Asymptotic Survival of Genuine Multipartite Entanglement in Noisy Quantum Networks Depends on the Topology

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The study of entanglement in multipartite quantum states plays a major role in quantum information theory and genuine multipartite entanglement signals one of its strongest forms for applications. However, its characterization for general (mixed) states is a highly nontrivial problem. We introduce a particularly simple subclass of multipartite states, which we term pair-entangled network (PEN) states, as those that can be created by distributing exclusively bipartite entanglement in a connected network. We show that genuine multipartite entanglement in a PEN state depends on both the level of noise and the network topology and, in sharp contrast to the case of pure states, it is not guaranteed by the mere distribution of mixed bipartite entangled states. Our main result is a markedly drastic feature of this phenomenon: the amount of connectivity in the network determines whether genuine multipartite entanglement is robust to noise for any system size or whether it is completely washed out under the slightest form of noise for a sufficiently large number of parties. This latter case implies fundamental limitations for the application of certain networks in realistic scenarios, where the presence of some form of noise is unavoidable. To illustrate the applicability of PEN states to study the complex phenomenology behind multipartite entanglement, we also use them to prove superactivation of genuine multipartite nonlocality for any number of parties.

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Introduction.—Entanglement is at the core of the foundations of quantum mechanics and it is a crucial resource for the applications of quantum information theory [1]. The analysis of many-body entanglement has provided relevant tools for condensed matter physics [2] and has given rise to several concrete multipartite applications such as secret sharing [3], conference key agreement [4], and measurement-based quantum computation [5]. Studying the complex ways in which multipartite entanglement manifests itself is thus interesting both theoretically and to come up with new applications. Of particular interest is the class of genuine multipartite entangled (GME) states, which are those that cannot be obtained by mixing only partially separable states and, therefore, entanglement spreads among all parties and not just a subset. GME is known to play a nontrivial role in certain quantum algorithms [6] and multipartite quantum key distribution schemes [7]. Moreover, remarkably, it has been shown to be a necessary condition to achieve maximum sensitivity in quantum metrology [8] and to obtain a multipartite private state and, hence, to establish a secret key [9]. Thus, the certification of GME states has been studied in detail [10], although the characterization of entanglement in general is known to be computationally hard [11]. On the other hand, the preparation, control, and distribution of GME states is a major experimental challenge. Arguably, the most feasible way to achieve this (e.g., in quantum optics applications) is by distributing exclusively bipartite entanglement among different pairs of parties giving rise to a connected network. In fact, quantum networks are currently being actively investigated as a realistic platform for quantum information processing. This includes establishing long-range entanglement starting from smaller entanglement links or harnessing node-to-node entanglement in order to achieve on-demand quantum communication between different possible subsets of parties (see [12] and references therein).

In this Letter, we intend to put forward a theoretical analysis, from the point of view of entanglement theory, of the properties of the underlying states that arise in quantum networks, which we term pair-entangled network (PEN) states. It should be noticed that such a state is a universal resource under local operations and classical communication (LOCC) provided that the underlying graph is connected and the bipartite entanglement shared by the nodes is of sufficient quality. This is because by means of local preparation and teleportation the parties can then end up sharing *any* quantum state of a given local dimension. Recent work has considered the limitations arising from the distribution of arbitrary bipartite entanglement in networks when state manipulation is bound to a class of operations that is a strict subset of LOCC and it has been shown that certain GME states cannot be prepared in this way [13,14]. However, here we study the entanglement properties of PEN states within the LOCC paradigm depending on the type of entanglement shared and the topology of the network [15]. From this point of view, to ask when a PEN state is GME seems a very relevant question as this makes it possible to benchmark the quality of the quantum network. If the corresponding PEN state is not GME, then it cannot be transformed by LOCC into a GME state and, therefore, as pointed out above, this leads to fundamental limitations in applications.

A simple argument shows that all pure PEN states are GME independently of the amount of entanglement shared and the geometry of the network (if it is connected) and, actually, we have recently shown that they are even genuine multipartite nonlocal (GMNL) [16]. However, in realistic implementations noise is unavoidable and mixed PEN states must be considered. The previous property of pure PEN states directly implies that, for a fixed network, GME should be robust to some noise, but the extent to which this holds is unclear. In fact, to our knowledge it is not known whether sharing arbitrary bipartite entanglement is enough to guarantee that a PEN state is GME. Here, we consider a simple and realistic model in which the nodes share isotropic states, i.e., maximally entangled states mixed with white noise, and show that the answer to the above question is negative. The mere fact that the nodes share bipartite entanglement does not imply that a connected network is GME. Furthermore, this not only depends on the level of noise but also on the topology of the network. However, our main result is a more extreme feature of this phenomenon. Instead of asking for which value of the noise parameter a given network is GME, we consider a more realistic approach in which the noise parameter is fixed and we ask which networks display GME under this constraint. It turns out that, for any nonzero value of the noise, any tree network of sufficiently many parties is no longer GME and, on the contrary, the GME of a completely connected network persists for any number of parties if the noise is below some threshold. Thus, asymptotic survival of GME depends drastically on the geometry of the network. While any unavoidable limitation in the ability to prepare entangled states upper bounds the number of parties that can share GME in some topologies, a larger connectivity guarantees GME for any size provided that a certain level of quality in the prepared entangled states can be achieved.

In addition to this, the overwhelming complexity of multipartite state space has often led to constrain the study of entanglement to subsets of states with relevant physical and/or mathematical properties such as graph states [17], locally maximally entangleable states [18] or tensor network states [19]. We believe that the class of PEN states is a promising platform endowed with a clear operational motivation in order to study the rich phenomenology of multipartite entanglement. Based on this, using PEN states we provide examples of GME states which are not GMNL, different from those known before [20,21]. The tensor product structure of quantum theory enables the fact that objects that are not resourceful may display this resource when several copies of them are taken together, a phenomenon known as superactivation. This is the case of nonlocality, where it has been proven that superactivation is possible in the bipartite case [22]. Here, building on this construction, we prove superactivation of GMNL for any number of parties using the aforementioned states.

Preliminaries.—As mentioned above, we will consider networks where the nodes share isotropic states on $\mathbb{C}^d \otimes \mathbb{C}^d$:

$$\rho(p) = p\phi_d^+ + (1-p)\tilde{\mathbb{1}}, \tag{1}$$

where $|\phi_d^+\rangle = (1/\sqrt{d}) \sum_{i=0}^{d-1} |ii\rangle$ is the d-dimensional maximally entangled state, $\phi_d^+ = |\phi_d^+\rangle\langle\phi_d^+|$ and $\tilde{\mathbb{1}} = \mathbb{1}/d^2$. Isotropic states not only represent a standard noise model but they also possess nontrivial symmetry properties. This has led to an in-depth study of these states and they appear as an intermediate step in several protocols [23]. In particular, isotropic states are entangled if and only if $p > 1/(d+1)$ [23].

PEN states are defined by selecting an undirected graph $G = (V, E)$ that encodes the structure of the network. The vertices $V = [n] \coloneqq \{1, 2, \ldots, n\}$ represent the parties and the edges $E \subseteq \{(i,j) : i, j \in V, i < j\}$ represent when two nodes share a bipartite state. In order to specify the PEN state, one must specify G as well as which state is associated to every edge in E. In our case, we will always consider isotropic states $\rho_{ij}(p)$ shared by parties i and j [24] and, for simplicity, we will often consider that all edges are given by the same isotropic state. Thus, given the graph G and the noise parameter p, the corresponding isotropic PEN state is

$$\sigma_G(p) = \bigotimes_{(i,j)\in E} \rho_{ij}(p). \tag{2}$$

Here, the indices in the tensor product indicate to which local Hilbert space each qudit of the isotropic states belongs [25]. Thus, given G, party i holds $\deg(i)$ qudits [where $\deg(i)$ is the degree of vertex i] and the local dimension of $\sigma_G(p)$ for each party i is $d^{\deg(i)}$. We will focus on some particular graphs: a tree graph is a graph with no cycles, such as the star graph in which a central node is connected to all other vertices and there are no more edges. These are graphs with the lowest connectivity. On the other hand, a completely connected graph is that for which $E = \{(i,j) : i, j \in V, i < j\}$. Sometimes it will be convenient to alter the notation for vertices in order to label the different particles held by one party. For instance, for three parties A, B, and C the star and completely connected PEN states can be also respectively denoted by

$$\sigma_{\rm star}(p) = \rho_{A_1B}(p) \otimes \rho_{A_2C}(p),$$
$$\sigma_{\rm cc}(p) = \rho_{A_1B_1}(p) \otimes \rho_{A_2C_1}(p) \otimes \rho_{B_2C_2}(p). \quad (3)$$

Last, we provide the definition of GME. Given the n-partite Hilbert space $H = \otimes_{i=1}^{n} H_i$, a pure state $|\psi\rangle \in H$ is biseparable (otherwise GME) if $|\psi\rangle = |\psi_M\rangle \otimes |\psi_{\bar{M}}\rangle$ for some $M \subsetneq [n]$ and its complement $\overline{M}$, where $|\psi_M\rangle \in \otimes_{i\in M} H_i$ and $|\psi_{\overline{M}}\rangle \in \otimes_{i\in\overline{M}} H_i$. The definition extends to mixed states by taking the convex hull: the set of biseparable states is $\mathrm{conv}\{|\psi\rangle\langle\psi| : |\psi\rangle \text{is biseparable}\}$ and a state that does not belong to it is GME. It follows from this definition that the set of biseparable states is closed under LOCC. Notice that, for PEN states, it is immediate that if a subset of the network only shares separable states with its complement, then the PEN state is biseparable. It is worth pointing out that studying GME in PEN states built from isotropic states is not only a standard noise model but also quite general, since all states with entangled fraction larger than $1/d$ can be transformed by LOCC into an entangled isotropic state [23].

Robustness of GME for isotropic PEN states.—The fact that any PEN state is GME for any connected network sharing arbitrary bipartite pure entangled states follows by noticing that the reduced state corresponding to any subset of parties $M \subsetneq [n]$ will in this case be mixed. Notice that, since the set of biseparable states is closed, any given fixed PEN state will then tolerate some noise in its edges so as to remain GME. However, this still leaves open the question of whether sharing arbitrary bipartite entanglement in any connected network is enough to generate GME. We start by observing that already the simplest case of tripartite PEN states with two-qubit isotropic edges [cf. Eq. (3)] shows that this is not the case (disproving, moreover, a conjecture in [35]). Although we did not compute the exact thresholds, by explicitly constructing biseparable decompositions and using the techniques of [36] to build fully decomposable witnesses for these states, in [25] we prove bounds on the noise parameter p that guarantee biseparability or GME for $\sigma_{\rm star}(p)$ and $\sigma_{\rm cc}(p)$. The results are summarized in Table I. Notice that for $0.491 < p \leq 0.547$, $\sigma_{\rm cc}(p)$ is GME while $\sigma_{\rm star}(p)$ is biseparable. Thus, this proves the intuitive fact that increasing the connectivity by producing more links makes GME more robust to noise.

We now move onto our main result. The above observations show that if an experimental implementation is bound to a certain visibility in the preparation of isotropic states, the ability to display GME may depend on the network configuration. Nevertheless, improving the apparatuses to produce isotropic states with $p > 0.577$ will suffice in any connected tripartite configuration. Increasing this visibility will ensure GME for any network of a *fixed* number of parties. However, this does not necessarily imply that GME asymptotically survives, i.e., that there is a threshold in the visibility an experimentalist can aim at above which GME is guaranteed independently of the number of parties. This is indeed a more realistic situation, where a certain quantum state can be obtained in experiments and one wants to use it in a large network. Since deleting edges is LOCC (as this amounts to tracing out subsystems), asymptotic survival of GME in one configuration ensures it for those with more links; however, it is not at all clear in principle whether this phenomenon is universal, impossible, or whether it depends on the network.

TABLE I. Bounds for biseparability and GME for tripartite PEN states with two-qubit isotropic edges. Notice that both states can be biseparable above the threshold $p > 1/3$ that determines that the edges are entangled.

	Biseparable for $p \leq$	GME for $p >$
$\sigma_{\rm star}(p)$	$(1+2\sqrt{2})/7 \simeq 0.547$	$1/\sqrt{3} \simeq 0.577$
$\sigma_{\rm cc}(p)$	$3/7 \simeq 0.429$	$(2\sqrt{5}-3)/3 \simeq 0.491$

We first focus on a general class of PEN states which covers the networks of lowest connectivity: tree graphs. For this family, we find a negative answer to the question of asymptotic survival of GME.

Theorem 1.—Let $G = (V, E)$ be a tree graph with n vertices and let $\sigma_G(p)$ denote the corresponding n-partite isotropic PEN state as given by Eq. (2). Then, $\sigma_G(p)$ is biseparable if $|E| \geq dp/(1-p)$.

The proof (cf. Supplemental Material [25]) is obtained by obtaining an explicit biseparable decomposition of $\sigma_G(p)$ using the separability properties of isotropic states. To illustrate the previous result, note that the n-partite PEN state $\sigma_{\rm star}(p)$ with two-qubit isotropic edges is biseparable when $n \geq (1+p)/(1-p)$. Thus, a visibility $p = 0.6$ precludes GME for more than three parties, while the already experimentally demanding value of $p = 0.95$ bounds the size to 38 parties. This shows a fundamental limitation to GME distribution in practical scenarios such as the star configuration in which a powerful central laboratory prepares entangled states for satellite nodes.

It should be stressed that the proof of Theorem 1 can be easily generalized to other noise models and, more importantly, to other networks. In this sense, in Supplemental Material, Theorem 4 [25], we prove a similar result for polygonal networks, i.e., those based on a cycle graph. At this point one may wonder whether asymptotic survival of GME is at all possible. Our next result shows that this is indeed the case by considering the network of highest connectivity.

Theorem 2.—Let G be a completely connected graph of n vertices and let $\sigma_{\rm cc}(p)$ denote the corresponding n-partite isotropic PEN state as given by Eq. (2). Then, there exists a value of $p_0 < 1$, which is independent of n (i.e., depends only on d), such that $\sigma_{\rm cc}(p)$ is GME for every n and for all $p > p_0$.

Hence, our results uncover a fundamental property of entanglement in quantum networks: asymptotic survival of

GME depends on the topology. The proof of Theorem 2, which is given in Supplemental Material [25], relies on two parts. First, we establish an upper bound on the sum over all pairs of parties of the fidelity with the maximally entangled state ϕ_2^+ that can be achieved after any LOCC protocol starting with a biseparable state. Then, we show that, above a certain threshold in the visibility, $\sigma_{\rm cc}(p)$ can overcome this bound by edge teleportation and entanglement distillation when n is large. Once GME is ensured to persist for a large number of parties, it follows that, for a fixed, large enough visibility, GME can be guaranteed for completely connected networks of any size. The precise value of the threshold p_0 can be explicitly given (at least in the limit of large size) as this is controlled by the success of the particular entanglement distillation protocol that is implemented. We used the one-way distillation protocol of [37], which in the particular case where the nodes share two-qubit isotropic states yields $p_0 \simeq 0.865$. We did not attempt any optimization in this direction.

Constructing PEN states with relevant entanglement properties.—In addition to the relevance of PEN states in the context of networks, we find this family extremely versatile to study general properties of multipartite entanglement. Here, we will focus on the relation between quantum entanglement and nonlocality. The latter concept refers to the possibility of obtaining certain correlations when performing separate measurements on multipartite quantum states which cannot be explained classically, and it is crucial in many applications in quantum information theory [38]. In precise terms, a given n-partite probability distribution $P = \{P(\alpha_1\alpha_2\ldots\alpha_n|\chi_1\chi_2\ldots\chi_n)\}_{\alpha_1,\ldots,\alpha_n,\chi_1,\ldots,\chi_n}$ (with input χ_i and output α_i for party i) is said to be GMNL if it is not of the form

$$\begin{aligned} &P(\alpha_1\alpha_2\ldots\alpha_n|\chi_1\chi_2\ldots\chi_n) \\ &= \sum_{M\subsetneq[n]}\sum_{\lambda} q_M(\lambda)P_M(\{\alpha_i\}_{i\in M}|\{\chi_i\}_{i\in M},\lambda) \\ &\quad \times P_{\overline{M}}(\{\alpha_i\}_{i\in\overline{M}}|\{\chi_i\}_{i\in\overline{M}},\lambda), \end{aligned} \tag{4}$$

where $q_M(\lambda) \geq 0 \;\forall\; \lambda, M$ and $\sum_{\lambda,M} q_M(\lambda) = 1$. Otherwise, we say that P is bilocal. The distributions $P_M, P_{\overline{M}}$ will be assumed to be nonsignaling as this captures most physical situations better than unrestricted $P_M, P_{\overline{M}}$ [39–42]. An n-partite state ρ is GMNL if local measurements $\{E^{(i)}_{\alpha_i|\chi_i} \geq 0\}$ ($\sum_{\alpha_i} E^{(i)}_{\alpha_i|\chi_i} = \mathbb{1}\;\forall\,\chi_i, i$) exist which give rise to a GMNL distribution

$$P(\alpha_1\alpha_2\ldots\alpha_n|\chi_1\chi_2\ldots\chi_n) = \mathrm{tr}(\rho \bigotimes_{i=1}^{n} E^{(i)}_{\alpha_i|\chi_i}). \tag{5}$$

While GMNL states are GME, as mentioned in the introduction, the converse implication is not true for any number of parties [20,21]. Finding more examples of GME states that are bilocal and the conditions under which this might happen is crucial to fully understand the relation between entanglement and nonlocality in the multipartite setting. In fact, the first such example found in the bipartite case [43,44] is a cornerstone in the field.

It is worth mentioning that many copies of isotropic PEN states are always GME [as long as $p > 1/(d+1)$ for the underlying isotropic states]. The fact that this holds even for biseparable PEN states is possible because the set of biseparable states is not closed under tensor products. Indeed, taking many copies of an isotropic PEN state can be understood as having another PEN state with the same topology but where each edge represents many copies of an isotropic state and, thus, whose edges are more entangled. By means of the LOCC protocol of [23], starting from sufficiently many copies of any isotropic PEN state one can distill another PEN state where each edge represents a state arbitrarily close to a maximally entangled state. However, this new state is GME (in fact, GMNL by [16]) and, therefore, the original state must be GME as well.

The situation is not so clear when looking at nonlocality since LOCC transformations do not preserve the set of bilocal states. Being able to obtain a GMNL state by taking many copies of a bilocal one would yield GMNL superactivation. While this has been shown in the bipartite scenario [22], to our knowledge, it has not been studied for more than two parties. Our last result tackles the previous two questions. It provides new families of bilocal GME states and, moreover, it shows that superactivation can also hold in the multipartite setting.

Theorem 3.—Let $\tau(p)$ denote the n-partite PEN state corresponding to a star graph in which all edges represent the maximally entangled state except one, which is given by the isotropic state $\rho(p)$. Then, if

$$\frac{1}{d+1} < p \leq \frac{(3d-1)(d-1)^{d-1}}{(d+1)d^d}, \tag{6}$$

(i) $\tau(p)$ is GME $\forall\, n \geq 3$; (ii) $\tau(p)$ is not GMNL $\forall\, n \geq 3$; and (iii) $\tau(p)^{\otimes k}$ is GMNL $\forall\, n \geq 3$ if k is large enough.

To obtain this result (see [25]), we prove that any star network with an entangled isotropic state on one edge and maximally entangled states on the rest is GME. We also establish a connection between having bilocality of PEN states and the edges being nonsteerable—a well-studied property of bipartite quantum states, intermediate between entanglement and nonlocality [45]. We show that any star network with a nonsteerable state on one edge is automatically bilocal. Combining the previous two results we can obtain a network $\tau(p)$ verifying conditions (i) and (ii) above, where the bounds in Eq. (6) guarantee that the edge with the isotropic state is entangled but nonsteerable [46]. Finally, using the ideas of [47], we extend the Bell inequality used to prove bipartite superactivation in [22,48] to a multipartite one in order to show that $\tau(p)^{\otimes k}$ is GMNL for a large enough k.

Conclusions.—In this Letter we have introduced the class of multipartite PEN states as those underlying the current proposals of quantum networks and we have investigated their GME properties. We have shown that sharing bipartite entanglement in a connected network does not guarantee GME, but that both a higher quality of node-to-node entanglement and a larger connectivity play in favor of displaying this property. Our main result is a drastically contrasting behavior with respect to this feature: while tree isotropic PEN states cannot be GME for any value of the visibility $p < 1$ for sufficiently many parties, the GME of the completely connected PEN state is robust for all visibilities above a fixed threshold for any system size. Furthermore, the class of PEN states is an operationally motivated subset of multipartite states with a clear mathematical structure in which the well-developed theory of bipartite entanglement can be exploited to analyze entanglement in the multipartite scenario. Thus, we have provided a construction of GME but non-GMNL PEN states for any number of parties that lead to superactivation of GMNL.

Besides these particular results, we believe that PEN states might find applications in different contexts and that this work can be continued in several directions. We conclude by posing two such possibilities. First, tree graphs and the completely connected graph represent the two most extreme cases in terms of connectivity. What is the minimal amount of connectivity that enables asymptotic survival of GME? Second, the aforementioned property of tree networks implies that their GMNL cannot asymptotically survive either. However, can the asymptotic survival of GME in completely connected PEN states be extended to GMNL? The dependence of these features on the geometry of the network suggests that there might be a fruitful interplay between these problems and the theory of complex networks.

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[25] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.128.220501 for more details on the definition of PEN states and the proofs of Theorems 1, 2, and 3 and the bounds given in Table I, which includes Refs. [26–34].

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