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Title

Nash equilibrium strategies of an inconsistent stochastic control problem

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DEDICATION

To my parents,

To my brothers, sisters and their kids,

To all my family,

To all those who are dear to me.

Acknowledgments

First and foremost, I thank God Almighty, who enabled me to do this work.

I would like to express my deepest gratitude to my supervisor, **Prof. Farid Chighoub**, for his valuable support, quality of supervision, patience, rigor, and willingness. Without his guidance and assistance, this thesis would not have been possible. I would like to especially thank him for his noble and good manners. I honestly couldn't have wished for a better supervisor.

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Abstract

In this thesis we study two research topics by using stochastic control methods in order to solve, in distinct contexts. The first topic presents a characterization of equilibrium in a game-theoretic description of discounting conditional stochastic linear-quadratic (LQ for short) optimal control problem, in which the controlled state process evolves according to a multidimensional linear stochastic differential equation, when the noise is driven by a Poisson process and an independent Brownian motion under the effect of a Markovian regime-switching. The running and the terminal costs in the objective functional are explicitly dependent on several quadratic terms of the conditional expectation of the state process as well as on a nonexponential discount function, which create the time-inconsistency of the considered model. Open-loop Nash equilibrium controls are described through some necessary and sufficient equilibrium conditions. A state feedback equilibrium strategy is achieved via certain differential-difference system of ODEs. As an application, we study an investment–consumption and equilibrium reinsurance/new business strategies for mean-variance utility for insurers when the risk aversion is a function of current wealth level. The financial market consists of one riskless asset and one risky asset whose price process is modeled by geometric Lévy processes and the surplus of the insurers is assumed to follow a jump-diffusion model, where the values of parameters change according to continuous-time Markov chain. A numerical example is provided to demonstrate the efficacy of theoretical results.

In the second topic, we investigate the Merton portfolio management problem with non-exponential discount function and general utility function. We consider that the market coefficients according to a finite state Markov chain. The non-exponential discount in the objective function is the reason for the time-inconsistency in our topic. Since this problem is time-inconsistent we treat it by placing within a game theoretic framework and look for subgame perfect Nash equilibrium strategies. Using a variational technical approach, we derive the necessary and sufficient equilibrium condition, also we provide a verification theorem for an open-loop equilibrium strategies.

Keys words. Stochastic Maximum Principle, time inconsistency, LQ control problem, equilibrium control, variational inequality, investment-consumption and reinsurance problem, Merton portfolio problem, non-exponential discounting.

Résumé

Cette thèse présente deux sujets de recherche. Ces deux sujets utilisent des méthodes de contrôle stochastique afin de résoudre, dans des contextes distincts, des problèmes de contrôle optimal stochastique inconsistants. Le premier sujet présente une caractérisation de l'équilibre dans une description théorique des jeux d'actualisation d'un problème de contrôle optimal stochastique linéaire-quadratique conditionnel (LQ), dans lequel le processus d'état contrôlé évolue selon une équation différentielle stochastique linéaire multidimensionnelle, lorsque le bruit est entraîné par un processus de Poisson et un mouvement brownien indépendant sous l'effet d'un changement de régime markovien. Les coûts de fonctionnement et les coûts terminaux dans la fonctionnelle objective dépendent explicitement de plusieurs termes quadratiques de l'espérance conditionnelle du processus d'état ainsi que d'une fonction d'actualisation non exponentielle, qui créent l'inconsistance temporelle du modèle considéré. Les contrôles d'équilibre de Nash en boucle ouverte sont décrits à travers certaines conditions d'équilibre nécessaires et suffisantes. Une stratégie d'équilibre de feedback d'état est obtenue via un certain système de différence différentielle d'EDO. Comme application, nous étudions une stratégie d'équilibre d'investissement-consommation et de réassurance/nouvelles affaires pour l'utilité moyenne-variance pour les assureurs lorsque l'aversion au risque est une fonction du niveau de richesse actuel. Le marché financier se compose d'un actif sans risque et d'un actif risqué dont le processus de prix est modélisé par des processus de Lévy géométriques et le surplus des assureurs est supposé suivre un modèle de saut-diffusion, où les valeurs des paramètres changent selon la chaîne de Markov en temps continu. Un exemple numérique est fourni pour démontrer l'efficacité des résultats théoriques.

Dans le deuxième sujet, nous étudions le problème de gestion de portefeuille de Merton avec une fonction d'actualisation non exponentielle et une fonction d'utilité générale. Nous considérons que les coefficients du marché selon une chaîne de Markov à états finis. La fonction d'actualisation non exponentielle dans la fonction objectif est la raison de l'inconsistance temporelle dans notre sujet.

Comme ce problème est inconsistant, nous le traitons en le plaçant dans un cadre théorique des jeux et en recherchant des stratégies d'équilibre de Nash parfait en sous-jeu. En utilisant une approche technique variationnelle, nous dérivons la condition d'équilibre nécessaire et suffisante, nous fournissons également un théorème de vérification pour une stratégie d'équilibre en boucle ouverte.

Mots Clés. Principe du maximum stochastique, inconsistance, problème de contrôle LQ, contrôle d'équilibre, inégalité variationnelle, problème d'investissement-consommation et de réassurance, problème de portefeuille de Merton, actualisation non exponentielle.

Contents

Introduction

For usual optimal control problems, by the dynamic principle of optimality [69] one may check that an optimal control remains optimal when it is restricted to a later time interval, meaning that optimal controls are time-consistent. The time consistency feature provides a powerful advance to deal with optimal control problems. The dynamic principle of optimality consists to establish relationships among a family of time-consistent optimal control problems parameterized by initial pairs (of time and state) through the so-called Hamilton-Jacobi-Bellman equation (HJB), which is a nonlinear partial differential equation. If the HJB equation is solvable, then one can find an optimal feedback control by taking the optimizer of the general Hamiltonian involved in the HJB equation.

However, in reality, the time consistency can be lost in various ways, meaning that, as time goes, an optimal control might not remain optimal. Among several possible reasons causing the time-inconsistency, there are three playing some important roles:

- The appearance of conditional expectations for the state data in the objective functional [9],
- The presence of a state-dependent risk aversion in the objective functional [10],
- The non-exponential discounting situation [1] and [32].

The portfolio optimization problem with a hyperbolic discount function [23] and the risk aversion attitude in mean-variance models [38] and [70] are two well-known cases of time-inconsistency in mathematical finance.

Time-inconsistent problems are usually studied using one of the following approaches:

- The game-theoretic approach studied in this thesis, which means formulating the problem as a game and look for equilibrium strategy.
- The pre-commitment approach, which means formulating the problem for a fixed initial point (t_0, x_0) and finds the optimal control law $\hat{u}$ that maximizes the objective functional at time t_0 with wealth x_0, $J(t_0, x_0, u)$, despite the fact that at future time $t > t_0$ the control law $\hat{u}$ will not be the maximizer of the objective functional at time t with wealth x, $J(t, x, u)$; consequently, it precommits to follow the initial strategy $\hat{u}$, despite the fact that at future times it will no longer be optimal according to its criterion. To do this, it must be able to precommit its future selves to the strategy selected at time t_0.
- The dynamic optimality approach, developed in [47]. See also [15] for a short description.

The game theoretic approach is to consider the time-inconsistent problems as non-cooperative games, in which decisions at every moment of time are taken as multiple players at each moment of time and intended to maximize or minimize their own objective functions. As a result, Nash equilibriums are considered rather than optimal solutions, see e.g. [2], [9], [17], [23], [29], [32], [51], [57], [66], [67] and [68]. Strotz [57] was the first to apply this game perspective to dealing with the dynamic time-inconsistent decision problem posed by the deterministic Ramsay problem. He then proposed a rudimentary notion of Nash equilibrium strategy by capturing the concept of non-commitment and allowing the commitment period to be infinitesimally small. Further references which extend [57] are [32] and [51]. Ekeland and Pirvu [23] gave a formal definition of feedback Nash equilibrium controls in a continuous time setting in order to investigate the optimal investment-consumption problem under general discount functions in both deterministic and stochastic frameworks. Björk and Murguci [9] and Ekeland et al. [22] are two further expansions of Ekeland and Pirvu's work. Yong [68] proposed an alternative method for analyzing general discounting time-inconsistent optimal control problem in continuous time setting by taking into account a discrete time counterpart. Zhao et al. [72] investigated the consumption-investment problem under a general discount function and a logarithmic utility function using Yong's method. Wang and Wu investigated a partially observed time inconsistent recursive optimization issue in [62] . Basak and Chabakauri [6] touched upon the continuous-time Markowitz's mean-variance portfolio selection problem, while Bojrk et al [10] addressed the mean-variance portfolio selection with state-dependent risk aversion. Hu et al [29], followed by Czichowski [17], found a time-consistent strategy for mean-variance portfolio selection in a non-Markovian framework. Yong worked on a general discounted time-inconsistent deterministic LQ model in [66] and he cosider a forward ordinary differential equation coupled with a backward Riccati-Volterra integral equation to obtain closed-loop equilibrium strategies. Hu et al. [29] presented a specific definition of open loop Nash equilibrium controls in a continuous time setting, which is distinct from the feedback one provided in [23], in order to analyze a time-inconsistent stochastic linear-quadratic optimal control problem with stochastic coefficients. Yong [68] studied a time-inconsistent stochastic LQ problem for mean-field type stochastic differential equation. Hu et al. [28] looked into the uniqueness of the equilibrium solution found in [29]. They are, the first to give a positive result regarding the uniqueness of the solution to a time-inconsistent problem.

Concerning time-inconsistence problems under the Markov regime switching model, see, for example, [74], [13], [14], [63] and [40]. Zhou and Yin [74] are the first who studied the problem of mean-variance optimization under a continuous time Markov regime-switching financial market. By applying stochastic linear-quadratic control methods, they obtained mean-variance efficient portfolios and efficient frontiers via solving two systems of ordinary linear differential equations. In the context of continuous and multi-

period time models, Chen et al. [13] and Chen and Yang [14] studied the mean-variance asset-liability management problem, respectively. Mean-variance asset-liability management problems with a continuous time Markov regime-switching setup have been studied by Wei et al. [63]. They explicitly deduced a time consistent investment strategy using the method described in [9]. Liang and Song [40] investigated optimal investment and reinsurance problems for insurers with mean-variance utility under partial information, where the stock's drift rate and the risk aversion of the insurer are both Markov-modulated.
This thesis is organised as follows:

- **Chapter 1:** This introductory chapter, we give a short introduction to stochastic control problem.

- **Chapter 2:** In this chapter, we formulate a time-inconsistent conditional linear-quadratic (LQ) control problem in which the controlled state process evolves according to a multidimensional linear stochastic differential equation, when the noise is driven by a Poisson process and an independent Brownian motion under the impact of a Markovian regime-switching. The time-inconsistency arises from the presence of a quadratic terms of the conditional expectation of the state process as well as a nonexponential discount function in the objective functional. We define an equilibrium, instead of an optimal, solution within the class of open-loop controls, in which equilibrium controls are characterized throught some necessary and sufficient equilibrium conditions. An equilibrium strategy is obtained via a certain differential-difference system of ODEs. As an application, we then consider an investment-consumption and equilibrium reinsurance/new business strategies for mean-variance utility for insurers with state-dependent risk aversion under a continuous-time Markov regime-switching model. A numerical example is provided to illustrate our results.

- **Chapter 3:** In this chapter, we investigate the Merton portfolio management problem under general discount functions and the impact of a Markovian regime-switching. In terms of stochastic system consisting of a flow of forward-backward stochastic differential equations and an equilibrium condition, we present a necessary and sufficient condition for open-loop equilibrium strategies.

Achieved Works

Papers

- N.E.H. Bouaicha, F. Chighoub, I. Alia, A. Sohail, Conditional LQ time-inconsistent Markov-switching stochastic optimal control problem for diffusion with jumps, Modern Stochastics: Theory and Applications, **9**(2) (2022),157-205.
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- N.E.H. Bouaicha, F. Chighoub, Optimal control for mean variance insurers with delay, The first national conference of mathematics and applications NCMA'21 online (webinar), University of Mohamed El Bachir El Ibrahimi Bordj Bou Arréridj, Algeria, (December 13-14, 2021).

Notation

The following notation is frequently used in this thesis

- S^n : the set of $n \times n$ symmetric real matrices.
- $C^\top$: the transpose of the vector (or matrix) C.
- $\langle \cdot, \cdot \rangle$: the inner product in some Euclidean space.

For any Euclidean space $H = \mathbb{R}^n$, or S^n with Frobenius norm $|\cdot|$, and $p, l, d \in \mathbb{N}$ we let for any $t \in [0, T]$

- $\mathbb{L}^p (\Omega, \mathcal{F}_t, \mathbb{P}; H) = \{\xi : \Omega \to H \mid \xi \text{ is } \mathcal{F}_t - \text{measurable, s.t. } \mathbb{E}[|\xi|^p] < \infty\}$, for any $p \geq 1$;
- $$\mathbb{L}^2 \left(\mathbb{R}^*, \mathcal{B}(\mathbb{R}^*), \theta; H^l\right) = \Big\{ r(\cdot) : \mathbb{R}^* \to H^l | r(\cdot) = (r_k(\cdot))_{k=1,2,\ldots,l} \text{ is}$$
$$\mathcal{B}(\mathbb{R}^*) - \text{measurable with } \sum_{k=1}^{l} \int_{\mathbb{R}^*} |r_k(z)|^2 \theta_\alpha^k (dz)\, ds < \infty \Big\};$$
- $$\mathcal{S}_\mathcal{F}^2 (t, T; H) = \Big\{ \mathcal{Y}(\cdot) : [t, T] \times \Omega \to H \mid \mathcal{Y}(\cdot) \text{ is } (\mathcal{F}_s)_{s \in [t,T]} - \text{adapted},$$
$$s \mapsto \mathcal{Y}(s) \text{ is càdlàg, with } \mathbb{E}\left[\sup_{s \in [t,T]} |\mathcal{Y}(s)|^2 ds\right] < \infty \Big\};$$
- $$\mathcal{C}_\mathcal{F}^2 (t, T; H) : \Big\{ \mathcal{Y}(\cdot) : [t, T] \times \Omega \to H \mid \mathcal{Y}(\cdot) \text{ is } (\mathcal{F}_s)_{s \in [t,T]} - \text{adapted},$$
$$s \mapsto \mathcal{Y}(s) \text{ is continuous, with } \mathbb{E}\left[\sup_{s \in [t,T]} |\mathcal{Y}(s)|^2 ds\right] < \infty \Big\};$$
- $$L_\mathcal{F}^p (t, T; H) : \Big\{ \mathcal{Y}(\cdot) : [t, T] \times \Omega \to H \mid \mathcal{Y}(\cdot) \text{ is } (\mathcal{F}_s)_{s \in [t,T]} - \text{adapted},$$
$$s \mapsto \mathcal{Y}(s), \text{ with } \mathbb{E}\left[\sup_{s \in [t,T]} |\mathcal{Y}(s)|^p ds\right] < \infty \Big\} \text{,for any } p \geq 1;$$
- $$\mathcal{L}_\mathcal{F}^2 (t, T; H^p) = \Big\{ \mathcal{Y}(\cdot) : [t, T] \times \Omega \to H^p | \mathcal{Y}(\cdot) \text{ is } (\mathcal{F}_s)_{s \in [t,T]} - \text{adapted},$$
$$\text{with } \mathbb{E}\left[\int_t^T |\mathcal{Y}(s)|^2 ds\right] < \infty \Big\};$$
- $$\mathcal{L}_{\mathcal{F},p}^2 (t, T; H) = \Big\{ \mathcal{Y}(\cdot) : [t, T] \times \Omega \to H | \mathcal{Y}(\cdot) \text{ is } (\mathcal{F}_s)_{s \in [t,T]} - \text{predictable},$$
$$\text{with } \mathbb{E}\left[\int_t^T |\mathcal{Y}(s)|^2 ds\right] < \infty \Big\};$$
- $$\mathcal{L}_{\mathcal{F},p}^{\theta,2} \left([t, T] \times \mathbb{R}^*; H^l\right) = \Big\{ \mathcal{R}(\cdot, \cdot) : [t, T] \times \Omega \times \mathbb{R}^* \to H^l \mid \mathcal{R}(\cdot, \cdot) \text{ is}$$
$$(\mathcal{F}_s)_{s \in [t,T]} - \text{predictable, with } \sum_{k=1}^{l} \mathbb{E}\left[\int_t^T \int_{\mathbb{R}^*} |R_k(s, z)|^2 \theta_\alpha^k (dz)\, ds\right] < \infty \Big\};$$
- $$\mathcal{L}_{\mathcal{F},p}^{\lambda,2} \left(t, T; H^d\right) = \Big\{ \mathcal{Y}(\cdot) : [t, T] \times \Omega \to H^d | \mathcal{Y}(\cdot) = (\mathcal{Y}_j(\cdot))_{j=1,\ldots,d}$$

is $(\mathcal{F}_s)_{s\in[t,T]}$ $-$predictable, with $\mathbb{E}\left[\int_t^T \sum_{j=1}^{d} |\mathcal{Y}_j(s)|^2 \lambda_j(s)\, ds\right] < \infty\Bigg\}$;

- $\mathcal{C}([0,T];H) = \{f : [0,T] \to H|\ f(\cdot)\ is \text{ continuous}\}$;
- $\mathcal{C}^1([0,T];H) = \left\{f : [0,T] \to H|\ f(\cdot) \text{ and } \frac{df}{ds}(\cdot) \text{ are continuous}\right\}$;
- $\mathcal{D}[0,T] = \{(t,s) \in [0,T] \times [0,T] \text{ such that } s \geq t\}$;
- $\mathcal{C}(\mathcal{D}[0,T];H) := \{f(.,.) : \mathcal{D}[0,T] \to H|\ f(.,.) \text{ is continuous}\}$.

Chapter 1

Stochastic Control Problem

This chapter will be organized as follows. In section 1, we give the classical control theory. In section 2, we study time inconsistent control problem.

1.1 Classical control problem

Optimal control theory is a branch of mathematical optimization that deals with finding a control for a dynamical system over a period of time such that an objective function is optimized. It has numerous applications in science, engineering, and operations research. The optimal control can be derived using one of two methods: Bellman's Dynamic Programming or Pontryagin's Maximum Principle.

1.1.1 Formulation of the control problem

In this subsection, we present two mathematical formulations (strong and weak formulations) of stochastic optimal control problems in the following two subsections, respectively.

Strong formulation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$, satisfying the usual condition, on which an d-dimensional standard Brownian motion $W(\cdot)$ is defined, denote by U the separable metric space and $T \in (0, +\infty)$ being fixed. Consider the following controlled stochastic differential equation

$$\begin{cases} dX(t) = b(t, X(t), u(t))dt + \sigma(t, X(t), u(t))dW(t), \\ X(0) = x_0 \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $b : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$ and $\sigma : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^{n \times d}$.

The function $u(\cdot)$ is called the control expressing the action of the decision-makers (controller). At any time instant the controller has some information (as specified by the information field $\{\mathcal{F}_t\}_{t \geq 0}$) of what has occurred up to that moment, but not able to predict what is going to happen afterwards due to the uncertainty of the system (as a consequence, for any t the controller cannot exercise his/her decision $u(t)$ before the time t really comes). This nonanticipative restriction in mathematical terms can be expressed as "$u(\cdot)$ is $\{\mathcal{F}_t\}_{t \geq 0}$ −adapted".

The control $u(\cdot)$ is an element of the set

$$\mathcal{U}[0, T] = \left\{ u : [0, T] \times \Omega \to U \text{ such that } u(\cdot) \text{ is } \{\mathcal{F}_t\}_{t \geq 0} - \text{adapted} \right\}.$$

We define the cost functional

$$J(u(\cdot)) = \mathbb{E}\left[\int_0^T f(t, X(t), u(t)) dt + h(X(T)) \right], \tag{1.2}$$

where $f : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}$.

Definition 1.1.1 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration $\{\mathcal{F}_t\}_{t \in [0,T]}$, satisfying the usual conditions and let $W(\cdot)$ be a given d-dimensional standard $\{\mathcal{F}_t\}_{t \in [0.T]}$-Brownian motion. A control $u(\cdot)$ is called an s-admissible control, and $(x(\cdot), u(\cdot))$ an s-admissible pair, if*

1. $u(\cdot) \in \mathcal{U}[0, T]$;
2. $x(\cdot)$ *is the unique solution of equation* (1.1);
3. $f(\cdot, X(\cdot), u(\cdot)) \in \mathcal{L}^1_{\mathcal{F}}(0, T; \mathbb{R})$ *and* $h(X(T)) \in \mathbb{L}^1(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$.

We denote by $\mathcal{U}^s_{ad}[0, T]$ the set of all admissible controls.

Our stochastic optimal control problem under strong formulation can be stated as follows:

Probleme 1.1.1 *Minimize* (1.2) *over $\mathcal{U}^s_{ad}[0, T]$. The goal is to find $\hat{u}(\cdot) \in \mathcal{U}^s_{ad}[0, T]$ (if it ever exists), such that*

$$J(\hat{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}^s_{ad}[0,T]} J(u(\cdot)). \tag{1.3}$$

Any $\hat{u}(\cdot) \in \mathcal{U}^s_{ad}[0, T]$ satisfying (1.3) is called an s-optimal control. The corresponding state process $\hat{X}(\cdot)$ and the state-control pair $\left(\hat{X}(\cdot), \hat{u}(\cdot)\right)$ are called an s-optimal state process and an s-optimal pair, respectively.

Weak formulation

We note that in the strong formulation the filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ along with the Brownian motion $W(\cdot)$ are all fixed, however it is not the case in the weak formulation, where we consider them as a parts of the control.

Definition 1.1.2 *A 6-tuple* $\pi = \left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, W(\cdot), u(\cdot)\right)$ *is called a w-admissible control, and* $(X(\cdot), u(\cdot))$ *a w-admissible pair, if*

1. $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ *is a filtered probability space satisfying the usual conditions*;
2. $W(\cdot)$ *is a d-dimensional standard Brownian motion defined on* $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$;
3. $u(\cdot)$ *is an* $\{\mathcal{F}_t\}_{t\geq 0}$*-adapted process on* $(\Omega, \mathcal{F}, \mathbb{P})$ *taking values in* U;
4. $X(\cdot)$ *is the unique solution of equation* (1.1) *on* $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ *under* $u(\cdot)$ *and some prescribed state constraints are satisfied*;
5. $f(\cdot, X(\cdot), u(\cdot)) \in \mathcal{L}^1_{\mathcal{F}}(0, T; \mathbb{R})$ *and* $h(X(T)) \in \mathbb{L}^1(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$. *The spaces* $\mathcal{L}^1_{\mathcal{F}}(0, T; \mathbb{R})$ *and* $\mathbb{L}^1(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ *are defined on the given filtered probability space* $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ *associated with the 6-tuple* π.

The set of all w-admissible controls is denoted by $\mathcal{U}^w_{ad}[0,T]$. Sometimes, might write $u(\cdot) \in \mathcal{U}^w_{ad}[0,T]$ instead of $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, W(\cdot), u(\cdot)\right) \in \mathcal{U}^w_{ad}[0,T]$.
Our stochastic optimal control problem under weak formulation can be stated as follows:

Probleme 1.1.2 *The objective is to minimize the cost functional given by equation* (1.2) *over the set of admissible controls* $\mathcal{U}^w_{ad}[0,T]$. *Namely, one seeks* $\hat{\pi}(\cdot) \in \mathcal{U}^w_{ad}[0,T]$ *such that*

$$J(\hat{\pi}(\cdot)) = \inf_{\pi(\cdot)\in\mathcal{U}^w_{ad}[0,T]} J(\pi(\cdot)).$$

1.1.2 Methods to solving optimal control problem

Two major methods for study an optimal control are Bellman's dynamic programming method and Pontryagin's maximum principle.

Dynamic Programming Method

In this subsection, we study an approach to solving optimal control problems, namely, the method of dynamic programming. Dynamic programming, originated by R. Bellman [7] in the early 1950's, is a mathematical method for making a sequence of interrelated decisions, which can be applied to numerous optimization problems (including optimal control problems). The basic idea of this technique applied to optimal controls is to consider a family of optimal control problems with different initial times and states, to establish relationships between these problems via the so-called Hamilton-Jacobi-Bellman equation (HJB, for short), which is a nonlinear first-order (in the deterministic case) or second- order (in the stochastic case) partial differential equation. If the HJB equation is solvable (either analytically or numerically), then one can get an optimal feedback control by taking the maximize/minimize of the Hamiltonian or generalized Hamiltonian involved in the HJB equation. This is the so-called verification technique. Mentioning that this approach actually solve a whole family of problems (with different initial times and states).

The Bellman principle Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$, satisfying the usual conditions, $T > 0$ a finite time, and W a d-dimensional Brownian motion defined on the filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}\right)$.
We consider the state stochastic differential equation

$$dX(s) = b(s, X(s), u(s))ds + \sigma(s, X(s), u(s))dW(s), s \in [0, T]. \tag{1.4}$$

The control $u = u(s)_{0\leq s\leq T}$ is a progressively measurable process valued in the control set $U \subset \mathbb{R}^k$, satisfies a square integrability condition. We denote by A the set of control processes u.
The Borelian functions $b : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$ and $\sigma : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^{n\times d}$ satisfying, for some constant $C > 0$ the following conditions:

$$|b(t, x, u) - b(t, y, u)| + |\sigma(t, x, u) - \sigma(t, y, u)| \leq C|x - y|, \tag{1.5}$$

$$|b(t, x, u)| + |\sigma(t, x, u)| \leq C\left[1 + |x|\right]. \tag{1.6}$$

Under (1.5) and (1.6) the SDE (1.4) has a unique solution x.
The cost functional associated with (1.4) is the following:

$$J(t, x, u) = \mathbb{E}^{t,x}\left[\int_t^T f(s, X(s), u(s))ds + h(X(T))\right], \tag{1.7}$$

where $\mathbb{E}^{t,x}$ is the expectation operator conditional on $X(t) = x$, $f : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}$, be given functions, we assume that

$$|f(t, x, u)| + |h(x)| \leq C\left[1 + |x|^2\right], \tag{1.8}$$

for some constant C. The quadratic growth condition (1.8), ensure that J is well defined.

The objective is to minimize the cost functional

$$V(t,x) = \inf_{u \in U} J(t,x,u), \text{ for } (t,x) \in [0,T] \times \mathbb{R}^n, \tag{1.9}$$

which is called the value function of the problem (1.4) and (1.7).

The dynamic programming is a fundamental principle in the theory of stochastic control, we present a version of the stochastic Bellman's principle of optimality.

Theorem 1.1.1 *Let $(t,x) \in [0,T] \times \mathbb{R}^n$ be given. Then we have*

$$V(t,x) = \inf_{u \in U} \mathbb{E}^{t,x} \left[\int_t^{t+h} f(s, X(s), u(s)) ds + V(t+h, X(t+h)) \right], \; for\ t \leq t+h \leq T. \tag{1.10}$$

Proof. The proof of the dynamic programming principle is found in the book by Yong and Zhou [69]. ■

The Hamilton-Jacobi-Bellman equation The HJB equation is the infinitesimal version of the dynamic programming principle. It is derived under smoothness assumptions on the value function.

We define the generalized Hamiltonian $\forall (t,x,u,p,P) \in [0,T] \times \mathbb{R}^n \times U \times \mathbb{R}^n \times \mathbb{R}^{n \times d}$

$$G(t,x,u,p,P) = \quad \tfrac{1}{2} tr \left(\sigma(t,x,u)\sigma(t,x,u)^\top P + b(t,x,u)^\top p + f(t,x,u), \right) \tag{1.11}$$

We also need to introduce the second-order infinitesimal generator $\mathcal{L}^u$ associated to the diffusion x with control u

$$\mathcal{L}^u \varphi(t,x) = b(t,x,u).D_x \varphi(t,x) + \frac{1}{2} tr \left(\sigma(t,x,u)\sigma(t,x,u)^\top D_x^2 \varphi(t,x) \right). \tag{1.12}$$

The classical HJB equation associated to the stochastic control problem (1.9) is

$$-V_t(t,x) - \inf_{u \in A} \left[\mathcal{L}^u V(t,x) + f(t,x,u) \right] = 0, \text{on } [0,T] \times \mathbb{R}^n. \tag{1.13}$$

We give sufficient conditions which enable to conclude that the smooth solution of the HJB equation coincides with the value function this is the so-called verification result.

Theorem 1.1.2 *Let W be a $C^{1,2}([0,T],\mathbb{R}^n) \cap C([0,T],\mathbb{R}^n)$ function. Assume that f and h are quadratic growth, i.e. there is a constant C such that*

$$|f(t,x,u)| + |h(x)| \leq C \left[1 + |x|^2 \right], \; for\ all\ (t,x,u) \in [0,T) \times \mathbb{R}^n \times U.$$

1. *Suppose that* $W(T,.) \leq h$, *and*

$$W_t(t,x) + G(t,x,W(t,x),D_xW(t,x),D_x^2W(t,x)) \geq 0, \tag{1.14}$$

on $[0,T) \times \mathbb{R}^n$, *then* $W \leq V$ *on* $[0,T) \times \mathbb{R}^n$.

2. *Assume further that* $W(T,\cdot) = h$, *and there exists a minimizer* $\hat{u}(t,x)$ *of* $\mathcal{L}^u V(t,x) + f(t,x,u)$, *such that*

$$\begin{aligned} 0 &= W_t(t,x) + G(t,x,W(t,x),D_xW(t,x),D_x^2W(t,x)) \\ &= W_t(t,x) + \mathcal{L}^{\hat{u}(t,x)}W(t,x) + f(t,x,u), \end{aligned} \tag{1.15}$$

the stochastic differential equation

$$dX(s) = b(s,X(s),\hat{u}(s,x))ds + \sigma(s,X(s),\hat{u}(s,x))dW(s),$$

defines a unique solution $X(t)$ *for each given initial data* $X(t) = x$, *and the process* $\hat{u}(s,x)$ *is a well-defined control process in* U. *Then* $W = V$, *and* $\hat{u}$ *is an optimal Markov control process.*

Proof. The proof of this verification theorem is found in the book by Yong and Zhou [69]. ■

Peng's maximum principle

In this subsection, we consider the stochastic maximum principle in stochastic control problems of systems governed by a SDE with controlled diffusion coefficient and also the control domain U is not necessarily convex.

Problem formulation and assumptions Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space with filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$, let $W(t)$ be an $\mathbb{R}^n$-valued standard Wiener process.
We assume that $\{\mathcal{F}_t\}_{t\in[0,T]} = \sigma(W(s), 0 \leq s \leq t)$. Consider the following stochastic controlled system:

$$\begin{cases} dX(t) = b(t,X(t),u(t))dt + \sigma(t,X(t),u(t))dW(t), \ t \in [0,T], \\ X(0) = x_0, \end{cases} \tag{1.16}$$

where $b : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$ and $\sigma : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^{n\times d}$. An admissible control u is an $\mathcal{F}_t$-adapted process with values in U such that

$$\sup_{0\leq t\leq T} \mathbb{E}\left|u(t)\right|^m < \infty, \quad m \geq 1,$$

where U is a nonempty subset of $\mathbb{R}^k$. We denote the set of all admissible controls by $\mathcal{U}$.

Our optimal control problem is to minimize the following cost functional over $\mathcal{U}$

$$J(u(\cdot)) = \mathbb{E}\left[\int_0^T f(t, X(t), u(t))dt + h(X(T))\right], \tag{1.17}$$

where $f : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}$.

Any $\hat{u}(\cdot) \in \mathcal{U}$ satisfying

$$J(\hat{u}(\cdot)) = \inf_{u\in\mathcal{U}} J(u(\cdot)), \tag{1.18}$$

is called an optimal control.

Consider the following assumptions about the coefficients b, σ, f and h.

(H1) b, σ, f and h are twice continuously differentiable with respect to x.

(H2) The derivatives $b_x, b_{xx}, \sigma_x, \sigma_{xx}, f_x, f_{xx}, h_x, h_{xx}$ are continuous in (x, u).

(H3) The derivatives $b_x, b_{xx}, \sigma_x, \sigma_{xx}, f_x, f_{xx}, h_x, h_{xx}$ are bounded and b, σ, f_x, h_x are bounded by $C(1 + |x| + |u|)$.

Adjoint equations and the maximum principle In this subsection we will introduce adjoint equations involved in a stochastic maximum principle and the associated stochastic Hamiltonian system.

The first-order adjoint equation satisfied by the processes $(p(\cdot), q(\cdot))$ as follows

$$\begin{cases} dp(t) = & -\Big\{ b_x(t, \hat{X}(t), \hat{u}(t))^\top p(t) + \sum_{j=1}^{d} \sigma_x^j(t, \hat{X}(t), \hat{u}(t))^\top q_j(t) \\ & -f_x(t, \hat{X}(t), \hat{u}(t)) \Big\} dt + q(t)\, dW(t), t \in [0.T] \\ p(T) = & -h_x(\hat{X}(T)). \end{cases} \tag{1.19}$$

The second-order adjoint equation satisfied by the processes $(P(\cdot), Q(\cdot))$ as follows

$$\begin{cases} dP(t) = & -\Big\{ b_x(t, \hat{X}(t), \hat{u}(t))^\top P(t) + P(t)\, b_x(t, \hat{X}(t), \hat{u}(t)) \\ & + \sum_{j=1}^{d} \sigma_x^j(t, \hat{X}(t), \hat{u}(t))^\top P(t)\, \sigma_x^j(t, \hat{X}(t), \hat{u}(t)) \\ & + \sum_{j=1}^{d} \sigma_x^j(t, \hat{X}(t), \hat{u}(t))^\top Q_j(t) + Q_j(t)\, \sigma_x^j(t, \hat{X}(t), \hat{u}(t)) \\ & + H_{xx}(t, \hat{X}(t), \hat{u}(t), p(t), q(t)) \Big\} dt + \sum_{j=1}^{d} Q_j(t)\, dW^j(t) \\ P(T) = & -h_{xx}(\hat{X}(T)), \end{cases} \tag{1.20}$$

where $H : [0,T] \times \mathbb{R}^n \times U \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \longrightarrow \mathbb{R}$ is the so-called Hamiltonian, which is given by

$$H(t, x, u, p, q) = \langle p, b(t, X, u)\rangle + tr[q^\top b(t, X, u)] - f(t, X, u). \tag{1.21}$$

Noting that under assumptions $(H1)-(H3)$, the equations (1.19) and (1.20) admis a unique solution $(p(\cdot),q(\cdot))\in\mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^n)\times\left(\mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^n)\right)^d$ and $(P(\cdot),Q(\cdot))\in\mathcal{L}^2_{\mathcal{F}}(0,T;S^n)\times\left(\mathcal{L}^2_{\mathcal{F}}(0,T;S^n)\right)^d$ respectively, where $S^n=\left\{A\in\mathbb{R}^{n\times n}/A^\top=A\right\}$.
Next, associated with an optimal 6-tuple $\left(\hat{X}(\cdot),\hat{u}(\cdot),p(\cdot),q(\cdot),P(\cdot),Q(\cdot)\right)$, we define an $\mathcal{H}$-function

$$\begin{aligned}\mathcal{H}(t,X,u)&=H(t,X,u,p(t),q(t))-\frac{1}{2}tr\left[\sigma(t,\hat{X}(t),\hat{u}(t))^\top P(t)\,\sigma(t,\hat{X}(t),\hat{u}(t))\right]\\&+\frac{1}{2}tr\left\{\left[\sigma(t,X,u)-\sigma(t,\hat{X}(t),\hat{u}(t))\right]^\top P(t)\left[\sigma(t,X,u)-\sigma(t,\hat{X}(t),\hat{u}(t))\right]\right\}.\end{aligned}\tag{1.22}$$

The stochastic maximum principle is given by the following theorem.

Theorem 1.1.3 *Let* $(H1)-(H3)$ *hold. Let* $\left(\hat{X}(\cdot),\hat{u}(\cdot)\right)$ *be an optimal pair. Then there are pairs of processes*

$$\begin{cases}(p(\cdot),q(\cdot))\in\mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^n)\times\left(\mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^n)\right)^d,\\(P(\cdot),Q(\cdot))\in\mathcal{L}^2_{\mathcal{F}}(0,T;S^n)\times\left(\mathcal{L}^2_{\mathcal{F}}(0,T;S^n)\right)^d,\end{cases}$$

satisfying the first-order and second-order adjoint equations (1.19) *and* (1.20), *respectively, such that*

$$\begin{aligned}&H(t,\hat{X}(t),\hat{u}(t),p(t),q(t))-H(t,\hat{X}(t),u,p(t),q(t))\\&-\frac{1}{2}tr\left\{\left[\sigma(t,\hat{X}(t),\hat{u}(t))-\sigma(t,\hat{X}(t),u)\right]^\top P(t)\left[\sigma(t,\hat{X}(t),\hat{u}(t))-\sigma(t,\hat{X}(t),u)\right]\right\}\\&\geq 0,\ \forall u\in U,\ a.e.\ t\in[0,T],\quad \mathbb{P}-a.s.,\end{aligned}\tag{1.23}$$

or, equivalently

$$\mathcal{H}(t,\hat{X}(t),\hat{u}(t))=\max_{u\in U}\mathcal{H}(t,\hat{X}(t),u),\ a.e.\ t\in[0,T],\quad \mathbb{P}-a.s.\tag{1.24}$$

The inequality (1.23) is called the variational inequality, and (1.24) is called the maximum condition.

Proof. The proof of this theorem is found in the book by Yong and Zhou [69]. ■

The system (1.16) along with its first-order adjoint system can be written as follows:

$$\begin{cases}dX(t)=H_p(t,X(t),u(t),p(t),q(t))dt+H_q(t,x(t),u(t),p(t),q(t))dW(t),\\dp(t)=-H_x(t,X(t),u(t),p(t),q(t))dt+q(t)dW(t),\quad t\in[0,T],\\X(0)=x,\\p(T)=-h_x(X(T)).\end{cases}\tag{1.25}$$

Definition 1.1.3 *The combination of* (1.25), (1.20) *and* (1.23) *(or* (1.24)*) is called an (extended) stochastic*

Hamiltonian system, with its solution being a 6-tuple $(X(\cdot), u(\cdot), p(\cdot), q(\cdot), P(\cdot), Q(\cdot))$.

Therefore, we can rephrase Theorem 1.1.3 as the following.

Theorem 1.1.4 *Let* $(H1)-(H3)$ *hold. Let* $\left(\hat{X}(\cdot), \hat{u}(\cdot)\right)$ *be an optimal pair. Then the optimal 6-tuple* $(X(\cdot), u(\cdot), p(\cdot), q(\cdot), P(\cdot), Q(\cdot))$ *solves the stochastic Hamiltonian system* (1.25), (1.20) *and* (1.23) (*or* (1.24))

1.2 Time-inconsistent problem

1.2.1 Introduction

In a standard dynamic programming problem setup, when a controler desires to optimize an objective function by choosing the best plan, he only has to decide his current action. This is because dynamic programming principle, also known as Belman's optimality principle, assumes that the future incarnations of the controler will solve the remaining part of today's problem and act optimally when future comes. However, the DPP does not hold in many problems, meaning that an optimal control chosen at some initial pair (of time and state) might not remain optimal as time goes. In such cases, the future incarnations of the controler may have modified preferences or tastes, or might prefer to make decisions based on different objective functions, essentially acting as opponents of the current self of the controler.

The dilemma stated above is called dynamic inconsistency, which has been observed and studied by economists for many years, especially in the context of non-exponential (or hyperbolic) type discount functions. In [57], Strotz proved that when a discount function was applied to consumption plans, one may favor a certain plan at first, but later change preference to another plan. This would hold true for vast types of discount functions, the only exception is the exponential. However, exponential discounting is the default setup in most literatures since none of the other types could give explicit solutions. Results from experimental researches contradict this assumption implying that the discount rates for the near future are much lower than the discount rates for the time further away in future, and therefore a discount function of the hyperbolic type would be more realistic; see, for example, Loewenstein and Prelec [41].

In addition to the non-exponential discounted utility maximization, the mean-variance optimization problems, presented by Markowitz [43], is another important exemple of time inconsistent problems. The idea of mean-variance criterion is that it uses variance to quantify the risk, which allows decision makers to achieve the highest return after evaluating their acceptable risk level. Nevertheless, due to the inclusion of a non-linear function of the expectation in the objective functional, the mean-variance criterion lacks the iterated expectation property. As a result, continuous-time and multi-period mean-variance problems are time-inconsistent.

There are other types of time inconsistency as well. In the literature, there have been listed three probable scenarios where time inconsistency would happen in stochastic continuous time control problems. More particularly, given an objective function of the following form

$$J\left(t,x,u\left(.\right)\right)=\mathbb{E}^{t}\left[\int_{t}^{T}\lambda\left(s-t\right)f\left(x,s,X\left(s\right),\mathbb{E}^{t}\left[X\left(s\right)\right],u\left(s\right)\right)ds+\lambda\left(T-t\right)h\left(x,X\left(T\right),\mathbb{E}^{t}\left[X\left(T\right)\right]\right)\right], \tag{1.26}$$

where $T>0$, $(t,x)\in[0,T]\times\mathbb{R}^{n}$, $\lambda\left(.\right)$, $f\left(.\right)$ and $h\left(.\right)$ are given functions, $u\left(s\right)\in U$ is the control action applied at time s, and $X\left(.\right)=X\left(.;u\left(.\right)\right)$ is some controlled state process which solves the following SDE, driven by a standard Brownian motion defined on some filtred probability space $\left(\Omega,\mathcal{F},\left\{\mathcal{F}_{t}\right\}_{t\geq0},\mathbb{P}\right)$

$$\begin{cases} dX\left(s\right)=b\left(s,X\left(s\right),\mathbb{E}^{t}\left[X\left(s\right)\right],u\left(s\right)\right)ds+\sigma\left(s,X\left(s\right),\mathbb{E}^{t}\left[X\left(s\right)\right],u\left(s\right)\right)dW\left(s\right),\ s\in\left[t,T\right], \\ X\left(t\right)=x, \end{cases} \tag{1.27}$$

the optimization for $J\left(t,x,.\right)$ is a time-inconsistent problem if:

- The discount function $\lambda\left(.\right)$ is not of exponential type, for example; a hyperbolic discount function;
- the coefficients b, σ, f and/or h are non linear functions of the marginal conditional probability law of the controlled state process, for example; mean field control problems;
- initial state x exists in the objective function, for example; a utility function that depends on the initial state x.

The mean-field models were first proposed to study the aggregate behavior of a huge number of mutually interacting particles in diverse areas of physical science, such as quantum mechanics, statistical mechanics and quantum chemistry. Roughly speaking, the mean-field models represent the complex interactions of individual “agents” (or particles) through a medium, specifically the mean-field term, which explains the action and reaction between the “agents”. In a recent study Lasry and Lions [34] extended the application of the mean-field models to finance and economics , where they considered N-player stochastic differential games, demonstrated the existence of the Nash equilibrium points and obtained rigorously the mean-field limit equations as N goes to infinity.

In all the three cases, stated above, the standard HJB equations cannot be obtained since the usual formulation needs an argument about the value function (process) being a supermartingale for arbitrary controls and being a martingale at optimum, which does not hold here.

1.2.2 An example of time-inconsistent optimal control problem

We present a simple illustration of stochastic optimal control problem which is time-inconsistent. Our aim is to show that the classical DPP approach is not efficient in the study of this problem. Let us consider the following controlled SDE starting from $(t, \xi) \in [0, T] \times \mathbb{R}$

$$\begin{cases} dX(s) = \left\{ r_0(s) X(s) + u(s)^\top B(s) \right\} ds + u(s)^\top D(s) dW^\star(s), \text{ for } s \in [t, T], \\ X(t) = \xi, \end{cases} \tag{1.28}$$

where $W^\star(s) = (0, W_1(s), ..., W_d(s))^\top$, $B(s) = \left(-1, r(s)^\top\right)^\top$ and $D(s) = \begin{pmatrix} 0 & 0_{\mathbb{R}^d}^\top \\ 0_{\mathbb{R}^d} & \sigma(s) \end{pmatrix}$, such that $r_0(\cdot)$, $r(\cdot)$ and $\sigma(\cdot)$ are deterministic functions. A stochastic process $u(\cdot) = (c(\cdot), u_1(\cdot), \ldots, u_d(\cdot))^\top$ is called a consumption-investment strategy, where $c(s)$ represents the consumption rate at time $s \in [0, T]$ and $u_i(s)$, for $i = 1, 2, .., d$, represents the amount invested in the i-th risky stock at time $s \in [0, T]$. The process $u_I(\cdot) = (u_1(\cdot), \ldots, u_d(\cdot))^\top$ is called an investment strategy.

The objective is to maximize the cost functional given by

$$J(t, \xi, u(\cdot)) = \mathbb{E}^t \left[\int_t^T \lambda(s-t) f\left(\Gamma^\top u(\cdot)\right) ds + \lambda(T-t) h(X(T)) \right], \tag{1.29}$$

where $\lambda(.) : [0, T] \to (0, \infty)$, is a general deterministic non-exponential discount function satisfying $\lambda(0) = 1$, $\lambda(s) \geq 0$ and $\int_0^T \lambda(t) dt < \infty$, $f(\cdot)$ and $h(\cdot)$ are the utility functions, $\Gamma = \left(1, 0_{\mathbb{R}^d}^\top\right)^\top$. In this illustation, we consider a logarithmic utility functions.

We assume that the financial market consists of one riskless asset and d risky assets. Arguing as in [23], we can demonstrate that, if the agent is naive and starts with a given positive wealth x, at some instant t, then by the standard dynamic programming approach, the value function associated with this stochastic control problem solves the following Hamilton–Jacobi–Bellman equation

$$\begin{cases} V_s^t(s,x) + \sup\limits_{(c,u_I) \in \mathbb{R}^{d+1}} \Big\{ \left(r_0(s) X(s) + u_I{}^\top r(s) - c\right) V_x^t(s,x) + \dfrac{1}{2} u_I^\top \sigma(s) \sigma(s)^\top u_I V_{xx}^t(s,x) \\ \qquad\qquad + \dfrac{\lambda'(s-t)}{\lambda(s-t)} V^t(s,x) + f(c) \Big\} = 0, \text{ for } s \in [t, T], \\ V^t(T, x) = h(x). \end{cases} \tag{1.30}$$

The HJB equation contains the term $\dfrac{\lambda'(s-t)}{\lambda(s-t)}$, which depends not only on the current time s but also on initial time t, thus, the optimal policy will depend on t as well. Indeed, the first order necessary conditions yield the $t-$optimal policy

$$\overline{u}_I^t(s) = r(s)\left(\sigma(s)\sigma(s)^\top\right)^{-1}\frac{V_x^t(s,x)}{V_{xx}^t(s,x)},$$
$$\overline{c}^t(s) = f^{-1}\left(V_x^t(s,x)\right).$$

Consider the following example: $f(x) = h(x) = \log x$. The naive agent for the initial pair $(0, x_0)$ solves the problem, supposing that the discount rate of time preference will be $\lambda(s)$, for $s \in [0,T]$, and the optimal consumption strategy will be

$$\overline{c}^{0,x_0}(s) = \left[1 + \int_s^T \exp\left\{\lambda(r-s) + \log\left(\frac{\lambda(r)}{\lambda(s)}\right)\right\} dr\right]^{-1}, \text{ for } s \in [0,T].$$

This solution corresponds to the so-called pre-commitment solution, meaning that it is optimal as long as the agent can precommit (by signing a contract, for example) his or her future behavior at time $t = 0$. If there is no commitment, the 0-agent will take the action $\overline{c}^{0,x_0}(s)$ but, in the near future, the ϵ-agent will change his decision rule (time-inconsistency) to the solution of the HJB equation (1.30) with $t = \epsilon$. In this case the optimal control trajectory for $s > \epsilon$ will be changed to $\overline{c}^{\epsilon,x_\epsilon}(s)$ given by

$$\overline{c}^{\epsilon,x_\epsilon}(s) = \overline{c}^{\epsilon,\bar{X}(\epsilon)}(s) = \left[1 + \int_s^T \exp\left\{\lambda(r-s) + \log\left(\frac{\lambda(r-\epsilon)}{\lambda(s-\epsilon)}\right)\right\} dr\right]^{-1}, \text{ for } s \in [\epsilon,T].$$

If $\lambda(t) = e^{-\delta t}$ where $\delta > 0$ is the constant discount rate, then

$$\overline{c}^{0,x_0}_{|[\epsilon,T]}(s) = \overline{c}^{\epsilon,x_\epsilon}(s), \text{ for } s \in [\epsilon,T],$$

thus the optimal consumption plan is time consistent. Once the discount function is non-exponential

$$\overline{c}^{0,x_0}_{|[\epsilon,T]}(s) \neq \overline{c}^{\epsilon,x_\epsilon}(s), \text{ for } s \in [\epsilon,T].$$

So the optimal consumption plan is not time consistent. generally, the solution for the naive agent will be obtained by solving the family of HJB equations (1.30) for $t \in [0,T]$, and patching together the "optimal" solutions $\overline{c}^{t,x_t}(t)$. If the agent is sophisticated, things become more complicated. The standard HJB equation cannot be used to derive the solution.

1.2.3 Approaches to handle time inconsistency

Given the inapplicability of standard DPP on these problems, there are three approaches of handling (various forms of) time inconsistency in optimal control problems.

Pre-committed optimal strategies

One possibility is to investigate the pre-committed problem: we fix one initial point, such as $(0, x_0)$, and then try to find the control process $\bar{u}(.)$ that optimizes $J(0, x_0, .)$. We then simply ignore the fact that at a later points in time like as $(s, X(s; 0, x_0, \bar{u}(.)))$ the control $\bar{u}(.)$ will not be optimal for the functional $J(s, X(s; \bar{u}(.)), .)$. Kydland and Prescott [33] argue that a pre-committed strategy may be economically meaningful in certain cases. In the context of MV optimization problem, pre-committed optimal solution have been widely investigated in different situations. [53] is probably the first paper that studies a pre-committed MV model in a continuous-time setting (although he only considers one single stock with a constant risk-free rate), followed by [5]. In a discrete-time setting, [37] developed an embedding approach to change the originally time-inconsistent MV problem into a stochastic LQ control problem. This approach was extended in [73], together with an indefinite stochastic linear– quadratic control approach, to the continuous-time case. Further extensions and modifications are carried out in, among many others, [35] and [8]. Markowitz's problem with transaction cost has recently solved in [18]. For general mean field control problems, Andersson & Djehiche [3] and Li [36] proposed a mean field type stochastic maximum principle to characterize "pre-commited" optimal control when both the state dynamics and the cost functional are of a mean-field type. The linear-quadratic optimal control problem for mean-field SDEs has been investigated by Yong [67]. The maximum principle for a jump-diffusion mean-field model have been studied in Shen and Siu [55].

Game theoretic approach

We use the game theoretic approach to handle the time inconsistency in the identical viewpoint as Ekeland et al. [23] and Bjork and Murgoci [9]. Let us briefly explain the game perspective that we will consider as follows:

- We consider a game with one player at every point t in the interval $[0, T)$. This player corresponds to the incarnation of the controller on instant t and referred to "player t".
- The $t-th$ player can control the scheme just at time t by taking his/her policy $u(t, \cdot) : \Omega \to \mathbb{R}^m$.

- A control process $u(\cdot)$ is then viewed as a complete explanation of the selected strategies of all players in the game.
- The reward to the player t is specified by the functional $J(t, \xi; u(\cdot))$.

We give the concept of a "Nash equilibrium strategy" of the game for the above description: This is an admissible control process $\hat{u}(\cdot)$ fulfilling the following criteria. Suppose that every player s, with $s > t$, will apply the strategy $\hat{u}(s)$. Then the optimal decision for player t is that, he/she also uses the strategy $\hat{u}(t)$. However, the difficulty with this "definition", is that the individual player t does not have any effect on the result of the game. He/she just chooses the control at one point t, Furthermore, because this is a time set of Lebesgue measure zero, the control dynamics will not be affected. As a result, to identify open-loop Nash equilibrium controls, we follow [29], who propose the formal definition inspired by [23].

The dynamic optimality approach

alternative approach has been proposed by Pedersen and Peskir [48] for the mean-variance portfolio selection problem, called the dynamically optimal strategy. This is a novel approach to treating time inconsistency, although related work can be found in Karnam et al. [31]. The strategy introduced by Pedersen & Peskir [48] is time-consistent in the meaning that it does not depend on initial time and initial state variable, but varies from the subgame perfect equilibrium strategy. Furthermore, their policy is intuitive and formalizes a quite natural approach to time inconsistency: it describes the behavior of an optimizer who continuously revaluates his position and solves infinitely number of problems in an instantaneously optimal way. The dynamically optimal individual is similar to the continuous version of the naive individual presented by Pollak [51]. At each time t the dynamically optimal investor is the “reincarnation” of the precommitted investor, for at time t he plays the same strategy that the time-t precommitted investor would play, forgets about his past and ignores his future, and deviates from it instantly after, by wearing the clothes of the time t+ precommitted investor. Noting that the dynamically optimal approach has similarities also with the receding horizon procedure or the model predictive control — known as rolling horizon procedures, see Powell [52]— that are well established approaches of repeated optimization along a rolling horizon for engineering optimization problems with an infinite time horizon.

Chapter 2

Conditional LQ time-inconsistent Markov-switching stochastic optimal control problem for diffusion with jumps

In this chapter, we present a general time-inconsistent stochastic conditional LQ control problem. Different from most current studies [29], [68], [72], where the noise is driven by a Brownian motion, in our LQ system the state develops according to a SDE, in which the noise is driven by a multidimensional Brownian motion and an independent multidimensional Poisson point process under a Markov regime-switching setup. Cases of continuous time mean-variance criterion with state-dependent risk aversion are included in the objective function. We establish a stochastic system that describes open loop Nash equilibrium controls, using the variational technique proposed by Hu et al [28]. We emphasize that our model generalizes the ones investigated by Zeng and Li [70], Li et al [38] and Sun and Guo [59], in addition to some classes of time-inconsistent stochastic LQ optimal control problems introduced in [29].

The chapter is organized as follows: in the first section, we formulate the problem and provide essential notations and preliminaries. Section 2 is dedicated to presenting the necessary and sufficient conditions for equilibrium, which is our main result, and we get the unique equilibrium control in state feedback representation through a specific category of ordinary differential equations. In section 3, we apply the results of section 2 to find the unique equilibrium reinsurance, investment and consumption strategies for the mean-variance-utility portfolio problem, as well as discuss some special cases. In the last section, we present some fundamental results on SDEs and BSDEs with jumps that we have used in this chapter.

2.1 Problem setting

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space where $\mathbb{F} := \{\mathcal{F}_t | \, t \in [0,T]\}$ is a right-continuous, $\mathbb{P}$-completed filtration to which all of the processes outlined below are adapted, such as the Markov chain, the Brownian motions, and the Poisson random measures.

During the present chapter, we assume that the Markov chain $\alpha(\cdot)$ takes values in finite state space $\chi = \{e_1, e_2, ..., e_d\}$ where $d \in \mathbb{N}$, $e_i \in \mathbb{R}^d$ and the j-th component of e_i is the Kronecker delta δ_{ij} for each $(i,j) \in \{1, ..., d\}^2$. $\mathcal{H} := (\lambda_{ij})_{1 \leq i,j \leq d}$ represents the rate matrix of the Markov chain under $\mathbb{P}$. Note that, λ_{ij} is the constant transition intensity of the chain from state e_i to state e_j at time t , for each $(i,j) \in \{1, ..., d\}^2$. As a result for, $i \neq j$, we have $\lambda_{ij} \geq 0$ and $\sum_{j=1}^{d} \lambda_{ij} = 0$, thus $\lambda_{ii} \leq 0$. In the sequel, for each $i, j = 1, 2, ..., d$ with $i \neq j$, we assume that $\lambda_{ij} > 0$ consequently, $\lambda_{ii} < 0$. We have the following semimartingale representation of the Markov chain $\alpha(\cdot)$ obtained from Elliott et al. [24]

$$\alpha(t) = \alpha(0) + \int_0^t \mathcal{H}^\top \alpha(\tau) d\tau + \mathcal{M}(t),$$

where $\{\mathcal{M}(t) | t \in [0,T]\}$ is an $\mathbb{R}^d$-valued, $(\mathbb{F}, \mathbb{P})$-martingale.

First, we provide a set of Markov jump martingales linked with the chain $\alpha(\cdot)$, which will be used to model the controlled state process. For each $(i,j) \in \{1, ..., d\}^2$, with $i \neq j$, and $t \in [0,T]$, denote by $J^{ij}(t) := \lambda_{ij} \int_0^t \langle \alpha(\tau-), e_i \rangle d\tau + m_{ij}(t)$ the number of jumps from state e_i to state e_j up to time t, where $m_{ij}(t) := \int_0^t \langle \alpha(\tau-), e_i \rangle \langle d\mathcal{M}(\tau), e_j \rangle d\tau$ an $(\mathbb{F}, \mathbb{P})$-martingale. $\Phi_j(t)$ denotes the number of jumps into state e_j up to time t, for each fixed $j = 1, 2, ..., d$, then

$$\begin{aligned} \Phi_j(t) &= \sum_{i=1, i \neq j}^{d} J^{ij}(t) \\ &= \sum_{i=1, i \neq j}^{d} \lambda_{ij} \int_0^t \langle \alpha(\tau-), e_i \rangle d\tau + \tilde{\Phi}_j(t), \end{aligned}$$

with $\tilde{\Phi}_j(t) := \sum_{i=1, i \neq j}^{d} m_{ij}(t)$ is an $(\mathbb{F}, \mathbb{P})$-martingale for each $j = 1, 2, ..., d$. Set for each $j = 1, 2, ..., d$

$$\lambda_j(t) = \sum_{i=1, i \neq j}^{d} \lambda_{ij} \int_0^t \langle \alpha(\tau), e_i \rangle d\tau.$$

Noting that, the process $\tilde{\Phi}_j(t) = \Phi_j(t) - \lambda_j(t)$ is an $(\mathbb{F}, \mathbb{P})$-martingale, for each $j = 1, 2, ..., d$.

Now, we present the Markov regime-switching Poisson random measures. Assume that $N_i(dt, dz)$, $i = 1, 2, ..., l$ are independent Poisson random measures on $([0,T] \times \mathbb{R}_0, \mathcal{B}([0,T]) \otimes \mathcal{B}_0)$ under $\mathbb{P}$. Assume that the compensator for the Poisson random measure $N_i(dt, dz)$ is defined by

$$n_\alpha^i(dt, dz) := \theta_{\alpha(t-)}^i(dz)dt = \left\langle \alpha(t-), \theta^i(dz) \right\rangle dt,$$

in which $\theta^i(dz) := \left(\theta_{e_1}^i(dz), \theta_{e_2}^i(dz), ..., \theta_{e_d}^i(dz)\right)^\top \in \mathbb{R}^d$. The subscript α in n_α^i, for $i = 1, 2, ..., l$ represents the dependence of the probability law of the Poisson random measure on the Markov chain $\alpha(\cdot)$. In fact $\theta_{e_j}^i(dz)$ is the conditional Lévy density of jump sizes of the random measure $N_i(dt, dz)$ at time t when $\alpha(t-) = e_j$, for each $j = 1, 2, ..., d$. Furthermore, the compensated Poisson random measure $\tilde{N}_\alpha(dt, dz)$ is given by

$$\tilde{N}_\alpha(dt, dz) = \left(N_1(dt, dz) - n_\alpha^1(dt, dz), ..., N_l(dt, dz) - n_\alpha^l(dt, dz)\right)^\top.$$

2.1.1 Assumptions and problem formulation

Throughout this chapter, we consider a multi-dimensional non homogeneous linear controlled jump-diffusion system starting from the situation $(t, \xi, e_i) \in [0,T] \times \mathbb{L}^2(\Omega, \mathcal{F}_t^\alpha, \mathbb{P}; \mathbb{R}^n) \times \chi$, by

$$\begin{cases} dX(s) = \{A(s, \alpha(s)) X(s) + B(s, \alpha(s)) u(s) + b(s, \alpha(s))\} ds \\ \qquad + \sum_{i=1}^{p} \{C_i(s, \alpha(s)) X(s) + D_i(s, \alpha(s)) u(s) + \sigma_i(s, \alpha(s))\} dW^i(s) \\ \qquad + \sum_{k=1}^{l} \int_{\mathbb{R}^*} \{E_k(s, z, \alpha(s)) X(s-) + F_k(s, z, \alpha(s)) u(s) \\ \qquad + c_k(s, z, \alpha(s))\} \tilde{N}_\alpha^k(ds, dz), \ s \in [t, T], \\ X(t) = \xi, \ \alpha(t) = e_i. \end{cases} \tag{2.1}$$

The coefficients $A(\cdot,\cdot), C_i(\cdot,\cdot) : [0,T] \times \chi \to \mathbb{R}^{n\times n}$; $B(\cdot,\cdot), D_i(\cdot,\cdot) : [0,T] \times \chi \to \mathbb{R}^{n\times m}$; $b(\cdot,\cdot), \sigma_i(\cdot,\cdot) : [0,T] \times \chi \to \mathbb{R}^n$; $E_k(\cdot,\cdot,\cdot) : [0,T] \times \mathbb{R}^* \times \chi \to \mathbb{R}^{n\times n}$; $F_k(\cdot,\cdot,\cdot) : [0,T] \times \mathbb{R}^* \times \chi \to \mathbb{R}^{n\times m}$; $c_k(\cdot,\cdot,\cdot) : [0,T] \times \mathbb{R}^* \times \chi \to \mathbb{R}^n$ are deterministic matrix-valued functions. Here, for any $t \in [0,T)$, the class of admissible control processes over $[t,T)$ is restricted to $\mathcal{L}_{\mathcal{F},p}^2(t,T;\mathbb{R}^m)$. For any $u(.) \in \mathcal{L}_{\mathcal{F},p}^2(t,T;\mathbb{R}^m)$ we denote by $X(\cdot) = X^{t,\xi,e_i}(\cdot; u(\cdot))$ its solution. Different controls $u(\cdot)$ will lead to different solutions $X(\cdot)$.

Remark 2.1.1 *In practice, the observable switching process is followed to represent the interest rate processes over various market settings. For example, the market may be generally split into "bullish" and "bearish" states, with characteristics varying greatly between the two modes. The application of switching* *model in mathematical finance can be discovered, for example, in [13], [14] and references therein. To measure the performance of* $u(\cdot) \in \mathcal{L}^2_{\mathcal{F},p}(t,T;\mathbb{R}^m)$, *we introduce the following cost functional*

$$\begin{aligned} & J(t,\xi,e_i;u(\cdot)) \\ & = \mathbb{E}\Big[\int_t^T \frac{1}{2}\left\{\langle Q(s)X(s),X(s)\rangle + \langle \bar{Q}(s)\mathbb{E}[X(s)|\mathcal{F}^\alpha_s],\mathbb{E}[X(s)|\mathcal{F}^\alpha_s]\rangle\right. \\ & \quad \left. + \langle R(t,s)u(s),u(s)\rangle\right\} ds + \langle \mu_1\xi+\mu_2, X(T)\rangle + \frac{1}{2}\langle GX(T),X(T)\rangle \\ & \quad + \frac{1}{2}\langle \bar{G}\mathbb{E}[X(T)|\mathcal{F}^\alpha_T],\mathbb{E}[X(T)|\mathcal{F}^\alpha_T]\rangle\Big]. \end{aligned} \tag{2.2}$$

Remark 2.1.2 *Due to the general influence of the modulating switching process* $\alpha(\cdot)$, *the conditional expectation is employed rather than the expectation in (2.2). The presence of* $\alpha(\cdot)$ *in all coefficients of the state equation (2.1) can be makes the objective functional depends on the process's history. This type of cost functional is also motivated from practical problems such as conditional mean-variance portfolio selection problem which is considered in the section 3 in this chapter. A reader interested in this type of problems is refer to [50] and [45]. The term* $\langle \mu_1\xi+\mu_2, X(T)\rangle$ *stems from a state-dependent utility function in economics [10].*

We need to impose the following assumptions about the coefficients.

(H1) The functions $A(\cdot,\cdot)$, $B(\cdot,\cdot)$, $b(\cdot,\cdot)$, $C_i(\cdot,\cdot)$, $D_i(\cdot,\cdot)$, $\sigma_i(\cdot,\cdot)$, $E_k(\cdot,\cdot,\cdot)$, $F_k(\cdot,\cdot,\cdot)$ and $c_k(\cdot,\cdot,\cdot)$ are deterministic, continuous and uniformly bounded. The coefficients on the cost functional satisfy

$$\begin{cases} Q(\cdot), \bar{Q}(\cdot) \in C([0,T];S^n), \\ R(\cdot,\cdot) \in C(\mathcal{D}[0,T];S^m), \\ G, \bar{G} \in S^n, \mu_1 \in \mathbb{R}^{n\times n}, \mu_2 \in \mathbb{R}^n. \end{cases}$$

(H2) The functions $R(\cdot,\cdot), Q(\cdot)$ and G satisfy $R(t,t) \geq 0$, $Q(t) \geq 0$, $\forall t \in [0,T]$ and $G \geq 0$.

Based on [54] we can prove under **(H1)** that, for any $(t,\xi,e_i,u(\cdot)) \in [0,T] \times \mathbb{L}^2(\Omega,\mathcal{F}^\alpha_t,\mathbb{P};\mathbb{R}^n) \times \mathcal{X} \times \mathcal{L}^2_{\mathcal{F},p}(t,T;\mathbb{R}^m)$, the state equation (2.1) has a unique solution $X(\cdot) \in \mathcal{S}^2_{\mathcal{F}}(t,T;\mathbb{R}^n)$. Moreover, we have the following estimate

$$\mathbb{E}\left[\sup_{t\leq s\leq T}|X(s)|^2\right] \leq K\left(1+\mathbb{E}\left[|\xi|^2\right]\right), \tag{2.3}$$

for some positive constant K. In particular for $t=0$ and $u(\cdot) \in \mathcal{L}^2_{\mathcal{F},p}(0,T;\mathbb{R}^m)$, equation (2.1) starting from initial data $(0, x_0)$ and has a unique solution $X(\cdot) \in \mathcal{S}^2_{\mathcal{F}}(0, T; \mathbb{R}^n)$ with the following estimate holds

$$\mathbb{E}\left[\sup_{0 \leq s \leq T} |X(s)|^2\right] \leq K\left(1 + |x_0|^2\right). \tag{2.4}$$

Our optimal control problem can be formulated as follows.

Problem (N). *For any initial pair* $(t, \xi, e_i) \in [0, T] \times \mathbb{L}^2(\Omega, \mathcal{F}^\alpha_t, \mathbb{P}; \mathbb{R}^n) \times \chi$, *find a control* $\hat{u}(\cdot) \in \mathcal{L}^2_{\mathcal{F},p}(t, T; \mathbb{R}^m)$ *such that*

$$J(t, \xi, e_i; \hat{u}(\cdot)) = \min_{u(.) \in \mathcal{L}^2_{\mathcal{F},p}(t,T;\mathbb{R}^m)} J(t, \xi, e_i; u(\cdot)).$$

Any $\hat{u}(\cdot) \in \mathcal{L}^2_{\mathcal{F},p}(t, T; \mathbb{R}^m)$ satisfying the above is called a *pre-commitment optimal control.* Furthermore, the presence of some quadratic terms of the conditional expectation of the state process as well as a state-dependent term in the objective functional destroy the time-consistency of a pre-committed optimal solutions of problem (N). Hence, problem (N) is time-inconsistent and there are two different sources of time inconsistency.

2.2 The Main results: Characterization and uniqueness of equilibrium

In view of the fact that Problem (N) is time-inconsistent, the reason of this section is to characterize open-loop Nash equilibriums as an alternative of optimal strategies. We apply the game theoretic approach to handle the time inconsistency in the same viewpoint as Ekeland et al. [23] and Bjork and Murgoci [9].

Remark 2.2.1 *In the rest of the chapter, for brevity, we suppress the subscript* $(s, \alpha(s))$ *for the coefficients* $A(s, \alpha(s)), B(s, \alpha(s)), b(s, \alpha(s)), C_i(s, \alpha(s)), D_i(s, \alpha(s)), \sigma_i(s, \alpha(s))$, *in addition we suppress the subscripts* (s) *and* (s, t) *for the coefficients* $Q(s)$, $\bar{Q}(s)$, $R(s, t)$ *and we use the notation* $\varrho(z)$ *instead of* $\varrho(s, z, \alpha(s))$ *for* $\varrho = E_k, F_k$ *and* c_k. *Furthermore, sometimes we simply call* $\hat{u}(\cdot)$ *an equilibrium control instead of calling it an open-loop Nash equilibrium control, when there is no confusion.*

In this section, we provide the main results about the necessary and sufficient conditions for equilibrium of the control problem formulated in the preceding section. To proceed, we start with the definition of an equilibrium by local spike variation, for a given admissible control $\hat{u}(.) \in \mathcal{L}^2_{\mathcal{F},p}(t, T; \mathbb{R}^m)$, for any

$t\in[0,T)$, $v\in\mathbb{L}^{2}\left(\Omega,\mathcal{F}_{t-}^{\alpha},\mathbb{P};\mathbb{R}^{m}\right)$ and $\varepsilon\in(0,T-t)$, define

$$u^{\varepsilon}(s)=\begin{cases}\hat{u}(s)+v, & \text{for } s\in[t,t+\varepsilon),\\ \hat{u}(s), & \text{for } s\in[t+\varepsilon,T).\end{cases} \tag{2.5}$$

We have the following definition.

Definition 2.2.1 (Open-loop Nash equilibrium) *An admissible control $\hat{u}(.)\in\mathcal{L}_{\mathcal{F},p}^{2}(t,T;\mathbb{R}^{m})$ is an open-loop Nash equilibrium control for Problem (N) if for every sequence $\varepsilon_n\downarrow 0$, we have*

$$\lim_{\varepsilon_n\downarrow 0}\frac{1}{\varepsilon_n}\left\{J\left(t,\hat{X}(t),\alpha(t);u^{\varepsilon_n}(.)\right)-J\left(t,\hat{X}(t),\alpha(t);\hat{u}(\cdot)\right)\right\}\geq 0, \tag{2.6}$$

for any $t\in[0,T]$ and $v\in\mathbb{L}^{2}\left(\Omega,\mathcal{F}_{t-}^{\alpha},\mathbb{P};\mathbb{R}^{m}\right)$. The corresponding equilibrium dynamics solves the following SDE with jumps, for $s\in[0,T]$,

$$\begin{cases} d\hat{X}(s)=\left\{A\hat{X}(s)+B\hat{u}(s)+b\right\}ds\\ \qquad\qquad +\sum_{i=1}^{p}\left\{C_i\hat{X}(s)+D_i\hat{u}(s)+\sigma_i\right\}dW^{i}(s)\\ \qquad\qquad +\sum_{k=1}^{l}\int_{\mathbb{R}^*}\left\{E_k(z)\hat{X}(s-)+F_k(z)\hat{u}(s)+c_k(z)\right\}\widetilde{N}_{\alpha}^{k}(ds,dz),\\ \hat{X}_0=x_0,\alpha(0)=e_{i_0}.\end{cases}$$

As the coefficients are affected by random Markov switching and since we consider a family of a continuous of random variables (conditional expectations) parametrized by $\varepsilon>0$, the limit in (2.6) is taken with any sequence (ε_n) tending to 0, not ε tending to 0, see the definition 2.2.1. Due to the uncountable property of $\varepsilon>0$, the a.s. limit with respect to the whole $\varepsilon>0$ may not make sence and this is the reason of using ε_n instead. We should consider a subsequence for the limit procedures in the proofs. To do so, we use the following lemma.

Lemma 2.2.1 *If $f(\cdot)=(f_1(\cdot),...,f_m(\cdot))\in\mathcal{L}_{\mathcal{F}}^{p}(0,T;\mathbb{R}^{m})$ with $m\in\mathbb{N}$ and $p>1$, then for $dt-a.e.$, there exists a sequence $\{\varepsilon_n^t\}_{n\in\mathbb{N}}\subset(0,T-t)$ depending on t such that $\underset{n\to\infty}{lim}\,\varepsilon_n^t=0$ and*

$$\lim_{n\to\infty}\frac{1}{\varepsilon_n^t}\mathbb{E}\left[\int_t^{t+\varepsilon_n^t}|f_i(s)-f_i(t)|^p\,ds\right]=0,\ \text{for } i=1,...,m,\ d\mathbb{P}-a.s.$$

Proof. See Wang in [61], Lemma 3.3. ■

2.2.1 Flow of the adjoint equations and characterization of equilibrium controls

In this subsection, we provide a general necessary and sufficient conditions to characterize the equilibrium strategies of Problem (N). First, we consider the adjoint equations used within the characterisation of equilibrium controls. Let $\hat{u}(\cdot) \in \mathcal{L}^2_{\mathcal{F},p}(t,T;\mathbb{R}^m)$ be a fixed control and denote by $\hat{X}(\cdot) \in \mathcal{S}^2_{\mathcal{F}}(0,T;\mathbb{R}^n)$ its corresponding state process. For each $t \in [0,T]$, the first order adjoint equation defined on the time interval $[t,T]$ and satisfied by the 4-tuple of processes $(p(\cdot;t), q(\cdot;t), r(\cdot,\cdot;t), l(\cdot;t))$ is considered as follows:

$$\begin{cases} dp(s;t) = -\Big\{A^\top p(s;t) + \sum_{i=1}^{p} C_i^\top q_i(s;t) \\ \qquad + \sum_{k=1}^{l} \int_{\mathbb{R}^*} E_k(z)^\top r_k(s,z;t)\,\theta_\alpha^k(dz) - Q\hat{X}(s) - \bar{Q}\mathbb{E}\left[\hat{X}(s)\,|\mathcal{F}_s^\alpha\right]\Big\} ds \\ \qquad + \sum_{i=1}^{p} q_i(s;t)\,dW^i(s) + \sum_{k=1}^{l} \int_{\mathbb{R}^*} r_k(s,z;t)\,\widetilde{N}_\alpha^k(ds,dz) \\ \qquad + \sum_{j=1}^{d} l_j(s,t)\,d\tilde{\Phi}_j(s),\ \ s \in [t,T], \\ p(T;t) = -G\hat{X}(T) - \bar{G}\mathbb{E}\left[\hat{X}(T)\,|\mathcal{F}_T^\alpha\right] - \mu_1\hat{X}(t) - \mu_2. \end{cases} \tag{2.7}$$

Through this section, we will prove that we can get the equilibrium strategy by solving a system of FBSDEs which is not standard since the flow of the unknown process $(p(\cdot;t), q(\cdot;t), r(\cdot,\cdot;t), l(\cdot;t))$ for $t \in [0,T]$ is involved. To the best of our knowledge the explicit ability to solve this type of equation remains an open problem, except for a certain form of the objective function. However, by the separating variables approach we are able to completely solve this problem.

Lemma 2.2.2 *Consider the deterministic matrix-valued function $\phi(\cdot,\cdot)$ as the solution of the following ODE*

$$\begin{cases} d\phi(s,\alpha(s)) = \phi(s,\alpha(s))\,A^\top ds,\ \ s \in [0,T], \\ \phi(T,e_i) = I_n. \end{cases}$$

For any $t \in [0,T]$ and $s \in [t,T]$ the solution of the equation (2.7) have the following representation

$$\begin{aligned} p(s;t) &= -\phi(s,\alpha(s))^{-1}\left(\bar{p}(s) + \bar{G}\mathbb{E}\left[\hat{X}(T)\,|\mathcal{F}_T^\alpha\right] + \mu_1\hat{X}(t) + \mu_2\right) \\ &\quad - \phi(s,\alpha(s))^{-1}\int_s^T \phi(\tau,\alpha(\tau))\,\bar{Q}\mathbb{E}\left[\hat{X}(\tau)\,|\mathcal{F}_\tau^\alpha\right] d\tau, \end{aligned}$$

and $(q_i(s;t), r_k(s,z;t), l_j(s;t)) = -\phi(s,\alpha(s))^{-1}\left(\bar{q}_i(s), \bar{r}_k(s,z), \bar{l}_j(s)\right)$ *for* $i = 1,2,...,p;\ k = 1,2,...,l;$

$j=1,2,...,d,$ *where*

$$\begin{cases} d\bar{p}(s)=-\Big\{\sum_{i=1}^{p}\phi(s,\alpha(s))C_i^{\top}\phi(s,\alpha(s))^{-1}\bar{q}_i(s) \\ \qquad +\sum_{k=1}^{l}\int_{\mathbb{R}^*}\phi(s,\alpha(s))E_k(z)^{\top}\phi(s,\alpha(s))^{-1}\bar{r}_k(s,z)\theta_\alpha^k(dz) \\ \qquad +\phi(s,\alpha(s))Q\hat{X}(s)\Big\}ds+\sum_{i=1}^{p}\bar{q}_i(s)dW^i(s) \\ \qquad +\sum_{k=1}^{l}\int_{\mathbb{R}^*}\bar{r}_k(s-,z)\tilde{N}_\alpha^k(ds,dz)+\sum_{j=1}^{d}\bar{l}_j(s)d\tilde{\Phi}_j(s),\ \ s\in[t,T], \\ \bar{p}(T)=G\hat{X}(T). \end{cases} \tag{2.8}$$

Proof. It is clear that $\phi(s,\alpha(s))$ is invertible for $\forall s\in[0,T]$, we denote by $\phi(s,\alpha(s))^{-1}$ the inverse of $\phi(s,\alpha(s))$. Define for $t\in[0,T]$ and $s\in[t,T]$ the process

$$\bar{p}(s;t)\equiv-\phi(s,\alpha(s))p(s;t)-\bar{G}\mathbb{E}\left[\hat{X}(T)|\mathcal{F}_T^\alpha\right]-\mu_1\hat{X}(t)-\mu_2 \\ -\int_s^T\phi(\tau,\alpha(\tau))\bar{Q}\mathbb{E}\left[\hat{X}(\tau)|\mathcal{F}_\tau^\alpha\right]d\tau,$$

and $\left(\bar{q}_i(s;t),\bar{r}_k(s,z;t),\bar{l}_j(s;t)\right)=-\phi(s,\alpha(s))\left(q_i(s;t),r_k(s,z;t),l_j(s;t)\right)$, for $i=1,2,...,p;k=1,2,...,l$ and $j=1,2,...,d$. Then for any $t\in[0,T]$, in the interval $[t,T]$, the 4-tuple $\left(\bar{p}(\cdot;t),\bar{q}(\cdot;t),\bar{r}(\cdot,\cdot;t),\bar{l}(\cdot;t)\right)$ satisfies

$$\begin{cases} d\bar{p}(s;t)=-\Big\{\sum_{i=1}^{p}\phi(s,\alpha(s))C_i^{\top}\phi(s,\alpha(s))^{-1}\bar{q}_i(s;t) \\ \qquad +\sum_{k=1}^{l}\int_{\mathbb{R}^*}\phi(s,\alpha(s))E_k(z)^{\top}\phi(s,\alpha(s))^{-1}\bar{r}_k(s,z;t)\theta_\alpha^k(dz) \\ \qquad +\phi(s,\alpha(s))Q\hat{X}(s)\Big\}ds+\sum_{i=1}^{p}\bar{q}_i(s;t)dW^i(s) \\ \qquad +\sum_{k=1}^{l}\int_{\mathbb{R}^*}\bar{r}_k(s-,z;t)\tilde{N}_\alpha^k(ds,dz)+\sum_{j=1}^{d}\bar{l}_j(s,t)d\tilde{\Phi}_j(s), \\ \bar{p}(T;t)=G\hat{X}(T). \end{cases} \tag{2.9}$$

Moreover, it is clear that for any $t_1,t_2,s\in[0,T]$ such that $0<t_1<t_2<s<T$, we have

$$\left(\bar{p}(s;t_1),\bar{q}_i(s;t_1),\bar{r}_k(s,z;t_1),\bar{l}_j(s;t_1)\right) \\ =\left(\bar{p}(s;t_2),\bar{q}_i(s;t_2),\bar{r}_k(s,z;t_2),\bar{l}_j(s;t_2)\right).$$

Hence, the solution $\left(\bar{p}(\cdot;t),\bar{q}(\cdot;t),\bar{r}(\cdot,\cdot;t),\bar{l}(\cdot;t)\right)$ does not depend on t. Thus we denote the solution of (2.9) by $\left(\bar{p}(\cdot),\bar{q}(\cdot),\bar{r}(\cdot,\cdot),\bar{l}(\cdot)\right)$.

We have then, for any $t \in [0,T]$ and $s \in [t,T]$

$$\begin{aligned} p(s;t) = \quad & -\phi(s,\alpha(s))^{-1} \Big(\bar{p}(s) + \bar{G}\mathbb{E}\left[\hat{X}(T) | \mathcal{F}_T^{\alpha} \right] + \mu_1 \hat{X}(t) + \mu_2 \\ & + \int_s^T \phi(\tau,\alpha(\tau)) \bar{Q}\mathbb{E}\left[\hat{X}(\tau) | \mathcal{F}_\tau^{\alpha} \right] d\tau \Big), \end{aligned} \tag{2.10}$$

and $(q_i(s;t), r_k(s,z;t), l_j(s;t)) = -\phi(s,\alpha(s))^{-1} \left(\bar{q}_i(s), \bar{r}_k(s,z), \bar{l}_j(s) \right)$ for $i = 1,2,...,p, k = 1,2,...,l,$ and $j = 1,2,...,d.$ ■

Remark 2.2.2 *1) We remark that neither the coefficients nor the terminal condition of (2.8) are affected by the starting time t, so it may be considered as a standard BSDE over the entire time period* $[0,T]$, *then, by the same manner of [56] we can verify that the equation* (2.8) *admis a unique solution.*

2) From the representation of $(p(\cdot;t), q(\cdot;t), r(\cdot,\cdot;t), l(\cdot;t))$, for $t \in [0,T]$ given by Lemma 2.2.2, we can check that under (H1) the equation (2.7) admits a unique solution

$$\begin{aligned} & (p(\cdot;t), q(\cdot;t), r(\cdot,\cdot;t), l(\cdot;t)) \in \mathcal{S}_{\mathcal{F}}^2(t,T;\mathbb{R}^n) \\ & \times \mathcal{L}^2(t,T;(\mathbb{R}^n)^p) \times \mathcal{L}_{\mathcal{F},p}^{\theta,2}\left([t,T] \times \mathbb{R}^*; (\mathbb{R}^n)^l \right) \times \mathcal{L}_{\mathcal{F},p}^{\lambda,2}\left(t,T; (\mathbb{R}^n)^d \right). \end{aligned}$$

The second order adjoint equation is defined on the time interval $[t,T]$ and satisfied by the 4-tuple of processes $(P(\cdot), \Lambda(\cdot), \Gamma(\cdot;\cdot), L(\cdot))$ as follows:

$$\begin{cases} dP(s) = -\Big\{ A^\top P(s) + P(s) A + \sum\limits_{i=1}^{p} \left(C_i^\top P(s) C_i + \Lambda_i(s) C_i + C_i^\top \Lambda_i(s) \right) \\ \qquad + \sum\limits_{k=1}^{l} \int_{\mathbb{R}^*} \Big\{ \Gamma_k(s,z) E_k(z) \theta_\alpha^k(dz) + E_k(z)^\top \Gamma_k(s,z) \Big\} \theta_\alpha^k(dz) \\ \qquad + \sum\limits_{k=1}^{l} \int_{\mathbb{R}^*} E_k(z)^\top (\Gamma_k(s,z) + P(s)) E_k(z) \theta_\alpha^k(dz) - Q \Big\} ds \\ \qquad + \sum\limits_{i=1}^{p} \Lambda_i(s) dW^i(s) + \sum\limits_{k=1}^{l} \int_{\mathbb{R}^*} \Gamma_k(s,z) \tilde{N}_\alpha^k(ds,dz) \\ \qquad + \sum\limits_{j=1}^{d} L_j(s) d\tilde{\Phi}_j(s), \ \ s \in [t,T], \\ P(T) = -G. \end{cases} \tag{2.11}$$

Noting that (2.11) is a standard BSDE over the entire time period $[0,T]$, then, by the same manner of [56] we can verify that the equation (2.11) admis a unique solution

$$\begin{aligned} & (P(\cdot), \Lambda(\cdot), \Gamma(\cdot;\cdot), L(\cdot)) \in \mathcal{S}_{\mathcal{F}}^2(t,T;S^n) \\ & \times \mathcal{L}^2(t,T;(S^n)^p) \times \mathcal{L}_{\mathcal{F},p}^{\theta,2}\left([t,T] \times \mathbb{R}^*; (S^n)^l \right) \times \mathcal{L}_{\mathcal{F},p}^{\lambda,2}\left(t,T; (S^n)^d \right). \end{aligned}$$

Now, associated to $\left(\hat{u}(\cdot), \hat{X}(\cdot), p(\cdot;\cdot), q(\cdot;\cdot), r(\cdot,\cdot;\cdot), P(\cdot), \Gamma(\cdot;\cdot)\right)$ we define for $(s,t) \in \mathcal{D}([0,T])$

$$\mathcal{U}(s;t) = B^{\top} p(s;t) + \sum_{i=1}^{p} D_i{}^{\top} q_i(s;t) + \sum_{k=1}^{l} \int_{\mathbb{R}^*} F_k(z)^{\top} r_k(s,z;t)\,\theta_{\alpha}^{k}(dz) - R\hat{u}(s), \tag{2.12}$$

and

$$\mathcal{V}(s;t) = \sum_{i=1}^{p} D_i^{\top} P(s) D_i + \sum_{k=1}^{l} \int_{\mathbb{R}^*} F_k(z)^{\top} (P(s) + \Gamma(s,z)) F_k(z)\,\theta_{\alpha}^{k}(dz) - R. \tag{2.13}$$

Remark 2.2.3 *The definition 2.2.1 is slightly diferent from the original definition provided by [29] and [28], where the open-loop equilibrium control is given by*

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \left\{ J\left(t, \hat{X}(t), \alpha(t); u^{\varepsilon}(.)\right) - J\left(t, \hat{X}(t), \alpha(t); \hat{u}(\cdot)\right) \right\} \geq 0. \tag{2.14}$$

Although the limit (2.14) *already provides a characterizing condition, however, it is not very useful because it involves an a.s. limit with respect to uncountable $\varepsilon > 0$. Thus, in this case by using the property of RCLL of state process $X(\cdot)$ we can deduce an equivalent condition for the equilibrium, see Hu et al, [29]. In this chapter, we defined an open-loop equilibrium control by sense* (2.6)*, which is well-defined in general.*

The following lemma will be used later in this study, it's provides some important property about the flow of adapted processes.

Lemma 2.2.3 *Under assumptions (H1)-(H2), for any $\hat{u}(\cdot) \in \mathcal{L}^2_{\mathcal{F},p}(t,T;\mathbb{R}^m)$, there exists a sequence $(\varepsilon_n^t)_{n \in \mathbb{N}} \subset (0, T-t)$ satisfying $\varepsilon_n^t \to 0$ as $n \to \infty$, such that*

$$\lim_{n \to \infty} \frac{1}{\varepsilon_n^t} \int_t^{t+\varepsilon_n^t} \mathbb{E}[\mathcal{U}(s;t)]\,ds = \mathcal{U}(t;t), \; d\mathbb{P}-a.s, dt-a.e. \tag{2.15}$$

Proof. From the representation (2.10) we have, for any $t \in [0,T]$ and $s \in [t,T]$

$$\begin{aligned} \mathcal{U}(s;t) - \mathcal{U}(s;s) &= B^{\top}[p(s;t) - p(s;s)] \\ &= B^{\top} \phi(s, \alpha(s))^{-1} \mu_1 \left[\hat{X}(s) - \hat{X}(t)\right]. \end{aligned} \tag{2.16}$$

Moreover, since B and $\phi(s,\alpha(s))^{-1}$ are uniformly bounded, for any $a > 0$, $t \in [0,T]$ and $\varepsilon \in (0, T-t)$, we obtain

$$\begin{aligned}
&\mathbb{P}\left(\left|\frac{1}{\varepsilon}\mathbb{E}\left[\int_t^{t+\varepsilon}\mathcal{U}\left(s;t\right)ds\right]-\frac{1}{\varepsilon}\mathbb{E}\left[\int_t^{t+\varepsilon}\mathcal{U}\left(s;s\right)ds\right]\right|\geq a\right),\\
&\leq\frac{1}{a}\mathbb{E}\left|\frac{1}{\varepsilon}\mathbb{E}\left[\int_t^{t+\varepsilon}\mathcal{U}\left(s;t\right)ds\right]-\frac{1}{\varepsilon}\mathbb{E}\left[\int_t^{t+\varepsilon}\mathcal{U}\left(s;s\right)ds\right]\right|ds,\\
&\leq K\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\mathbb{E}\left|\hat{X}\left(s\right)-\hat{X}\left(t\right)\right|ds=0,
\end{aligned}$$

where the last equality is due to $\hat{X}\left(\cdot\right)$ is a right continuous with finite left limit.

Therefore

$$\lim_{\varepsilon\downarrow 0}\mathbb{P}\left(\left|\frac{1}{\varepsilon}\mathbb{E}\left[\int_t^{t+\varepsilon}\mathcal{U}\left(s;t\right)ds\right]-\frac{1}{\varepsilon}\mathbb{E}\left[\int_t^{t+\varepsilon}\mathcal{U}\left(s;s\right)ds\right]\right|\geq a\right)=0.$$

Hence, for each t there exists a sequence $\left(\varepsilon_n^t\right)_{n\geq 0}\subset\left(0,T-t\right)$ such that $\lim_{n\to\infty}\varepsilon_n^t=0$ and

$$\lim_{n\to\infty}\left|\frac{1}{\varepsilon_n^t}\mathbb{E}\left[\int_t^{t+\varepsilon_n^t}\mathcal{U}\left(s;t\right)ds\right]-\frac{1}{\varepsilon_n^t}\mathbb{E}\left[\int_t^{t+\varepsilon_n^t}\mathcal{U}\left(s;s\right)ds\right]\right|=0,\ d\mathbb{P}-a.s.$$

Moreover, we get from Lemma 2.2.1 that there exists a subsequence of $\left(\varepsilon_n^t\right)_{n\geq 0}$, which we also denote by $\left(\varepsilon_n^t\right)_{n\geq 0}$ such that

$$\lim_{n\to\infty}\frac{1}{\varepsilon_n^t}\mathbb{E}\left[\int_t^{t+\varepsilon_n^t}\mathcal{U}\left(s;s\right)ds\right]=\mathcal{U}\left(t;t\right),\ dt-a.e,\ d\mathbb{P}-a.s.$$

■

Now we introduce the following space

$$\mathcal{L}=\left\{\Lambda\left(\cdot;t\right)\in\mathcal{S}_{\mathcal{F}}^2\left(t,T;\mathbb{R}^n\right)\text{ such that }\sup_{t\in[0,T]}\mathbb{E}\left[\sup_{s\in[t,T]}\left|\Lambda\left(s;t\right)\right|^2\right]<+\infty\right\}.\tag{2.17}$$

Clearly, for any $\hat{u}\left(\cdot\right)\in\mathcal{L}_{\mathcal{F},p}^2\left(0,T;\mathbb{R}^m\right)$, it's associated flow of adjoint processes $p\left(\cdot;\cdot\right)\in\mathcal{L}$.

The following theorem is the first main result of this chapter, it's provides a necessary and sufficient conditions for equilibrium controls to the time-inconsistent Problem (N).

Theorem 2.2.1 (Characterization of equilibrium) *Let (H1) holds. Given an admissible control* $\hat{u}\left(\cdot\right)\in\mathcal{L}_{\mathcal{F},p}^2\left(0,T;\mathbb{R}^m\right)$, *let*

$$\begin{aligned}
&\left(p\left(\cdot;\cdot\right),q\left(\cdot;t\right),r\left(\cdot,\cdot;t\right),l\left(\cdot;t\right)\right)\\
&\in\mathcal{L}\times\mathcal{L}_{\mathcal{F}}^2\left(0,T;\left(\mathbb{R}^n\right)^p\right)\times\mathcal{L}_{\mathcal{F},p}^{\theta,2}\left(\left[0,T\right]\times\mathbb{R}^*;\left(\mathbb{R}^n\right)^l\right)\times\mathcal{L}_{\mathcal{F},p}^{\lambda,2}\left(0,T;\left(\mathbb{R}^n\right)^d\right),
\end{aligned}$$

be the unique solution to the BSDE (2.7) *and let*

$$\begin{aligned}&\left(P\left(\cdot\right),\Lambda\left(\cdot\right),\Gamma\left(\cdot,\cdot\right),L\left(\cdot\right)\right)\in\mathcal{S}_{\mathcal{F}}^{2}\left(t,T;S^{n}\right)\\&\times\mathcal{L}^{2}\left(t,T;\left(S^{n}\right)^{p}\right)\times\mathcal{L}_{\mathcal{F},p}^{\theta,2}\left(\left[t,T\right]\times\mathbb{R}^{*};\left(S^{n}\right)^{l}\right)\times\mathcal{L}_{\mathcal{F},p}^{\lambda,2}\left(t,T;\left(S^{n}\right)^{d}\right),\end{aligned}$$

be the unique solution to the BSDE (2.11). *Then $\hat{u}\left(.\right)$ is an open-loop Nash equilibrium if and only if the following two conditions hold: The first order equilibrium condition*

$$\mathcal{U}\left(t;t\right)=0,\ d\mathbb{P}-a.s,\ dt-a.e., \tag{2.18}$$

and the second order equilibrium condition

$$\mathcal{V}\left(t;t\right)\leq 0,\ d\mathbb{P}-a.s,\ \forall t\in\left[0,T\right], \tag{2.19}$$

where, $\mathcal{U}\left(t;t\right)$ and $\mathcal{V}\left(t;t\right)$ are given by (2.12) *and* (2.13) *respectively.*

In order to give a proof for the above theorem, the main idea is still based on the variational techniques in the same spirit of proving the characterization of equilibria [28] and [29] in the absence of random jumps. Let $\hat{u}\left(\cdot\right)\in\mathcal{L}_{\mathcal{F},p}^{2}\left(0,T;\mathbb{R}^{m}\right)$ be an admissible control and $\hat{X}\left(\cdot\right)$ the corresponding controlled state process. Consider the perturbed control $u^{\varepsilon}\left(\cdot\right)$ defined by the spike variation (2.5) for some fixed arbitrary $t\in\left[0,T\right],\ v\in\mathbb{L}^{2}\left(\Omega,\mathcal{F}_{t-}^{\alpha},\mathbb{P};\mathbb{R}^{m}\right)$ and $\varepsilon\in\left(0,T-t\right)$. Denote by $\hat{X}^{\varepsilon}\left(\cdot\right)$ the solution of the state equation corresponding to $u^{\varepsilon}\left(\cdot\right)$. It follows from the standard perturbation approach, see, for example, [60] and [71], that $\hat{X}^{\varepsilon}\left(\cdot\right)-\hat{X}\left(\cdot\right)=y^{\varepsilon,v}\left(\cdot\right)+Y^{\varepsilon,v}\left(\cdot\right)$, where $y^{\varepsilon,v}\left(\cdot\right)$ and $Y^{\varepsilon,v}\left(\cdot\right)$ solve the following SDEs, respectively, for $s\in\left[t,T\right]$

$$\left\{\begin{aligned}&dy^{\varepsilon,v}\left(s\right)=Ay^{\varepsilon,v}\left(s\right)ds+\sum_{i=1}^{p}\left\{C_{i}y^{\varepsilon,v}\left(s\right)+D_{i}v1_{\left[t,t+\varepsilon\right)}\left(s\right)\right\}dW^{i}\left(s\right)\\&\qquad+\sum_{k=1}^{l}\int_{\mathbb{R}^{*}}\left\{E_{k}\left(z\right)y^{\varepsilon,v}\left(s-\right)+F_{k}\left(z\right)v1_{\left[t,t+\varepsilon\right)}\left(s\right)\right\}\tilde{N}_{\alpha}^{k}\left(ds,dz\right),\\&y^{\varepsilon,v}\left(t\right)=0,\end{aligned}\right. \tag{2.20}$$

$$\left\{\begin{aligned}&dY^{\varepsilon,v}\left(s\right)=\left\{AY^{\varepsilon,v}\left(s\right)+Bv1_{\left[t,t+\varepsilon\right)}\left(s\right)\right\}ds+\sum_{i=1}^{p}C_{i}Y^{\varepsilon,v}\left(s\right)dW^{i}\left(s\right)\\&\qquad+\sum_{k=1}^{l}\int_{\mathbb{R}^{*}}E_{k}\left(z\right)Y^{\varepsilon,v}\left(s-\right)\tilde{N}_{\alpha}^{k}\left(ds,dz\right),\\&Y^{\varepsilon,v}\left(t\right)=0.\end{aligned}\right. \tag{2.21}$$

We need to the following lemma

Lemma 2.2.4 *Under assumption (H1), the following estimates hold*

$$\sup_{s\in[t,T]} \mathbb{E}\left[\left|y^{\varepsilon,v}(s)\right|^2\right] = O(\varepsilon), \tag{2.22}$$

$$\sup_{s\in[t,T]} \mathbb{E}\left[\left|Y^{\varepsilon,v}(s)\right|^2\right] = O\left(\varepsilon^2\right). \tag{2.23}$$

We have also

$$\sup_{s\in[t,T]} \left|\mathbb{E}\left[y^{\varepsilon,v}(s)\,|\mathcal{F}_s^{\alpha}\right]\right|^2 = O\left(\varepsilon^2\right). \tag{2.24}$$

Moreover, we have the following equality

$$\begin{aligned} &J\left(t,\hat{X}(t),\alpha(t);u^{\varepsilon}(\cdot)\right) - J\left(t,\hat{X}(t),\alpha(t);\hat{u}(\cdot)\right) \\ &= -\int_t^{t+\varepsilon} \mathbb{E}\left\{\langle\mathcal{U}(s;t),v\rangle + \tfrac{1}{2}\langle\mathcal{V}(s;t)v,v\rangle\right\} ds + o(\varepsilon). \end{aligned} \tag{2.25}$$

Proof. Proceeding with standard arguments by using Gronwall's lemma and the moment inequalities for diffusion processes with jump (see, e.g., Lemma 4.1 in [58]), we obtain (2.22) and (2.23).

Moreover, it follows from the dynamics of $y^{\varepsilon,v}(\cdot)$ in (2.20) that

$$\mathbb{E}\left[y^{\varepsilon,v}(s)\,|\mathcal{F}_s^{\alpha}\right] = \int_t^s \mathbb{E}[A(r,\alpha(r))y^{\varepsilon,v}(r)\,|\mathcal{F}_r^{\alpha}]dr$$

for all $s\in[t,T]$. By setting $\Psi(s)=A(s,\alpha(s))$ in lemma A.1 in [59], we get for some positive constants C that

$$\begin{aligned}\left|\int_t^s \mathbb{E}[A(r,\alpha(r))y^{\varepsilon,v}(r)\,|\mathcal{F}_r^{\alpha}]dr\right|^2 &\leq C\int_t^s \left|\mathbb{E}[A(r,\alpha(r))y^{\varepsilon,v}(r)\,|\mathcal{F}_r^{\alpha}]\right|^2 dr, \\ &\leq C\varepsilon\xi(\varepsilon),\end{aligned}$$

where $\xi:\Omega\times]0,\infty[\to]0,\infty[$ satisfies $\xi(\varepsilon)\downarrow 0$ as $\varepsilon\downarrow 0$, a.s., which prove (2.24).

Now, we consider the difference

$$\begin{aligned}
&J\left(t,\hat{X}(t),\alpha(t);u^{\varepsilon}(.)\right)-J\left(t,\hat{X}(t),\alpha(t);\hat{u}(.)\right)\\
&=\mathbb{E}\Big[\int_t^T\Big\{\Big\langle Q\hat{X}(s)+\bar{Q}\mathbb{E}\left[\hat{X}(s)|\mathcal{F}_s^{\alpha}\right],y^{\varepsilon,v}(s)+Y^{\varepsilon,v}(s)\Big\rangle\\
&\qquad+\frac{1}{2}\left\langle Q\left(y^{\varepsilon,v}(s)+Y^{\varepsilon,v}(s)\right),y^{\varepsilon,v}(s)+Y^{\varepsilon,v}(s)\right\rangle\\
&\qquad+\frac{1}{2}\left\langle\bar{Q}\mathbb{E}\left[y^{\varepsilon,v}(s)+Y^{\varepsilon,v}(s)|\mathcal{F}_s^{\alpha}\right],\mathbb{E}\left[y^{\varepsilon,v}(s)+Y^{\varepsilon,v}(s)|\mathcal{F}_s^{\alpha}\right]\right\rangle\\
&\qquad+\left\langle R\hat{u}(s),v\right\rangle 1_{[t,t+\varepsilon)}(s)+\frac{1}{2}\left\langle Rv,v\right\rangle 1_{[t,t+\varepsilon)}(s)\Big\}ds\\
&\qquad+\frac{1}{2}\left\langle G\left(y^{\varepsilon,v}(T)+Y^{\varepsilon,v}(T)\right),y^{\varepsilon,v}(T)+Y^{\varepsilon,v}(T)\right\rangle\\
&\qquad+\Big\langle G\hat{X}(T)+\bar{G}\mathbb{E}\left[\hat{X}(T)|\mathcal{F}_T^{\alpha}\right]+\mu_1\hat{X}(t)+\mu_2,y^{\varepsilon,v}(T)+Y^{\varepsilon,v}(T)\Big\rangle\\
&\qquad+\frac{1}{2}\left\langle\bar{G}\mathbb{E}\left[y^{\varepsilon,v}(T)+Y^{\varepsilon,v}(T)|\mathcal{F}_T^{\alpha}\right],\mathbb{E}\left[y^{\varepsilon,v}(T)+Y^{\varepsilon,v}(T)|\mathcal{F}_T^{\alpha}\right]\right\rangle\Big].
\end{aligned}\tag{2.26}$$

In an other hand, from **(H1)** and (2.22) – (2.24) the following estimate holds

$$\begin{aligned}
&\mathbb{E}\left[\int_t^T\frac{1}{2}\left\langle\bar{Q}\mathbb{E}\left[y^{\varepsilon,v}(s)+Y^{\varepsilon,v}(s)|\mathcal{F}_s^{\alpha}\right],\mathbb{E}\left[y^{\varepsilon,v}(s)+Y^{\varepsilon,v}(s)|\mathcal{F}_s^{\alpha}\right]\right\rangle ds\right.\\
&\left.+\frac{1}{2}\left\langle\bar{G}\mathbb{E}\left[y^{\varepsilon,v}(T)+Y^{\varepsilon,v}(T)|\mathcal{F}_T^{\alpha}\right],\mathbb{E}\left[y^{\varepsilon,v}(T)+Y^{\varepsilon,v}(T)|\mathcal{F}_T^{\alpha}\right]\right\rangle\right]=o(\varepsilon).
\end{aligned}$$

Then, from the terminal conditions in the adjoint equations, it follows that

$$\begin{aligned}
&J\left(t,\hat{X}(t),\alpha(t);u^{\varepsilon}(.)\right)-J\left(t,\hat{X}(t),\alpha(t);\hat{u}(.)\right)\\
&=\mathbb{E}\Big[\int_t^T\Big\{\Big\langle Q\hat{X}(s)+\bar{Q}\mathbb{E}\left[\hat{X}(s)|\mathcal{F}_s^{\alpha}\right],y^{\varepsilon,v}(s)+Y^{\varepsilon,v}(s)\Big\rangle\\
&\qquad+\frac{1}{2}\left\langle Q\left(y^{\varepsilon,v}(s)+Y^{\varepsilon,v}(s)\right),y^{\varepsilon,v}(s)+Y^{\varepsilon,v}(s)\right\rangle\\
&\qquad+\left\langle R\hat{u}(s),v\right\rangle 1_{[t,t+\varepsilon)}(s)+\frac{1}{2}\left\langle Rv,v\right\rangle 1_{[t,t+\varepsilon)}(s)\Big\}ds\\
&\qquad-\left\langle p(T;t),y^{\varepsilon,v}(T)+Y^{\varepsilon,v}(T)\right\rangle\\
&\qquad-\frac{1}{2}\left\langle P(T)\left(y^{\varepsilon,v}(T)+Y^{\varepsilon,v}(T)\right),y^{\varepsilon,v}(T)+Y^{\varepsilon,v}(T)\right\rangle\Big]+o(\varepsilon).
\end{aligned}\tag{2.27}$$

Now, by applying Ito's formula to $s\mapsto\left\langle p(s;t),y^{\varepsilon,v}(s)+Y^{\varepsilon,v}(s)\right\rangle$ on $[t,T]$ and by taking the expectation, we get

$$\begin{aligned}
&\mathbb{E}\left[\left\langle p(T;t),y^{\varepsilon,v}(T)+Y^{\varepsilon,v}(T)\right\rangle\right]\\
&=\mathbb{E}\Big[\int_t^T\Big\{v^{\top}B^{T}p(s;t)1_{[t,t+\varepsilon)}(s)\\
&\qquad+\left(y^{\varepsilon,v}(s)+Y^{\varepsilon,v}(s)\right)^{\top}\left(Q\hat{X}(s)+\bar{Q}\mathbb{E}\left[\hat{X}(s)|\mathcal{F}_s^{\alpha}\right]\right)\\
&\qquad+\sum_{i=1}^{p}v^{\top}D_i^{T}q_i(s)1_{[t,t+\varepsilon)}(s)\\
&\qquad+\sum_{k=1}^{l}\int_{\mathbb{R}^*}v^{\top}F_k(z)^{T}r_k(s,z)\theta_{\alpha}^{k}(dz)1_{[t,t+\varepsilon)}(s)\Big\}ds\Big].
\end{aligned}\tag{2.28}$$

By applying Ito's formula to $s \mapsto \langle P(s)(y^{\varepsilon,v}(s)+Y^{\varepsilon,v}(s)), y^{\varepsilon,v}(s)+Y^{\varepsilon,v}(s)\rangle$ on $[t,T]$, we conclude from **(H1)** together with (2.22) – (2.24) and by taking the conditional expectation

$$\begin{aligned}
&\mathbb{E}\left[\langle P(T)(y^{\varepsilon,v}(T)+Y^{\varepsilon,v}(T)), y^{\varepsilon,v}(T)+Y^{\varepsilon,v}(T)\rangle\right]\\
&=\mathbb{E}\left[\int_t^T\left\{(y^{\varepsilon,v}(s)+Y^{\varepsilon,v}(s))^\top Q(s)\,y^{\varepsilon,v}(s)\;+\sum_{i=1}^p v^\top D_i^\top P(s)\,D_i v 1_{[t,t+\varepsilon)}(s)\right.\right.\\
&\left.\left.+\sum_{k=1}^l\int_{\mathbb{R}^*} v^\top F_k(z)^\top(P(s)+\Gamma(s,z))\,F_k(z)\,v1_{[t,t+\varepsilon)}(s)\,\theta_\alpha^k(dz)\right\}ds\right]+o(\varepsilon). \qquad (2.29)
\end{aligned}$$

By taking (2.28) and (2.29) in (2.27), it follows that

$$\begin{aligned}
&J\left(t,\hat{X}(t),\alpha(t);u^\varepsilon(.)\right)-J\left(t,\hat{X}(t),\alpha(t);\hat{u}(.)\right)\\
&=-\mathbb{E}\left[\int_t^{t+\varepsilon}\left\{v^\top B^\top p(s;t)+\sum_{i=1}^p v^\top D_i^\top q_i(s)\right.\right.\\
&\quad+\frac{1}{2}\sum_{i=1}^p v^\top D_i^\top P(s)\,D_i v-v^\top R\hat{u}(s)-\frac{1}{2}v^\top Rv\\
&\quad\left.\left.+\sum_{k=1}^l\int_{\mathbb{R}^*} v^\top F_k(z)^\top\left(r_k(s,z)+\frac{1}{2}(P(s)+\Gamma(s))\,F_k(z)\,v\right)\theta_\alpha^k(dz)\right\}ds\right]\\
&\quad+o(\varepsilon), \qquad (2.30)
\end{aligned}$$

which is equivalent to (2.25). ■

Now, we are ready to give the proof of the Theorem 2.2.1**.**

Proof of Theorem 2.2.1. Given an admissible control $\hat{u}(\cdot)\in\mathcal{L}^2_{\mathcal{F},p}(0,T;\mathbb{R}^m)$, for which (2.18) and (2.19) holds, according to Lemma 2.2.3, we have from (2.25) that for any $t\in[0,T]$ and for any $\mathbb{R}^m-$valued, $\mathcal{F}_t^\alpha-$measurable and bounded random variable v, there exists a sequence $(\varepsilon_n^t)_{n\in\mathbb{N}}\subset(0,T-t)$ satisfying $\varepsilon_n^t\to 0$ as $n\to\infty$, such that

$$\begin{aligned}
&\lim_{n\to 0}\frac{1}{\varepsilon_n^t}\left\{J\left(t,\hat{X}(t),\alpha(t);u^\varepsilon(\cdot)\right)-J\left(t,\hat{X}(t),\alpha(t);\hat{u}(\cdot)\right)\right\}\\
&=-\left\{\langle\mathcal{U}(t;t),v\rangle+\frac{1}{2}\langle\mathcal{V}(t;t)\,v,v\rangle\right\},\\
&=-\frac{1}{2}\langle\mathcal{V}(t;t)\,v,v\rangle,\\
&\geq 0,\ d\mathbb{P}-a.s.
\end{aligned}$$

Hence $\hat{u}(\cdot)$ is an equilibrium strategy.

Conversely, assume that $\hat{u}(\cdot)$ is an equilibrium strategy. Then, by (2.6) together with (2.25) and Lemma

2.2.3, for any $(t,u) \in [0,T] \times \mathbb{R}^m$, the following inequality holds

$$\langle \mathcal{U}(t;t), u \rangle + \frac{1}{2} \langle \mathcal{V}(t;t) u, u \rangle \leq 0. \tag{2.31}$$

Now, we define $\forall (t,u) \in [0,T] \times \mathbb{R}^m$, $\Phi(t,u) = \langle \mathcal{U}(t;t), u \rangle + \frac{1}{2} \langle \mathcal{V}(t;t) u, u \rangle$. Easy manipulations show that the inequality (2.31) is equivalent to $\Phi(t,0) = \max_{u \in \mathbb{R}^m} \Phi(t,u)$, $d\mathbb{P} - a.s, \forall t \in [0,T]$. So it is easy to prove that the maximum condition is equivalent to the following two conditions

$$\Phi_u(t,0) = \mathcal{U}(t;t) = 0, \ \forall t \in [0,T], \ d\mathbb{P} - a.s, \tag{2.32}$$

$$\Phi_{uu}(t,0) = \mathcal{V}(t;t) \leq 0, \ \forall t \in [0,T], \ d\mathbb{P} - a.s. \tag{2.33}$$

This completes the proof. ■

Remark 2.2.4 *It is worth noting that from the positive semi-definite conditions on the coefficients $Q(\cdot)$, G and $R(\cdot,\cdot)$, the corresponding process $P(\cdot)$ in [29] and [28] is indeed positive semi-definite due to the comparison principles of BSDEs. Thus, as a result of Theorem 2.2.1, a necessary and sufficient condition for a control being an equilibrium strategy is only the first order equilibrium condition* (2.18)*. However, there is a significant difference between the estimate for the cost functional presented and that in [29] and [28]. Because stochastic coefficients and random jumps of the controlled system are taken into account, an additional term $\Gamma(\cdot,\cdot)$ occurs in the formulation of $P(\cdot)$. So in this chapter, $P(\cdot)$ is not necessarily positive semi-definite. This is why we modify the methodology of deriving the sufficient condition for equilibrium controls. Therefore, we have the following corollary, the proof follows the same arguments as the proof of Proposition 3.2 of [59].*

Corollary 2.2.1 *Let (H1)-(H2) hold. Given an admissible control $\hat{u}(\cdot) \in \mathcal{L}^2_{\mathcal{F},p}(0,T;\mathbb{R}^m)$. Let*

$$\begin{aligned}&(p(\cdot;\cdot), q(\cdot;\cdot), r(\cdot,\cdot;\cdot), l(\cdot;\cdot)) \\ &\in \mathcal{L} \times \mathcal{L}^2_{\mathcal{F}}(0,T;(\mathbb{R}^n)^p) \times \mathcal{L}^{\theta,2}_{\mathcal{F},p}\left([0,T] \times \mathbb{R}^*; (\mathbb{R}^n)^l\right) \times \mathcal{L}^{\lambda,2}_{\mathcal{F},p}\left(0,T;(\mathbb{R}^n)^d\right)\end{aligned}$$

be the unique solution to the BSDE (2.7)*. Then $\hat{u}(\cdot)$ is an equilibrium, if the following condition holds $dP - a.s$, $dt - a.e.$*

$$B^\top p(t;t) + \sum_{i=1}^{p} D_i^\top q_i(t) + \sum_{k=1}^{l} \int_{\mathbb{R}^*} F_k(z)^\top r_k(t,z) \theta_\alpha^k(dz) - R\hat{u}(t) = 0. \tag{2.34}$$

Proof. First, we have

$$
\begin{aligned}
&J\left(t,\hat{X}\left(t\right),\alpha\left(t\right);u^{\varepsilon}\left(\cdot\right)\right)-J\left(t,\hat{X}\left(t\right),\alpha\left(t\right);\hat{u}\left(\cdot\right)\right)\\
&=\mathbb{E}\left[\int_t^T\left\{\frac{1}{2}\left\langle Q\left(X^{\varepsilon}\left(s\right)+\hat{X}\left(s\right)\right)+\bar{Q}\mathbb{E}\left[X^{\varepsilon}\left(s\right)+\hat{X}\left(s\right)|\mathcal{F}_s^{\alpha}\right]\right.\right.\right.\\
&\qquad\left.\left.,X^{\varepsilon}\left(s\right)-\hat{X}\left(s\right)\right\rangle+\frac{1}{2}\left\langle R\left(u^{\varepsilon}\left(s\right)+\hat{u}\left(s\right)\right),u^{\varepsilon}\left(s\right)-\hat{u}\left(s\right)\right\rangle\right\}ds\\
&+\frac{1}{2}\left\langle G\left(X^{\varepsilon}\left(T\right)+\hat{X}\left(T\right)\right)+\overline{G}\mathbb{E}\left[X^{\varepsilon}\left(T\right)+\hat{X}\left(T\right)|\mathcal{F}_T^{\alpha}\right]+2\left(\mu_1\hat{X}\left(t\right)+\mu_2\right)\right.\\
&\qquad\left.\left.,X^{\varepsilon}\left(T\right)-\hat{X}\left(T\right)\right\rangle\right].
\end{aligned}
$$

Noting that by applying Itô's formula to $s\mapsto\left\langle p\left(s;t\right),X^{\varepsilon}\left(s\right)-\hat{X}\left(s\right)\right\rangle$

$$
\begin{aligned}
&\mathbb{E}\left[\left\langle G\hat{X}\left(T\right)+\overline{G}\mathbb{E}\left[\hat{X}\left(T\right)|\mathcal{F}_T^{\alpha}\right]+\mu_1\hat{X}\left(t\right)+\mu_2,X^{\varepsilon}\left(T\right)-\hat{X}\left(T\right)\right\rangle\right]\\
&=-\mathbb{E}\left[\int_t^T\left\{\left\langle B^{\top}p\left(s;t\right)+\sum_{i=1}^{p}v^{\top}D_i^{\top}q_i\left(s\right)+\sum_{k=1}^{l}\int_{\mathbb{R}^*}F_k\left(z\right)^{\top}r_k\left(s,z\right)\theta_{\alpha}^{k}\left(dz\right)\right.\right.\right.,\\
&\qquad\left.\left.\left.u^{\varepsilon}\left(s\right)-\hat{u}\left(s\right)\right\rangle+\left\langle Q\hat{X}\left(s\right)+\bar{Q}\mathbb{E}\left[\hat{X}\left(s\right)|\mathcal{F}_T^{\alpha}\right],X^{\varepsilon}\left(s\right)-\hat{X}\left(s\right)\right\rangle\right\}ds\right].
\end{aligned}
$$

Consequently

$$
\begin{aligned}
&J\left(t,\hat{X}\left(t\right),\alpha\left(t\right);u^{\varepsilon}\left(\cdot\right)\right)-J\left(t,\hat{X}\left(t\right),\alpha\left(t\right);\hat{u}\left(\cdot\right)\right)\\
&=\mathbb{E}\left[\int_t^T\left\{\frac{1}{2}\left\langle Q\left(X^{\varepsilon}\left(s\right)+\hat{X}\left(s\right)\right)-2Q\hat{X}\left(s\right)+\bar{Q}\mathbb{E}\left[X^{\varepsilon}\left(s\right)+\hat{X}\left(s\right)|\mathcal{F}_s^{\alpha}\right]\right.\right.\right.\\
&\qquad\left.-2\bar{Q}\mathbb{E}\left[\hat{X}\left(s\right)|\mathcal{F}_s^{\alpha}\right],X^{\varepsilon}\left(s\right)-\hat{X}\left(s\right)\right\rangle\\
&+\frac{1}{2}\left\langle R\left(u^{\varepsilon}\left(s\right)+\hat{u}\left(s\right)\right)-2(B^{\top}p\left(s;t\right)+\sum_{i=1}^{p}D_i^{\top}q_i\left(s\right)\right.\\
&\qquad\left.\left.+\sum_{k=1}^{l}\int_{\mathbb{R}^*}F_k\left(z\right)^{\top}r_k\left(s,z\right)\theta_{\alpha}^{k}\left(dz\right)),u^{\varepsilon}\left(s\right)-\hat{u}\left(s\right)\right\rangle\right\}ds\\
&+\frac{1}{2}\left\langle G\left(X^{\varepsilon}\left(T\right)+\hat{X}\left(T\right)\right)+\overline{G}\mathbb{E}\left[X^{\varepsilon}\left(T\right)+\hat{X}\left(T\right)|\mathcal{F}_T^{\alpha}\right],X^{\varepsilon}\left(T\right)-\hat{X}\left(T\right)\right\rangle\\
&\left.-\left\langle G\hat{X}\left(T\right)+\overline{G}\mathbb{E}\left[\hat{X}\left(T\right)|\mathcal{F}_T^{\alpha}\right],X^{\varepsilon}\left(T\right)-\hat{X}\left(T\right)\right\rangle\right].
\end{aligned}
$$

By completing the square we get

$$
\begin{aligned}
&= \mathbb{E}\left[\int_t^T \left\{ \left| \sqrt{\frac{Q}{2}} \left(X^{\varepsilon}(s) - \hat{X}(s) \right) \right|^2 + \left| \sqrt{\frac{\bar{Q}}{2}} \left(\mathbb{E}\left[X^{\varepsilon}(s) + \hat{X}(s) \,|\mathcal{F}_T^{\alpha} \right] \right) \right|^2 \right\} ds \right. \qquad (2.35)\\
&+ \frac{1}{2}\int_t^{t+\varepsilon} \left\langle R\left(v + 2\hat{u}(s)\right) - 2(B^{\top} p(s;t) + \sum_{i=1}^{p} D_i^{\top} q_i(s) \right.\\
&\qquad \left. + \sum_{k=1}^{l} \int_{\mathbb{R}^*} F_k(z)^{\top} r_k(s,z)\, \theta_{\alpha}^k(dz)), v \right\rangle ds \Bigg\}\\
&\left. + \left| \sqrt{\frac{G}{2}} \left(X^{\varepsilon}(T) - \hat{X}(T) \right) \right|^2 + \left| \sqrt{\frac{\bar{G}}{2}} \left(\mathbb{E}\left[X^{\varepsilon}(T) + \hat{X}(T) \,|\mathcal{F}_T^{\alpha} \right] \right) \right|^2 \right],\\
&\geq \frac{1}{2}\mathbb{E}\left[\int_t^{t+\varepsilon} \langle Rv - 2\mathcal{U}(s;t), v \rangle\, ds \right] \geq -\int_t^{t+\varepsilon} \langle \mathbb{E}\left[\mathcal{U}(s;t)\right], v\rangle\, ds.
\end{aligned}
$$

Dividing by ε_n and sending ε_n to 0. Therefore, it follows from Lemma 2.2.3 that $\hat{u}(\cdot)$ is an equilibrium control. ■

2.2.2 Linear feedback stochastic equilibrium control

In this subsection, our goal is to obtain a state feedback representation of an equilibrium control for problem (N) via some class of ordinary differential equations.

Now, suppressing the subscript (s, e_i) for the coefficients $A, B, b, C_i, D_i, \sigma_i$ and we use the notation $\varrho(z)$ instead of $\varrho(s, z, e_i)$ for $\varrho = E_k, F_k$ and c_k. First, for any deterministic, differentiable function $\eta \in C([0,T] \times \chi; \mathbb{R}^{n\times n})$ consider the differential-difference operator

$$
\mathcal{L}(\eta(s,\cdot)) = \eta'(s,\cdot) + \sum_{j=1}^{d} \lambda_{ij} \left\{ \eta(s,e_j) - \eta(s,\cdot) \right\}.
$$

Then we introduce the following system of differential-difference equations for $s \in [0,T]$,

$$\begin{cases} 0 = \mathcal{L}\left(M\left(s,e_i\right)\right) + M\left(s,e_i\right)A + A^\top M\left(s,e_i\right) + \sum_{i=1}^{p} C_i^\top M\left(s,e_i\right)C_i \\ \qquad -\Big(M\left(s,e_i\right)B + \sum_{i=1}^{p} C_i^\top M\left(s,e_i\right)D_i \\ \qquad + \sum_{k=1}^{l}\int_{\mathbb{R}^*} E_k\left(z\right)^\top M\left(s,e_i\right)F_k\left(z\right)\theta_\alpha^k\left(dz\right)\Big)\Psi\left(s,e_i\right) \\ \qquad + \sum_{k=1}^{l}\int_{\mathbb{R}^*} E_k\left(z\right)^\top M\left(s,e_i\right)E_k\left(z\right)\theta_\alpha^k\left(dz\right) + Q, \\ 0 = \mathcal{L}\left(\bar{M}\left(s,e_i\right)\right) + \bar{M}\left(s,e_i\right)A + A^\top\bar{M}\left(s,e_i\right) - \bar{M}\left(s,e_i\right)B\Psi\left(s,e_i\right) + \bar{Q}, \\ 0 = \mathcal{L}\left(\Upsilon\left(s,e_i\right)\right) + A^\top\Upsilon\left(s,e_i\right), \\ 0 = \mathcal{L}\left(\varphi\left(s,e_i\right)\right) + A^\top\varphi\left(s,e_i\right) + \left(M\left(s,e_i\right) + \bar{M}\left(s,e_i\right)\right)\left(b - B\psi\left(s,e_i\right)\right) \\ \qquad + \sum_{i=1}^{p} C_i^\top M\left(s,e_i\right)\left(\sigma_i - D_i\psi\left(s,e_i\right)\right) \\ \qquad + \sum_{k=1}^{l}\int_{\mathbb{R}^*} E_k\left(z\right)^\top M\left(s,e_i\right)\left(c_k\left(z\right) - F_k\left(z\right)\psi\left(s,e_i\right)\right)\theta_\alpha^k\left(dz\right), \\ M\left(T,e_i\right) = G; \bar{M}\left(T,e_i\right) = \bar{G}; \Upsilon\left(T,e_i\right) = \mu_1; \varphi\left(T,e_i\right) = \mu_2, \end{cases} \tag{2.36}$$

where $\Psi(\cdot,\cdot)$ and $\psi(\cdot,\cdot)$ are given by

$$\begin{cases} \Psi\left(s,e_i\right) \triangleq \Theta\left(s,e_i\right)\Big(B^\top\left(M\left(s,e_i\right) + \bar{M}\left(s,e_i\right) + \Upsilon\left(s,e_i\right)\right) \\ \qquad + \sum_{i=1}^{p} D_i^\top M\left(s,e_i\right)C_i + \sum_{k=1}^{l}\int_{\mathbb{R}^*} F_k\left(z\right)^\top M\left(s,e_i\right)E_k\left(z\right)\theta_\alpha^k\left(dz\right)\Big), \\ \psi\left(s,e_i\right) \triangleq \Theta\left(s,e_i\right)\Big(B^\top\varphi\left(s,e_i\right) + \sum_{i=1}^{p} D_i^\top M\left(s,e_i\right)\sigma_i \\ \qquad + \sum_{k=1}^{l}\int_{\mathbb{R}^*} F_k\left(z\right)^\top M\left(s,e_i\right)c_k\left(z\right)\theta_\alpha^k\left(dz\right)\Big), \end{cases} \tag{2.37}$$

with

$$\Theta\left(s,\cdot\right) = \left(R + \sum_{i=1}^{p} D_i^\top M\left(s,\cdot\right)D_i + \sum_{k=1}^{l}\int_{\mathbb{R}^*} F_k\left(z\right)^\top M\left(s,\cdot\right)F_k\left(z\right)\theta_\alpha^k\left(dz\right)\right)^{-1}.$$

The following theorem presents the existence condition for a linear feedback equilibrium control.

Theorem 2.2.2 *Let (H1)-(H2) hold. Suppose that the system of equations* (2.36) *admit a solution* $M(\cdot,e_i)$, $\bar{M}(\cdot,e_i)$, $\Upsilon(\cdot,e_i)$ *and* $\varphi(.,e_i)$, *for any* $e_i \in \mathcal{X}$, *on* $C([0,T];\mathbb{R}^{n\times n})$. *Then the time-inconsistent LQ problem (N) has an equilibrium control that can be represented by the state feedback form*

$$\hat{u}\left(t\right) = -\Psi\left(t,\alpha\left(t\right)\right)\hat{X}\left(t-\right) - \psi\left(t,\alpha\left(t\right)\right), \tag{2.38}$$

where $\Psi(\cdot,\cdot)$ *and* $\psi(\cdot,\cdot)$ *are given by* (2.37).

Proof. Suppose that $\hat{u}(\cdot)$ is an admissible control and denote by $\hat{X}(\cdot)$ it corresponding controlled process. According to Corollary 2.2.1, suppose that there exists a flow of 4-tuple of adapted processes for which the processes $\left(\hat{X}(\cdot), p(\cdot;\cdot), q(\cdot;\cdot), r(\cdot,\cdot;\cdot), l(\cdot;\cdot)\right)$ satisfies the following system of regime switching forward-backward stochastic differential equations

$$\begin{cases} d\hat{X}(s) = \left\{A\hat{X}(s) + B\hat{u}(s) + b\right\} ds + \sum_{i=1}^{p} \left\{C_i\hat{X}(s) + D_i\hat{u}(s) + \sigma_i\right\} dW^i(s) \\ \qquad + \sum_{k=1}^{l} \int_{\mathbb{R}^*} \left\{E_k(z)\hat{X}(s-) + F_k(z)\hat{u}(s) + c_k(z)\right\} \widetilde{N}_\alpha^k(ds,dz), \ s \in [0,T], \\ dp(s;t) = -\left\{A^\top p(s;t) + \sum_{i=1}^{p} C_i^\top q_i(s;t) + \sum_{k=1}^{l} \int_{\mathbb{R}^*} E_k(z)^\top r_k(s,z;t)\theta_\alpha^k(dz) \right. \\ \qquad \left. - Q\hat{X}(s) - \bar{Q}\mathbb{E}\left[\hat{X}(s)|\mathcal{F}_s^\alpha\right]\right\} ds + \sum_{i=1}^{p} q_i(s;t) dW^i(s) \\ \qquad + \sum_{k=1}^{l} \int_{\mathbb{R}^*} r_k(s,z;t)\widetilde{N}_\alpha^k(ds,dz) + \sum_{j=1}^{d} l_j(s,t) d\tilde{\Phi}_j(s), \ s \in [t,T], \\ \hat{X}_0 = x_0, \alpha(0) = e_{i_0}, \\ p(T;t) = -G\hat{X}(T) - \bar{G}\mathbb{E}\left[\hat{X}(T)|\mathcal{F}_T^\alpha\right] - \mu_1\hat{X}(t) - \mu_2, \end{cases} \tag{2.39}$$

with the equilibrium condition $d\mathbb{P}-a.s, dt-a.e.$

$$B^\top p(t;t) + \sum_{i=1}^{p} D_i^\top q_i(t) + \sum_{k=1}^{l} \int_{\mathbb{R}^*} F_k(z)^\top r_k(t,z)\theta_\alpha^k(dz) - R\hat{u}(t) = 0. \tag{2.40}$$

Now, to solve the above system, we assume the following ansatz : for $0 \le t \le s \le T$, we put

$$\begin{aligned} p(s;t) = \ & -M(s,\alpha(s))\hat{X}(s) - \bar{M}(s,\alpha(s))\mathbb{E}\left[\hat{X}(s)|\mathcal{F}_s^\alpha\right] \\ & -\Upsilon(s,\alpha(s))\hat{X}(t) - \varphi(s,\alpha(s)), \end{aligned} \tag{2.41}$$

where $M(\cdot,\cdot)$, $\bar{M}(\cdot,\cdot)$, $\Upsilon(\cdot,\cdot)$ and $\varphi(\cdot,\cdot)$ are deterministic, differentiable functions which are to be determined. From the terminal condition of the adjoint process, $M(\cdot,\cdot)$, $\bar{M}(\cdot,\cdot)$, $\Upsilon(\cdot,\cdot)$ and $\varphi(\cdot,\cdot)$ must satisfy the following terminal boundary condition, for all $e_i \in \chi$

$$M(T,e_i) = G, \ \bar{M}(T,e_i) = \bar{G}, \ \Upsilon(T,e_i) = \mu_1, \ \varphi(T,e_i) = \mu_2. \tag{2.42}$$

Applying Itô's formula to (2.41) and using (2.39) , it yields

$$\begin{aligned}
&dp\left(s;t\right)\\
&=-\Big\{\mathcal{L}\left(M\left(s,\alpha\left(s\right)\right)\right)\hat{X}\left(s\right)+\mathcal{L}\left(\overline{M}\left(s,\alpha\left(s\right)\right)\right)\mathbb{E}\left[\hat{X}\left(s\right)|\mathcal{F}_{s}^{\alpha}\right]\\
&+\mathcal{L}\left(\Upsilon\left(s,\alpha\left(s\right)\right)\right)\hat{X}\left(t\right)+\mathcal{L}\left(\varphi\left(s,\alpha\left(s\right)\right)\right)\\
&+M\left(s,\alpha\left(s\right)\right)\Big(A\hat{X}\left(s\right)+B\hat{u}\left(s\right)+b\Big)\\
&+\bar{M}\left(s,\alpha\left(s\right)\right)\Big(A\mathbb{E}\left[\hat{X}\left(s\right)|\mathcal{F}_{s}^{\alpha}\right]+B\mathbb{E}\left[\hat{u}\left(s\right)|\mathcal{F}_{s}^{\alpha}\right]+b\Big)\Big\}\,ds\\
&-M\left(s,\alpha\left(s\right)\right)\sum_{i=1}^{p}\Big\{C_{i}\hat{X}\left(s\right)+D_{i}\hat{u}\left(s\right)+\sigma_{i}\Big\}\,dW^{i}\left(s\right)\\
&-M\left(s,\alpha\left(s\right)\right)\sum_{k=1}^{l}\int_{\mathbb{R}^{*}}\Big\{E_{k}\left(z\right)\hat{X}\left(s-\right)+F_{k}\left(z\right)\hat{u}\left(s\right)+c_{k}\left(z\right)\Big\}\,\widetilde{N}_{\alpha}^{k}\left(ds,dz\right)\\
&-\sum_{j=1}^{d}\Big\{\left(M\left(s,e_{j}\right)-M\left(s,\alpha\left(s-\right)\right)\right)\hat{X}\left(s\right)\\
&+\left(\bar{M}\left(s,e_{j}\right)-\bar{M}\left(s,\alpha\left(s-\right)\right)\right)\mathbb{E}\left[\hat{X}\left(s\right)|\mathcal{F}_{s}^{\alpha}\right]\\
&+\left(\Upsilon\left(s,e_{j}\right)-\Upsilon\left(s,\alpha\left(s-\right)\right)\right)\hat{X}\left(t\right)+\left(\varphi\left(s,e_{j}\right)-\varphi\left(s,\alpha\left(s-\right)\right)\right)\Big\}\,d\tilde{\Phi}_{j}(s).
\end{aligned} \tag{2.43}$$

Compared to (2.39) we deduce that for $i=1,2,...,p$, $k=1,2,...,l$, and $j=1,2,...,d$

$$\begin{cases}
q_{i}\left(s;t\right)=q_{i}\left(s\right)=-M\left(s,\alpha\left(s\right)\right)\Big(C_{i}\hat{X}\left(s\right)+D_{i}\hat{u}\left(s\right)+\sigma_{i}\Big),\\
r_{k}\left(s,z;t\right)=r_{k}\left(s,z\right)=-M\left(s,\alpha\left(s\right)\right)\Big(E_{k}\left(z\right)\hat{X}\left(s-\right)+F_{k}\left(z\right)\hat{u}\left(s\right)+c_{k}\left(z\right)\Big),\\
l_{j}\left(s;t\right)=l_{j}\left(s\right)=-\left(M\left(s,e_{j}\right)-M\left(s,\alpha\left(s\right)\right)\right)\hat{X}\left(s\right)\\
\qquad-\left(\bar{M}\left(s,e_{j}\right)-\bar{M}\left(s,\alpha\left(s\right)\right)\right)\mathbb{E}\left[\hat{X}\left(s\right)|\mathcal{F}_{s}^{\alpha}\right]\\
\qquad-\left(\Upsilon\left(s,e_{j}\right)-\Upsilon\left(s,\alpha\left(s\right)\right)\right)\hat{X}\left(t\right)-\left(\varphi\left(s,e_{j}\right)-\varphi\left(s,\alpha\left(s\right)\right)\right).
\end{cases} \tag{3.44}$$

Moreover, by taking (2.41) and (2.44) in (2.40), we obtain

$$\begin{aligned}
&R\hat{u}\left(t\right)+B^{\top}\Big(\left(M\left(t,\alpha\left(t\right)\right)+\bar{M}\left(t,\alpha\left(t\right)\right)+\Upsilon\left(t,\alpha\left(t\right)\right)\right)\hat{X}\left(t\right)\Big)\\
&\quad+B^{\top}\varphi\left(t,\alpha\left(t\right)\right)+\sum_{i=1}^{p}D_{i}^{\top}M\left(t,\alpha\left(t\right)\right)\Big\{C_{i}\hat{X}\left(t\right)+D_{i}\hat{u}\left(t\right)+\sigma_{i}\Big\}\\
&\quad+\sum_{k=1}^{l}\int_{\mathbb{R}^{*}}F_{k}\left(z\right)^{\top}M\left(t,\alpha\left(t\right)\right)\Big(E_{k}\left(z\right)\hat{X}\left(t-\right)+F_{k}\left(z\right)\hat{u}\left(t\right)+c_{k}\Big)\,\theta_{\alpha}^{k}\left(dz\right)\\
&=0.
\end{aligned}$$

Subsequently, we obtain that $\hat{u}\left(\cdot\right)$ admits the following representation

$$\hat{u}\left(s\right)=-\Psi\left(s,\alpha\left(s\right)\right)\hat{X}\left(s\right)-\psi\left(s,\alpha\left(s\right)\right),$$

where $\Psi\left(\cdot,\cdot\right)$ and $\psi\left(\cdot,\cdot\right)$ are given by (2.37) .

Hence (2.38) holds, and for $s \in [0,T]$, we have

$$\mathbb{E}\left[\hat{u}(s)\,|\mathcal{F}_s^{\alpha}\right] = -\Psi(s,\alpha(s))\,\mathbb{E}\left[\hat{X}(s)\,|\mathcal{F}_s^{\alpha}\right] - \psi(s,\alpha(s))\,. \tag{2.45}$$

Next, comparing the ds term in (2.43) by the ones in the second equation in (2.39), then by using the expressions (2.38) and (2.37), we obtain

$$\begin{aligned}
0 = \; & \Big\{\mathcal{L}(M) + MA + A^{\top}M + \sum_{i=1}^{p} C_i^{\top} M C_i \\
& - \Big(MB + \sum_{i=1}^{p} C_i^{\top} M D_i + \sum_{k=1}^{l} \int_{\mathbb{R}^*} E_k(z)^{\top} M F_k(z)\,\theta_{\alpha}^{k}(dz)\Big)\Psi(s,\alpha(s)) \\
& + \sum_{k=1}^{l} \int_{\mathbb{R}^*} E_k(z)^{\top} M E_k(z)\,\theta_{\alpha}^{k}(dz) + Q\Big\}\hat{X}(s) \\
& + \Big\{\mathcal{L}(\bar{M}) + \bar{M}(A - B\Psi) + A^{\top}\bar{M} + \bar{Q}\Big\}\mathbb{E}\left[\hat{X}(s)\,|\mathcal{F}_s^{\alpha}\right] \\
& + \Big\{\mathcal{L}(\Upsilon) + A^{\top}\Upsilon\Big\}\hat{X}(t) + \mathcal{L}(\varphi) + A^{\top}\varphi \\
& + \big(M(s,\alpha(s)) + \bar{M}(s,\alpha(s))\big)\left(b - B\psi(s,\alpha(s))\right) + \sum_{i=1}^{p} C_i^{\top} M\left(\sigma_i - D_i\psi\right) \\
& + \sum_{k=1}^{l} \int_{\mathbb{R}^*} E_k(z)^{\top} M\left(c_k(z) - F_k(z)\,\psi\right)\theta_{\alpha}^{k}(dz)\,.
\end{aligned}$$

This suggests that the functions $M(\cdot,\cdot)$, $\bar{M}(\cdot,\cdot)$, $\Upsilon(\cdot,\cdot)$ and $\varphi(\cdot,\cdot)$ solve the system of equations (2.36). In addition, we can check that $\Psi(\cdot,\cdot)$ and $\psi(\cdot,\cdot)$ in (2.37) are both uniformly bounded. Then for $s \in [0,T]$ the following linear SDEJ

$$\begin{cases}
d\hat{X}(s) = \Big\{\left(A - B\Psi(s,\alpha(s))\right)\hat{X}(s) + b - B\psi(s,\alpha(s))\Big\}ds \\
\qquad + \sum_{i=1}^{p} \Big\{\left(C_i - D_i\Psi(s,\alpha(s))\right)\hat{X}(s) + \sigma_i - D_i\psi(s,\alpha(s))\Big\}dW^{i}(s) \\
\qquad + \sum_{k=1}^{l} \int_{\mathbb{R}^*} \Big\{\left(E_k(z) - F_k(z)\,\Psi(s,\alpha(s))\right)\hat{X}(s-) + c_k(z) \\
\qquad - F_k(z)\,\psi(s,\alpha(s))\}\,\widetilde{N}_{\alpha}^{k}(ds,dz)\,, \\
\hat{X}(0) = x_0, \alpha(0) = e_{i_0},
\end{cases}$$

has a unique solution $\hat{X}(\cdot) \in \mathcal{S}_{\mathcal{F}}^{2}(0,T;\mathbb{R}^n)$, and the following estimate holds

$$\mathbb{E}\left[\sup_{s\in[0,T]}\left|\hat{X}(s)\right|^2\right] \leq K\left(1 + |x_0|^2\right).$$

Hence the control $\hat{u}(\cdot)$ defined by (2.38) is admissible. ■

2.2.3 Uniqueness of the equilibrium control

In this subsection, we prove that if the system of equations (2.36) is solvable, then the state feedback equilibrium control given by (2.38) is the unique open loop Nash equilibrium control of Problem (N).

Theorem 2.2.3 *Let (H1)-(H2) hold. Suppose that* $M(\cdot,\cdot)$, $\bar{M}(\cdot,\cdot)$, $\Upsilon(\cdot,\cdot)$ *and* $\varphi(\cdot,\cdot)$ *are solutions to the system* (2.36). *Then* $\hat{u}(\cdot)$ *given by* (2.38) *is the unique open loop Nash equilibrium control for problem (N).*

Proof. Suppose that, there is another equilibrium control $\tilde{u}(\cdot) \in \mathcal{L}^2_{\mathcal{F},p}(0,T;\mathbb{R}^m)$ and denote by $\tilde{X}(\cdot)$ it's corresponding controlled sate equation, and $\left(\tilde{p}(\cdot;\cdot), \tilde{q}(\cdot), \tilde{r}(\cdot,\cdot), \tilde{l}(\cdot)\right)$ it's corresponding unique solution to the BSDE (2.7) with $\hat{X}(\cdot)$ replaced by $\tilde{X}(\cdot)$, then by Corollary 2.2.1 the 5-tuple $\left(\tilde{p}(\cdot;\cdot), \tilde{q}(\cdot), \tilde{r}(\cdot,\cdot), \tilde{l}(\cdot), \tilde{u}(\cdot)\right)$ satisfies $d\mathbb{P}-a.s, dt-a.e.$

$$B^\top \tilde{p}(t;t) + \sum_{i=1}^{p} D_i^\top \tilde{q}_i(t) + \sum_{k=1}^{l} \int_{\mathbb{R}^*} F_k(z)^\top \tilde{r}_k(t,z)\, \theta_\alpha^k(dz) - R\tilde{u}(t) = 0, \tag{2.46}$$

Now, we define for $t \in [0,T]$, $s \in [t,T]$, $i = 1,...,p$, $k = 1,...,l$, $j = 1,2,...,d$.

$$\begin{cases} \hat{p}(s;t) = \tilde{p}(s;t) + M(s,\alpha(s))\tilde{X}(s) + \bar{M}(s,\alpha(s))\mathbb{E}\left[\tilde{X}(s)\,|\mathcal{F}_s^\alpha\right] \\ \qquad\qquad + \Upsilon(s,\alpha(s))\tilde{X}(t) + \varphi(s,\alpha(s)), \\ \hat{q}_i(s) = \tilde{q}_i(s) + M(s,\alpha(s))\left(C_i\tilde{X}(s) + D_i\tilde{u}(s) + \sigma_i(s)\right), \\ \hat{r}_k(s,z) = \tilde{r}_k(s,z) + M(s,\alpha(s))\left(E_k(z)\tilde{X}(s-) + F_k(z)\tilde{u}(s-) + c_k(z)\right), \\ \hat{l}_j(s) = \tilde{l}_j(s) + \left(M(s,e_j) - M(s,\alpha(s))\right)\tilde{X}(s) \\ \qquad + \left(\bar{M}(s,e_j) - \bar{M}(s,\alpha(s))\right)\mathbb{E}\left[\tilde{X}(s)\,|\mathcal{F}_s^\alpha\right] \\ \qquad + \left(\Upsilon(s,e_j) - \Upsilon(s,\alpha(s))\right)\tilde{X}(t) + \left(\varphi(s,e_j) - \varphi(s,\alpha(s))\right). \end{cases}$$

It is easy to prove that

$$\begin{aligned} &\left(\hat{p}(\cdot;t), \hat{q}(\cdot), \hat{r}(\cdot,\cdot), \hat{l}(\cdot)\right) \in \mathcal{L} \times \mathcal{L}^2_{\mathcal{F}}\left(0,T;(\mathbb{R}^n)^p\right) \\ &\times \mathcal{L}^{\theta,2}_{\mathcal{F},p}\left([0,T]\times\mathbb{R}^*;(\mathbb{R}^n)^l\right) \times \mathcal{L}^{\lambda,2}_{\mathcal{F},p}\left(0,T;(\mathbb{R}^n)^d\right). \end{aligned}$$

By (2.46) we have $d\mathbb{P}-a.s, dt-a.e.$

$$
\begin{aligned}
&- B^{\top}\left\{\hat{p}(t;t)-\left(M(t,\alpha(t))+\bar{M}(t,\alpha(t))+\Upsilon(t,\alpha(t))\right)\tilde{X}(t-)-\varphi(t,\alpha(t))\right\} \\
&- \sum_{i=1}^{p} D_i^{\top}\left\{\hat{q}_i(t)-M(t,\alpha(t))\left(C_i(t)\tilde{X}(t-)+D_i\tilde{u}(t)+\sigma_i\right)\right\} \\
&- \sum_{k=1}^{l}\int_{\mathbb{R}^*} F_k(z)^{\top}\left\{\hat{r}_k(t,z)\right. \\
&\left.- M(t,\alpha(t))\left(E_k(z)\tilde{X}(t-)+F_k(z)\tilde{u}(t)+c_k(z)\right)\right\}\theta_{\alpha}^{k}(dz)+R\tilde{u}(t)=0,
\end{aligned}
$$

since $\Theta(t,\alpha(t))$ exists $d\mathbb{P}-a.s, dt-a.e.$, using (2.37), we get

$$
\begin{aligned}
\tilde{u}(t) &= \Theta(t,\alpha(t))\left\{B^{\top}\hat{p}(t;t)+\sum_{i=1}^{p}D_i^{\top}\hat{q}_i(t)+\sum_{k=1}^{l}\int_{\mathbb{R}^*}F_k(z)^{\top}\hat{r}_k(t,z)\,\theta_{\alpha}^{k}(dz)\right\} \\
&\quad - \Psi(t,\alpha(t))\tilde{X}(t-)-\psi(t,\alpha(t)).
\end{aligned} \tag{2.47}
$$

From the above equality, we remark that if $\hat{p}(t;t)=\hat{q}(t)=\hat{r}(t,z)=0$, $d\mathbb{P}-a.s, dt-a.e.$, we get then $\tilde{u}(\cdot)$ being in the same form of feedback control law as the one specified by (2.38) and hence the uniqueness of the equilibrium control given by (2.38) holds. Moreover, since for any $t\in[0,T]$ and for any $s\in[t,T]$ we have

$$
\begin{aligned}
d\hat{p}(s;t) &= d\tilde{p}(s;t)+d\Big(M(s,\alpha(s))\tilde{X}(s)+\bar{M}(s,\alpha(s))\mathbb{E}\left[\tilde{X}(s)\,|\mathcal{F}_s^{\alpha}\right] \\
&\qquad + \Upsilon(s,\alpha(s))\tilde{X}(t)+\varphi(s,\alpha(s))\Big).
\end{aligned}
$$

Using the equations for $\tilde{p}(\cdot;t)$, $\tilde{X}(\cdot)$, $M(\cdot,\cdot)$, $\bar{M}(\cdot,\cdot)$, $\Upsilon(\cdot,\cdot)$ and $\varphi(\cdot,\cdot)$ respectively and by using equality (2.47) we find then $\left(\hat{p}(\cdot;\cdot),\hat{q}(\cdot),\hat{r}(\cdot,\cdot),\hat{l}(\cdot)\right)$ satisfies

$$
\begin{cases}
d\hat{p}(s;t) = -g\left(s,\hat{p}(s;t),\hat{q}(s),\hat{r}(s,z),\hat{p}(s;s),\mathbb{E}\left[\hat{p}(s;s)\,|\mathcal{F}_s^{\alpha}\right],\right. \\
\qquad\qquad \left.\mathbb{E}\left[\hat{q}(s)\,|\mathcal{F}_s^{\alpha}\right],\mathbb{E}\left[\hat{r}(s,z)\,|\mathcal{F}_s^{\alpha}\right]\right)ds+\sum_{i=1}^{p}\hat{q}_i(s)\,dW^i(s) \\
\qquad\qquad +\sum_{k=1}^{l}\int_{\mathbb{R}^*}\hat{r}_k(s-,z)\,\tilde{N}_{\alpha}^{k}(ds,dz)+\sum_{j=1}^{d}\hat{l}_j(s)\,d\tilde{\Phi}_j(s),\ 0\le t\le s\le T, \\
\hat{p}(T;t)=0,\ t\in[0,T],
\end{cases} \tag{2.48}
$$

where

$$
\begin{aligned}
& g\left(s, \hat{p}\left(s;t\right), \hat{q}\left(s\right), \hat{r}\left(s,z\right), \hat{p}\left(s;s\right), \mathbb{E}\left[\hat{p}\left(s;s\right)|\mathcal{F}_s^{\alpha}\right], \mathbb{E}\left[\hat{q}\left(s\right)|\mathcal{F}_s^{\alpha}\right], \mathbb{E}\left[\hat{r}\left(s,z\right)|\mathcal{F}_s^{\alpha}\right]\right) \\
& = \Bigg\{ A^{\top}\hat{p}\left(s;t\right) + \sum_{i=1}^{p} C_i^{\top}\hat{q}_i\left(s\right) + \sum_{k=1}^{l}\int_{\mathbb{R}^*} E_k\left(z\right)^{\top}\hat{r}_k\left(s,z\right)\theta_{\alpha}^{k}\left(dz\right) \\
& - \Bigg(M\left(s,\alpha\left(s\right)\right)B + \sum_{i=1}^{p} C_i^{\top}M\left(s,\alpha\left(s\right)\right)D_i \\
& + \sum_{k=1}^{l}\int_{\mathbb{R}^*} E_k\left(z\right)^{\top}M\left(s,\alpha\left(s\right)\right)F_k\left(z\right)\theta_{\alpha}^{k}\left(dz\right)\Bigg)\Theta\left(s,\alpha\left(s\right)\right) \\
& \times \left(B^{\top}\hat{p}\left(s;s\right) + \sum_{i=1}^{p} D_i^{\top}\hat{q}_i\left(s\right) + \sum_{k=1}^{l}\int_{\mathbb{R}^*} F_k\left(z\right)^{\top}\hat{r}_k\left(s,z\right)\theta_{\alpha}^{k}\left(dz\right)\right) \\
& - \bar{M}\left(s,\alpha\left(s\right)\right)B\Theta\left(s,\alpha\left(s\right)\right)\Bigg(B^{\top}\mathbb{E}\left[\hat{p}\left(s;s\right)|\mathcal{F}_s^{\alpha}\right] + \sum_{i=1}^{p} D_i^{\top}\mathbb{E}\left[\hat{q}_i\left(s\right)|\mathcal{F}_s^{\alpha}\right] \\
& + \sum_{k=1}^{l}\int_{\mathbb{R}^*} F_k\left(z\right)^{\top}\mathbb{E}\left[\hat{r}_k\left(s,z\right)|\mathcal{F}_s^{\alpha}\right]\theta_{\alpha}^{k}\left(dz\right)\Bigg)\Bigg\}.
\end{aligned}
\tag{2.49}
$$

We will prove in the next lemma that equation (2.48) admits at most one solution in $\mathcal{L}\times\mathcal{L}_{\mathcal{F}}^{2}\left(0,T;\left(\mathbb{R}^n\right)^p\right)\times \mathcal{L}_{\mathcal{F},p}^{\theta,2}\left(\left[0,T\right]\times\mathbb{R}^*;\left(\mathbb{R}^n\right)^l\right)\times\mathcal{L}_{\mathcal{F},p}^{\lambda,2}\left(0,T;\left(\mathbb{R}^n\right)^d\right)$. Thus $\hat{p}\equiv 0$, $\hat{q}\equiv 0$, $\hat{r}\equiv 0$ and $\hat{l}\equiv 0$, hence the uniqueness of the equilibrium control given by (2.38) holds. ■

For the uniqueness of solution to (2.48), we have the following Lemma.

Lemma 2.2.5 *Equation* (2.48) *admits at most one solution in*

$$
\mathcal{L}\times\mathcal{L}_{\mathcal{F}}^{2}\left(0,T;\left(\mathbb{R}^n\right)^p\right)\times\mathcal{L}_{\mathcal{F},p}^{\theta,2}\left(\left[0,T\right]\times\mathbb{R}^*;\left(\mathbb{R}^n\right)^l\right)\times\mathcal{L}_{\mathcal{F},p}^{\lambda,2}\left(0,T;\left(\mathbb{R}^n\right)^d\right).
$$

Proof. For any $t \in [0,T]$ and $s \in [t,T]$, by Itô's formula we have by taking expectations, there exists a constant $K_1 > 0$, such that

$$\begin{aligned}
&\mathbb{E}\left[\left|\hat{p}(s;t)\right|^2 + \sum_{i=1}^{p}\int_s^T \left|\hat{q}_i(\tau)\right|^2 d\tau + \sum_{k=1}^{l}\int_s^T\int_{\mathbb{R}^*}\left|\hat{r}_k(\tau,z)\right|^2 \theta_\alpha^k(dz)\,d\tau \right.\\
&\quad\left. + \sum_{j=1}^{d}\int_s^T \left|\hat{l}_j(\tau)\right|^2 \lambda_j(\tau)\,d\tau\right]\\
&\leq K_1\mathbb{E}\left[\int_s^T \left|\hat{p}(\tau;t)\right|\left(\left|\hat{p}(\tau;t)\right| + \sum_{i=1}^{p}\left|\hat{q}_i(\tau)\right| + \sum_{k=1}^{l}\int_{\mathbb{R}^*}\left|\hat{r}_k(\tau,z)\right|\theta_\alpha^k(dz)\right.\right.\\
&\quad + \sum_{j=1}^{d}\left|\hat{l}_j(\tau)\right|\lambda_j(\tau) + \left|\hat{p}(\tau;\tau)\right| + \left|\mathbb{E}\left[\hat{p}(\tau;\tau)\,|\mathcal{F}_\tau^\alpha\right]\right| + \sum_{i=1}^{p}\left|\mathbb{E}\left[\hat{q}_i(\tau)\,|\mathcal{F}_\tau^\alpha\right]\right|\\
&\quad\left.\left. + \sum_{k=1}^{l}\int_{\mathbb{R}^*}\left|\mathbb{E}\left[\hat{r}_k(\tau,z)\,|\mathcal{F}_\tau^\alpha\right]\right|\theta_\alpha^k(dz) + \sum_{j=1}^{d}\left|\mathbb{E}\left[\hat{l}_j(\tau)\,|\mathcal{F}_\tau^\alpha\right]\right|\lambda_j(\tau)\right)d\tau\right]\\
&\leq K_2\mathbb{E}\int_s^T\left[\left(\left|\hat{p}(\tau;t)\right|^2 + \left|\hat{p}(\tau;\tau)\right|^2\right)d\tau\right]\\
&\quad + \frac{1}{2}\mathbb{E}\left[\sum_{i=1}^{p}\int_s^T\left|\hat{q}_i(\tau)\right|^2 d\tau + \sum_{k=1}^{l}\int_s^T\int_{\mathbb{R}^*}\left|\hat{r}_k(\tau,z)\right|^2\theta_\alpha^k(dz)\,d\tau\right.\\
&\quad\left. + \sum_{j=1}^{d}\int_s^T\left|\hat{l}_j(\tau)\right|^2\lambda_j(\tau)\,d\tau\right],
\end{aligned}$$

where we have used the inequality $cab \leq \beta c^2 a^2 + \frac{1}{\beta}b^2$, $\forall\beta > 0$, $a > 0$, $b > 0$. Hence there exists a $K_3 > 0$ such that

$$\begin{aligned}
&\mathbb{E}\left[\left|\hat{p}(s;t)\right|^2\right] + \sum_{i=1}^{p}\mathbb{E}\left[\int_s^T\left|\hat{q}_i(\tau)\right|^2 d\tau\right] + \sum_{k=1}^{l}\mathbb{E}\left[\int_s^T\int_{\mathbb{R}^*}\left|\hat{r}_k(\tau,z)\right|^2\theta_\alpha^k(dz)\,d\tau\right]\\
&+ \sum_{j=1}^{d}\mathbb{E}\left[\int_s^T\left|\hat{l}_j(\tau)\right|^2\lambda_j(\tau)\,d\tau\right] \leq K_3\mathbb{E}\left[\int_s^T\left(\left|\hat{p}(\tau;t)\right|^2 + \left|\hat{p}(\tau;\tau)\right|^2\right)d\tau\right]. \qquad (2.50)
\end{aligned}$$

Then we have for any $t \in [0,T]$, and $s \in [t,T]$

$$\mathbb{E}\left[\left|\hat{p}(s;t)\right|^2\right] \leq K_3\mathbb{E}\left[\int_s^T\left(\left|\hat{p}(\tau;t)\right|^2 + \left|\hat{p}(\tau;\tau)\right|^2\right)d\tau\right], \qquad (2.51)$$

thus

$$\begin{aligned}
\mathbb{E}\left[\left|\hat{p}(s;t)\right|^2\right] &\leq K_3(T-t)\left(\sup_{\tau\in[t,T]}\mathbb{E}\left[\left|\hat{p}(\tau;t)\right|^2\right] + \sup_{\tau\in[t,T]}\mathbb{E}\left[\left|\hat{p}(\tau;\tau)\right|^2\right]\right)\\
&\leq 2K_3(T-t)\sup_{t\leq\tau\leq s\leq T}\mathbb{E}\left[\left|\hat{p}(s;\tau)\right|^2\right],
\end{aligned}$$

hence

$$\sup_{t\leq\tau\leq s\leq T}\mathbb{E}\left[\left|\hat{p}\left(s;\tau\right)\right|^{2}\right]\leq 2K_{3}\left(T-t\right)\sup_{t\leq\tau\leq s\leq T}\mathbb{E}\left[\left|\hat{p}\left(s;\tau\right)\right|^{2}\right].$$

If we take $\epsilon=\dfrac{1}{8K_{3}}$, we get then for $t\in[T-\epsilon,T]$ and $s\in[t,T]$,

$$\sup_{t\leq\tau\leq s\leq T}\mathbb{E}\left[\left|\hat{p}\left(s;\tau\right)\right|^{2}\right]\leq\frac{1}{4}\sup_{t\leq\tau\leq s\leq T}\mathbb{E}\left[\left|\hat{p}\left(s;\tau\right)\right|^{2}\right],$$

hence

$$\sup_{t\leq\tau\leq s\leq T}\mathbb{E}\left[\left|\hat{p}\left(s;\tau\right)\right|^{2}\right]=0,$$

which means that $\hat{p}\left(s;\tau\right)=0$, $\mathbb{P}-a.s.$ $\forall\left(\tau,s\right)\in\left\{\left(\tau,s\right):t\leq\tau\leq s\leq T\right\}$. For $t\in[T-2\epsilon,T-\epsilon]$ and $s\in[T-\epsilon,T]$, since we have $\hat{p}\left(\tau;\tau\right)=0$ for $\tau\in[s,T]$, by (2.51), we have

$$\mathbb{E}\left[\left|\hat{p}\left(s;t\right)\right|^{2}\right]\leq K_{3}\mathbb{E}\left[\int_{s}^{T}\left|\hat{p}\left(\tau;t\right)\right|^{2}d\tau\right],$$

by Gronwall inequality we conclud that $\hat{p}\left(s;t\right)=0$.

Now for $t\in[T-2\epsilon,T-\epsilon]$ and $s\in[t,T-\epsilon]$, since we have $\hat{p}\left(T-\epsilon;t\right)=0$, we apply the above analysis for the region $t\in[T-\epsilon,T]$ and $s\in[t,T]$, to confirm that $\hat{p}\left(s;\tau\right)=0$, $\mathbb{P}-a.s.$ $\forall\left(\tau,s\right)\in\left\{\left(\tau,s\right):t\leq\tau\leq s\leq T-\epsilon\right\}$. We reiterate the same analysis for $t\in[T-3\epsilon,T-2\epsilon]$, and again and again up time $t=0$. Hence $\hat{p}\left(s;t\right)=0$, $\mathbb{P}-a.s.$ For every $\left(t,s\right)\in\mathcal{D}[0,T]$.

Finally, by (2.50) we obtain

$$\begin{aligned}&\mathbb{E}\left[\int_{0}^{T}\left(\sum_{i=1}^{p}\left|\hat{q}_{i}\left(\tau\right)\right|^{2}+\sum_{k=1}^{l}\int_{\mathbb{R}^{*}}\left|\hat{r}\left(\tau,z\right)\right|^{2}\theta_{\alpha}^{k}\left(dz\right)+\sum_{j=1}^{d}\left|\hat{l}_{j}\left(\tau\right)\right|^{2}\lambda_{j}\left(\tau\right)\right)d\tau\right]\\&\leq K_{3}\mathbb{E}\left[\int_{0}^{T}\left(\left|\hat{p}\left(\tau;t\right)\right|^{2}+\left|\hat{p}\left(\tau;\tau\right)\right|^{2}\right)d\tau\right]\\&=0,\end{aligned}$$

which yields that $\bar{q}\equiv 0,\bar{r}\equiv 0$ and $\bar{l}\equiv 0$. ■

2.3 Applications

In this section, we discuss an extention of a new class of optimization problems [65], in which the investor manages her/his wealth by consuming and investing in a financial market subject to a mean variance criterion controlling the final risk of the portfolio. This problem can be eventually formulated as a time-inconsistent stochastic LQ problem and solved by the results presented in the preceding sections.

2.3.1 Conditional mean-variance-utility consumption-investment and reinsurance problem

We study equilibrium reinsurance (eventually new business), investment and consumption strategies for mean-variance-utility portfolio problem where the surplus of the insurers is assumed to follow a jump-diffusion model. The financial market consists of one riskless asset and one risky asset whose price processes are described by regime-switching SDEs. The problem is formulated as follows. Consider an insurer whose surplus process is described by the following jump-diffusion model

$$d\Lambda(s) = cds + \beta_0 dW^1(s) - d\sum_{i=1}^{N_\alpha(s)} Y_i,\ s \in [0,T], \tag{2.52}$$

where $c > 0$ is the premium rate, β_0 is a positive constant, W^1 is a one-dimensional standard Brownian motion, N_α is a Poisson process with intensity $\lambda > 0$ and $\{Y_i\}_{i\in\mathbb{N}-\{0\}}$ is a sequence of independent and identically distributed positive random variables with common distribution $\mathbb{P}_Y$ having finite first and second moments $\mu_Y = \int_0^\infty z\mathbb{P}_Y(dz)$ and $\sigma_Y = \int_0^\infty z^2\mathbb{P}_Y(dz)$. We assume that W^1, N_α, and $\left\{\sum_{i=1}^{N_\alpha(.)} Y_i\right\}$ are independent. Let Y be a generic random variable which has the same distribution as Y_i. The premium rate c is assumed to be calculated via the expected value principle, i.e. $c = (1+\eta)\lambda\mu_Y$ with safety loading $\eta > 0$.

Note that, the process $\sum_{i=1}^{N_\alpha(s)} Y_i$ can also be defined through a random measure $N_\alpha^1(ds,dz)$ as

$$\sum_{i=1}^{N_\alpha(s)} Y_i = \int_0^s \int_0^\infty zN_\alpha^1(dr,dz),$$

where N_α^1 is a finite Poisson random measure with a random compensator having the form $\theta_\alpha^1(dz)\,ds = \lambda\mathbb{P}_Y(dz)\,ds$. We recall that $\tilde{N}_\alpha^1(ds,dz) = N_\alpha^1(ds,dz) - \theta_\alpha^1(dz)\,ds$ define the compensated jump martingale random measure of N_α^1. Obviously, we have

$$\int_0^{+\infty} z\theta_\alpha^1(dz)\,ds = \lambda\int_0^{+\infty} z\mathbb{P}_Y(dz)\,ds = \lambda\mu_Y ds.$$

Hence (2.52) is equivalent to

$$d\Lambda(s) = \eta\lambda\mu_Y ds + \beta_0 dW^1(s) - \int_0^{+\infty} z\tilde{N}_\alpha^1(ds,dz). \tag{2.53}$$

Suppose that the insurer is allowed to invest its wealth in a financial market, in which two securities are traded continuously. One of them is a bond, with price $S^0(s)$ at time $s \in [0,T]$ governed by

$$dS^0(s) = r_0(s,\alpha(s)) S^0(s) ds, \ S^0(0) = s_0 > 0. \tag{2.54}$$

There is also a risky asset with unit price $S^1(s)$ at time $s \in [0,T]$ governed by

$$\begin{aligned} dS^1(s) = \ & S^1(s-) \big(\sigma(s,\alpha(s)) ds + \beta(s,\alpha(s)) dW^2(s) \\ & + \textstyle\int_{-1}^{+\infty} z \left(N_\alpha^2(ds,dz) - \theta_\alpha^2(dz) ds\right)\big), \ S^1(0) = s^1 > 0, \end{aligned} \tag{2.55}$$

where $r_0, \sigma, \beta : [0,T] \times \mathcal{X} \to (0,\infty)$ are assumed to be deterministic and continuous functions such that $\sigma(s,\alpha(s)) > r_0(s,\alpha(s)) > 0$, $W^2(\cdot)$ is a one-dimensional standard Brownian motion, N_α^2 is a finite Poisson random measure with random compensator having the form $n_\alpha^2(ds,dz) = \theta_\alpha^2(dz) ds$. We assume that $W^1(\cdot), W^2(\cdot), N_\alpha^1(\cdot,\cdot)$ and $N_\alpha^2(\cdot,\cdot)$ are independent and $\theta_\alpha^2(\cdot)$ is a Lévy measure on $(-1,+\infty)$ such that $\int_{-1}^{+\infty} |z|^2 \theta_\alpha^2(dz) < \infty$.

The insurer, starting from an initial capital $x_0 > 0$ at time 0, is allowed to dynamically purchase proportional reinsurance (acquire new business) , invest in the financial market and consuming. A trading strategy $u(\cdot)$ is described by a three-dimensional stochastic processes $(u_1(\cdot), u_2(\cdot), u_3(\cdot))^\top$. The strategy $u_1(s) \geq 0$ represents the retention level of reinsurance or new business acquired at time $s \in [0,T]$. We point that $u_1(s) \in [0,1]$ corresponds to a proportional reinsurance cover and shows that the cedent should divert part of the premium to the reinsurer at the rate of $(1 - u_1(t)))(\theta_0 + 1)\lambda\mu_Y$, where θ_0 is the relative safety loading of the reinsurer satisfying $\theta_0 \geq \eta$. Meanwhile, for each claim Y occurring at time s, the reinsurer pays $(1 - u_1(t)))Y$ of the claim, and the cedent pays the rest. $u_1(s) \in (1,+\infty)$ corresponds to acquiring new business. $u_2(s) \geq 0$ represents the amount invested in the risky stock at time s. The dollar amount invested in the bond at time s is $X^{x_0,e_{i_0},u(\cdot)}(s) - u_2(s)$, where $X^{x_0,e_{i_0},u(\cdot)}(\cdot)$ is the wealth process associated with strategy $u(\cdot)$ and the initial states (x_0, e_{i_0}), $u_3(s)$ represents the consumption rate at time $s \in [0,T]$. Thus, incorporating reinsurance/new business, and investment strategies into the surplus process and the risky asset, respectively. As time evolves, we consider the evolution of the controlled stochastic differential equation parametrized by $(t,\xi,e_i) \in [0,T] \times \mathbb{L}^2(\Omega, \mathcal{F}_t^\alpha, \mathbb{P}; \mathbb{R}) \times \chi$ and satisfied by $X(\cdot)$, for $s \in [0,T]$

$$\begin{cases} dX(s) = \{r_0(s,\alpha(s)) X(s) + (\delta + \theta_0 u_1(s)) \lambda\mu_Y + r(s,\alpha(s)) u_2(s)\} ds \\ \qquad\qquad - u_3(s) ds + \beta_0 u_1(s) dW^1(s) + \beta(s,\alpha(s)) u_2(s) dW^2(s) \\ \qquad\qquad - u_1(s-) \int_0^{+\infty} z\tilde{N}_\alpha^1(ds,dz) + u_2(s-) \int_{-1}^{+\infty} z\tilde{N}_\alpha^2(ds,dz), \\ X(t) = \xi, \alpha(t) = e_i, \end{cases} \tag{2.56}$$

where $r(s,\alpha(s)) = (\sigma(s,\alpha(s)) - r_0(s,\alpha(s)))$ and $\delta = \eta - \theta_0$. Then, for any $(t,\xi,e_i) \in [0,T] \times \mathbb{L}^2(\Omega,\mathcal{F}_t^\alpha,\mathbb{P};\mathbb{R}) \times \chi$ the mean-variance-utility consumption-investment and reinsurance optimization problem is reduced to maximize the utility function $J(t,\xi,e_i;\cdot)$ given by

$$\begin{aligned} J(t,\xi,e_i;u(.)) &= \mathbb{E}\Big[\int_t^T \tfrac{1}{2} h(s-t)\, u_3(s)^2 ds + \tfrac{1}{2}\mathrm{Var}[X(T)\,|\mathcal{F}_T^\alpha] \\ &\qquad - (\mu_1\xi + \mu_2)\,\mathbb{E}[X(T)\,|\mathcal{F}_T^\alpha]\Big], \end{aligned} \tag{2.57}$$

subject to (2.56), where $h(\cdot):[0,T] \to \mathbb{R}$ is a general deterministic non-exponential discount function satisfying $h(0)=1$, $h(s)>0$ $ds-a.e.$ and $\int_0^T h(s)ds < \infty$. In this chapter we consider general discount functions satisfying the above standing assumptions. Some possible examples of discount functions are considered in the literatures [72] and [22].

Remark 2.3.1 *Similar to [45] and [50], due to the presence of the observable random factor $\alpha(\cdot)$, we consider the expectation of a conditional mean-variance criterion in the above cost functional. This is different from the mean-variance portfolio selection problem with regime switching considered in [71] and [13]. In [50], a conditional mean-variance portfolio selection problem with common noise is proposed and solved using linear quadratic optima control of conditional McKean-Vlasov equation with random coefficients and dynamic programming approach.*

With $n=1$, $p=l=m=3$, the optimal control problem associated with (2.56) and (2.57) is equivalent to maximize

$$\begin{aligned} J(t,\xi,e_i;u(.)) = \ &\mathbb{E}\Big[\int_t^T \tfrac{1}{2}\left\langle h(s-t)\,\Gamma^\top\Gamma u(s), u(s)\right\rangle ds + \tfrac{1}{2}\mathrm{Var}[X(T)\,|\mathcal{F}_T^\alpha] \\ &\qquad - (\mu_1\xi + \mu_2)\,\mathbb{E}[X(T)\,|\mathcal{F}_T^\alpha]\Big], \end{aligned} \tag{2.58}$$

subject to (2.1). Here $A = r_0(s,\alpha(s))$, $B = \begin{pmatrix} \lambda\mu_Y\theta_0 & r(s,\alpha(s)) & -1 \end{pmatrix}$, $b = \delta\lambda\mu_Y$, $D_1 = \begin{pmatrix} \beta_0 & 0 & 0 \end{pmatrix}$, $D_2 = \begin{pmatrix} 0 & \beta(s,\alpha(s)) & 0 \end{pmatrix}$, $Q=0$, $\bar{Q}=0$, $F_1(z) = \begin{pmatrix} -z1_{(0,\infty)}(z) & 0 & 0 \end{pmatrix}$, $F_2(z) = \begin{pmatrix} 0 & z1_{(-1,\infty)}(z) & 0 \end{pmatrix}$, $\Gamma = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$, $R(t,s) = h(s-t)\,\Gamma^\top\Gamma$, $G=1$, $\bar{G}=-1$, $C_i=0$, $\sigma_i=0$, $E_k(z)=0$ and $c_k(z)=0$. Thus, the above model is a special case of the general time inconsistent LQ problem formulated earlier in this chapter. Then we apply corollary 2.2.1 and theorem 2.2.2 to obtain the unique Nash equilibrium trading strategy. Define

$$\rho(s,\alpha(s)) \triangleq \left(\frac{(\lambda\mu_Y\theta_0)^2}{\left(\beta_0^2 + \int_0^{+\infty} z^2\theta_\alpha^1(dz)\right)} + \frac{r(s,\alpha(s))^2}{\left(\beta(s,\alpha(s))^2 + \int_{-1}^{+\infty} z^2\theta_\alpha^2(dz)\right)} \right). \tag{2.59}$$

Then the system (2.36) reduced to, for $s \in [0, T]$

$$\begin{cases} M'(s,e_i) + M(s,e_i)(2r_0(s,e_i) - \Upsilon(s,e_i) + \lambda_{ii}) - \rho(s,e_i)\Upsilon(s,e_i) \\ \quad + \sum_{j\neq i}^{d} \lambda_{ij} M(s,e_j) = 0, \\ \bar{M}'(s,e_i) + \bar{M}(s,e_i)(2r_0(s,e_i) - \Upsilon(s,e_i) + \lambda_{ii}) - \rho(s,e_i)\Upsilon(s,e_i) \\ \quad + \sum_{j\neq i}^{d} \lambda_{ij} \bar{M}(s,e_j) = 0, \\ \Upsilon'(s,e_i) + \Upsilon(s,e_i)(r_0(s,e_i) + \lambda_{ii}) + \sum_{j\neq i}^{d} \lambda_{ij}\Upsilon(s,e_j) = 0, \\ \varphi'(s,e_i) + \varphi(s,e_i)(r_0(s,e_i) + \lambda_{ii}) + \sum_{j\neq i}^{d} \lambda_{ij}\varphi(s,e_j) = 0, \\ M(T,e_i) = 1,\ \bar{M}(T,e_i) = -1,\ \Upsilon(T,e_i) = -\mu_1,\ \varphi(T,e_i) = -\mu_2. \end{cases} \tag{2.60}$$

By standard arguments, we obtain for $s \in [0, T]$ and $e_i \in \mathcal{X}$

$$\begin{aligned} M(s,e_i) &= e^{\int_s^T (2r_0(\tau,e_i) - \Upsilon(\tau,e_i) + \lambda_{ii}) d\tau} \\ &\quad \Big(1 + \int_s^T e^{-\int_\tau^T (2r_0(u,e_i) - \Upsilon(u,e_i) + \lambda_{ii}) du} \{-\rho(\tau,e_i)\Upsilon(\tau,e_i) \\ &\quad + \sum_{j\neq i}^{d} \lambda_{ij} M(\tau,e_j)\} d\tau\Big), \\ &= -\bar{M}(s,e_i), \end{aligned}$$

also we have for $e_i \in \mathcal{X}$,

$$\begin{aligned} \Upsilon(s,e_i) &= e^{\int_s^T (r_0(\tau,e_i) + \lambda_{ii}) d\tau} \\ &\quad \times \left(-\mu_1 + \int_s^T e^{\int_\tau^T -(r_0(u,e_i) + \lambda_{ii}) du} \sum_{j\neq i}^{d} \lambda_{ij}\Upsilon(\tau,e_j)\, d\tau\right) \end{aligned}$$

and

$$\begin{aligned} \varphi(s,e_i) &= e^{\int_s^T (r_0(\tau,e_i) + \lambda_{ii}) d\tau} \\ &\quad \times \left(-\mu_2 + \int_s^T e^{\int_\tau^T -(r_0(u,e_i) + \lambda_{ii}) du} \sum_{j\neq i}^{d} \lambda_{ij}\varphi(\tau,e_j)\, d\tau\right). \end{aligned}$$

In view of theorem 2.2.2, the Nash equilibrium control (2.38) gives, for $s \in [0, T]$

$$\hat{u}_1(s) = -\sum_{i=1}^{d} \langle \alpha(s-), e_i \rangle \frac{(\lambda \mu_Y \theta_0)}{\left(\beta_0^2 + \int_0^{+\infty} z^2 \theta_\alpha^1(dz)\right)} \Big(\Phi_1(s, e_i) \hat{X}(s) + \Phi_2(s, e_i) \Big), \quad (2.61)$$

$$\hat{u}_2(s) = -\sum_{i=1}^{d} \langle \alpha(s-), e_i \rangle \frac{r(s,e_i)}{\left(\beta(s,e_i)^2 + \int_{-1}^{+\infty} z^2 \theta_\alpha^2(dz)\right)} \Big(\Phi_1(s, e_i) \hat{X}(s) + \Phi_2(s, e_i) \Big), \quad (2.62)$$

$$\hat{u}_3(s) = \sum_{i=1}^{d} \langle \alpha(s-), e_i \rangle \Big(\Upsilon(s, e_i) \hat{X}(s) + \varphi(s, e_i) \Big), \quad (2.63)$$

where $\forall (s, e_i) \in [0, T] \times \mathcal{X}$

$$\Phi_1(s, e_i) = \frac{e^{\int_s^T (-r_0(\tau,e_i) + \Upsilon(\tau,e_i))d\tau} \left(-\mu_1 + \int_s^T e^{\int_\tau^T -(r_0(u,e_i)+\lambda_{ii})du} \sum_{j\neq i}^{d} \lambda_{ij} \Upsilon(\tau,e_j) d\tau \right)}{1 + \int_s^T e^{-\int_\tau^T (2r_0(u,e_i) - \Upsilon(u,e_i) + \lambda_{ii})du} \left\{ -\rho(\tau,e_i)\Upsilon(\tau,e_i) + \sum_{j\neq i}^{d} \lambda_{ij} M(\tau,e_j) \right\} d\tau}, \quad (2.64)$$

and

$$\Phi_2(s, e_i) = \frac{e^{\int_s^T (-r_0(\tau,e_i) + \Upsilon(\tau,e_i))d\tau} \left(-\mu_2 + \int_s^T e^{\int_\tau^T -(r_0(u,e_i)+\lambda_{ii})du} \sum_{j\neq i}^{d} \lambda_{ij} \varphi(\tau,e_j) d\tau \right)}{1 + \int_s^T e^{-\int_\tau^T (2r_0(u,e_i) - \Upsilon(u,e_i) + \lambda_{ii})du} \left\{ -\rho(\tau,e_i)\Upsilon(\tau,e_i) + \sum_{j\neq i}^{d} \lambda_{ij} M(\tau,e_j) \right\} d\tau}. \quad (2.65)$$

The conditional expectation of the corresponding equilibrium wealth process solves the equation

$$\begin{cases} d\mathbb{E}\left[\hat{X}(s) | \mathcal{F}_T^\alpha\right] = \left\{ \mathcal{P}_1(s, \alpha(s)) \mathbb{E}\left[\hat{X}(s) | \mathcal{F}_T^\alpha\right] + \mathcal{P}_2(s, \alpha(s)) \right\} ds, \\ \mathbb{E}\left[\hat{X}(0) | \mathcal{F}_T^\alpha\right] = x_0, \end{cases}$$

where

$$\begin{cases} \mathcal{P}_1(s, \alpha(s)) = r_0(s, \alpha(s)) - \rho(s, \alpha(s)) \Phi_1(s, \alpha(s)) - \Upsilon(s, \alpha(s)), \\ \mathcal{P}_2(s, \alpha(s)) = -\rho(s, \alpha(s)) \Phi_2(s, \alpha(s)) - \varphi(s, \alpha(s)) + b(s, \alpha(s)). \end{cases}$$

A technical computations show that

$$\begin{cases} d\mathbb{E}\left[\hat{X}(s)^2 | \mathcal{F}_T^\alpha\right] = \Big(\{ 2\mathcal{P}_1(s, \alpha(s)) + \mathcal{P}_3(s, \alpha(s)) \} \mathbb{E}\left[\hat{X}(s)^2 | \mathcal{F}_T^\alpha\right] \\ \qquad + 2\left(\mathcal{P}_2(s, \alpha(s)) + \mathcal{P}_4(s, \alpha(s))\right) \mathbb{E}\left[\hat{X}(s) | \mathcal{F}_T^\alpha\right] \\ \qquad + \mathcal{P}_5(s, \alpha(s)) \Big) ds, \\ \mathbb{E}\left[\hat{X}(0)^2 | \mathcal{F}_T^\alpha\right] = x_0^2, \end{cases}$$

and

$$\begin{cases} d\text{Var}\left[\hat{X}(s)|\mathcal{F}_T^\alpha\right] \\ \quad =\left\{2\mathcal{P}_1(s,\alpha(s))\text{Var}\left[\hat{X}(s)|\mathcal{F}_T^\alpha\right]+\mathcal{P}_3(s,\alpha(s))\mathbb{E}\left[\hat{X}(s)^2|\mathcal{F}_T^\alpha\right]\right. \\ \quad\quad \left.+2\mathcal{P}_4(s,\alpha(s))\mathbb{E}\left[\hat{X}(s)|\mathcal{F}_T^\alpha\right]+\mathcal{P}_5(s,\alpha(s))\right\}ds, \\ \text{Var}\left[\hat{X}(0)|\mathcal{F}_T^\alpha\right]=0, \end{cases}$$

where

$$\begin{cases} \mathcal{P}_3(s,\alpha(s))=\rho(s,\alpha(s))\Phi_1(s,\alpha(s))^2, \\ \mathcal{P}_4(s,\alpha(s))=\rho(s,\alpha(s))\Phi_1(s,\alpha(s))\Phi_2(s,\alpha(s)), \\ \mathcal{P}_5(s,\alpha(s))=\rho(s,\alpha(s))\Phi_2(s,\alpha(s))^2. \end{cases}$$

Then

$$\begin{aligned} \mathbb{E}\left[\hat{X}(s)|\mathcal{F}_T^\alpha\right]= & \sum_{i=1}^d\langle\alpha(s-),e_i\rangle e^{\int_0^s \mathcal{P}_1(\tau,e_i)d\tau} \\ & \times\left(x_0+\int_0^s e^{\int_0^\tau -\mathcal{P}_1(u,e_i)du}\mathcal{P}_2(\tau,e_i)d\tau\right), \\ \mathbb{E}\left[\hat{X}(s)^2|\mathcal{F}_T^\alpha\right]= & \sum_{i=1}^d\langle\alpha(s-),e_i\rangle e^{\int_0^s\{2\mathcal{P}_1(\tau,e_i)+\mathcal{P}_3(\tau,e_i)\}d\tau} \\ & \times\left\{x_0^2+\int_0^s e^{\int_0^\tau -\{2\mathcal{P}_1(u,e_i)+\mathcal{P}_3(u,e_i)\}du}\right. \\ & \left.\times\left(2(\mathcal{P}_2(\tau,e_i)+\mathcal{P}_4(\tau,e_i))\mathbb{E}\left[\hat{X}(\tau)|\mathcal{F}_T^\alpha\right]+\mathcal{P}_5(\tau,e_i)\right)d\tau\right\}, \end{aligned}$$

and

$$\begin{aligned} \text{Var}\left[\hat{X}(s)|\mathcal{F}_T^\alpha\right]= & \sum_{i=1}^d\langle\alpha(s-),e_i\rangle e^{\int_0^s 2\mathcal{P}_1(\tau,e_i)d\tau}\int_0^s e^{\int_0^\tau -2\mathcal{P}_1(u,e_i)du}\left\{\mathcal{P}_3(\tau,e_i)\mathbb{E}\left[\hat{X}(\tau)^2|\mathcal{F}_T^\alpha\right]\right. \\ & \left.+2\mathcal{P}_4(\tau,e_i)\mathbb{E}\left[\hat{X}(\tau)|\mathcal{F}_T^\alpha\right]+\mathcal{P}_5(\tau,e_i)\right\}d\tau. \end{aligned}$$

Hence the objective function value of the equilibrium trading strategy $\hat{u}(\cdot)$ is

$$\begin{aligned} J(0,x_0,e_{i_0};\hat{u}(\cdot))= & \mathbb{E}\left[\sum_{i=1}^d\langle\alpha(T),e_i\rangle\left\{\int_0^T\frac{1}{2}h(s)\left(\Upsilon(s,e_i)\hat{X}(s)+\varphi(s,e_i)\right)^2ds\right.\right. \\ & +\frac{1}{2}e^{\int_0^T 2\mathcal{P}_1(\tau,e_i)d\tau}\int_0^T e^{\int_0^\tau -2\mathcal{P}_1(u,e_i)du}\left\{\mathcal{P}_3(\tau,e_i)\mathbb{E}\left[\hat{X}(\tau)^2|\mathcal{F}_T^\alpha\right]\right. \\ & \left.+2\mathcal{P}_4(\tau,e_i)\mathbb{E}\left[\hat{X}(\tau)|\mathcal{F}_T^\alpha\right]+\mathcal{P}_5(\tau,e_i)\right\}d\tau \\ & \left.\left.-(\mu_1x_0+\mu_2)e^{\int_0^T\mathcal{P}_1(\tau,e_i)d\tau}\left(x_0+\int_0^T e^{\int_0^\tau -\mathcal{P}_1(u,e_i)du}\mathcal{P}_2(\tau,e_i)d\tau\right)\right\}\right]. \end{aligned}$$

2.3.2 Conditional mean-variance investment and reinsurance strategies

In this paragraph, we will address a special case where the insurers does not take into account the consumption strategy. The objective is to maximize the conditional expectation of terminal wealth $\mathbb{E}[X(T)|\mathcal{F}_T^\alpha]$ and at the same time to minimize the conditional variance of the terminal wealth $\text{Var}[X(T)|\mathcal{F}_T^\alpha]$, over controls $u(\cdot)$ valued in $\mathbb{R}^2$. Then, the mean-variance investment and reinsurance optimization problem is denoted as: minimizing the cost $J(t,\xi,e_i;.)$ given by

$$J(t,\xi,e_i;u(\cdot)) = \frac{1}{2}\mathbb{E}\left[\operatorname{Var}\left[X(T)|\mathcal{F}_T^\alpha\right] - (\mu_1\xi+\mu_2)\,\mathbb{E}\left[X(T)|\mathcal{F}_T^\alpha\right]\right], \tag{2.66}$$

subject to, for $s\in[0,T]$

$$\begin{cases} dX(s) = \{r_0(s,\alpha(s))X(s) + (\delta+\theta_0 u_1(s))\lambda\mu_Y + r(s,\alpha(s))u_2(s)\}\,ds \\ \qquad\qquad + \beta_0 u_1(s)\,dW^1(s) + \beta(s,\alpha(s))u_2(s)\,dW^2(s) \\ \qquad\qquad - u_1(s-)\int_0^{+\infty} z\tilde{N}_\alpha^1(ds,dz) + u_2(s-)\int_{-1}^{+\infty} z\tilde{N}_\alpha^2(ds,dz) \\ X(t)=\xi, \alpha(t)=e_i, \end{cases} \tag{2.67}$$

where $(t,\xi,e_i)\in[0,T]\times\mathbb{L}^2(\Omega,\mathcal{F}_t^\alpha,\mathbb{P};\mathbb{R})\times\chi$ and $u(\cdot)=(u_1(\cdot),u_2(\cdot))^\top$ is an admissible trading strategy. In this case, the equilibrium strategy given by the expressions (2.61) and (2.62) change to, for $s\in[0,T]$

$$\hat{u}_1(s) = -\sum_{i=1}^d \langle\alpha(s-),e_i\rangle \frac{(\lambda\mu_Y\theta_0)}{\left(\beta_0^2+\int_0^{+\infty} z^2\theta_\alpha^1(dz)\right)}\left(\Phi_1(s,e_i)\hat{X}(s)+\Phi_2(s,e_i)\right), \tag{2.68}$$

$$\hat{u}_2(s) = -\sum_{i=1}^d \langle\alpha(s-),e_i\rangle \frac{r(s,e_i)}{\left(\beta(s,e_i)^2+\int_{-1}^{+\infty} z^2\theta_\alpha^2(dz)\right)}\left(\Phi_1(s,e_i)\hat{X}(s)+\Phi_2(s,e_i)\right), \tag{2.69}$$

where $\forall(s,e_i)\in[0,T]\times\mathcal{X}$

$$\Phi_1(s,e_i) = \frac{e^{\int_s^T -r_0(\tau,e_i)d\tau}\left(-\mu_1+\int_s^T e^{\int_\tau^T -r_0(u,e_i)du}\sum_{j\neq i}^d \lambda_{ij}\Upsilon(\tau,e_j)d\tau\right)}{1+\int_s^T e^{-\int_\tau^T(2r_0(u,e_i)+\lambda_{ii})du}\left\{-\rho(\tau,e_i)\Upsilon(\tau,e_i)+\sum_{j\neq i}^d\lambda_{ij}M(\tau,e_j)\right\}d\tau}, \tag{2.70}$$

$$\Phi_2(s,e_i) = \frac{e^{\int_s^T -r_0(\tau,e_i)d\tau}\left(-\mu_2+\int_s^T e^{\int_\tau^T -r_0(u,e_i)du}\sum_{j\neq i}^d \lambda_{ij}\varphi(\tau,e_j)d\tau\right)}{1+\int_s^T e^{-\int_\tau^T(2r_0(u,e_i)+\lambda_{ii})du}\left\{-\rho(\tau,e_i)\Upsilon(\tau,e_i)+\sum_{j\neq i}^d\lambda_{ij}M(\tau,e_j)\right\}d\tau}. \tag{2.71}$$

Numerical Example. In this section, by providing some numerical examples, we demonstrate the validity and good performance of our proposed study to solving the mean-variance problem with Markov switching. For simplicity, let us consider equation (2.67) in which the Markov chain takes two possible states e_1 =1 and e_2 =2, i.e. χ ={1, 2}, with the generator of the Markov chain being

$$\mathcal{H} = \begin{pmatrix} 2 & -2 \\ -4 & 4 \end{pmatrix}$$

and the initial condition $X(0)=1.1$. For illustration purpose, we assume the finite time horizon is given with $T = 60$ and that the coefficients of the dynamic equation are given below

	$r_0(\alpha(t))$	$r(\alpha(t))$	$\beta(\alpha(t))$	δ	θ_0	β_0	λ	μ_Y
$\alpha(t)$ =1	0.35	0.20	0.30	0.09	1.5	0.5	0.65	0.6
$\alpha(t)$ =2	0.40	0.25	0.55	0.09	1.5	0.5	0.65	0.6

We consider the cost function defined by equation (2.66) with $\mu_1 = \mu_2 = 1$. Without loss of generality we use the notation $\mathbf{E}[\mathbf{X}(\mathbf{t}, \mathbf{i})]$ for $\mathbb{E}\left[\hat{X}(t) \left| \mathcal{F}_T^i \right.\right]$ where i=1,2 and α.

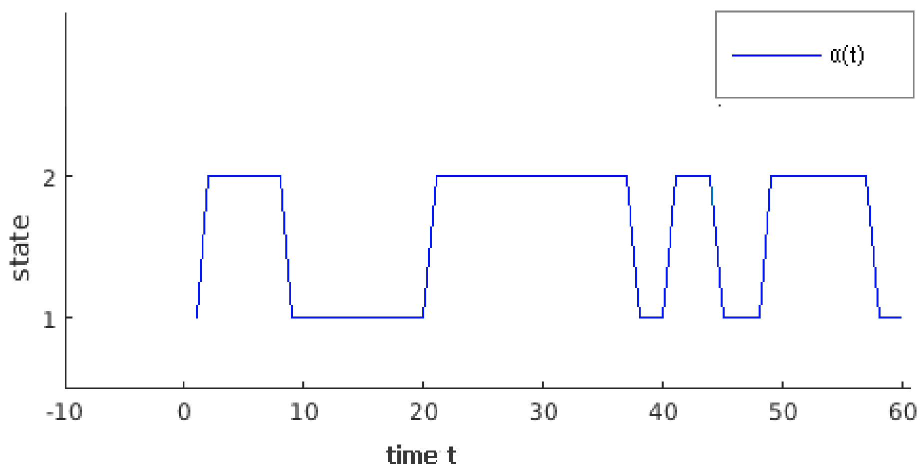

Fig. 1. The state change of the Markov chain

Figure 1 depicts the state change of the Markov chain $\alpha(\cdot)$ between 0 and 60 unit of time, where the initial state is assume to be $\alpha(0)$ =1.

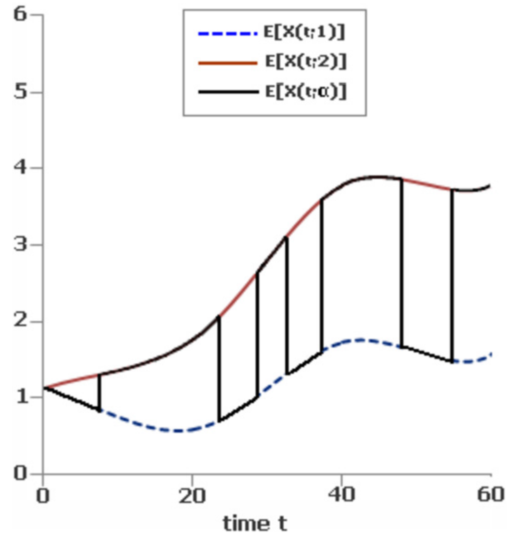

Fig. 2. Expected equilibrium wealth in the three modes for i = 1, 2 and alpha

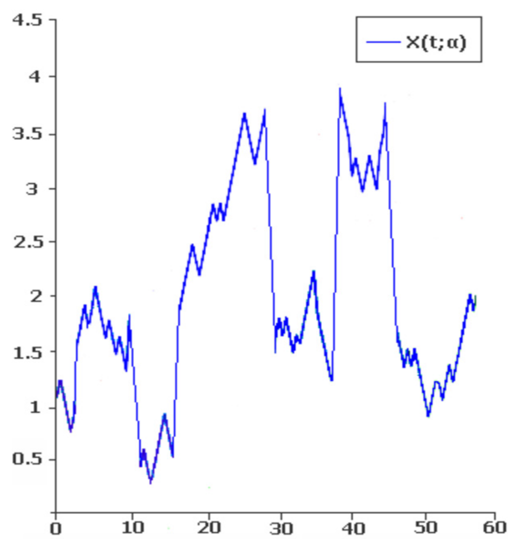

Fig. 3. Trajectories of the equilibrium wealth correspond to the Markov chain

Figure 2 presents the curves of the different state trajectories of the equilibrium expected wealth $\mathbf{E}[\mathbf{X}(\mathbf{t}, \mathbf{i})]$, in the three mods, $i = 1$, $i = 2$ and $i = \alpha(t)$. By using Matlab's advanced ODE solvers (particularly the function ode45) and Markov chain $\alpha(\cdot)$, we can achieve trajectories of $\mathbf{E}[\mathbf{X}(\mathbf{t}, \mathbf{1})]$, $\mathbf{E}[\mathbf{X}(\mathbf{t}, \mathbf{2})]$ and

$\mathbf{E}[\mathbf{X}(\mathbf{t},\alpha(t))]$ and their graphs, the dashed blue line is the graph of $\mathbf{E}[\mathbf{X}(\mathbf{t},\mathbf{1})]$, the continuous brouwn line is the graph of $\mathbf{E}[\mathbf{X}(\mathbf{t},\mathbf{2})]$, and the solid black line is the graph of $\mathbf{E}[\mathbf{X}(\mathbf{t},\alpha(t))]$, whose values are switched between the dashed blue line and the continuous brouwn line.

Figure 3 shows the state trajectorie of the equilibrium wealth X($\cdot$). In fact, when $\alpha(0)=1$, X(0)=1.1 is the initial state trajectory. Then the values are also switched between two paths which are the trajectories of the equilibrium wealth correspond to the different states of the Markov chain $\alpha(t)=1$ and $\alpha(t)=2$. As a result, by comparing with Figure 1, we can clearly see how the Markovian switching influences the overall behaviour of the state trajectories of the equilibrium wealth.

2.3.3 Special cases and relationship to other works

Classical Cramér–Lundberg

Now, assume that the insurer's surplus is modelled the classical Cramér–Lundberg (CL) model (i.e. the model (2.53) with $\beta_0=0$), and that the financial market consists of one risk-free asset whose price process is given by (2.54), and only one risky asset whose price process do not have jumps and is modelled by a diffusion process (i.e. the model (2.55) with $z=0, ds-a.e.$). Then the dynamics of the wealth process $X(\cdot)=X^{t,\xi,e_i}(\cdot;u(\cdot))$ which corresponds to an admissible strategy $u(\cdot)=(u_1(\cdot),u_2(\cdot))^\top$ and initial pair $(t,\xi,e_i)\in[0,T]\times\mathbb{L}^2(\Omega,\mathcal{F}_t^\alpha,\mathbb{P};\mathbb{R})\times\mathcal{X}$ can be described for $s\in[t,T]$, by

$$\begin{cases} dX(s)=\{r_0(s,\alpha(s))X(s)+(\delta+\theta_0u_1(s))\lambda\mu_Y+r(s,\alpha(s))u_2(s)\}ds \\ \qquad\qquad +\beta(s,\alpha(s))u_2(s)dW^2(s)-u_1(s-)\int_0^{+\infty}z\tilde{N}_\alpha^1(ds,dz), \\ X(t)=\xi,\alpha(t)=e_i. \end{cases} \tag{2.72}$$

We derive the equilibrium strategy which is described as in the following two cases.

Case 1: $\mu_1=0$ We suppose that $\mu_1=0$ and $\mu_2=\dfrac{1}{\gamma}$, such that $\gamma>0$. Then the minimisation problem (2.66) reduces to

$$\min J(t,\xi,e_i;u(\cdot))=\mathbb{E}\left[\frac{1}{2}\mathrm{Var}\left[X(T)|\mathcal{F}_T^\alpha\right]-\frac{1}{\gamma}\mathbb{E}\left[X(T)|\mathcal{F}_T^\alpha\right]\right], \tag{2.73}$$

subject to $u(\cdot)\in\mathcal{L}^2_{\mathcal{F},p}\left(0,T;\mathbb{R}^2\right)$, where $X(\cdot)=X^{t,\xi,e_i}(\cdot;u(\cdot))$ satisfies (2.72), for every $(t,x_t,e_i)\in[0,T]\times\mathbb{L}^2(\Omega,\mathcal{F}_t^\alpha,\mathbb{P};\mathbb{R})\times\chi$. In this case the equilibrium reinsurance-investment strategy given by (2.68) and (2.69) for $s \in [0,T]$ becomes

$$\hat{u}_1(s) = -\sum_{i=1}^{d} \langle \alpha(s-), e_i \rangle \frac{(\lambda \mu_Y \theta_0)}{\int_0^{+\infty} z^2 \theta_\alpha^1(dz)} \Big(\Phi_1(s,e_i) \hat{X}(s) + \Phi_2(s,e_i) \Big), \tag{2.74}$$

$$\hat{u}_2(s) = -\sum_{i=1}^{d} \langle \alpha(s-), e_i \rangle \frac{r(s,e_i)}{\beta(s,e_i)^2} \Big(\Phi_1(s,e_i) \hat{X}(s) + \Phi_2(s,e_i) \Big), \tag{2.75}$$

where $\Phi_1(s,e_i)$ and $\Phi_2(s,e_i)$ are given by (2.70) and (2.71) for $\mu_1 = 0$ and $\mu_2 = \frac{1}{\gamma}$.

In the absence of the Markov chain i.e. $d = 1$, $\ell(s,\alpha(s)) \equiv \ell(s)$ for $\ell = r_0, r$ and β, the equilibrium solution (2.74) and (2.75) for $s \in [0,T]$, reduces to

$$\hat{u}_1(s) = \frac{(\lambda \mu_Y \theta_0) e^{\int_s^T -r_0(\tau)d\tau}}{\gamma \Big(\int_0^{+\infty} z^2 \theta^1(dz) \Big)},$$

$$\hat{u}_2(s) = \frac{r(s) e^{\int_s^T -r_0(\tau)d\tau}}{\gamma \beta(s)^2}.$$

It is worth pointing out that the above equilibrium solutions are identical to the ones found in Zeng and Li [70] by solving some extended HJB equations.

Case 2: $\mu_2 = 0$ Now, suppose that $\mu_1 = \frac{1}{\gamma}$ and $\mu_2 = 0$, such that $\gamma > 0$. Then the minimisation problem (2.66) reduces to

$$\min J(t,\xi,e_i;u(\cdot)) = \mathbb{E}\left[\frac{1}{2} \mathrm{Var}\left[X(T) | \mathcal{F}_T^\alpha \right] - \frac{\xi}{\gamma} \mathbb{E}\left[X(T) | \mathcal{F}_T^\alpha \right] \right],$$

for any $(t, x_t, e_i) \in [0,T] \times \mathbb{L}^2(\Omega, \mathcal{F}_t^\alpha, \mathbb{P}; \mathbb{R}) \times \chi$. This is the case of the mean-variance problem with state dependent risk aversion. For this case the equilibrium reinsurance-investment strategy given by (2.68) and (2.69) for $s \in [0,T]$, reduces to

$$\hat{u}_1(s) = -\sum_{i=1}^{d} \langle \alpha(s-), e_i \rangle \frac{(\lambda \mu_Y \theta_0)}{\int_0^{+\infty} z^2 \theta_\alpha^1(dz)} \Big(\Phi_1(s,e_i) \hat{X}(s) + \Phi_2(s,e_i) \Big), \tag{2.76}$$

$$\hat{u}_2(s) = -\sum_{i=1}^{d} \langle \alpha(s-), e_i \rangle \frac{r(s,e_i)}{\beta(s,e_i)^2} \Big(\Phi_1(s,e_i) \hat{X}(s) + \Phi_2(s,e_i) \Big), \tag{2.77}$$

where $\Phi_1(s,e_i)$ and $\Phi_2(s,e_i)$ are given by (2.70) and (2.71) for $\mu_1 = \frac{1}{\gamma}$ and $\mu_2 = 0$.

In the absence of the Markov chain the equilibrium solution reduces for $s \in [0, T]$, to

$$\hat{u}_1(s) = \frac{(\lambda \mu_Y \theta_0) e^{\int_s^T -r_0(\tau)d\tau} \hat{X}(s)}{\left(\int_0^{+\infty} z^2 \theta^1(dz)\right)\left(\gamma + \int_s^T e^{-\int_\tau^T r_0(u)du} \rho(\tau) d\tau\right)}, \quad (2.78)$$

$$\hat{u}_2(s) = \frac{r(s) e^{\int_s^T -r_0(\tau)d\tau} \hat{X}(s)}{\beta(s)^2 \left(\gamma + \int_s^T e^{-\int_\tau^T r_0(u)du} \rho(\tau) d\tau\right)}. \quad (2.79)$$

The equilibrium reinsurance-investment solution presented above is comparable to that found in Li and Li [38] in which the equilibrium is however defined within the class of feedback controls. Note that in [38] the authors adopted the approach developed by Bjork et al [10] and they have obtained feedback equilibrium solutions via some well posed integral equations.

The investment only

In this subsection, we consider the investment-only optimization problem. In this case the insurer does not purchase reinsurance or acquire new business, which means that $u_1(s) \equiv 1$, and his consumption does not take into account. We assume that the financial market consists of one risk-free asset whose price process is given by (2.54), and only one risky asset whose price process do not have jumps. A trading strategy $u(\cdot)$ reduces to a one-dimensional stochastic processes $u_2(\cdot)$ in this case, where $u_2(s)$ represents the amount invested in the risky stock at time s. The dynamics of the wealth process $X(\cdot)$ which corresponds to an admissible investment strategy $u_2(\cdot)$ and initial pair $(t, \xi, e_i) \in [0, T] \times \mathbb{L}^2(\Omega, \mathcal{F}_t^\alpha, \mathbb{P}; \mathbb{R}) \times \mathcal{X}$ can be described by

$$\begin{cases} dX(s) = \{r_0(s, \alpha(s)) X(s) + \delta\lambda\mu_Y + r(s, \alpha(s)) u_2(s)\} ds + \beta_0 dW^1(s) \\ \qquad + \beta(s, \alpha(s)) u_2(s) dW^2(s) - \int_0^{+\infty} z \tilde{N}_\alpha^1(ds, dz), \text{ for } s \in [t, T], \\ X(t) = \xi, \alpha(t) = e_i. \end{cases}$$

Similar to the previous subsection, for the investment-only case we derive the equilibrium strategy which is described as in the following two cases.

Case 1: $\mu_1 = 0$ We suppose that $\mu_1 = 0$ and $\mu_2 = \frac{1}{\gamma}$, such that $\gamma > 0$. In this case the equilibrium investment strategy given by (2.68) becomes

$$\hat{u}_2(s) = -\sum_{i=1}^d \langle \alpha(s-), e_i \rangle \frac{r(s,e_i)}{\beta(s,e_i)^2} \left(\Phi_1(s, e_i) \hat{X}(s) + \Phi_2(s, e_i)\right), \ s \in [0, T],$$

where $\Phi_1(s, e_i)$ and $\Phi_2(s, e_i)$ are given by (2.70) and (2.71) for $\mu_1 = 0$ and $\mu_2 = \frac{1}{\gamma}$.

In the absence of the Markov chain the equilibrium solution reduces to

$$\hat{u}_2(s) \quad = \frac{r(s)\, e^{\int_s^T -r_0(\tau)d\tau}}{\gamma\beta(s)^2}, \;\; s \in [0,T].$$

This essentially covers the solution obtained by Bjök and Murgoci [9] by solving some extended HJB equations.

Case 2: $\mu_2 = 0$ Now, suppose that $\mu_1 = \dfrac{1}{\gamma}$ and $\mu_2 = 0$, such that $\gamma > 0$. This is the case of the mean-variance problem with state dependent risk aversion. For this case the equilibrium investment strategy given by (2.68) reduces to

$$\hat{u}_2(s) \quad = -\sum_{i=1}^{d} \langle \alpha(s-), e_i \rangle \frac{r(s,e_i)}{\beta(s,e_i)^2} \Big(\Phi_1(s,e_i)\hat{X}(s) + \Phi_2(s,e_i) \Big), \;\; s \in [0,T],$$

where $\Phi_1(s,e_i)$ and $\Phi_2(s,e_i)$ are given by (2.70) and (2.71) for $\mu_1 = \dfrac{1}{\gamma}$ and $\mu_2 = 0$.
In the absence of the Markov chain the equilibrium solution reduces to

$$\hat{u}_2(s) \quad = \frac{r(s)e^{\int_s^T -r_0(\tau)d\tau}\hat{X}(s)}{\beta(s)^2\left(\gamma + \int_s^T e^{-\int_\tau^T r_0(u)du}\rho(\tau)d\tau\right)}, \;\; s \in [0,T].$$

This essentially covers the solution obtained by Hu et al [29].

2.4 Existence and uniqueness of SDE and BSDE

In what follows, we will state some basic results on SDEs and BSDEs with jumps which we have used in this chapter.
Let $t \in [0,T]$, denote by $\mathfrak{P}$ the $\mathcal{F}_t$-predictable σ-field on $[0,T] \times \mathcal{F}$ and by $\mathcal{B}(H)$ the Borel σ-algebra of any topological space H. For any given $s \in [0,T]$, consider SDE with jumps

$$\begin{aligned} X(t) = \quad & \xi + \int_s^t b(r, X(r), \alpha(r))dr + \int_s^t \sigma(r, X(r), \alpha(r))dW(r) \\ & + \iint_{\mathbb{R}^* \times (s,t]} c(r, z, X(r-), \alpha(r))\tilde{N}_\alpha(\mathrm{d}r, \mathrm{d}z), \end{aligned} \tag{2.80}$$

where $s \leq t \leq T$. Here the coefficients (ξ, b, σ, c) are given mappings $\xi : \Omega \longrightarrow \mathbb{R}^n$, $b : [0,T] \times \Omega \times \mathbb{R}^n \times \chi \longrightarrow \mathbb{R}^n$, $\sigma \equiv (\sigma^1, \sigma^2, \ldots, \sigma^p) : [0,T] \times \Omega \times \mathbb{R}^n \times \chi \longrightarrow \mathbb{R}^{n \times p}$, $\quad c \equiv (c^1, c^2, \ldots, c^l) : [0,T] \times \Omega \times \mathbb{R}^* \times \mathbb{R}^n \times \chi \longrightarrow \mathbb{R}^{n \times l}$ satisfying the assumptions below

(H'1) $\xi \in \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^n)$, the coefficients b, σ are $\mathfrak{P} \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\chi)$ measurable and c is $\mathfrak{P} \otimes \mathcal{B}(\mathbb{R}^n) \otimes$

$\mathcal{B}(\mathbb{R}^*) \otimes \mathcal{B}(\chi)$ measurable with: for all $e_i \in \chi$

$$\mathbb{E}\left[\int_0^T \left(b(t,0,e_i) + \sigma(t,0,e_i) + \int_{\mathbb{R}^*} c(t,z,0,e_i)\theta_\alpha(dz)\right) dt\right] < \infty.$$

(H'2) b, σ and c are uniformly Lipschitz continuous w.r.t. x, that is, there exists a constant $C > 0$ s.t. for all $(t, x, \bar{x}, e_i) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \chi$ and a.s. $\omega \in \Omega$,

$$\begin{aligned} &|b(t,x,e_i) - b(t,\bar{x},e_i)|^2 + |\sigma(t,x,e_i) - \sigma(t,\bar{x},e_i)|^2 \\ &+ \int_{\mathbb{R}^*} |c(t,z,x,e_i) - c(t,z,\bar{x},e_i)|^2 \theta_\alpha(dz) \leqslant C|x-\bar{x}|^2. \end{aligned}$$

Theorem 2.4.1 *If the coefficients (ξ, b, σ, c) satisfy Assumption H'1)-H'2), then the SDE* (2.80) *has a unique solution $X(\cdot) \in S^2_{\mathcal{F}}(s,T;\mathbb{R}^n)$. Moreover, the following estimate holds*

$$\mathbb{E}\left[\sup_{s\leq t\leq T} |X(s)|^2\right] \leq K\left(1 + \mathbb{E}\left[|\xi|^2\right]\right),$$

Proof. Let $0 = \tau_0 < \tau_1 < \tau_2 < \ldots, < \tau_n < \ldots$ be the jump times of the Markov chain $\alpha(\cdot)$, and let $e_1 \in \chi$ be the starting state. Thus $\alpha(t) = e_1$ on $[\tau_0, \tau_1)$, and the system (2.80) for $t \in [\tau_0, \tau_1)$ has the following form:

$$dX(t) = b(t, X(t), e_1)dt + \sigma(t, X(t), e_1)dW(t) + \int_{\mathbb{R}^*} c(t,z,X(t-),e_1)\tilde{N}_\alpha(\mathrm{d}t, \mathrm{d}z).$$

By theorem 117 in [54], the above SDE has the unique solution $X(\cdot)$ on the space $\mathcal{S}^2_{\mathcal{F}}([\tau_0,\tau_1);\mathbb{R}^n)$, and by continuity for $t = \tau_1$, as well. By considering $\alpha(\tau_1) = e_2$, the system for $t \in [\tau_1, \tau_2)$ becomes

$$dX(t) = b(t, X(t), e_2)dt + \sigma(t, X(t), e_2)dW(t) + \int_{\mathbb{R}^*} c(t,z,X(t-),e_2)\tilde{N}_\alpha(\mathrm{d}t, \mathrm{d}z).$$

Again, by theorem 117 in [54], the above SDE has a unique solution $X(\cdot) \in \mathcal{S}^2_{\mathcal{F}}([\tau_1,\tau_2);\mathbb{R}^n)$, and by continuity for $t = \tau_2$, as well. Repeating this process continuously, we obtain that the solution $X(\cdot)$ of system (2.80) remains in $\mathcal{S}^2_{\mathcal{F}}(0,T;\mathbb{R}^n)$ with probability one. ■

The form of linear BSDEs (2.8) and (2.11) is the motivation for us to study the following general BSDE with Markov switching

$$\begin{aligned} Y(t) = \quad & \varsigma + \int_t^T g(s, Y(s), Z(s), K(s,\cdot), V(s), \alpha(s))\mathrm{d}s - \int_t^T \sum_{i=1}^p Z_i(s)\mathrm{d}W^i(s) \\ & - \int_t^T \int_{\mathbb{R}^*} \sum_{r=1}^l K_r(s,z)\tilde{N}^r_\alpha(\mathrm{d}s,\mathrm{d}z) - \int_t^T \sum_{j=1}^d V_j(s)\, d\tilde{\Phi}_j(s), \quad t \in [0,T]. \end{aligned} \tag{2.81}$$

Here $g:\Omega\times[0,T]\times\mathbb{R}^n\times\mathbb{R}^{n\times d}\times\mathbb{L}^2\left(\mathbb{R}^*,\mathcal{B}\left(\mathbb{R}^*\right),\theta;\mathbb{R}^{n\times l}\right)\times\mathcal{L}_\lambda^2\times\chi\rightarrow\mathbb{R}^n$, where $\mathcal{L}_\lambda^2$ is the set of functions $\mathcal{I}(\cdot):\chi\rightarrow\mathbb{R}^{n\times d}$ such that $\|\mathcal{I}(\cdot)\|_\lambda^2:=\sum_{j=1}^d|\mathcal{I}_j(t)|^2\lambda_j(t)<\infty$. We make the following assumption

(H'3) $\varsigma\in\mathbb{L}^2\left(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^n\right)$.

(H'4) For all $(y,z,k,\upsilon)\in\mathbb{R}^n\times\mathbb{R}^{n\times d}\times\mathbb{L}^2\left(\mathbb{R}^*,\mathcal{B}\left(\mathbb{R}^*\right),\theta;\mathbb{R}^{n\times l}\right)\times\mathcal{L}_\lambda^2$ and $e_i\in\chi$, for $i=1,..,d$. $g(\cdot,y,z,k,\upsilon,e_i)\in\mathcal{L}_\mathcal{F}^2\left(0,T;\mathbb{R}^n\right)$.

(H'5) $\forall e_i\in\chi$, $g(t,y,z,k,\upsilon,e_i)$ is uniformly Lipschitz with respect to y, z, k and υ, i.e. there exists a constant $C>0$, such that for all $(\omega,t)\in\Omega\times[0,T]$, $y,y'\in\mathbb{R}^n$, $z,z'\in\mathbb{R}^{n\times d}$, $k,k'\in\mathbb{L}^2\left(\mathbb{R}^*,\mathcal{B}\left(\mathbb{R}^*\right),\theta;\mathbb{R}^{n\times l}\right)$, $\upsilon,\upsilon'\in\mathcal{L}_\lambda^2$

$$\begin{aligned}&|g(t,y,z,k,e_i)-g\left(t,y',z',k',e_i\right)|\\&\leq C\left(|y-y'|+|z-z'|+\|k-k'\|_\theta+\|\upsilon-\upsilon'\|_\lambda\right).\end{aligned}$$

Theorem 2.4.2 *Suppose that (H'3)-(H'5) holds. Then BSDE with Markov switching* (2.81) *admits a unique solution.*

Before proving this theorem, we give an extended martingale representation results by the folowing lamma, its proof follows from Lemma 3.1. in Cohen and Elliott [16], together with Proposition 3.2. in Shi and Wu [56].

Lemma 2.4.1 *Let $t\in[0,T]$, for $M\in\mathbb{L}^2\left(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^n\right)$, there exists a unique process*

$$\begin{aligned}&(Y,Z,K,V)\in\mathcal{S}_\mathcal{F}^2\left(0,T;\mathbb{R}^n\right)\times\mathcal{L}^2\left(0,T;\left(\mathbb{R}^n\right)^p\right)\\&\qquad\times\mathcal{L}_{\mathcal{F},p}^{\theta,2}\left([0,T]\times\mathbb{R}^*;\left(\mathbb{R}^n\right)^l\right)\times\mathcal{L}_{\mathcal{F},p}^{\lambda,2}\left(0,T;\left(\mathbb{R}^n\right)^d\right)\end{aligned}$$

such that

$$\begin{aligned}M\left(t\right)=\quad&M(0)+\int_0^t\sum_{i=1}^pZ_i(s)dW^i(s)+\int_0^t\int_{\mathbb{R}^*}\sum_{r=1}^lK_r(s,z)\tilde{N}_\alpha^r(\mathrm{d}s,\mathrm{d}z)\\&+\int_0^t\sum_{j=1}^dV_j\left(s\right)d\tilde{\Phi}_j\left(s\right).\end{aligned}$$

Proof of Theorem 2.4.2. First we noting that, for all

$$\begin{aligned}&(y,z,k,\upsilon)\in\mathcal{S}_\mathcal{F}^2\left(0,T;\mathbb{R}^n\right)\times\mathcal{L}^2\left(0,T;\left(\mathbb{R}^n\right)^p\right)\\&\qquad\times\mathcal{L}_{\mathcal{F},p}^{\theta,2}\left([0,T]\times\mathbb{R}^*;\left(\mathbb{R}^n\right)^l\right)\times\mathcal{L}_{\mathcal{F},p}^{\lambda,2}\left(0,T;\left(\mathbb{R}^n\right)^d\right),\end{aligned}$$

we have

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^T g(s,y(s),z(s),k(s,\cdot),\upsilon\left(s\right),\alpha(s))\mathrm{d}s\right]^2\\
&\leq 2\mathbb{E}\left[\int_0^T (g(s,y(s),z(s),k(s,\cdot),\upsilon\left(s\right),\alpha(s))-g(s,0,0,0,0,\alpha(s)))\mathrm{d}s\right]^2\\
&+2\mathbb{E}\left[\int_0^T g(s,0,0,0,0,\alpha(s))\mathrm{d}s\right]^2,\\
&\leq C\sum_{i=1}^{d}\mathbb{E}\int_0^T |g(s,0,0,0,0,e_i)|^2\mathrm{d}s\\
&+C\mathbb{E}\int_0^T \left[|y(s)|^2+|z(s)|^2+\|k(s,\cdot)\|_\theta^2+\|\upsilon(s)\|_\lambda^2\right]\mathrm{d}s\\
&<\infty.
\end{aligned}
$$

It follows that

$$
\varsigma+\int_0^T g(s,y(s),z(s),k(s,\cdot),\upsilon\left(s\right),\alpha(s))\mathrm{d}s\in\mathbb{L}^2\left(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^n\right).
$$

From assumption (H'3)-(H'5), it is clear that

$$
M(t)=\mathbb{E}\left[\xi+\int_0^T g(s,y(s),z(s),k(s,\cdot),\upsilon\left(s\right),\alpha(s))dt\mid\mathcal{F}_t\right],
$$

is a square integrable $\mathcal{F}_t$-martingale. By virtue of martingale representation theorem, there exists

$$
\begin{aligned}
(Y,Z,K,V)\in{}&\mathcal{S}_{\mathcal{F}}^2\left(0,T;\mathbb{R}^n\right)\times\mathcal{L}^2\left(0,T;(\mathbb{R}^n)^p\right)\\
&\times\mathcal{L}_{\mathcal{F},p}^{\theta,2}\left([0,T]\times\mathbb{R}^*;(\mathbb{R}^n)^l\right)\times\mathcal{L}_{\mathcal{F},p}^{\lambda,2}\left(0,T;(\mathbb{R}^n)^d\right),
\end{aligned}
$$

such that

$$
\begin{aligned}
M(t)={}&M\left(0\right)+\int_0^t\sum_{i=1}^{p}Z_i(s)\mathrm{d}W^i(s)+\int_0^t\int_{\mathbb{R}^*}\sum_{r=1}^{l}K_r(s,z)\tilde{N}_\alpha^r(\mathrm{d}s,\mathrm{d}z)\\
&+\int_0^t\sum_{j=1}^{d}V_j\left(s\right)d\tilde{\Phi}_j\left(s\right).
\end{aligned}
$$

Setting $Y(t) = M(t) - \int_0^t g(s, y(s), z(s), k(s), \upsilon(s), \alpha(s))ds$ gives

$$\begin{aligned} Y(t) &= \varsigma + \int_t^T g(s, y(s), z(s), k(s,\cdot), \upsilon(s), \alpha(s))ds - \int_t^T \sum_{i=1}^p Z_i(s)\mathrm{d}W^i(s) \\ &\quad - \int_t^T \int_{\mathbb{R}^*} \sum_{r=1}^l K_r(s,z)\tilde{N}_\alpha^r(\mathrm{d}s, \mathrm{d}z) - \int_t^T \sum_{j=1}^d V_j(s)\, d\tilde{\Phi}_j(s). \end{aligned}$$

From the argument given above, we define the mapping Δ from

$$\mathcal{S}_{\mathcal{F}}^2(0,T;\mathbb{R}^n) \times \mathcal{L}^2(0,T;(\mathbb{R}^n)^p) \times \mathcal{L}_{\mathcal{F},p}^{\theta,2}\left([0,T]\times\mathbb{R}^*;(\mathbb{R}^n)^l\right) \times \mathcal{L}_{\mathcal{F},p}^{\lambda,2}\left(0,T;(\mathbb{R}^n)^d\right),$$

into itself by $\Delta(y,z,k,\upsilon) := (Y,Z,K,V)$ and for

$$\begin{aligned} (y,z,k,\upsilon) &\in \mathcal{S}_{\mathcal{F}}^2(0,T;\mathbb{R}^n) \times \mathcal{L}^2(0,T;(\mathbb{R}^n)^p) \\ &\quad \times \mathcal{L}_{\mathcal{F},p}^{\theta,2}\left([0,T]\times\mathbb{R}^*;(\mathbb{R}^n)^l\right) \times \mathcal{L}_{\mathcal{F},p}^{\lambda,2}\left(0,T;(\mathbb{R}^n)^d\right), \end{aligned}$$

we introduce the norm defined by

$$\|(y,z,k,\upsilon)\|_{\beta,\theta,\lambda}^2 := \mathbb{E}\left[\int_0^T \mathrm{e}^{\beta s}\left\{|y(s)|^2 + |z(s)|^2 + \|k(s,\cdot)\|_\theta + \|\upsilon(s)\|_\lambda^2\right\} ds\right],$$

where $\beta > 0$ is to be determined later. We will prove that Δ is a contraction mapping under the norm $\|\cdot\|_{\beta,\theta,\lambda}$. For this purpose, let

$$\begin{aligned} (y,z,k,\upsilon), (y',z',k',\upsilon') &\in \mathcal{S}_{\mathcal{F}}^2(0,T;\mathbb{R}^n) \times \mathcal{L}^2(0,T;(\mathbb{R}^n)^p) \\ &\quad \times \mathcal{L}_{\mathcal{F},p}^{\theta,2}\left([0,T]\times\mathbb{R}^*;(\mathbb{R}^n)^l\right) \times \mathcal{L}_{\mathcal{F},p}^{\lambda,2}\left(0,T;(\mathbb{R}^n)^d\right), \end{aligned}$$

where $(Y,Z,K,V) = \Delta(y,z,k,\upsilon), (Y',Z',K',V') = \Delta(y',z',k',\upsilon')$. We set

$$\begin{aligned} (\hat{y},\hat{z},\hat{k},\hat{\upsilon}) &= (y-y', z-z', k-k', \upsilon-\upsilon'), \\ (\hat{Y},\hat{Z},\hat{K},\hat{V}) &= (Y-Y', Z-Z', K-K', V-V'), \end{aligned}$$

we know that

$$\begin{aligned} (\hat{Y},\hat{Z},\hat{K},\hat{V}) &\in \mathcal{S}_{\mathcal{F}}^2(0,T;\mathbb{R}^n) \times \mathcal{L}^2(0,T;(\mathbb{R}^n)^p) \\ &\quad \times \mathcal{L}_{\mathcal{F},p}^{\theta,2}\left([0,T]\times\mathbb{R}^*;(\mathbb{R}^n)^l\right) \times \mathcal{L}_{\mathcal{F},p}^{\lambda,2}\left(0,T;(\mathbb{R}^n)^d\right), \end{aligned}$$

and $\mathrm{E}\left[\sup_{0\leq t\leq T}|\hat{Y}(t)|^2\right]<\infty$. Note that

$$\begin{aligned}\hat{Y}(t)= & \int_t^T [g(s,y(s),z(s),k(s,\cdot),\upsilon\left(s\right),\alpha(s)) \\ & -g\left(s,y'(s),z'(s),k'(s,\cdot),\upsilon'\left(s\right),\alpha(s)\right)]\,\mathrm{d}s \\ & -\int_t^T \hat{Z}(s)\mathrm{d}W(s)-\int_t^T\int_{\mathbb{R}^*}\hat{K}(s,z)\tilde{N}_\alpha(\mathrm{d}s,\mathrm{d}z), t\in[0,T].\end{aligned}$$

Applying Ito's formula to $|\hat{Y}(s)|^2\mathrm{e}^{\beta s}$, we can get

$$\begin{aligned}&\mathbb{E}|\hat{Y}(0)|^2+\mathbb{E}\int_0^T\left(\beta|\hat{Y}(s)|^2+|\hat{Z}(s)|^2+\|\hat{K}(s,\cdot)\|_\theta^2+\|\hat{V}(s)\|_\lambda^2\right)\mathrm{e}^{\beta s}\mathrm{d}s\\ &=\mathbb{E}\int_0^T 2\hat{Y}(s)\left[g(s,y(s),z(s),k(s,\cdot),\upsilon\left(s\right),\alpha(s))\right.\\ &-\left.g\left(s,y'(s),z'(s),k'(s,\cdot),\upsilon'\left(s\right),\alpha(s)\right)\right]\mathrm{e}^{\beta s}\mathrm{d}s,\\ &\leq 2C\mathbb{E}\int_0^T\hat{Y}(s)\left(|\hat{y}(s)|+|\hat{z}(s)|+\|\hat{k}(s,\cdot)\|_\theta+\|\upsilon(s)\|_\lambda\right)\mathrm{e}^{\beta s}\mathrm{d}s,\\ &\leq\frac{1}{2}\mathbb{E}\int_0^T\left(|\hat{y}(s)|^2+|\hat{z}(s)|^2+\|\hat{k}(s,\cdot)\|_\theta^2+\|\upsilon(s)\|_\lambda^2\right)\mathrm{e}^{\beta s}\mathrm{d}s\\ &+6C^2\mathbb{E}\int_0^T|\hat{Y}\left(s\right)|^2\mathrm{e}^{\beta s}\mathrm{d}s.\end{aligned}$$

We choose $\beta=1+6C^2$, hence

$$\begin{aligned}&\mathbb{E}\int_0^T\left(|\hat{Y}(s)|^2+|\hat{Z}(s)|^2+\|\hat{K}(s,\cdot)\|_\theta^2+\|V(s)\|_\lambda^2\right)\mathrm{e}^{\beta s}\mathrm{d}s\\ &\leq\frac{1}{2}\mathbb{E}\int_0^T\left(|\hat{y}(s)|^2+|\hat{z}(s)|^2+\|\hat{k}(s,\cdot)\|_\theta^2+\|\upsilon(s)\|_\lambda^2\right)\mathrm{e}^{\beta s}\mathrm{d}s\end{aligned}$$

i.e.

$$\|(\hat{Y},\hat{Z},\hat{K},\hat{V})\|_{\beta,\theta,\lambda}\leq\frac{1}{\sqrt{2}}\|(\hat{y},\hat{z},\hat{k},\hat{\upsilon})\|_{\beta,\theta,\lambda}.$$

Then Δ is a strict mapping on

$$\mathcal{S}_{\mathcal{F}}^2\left(0,T;\mathbb{R}^n\right)\times\mathcal{L}^2\left(0,T;(\mathbb{R}^n)^p\right)\times\mathcal{L}_{\mathcal{F},p}^{\theta,2}\left([0,T]\times\mathbb{R}^*;(\mathbb{R}^n)^l\right)\times\mathcal{L}_{\mathcal{F},p}^{\lambda,2}\left(0,T;(\mathbb{R}^n)^d\right).$$

It follows from the fixed-point theorem that this mapping admits a fixed point which is the unique solution of (2.81). The proof is complete. ■

Chapter 3

Time-inconsistent consumption-investment and reinsurance problem under a Markovian regime-switching

This chapter presents a characterization of equilibrium strategies for a time-inconsistent consumption-investment and reinsurance problem with a non-exponential discount function and a general utility function. Different from [42] and [23], where the authors gives a characterization of equilibrium strategies for special forms of the discount factor, the non-exponential discount function in our model is in a fairly general form. Furthermore, we consider equilibrium strategies in the open-loop sense, as stated in [29] and [28], which is different from the majority of the existing literature on this topic. Also note that the time-inconsistency, in our chapter, arises from a non exponential discounting in the objective function, while the papers [29] and [28] are concerned with a quite different type of time-inconsistency which is caused by the presence of non linear terms of expectations in the terminal cost. However, the objective functional, in our chapter, is not reduced to the quadratic form as in [29] and [28].

We rely on a variational technique approach leading to a version of a necessary and sufficient condition for equilibrium, which includes a flow of forward-backward stochastic differential equations (FBSDEs) along with a certain equilibrium condition. We also provide a verification theorem that covers some possible examples of utility functions.

The rest of the chapter is organized as follows. In Section 1, we formulate the problem and presente the necessary notations and preliminaries. In Section 2 we give the main results of the chapter, Theorem 3.2.1 and Theorem 3.2.2, that characterizes the equilibrium decisions by some necessary and sufficient conditions.

3.1 Problem formulation

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space where $\mathbb{F} := \{\mathcal{F}_t | t \in [0,T]\}$ is a right-continuous, $\mathbb{P}$-completed filtration to which the Markov chain and the Brownian motions are adapted.

3.1.1 Risk process

The classical risk process of an insurer is represented by the following stochastic differential equation (SDE)

$$dR_1(s) = cds - d\sum_{i=1}^{L(s)} Y_i \tag{3.1}$$

where the premium rate c is a constant, meaning that the insurance company receives c units of money per unit of time deterministically. Meanwhile, the insurance company must pay out a random amount of money when a claim occurs. Assume that the counting process $\{L(s)\}_{s\geq 0}$ indicates the number of claims during the time interval $[0,t]$, Y_i is the i-th claim size, and $\{Y_i\}_{i\geq 1}$ are i.i.d. random variables which are independent of $L(s)$. We assume that $\{L(s)\}_{s\geq 0}$ is a Poisson process with intensity $\lambda_L > 0$, implying that $\mathbb{E}[L(s)] = \lambda_L s$. The distribution of the generic random variable Y is the same as that of $\{Y_i\}_{i\geq 1}$. $m_Y > 0$ and $\sigma_Y > 0$ are the first and second moments of Y, respectively. The premium rate c is assumed to be derived via the expected value principle, i.e., $c = (1+\eta_1)\dfrac{\mathbb{E}[L(s)]\mathbb{E}(Y)}{t} = (1+\eta_1)\lambda_L m_Y$ with safety loading $\eta_1 > 0$, then $\mathbb{E}[dX(s)] = (c - \lambda_L m_Y)ds = \eta_1 \lambda_L m_Y ds$ is the expected profit of the insurance company. According to Grandell [26], we investigate the diffusion approximation, that is, approximating the classical risk model by a Brownian motion with drift. This approach is mathematically based on the notion of weak convergence of probability measures. One way to describe this diffusion approximation is that if the classical risk model is considered as “large deviation”, the diffusion model is associated to the “central limit theorem”. In the literature, the diffusion approximation is commonly used on the optimal problems for insurers, such as Browne [12], Bai and Zhang [4], etc. In this chapter, we will consider the diffusion approximation of the classical risk process as follow

$$dR_2(s) = (1+\eta_1)\lambda_L m_Y ds - \lambda_L m_Y ds + \sqrt{\lambda_L \sigma_Y} dW_0(s), \tag{3.2}$$

where $W_0(s)$ is a standard Brownian motion. Grandel [26] provides more details about this diffusion approximation. The premium is paid to the insurance company, but it also face the risk of paying for claims. If the risk is too considerable, the insurer may seek to transfer a portion of the risk to another insurer. The process of transferring risks from one insurance company to another is known as reinsurance. The second insurance company is known as the reinsurer. The reinsurance company usually does the same, i.e., it transmit part of its own risk to a third company and etc. By transferring on parts of risks, Big risks are split into a number of smaller parts that are taken up by different risk carriers. This risk exchange procedure makes large claims less dangerous to the individual insurers. There are many different forms of reinsurance, as proportionate reinsurance, excess-loss reinsurance, and stop-loss reinsurance, among others. In this chapter, we consider proportional reinsurance, which is mostly used in practice. Let $a(s)$ denote the retention level of new business (specially, the reinsurance business) acquired at time s . It signifies that the insurer pays $a(s)Y$ for the claim Y occurring at time s and the new businessman (in this case, the reinsurer) pays $(1-a(s))Y$. The reinsurance premium is also assumed to be calculated via the expected value principle, i.e., the premium is to be paid at rate $(1-a(s))c_1 = (1-a(s))(1+\theta)\lambda_L m_Y$ for this business, where $\theta > 0$ is the safety loading of the new businessman, where we suppose that η_1 and θ are equal. As a result, the reinsurance company has the expected profit $\{(1-a(s))c_1 - (1-a(s))\lambda_L m_Y\}\, ds$ $= (1-a(s))\theta\lambda_L m_Y ds$ in $[s, s+ds)$. Mention that for the first insurance company, $a(s) \in [0,1]$ corresponds to a reinsurance cover, $a(s) > 1$ would mean that the company is able to take on more insurance business from other companies (i.e., act as a reinsurer for other cedents) and $a(s) < 0$ denotes other new businesses. The reserve process with new business before investment is represented by the following SDE

$$dR(s) = (1+\eta_1)\lambda_L m_Y ds - (1-a(s))(1+\theta)\lambda_L m_Y ds - \lambda_L m_Y a(s) ds + a(s)\sqrt{\lambda_L \sigma_Y} dW_0(s),$$

equivalently, we have

$$dR(s) = a(s)\theta\lambda_L m_Y ds + a(s)\sqrt{\lambda_L \sigma_Y} dW_0(s). \tag{3.3}$$

3.1.2 Financial market

Consider an individual dealing with the inter-temporal consumption and portfolio problem where the market environment including one riskless and N risky assets. The risky assets are stocks and their prices are represented as Itô processes. Namely, for $n = 1, 2, ..., N$, the price $S_n(s)$ are governed by the following Markov-modulated SDE

$$dS_n(s) = S_n(s)\left(r_n(s,\alpha(s))\,ds + \sum_{m=1}^{N}\sigma_{nm}(s,\alpha(s-))\,dW_m(s)\right), \text{ for } s\in[0,T], \tag{3.4}$$

with $S_n(0) > 0$, for $n = 1,2,...,N$, and the coefficients $r_n(.,.) : [0,T]\times\chi \to (0,\infty)$ and $\sigma_n(.,.) = (\sigma_{n1}(.,.),..,\sigma_{nN}(.,.))^\top : [0,T]\times\chi \to \mathbb{R}^N$ represent the appreciation rate and the volatility of the n-th stock, respectively. For brevity, we use $r(s,e_i) = (r_1(s,e_i), r_2(s,e_i),..,r_N(s,e_i))^\top$ to denote the drift rate vector and $\sigma(s,e_i) = (\sigma_{nm}(s,e_i))_{1\leq n,m\leq N}$ to denote the random volatility matrix.
The riskless asset has the price process $B(s)$, for $s\in[0,T]$, governed by

$$dB(s) = r_0(s)B(s)\,ds,\ B(0) = 1, \tag{3.5}$$

where $r_0(\cdot)$ is a deterministic function with values in $[0,\infty)$ which denoted the interest rate. We assume that $\mathbb{E}[r_n(t,e_i)] > r_0(t) \geq 0$, $dt - a.e.$, for $e_i\in\chi$ and $n = 1,2,..,N$. This is a reasonable assumption, since otherwise, no one would be willing to invest in risky stocks.

3.1.3 Consumption-reinsurance-investment policies and wealth process

In this chapter, we assume that the insurer is allowed to purchase proportional reinsurance, invest in the stocks as well as in the bond and consum. The trading strategy is represented by a (N + 2)-dimensional stochastic process $u(\cdot) = (c(\cdot), a(\cdot), \pi_1(\cdot), \ldots, \pi_N(\cdot))^\top$, where $c(s)$ denotes the consumption rate at time $s\in[0,T]$, $a(s)$ denotes the retention level of reinsurance or new business acquired at time $s\in[0,T]$ and $\pi_n(s)$, for $n = 1,2,..,N$, denotes the amount invested in the *n-th* risky stock at time $s\in[0,T]$. The process $\pi(\cdot) = (\pi_1(\cdot),\ldots,\pi_N(\cdot))^\top$ is known as an investment strategy. The amount invested in the bond at time s is

$$X^{x_0,e_{i_0},u}(s) - \sum_{n=1}^{N}\pi_n(s),$$

where $X^{x_0,e_{i_0},u}(\cdot)$ is the wealth process associated with the strategy $u(\cdot)$ and the initial capital $(x_{0,}e_{i_0})$. The evolution of $X^{x_0,e_{i_0},u}(\cdot)$ can be described as

$$\begin{cases} dX^{x_0,e_{i_0},u}(s) = dR(s) + \left(X^{x_0,e_{i_0},u}(s) - \sum\limits_{n=1}^{N}\pi_n(s)\right)\dfrac{dB(s)}{B(s)} + \sum\limits_{n=1}^{N}\pi_n(s)\dfrac{dS_n(s)}{S_n(s)} \\ \qquad\qquad - c(s)\,ds, \quad \text{for } s\in[0,T], \\ X^{x_0,u}(0) = x_0,\ \alpha(0) = e_{i_0}\in\chi. \end{cases}$$

Accordingly, the wealth process solves the SDE

$$\begin{cases} dX^{x_0,e_{i_0},u}(s) = \left\{ r_0(s) X^{x_0,e_{i_0},u}(s) + \pi(s)^\top \rho(s,\alpha(s)) + \theta a(s)\lambda_L m_Y - c(s) \right\} ds \\ \qquad + \sqrt{\lambda_L \sigma_Y} a(s) dW_0(s) + \pi(s)^\top \sigma(s,\alpha(s)) dW(s), \text{ for } s \in [0,T], \\ X^{x_0,u}(0) = x_0, \ \alpha(0) = e_{i_0} \in \chi, \end{cases} \tag{3.6}$$

where $\rho(s,e_i) = (r_1(s,e_i) - r_0(s), \ldots, r_N(s,e_i) - r_0(s))^\top$.

As time evolves, we can consider the controlled stochastic differential equation that parametrized by $(t,\xi,e_i) \in [0,T] \times \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}) \times \chi$ and satisfied by $X(\cdot) = X^{t,\xi,e_i}(\cdot; u(\cdot))$,

$$\begin{cases} dX(s) = \left\{ r_0(s) X(s) + \pi(s)^\top \rho(s,\alpha(s)) + \theta a(s)\lambda_L m_Y - c(s) \right\} ds + \sqrt{\lambda_L \sigma_Y} a(s) dW_0(s) \\ \qquad + \pi(s)^\top \sigma(s,\alpha(s)) dW(s), \text{ for } s \in [t,T], \\ X(t) = \xi, \ \alpha(t) = e_i. \end{cases} \tag{3.7}$$

Definition 3.1.1 (Admissible Strategy) *A strategy $u(\cdot) = \left(c(\cdot), a(\cdot), \pi(\cdot)^\top\right)^\top$ is said to be admissible over $[t,T]$ if $u(\cdot) \in L^1_{\mathcal{F}}(t,T;\mathbb{R}) \times L^2_{\mathcal{F}}\left(t,T;\mathbb{R}^{N+1}\right)$ and for any $(t,\xi,e_i) \in [0,T] \times \mathbb{L}^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}) \times \chi$, the equation* (3.7) *admits a unique solution $X(\cdot) = X^{t,\xi,e_i}(\cdot; u(\cdot))$.*

Regarding the coefficients, we adopt the following assumption.

(H1) Processes $r_0(\cdot), \rho(\cdot,\cdot)$ and $\sigma(\cdot,\cdot)$ are uniformly bounded. Also we suppose the following uniform ellipticity condition:

$$\sigma(s,e_i)\sigma(s,e_i)^\top \geq \epsilon I_N, \ \forall (s,e_i) \in [0,T] \times \chi,$$

for some $\epsilon > 0$, where I_N is the identity matrix on $\mathbb{R}^{N \times N}$.

Under (**H1**), for any $(t,\xi,e_i,u(\cdot)) \in [0,T] \times \mathbb{L}^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}) \times \chi \times L^1_{\mathcal{F}}(t,T;\mathbb{R}) \times L^2_{\mathcal{F}}\left(t,T;\mathbb{R}^{N+1}\right)$, the state equation (3.7) admits a unique solution $X(\cdot) \in \mathcal{C}^2_{\mathcal{F}}(t,T;\mathbb{R})$. Moreover, we have the following estimate

$$\mathbb{E}\left[\sup_{t \leq s \leq T} |X(s)|^2\right] \leq C\left(1 + \mathbb{E}\left[|\xi|^2\right]\right), \tag{3.8}$$

for some positive constant C. In particular for $t = 0$, $x_0 > 0$ and $u(\cdot) = \left(c(\cdot), a(\cdot), \pi(\cdot)^\top\right)^\top \in L^1_{\mathcal{F}}(0,T;\mathbb{R}) \times L^2_{\mathcal{F}}\left(t,T;\mathbb{R}^{N+1}\right)$, the state equation (3.6) has a unique solution $X^{x_0,e_{i_0},u}(\cdot) \in \mathcal{C}^2_{\mathcal{F}}(0,T;\mathbb{R})$. Thus the following estimate holds:

$$\mathbb{E}\left[\sup_{0 \leq s \leq T} |X^{x_0,e_{i_0},u}(s)|^2\right] \leq C\left(1 + |x_0|^2\right). \tag{3.9}$$

3.1.4 General discounted utility function

The majority of the literature on financial economics assumes that the rate of time preference is constant (exponential discounting). However, evidence is mounting that this may is not the case. In this subsection, we discuss the general discounting preferences. We also give the basic modeling framework of Merton's consumption and portfolio problem. We refer the reader to [25] and [44] for a detailed discussion of the classical Merton model.

Discount function

Once the discount is non-exponential, Many papers use a special form of the non-exponential discount factor. Different to these papers, we consider a general form of the discount factor.

Definition 3.1.2 *A discount function* $\mathfrak{D}(\cdot) : [0, T] \rightarrow \mathbb{R}$ *is a continuous and deterministic function satisfying* $\mathfrak{D}(0) = 1$, $\mathfrak{D}(s) > 0$ $ds-a.e.$ *and* $\int_0^T \mathfrak{D}(s)\, ds < \infty$.

Remark 3.1.1 *Some examples of discount functions are provided in many papers, such as exponential discount functions, see [44], mixture of exponential functions, see [23], and hyperbolic discount functions, see [72].*

Utility functions and objective

The decision maker derives utility from inter-temporal consumption and final wealth, in order to evaluate the performance of a consumption-investment and reinsurance strategy. Let $f(\cdot)$ be the utility of inter-temporal consumption and $h(\cdot)$ the utility of the terminal wealth at some non-random horizon T (which is a primitive of the model). Then, for any $(t, \xi, e_i) \in [0, T] \times \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}) \times \chi$ the objective of the consumption-investment and reinsurance optimization problem is to maximize the utility function $J(t, \xi, e_i; \cdot)$ given by

$$J(t, \xi, e_i; u(\cdot)) = \mathbb{E}^t \left[\int_t^T \mathfrak{D}(s-t) f(c(s))\, ds + \mathfrak{D}(T-t) h(X(T)) \right], \tag{3.10}$$

over $u(\cdot) \in L^1_{\mathcal{F}}(t, T; \mathbb{R}) \times L^2_{\mathcal{F}}\left(t, T; \mathbb{R}^{N+1}\right)$, subject to (3.7), where $\mathbb{E}^t[\cdot] = \mathbb{E}[\cdot\, |\mathcal{F}_t]$. We restrict ourselves to utility functions that satisfy the following conditions

(H2) The maps $f(\cdot), h(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$ are strictly increasing, strictly concave and satisfy the integrability condition

$$\mathbb{E}\left[\int_0^T |f(c(s))|\, ds + |h(X(T))| \right] < \infty.$$

(H3) The maps $f(\cdot), h(\cdot)$ are twice continuously differentiable functions, and so, all the derivatives $f_x(\cdot), h_x(\cdot), f_{xx}(\cdot)$ and $h_{xx}(\cdot)$ are continuous.

(H4) For all admissible strategy pairs, there exists a constant $p > 1$ sush that

$$\mathbb{E}\left[\int_0^T |f_x(c(s))|^p \, ds + |h_x(X(T))|^p\right] < \infty,$$
$$\mathbb{E}\left[\int_0^T \sup_{\eta \in \mathbb{R}, |\eta| \leq M} |f_{xx}(c(s) + \eta)|^p \, ds\right] < \infty, \text{ for } M \geq 0.$$

If we write $\bar{W}(s) = \left(0, W^\star(s)^\top\right)^\top$ where $W^\star(s) = \left(W_0(s), W(s)^\top\right)^\top$ and we denote $\mathbf{B}(s, \alpha(s)) = \left(-1, \theta\lambda_L m_Y, \rho(s, \alpha(s))^\top\right)^\top$, $\Gamma = \left(1, 0_{\mathbb{R}^{N+1}}^\top\right)^\top$. We also consider the following notations

$$\mathbf{D}(s, \alpha(s)) = \begin{pmatrix} 0 & 0_{\mathbb{R}^{N+1}}^\top \\ 0_{\mathbb{R}^{N+1}} & \bar{\sigma}(s, \alpha(s)) \end{pmatrix}, \text{ where } \bar{\sigma}(s, \alpha(s)) = \begin{pmatrix} \sqrt{\lambda_L \sigma_Y} & 0_{\mathbb{R}^N}^\top \\ 0_{\mathbb{R}^N} & \sigma(s, \alpha(s)) \end{pmatrix},$$

then the optimal control problem associated with (3.7) and (3.10) is equivalent to maximize

$$J(t, \xi, e_i; u(\cdot)) = \mathbb{E}^t\left[\int_t^T \mathfrak{D}(s-t) f\left(\Gamma^\top u(\cdot)\right) ds + \mathfrak{D}(T-t) h(X(T))\right], \tag{3.11}$$

subject to

$$\begin{cases} dX(s) = \left\{r_0(s) X(s) + u(s)^\top \mathbf{B}(s, \alpha(s))\right\} ds + u(s)^\top \mathbf{D}(s, \alpha(s)) \, d\bar{W}(s), \text{ for } s \in [t, T], \\ X(t) = \xi, \ \alpha(t) = e_i, \end{cases} \tag{3.12}$$

over $u(\cdot) \in L^1_{\mathcal{F}}(t, T; \mathbb{R}) \times L^2_{\mathcal{F}}\left(t, T; \mathbb{R}^{N+1}\right)$.

3.2 Equilibrium strategies

The problem given above by (3.11) – (3.12) is well known to be time inconsistent, meaning that it does not satisfy the Bellman optimality principle, since a restriction of an optimal control for a particular initial pair on a later time interval may not be optimal for the corresponding initial pair, for more details, see Ekeland and Pirvu [23] and Yong [66]. Due to the lack of time consistency, we consider open-loop Nash equilibrium controls instead of optimal controls. As in [29], we first consider an equilibrium by local spike variation, given, for $t \in [0, T]$, an admissible consumption-investment and reinsurance strategy $\hat{u}(\cdot) \in L^1_{\mathcal{F}}(t,T;\mathbb{R}) \times L^2_{\mathcal{F}}\left(t,T;\mathbb{R}^{N+1}\right)$. For any $\mathbb{R}^{N+2}-$valued, $\mathcal{F}_t-$measurable and bounded random variable v and for any $\varepsilon > 0$, define

$$u^{\varepsilon}(s) := \begin{cases} \hat{u}(s) + v, & \text{for } s \in [t, t+\varepsilon), \\ \hat{u}(s), & \text{for } s \in [t+\varepsilon, T]. \end{cases} \tag{3.13}$$

We have the following definition.

Definition 3.2.1 (Open-loop Nash equilibrium) *An admissible strategy* $\hat{u}(\cdot) \in L^1_{\mathcal{F}}(t,T;\mathbb{R}) \times L^2_{\mathcal{F}}\left(t,T;\mathbb{R}^{N+1}\right)$ *is an open-loop Nash equilibrium strategy if*

$$\lim_{\varepsilon_n \downarrow 0} \frac{1}{\varepsilon_n} \left\{ J\left(t, \hat{X}(t), \alpha(t); u^{\varepsilon_n}(\cdot)\right) - J\left(t, \hat{X}(t), \alpha(t); \hat{u}(\cdot)\right) \right\} \leq 0, \tag{3.14}$$

for every sequence $\varepsilon_n \downarrow 0$ *and any* $t \in [0,T]$, *where* $\hat{X}(\cdot)$ *is the equilibrium wealth process solution of the SDE*

$$\begin{cases} d\hat{X}(s) = \left\{ r_0(s)\hat{X}(s) + \hat{u}(s)^{\top}\mathbf{B}(s,\alpha(s)) \right\} ds + \hat{u}(s)^{\top}\mathbf{D}(s,\alpha(s))\, d\bar{W}(s), \text{ for } s \in [t,T], \\ \hat{X}(t) = \xi, \alpha(t) = e_i. \end{cases} \tag{3.15}$$

3.2.1 Necessary and sufficient condition for equilibrium controls

In this chapter, we follow an alternative method, which is effectively a necessary and sufficient condition for equilibrium. In the same manner of proving the stochastic Pontryagin's maximum principle for equilibrium in [29] for the case of linear quadratic models (LQ), we derive this condition by a second-order expansion in the spike variation. Now, we introduce the adjoint equations involved in the characterization of open-loop Nash equilibrium controls. Let $\hat{u}(\cdot) = \left(\hat{c}(\cdot), \hat{a}(\cdot), \hat{\pi}(\cdot)^{\top}\right)^{\top} \in L^1_{\mathcal{F}}(0,T;\mathbb{R}) \times L^2_{\mathcal{F}}\left(t,T;\mathbb{R}^{N+1}\right)$ an admissible strategy and denote by $\hat{X}(\cdot) \in \mathcal{C}^2_{\mathcal{F}}(0,T;\mathbb{R})$ the corresponding wealth process. For each $t \in [0,T]$, we introduce the first order adjoint equation defined on the time interval $[t,T]$, and satisfied by the processes $(p(\cdot;t), q(\cdot;t), l(\cdot;t))$ as follows

$$\begin{cases} dp(s;t) = -r_0(s)p(s;t)\,ds + \sum_{m=1}^{N+1} q_m(s;t)\,dW_m(s) + \sum_{j \neq i} l_{ij}(s,t)\,d\tilde{\Phi}_{ij}(s), \text{ for } s \in [t,T], \\ p(T;t) = \mathfrak{D}(T-t)\,h_x\left(\hat{X}(T)\right), \end{cases} \tag{3.16}$$

where $q(\cdot;t) = (q_0(\cdot;t), q_1(\cdot;t), \ldots, q_N(\cdot;t))^{\top}$ and $l(s;t) = (l_{ij}(s;t))_{1 \leq i,j \leq d} \in \mathbb{R}^{d \times d}$. According to Theorem 5.15 in [39], we deduce that equation (3.16) is uniquely solvable in $\left(\mathcal{C}^2_{\mathcal{F}}(t,T;\mathbb{R}) \times L^2_{\mathcal{F}}\left(t,T;\mathbb{R}^{N+1}\right) \times \mathcal{L}^{\lambda,2}_{\mathcal{F},p}\left(t,T;\mathbb{R}^{d \times d}\right)\right)$. Moreover there exists a constant $C > 0$ such that, for any $t \in [0,T]$, we have the following estimate

$$\|p(\cdot;t)\|^2_{\mathcal{C}^2_{\mathcal{F}}(t,T;\mathbb{R})}+\|q(\cdot;t)\|^2_{L^2_{\mathcal{F}}(t,T;\mathbb{R}^{N+1})}+\|l(\cdot;t)\|^2_{\mathcal{L}^{\lambda,2}_{\mathcal{F},p}(t,T;\mathbb{R}^{d\times d})}\leq C\left(1+\xi^2\right). \tag{3.17}$$

The second order adjoint equation is defined on the time interval $[t,T]$ and satisfied by the processes $(P(\cdot;t),Q(\cdot;t),L(\cdot;t))\in\left(\mathcal{C}^2_{\mathcal{F}}(t,T;\mathbb{R})\times L^2_{\mathcal{F}}\left(t,T;\mathbb{R}^{N+1}\right)\times\mathcal{L}^{\lambda,2}_{\mathcal{F},p}\left(t,T;\mathbb{R}^{d\times d}\right)\right)$as follows

$$\begin{cases} dP(s;t)=-2r_0(s)P(s;t)\,ds+\sum\limits_{m=1}^{N+1}Q_m(s;t)\,dW_m(s)+\sum\limits_{j\neq i}L_{ij}(s;t)\,d\tilde{\Phi}_{ij}(s)\,,\ \text{for } s\in[t,T]\,,\\ P(T;t)=\mathfrak{D}(T-t)\,h_{xx}\left(\hat{X}(T)\right), \end{cases} \tag{3.18}$$

where $Q(\cdot;t)=(Q_0(\cdot;t),Q_1(\cdot;t),\ldots,Q_N(\cdot;t))^\top$ and $L(s;t)=(L_{ij}(s;t))_{1\leq i,j\leq d}\in\mathbb{R}^{d\times d}$. According to Theorem 5.15 in [39], the above BSDE has a unique solution $(P(\cdot;t),Q(\cdot;t),L(\cdot;t))\in\big(\mathcal{C}^2_{\mathcal{F}}(t,T;\mathbb{R})\times L^2_{\mathcal{F}}\left(t,T;\mathbb{R}^{N+1}\right)\times\mathcal{L}^{\lambda,2}_{\mathcal{F},p}\left(t,T;\mathbb{R}^{d\times d}\right)\big)$. Moreover we have the following representation for $P(\cdot;t)$

$$P(s;t)=\mathbb{E}^s\left[\mathfrak{D}(T-t)\,e^{\int_s^T 2r_0(\tau)d\tau}h_{xx}\left(\hat{X}(T)\right)\right],\ \text{for } s\in[t,T]\,. \tag{3.19}$$

Indeed, if we define the function $\Theta(\cdot,t)$, for each $t\in[0,T]$, as the fundamental solution of the linear ODE

$$\begin{cases} d\Theta(\tau,t)=r_0(\tau)\,\Theta(\tau,t)\,d\tau,\ \text{for } \tau\in[t,T]\,,\\ \Theta(t,t)=1, \end{cases} \tag{3.20}$$

and we apply the Itô's formula to $\tau\to P(\tau;t)\,\Theta(\tau,t)^2$ on $[t,T]$, by taking conditional expectations, we obtain (3.19). Note that since $h_{xx}\left(\hat{X}(T)\right)\leq 0$, then $P(s;t)\leq 0$, $ds-a.e.$
Now we will present the theorem that represents the main result of this chapter, it provides a necessary and sufficient condition for equilibrium. First, we define the process $\tilde{q}(s;t)=\left(0,q(s;t)^\top\right)^\top$ and we introduce the following notations

$$\mathcal{U}(s;t)\triangleq p(s;t)\,\mathbf{B}(s,\alpha(s))+\mathbf{D}(s,\alpha(s))\,\tilde{q}(s;t)+\mathfrak{D}(s-t)\,f_x\left(\Gamma^\top\hat{u}(s)\right)\Gamma \tag{3.21}$$

and

$$\mathcal{V}^\varepsilon(s;t)\triangleq\begin{pmatrix}\mathfrak{D}(s-t)\,f_{xx}\left(\Gamma^\top\left(\hat{u}(s)+\kappa v 1_{[t,t+\varepsilon)}\right)\right)\Gamma\Gamma^\top & 0^\top_{\mathbb{R}^{N+1}}\\ 0_{\mathbb{R}^{N+1}} & \bar{\sigma}(s,\alpha(s))\,\bar{\sigma}(s,\alpha(s))^\top P(s;t)\end{pmatrix}. \tag{3.22}$$

Theorem 3.2.1 *Let **(H1)-(H4)** hold. Given an admissible strategy* $\hat{u}(\cdot) \in L^1_{\mathcal{F}}(0,T;\mathbb{R}) \times L^2_{\mathcal{F}}\left(t,T;\mathbb{R}^{N+1}\right)$, *let for any* $t \in [0,T]$, *the process*

$$\left(p(\cdot;t), q(\cdot;t), l(\cdot;t)\right) \in \left(\mathcal{C}^2_{\mathcal{F}}(t,T;\mathbb{R}) \times L^2_{\mathcal{F}}\left(t,T;\mathbb{R}^{N+1}\right) \times \mathcal{L}^{\lambda,2}_{\mathcal{F},p}\left(t,T;\mathbb{R}^{d\times d}\right)\right)$$

be the unique solution to the BSDE (3.16). *Then,* $\hat{u}(\cdot)$ *is an equilibrium trading strategy, if and only if, the following condition holds*

$$\mathcal{U}(t;t) = 0,\ d\mathbb{P}-a.s.,\ dt-a.e. \tag{3.23}$$

In order to prove this theorem, we first need to derive some technical results. First, denote by $\hat{X}^{\varepsilon}(\cdot)$ the solution of the state equation corresponding to $u^{\varepsilon}(\cdot)$. Since the coefficients of the controlled state equation are linear, using the standard perturbation approach, see e.g. [69], we have

$$\hat{X}^{\varepsilon}(s) - \hat{X}(s) = Y^{\varepsilon,v}(s) + Z^{\varepsilon,v}(s),\ \text{for } s \in [t,T], \tag{3.24}$$

where for any $\mathbb{R}^{N+2}-$valued, $\mathcal{F}_t-$measurable and bounded random variable v and for any $\varepsilon \in [0, T-t)$, $Y^{\varepsilon,v}(\cdot)$ and $Z^{\varepsilon,v}(\cdot)$ solve respectively the following linear stochastic differential equations:

$$\begin{cases} dY^{\varepsilon,v}(s) = r_0(s) Y^{\varepsilon,v}(s)\, ds + v^{\top}\mathbf{D}(s,\alpha(s))\, 1_{[t,t+\varepsilon)}(s)\, d\bar{W}(s),\ \text{for } s \in [t,T], \\ Y^{\varepsilon,v}(t) = 0, \end{cases} \tag{3.25}$$

and

$$\begin{cases} dZ^{\varepsilon,v}(s) = \left\{r_0(s) Z^{\varepsilon,v}(s) + v^{\top}\mathbf{B}(s,\alpha(s))\, 1_{[t,t+\varepsilon)}(s)\right\} ds,\ \text{for } s \in [t,T], \\ Z^{\varepsilon,v}(t) = 0. \end{cases} \tag{3.26}$$

Proposition 3.2.1 *Let **(H1)-(H4)** holds. For any* $t \in [0,T]$, *the following estimates hold for any* $k \geq 1$:

$$\mathbb{E}^t\left[\sup_{s\in[t,T]} \left|Y^{\varepsilon,v}(s)\right|^{2k}\right] = O\left(\varepsilon^k\right), \tag{3.27}$$

$$\mathbb{E}^t\left[\sup_{s\in[t,T]} \left|Z^{\varepsilon,v}(s)\right|^{2k}\right] = O\left(\varepsilon^{2k}\right), \tag{3.28}$$

$$\mathbb{E}^t\left[\sup_{s\in[t,T]} \left|Y^{\varepsilon,v}(s) + Z^{\varepsilon,v}(s)\right|^{2k}\right] = O\left(\varepsilon^k\right). \tag{3.29}$$

In addition, we have the following equality

$$\begin{aligned} & J\left(t, \hat{X}(t), \alpha(t) ; u^{\varepsilon}(\cdot)\right)-J\left(t, \hat{X}(t), \alpha(t) ; \hat{u}(\cdot)\right) \\ & =\int_t^{t+\varepsilon} \mathbb{E}^t\left[\langle\mathcal{U}(s ; t), v\rangle+\frac{1}{2}\left\langle\mathcal{V}^{\varepsilon}(s ; t) v, v\right\rangle\right] d s+o(\varepsilon) . \end{aligned} \tag{3.30}$$

Proof. The estimates (3.27) – (3.29) follow from Theorem 4.4 in [69]. Moreover the following representation holds for the objective functional

$$\begin{aligned} & J\left(t, \hat{X}(t), \alpha(t) ; u^{\varepsilon}(\cdot)\right)-J\left(t, \hat{X}(t), \alpha(t) ; \hat{u}(\cdot)\right) \\ & =\mathbb{E}^t\left[\int_t^T \mathfrak{D}(s-t)\left(f\left(\Gamma^{\top} u^{\varepsilon}(s)\right)-f\left(\Gamma^{\top} \hat{u}(s)\right)\right) d s+\mathfrak{D}(T-t)\left(h\left(X^{\varepsilon}(T)\right)-h\left(\hat{X}(T)\right)\right)\right] . \end{aligned} \tag{3.31}$$

From (3.24) and by applying the second order Taylor-Young expansion, we get

$$\begin{aligned} h\left(\hat{X}^{\varepsilon}(T)\right)-h\left(\hat{X}(T)\right)= & h_x\left(\hat{X}(T)\right)\left(Y^{\varepsilon, v}(s)+Z^{\varepsilon, v}(s)\right)+\frac{1}{2} h_{x x}\left(\hat{X}(T)\right)\left(Y^{\varepsilon, v}(s)+Z^{\varepsilon, v}(s)\right)^2 \\ & +o\left(\left(Y^{\varepsilon, v}(s)+Z^{\varepsilon, v}(s)\right)^2\right) . \end{aligned}$$

Again, by applying the second order Taylor-Lagrange expansion, we find that

$$f\left(\Gamma^{\top} u^{\varepsilon}(s)\right)-f\left(\Gamma^{\top} \hat{u}(s)\right)=\left\langle f_x\left(\Gamma^{\top} \hat{u}(s)\right) \Gamma, v\right\rangle+\frac{1}{2}\left\langle f_{x x}\left(\Gamma^{\top} \hat{u}(s)+\kappa v 1_{[t, t+\varepsilon)}\right) \Gamma \Gamma^{\top} v, v\right\rangle .$$

From (3.29), it follows that

$$\begin{aligned} & J\left(t, \hat{X}(t), \alpha(t) ; u^{\varepsilon}(\cdot)\right)-J\left(t, \hat{X}(t), \alpha(t) ; \hat{u}(\cdot)\right) \\ & =\mathbb{E}^t\left[\int_t^T \mathfrak{D}(s-t)\left\{\left\langle f_x\left(\Gamma^{\top} \hat{u}(s)\right) \Gamma, v\right\rangle+\frac{1}{2}\left\langle f_{x x}\left(\Gamma^{\top} \hat{u}(s)+\kappa v 1_{[t, t+\varepsilon)}\right) \Gamma \Gamma^{\top} v, v\right\rangle\right\} 1_{[t, t+\varepsilon)} d s\right. \\ & \quad\left.+\mathfrak{D}(T-t)\left(h_x\left(\hat{X}(T)\right)\left(Y^{\varepsilon, v}(T)+Z^{\varepsilon, v}(T)\right)+\frac{1}{2} h_{x x}\left(\hat{X}(T)\right)\left(Y^{\varepsilon, v}(T)+Z^{\varepsilon, v}(T)\right)^2\right)\right] \\ & \quad+o(\varepsilon) . \end{aligned} \tag{3.32}$$

Notice that

$$\begin{aligned} & \mathfrak{D}(T-t)\left(h_x\left(\hat{X}(T)\right)\left(Y^{\varepsilon, v}(T)+Z^{\varepsilon, v}(T)\right)+\frac{1}{2} h_{x x}\left(\hat{X}(T)\right)\left(Y^{\varepsilon, v}(T)+Z^{\varepsilon, v}(T)\right)^2\right) \\ & =p(T ; t)\left(Y^{\varepsilon, v}(T)+Z^{\varepsilon, v}(T)\right)+\frac{1}{2} P(T ; t)\left(Y^{\varepsilon, v}(T)+Z^{\varepsilon, v}(T)\right)^2 . \end{aligned}$$

Now, by applying Itô's formula to $s\mapsto p(s;t)(Y^{\varepsilon,v}(s)+Z^{\varepsilon,v}(s))$ on $[t,T]$, we get

$$\mathbb{E}^t\left[p(T;t)(Y^{\varepsilon,v}(T)+Z^{\varepsilon,v}(T))\right]=\mathbb{E}^t\left[\int_t^{t+\varepsilon}\left\{v^\top\mathbf{B}(s,\alpha(s))p(s;t)+v^\top\mathbf{D}(s,\alpha(s))\tilde{q}(s;t)\right\}ds\right].\quad(3.33)$$

Again, by applying Itô's formula to $s\mapsto P(s;t)(Y^{\varepsilon,v}(s)+Z^{\varepsilon,v}(s))^2$ on $[t,T]$, we obtain

$$\begin{aligned}&\mathbb{E}^t\left[P(T;t)(Y^{\varepsilon,v}(T)+Z^{\varepsilon,v}(T))^2\right]\\&=\mathbb{E}^t\left[\int_t^{t+\varepsilon}\Big\{2v^\top(Y^{\varepsilon,v}(s)+Z^{\varepsilon,v}(s))\Big(\mathbf{B}(s,\alpha(s))P(s,t)+\mathbf{D}(s,\alpha(s))\tilde{Q}(s,t)\Big)\right.\\&\qquad\left.+v^\top\Big(\mathbf{D}(s,\alpha(s))\mathbf{D}(s,\alpha(s))^\top\Big)vP(s,t)\Big\}ds\right],\end{aligned}\quad(3.34)$$

where $\tilde{Q}(s;t)=\left(0,Q(s;t)^\top\right)^\top$. On the other hand, we conclude from **(H1)** together with (3.29) that

$$\mathbb{E}^t\left[\int_t^{t+\varepsilon}(Y^{\varepsilon,v}(s)+Z^{\varepsilon,v}(s))\Big(\mathbf{B}(s,\alpha(s))P(s,t)+\mathbf{D}(s,\alpha(s))\tilde{Q}(s,t)\Big)ds\right]=o(\varepsilon).\quad(3.35)$$

By taking $(3.33),(3.34)$ and (3.35) in (3.32), it follows that

$$\begin{aligned}&J\left(t,\hat{X}(t),\alpha(t);u^\varepsilon(\cdot)\right)-J\left(t,\hat{X}(t),\alpha(t);\hat{u}(\cdot)\right)\\&=\mathbb{E}^t\left[\int_t^{t+\varepsilon}\Big\{\left\langle\mathbf{B}(s,\alpha(s))p(s;t)+\mathbf{D}(s,\alpha(s))\tilde{q}(s;t)+\mathfrak{D}(s-t)f_x\left(\Gamma^\top\hat{u}(s)\right)\Gamma,v\right\rangle\right.\\&\qquad\left.+\frac{1}{2}\left\langle\Big(\mathfrak{D}(s-t)f_{xx}\left(\left\langle\Gamma,\hat{u}(s)+\kappa v1_{[t,t+\varepsilon)}\right\rangle\right)\Gamma\Gamma^\top+P(s,t)\mathbf{D}(s,\alpha(s))\mathbf{D}(s,\alpha(s))^\top\Big)v,v\right\rangle\Big\}ds\right]+o(\varepsilon),\end{aligned}$$

which is equivalent to (3.30).

■

Now, we present the following technical lemma that will be needed later. The proof follows an argument adapted from Hamaguchi [27].

Lemma 3.2.1 *Under assumptions **(H1)-(H4)**, there exists a sequence* $(\varepsilon_n^t)_{n\in\mathbb{N}}\subset(0,T-t)$ *satisfying* $\varepsilon_n^t\to0$ *as* $n\to\infty$*, such that*

1) $\lim_{n\to\infty}\frac{1}{\varepsilon_n^t}\int_t^{t+\varepsilon_n^t}\mathbb{E}^t\left[\mathcal{U}(s;t)\right]ds=\mathcal{U}(t;t)$, $d\mathbb{P}-a.s,\ dt-a.e.$

2) $\lim_{n\to\infty}\frac{1}{\varepsilon_n^t}\int_t^{t+\varepsilon_n^t}\mathbb{E}^t\left[\mathcal{V}^{\varepsilon_n^t}(s;t)\right]ds=\mathcal{V}^0(t;t)$, $d\mathbb{P}-a.s,\ dt-a.e.$

Proof. We define, for $t\in[0,T]$ and $s\in[t,T]$,

$$\left(\bar{p}(s;t),\bar{q}(s;t),\bar{l}(s;t)\right):=\frac{1}{\mathfrak{D}(T-t)}e^{-\int_s^T r_0(\tau)d\tau}\left(p(s;t),q(s;t),l(s;t)\right).$$

Then, for any $t \in [0,T]$, in the interval $[t,T]$, the pair $\left(\bar{p}(\cdot;t), \bar{q}(\cdot;t), \bar{l}(\cdot;t)\right)$ satisfies

$$\begin{cases} d\bar{p}(s;t) = & \bar{q}(s;t)^{\top} dW^{*}(s) + \sum\limits_{j\neq i} \bar{l}_{ij}(s;t)\, d\tilde{\Phi}_{ij}(s), \; s \in [t,T], \\ \bar{p}(T;t) = & h_x\left(\hat{X}(T)\right). \end{cases} \tag{3.36}$$

Moreover, it is evident that from the uniqueness of solutions to (3.36), we have $\left(\bar{p}(s;t_1), \bar{q}(s;t_1), \bar{l}(s;t_1)\right) = \left(\bar{p}(s;t_2), \bar{q}(s;t_2), \bar{l}(s;t_2)\right)$, for any $t_1, t_2, s \in [0,T]$ such that $0 < t_1 < t_2 < s < T$. Hence, the solution $\left(\bar{p}(\cdot;t), \bar{q}(\cdot;t), \bar{l}(\cdot;t)\right)$ is independent of the variable t, allowing us to denote the solution of (3.36) by $\left(\bar{p}(\cdot), \bar{q}(\cdot), \bar{l}(\cdot)\right)$. We have then, for any $t \in [0,T]$, and $s \in [t,T]$,

$$(p(s;t), q(s;t), l(s;t)) = \mathfrak{D}(T-t)\, e^{\int_s^T r_0(\tau)d\tau} \left(\bar{p}(s), \bar{q}(s), \bar{l}(s)\right). \tag{3.37}$$

By using (3.37) and under **(H2)**, we have, for any $t \in [0,T]$ and $s \in [t,T]$,

$$|p(s;t) - p(s;s)| \leq \sup_{t\leq s\leq t+\varepsilon} |\mathfrak{D}(T-t) - \mathfrak{D}(T-s)|\, e^{-\int_s^T r_0(\tau)d\tau} |\bar{p}(s)|, \tag{3.38}$$

and

$$|q(s;t) - q(s;s)| \leq \sup_{t\leq s\leq t+\varepsilon} |\mathfrak{D}(T-t) - \mathfrak{D}(T-s)|\, e^{-\int_s^T r_0(\tau)d\tau} |\bar{q}(s)|. \tag{3.39}$$

From which, we have for any $a > 0$, $t \in [0,T]$, and $\varepsilon \in (0, T-t)$,

$$\begin{aligned} &\mathbb{P}\left(\left|\frac{1}{\varepsilon}\mathbb{E}^t\left[\int_t^{t+\varepsilon} \mathcal{U}(s;t)\, ds\right] - \frac{1}{\varepsilon}\mathbb{E}^t\left[\int_t^{t+\varepsilon} \mathcal{U}(s;s)\, ds\right]\right| \geq a\right), \\ &\leq \frac{1}{a}\mathbb{E}\left|\frac{1}{\varepsilon}\mathbb{E}^t\left[\int_t^{t+\varepsilon} \mathcal{U}(s;t)\, ds\right] - \frac{1}{\varepsilon}\mathbb{E}^t\left[\int_t^{t+\varepsilon} \mathcal{U}(s;s)\, ds\right]\right|, \\ &\leq C \sup_{t\leq s\leq t+\varepsilon} |\mathfrak{D}(T-t) - \mathfrak{D}(T-s)|\, \frac{1}{\varepsilon}\mathbb{E}\int_t^{t+\varepsilon} \left(|\bar{p}(s)| + |\bar{q}(s)|\right) ds \\ &\quad + \sup_{t\leq s\leq t+\varepsilon} |\mathfrak{D}(s-t) - 1|\, \frac{1}{\varepsilon}\int_t^{t+\varepsilon} \mathbb{E}\left[f_x\left(\Gamma^{\top}\hat{u}(s)\right)\right] ds. \end{aligned}$$

Noting that since $\mathfrak{D}(\cdot)$ is continuous, we get $\lim\limits_{\varepsilon\downarrow 0}\sup\limits_{t\leq s\leq t+\varepsilon} |\mathfrak{D}(T-t) - \mathfrak{D}(T-s)| = 0$ for $t \in [0,T]$. Moreover, since $(\bar{p}(\cdot), \bar{q}(\cdot)) \in \mathcal{C}^2_{\mathcal{F}}(0,T;\mathbb{R}) \times L^2_{\mathcal{F}}\left(0,T;\mathbb{R}^{N+1}\right)$ we get

$$\lim_{\varepsilon\downarrow 0}\sup_{t\leq s\leq t+\varepsilon} |\mathfrak{D}(T-t) - \mathfrak{D}(T-s)|\, \frac{1}{\varepsilon}\mathbb{E}\int_t^{t+\varepsilon} \left(|\bar{p}(s)| + |\bar{q}(s)|\right) ds = 0.$$

Noting that $\mathfrak{D}(0) = 1$ then $\lim\limits_{\varepsilon\downarrow 0}\sup\limits_{t\leq s\leq t+\varepsilon} |\mathfrak{D}(s-t) - 1| = 0$. According to **(H3)**, by using the dominated convergence theorem

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\mathbb{E}\left[f_x\left(\Gamma^\top\hat{u}\left(s\right)\right)\right]ds=\mathbb{E}\left[f_x\left(\Gamma^\top\hat{u}\left(t\right)\right)\right]<\infty,\ dt-a.e.$$

Therefore

$$\lim_{\varepsilon\downarrow 0}\mathbb{E}\left|\frac{1}{\varepsilon}\mathbb{E}^t\left[\int_t^{t+\varepsilon}\mathcal{U}\left(s;t\right)ds\right]-\frac{1}{\varepsilon}\mathbb{E}^t\left[\int_t^{t+\varepsilon}\mathcal{U}\left(s;s\right)ds\right]\right|=0.$$

Hence, for each t there exists a sequence $\left(\varepsilon_n^t\right)_{n\geq 0}\subset\left(0,T-t\right)$ such that $\lim\limits_{n\to\infty}\varepsilon_n^t=0$ and

$$\lim_{n\to\infty}\left|\frac{1}{\varepsilon_n^t}\mathbb{E}^t\left[\int_t^{t+\varepsilon_n^t}\mathcal{U}\left(s;t\right)ds\right]-\frac{1}{\varepsilon_n^t}\mathbb{E}^t\left[\int_t^{t+\varepsilon_n^t}\mathcal{U}\left(s;s\right)ds\right]\right|=0,\ d\mathbb{P}-a.s.$$

Moreover, since $f_x\left(\Gamma^\top\hat{u}\left(\cdot\right)\right)\in L^p_{\mathcal{F}}\left(0,T;\mathbb{R}\right)$ and

$$\left(\bar{p}\left(\cdot\right),\bar{q}\left(\cdot\right)\right)\in\mathcal{C}^2_{\mathcal{F}}\left(0,T;\mathbb{R}\right)\times L^2_{\mathcal{F}}\left(0,T;\mathbb{R}^{N+1}\right),$$

we get from Lemma 2.2.1 that, there exists a subsequence of $\left(\varepsilon_n^t\right)_{n\geq 0}$ which also denote by $\left(\varepsilon_n^t\right)_{n\geq 0}$ such that

$$\lim_{n\to\infty}\frac{1}{\varepsilon_n^t}\mathbb{E}^t\left[\int_t^{t+\varepsilon_n^t}\mathcal{U}\left(s;s\right)ds\right]=\mathcal{U}\left(t;t\right),\ dt-a.e,\ d\mathbb{P}-a.s.$$

To derive the second statement in the Lemma 3.2.1, it is sufficient to prove the following, for each t there exists a sequence $\left(\varepsilon_n^t\right)_{n\geq 0}\subset\left(0,T-t\right)$ such that $\lim\limits_{n\to\infty}\varepsilon_n^t=0$ and

$$\lim_{n\to\infty}\frac{1}{\varepsilon_n^t}\mathbb{E}^t\left[\int_t^{t+\varepsilon_n^t}\mathfrak{D}\left(s-t\right)f_{xx}\left(\Gamma^\top\left(\hat{u}\left(s\right)+\kappa v 1_{[t,t+\varepsilon)}\right)\right)ds\right]=f_{xx}\left(\Gamma^\top\left(\hat{u}\left(t\right)\right)\right),$$
$$\lim_{n\to\infty}\frac{1}{\varepsilon_n^t}\mathbb{E}^t\left[\int_t^{t+\varepsilon_n^t}\widetilde{\sigma}\left(s,\alpha\left(s\right)\right)\widetilde{\sigma}\left(s,\alpha\left(s\right)\right)^\top P\left(s;t\right)ds\right]=\widetilde{\sigma}\left(t,\alpha\left(t\right)\right)\widetilde{\sigma}\left(t,\alpha\left(t\right)\right)^\top P\left(t;t\right).$$

Let us prove the first limit. We have

$$\left|\frac{1}{\varepsilon}\mathbb{E}^t\left[\int_t^{t+\varepsilon}\mathfrak{D}\left(s-t\right)f_{xx}\left(\Gamma^\top\left(\hat{u}\left(s\right)+\kappa v 1_{[t,t+\varepsilon)}\right)\right)ds\right]-\frac{1}{\varepsilon}\mathbb{E}^t\left[\int_t^{t+\varepsilon}f_{xx}\left(\Gamma^\top\left(\hat{u}\left(s\right)\right)\right)ds\right]\right|$$
$$\leq\sup_{t\leq s\leq t+\varepsilon}\left|\mathfrak{D}\left(s-t\right)-1\right|\frac{1}{\varepsilon}\mathbb{E}^t\left[\int_t^{t+\varepsilon}\sup_{\eta\leq M}\left|f_{xx}\left(\Gamma^\top\left(\hat{u}\left(s\right)+\eta\right)\right)\right|ds\right].$$

Applying the same arguments used in the first limit, we get according to Lemma 2.2.1,

$$\lim_{n\to\infty}\frac{1}{\varepsilon_n^t}\mathbb{E}^t\left[\int_t^{t+\varepsilon_n^t}f_{xx}\left(\Gamma^\top\left(\hat{u}\left(s\right)\right)\right)ds\right]=f_{xx}\left(\Gamma^\top\left(\hat{u}\left(t\right)\right)\right),$$

at least for a subsequence. ■

Now, we are ready to give a proof of Theorem 3.2.1. The proof is inspired by [29] and [28].

Proof of Theorem 3.2.1. Given an admissible strategy

$$\hat{u}(\cdot) \in L^1_{\mathcal{F}}(0,T;\mathbb{R}) \times L^2_{\mathcal{F}}\left(t,T;\mathbb{R}^{N+1}\right),$$

for which (3.23) holds, according to Lemma 3.2.1, we have from (3.30), for any $t \in [0,T]$ and for any $\mathbb{R}^{N+2}-$valued, $\mathcal{F}_t-$measurable and bounded random variable v, there exists a sequence $(\varepsilon^t_n)_{n\in\mathbb{N}} \subset (0,T-t)$ satisfying $\varepsilon^t_n \to 0$ as $n \to \infty$, such that

$$\begin{aligned}\lim_{n\to 0}\frac{1}{\varepsilon^t_n}\left\{J\left(t,\hat{X}(t),\alpha(t);u^{\varepsilon}(\cdot)\right)-J\left(t,\hat{X}(t),\alpha(t);\hat{u}(\cdot)\right)\right\} &= \langle\mathcal{U}(t;t),v\rangle+\frac{1}{2}\left\langle\mathcal{V}^0(t;t)v,v\right\rangle,\\ &=\frac{1}{2}\left\langle\mathcal{V}^0(t;t)v,v\right\rangle,\\ &\leq 0,\end{aligned}$$

where the last inequality arises from the concavity condition of $f(\cdot)$ and $h(\cdot)$, it follows $\left\langle\mathcal{V}^0(t;t)v,v\right\rangle \leq 0$. Hence $\hat{u}(\cdot)$ is an equilibrium strategy.

Conversely, assume that $\hat{u}(\cdot)$ is an equilibrium strategy. Then, by (3.14) together with (3.30) and Lemma 3.2.1, for any $(t,u) \in [0,T] \times \mathbb{R}^{N+2}$, the following inequality holds:

$$\langle\mathcal{U}(t;t),u\rangle+\frac{1}{2}\left\langle\mathcal{V}^0(t;t)u,u\right\rangle \leq 0. \tag{3.40}$$

Now, we define $\forall (t,u) \in [0,T]\times\mathbb{R}^{N+2}$,

$$\Psi(t,u)=\langle\mathcal{U}(t;t),u\rangle+\frac{1}{2}\left\langle\mathcal{V}^0(t;t)u,u\right\rangle.$$

Clearly $\Psi(\cdot,\cdot)$ is well defined. In fact, it is a second order polynomial in terms of the components of vector u. Easy manipulations prove that the inequality (3.40) is equivalent to

$$\Psi(t,0)=\max_{u\in\mathbb{R}^{N+2}}\Psi(t,u),\ d\mathbb{P}-a.s,\forall t\in[0,T]. \tag{3.41}$$

So it is clear that the maximum condition (3.41) leads to the following condition: $\forall t \in [0,T]$,

$$\Psi_u(t,0)=\mathcal{U}(t;t)=0,\ d\mathbb{P}-a.s. \tag{3.42}$$

According to Lemma 3.2.1, the statement (3.23) follows immediately. ■

3.2.2 Characterization of equilibrium strategies by verification argument

In classical stochastic control theory the sufficient condition of optimality is important for computing optimal controls. It states that if an admissible control satisfies the maximum condition of the Hamiltonian function, it is indeed optimal for the stochastic control problem. This enables one to solve examples of optimal control problems in which a smooth solution to the associated adjoint equation may be found. The purpose of the following theorem is to characterize the open-loop equilibrium pair only by a sufficient condition of equilibrium. Let us introduce an alternative to **(H3)** hypothesis:

(H3') The maps $f(\cdot), h(\cdot)$ are continuously differentiable and the first order derivatives $f_x(\cdot), h_x(\cdot)$ are continuous.

Them we have the following theorem:

Theorem 3.2.2 *Let **(H1)**, **(H2)** and **(H3')** hold. Given an admissible strategy $\hat{u}(\cdot) \in L^1_{\mathcal{F}}(0,T;\mathbb{R}) \times L^2_{\mathcal{F}}\left(t,T;\mathbb{R}^{N+1}\right)$, let for any $t \in [0,T]$, the process*

$$(p(\cdot;t), q(\cdot;t), l(\cdot;t)) \in \left(\mathcal{C}^2_{\mathcal{F}}(t,T;\mathbb{R}) \times L^2_{\mathcal{F}}\left(t,T;\mathbb{R}^{N+1}\right) \times \mathcal{L}^{\lambda,2}_{\mathcal{F},p}\left(t,T;\mathbb{R}^{d\times d}\right)\right)$$

be the unique solution to the BSDE (3.16). Then, $\hat{u}(\cdot)$ is an equilibrium trading strategy, if the following condition holds

$$\mathcal{U}(t;t) = 0,\ d\mathbb{P}\text{-}a.s.,\ dt - a.e. \tag{3.43}$$

Proof. Suppose that $\hat{u}(\cdot)$ is an admissible control for which the condition (3.43) holds. In addition, for any $t \in [0,T]$ and $\varepsilon \in [0, T-t)$, we consider $u^{\varepsilon}(\cdot)$ by (3.13). Then, we have the following difference

$$\begin{aligned}&J\left(t,\hat{X}(t),\alpha(t);\hat{u}(\cdot)\right) - J\left(t,\hat{X}(t),\alpha(t);u^{\varepsilon}(\cdot)\right)\\ &= \mathbb{E}^t\left[\int_t^T \mathfrak{D}(s-t)\left(f\left(\Gamma^{\top}\hat{u}(s)\right) - f\left(\Gamma^{\top}u^{\varepsilon}(s)\right)\right)ds + \mathfrak{D}(T-t)\left(h\left(\hat{X}(T)\right) - h\left(\hat{X}^{\varepsilon}(T)\right)\right)\right].\end{aligned}$$

Mentioning that, by the concavity of $h(\cdot)$, we have

$$\mathbb{E}^t\left[\mathfrak{D}(T-t)\left(h\left(\hat{X}(T)\right) - h\left(\hat{X}^{\varepsilon}(T)\right)\right)\right] \geq \mathbb{E}^t\left[\mathfrak{D}(T-t)\left(\hat{X}(T) - \hat{X}^{\varepsilon}(T)\right)^{\top} h_x\left(\hat{X}(T)\right)\right].$$

Accordingly, by the terminal condition in the BSDE (3.16), we obtain that

$$\begin{aligned}
&J\left(t,\hat{X}\left(t\right),\alpha\left(t\right),\hat{u}\left(\cdot\right)\right)-J\left(t,\hat{X}\left(t\right),\alpha\left(t\right),u^{\varepsilon}\left(\cdot\right)\right)\\
&\geq\mathbb{E}^{t}\left[\int_{t}^{T}\mathfrak{D}\left(s-t\right)\left(f\left(\Gamma^{\top}\hat{u}\left(s\right)\right)-f\left(\Gamma^{\top}u^{\varepsilon}\left(s\right)\right)\right)ds+\left(\hat{X}\left(T\right)-\hat{X}^{\varepsilon}\left(T\right)\right)^{\top}p\left(T;t\right)\right]. \qquad (3.44)
\end{aligned}$$

By applying Ito's formula to $s\mapsto\left(\hat{X}\left(s\right)-\hat{X}^{\varepsilon}\left(s\right)\right)^{\top}p\left(s;t\right)$ on $[t,T]$, we get

$$\mathbb{E}^{t}\left[\left(\hat{X}\left(T\right)-\hat{X}^{\varepsilon}\left(T\right)\right)^{\top}p\left(T;t\right)\right]=\mathbb{E}^{t}\left[\int_{t}^{T}\left(\hat{u}\left(s\right)-u^{\varepsilon}\left(s\right)\right)^{\top}\left(\mathbf{B}\left(s,\alpha\left(s\right)\right)p\left(s;t\right)+\mathbf{D}\left(s,\alpha\left(s\right)\right)\widetilde{q}\left(s;t\right)\right)ds\right]. \qquad (3.45)$$

By the concavity of $f\left(\cdot\right)$, we get

$$\begin{aligned}
&\mathbb{E}^{t}\left[\int_{t}^{T}\mathfrak{D}\left(s-t\right)\left(f\left(\Gamma^{\top}\hat{u}\left(s\right)\right)-f\left(\Gamma^{\top}u^{\varepsilon}\left(s\right)\right)\right)ds\right]\\
&\geq\mathbb{E}^{t}\left[\int_{t}^{T}\mathfrak{D}\left(s-t\right)\left\langle f_{x}\left(\Gamma^{\top}\hat{u}\left(s\right)\right)\Gamma,\hat{u}\left(s\right)-u^{\varepsilon}\left(s\right)\right\rangle ds\right]. \qquad (3.46)
\end{aligned}$$

By taking (3.45) and (3.46) in (3.44) , it follows that

$$\begin{aligned}
&J\left(t,\hat{X}\left(t\right),\alpha\left(t\right);u^{\varepsilon}\left(\cdot\right)\right)-J\left(t,\hat{X}\left(t\right),\alpha\left(t\right);\hat{u}\left(\cdot\right)\right)\\
&\leq\mathbb{E}^{t}\left[\int_{t}^{T}\left\langle\mathbf{B}\left(s,\alpha\left(s\right)\right)p\left(s;t\right)+\mathbf{D}\left(s,\alpha\left(s\right)\right)\widetilde{q}\left(s;t\right)+\mathfrak{D}\left(s-t\right)f_{x}\left(\Gamma^{\top}\hat{u}\left(s\right)\right)\Gamma,u^{\varepsilon}\left(s\right)-\hat{u}\left(s\right)\right\rangle ds\right]\\
&=\mathbb{E}^{t}\left[\int_{t}^{t+\varepsilon}\left\langle\mathcal{U}\left(s;t\right),v\right\rangle ds\right].
\end{aligned}$$

Now, dividing both sides by ε and taking the limit when ε vanishes, by Lemma 3.2.1, we deduce tha $\hat{u}(\cdot)$ is an equilibrium control.

Remark 3.2.1 *The purpose of the sufficient condition of optimality is to obtain an optimal control by computing the difference* $J\left(\hat{u}\left(\cdot\right)\right)-J\left(u\left(\cdot\right)\right)$ *in terms of the Hamiltonian function, where* $u\left(\cdot\right)$ *is an arbitrary admissible control. Here, the spike variation perturbation* (3.13) *plays an important role in deriving the sufficient condition for equilibrium strategies, which reduces to calculating the difference* $J\left(t,\hat{X}\left(t\right),\alpha\left(t\right),\hat{u}\left(\cdot\right)\right)-J\left(t,\hat{X}\left(t\right),\alpha\left(t\right),u^{\varepsilon}\left(\cdot\right)\right)$*, without the need to achieve the second order expansion in the spike variation.*

Conclusion

In this thesis, we have investigated about two stochastic optimal control problems that, in various ways, are time inconsistent in the sense that they do not admit a Bellman optimality principle. In chapter 2, we considered a class of dynamic decision models of the conditional time-inconsistent LQ type under the effect of a Markovian regime-switching. We have employed the game theoretic approach to handle the time inconsistency. Throughout this study, open-loop Nash equilibrium strategies are established as an alternative to optimal strategies. This was achieved using a stochastic system that includes a flow of forward-backward stochastic differential equations under equilibrium conditions. The inclusion of concrete examples in mathematical finance confirms the validity of our proposed study. The work may be developed in different ways:

(1) The methodology may be extended, for example, to a non-Markovian framework, implying that the coefficients of the controlled SDE as well as the coefficients of the objective functional are random.

(2) The model discussed in this chapter may be extended to "progressive measurable" as an alternative of "predictable" control problem, and a research problem on how to obtain the corresponding state feedback equilibrium strategy is a very interesting and challenging one (see [58] for mor detail). Some further investigations will be carried out in our future publications.

In chapter 3, we revisited the equilibrium consumption-investment and reinsurance for Merton's portfolio problem with a general discount function and a general utility function in a Markovian framework. We assumed that the coefficients in our model, including the appreciation rate and volatility of the stock, were Markov modulated processes. The insurers received a deterministic income, invested in risky assets, consumed continuously, and purchased proportional reinsurance or acquired new business. The objective was to maximize the terminal wealth and the accumulated consumption utility. The non-exponential discounting makes the optimal strategy adopted time-inconsistent. Consequently, the Bellman's optimality principle no longer holds. By formulating the problem in the game theoretic framework and using a variational technical approach, we derived the necessary and sufficient equilibrium condition. Possible extensions of the results in the chapter include many financial and actuarial applications, such as contribution and portfolio selection in pension funding (see, e.g., Josa-Fombellida and Rincón-Zapatero [30] and references therein).

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