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LOWER BOUNDS FOR SELF-ADJOINT STURM–LIOUVILLE OPERATORS

Abstract. In this note we provide estimates for the lower bound of the self-adjoint operator associated with the three-coefficient Sturm–Liouville differential expression

$$\frac{1}{r}\left(-\frac{\mathrm{d}}{\mathrm{d}x}p\frac{\mathrm{d}}{\mathrm{d}x}+q\right)$$

in the weighted L^2-Hilbert space $L^2(\mathbb{R}; r dx)$.

1. Introduction

One-dimensional Schrödinger operators of the form

$$H=-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+q \tag{1.1}$$

a with real-valued potential q have been studied in the mathematical and physical literature intensively in the last century due to their particular importance in quantum mechanics. Typically one is interested in a suitable self-adjoint realization in $L^2(\mathbb{R})$ and its spectral properties, among them estimates for lower bounds, numbers of negative eigenvalues, and Lieb–Thirring inequalities are particularly important, see, for instance, the recent survey [9].

The main objective of this note is to derive estimates on the lower bound of more general Sturm-Liouville operators of the type

$$T=\frac{1}{r}\left(-\frac{\mathrm{d}}{\mathrm{d}x}p\frac{\mathrm{d}}{\mathrm{d}x}+q\right) \tag{1.2}$$

with real-valued coefficients under the standard assumptions $r, 1/p, q \in L^1_{\mathrm{loc}}(\mathbb{R})$ and r, p positive almost everywhere. We refer the reader to the textbooks [6], [11], [13], [15], [18], [21], [22], and [23] for an overview and detailed study of Sturm-Liouville (resp., Schrödinger) operators. The natural Hilbert space in this context is the weighted L^2-space $L^2_r(\mathbb{R}) := L^2(\mathbb{R}; r dx)$ and under some mild additional assumptions on the coefficients one concludes that T is a semibounded self-adjoint operator in $L^2_r(\mathbb{R})$. As mentioned above, lower bounds for the spectrum of T are known for the special case $r = p = 1$, that is, $T = H$, and for completeness we provide a straightforward estimate as a warm up in Section 2.

In the general setting it seems that a systematic study is missing and it is the aim of this note to initiate and contribute to this circle of problems. It is clear that the coefficients r and p have an essential influence on the lower bound. If, for instance, the weight function $r = r_0$ is constant and $p = 1$ then formally $T = (1/r_0)H$ and the lower bound $\min\sigma(T)$ of T is simply given by $(1/r_0)\min\sigma(H)$. This already indicates that for a nonconstant weight function r the L^∞-norm of $1/r$ will appear in the lower bounds, and the situation becomes much more difficult if $1/r \notin L^\infty(\mathbb{R})$, in which case we require the existence of a function g that neutralizes the behaviour of the weight function r on subsets of $\mathbb{R}$ where r is small. Furthermore, the norm of the coefficient p will enter in lower bound estimates and very roughly speaking $1/p$ has to be considered in conjunction with the potential q. The methods and proofs in this paper are strongly inspired by [2], where bounds on nonreal eigenvalues of indefinite Sturm-Liouville operators are obtained.

2. One-dimensional Schrödinger operators

As a warm up we discuss in this short section the special case $p = r = 1$ and $q \in L^s(\mathbb{R})$ real-valued a.e., $s \in [1,\infty]$, and derive a lower bound for the self-adjoint Schrödinger operator H in (1.1) using the argument presented in [20, (3.5.30), p. 155–156].

We start by recalling that $q \in L^s(\mathbb{R})$, $s \in [1,\infty)$, implies that q is relatively form compact with respect to the free Hamiltonian H_0 in $L^2(\mathbb{R})$, where

$$H_0 f = -f'', \quad f \in D(H_0) = H^2(\mathbb{R}), \tag{2.1}$$

with $H^\ell(\mathbb{R})$, $\ell \in [0,\infty)$, the standard scale of Sobolev spaces. This follows from the stronger statement that $|q|^{1/2}(H_0+I)^{-1/2}$ satisfies (see, e.g., [16, Theorem XI.20])

$$|q|^{1/2}(H_0+I)^{-1/2} \in \mathcal{B}_{2s}\big(L^2(\mathbb{R})\big) \ \text{ if } \ q \in L^s(\mathbb{R}),\ s \in [1,\infty), \tag{2.2}$$

where $\mathcal{B}_t(\mathcal{H})$ represent the $\ell^t(\mathbb{N})$-based trace ideals of compact operators in the complex, separable Hilbert space $\mathcal{H}$. In particular,

$$|q|^{1/2}(H_0+I)^{-1/2} \ \text{ is compact}, \tag{2.3}$$

and hence the form sum H of H_0 and q is self-adjoint in $L^2(\mathbb{R})$ and bounded from below. By a result of Hartman [12] and Rellich [17] (see also [11, Theorem 8.5.2]), the boundedness from below of the minimal operator associated with the differential expression $-(d^2/dx^2)+q$ implies that the latter is in the limit point case at $\pm\infty$ and hence the maximal operator associated with $-(d^2/dx^2)+q$ is self-adjoint in $L^2(\mathbb{R})$, and thus necessarily coincides with H. It is clear that for $s=\infty$ the same is true as $q \in L^\infty(\mathbb{R})$ is a bounded perturbation of H_0. Consequently, H is given by

$$\begin{aligned}
&Hf = -f'' + qf, \\
&f \in D(H) = \big\{ g \in L^2(\mathbb{R}) \,\big|\, g, g' \in AC_{\text{loc}}(\mathbb{R});\ (-g''+qg) \in L^2(\mathbb{R}) \big\}.
\end{aligned}$$

Property (2.3) then implies

$$\sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(H_0) = [0,\infty), \tag{2.4}$$

and hence it suffices to consider negative eigenvalues, which turn out to be simple as $-(d^2/dx^2)+q$ is in the limit point case at $\pm\infty$. We consider an eigenvalue $\lambda<0$ of H and denote the corresponding eigenfunction by f_λ. From $-f_\lambda''+qf_\lambda = \lambda f_\lambda$ one concludes

$$f_\lambda = -\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \lambda\right)^{-1} q f_\lambda$$

and using the corresponding Green's function we obtain

$$\|f_\lambda\|_2 = \frac{1}{2\sqrt{-\lambda}} \big\| e^{-\sqrt{-\lambda}|\,\cdot\,|} * q f_\lambda \big\|_2 \le \frac{1}{2\sqrt{-\lambda}} \big\| e^{-\sqrt{-\lambda}|\,\cdot\,|} \big\|_t \|q f_\lambda\|_{t'}, \tag{2.5}$$

where Young's inequality,[1] with $1/t + 1/t' = 1 + 1/2$ was applied in the last step. Hölder's inequality then yields $\|qf_\lambda\|_{t'} \leq \|q\|_s \|f_\lambda\|_2$ for $1/t' = 1/s + 1/2$ and hence,

$$\sqrt{-\lambda} \leq \frac{1}{2}\big\|e^{-\sqrt{-\lambda}|\,\cdot\,|}\big\|_t \|q\|_s = \frac{1}{2}\left(\frac{2}{t\sqrt{-\lambda}}\right)^{\frac{1}{t}} \|q\|_s \tag{2.6}$$

if $t \in (1,\infty)$, that is, $s \in (1,\infty)$. As $1/t = 1 - 1/s$ it follows for $s \in (1,\infty)$ that

$$(-\lambda)^{\frac{2s-1}{2s}} \leq 2^{-\frac{1}{s}} \left(\frac{s-1}{s}\right)^{\frac{s-1}{s}} \|q\|_s$$

and hence

$$\min \sigma(H) \geq -2^{-\frac{2}{2s-1}} \left(\frac{s-1}{s}\right)^{\frac{2(s-1)}{2s-1}} \|q\|_s^{\frac{2s}{2s-1}} \quad \text{for } s \in (1,\infty). \tag{2.7}$$

It is easy to see that the lower bound (2.7) also remains valid for $s = 1$ (indeed, inequality (2.5) and the first inequality in (2.6) apply with $s = 1$, $t = \infty$, $1/t' = 3/2$) in which case one obtains

$$\min \sigma(H) \geq -(1/4)\|q\|_1^2, \tag{2.8}$$

and obviously applies to $s = \infty$, implying

$$\min \sigma(H) \geq -\|q\|_\infty. \tag{2.9}$$

We mention that the bound (2.7) and (2.8) coincide with [5, Corollary 14.3.11 and Corollary 14.3.12] and that the above argument also leads to bounds for Schrödinger operators with complex potentials $q \in L^s(\mathbb{R})$, $s \in [1,\infty)$ (see, in particular, [1] for the case $s = 1$). In the context of Schrödinger operators with complex-valued potentials we also refer, for instance, to [4], [7], [8], and [10].

Remark 2.1. We mention that Lieb–Thirring inequalities (see, e.g., [9], [14] and the extensive literature cited therein) also lead to lower bounds of H. More specifically, for $s = 3/2$ one can compare with the one-particle constant $L_1^{(1)} = 4/\big[3^{3/2}\pi\big] = 0.24503$ in [9, Section 3.2]: The corresponding constant in (2.7) equals $2^{-1}3^{-1/2} = 0.28867$. Historically, we note that Barnes, Brascamp, and Lieb [3] derived a lower bound for the ground state energy of (multi-dimensional) Schrödinger operators already in 1976.

3. Main results

Now we consider the general Sturm-Liouville differential expression

$$\tau = \frac{1}{r}\left(-\frac{\mathrm{d}}{\mathrm{d}x}p\frac{\mathrm{d}}{\mathrm{d}x} + q\right) \tag{3.1}$$

on $\mathbb{R}$ with real-valued coefficients under the standard assumptions,

$$r, 1/p, q \in L^1_{\mathrm{loc}}(\mathbb{R}), \tag{3.2}$$

and we assume from now on that Hypothesis 3.1 below is satisfied. In the following $L^1_{\mathrm{u}}(\mathbb{R})$ denotes the normed space of uniformly locally integrable functions, that is,

$$L^1_{\mathrm{u}}(\mathbb{R}) = \big\{h \in L^1_{\mathrm{loc}}(\mathbb{R}) : \|h\|_{\mathrm{u}} < \infty\big\}, \qquad \|h\|_{\mathrm{u}} = \sup_{n\in\mathbb{Z}} \int_n^{n+1} |h(t)|\,\mathrm{d}t.$$

[1] Explicitly, $\|f * g\|_\alpha \leq \|f\|_\beta \|g\|_\gamma$, $1 \leq \alpha, \beta, \gamma \leq \infty$, $1 + \alpha^{-1} = \beta^{-1} + \gamma^{-1}$.

Hypothesis 3.1. The real coefficients p, q and r of τ satisfy the following:

(a) $p(x) > 0$ for a. a. $x \in \mathbb{R}$ and $1/p \in L^\eta(\mathbb{R})$ for some $\eta \in [1,\infty]$;
(b) $q \in L^1_{\mathrm{u}}(\mathbb{R})$;
(c) $r(x) > 0$ for a. a. $x \in \mathbb{R}$ and there exist a, $b \in \mathbb{R}$ with $a < b$ such that

$$\operatorname{ess\,inf}_{t\in\mathbb{R}\setminus[a,b]} r(t) > 0. \tag{3.3}$$

It is known that the differential expression τ is in the limit-point case at both singular endpoints $\pm\infty$, and the corresponding maximal operator

$$Tf = \tau f = \frac{1}{r}\left(-(pf')' + qf\right),$$
$$f \in D(T) = \left\{ g \in L^2_r(\mathbb{R}) \,\middle|\, g, pg' \in AC_{\mathrm{loc}}(\mathbb{R});\ \tau g \in L^2_r(\mathbb{R}) \right\},$$

is self-adjoint in the weighted L^2-space $L^2_r(\mathbb{R})$ and semibounded from below; cf. [2, Lemma A.2]. Our main goal is to derive estimates for the lower bound $\min\sigma(T)$ of T. For a nonnegative function $g \in L^\infty(\mathbb{R})$ the set

$$\Omega_g := \{x \in \mathbb{R} \,|\, r(x)g(x) < 1\}$$

and its Lebesgue measure $\mu(\Omega_g)$ will appear in the lower bound estimates in our main results below. In the particular case $1/r \in L^\infty(\mathbb{R})$ one can choose $g = 1/r$ and hence Ω_g becomes a Lebesgue null set, which leads to more explicit lower bound estimates. We also mention that for any $\varepsilon > 0$ there exists a (constant) nonnegative function $g_\varepsilon \in L^\infty(\mathbb{R})$ such that $\mu(\Omega_{g_\varepsilon}) < \varepsilon$; this follows from

$$\begin{aligned}\lim_{n\to\infty} \mu(\{x \in \mathbb{R} \,|\, r(x) < 1/n\}) &= \mu\left(\bigcap_{n=1}^{\infty} \{x \in \mathbb{R} \,|\, r(x) < 1/n\}\right) \\ &= \mu(\{x \in \mathbb{R} \,|\, r(x) = 0\}) = 0.\end{aligned}$$

The first main result requires only minimal assumptions on the potential in Hypothesis 3.1, that is, $q \in L^1_{\mathrm{u}}(\mathbb{R})$, but we have to assume that $1/p \in L^\infty(\mathbb{R})$. We shall denote the negative part of the potential q by q_-.

Theorem 3.2. *In addition to Hypothesis 3.1, assume that* $1/p \in L^\infty(\mathbb{R})$, *let*

$$\alpha = 2\|q_-\|_{\mathrm{u}} + 4\|1/p\|_\infty \|q_-\|^2_{\mathrm{u}} \quad \textit{and} \quad \beta = \left(4\|1/p\|_\infty \alpha\right)^{1/2}, \tag{3.4}$$

and choose a nonnegative function $g \in L^\infty(\mathbb{R})$ *such that* $\mu(\Omega_g)\beta < 1$. *Then*

$$\min\sigma(T) \geq \frac{-\alpha\|g\|_\infty}{1 - \mu(\Omega_g)\beta}$$

and in the special case $1/r \in L^\infty(\mathbb{R})$ *the choice* $g = 1/r$ *implies* $\mu(\Omega_g) = 0$ *and*

$$\min\sigma(T) \geq -\left(2\|q_-\|_{\mathrm{u}} + 4\|1/p\|_\infty\|q_-\|^2_{\mathrm{u}}\right)\|1/r\|_\infty.$$

In the next result we consider the case $q_- \in L^s(\mathbb{R})$, $s \in [1,\infty]$, and $1/p \in L^\infty(\mathbb{R})$.

Theorem 3.3. *In addition to Hypothesis 3.1, assume that* $1/p \in L^\infty(\mathbb{R})$ *and* $q_- \in L^s(\mathbb{R})$ *for some* $s \in [1,\infty]$, *let*

$$\alpha = \begin{cases} \|q_-\|_s \beta^{\frac{1}{s}} & \textit{if } s \in [1,\infty), \\ \|q_-\|_\infty & \textit{if } s = \infty, \end{cases} \quad \textit{and} \quad \beta = \begin{cases} \left(4\|1/p\|_\infty\|q_-\|_s\right)^{\frac{s}{2s-1}} & \textit{if } s \in [1,\infty), \\ \left(4\|1/p\|_\infty\|q_-\|_\infty\right)^{1/2} & \textit{if } s = \infty, \end{cases} \tag{3.5}$$

and choose a nonnegative function $g \in L^\infty(\mathbb{R})$ such that $\mu(\Omega_g)\beta < 1$. Then

$$\min \sigma(T) \geq \frac{-\alpha \|g\|_\infty}{1 - \mu(\Omega_g)\beta}$$

and in the special case $1/r \in L^\infty(\mathbb{R})$ the choice $g = 1/r$ implies $\mu(\Omega_g) = 0$ and

$$\min \sigma(T) \geq -\big(4\|1/p\|_\infty\big)^{\frac{1}{2s-1}} \|q_-\|_s^{\frac{2s}{2s-1}} \|1/r\|_\infty \quad \text{if } s \in [1, \infty). \tag{3.6}$$

If $s = \infty$ we have

$$\min \sigma(T) \geq -\|q_-\|_\infty \|1/r\|_\infty.$$

Remark 3.4. The bounds in Theorem 3.3 above are not optimal. In fact, in the special case $p = r = 1$ and $q_- \in L^s(\mathbb{R})$ for some $s \in [1, \infty)$ the bound in (3.6) becomes

$$\min \sigma(T) \geq \begin{cases} -4\|q_-\|_1^2 & \text{if } s = 1, \\ -2^{\frac{2}{2s-1}} \|q_-\|_s^{\frac{2s}{2s-1}} & \text{if } s \in (1, \infty), \end{cases}$$

while (2.7) (or [5, Corollary 14.3.11 and Corollary 14.3.12]) show that

$$\min \sigma(T) \geq \begin{cases} -\|q\|_1^2/4 & \text{if } s = 1, \\ -2^{-\frac{2}{2s-1}} \left(\frac{s-1}{s}\right)^{\frac{2(s-1)}{2s-1}} \|q\|_s^{\frac{2s}{2s-1}} & \text{if } s \in (1, \infty). \end{cases} \tag{3.7}$$

◇

In the following theorem we deal with $q_- \in L^s(\mathbb{R})$, $s \in [1, \infty]$, and $1/p \in L^\eta(\mathbb{R})$, $\eta \in [1, \infty)$.

Theorem 3.5. *In addition to Hypothesis 3.1, assume that $1/p \in L^\eta(\mathbb{R})$ for some $\eta \in [1, \infty)$ and $q_- \in L^s(\mathbb{R})$ for some $s \in [1, \infty]$ such that $\eta + s > 2$ if $s \neq \infty$. Let α be as in (3.5) and*

$$\beta = \begin{cases} \left(\left(\frac{2\eta-1}{\eta}\right)^2 \|1/p\|_\eta \|q_-\|_s\right)^{\frac{\eta s}{2\eta s - \eta - s}} & \text{if } s \in [1, \infty), \\ \left(\left(\frac{2\eta-1}{\eta}\right)^2 \|1/p\|_\eta \|q_-\|_\infty\right)^{\frac{\eta}{2\eta - 1}} & \text{if } s = \infty, \end{cases} \tag{3.8}$$

and choose a nonnegative function $g \in L^\infty(\mathbb{R})$ such that $\mu(\Omega_g)\beta < 1$. Then

$$\min \sigma(T) \geq \frac{-\alpha \|g\|_\infty}{1 - \mu(\Omega_g)\beta}$$

and in the special case $1/r \in L^\infty(\mathbb{R})$ the choice $g = 1/r$ implies $\mu(\Omega_g) = 0$ and

$$\min \sigma(T) \geq -\left(\left(\frac{2\eta - 1}{\eta}\right)^2 \|1/p\|_\eta\right)^{\frac{\eta}{2\eta s - \eta - s}} \|q_-\|_s^{\frac{2\eta s - s}{2\eta s - \eta - s}} \|1/r\|_\infty \quad \text{if } s \in [1, \infty).$$

If $s = \infty$ we have

$$\min \sigma(T) \geq -\|q_-\|_\infty \|1/r\|_\infty.$$

The following proposition provides a simple condition for nonnegativity of T.

Proposition 3.6. *Suppose that Hypothesis 3.1 holds and assume that $1/p, q_- \in L^1(\mathbb{R})$. If $\|1/p\|_1 \|q_-\|_1 < 1$ then $\min \sigma(T) \geq 0$.*

Remark 3.7. If the coefficients p, q and r in Hypothesis 3.1 are restricted to the half line $(0,\infty)$ and T_+ denotes the self-adjoint realization of the (restricted) differential expression τ in $L^2((0,\infty))$ with Dirichlet boundary conditions at the regular endpoint 0, then the lower bound estimates above remain valid for T_+. In fact, the Dirichlet boundary condition at 0 ensures that the boundary term in the integration by parts formula for $f \in D(T_+)$ vanishes and hence the proofs in the next section (see, e.g. (4.4)) extend directly to the half line case.

4. Proofs

In this section we prove our main results. It will always be assumed that the coefficients p, q, r satisfy Hypothesis 3.1. The first three items of the following useful statement can be found, for instance, in [2, Lemma A.2]. The last item follows from [2, Lemma A.1] and the first item.

Lemma 4.1. *Assume Hypothesis 3.1, then the following assertions hold for all* $f,\ g \in D(T)$*:*

(i) $f,\ \sqrt{p}f' \in L^2(\mathbb{R})$ *and* $qf^2 \in L^1(\mathbb{R})$*;*
(ii) *there exists a sequence* $(x_n)_{n\in\mathbb{Z}}$ *in* $\mathbb{R}$ *satisfying* $\lim_{n\to\infty} x_n = \infty$ *and* $\lim_{n\to-\infty} x_n = -\infty$ *such that* $\lim_{|n|\to\infty} f(x_n) = 0$*;*
(iii) $\lim_{|x|\to\infty} (pf')(x)\overline{g(x)} = 0$.
(iv) $f \in L^\infty(\mathbb{R})$.

For our estimates of the lower bound of T it is convenient to reduce the considerations to the set

$$D_-(T) := \{f \in D(T) \,|\, (Tf, f)_r \leq 0\}, \tag{4.1}$$

where $(\cdot,\cdot)_r$ stands for the weighted inner product corresponding to $L^2_r(\mathbb{R})$. The potential q is decomposed in its positive part q_+ and negative part q_-, i. e.

$$q = q_+ - q_-, \quad \text{where} \quad q_+ := \frac{|q| + q}{2} \quad \text{and} \quad q_- := \frac{|q| - q}{2}. \tag{4.2}$$

Lemma 4.2. *Assuming Hypothesis 3.1, every function* $f \in D_-(T)$ *satisfies*

$$\|\sqrt{p}f'\|_2^2 \leq \|q_- f^2\|_1 \quad \text{and} \quad \|qf^2\|_1 \leq 2\|q_- f^2\|_1. \tag{4.3}$$

Moreover, the inequality $\|q_- f^2\|_1 \leq \|q_+ f^2\|_1$ *implies* $\|\sqrt{p}f'\|_2 = 0$.

Proof. For $f \in D_-(T)$ integration by parts together with Lemma 4.1 (i) and (iii) yields

$$\begin{aligned} 0 \geq (Tf, f)_r &= \int_{\mathbb{R}} p(t)|f'(t)|^2 \,\mathrm{d}t + \int_{\mathbb{R}} q(t)|f(t)|^2 \,\mathrm{d}t \\ &= \|\sqrt{p}f'\|_2^2 + \|q_+ f^2\|_1 - \|q_- f^2\|_1. \end{aligned} \tag{4.4}$$

This implies $\|\sqrt{p}f'\|_2^2 \leq \|q_- f^2\|_1$ and $\|q_+ f^2\|_1 \leq \|q_- f^2\|_1$. Therefore, with $|q| = q_+ + q_-$ we have

$$\|qf^2\|_1 = \|q_+ f^2\|_1 + \|q_- f^2\|_1 \leq 2\|q_- f^2\|_1.$$

If $\|q_- f^2\|_1 \leq \|q_+ f^2\|_1$ holds, then (4.4) implies $\|\sqrt{p}f'\|_2 = 0$. □

Lemma 4.3. *In addition to Hypothesis 3.1, assume that there are constants* $\alpha \geq 0$, $\beta \geq 0$ *and a nonnegative function* $g \in L^\infty(\mathbb{R})$ *such that*

(i) $\|q_- f^2\|_1 \leq \alpha\|f\|_2^2$ *and* $\|f\|_\infty^2 \leq \beta\|f\|_2^2$ *for all* $f \in D_-(T)$*;*
(ii) $\mu(\Omega_g)\beta < 1$.

Then one has

$$\min \sigma(T) \geq \frac{-\alpha\|g\|_\infty}{1-\mu(\Omega_g)\beta}. \tag{4.5}$$

Proof. Let $f \in D_-(T)$. Then one has

$$\begin{aligned} \|g\|_\infty (f,f)_r &= \|g\|_\infty \int_{\mathbb{R}} |f(t)|^2 r(t)\,\mathrm{d}t \geq \int_{\mathbb{R}} |f(t)|^2 r(t) g(t)\,\mathrm{d}t \\ &\geq \int_{\mathbb{R}\setminus\Omega_g} |f(t)|^2 r(t) g(t)\,\mathrm{d}t \geq \|f\|_2^2 - \int_{\Omega_g} |f(t)|^2\,\mathrm{d}t \\ &\geq \|f\|_2^2 - \mu(\Omega_g)\|f\|_\infty^2 \geq \big(1-\mu(\Omega_g)\beta\big)\|f\|_2^2. \end{aligned} \tag{4.6}$$

Further, we have by (4.4)

$$(Tf,f)_r = \|\sqrt{p}f'\|_2^2 + \|q_+ f^2\|_1 - \|q_- f^2\|_1 \geq -\|q_- f^2\|_1 \geq -\alpha\|f\|_2^2.$$

This together with (4.6) yields

$$(Tf,f)_r \geq -\frac{\alpha\|g\|_\infty}{1-\mu(\Omega_g)\beta}(f,f)_r. \tag{4.7}$$

Obviously, the inequality in (4.7) holds also for $f \in D(T)\setminus D_-(T)$ and, thus, for all $f \in D(T)$. This implies (4.5) □

Next we recall estimates on the L^∞-norm of functions in $D(T)$ from [2].

Lemma 4.4. *Assume Hypothesis 3.1. Then the following assertions hold for all* $f \in D(T)$*.*

(i) *If* $1/p \in L^\eta(\mathbb{R})$*, where* $\eta \in [1,\infty)$*, then*

$$\|f\|_\infty \leq \left(\frac{2\eta-1}{\eta}\sqrt{\|1/p\|_\eta}\|\sqrt{p}f'\|_2\right)^{\frac{\eta}{2\eta-1}} \|f\|_2^{\frac{\eta-1}{2\eta-1}}. \tag{4.8}$$

(ii) *If* $1/p \in L^\infty(\mathbb{R})$ *then*

$$\|f\|_\infty \leq \left(2\sqrt{\|1/p\|_\infty}\|\sqrt{p}f'\|_2\|f\|_2\right)^{1/2}. \tag{4.9}$$

Moreover, for every $\varepsilon > 0$ *and all* $n \in \mathbb{Z}$ *one has*

$$\sup_{t\in[n,n+1]} |f(t)|^2 \leq \varepsilon\|1/p\|_\infty \int_n^{n+1} p(t)|f'(t)|^2\,\mathrm{d}t + \left(1+\frac{1}{\varepsilon}\right)\int_n^{n+1} |f(t)|^2\,\mathrm{d}t. \tag{4.10}$$

Proof. The estimates (4.8) and (4.9) are proved in [2, Lemma 4.1]. For the convenience of the reader we verify the estimate (4.10), which is a variant of [19, Lemma 9.32]. Let $\varepsilon > 0$ and $n \in \mathbb{Z}$. Then for $f \in D(T)$ and x, $y \in [n,n+1]$

$$|f(x)|^2 = |f(y)|^2 + 2\operatorname{Re}\int_y^x f'(t)\overline{f(t)}\,\mathrm{d}t.$$

By the mean value theorem we can choose y such that $|f(y)|^2 = \int_n^{n+1}|f(t)|^2\,\mathrm{d}t$. Thus, by the Cauchy–Schwarz inequality and $2\alpha\beta \leq \alpha^2+\beta^2$ for α, $\beta \in \mathbb{R}$ we obtain

$$
\begin{aligned}
|f(x)|^2 \leq & \int_n^{n+1} |f(t)|^2 \,\mathrm{d}t \\
& + 2\left(\frac{1}{\varepsilon}\int_n^{n+1} |f(t)|^2 \,\mathrm{d}t\right)^{1/2} \cdot \left(\|1/p\|_\infty \varepsilon \int_n^{n+1} p(t)|f'(t)|^2 \,\mathrm{d}t\right)^{1/2} \\
& \leq \varepsilon \|1/p\|_\infty \int_n^{n+1} p(t)|f'(t)|^2 \,\mathrm{d}t + \left(1+\frac{1}{\varepsilon}\right)\int_n^{n+1} |f(t)|^2 \,\mathrm{d}t,
\end{aligned}
$$

which leads to (4.10). □

For the proofs of Theorems 3.2 – 3.5 it is no restriction to consider $f \in D_-(T) \setminus \{0\}$ and to assume that q_- is positive on a set of positive Lebesgue measure.

Proof of Theorem 3.2. Let $1/p \in L^\infty(\mathbb{R})$ and consider α, β as in (3.4). Choose $\varepsilon = (2\|q_-\|_{\mathrm{u}}\|1/p\|_\infty)^{-1} > 0$. The estimate in (4.10) of Lemma 4.4 yields

$$
\begin{aligned}
\|q_- f^2\|_1 &= \int_{\mathbb{R}} q_-(t)|f(t)|^2 \,\mathrm{d}t \\
&\leq \|q_-\|_{\mathrm{u}} \sum_{n\in\mathbb{Z}} \sup_{t\in[n,n+1]} |f(t)|^2 \\
&\leq \|q_-\|_{\mathrm{u}} \left(\varepsilon\|1/p\|_\infty \|\sqrt{p}f'\|_2^2 + \left(1+\frac{1}{\varepsilon}\right)\|f\|_2^2\right) \qquad (4.11)\\
&= \frac{1}{2}\|\sqrt{p}f'\|_2^2 + \left(\|q_-\|_{\mathrm{u}} + 2\|1/p\|_\infty\|q_-\|_{\mathrm{u}}^2\right)\|f\|_2^2 \\
&= \frac{1}{2}\|\sqrt{p}f'\|_2^2 + \frac{\alpha}{2}\|f\|_2^2.
\end{aligned}
$$

Together with Lemma 4.2 we obtain

$$
\|\sqrt{p}f'\|_2^2 = 2\|\sqrt{p}f'\|_2^2 - \|\sqrt{p}f'\|_2^2 \leq 2\|q_-f^2\|_1 - \|\sqrt{p}f'\|_2^2 \leq \alpha\|f\|_2^2.
$$

With (4.9) in Lemma 4.4 and (4.11) we see

$$
\|f\|_\infty^2 \leq 2\sqrt{\|1/p\|_\infty \alpha}\,\|f\|_2^2 = \beta\|f\|_2^2 \quad \text{and} \quad \|q_-f^2\|_1 \leq \alpha\|f\|_2^2
$$

and hence Lemma 4.3 leads to the statements in Theorem 3.2. □

Proof of Theorem 3.5. Suppose that $1/p \in L^\eta(\mathbb{R})$ and $q_- \in L^s(\mathbb{R})$, where η, $s \in [1,\infty)$ with $\eta + s > 2$. Since $\eta + s > 2$ we obtain

$$
2\eta s - \eta - s = \eta(s-1) + s(\eta-1) \geq s - 1 + \eta - 1 > 0.
$$

Let α and β as in (3.5) and (3.8), respectively. From Hölder's inequality we obtain

$$\begin{aligned}\|q_-f^2\|_1 &\leq \|f\|_\infty^{\frac{2}{s}} \int_{\mathbb{R}} |q_-(t)||f(t)|^{\frac{2(s-1)}{s}}\,\mathrm{d}t \\ &\leq \|f\|_\infty^{\frac{2}{s}} \left(\int_{\mathbb{R}} |q_-(t)|^s\,\mathrm{d}t\right)^{\frac{1}{s}} \left(\int_{\mathbb{R}} |f(t)|^2\,\mathrm{d}t\right)^{\frac{s-1}{s}} \\ &= \|q_-\|_s \|f\|_\infty^{\frac{2}{s}} \|f\|_2^{\frac{2(s-1)}{s}}.\end{aligned} \tag{4.12}$$

Thus, together with Lemma 4.4 (i) and Lemma 4.2 we obtain

$$\begin{aligned}\|f\|_\infty^2 &= \left(\frac{\|f\|_\infty^{\frac{2(2\eta-1)}{\eta}}}{\|f\|_\infty^{\frac{2}{s}}}\right)^{\frac{\eta s}{2\eta s-\eta-s}} \leq \left(\frac{\left(\frac{2\eta-1}{\eta}\right)^2 \|1/p\|_\eta \|\sqrt{p}f'\|_2^2 \|f\|_2^{\frac{2(\eta-1)}{\eta}}}{\|f\|_\infty^{\frac{2}{s}}}\right)^{\frac{\eta s}{2\eta s-\eta-s}} \\ &\leq \left(\left(\frac{2\eta-1}{\eta}\right)^2 \|1/p\|_\eta \|q_-\|_s\right)^{\frac{\eta s}{2\eta s-\eta-s}} \|f\|_2^2 = \beta\|f\|_2^2.\end{aligned}$$

The estimate from (4.12) yields

$$\|q_-f^2\|_1 \leq \|q_-\|_s \beta^{\frac{1}{s}} \|f\|_2^2 = \alpha\|f\|_2^2.$$

Now consider the case $q_- \in L^\infty(\mathbb{R})$. Choose α and β as in (3.5) and (3.8), respectively. Observe that

$$\|q_-f^2\|_1 \leq \|q_-\|_\infty \|f\|_2^2 = \alpha\|f\|_2^2. \tag{4.13}$$

Lemma 4.4 (i) in combination with Lemma 4.2 and (4.13) leads to

$$\begin{aligned}\|f\|_\infty^2 &\leq \left(\left(\frac{2\eta-1}{\eta}\right)^2 \|1/p\|_\eta \|\sqrt{p}f'\|_2^2\right)^{\frac{\eta}{2\eta-1}} \|f\|_2^{\frac{2(\eta-1)}{2\eta-1}} \\ &\leq \left(\left(\frac{2\eta-1}{\eta}\right)^2 \|1/p\|_\eta \|q_-\|_\infty\right)^{\frac{\eta}{2\eta-1}} \|f\|_2^2 = \beta\|f\|_2^2.\end{aligned}$$

Now Theorem 3.5 follows from Lemma 4.3. □

Proof of Theorem 3.3. Consider first the case $1/p \in L^\infty(\mathbb{R})$ and $q_- \in L^s(\mathbb{R})$, where $s \in [1,\infty)$. Let α and β be as in (3.5). Again Hölder's inequality yields (4.12). Lemma 4.4 (ii), (4.12) and Lemma 4.2 imply

$$\begin{aligned}\|f\|_\infty^2 &= \left(\frac{\|f\|_\infty^4}{\|f\|_\infty^{\frac{2}{s}}}\right)^{\frac{s}{2s-1}} \leq \left(\frac{4\|1/p\|_\infty \|\sqrt{p}f'\|_2^2 \|f\|_2^2}{\|f\|_\infty^{\frac{2}{s}}}\right)^{\frac{s}{2s-1}} \\ &\leq (4\|1/p\|_\infty \|q_-\|_s)^{\frac{s}{2s-1}} \|f\|_2^2 = \beta\|f\|_2^2.\end{aligned}$$

By applying this to the estimate in (4.12) we arrive at

$$\|q_-f^2\|_1 \leq \|q_-\|_s \beta^{\frac{1}{s}} \|f\|_2^2 = \alpha\|f\|_2^2,$$

and again the statements in Theorem 3.3 follow from Lemma 4.3.

The assertion for $1/p$, $q_- \in L^\infty(\mathbb{R})$ follows in a similar way. Consider α, β in (3.5). As before (4.13) holds. Lemma 4.4 (ii) in combination with Lemma 4.2 and (4.13) implies

$$\|f\|_\infty^2 \le 2\sqrt{\|1/p\|_\infty}\|\sqrt{p}f'\|_2\|f\|_2 \le 2\sqrt{\|1/p\|_\infty\|q_-\|_\infty}\|f\|_2^2 = \beta\|f\|_2^2. \qquad \square$$

Proof of Proposition 3.6. Let $f \in D_-(T)$. In the case $1/p$, $q_- \in L^1(\mathbb{R})$ Lemma 4.2 and Lemma 4.4 (i) yield

$$\|f\|_\infty^2 \le \|1/p\|_1\|\sqrt{p}f'\|_2^2 \le \|1/p\|_1\|q_-f^2\|_1 \le \|1/p\|_1\|q_-\|_1\|f\|_\infty^2.$$

If $\|1/p\|_1\|q_-\|_1 < 1$, then $\|f\|_\infty = 0$ and hence $D_-(T) = \{0\}$. This implies $(Tf, f)_r \ge 0$ for all $f \in D(T)$ and hence $\min\sigma(T) \ge 0$. $\square$

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