperspaces of V. Then $W_1 \cup W_2$ is a subhyperspace of V if and only if $W_1 \subseteq W_2$ or $W_1 \subseteq W_2$.*

Proof. Suppose without loss of generality that $W_1 \subseteq W_2$, where $W_2 = \langle AX \rangle$ with X an independent subset of $Q(V)$. Then $W_1 \cup W_2 = W_2$ so we are done.

Conversely, suppose that $W_1 \not\subseteq W_2$ and $W_2 \not\subseteq W_1$. Then there exist $x, y \in V$ such that $x \in W_1, y \in W_2, x \notin W_2, y \notin W_1$. Since $W_1 \cup W_2$ is assumed to be a subhyperspace, having $x + y \subseteq W_1 \cup W_2$ implies that for all $z \in x + y$, we have $z \in W_1 \cup W_2$. Without loss of generality, suppose that $z \in W_1$. Then $z \in x + y$ implies that $y \in z - x \subseteq W_1$, a contradiction. ∎

We end off with the analogue of the Decomposition Theorem for hyper-near vector spaces. André proves in [1] that every near-vector space is isomorphic to the direct sum of its maximal regular subspaces — this is the Decomposition Theorem of Section 2.4. However, this result does not generalise to hyper near-vector spaces; in fact, the direct sum of hyper near-vector spaces is not defined in the category theoretical sense. Instead, we show that any finite-dimensional hyper near-vector space can be expressed as the direct product of its maximal regular subhyperspaces. This result does not generalise to arbitrary hyper near-vector spaces as arbitrary direct products are not defined even for near-vector spaces. First, we show that finite direct products are defined for hypersubspaces of hyper near-vector spaces.

Theorem 4.40. *Let (V, A) be a near-vector space, $I = \{1, \dots, n\}$ and suppose $\{V_i \,|\, i \in I\}$ is a set of subhyperspaces of V. Define*

$$\prod_{i=1}^{n} V_i = \{(v_i)_{i \in I} \,|\, \forall i \in I\, [v_i \in V_i]\},$$

with addition defined as $(v_i)_{i\in I} + (w_i)_{i\in I} = \{(u_i)_{i\in I} \,|\, \forall i \in I[u_i \in v_i + w_i]\}$ and scalar multiplication defined componentwise. Then $(\prod_{i=1}^{n} V_i, A)$ is a hyper near-vector space, and it is a direct product of $\{V_i \,|\, i \in I\}$, with projection maps $\pi_j : \prod_{i=1}^{n} V_i \to V_j$ defined by $\pi_j((v_i)_{i\in I}) = v_j$ for all $j \in I$.

Proof. It is routine to show $\prod_{i=1}^{n} V_i$ is a hyper near-vector space with neutral element $(0)_{i\in I}$, and that π_i is a good homomorphism for each $i \in I$. To show that $\prod_{i=1}^{n} V_i$ is indeed the direct product of $\{V_1, \dots, V_n\}$, let W be a hyper near-vector space, and let $f_i : W \to V_i$ be homomorphisms. Define $f : W \to \prod_{i=1}^{n} V_i$ such that $f(w) = (f_i(w))_{i\in I}$. Suppose for some $x + y \in W$ that $(u_i)_{i\in I} \in f(x + y)$. Then there exists some $w \in x + y$ such that $f(w) = (u_i)_{i\in I}$. It follows that $u_i = f_i(w) \in f_i(x + y) \subseteq f_i(x) + f_i(y)$, so that $(u_i)_{i\in I} \in f(x) + f(y)$. Furthermore, $f(\alpha w) = (f_i(\alpha w))_{i\in I} = (\alpha f_i(w))_{i\in I} = \alpha(f_i(w))_{i\in I} = \alpha f(w)$ for all $\alpha \in A$ and $w \in W$. Hence f is a homomorphism. Furthermore, $(\pi_j \circ f)(w) = \pi_j((f_i(w))_{i\in I}) = f_j(w)$, so that $\pi_j \circ f = f_j$ for all $j \in I$. Finally, to show uniqueness, suppose $g : W \to \prod_{i=1}^{n} V_i$

such that$\pi_j \circ g = f_j$ for all $j \in I$. Then, for $w \in W$, $g(w) = (f_i(w))_{i\in I} = f(w)$, so that $g = f$. ■

Theorem 4.41. *Let (V, A) be a finite-dimensional hyper–near-vector space. Then V is isomorphic to the direct product of maximal regular subhyperspaces, with each $u \in Q^*$ being in exactly one of these maximal regular subhyperspaces.*

Proof. Let $\{Q_i \,|\, i \in I\}$ be the partition of Q^* into its compatible elements, and define $B_i = B \cap Q_i$, where B is a basis of V. Define $V_i = \langle AB_i \rangle$. By definition V_i is a subhyperspace of V with basis B_i. Since $B_i \subseteq Q_i$, it follows that B_i consists of mutally pairwise compatible vectors, so that V_i is a regular subhyperspace for all $i \in I$. Furthermore, if $V_i \subset W \subseteq V$, where W is a regular subhyperspace of V, then W has a basis of mutually pairwise compatible vectors (by Theorem 4.34) properly containing B_i and properly contained in B, a contradiction, since B_i contains all vectors of B that lie in the partition Q_i. Hence the V_i subhyperspaces are maximal.

Let $u \in Q^*$. Then, since Q^* is partitioned by Q_i's, $i \in I$, it follows that $u \in Q_j$ for exactly one $j \in I$. We wish to show that $u \in V_j$. Let $u \in \sum_{i=1}^{n} \lambda_i b_i$ for some $b_1, \ldots, b_n \in B$ and $\lambda_1, \ldots, \lambda_n \in A^*$. Then by Lemma 4.31, u cp b_i for each $i \in \{1, \ldots, n\}$. It follows that $b_i \in Q_j$ for each $i \in \{1, \ldots, n\}$, so $b_i \in B \cap Q_j = B_j$ for all $i \in \{1, \ldots, n\}$. It follows $u \in \sum_{i=1}^{n} \lambda_i b_i \subseteq \langle AB_j \rangle = V_j$.

Now, suppose $u \in V_k$ for some $k \in I$ such that $j \neq k$. Then, because the unique expression (by Lemma 4.23) for u in terms of the basis B is $u \in \sum_{i=1}^{n} \lambda_i b_i$, $b_1, \ldots, b_n \in B_k$, we have that $b_1, \ldots, b_n \in Q_k$ — a contradiction, since $b_1, \ldots, b_n \in Q_j$ and $Q_j \cap Q_k = \varnothing$. Hence u lies in exactly one V_i, $i \in I$.

Define now $f : \prod_{i=1}^{n} V_i \to V$ such that $f((u_i)_{i\in I}) \in \sum_{i\in I} u_i$. Since V is finite-dimensional, I is finite, so that the sum $\sum_{i\in I} u_i$ is defined. To show that f is well-defined, note that, for each $i \in I$, u_i is the unique element such that $u_i \in \sum_{b_{ij}\in B_i} \lambda_{ij} b_{ij}$ for some distinct $b_{ij} \in B_i$ and $\lambda_{ij} \in A$ (see paragraph below Corollary 4.26). It follows that $\sum_{i\in I} u_i = \sum_{i\in I} \sum_{b_{ij}\in B_i} \lambda_{ij} b_{ij}$. But $B_i \cap B_k = \varnothing$ for all $i, k \in I$ where $i \neq k$, so $\sum_{i\in I} \sum_{b_{ij}\in B_i} \lambda_{ij} b_{ij}$ is a linear combination of distinct basis elements and so contains only one element.

Now suppose $x \in f((u_i)_{i\in I} + (v_i)_{i\in I})$. It follows there exist $x_i \in u_i + v_i$ for all $i \in I$ such that $x = f((x_i)_{i\in I}) \in \sum_{i\in I} x_i \subseteq \sum_{i\in I}(u_i + v_i) = \sum_{i\in I} u_i + \sum_{i\in I} v_i = f((u_i)_{i\in I}) + f((v_i)_{i\in I})$. Conversely, if $x \in f((u_i)_{i\in I}) + f((v_i)_{i\in I})$, then $x \in \sum_{i\in I} u_i + \sum_{i\in I} v_i = \sum_{i\in I}(u_i + v_i)$, therefore for all $i \in I$ there exists $x_i \in u_i + v_i$

such that $x \in \sum_{i\in I} x_i$. But then $x = f((x_i)_{i\in I}) \in f((u_i)_{i\in I} + (v_i)_{i\in I})$. Finally, $f(\alpha(u_i)_{i\in I}) = f((\alpha u_i)_{i\in I}) \in \sum_{i\in I} \alpha u_i = \alpha \sum_{i\in I} u_i$. But $\alpha f((u_i)_{i\in I}) \in \alpha \sum_{i\in I} u_i$, so $f(\alpha(u_i)_{i\in I}) = \alpha f((u_i)_{i\in I})$, and so f is a good homomorphism.

Furthermore, f is bijective: if $u \in V$, suppose it has decomposition $\sum_{i\in I} \sum_{b_{ij}\in B_i} \lambda_{ij} b_{ij}$. Define u_i to be the unique element with decomposition $\sum_{b_{ij}\in B_i} \lambda_{ij} b_{ij}$. Then $u_i \in V_i$, and $f((u_i)_{i\in I}) \in \sum_{i\in I} u_i \subseteq \sum_{i\in I} \sum_{b_{ij}\in B_i} \lambda_{ij} b_{ij}$. But $u \in \sum_{i\in I} \sum_{b_{ij}\in B_i} \lambda_{ij} b_{ij}$, so $u = f((u_i)_{i\in I})$ and f is surjective. Furthermore, if $f((u_i)_{i\in I}) = f((v_i)_{i\in I})$, then $\sum_{i\in I} u_i = \sum_{i\in I} v_i$, so that $0 \in \sum_{i\in I} u_i - \sum_{i\in I} v_i = \sum_{i\in I}(u_i - v_i)$. It follows there exists $w_i \in u_i - v_i$ such that $0 \in \sum_{i\in I} w_i$. Let w_i have decomposition $\sum_{b_{ij}\in B_i} \lambda_{ij} b_{ij}$. Then $0 \in \sum_{i\in I} w_i \subseteq \sum_{i\in I} \sum_{b_{ij}\in B_i} \lambda_{ij} b_{ij}$. It follows $\lambda_{ij} = 0$ for all $i \in I$ and $b_{ij} \in B_i$. But then $w_i \in \sum_{b_{ij}\in B_i} = \{0\}$, so $w_i = 0$ for all $i \in I$. Hence $0 \in u_i - v_i$, and so $u_i = v_i$ for all $i \in I$. It follows $(u_i)_{i\in I} = (v_i)_{i\in I}$ and hence f is injective.

It follows f is bijective and a good homomorphism, so that f is an isomorphism. Hence $V \cong \prod_{i=1}^n V_i$.

■

The above decomposition is unique up to the order of the subhyperspaces, as will be shown in the next result.

Theorem 4.42. *Let V be a hyper near-vector space, and suppose*

$$\prod_{i=1}^{n} V_i \cong V \cong \prod_{j=1}^{m} V_j'$$

where V_i and V_j' are maximal regular subspaces for all $i \in \{1,\dots,n\}$ and $j \in \{1\dots,m\}$. Then $m = n$ and there exists $\sigma \in S_n$ such that $V_i = V_{\sigma(i)}'$.

Proof. Let $I = \{1,\dots,n\}$ and $J = \{1,\dots,m\}$, and let $Q_i = Q(V_i)^*$ for some $i \in I$. Then Q_i is a maximal set of compatible vectors of $Q(V)$. If not, there exists $u \in Q(V)^* \backslash Q_i$ such that u cp v for all $v \in Q_i$. But then $u \notin Q(V_i) = V_i \cap Q(V)$, so it follows $u \notin V_i$. But then $V_i = \langle Q_i \rangle \subsetneq \langle Q_i \cup \{u\}\rangle$, contradicting its maximality. It follows $Q_i \in Q^*/\text{cp}$. To show $\{Q_i \,|\, i \in I\} = Q^*/\text{cp}$, suppose $u \in Q^*$ such that $u \notin Q_i$ for any $i \in \{1,\dots,n\}$. Since $V \cong \prod_{i=1}^n V_i$, it follows that there exists an isomorphism $\phi : V \to \prod_{i=1}^n V_i$. Let $\phi(u) = (u_1,\dots,u_n)$, then $\pi_i(\phi(u)) = u_i$ for all $i \in I$.

Consider the sum $\sum_{i\in I} v_i$ for some $v_i \in V_i$ for each $i \in \{i\dots,n\}$. Suppose v_i has decomposition $\sum_{j=1}^{m_i} \lambda_{ij} b_{ij}$ where $B_i = \{b_{i1}, \ldots, b_{im_i}\}$ is some independent subset of $Q(V_i)^* = Q_i$. Then $B = \{b_{ij} \,|\, i \in I, 1 \leq j \leq m_i\}$ is independent. If not, then there is some minimal dependent subset of B, say $B' = \{b_k \,|\, k \in K\}$, such that B' is dependent. It follows that there exist $k' \in K$ such that $b_{k'} \triangleleft B' \setminus \{b_{k'}\}$, i.e. there exist some $K' \subseteq K \setminus \{k'\}$ and $\lambda_k \in A^*$ for each $k \in K'$ such that $b_{k'} \in \sum_{k \in K'} \lambda_k b_k$. Since B' is a minimally dependent set, $\{b_k \,|, k \in K'\}$ is independent, so that $b_{k'}$ cp b_k for each $k \in K'$ by Lemma 4.31. It follows there is some $i \in I$ such that $\{b_k \,|\, k \in K'\} \cup \{b_{k'}\} \subseteq B_i$ — a contradiction, since B_i is independent and therefore has no dependent subsets. Hence B is independent, and so $\sum_{i=1}^{n} \sum_{j=1}^{m_i} \lambda_{ij} b_{ij}$ contains a unique element, say v. But $\sum_{i=1}^{n} v_i \subseteq \sum_{i=1}^{n} \sum_{j=1}^{m_i} \lambda_{ij} b_{ij} = \{v\}$, so $\sum_{i=1}^{n} v_i = \{v\}$.

Define then the map $f : V \to V$ such that, if $\pi_i(\phi(v)) = v_i$, then $f(v)$ is the unique element in $\sum_{i=1}^{n} v_i$.

We show f is a good homomorphism. Let $x, y, z \in V$ such that $x \in y + z$ and $\{f(x)\} = \sum_{i=1}^{n} x_i$, $\{f(y)\} = \sum_{i=1}^{n} y_i$ and $\{f(z)\} = \sum_{i=1}^{n} z_i$, where $x_i, y_i, z_i \in V_i$ for each $i \in \{1, \ldots, n\}$. Since $x \in y + z$, it follows $x_i = \pi_i(\phi(x)) \in \pi_i(\phi(y + z)) = \pi_i(\phi(y)) + \pi_i(\phi(z)) = y_i + z_i$. It follows $\{f(x)\} = \sum_{i=1}^{n} x_i \subseteq \sum_{i=1}^{n}(y_i + z_i) = \sum_{i=1}^{n} y_i + \sum_{i=1}^{n} z_i = f(y) + f(z)$. It follows $f(x) \in f(y) + f(z)$, so that $f(y+z) \subseteq f(y) + f(z)$.

Conversely, if $x \in f(y)+f(z)$, with $\{f(y)\} = \sum_{i=1}^{n} y_i$ and $\{f(z)\} = \sum_{i=1}^{n} z_i$, where $y_i, z_i \in V_i$ for each $i \in \{1, \ldots, n\}$. Then $x \in \sum_{i=1}^{n} y_i + \sum_{i=1}^{n} z_i = \sum_{i=1}^{n}(y_i + z_i)$. It follows there exists $x_i \in y_i + z_i$ such that $x \in \sum_{i=1}^{n} x_i$. Since $y_i, z_i \in V_i$, it follows $x_i \in V_i$. Moreover, $y_i + z_i = \pi_i(\phi(y)) + \pi_i(\phi(z)) = \pi_i(\phi(y + z))$, since π_i and ϕ are good homomorphisms. It follows $x_i \in \pi_i(\phi(y + z))$, so that $\{x\} = \sum_{i=1}^{n} x_i \subseteq \sum_{i=1}^{n} \pi_i(\phi(y+z)) = \bigcup \{\sum_{i=1}^{n} \pi_i(\phi(x')) \,|\, x' \in y + z\} = \{f(x') \,|\, x' \in y + z\} = f(y+z)$. Hence $x \in f(y + z)$, and so f is a good homomorphism.

Consider now $f(u)$. Since f is a good homomorphism, $f(u) \in Q(V)$. Suppose $f(u) \neq 0$. Then u cp $f(u)$ by Lemma 4.30. Let u_i have decomposition $\sum_{j=1}^{m_i} \lambda_{ij} b_{ij}$ where each $b_{ij} \in V_i$. Then $f(u) \in \sum_{i=1}^{n} \sum_{j=1}^{m_i} \lambda_{ij} b_{ij}$. Since $f(u) \neq 0$, it follows $\lambda_{ij} \neq 0$ for at least one $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m_i\}$. Therefore $f(u)$ cp b_{ij} by Lemma 4.31. But then u cp $f(u)$ cp $b_{ij} \in Q(V_i)^* = Q_i$, so that $u \in Q_i$ — a contradiction. Suppose then that $f(u) = 0$. Then $u_i = 0$ for each $i \in \{1, \ldots, n\}$, so that $\phi(u) = (0, \ldots, 0)$. It follows $u = 0$, since ϕ is an isomorphism — a contradiction. Hence there is no element $u \in Q(V)^*$ such that $u \notin Q_i$ for each $i \in \{1, \ldots, n\}$. It follows that $\{Q_i \,|1 \leq i \leq n\} = Q^*/\text{cp}$.

By a symmetric argument, $\left\{Q'_j \,|\, j \in J\right\} = Q^*/\text{cp}$, so that $\{Q_i \,|\, i \in I\} = \left\{Q'_j \,|\, j \in J\right\}$. It follows $n = m$ and for each $i \in I$ there is some $j \in I$ such that $Q_i = Q'_j$, so that $V_i = \langle Q_i \rangle = \langle Q'_j \rangle = V'_j$.

■

Example 4.43. Returning to Example 4.12 we have that $V \cong V$ (because V is regular) while for Example 4.29, we have that $V \cong \mathbb{Z}^2 \times X$.

Chapter 5

Conclusion

The expansion of the geometric theory behind André's near-vector spaces has answered a number of questions about the intuition behind these spaces that have always lurked in the minds of those that have researched them, and at the same time, has opened brand new avenues for research and exploration.

In the case of the nearaffine space, one problem that has as of yet not been attempted, has been to answer whether every subspace of a nearaffine space also meets the stronger condition of being a flat for a general nearaffine space — that is, that for any two points x and y of the subspace, if L is a line such that $x, y \in L$, then L is contained in the subspace, independently of whether either of the points is a base point of L. The problem has been solved specifically for the finite case in [2] by André. Of course, for the standard affine space, in which each line is straight, this condition is certainly satisfied; however, the property is not trivial when the proper lines of nearaffine spaces are considered. Undoubtably, any proof that the condition holds would rest upon arguments unique to the geometry of near-structures. The consequences of this property on the algebraic understanding of near-vector spaces could also be worth examining.

Beyond this, the new geometries defined in Chapter 3, in particular the near-linear space and the projections of nearaffine spaces, are themselves new topics worth exploring. In the case of the latter, it has already been shown which properties of a projective space these geometries capture and which they do not, and an interesting future research project would be to axiomatise these properties as a general definition of a near-projective space, with the inclusion of other potentially necessary properties that the projection of a nearaffine space satisfy, but that are as of yet undiscovered. Whether these potential near-projective spaces satisfy the necessary conditions to also then emit notions of independence, dimension, and a basis are also of interest.

While the theory of hyper near-vector spaces has been developed to a degree in this thesis, the mathematical structure remains in its infancy, and has a significant potential for future research. In particular, an interesting question to ask about these structures is whether a coherent notion of their geometry can be developed, much in the same way as the geometry of near-vector spaces has been developed.

Algebraically, there is also possible avenues of exploration. In particular, it is unknown whether the subspace test for vector spaces and near-vector spaces could be generalised to these structures; that is, whether any canonical subhypergoup of a hyper near-vector space (V, A) that is closed under scalar multiplication would also be subhyperspace.

Furthermore, since there is more than one way to define the notion of a hyper vector space; one further avenue for future exploration, would be to look at what the corresponding near-vector spaces should be.

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