Working paper

2023-03

Statistics and Econometrics

ISSN 2387-0303

Data cloning for a threshold asymmetric stochastic volatility model

J. Miguel Marín, Helena Veiga

uc3m | Universidad Carlos III de Madrid

Data cloning for a threshold asymmetric stochastic volatility model

J. Miguel Marin

Department of Statistics Universidad Carlos III de Madrid, Spain.

Helena Veiga*

Department of Statistics and Instituto Flores de Lemus, Universidad Carlos III de Madrid, Spain and BRU-IUL, Instituto Universitário de Lisboa, Portugal.

Abstract

In this paper, we propose a new asymmetric stochastic volatility model whose asymmetry parameter can change depending on the intensity of the shock and is modeled as a threshold function whose threshold depends on past returns. We study the model in terms of leverage and propagation using a new concept that has recently appeared in the literature. We find that the new model can generate more leverage and propagation than a well-known asymmetric volatility model. We also propose to estimate the parameters of the model by cloning data. We compare the estimates in finite samples of data cloning and a Bayesian approach and find that data cloning is often more accurate. Data cloning is a general technique for computing maximum likelihood estimators and their asymptotic variances using a Markov chain Monte Carlo (MCMC) method. The empirical application shows that the new model often improves the fit compared to the benchmark model. Finally, the new proposal together with data cloning estimation often leads to more accurate 1-day and 10-day volatility forecasts, especially for return series with high volatility.

Keywords: Asymmetric stochastic volatility, Data cloning, Leverage effect, Propagation, Volatility forecasting

1. Introduction

In the literature on stochastic volatility, asymmetric volatility effects have traditionally been modeled either by assuming a negative correlation between returns and future volatility (Harvey and Shephard, 1996) or by allowing the parameters of the log- volatility equation to differ as a function of the sign of lagged returns (Breidt, 1996; So et al., 2002). This asymmetric response of volatility to price changes is also known as the leverage effect.

Catania (2020) proposes a different way of measuring the leverage effect that does not consist of the correlation between volatility and past returns. The new measure has advantages in that it does not require the existence of the fourth moment of standardized returns and the variance process, it can be calculated using only returns, and it allows quantifying the contribution of leverage to the total variance of the process. Moreover, Catania (2020) shows that the estimator

*Corresponding author

Email address: mhveiga@est-econ.uc3m.es. (Helena Veiga)

of the correlation between volatility and past returns for the leverage effect is inconsistent, i.e. it often underestimates this effect.

In this paper, we propose a new asymmetric stochastic volatility model whose asymmetric volatility response parameter depends on the intensity of the shock. It is modeled as a threshold function whose threshold depends on past returns. If the lagged return is more negative than the average of the lagged returns of the last 5 days (weekly effect), we consider this as a strong shock and therefore allow the leverage coefficient to be more negative than that when the lagged return is less negative than the average of the last 5 returns.

The model is called the threshold asymmetric autoregressive stochastic volatility (TAARSV) model and is an extension of AARSV, which belongs to the family of generalized asymmetric stochastic volatility (GASV) proposed by Mao et al. (2020).

We apply the new measure of leverage and propagation of Catania (2020) to the TAARSV and AARSV models and find that the new proposal is able to produce more leverage and propagation than AARSV. We also prove mathematically that AARSV is able to generate leverage and propagation.

Catania (2022) also proposes a new stochastic volatility model that establishes a general correlation structure between the shock of the return equation and log-volatility at different lags, and he shows that it improves the fit of leverage and propagation compared to other existing models in the literature.

Although related to modelling leverage in stochastic volatility models, from a different perspective, some work in the literature allows leverage to vary over time. For example, Bandi and Renò (2012) define leverage as a flexible function that depends on the state of the firm and is thus variable over time. The state of the firm is measured by the level of spot variance. Yu (2012) proposes a semiparametric stochastic volatility model in which the correlation between the return and the volatility innovation depends on the type of news that arrived in the market. The model is based on a linear spline. Others, such as Bretó (2014), use the Fisher transformation to allow leverage to change over time. Motivated by Yu (2012), Wu and Wang (2020) model volatility innovation as a linear spline. On the other hand, Nguyen et al. (2023) assume that volatility innovation follows a generalized autoregressive score process.

The estimation of the TAARSV is nonstandard. Bermudez et al. (2020) use data cloning (DC) to estimate several asymmetric stochastic volatility models that assume different distributions for standardized returns. They find that DC is quite reliable and computationally efficient for finite samples and can be an effective alternative to existing estimation methods for general asymmetric volatility models. Therefore, in this paper we also propose the use of DC to estimate the parameters of the TAARSV model.

Data cloning is a general technique to obtain approximate maximum likelihood estimators along with their asymptotic variances using an MCMC procedure (Lele et al., 2007, 2010). Its greatest strength is that changes to the model specification can be easily made. Moreover, the method is hardly affected by the choice of the prior (proper) distributions of the parameters, assuming a reasonable number of clones (Lele et al., 2007, 2010).

For the implementation of DC we use the free software `dclone` in an R framework and for the Bayesian approach (BA) we use JAGS (Plummer, 2003). The main weakness of the Bayesian MCMC implementation is its slow convergence and inefficiency in simulation, since it is based on a single-move Gibbs sampling algorithm as in WinBUGS (see Meyer and Yu, 2000; Yu, 2005, 2012, for the WinBUGS implementation of SV models). DC often improves the accuracy of parameter estimates of the models considered in this paper compared to the standard BA, making it an effective alternative estimation method for more flexible SV models.

The main contributions of this paper are first to propose a new asymmetric stochastic volatility model that can generate higher leverage. It is a nesting of other well-known models that already exist in the literature. Second, we use DC to estimate the new model and compare its performance with that of a standard Bayesian approach. Third, we use a new concept of leverage and propagation from Catania (2020) and study the new proposal and the benchmark in terms of leverage and propagation. The unconditional and conditional tests of superior predictive ability for volatility forecasting show that the new model and the DC estimation method often perform better for 1-day volatility forecasts. For 10-day volatility forecasts, the new proposal for Bitcoin remains the preferred specification.

The paper proceeds as follows. Section 2 introduces the new model and the concept of leverage and propagation that we follow in this paper. We also prove mathematically that a nested model is able to generate leverage and propagation and show through simulations that the new model is able to generate more leverage and propagation than the benchmark. Section 3 describes the DC estimation method and presents a simulation study to analyze the performance of the estimator in finite samples. Section 4 presents the empirical application and discusses the prediction results. Section 5 shows the main final conclusions of the paper. The proof and some empirical results are in the Appendix.

2. Model description

This section proposes a new specification that allows for high leverage and propagation. Volatility asymmetry depends on the intensity (int) of past returns and the specification is an extension of the AARSV proposed by Mao et al. (2020).

Let y_t be the return at time t, σ_t^2 its volatility, which is unobservable, $h_t \equiv \log \sigma_t^2$, ϵ_t an independent and identically distributed (IID) sequence with mean zero and variance one, η_t follows a normal distribution with mean zero and variance σ_η^2, and ϵ_t and η_t are uncorrelated for all leads and lags. The univariate threshold asymmetric autoregressive stochastic volatility model (TAARSV) is given by

$$y_t = exp\left(\frac{h_t}{2}\right)\epsilon_t,$$

$$h_t - \mu = \phi(h_{t-1} - \mu) + \gamma_{int}\epsilon_{t-1} + \eta_t,$$

where

$$\gamma_{int} = \begin{cases} \gamma_1 & \text{if } y_{t-1} < 0 \text{ and } y_{t-1} \leq mean\,(y_{t-1}, ..., y_{t-5}) \\ \gamma_2 & \text{if } y_{t-1} < 0 \text{ and } y_{t-1} > mean\,(y_{t-1}, ..., y_{t-5}) \\ 0 & \text{if } y_{t-1} \geq 0, \end{cases} \tag{1}$$

with $\gamma_1 \leq \gamma_2$ and $\gamma_1, \gamma_2 < 0$.

Volatility is expected to increase more with the size of the shock when the stock market declines. We assume that $|\phi| < 1$, so that the log variance is weakly stationary.

If we set $\gamma_{int} = \gamma$, the model is equivalent to the AARSV model from the GASV family. By recursive substitution, h_t can be written as follows:

$$h_t - \mu = \sum_{i=1}^{\infty} \phi^{i-1} \gamma_{int} \epsilon_{t-i} + \sum_{i=1}^{\infty} \phi^{i-1} \eta_{t-i}. \tag{2}$$

If instead we analyze commodities and are faced with "reverse" leverage, the model can be adjusted very easily.[1]

2.1. Leverage and propagation

Bollerslev et al. (2006) define the existence of a "prolonged leverage effect" whenever the correlation between the variance and past returns is nonzero, i.e., $corr(y_t^2, y_{t-k}) \neq 0$ for $k > 1$. However, Catania (2020) shows that this estimator for leverage is generally inconsistent, leading to its underestimation, and proposes a new measure of leverage and propagation given by:

$$v(s) = Var[y_t | y_{t-s} < 0] - Var[y_t],$$

so that whenever $v(1) > 0$, there is leverage.

The existence of leverage propagation at lag s requires that $v(s) > 0$ for $s > 1$. The idea according to Catania (2020) is that if a negative shock occurs at time $t - s$, volatility at time t increases more than its unconditional value, and then there is leverage and propagation.

In Theorem 1, we investigate whether a nested model in the system (1) is able to handle leverage and propagation as defined by Catania (2020).

Theorem 1. *Consider the stationary process y_t defined by system (1) with $|\phi| < 1$. If ϵ_t follows a normal distribution and $\gamma_{int} = \gamma$, then the leverage effect and the propagation till time s is given by*

$$v(1) = var[y_t] \left[2\Phi(-\gamma) - 1 \right] \tag{3}$$

where $\Phi(.)$ is the accumulative normal distribution,

$$var[y_t] = \exp\left(\frac{h_t}{2}\right) = \exp\left(\mu + \frac{\gamma^2 + \sigma_\eta^2}{2(1-\phi^2)}\right)$$

and

$$v(s) = var[y_t] \left[2\Phi(-\gamma\phi^{(s-1)}) - 1 \right] \quad \textit{for } s > 1, \tag{4}$$

respectively.

Proof in the Appendix.

[1] In this case, we can define γ_{int} as follows

$$\gamma_{int} = \begin{cases} \gamma_1 & \text{if } y_{t-1} > 0 \text{ and } y_{t-1} \geq mean\,(y_{t-1}, ..., y_{t-5}) \\ \gamma_2 & \text{if } y_{t-1} > 0 \text{ and } y_{t-1} < mean\,(y_{t-1}, ..., y_{t-5}) \\ 0 & \text{if } y_{t-1} \leq 0, \end{cases}$$

where $\gamma_1 \geq \gamma_2$ and $\gamma_1, \gamma_2 > 0$.

To illustrate the results of Theorem 1, we consider five cases involving the parameter set $(\mu, \phi, \gamma, \sigma_\eta)$:

- Case 1: set of parameters $(0, 0.98, -0.07, 0.05)$,
- Case 2: set of parameters $(0, 0.70, -0.07, 0.05)$,
- Case 3: set of parameters $(0, 0.98, -0.15, 0.05)$,
- Case 4: set of parameters $(0, 0.98, -0.07, 0.10)$,
- Case 5: set of parameters $(0, 0.98, 0.07, 0.05)$.

Figure 1 shows that the leverage effect and propagation increase with the absolute value of γ and also with σ_η^2, as expected. In contrast, they decrease with the decrease of volatility persistence (parameter ϕ).

Therefore, Figure 1 shows that the model proposed by Mao et al. (2020) can generate leverage and propagation already when γ is negative. When γ is positive, $v(s)$ is negative for several periods and then converges to zero as s increases. A phenomenon corresponding to the "inverted" leverage effect and "inverted" propagation.

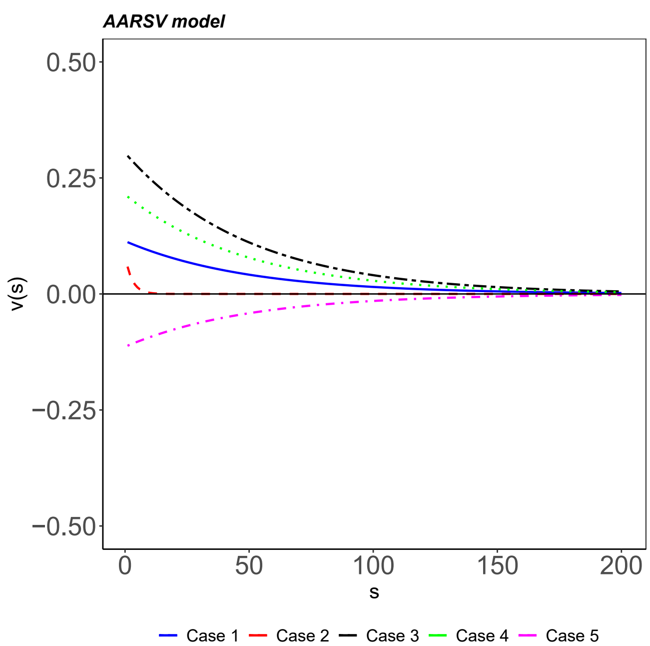


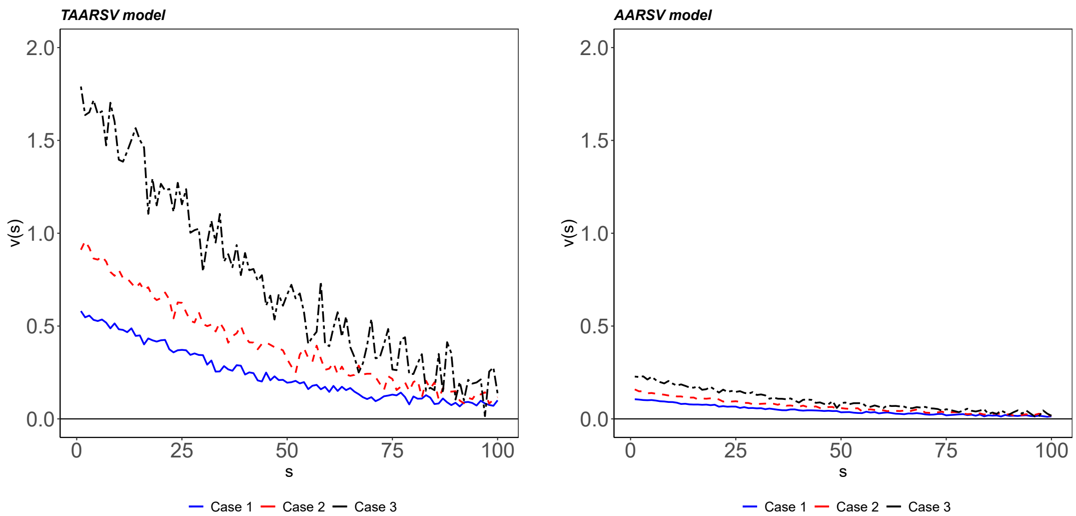
Figure 1. Leverage effect and propagation.

In the case where γ_{int} follows the process in the system (1), the leverage and propagation are analyzed by simulation for two distributions, i.e., we assume that ϵ_t follows either a Gaussian or a Student-t, and we compare the leverage and propagation with those obtained with the AARSV model.

Thus, for a sample of T observations, $v(s)$ is estimated as

$$\widehat{v_T(s)} = \hat{\omega}_T^{-1} \sum_{t=s+1}^{T} y_t^2 I(y_{t-s} < 0) - (T-s)^{-1} \sum_{t=s+1}^{T} y_t^2, \tag{5}$$

where $I(.)$ is an indicator function that is one when the argument is true and zero otherwise, and $\hat{\omega}_T^{-1} = \sum_{t=s+1}^{T} I(y_{t-s} < 0)$; see Catania (2020).



(a) (b)

Figure 2. Simulations: Leverage effect and propagation.

Figure 2, panel (a), shows the leverage effect and propagation for the TAARSV model considering three cases in terms of the parameter set $(\mu, \phi, \gamma_{int}, \sigma_\eta) = (0, 0.98, (-0.10,\ -0.03, 0), 0.05)$ and the distribution assumed for ϵ_t.

Case 1 corresponds to the Gaussian distribution, and cases 2 and 3 correspond to Student-$t(10)$ and Student-$t(5)$, respectively. The values of the parameters are chosen so that the average of γ_{int} for each generated series is close to -0.07.

Panel (b), on the other hand, corresponds to the AARSV model with the parameter set $(\mu, \phi, \gamma, \sigma_\eta) = (0, 0.98, -0.07, 0.05)$ and different distributions for ϵ_t corresponding to those of the previous two cases. Panel (b) is available for comparison purposes.

We simulate 1000 series of size 5000 with the previously given parameters and calculate $\widehat{v_T(s)}$ for each series. The lines in the figure are obtained by averaging the values of $\widehat{v_T(s)}$ at each s.

Figure 2 shows that the TAARSV model is able to produce larger values for leverage and propagation for each s, especially compared to the AARSV model. Moreover, the more fat-tailed the distribution of standardized returns, the larger the values of leverage and propagation for each s.

All in all, this may suggest that more flexible models may perform better in fitting real financial time series and consequently in estimating and predicting volatility.

3. Data cloning estimation

The method of data cloning is a computational technique that allows obtaining approximate estimators of the maximum likelihood (ML) of parameters and their asymptotic variances using a MCMC procedure (see, e.g, Lele et al., 2007, 2010). It is based on the intuitive idea of running an experiment several times, always obtaining the same observations.

In this way, given some observed data $\mathbf{y} = (y_1, \ldots, y_n)$, a new data set is formed by creating K clones of the original observations: $\mathbf{y}^{(K)} = (\mathbf{y}, \ldots, \mathbf{y})$. It is assumed that the clones are independent and K is large enough. Accordingly, the likelihood of $\mathbf{y}^{(K)}$ is equal to the K-th power of the likelihood of the original data $[L(\boldsymbol{\theta}|\mathbf{y})]^K$, where $\boldsymbol{\theta}$ is the vector of parameters.

In general, with data cloning, a MCMC procedure is applied by taking the cloned data $\mathbf{y}^{(K)}$ and multiplying the corresponding likelihood $[L(\boldsymbol{\theta}|\mathbf{y})]^K$ by a given prior distribution $\pi(\boldsymbol{\theta})$ of $\boldsymbol{\theta}$. Then samples are generated from the posterior distributions $\pi^{(K)}(\boldsymbol{\theta}|\mathbf{y})$. The selection or elicitation of a proper prior distribution is not very important. The mean of the posterior distribution of $\boldsymbol{\theta}$ approximates the ML estimator ($\hat{\boldsymbol{\theta}}$) as the number of clones K increases. However, as Lele et al. (2010) notes, the convergence of the mean of the posterior distribution to the ML estimator is faster when the prior distribution is more informative, and then a smaller number of clones K is required.

For K large enough, $\pi^{(K)}(\boldsymbol{\theta}|\mathbf{y})$ converges to a multivariate normal distribution whose mean is equal to the ML estimator of $\boldsymbol{\theta}$, and the covariance matrix equal to $1/K$ times the inverse of the Fisher information matrix of the ML estimator of $\boldsymbol{\theta}$ (see the appendix of Lele et al., 2007, for details).

In summary, the data cloning algorithm consists of 3 steps:

Step 1: Generate a K-cloned data set $\mathbf{y}^{(K)} = (\mathbf{y}, \mathbf{y}, \ldots, \mathbf{y})$ in such a way that the observed data vector is repeated K times.

Step 2: Generate random deviates from the posterior distribution by means of an MCMC algorithm, with any proper prior distribution $\pi(\boldsymbol{\theta})$ and the cloned data vector $\mathbf{y}^{(K)} = (\mathbf{y}, \mathbf{y}, \ldots, \mathbf{y})$. The K copies of $\mathbf{y}$ are assumed to be independent of each other.

Step 3: Compute sample means and variances of the MCMC chains generated from the posterior distribution of the vector of parameters $\boldsymbol{\theta}$. The means of the posterior distributions approximate the ML estimates of parameters, and K times the variances of the posterior distributions coincide with the estimated asymptotic variances of the ML estimators of the parameters.

One of the main advantages of data cloning is its simple and friendly implementation in R with the packages `dclone` (Solymos, 2010) and `dcmle` (Solymos, 2016). The syntax used in `dclone` is similar to the language BUGS.[2]

On the other hand, this technique is related to the *Simulated Annealing* technique, where the power is increased gradually and progressively. In data cloning, the number of clones is fixed, and although this can lead to a very accurate ML estimate, it can lead to problems such as difficulty in exploring the entire parameter space and finding global modes if the number of clones is not large enough. This fact can take a lot of computational time, especially when the number of clones is large, and therefore makes the method less attractive from a practical point of view. However, in our work we do not find any problems with convergence, since the sample sizes in the simulations and the real data are large enough. However, in other cases, when large data sets are involved, the computation time can be quite high.

[2]See code example in Appendix A.

First, the prior distributions of the parameters of the models must be defined. A suitable set of prior distributions can be assumed. We consider weakly informative prior distributions based on the literature on the estimation of SV models:

$$\begin{array}{lll} \mu \sim \mathrm{N}(0,\ 10^3) & \frac{1+\phi}{2} \sim \mathrm{Beta}(1,\ 1) & \sigma_\eta^{-2} \sim \mathrm{Gamma}(10^{-3},\ 10^{-3}) \\ \gamma \sim \mathrm{N}(0,\ 10^2) & \nu \sim \mathrm{Gamma}(2,\ 0.1), & \end{array}$$

where ν are the degrees of freedom of the Student-t distribution.

Second, we have determined the optimal number of clones to use. The library `dclone` contains several diagnostic measures, such as the function `dcdiag`, which computes some statistics to help the user make this decision. One of these statistics is the maximum eigenvalue of the posterior covariance matrix, provided by Lele et al. (2010), which gives us information about the degeneracy of the posterior distribution; accordingly, when it is close to zero, the distributions have a small impact on the results. In our simulations and applications, we use 5 clones.

We have also experimented with a higher number of clones, but the improvement in the estimates of the parameters is irrelevant. Other measures related to the selection of the optimal number of clones are the mean square error (*MSE*) and the R^2 statistic. They are both based on a χ^2 approximation and should converge to zero as the number of clones increases. The optimal number of clones selected by these measures is 5, as before.

3.1. Simulation study

In this subsection, we conduct simulation experiments to evaluate the performance of the approximated ML estimators for the parameters of the TAARSV model using data cloning, hereafter referred to as DC estimator, and a standard Bayesian estimator implemented in JAGS, referred to as BA estimator.

We consider two possible designs for the Monte Carlo experiments, depending on the error distribution of the standardized returns. They are assumed to follow either a $N(0,1)$ or a Student-t with v degrees of freedom. The number of replicates is 200 and the full set of parameter values for the TAARSV model is $(\mu, \phi, \gamma_1, \gamma_2, \sigma_\eta^2, v) = (0.000, 0.980, -0.200, -0.100, 0.050, 5)$. Finally, we consider three sample sizes $T = 1000,\ 3000$, and 5000.

To determine the parameters of the TAARSV model, we need to make some restrictions. We assume that the propagation pattern tends to decrease slowly over time. So the constraints are $\gamma_1 \leq \gamma_2$ and $\gamma_1, \gamma_2 < 0$ imposed by the priors of these parameters, which are:

$$\gamma_1 \sim U(-1, 0), \quad \gamma_2 \sim U(\gamma_1, 0).$$

The performance of the DC estimator in finite samples for the AARSV model is very accurate and can be found in Bermudez et al. (2020). With respect to the new model, we find that the BA estimator is less reliable than the DC estimator in small samples, except for γ_2. For this parameter, both estimators do not perform as well. When the standardized returns follow a Student-t distribution, we find that the DC estimator provides more accurate estimates of μ, σ_η^2, and the degrees of freedom of the Student-t distribution, even for large sample sizes. All in all, we conclude that the DC estimator performs quite reasonably for finite samples.

Table 1: TAARSV simulation results

True values	$\mu = 0.000$		$\phi = 0.980$		$\gamma_1 = -0.200$		$\gamma_2 = -0.100$		$\sigma^2_\eta = 0.050$		$v = 5$	
	Bayes	DC	Bayes	DC	Bayes	DC	Bayes	DC	Bayes	DC	Bayes	DC
Normal Standardized Returns												
$T = 1000$												
Mean	0.026	0.016	0.972	0.975	-0.214	-0.212	-0.111	-0.117	0.063	0.052		
Sd	(0.035)	(0.033)	(0.008)	(0.007)	(0.054)	(0.053)	(0.068)	(0.114)	(0.016)	(0.014)		
$T = 3000$												
Mean	0.007	0.004	0.978	0.979	-0.207	-0.206	-0.116	-0.135	0.053	0.050		
Sd	(0.019)	(0.019)	(0.004)	(0.004)	(0.030)	(0.030)	(0.057)	(0.081)	(0.008)	(0.008)		
$T = 5000$												
Mean	0.004	0.002	0.979	0.979	-0.208	-0.207	-0.123	-0.144	0.052	0.050		
Sd	(0.015)	(0.015)	(0.003)	(0.003)	(0.023)	(0.024)	(0.054)	(0.069)	(0.006)	(0.006)		
Student-t Standardized Returns												
$T = 1000$												
Mean	0.045	0.023	0.971	0.975	-0.207	-0.209	-0.109	-0.122	0.071	0.053	7.232	5.675
Sd	(0.044)	(0.038)	(0.009)	(0.007)	(0.050)	(0.047)	(0.065)	(0.102)	(0.023)	(0.019)	(2.858)	(1.452)
$T = 3000$												
Mean	0.012	0.005	0.978	0.979	-0.205	-0.204	-0.117	-0.136	0.055	0.050	5.504	5.227
Sd	(0.023)	(0.022)	(0.004)	(0.004)	(0.027)	(0.026)	(0.056)	(0.078)	(0.011)	(0.010)	(0.701)	(0.609)
$T = 5000$												
Mean	0.004	0.001	0.979	0.980	-0.204	-0.204	-0.123	-0.143	0.053	0.050	5.282	5.151
Sd	(0.017)	(0.017)	(0.003)	(0.003)	(0.020)	(0.021)	(0.052)	(0.067)	(0.008)	(0.008)	(0.489)	(0.452)

Monte Carlo results of a standard Bayesian approach and DC estimators of the parameters of the TAARSV model with Gaussian and Student-t errors. Monte Carlo averages of the estimates and standard deviations are given (in parentheses). The first column of each parameter corresponds to estimates obtained with a standard Bayesian approach (Bayes), while the second column corresponds to estimates obtained with DC.

4. Empirical application

We estimate the new TAARSV proposed in this paper and the AARSV of Mao et al. (2020) using daily returns for three financial series, namely: Bitcoin (BTC), Nasdaq and S&P 500 (SP500). Bitcoin returns cover the period from September 18, 2014, to August 15, 2022, while Nasdaq and SP500 returns cover the period from January 4, 2000, to August 12, 2022. The total number of observations is 2889 and 5689, respectively.

Figure 3 displays the time series of returns. The stock market returns cover the period of the dot.com crisis, the global financial crisis, and the pandemic crisis, which correspond to periods of high volatility.

Table 2 shows the descriptive statistics of the returns. The empirical distribution of returns is leptokurtic, which supports the use of the Student-t distribution to model the standardized returns. In addition, all series exhibit negative skewness.

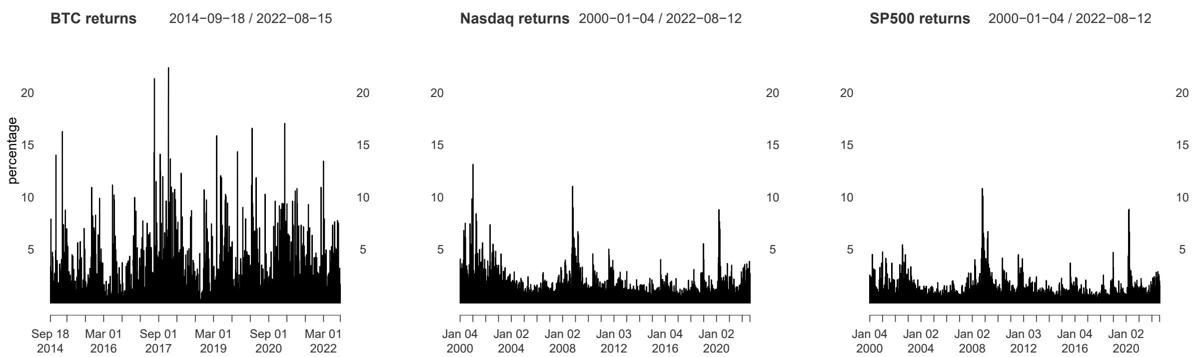


Figure 3. Daily returns in percentage.

Table 2
Descriptive statistics of returns

Data	Mean	Std. Dev.	Skewness	Kurtosis
BTC	0.137	3.904	−0.771	10.798
Nasdaq	0.020	1.599	−0.146	6.218
SP500	0.019	1.247	−0.398	10.477

Table 3 presents estimates of the TAARSV and AARSV models using either BA or DC. We find that estimates of the standard errors of parameters using the DC estimation method are often smaller than those using BA, suggesting that estimates of parameters using this estimation method may be more reliable.

When using BA, we use the Deviance Information Criterion (DIC) to compare the TAARSV and the AARSV. According to this criterion, the TAARSV model fits the BTC and SP500 returns better, while the AARSV model seems to fit the Nasdaq returns best. Moreover, the estimates obtained with BA and DC are not significantly different for each model. The parameters are often statistically different from zero.

Figure 4 shows the estimated volatilities for each return series and each estimation method. We find that both estimation methods provide fairly similar volatility estimates, although BA tends to slightly overestimate volatility.

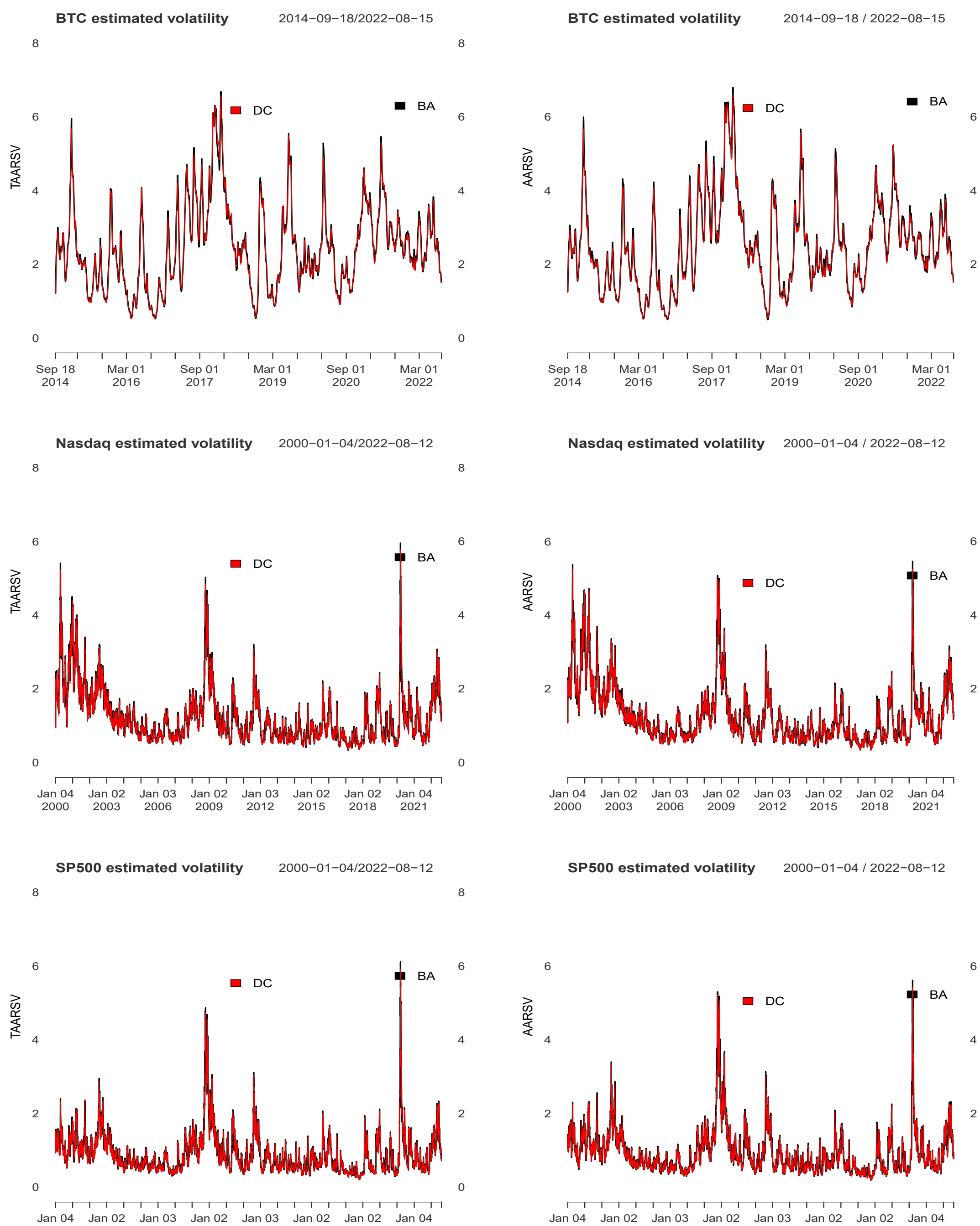


Figure 4. Estimated volatilities obtained with the TAARSV and AARSV models.

Table 3
Estimates obtained using BA and DC

Series	Param	BA TAARSV	BA AARSV	DC TAARSV	DC AARSV
BTC	$\mu(1-\phi)$	0.035	0.049	0.037	0.043
		(0.014)	(0.014)	(0.015)	(0.012)
	ϕ	0.969	0.969	0.972	0.972
		(0.007)	(0.008)	(0.007)	(0.007)
	γ		-0.001		0.001
			(0.014)		(0.013)
	γ_1	-0.026		-0.014	
		(0.019)		(0.020)	
	γ_2	-0.015		-0.010	
		(0.015)		(0.018)	
	σ_η^2	0.075	0.078	0.068	0.068
		(0.015)	(0.018)	(0.013)	(0.014)
	ν	3.458	3.578	3.413	3.436
		(0.276)	(0.319)	(0.280)	(0.283)
	DIC	15.074	15.161		
Nasdaq	$\mu(1-\phi)$	-0.099	0.007	-0.097	0.006
		(0.006)	(0.002)	(0.006)	(0.002)
	ϕ	0.978	0.977	0.980	0.978
		(0.003)	(0.003)	(0.002)	(0.003)
	γ		-0.149		-0.145
			(0.010)		(0.010)
	γ_1	-0.246		-0.236	
		(0.013)		(0.015)	
	γ_2	-0.205		-0.225	
		(0.035)		(0.028)	
	σ_η^2	0.011	0.021	0.007	0.018
		(0.002)	(0.003)	(0.002)	(0.003)
	ν	11.915	22.468	10.464	18.292
		(1.988)	(6.753)	(1.321)	(4.277)
	DIC	18.572	18.499		
SP500	$\mu(1-\phi)$	-0.134	-0.011	-0.130	-0.010
		(0.010)	(0.030)	(0.008)	(0.002)
	ϕ	0.972	0.972	0.975	0.974
		(0.003)	(0.003)	(0.003)	(0.003)
	γ		-0.187		-0.180
			(0.009)		(0.011)
	γ_1	-0.281		-0.270	
		(0.020)		(0.016)	
	γ_2	-0.233		-0.257	
		(0.044)		(0.030)	
	σ_η^2	0.013	0.023	0.008	0.021
		(0.003)	(0.004)	(0.003)	(0.003)
	ν	8.701	14.670	8.077	12.922
		(0.916)	(3.366)	(0.906)	(2.434)
	DIC	15.484	15.493		

Note: Standard errors are in parentheses. The values of the DIC are divided by 10^3.

4.1. *Forecasting results*

In this subsection, we present the volatility proxy used to evaluate the forecasting performance of the models and estimation methods, the loss functions used and their evolution over time obtained using a rolling window of 90 days, and the results of the unconditional and conditional tests of superior predictive ability for 1- and 10-day volatility forecasts. Recall that the Basel Committee requires institutions to calculate value-at-risk for a time horizon of at least 10 days and, consequently, volatility forecasts for that time horizon.

4.1.1. *Proxy of volatility and loss functions*

The out-of-sample periods include approximately 252 daily observations from December 7, 2021 to August 15, 2022 for BTC returns and from August 13, 2021 to August 12, 2022 for Nasdaq and SP500 returns.

As a proxy for volatility, we chose realized volatility (RV) calculated from the 5-minute returns. Realized volatility on day t is calculated as $RV_t = \sum_{i=1}^{M} r_{i,t}^2$, where M is the total number of intraday observations and $r_{i,t}$ is the returns calculated from the 5-minute prices on day t.[3]

Figure 5 represents the RV for BTC, Nasdaq and SP500. The out-of-sample period is very volatile. Recall that during this period Russia invaded Ukraine and the Omicron variant emerged, leading to a rapid and strong spread of the Covid pandemic around the world.

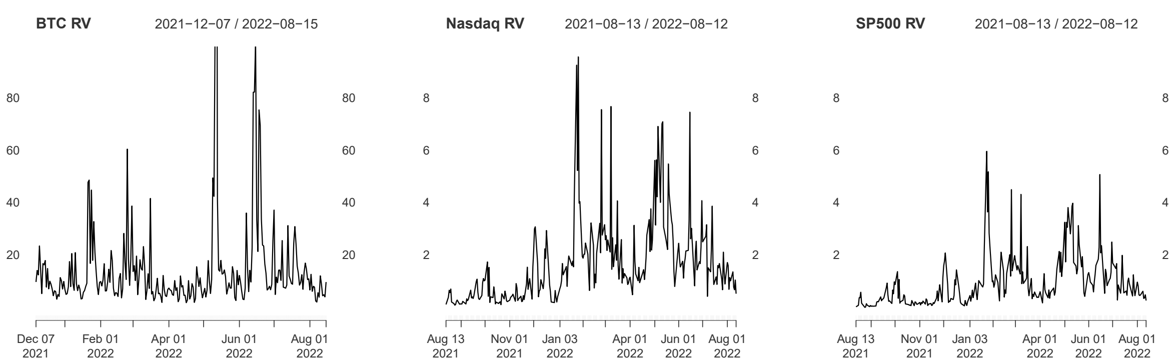


Figure 5. Realized volatility in the out-of-sample period.

Figures 6 and 7 show the rolling window mean square errors (MSE) and QLIKEs for 1-day forecasts proposed by Patton (2011) that are robust to the presence of noise in the volatility proxy.

We first focus on the MSE and find that the patterns are quite similar for the TAARSV and AARSV models, regardless of the estimation method used. We observe a huge spike around May 2022 that continues to increase throughout the rest of the out-of-sample period. The week of May 12, 2022 saw a total meltdown in the crypto world with a sell-off that highlighted the risks of experimental and unregulated digital currencies. This, along with the impact of higher interest rates, makes the crypto world nervous. These events increase Bitcoin's volatility and models show more difficulty in forecasting.

[3]Data are from www.firstratedata.com.

For Nasdaq, we also observe an increase in rolling window MSEs, especially when using the BA approach. When we estimate the models using DC, we find that the MSEs are on average smaller than those of the BA approach. We observe an increase in MSEs around February 2022, corresponding to the uncertainty caused by Russia's invasion of Ukraine, and then they increase again until June 2022, perhaps caused by the collapse of cryptocurrencies and the deterioration of economic forecasts.

For the SP500 and the TAARSV-DC, we find that the rolling window MSEs tend to decrease after March 2022, while they are larger using BA and peak around May-June 2022. The AARSV model also shows a peak around May-June 2022 and the MSEs are much lower when using the DC estimation method.

The rolling window QLIKEs show different patterns than the rolling window MSEs for both models. For BTC, we observe a spike in QLIKEs around May 2022 and then another spike starting in June 2022 due to the successive turmoil in the crypto world.

For Nasdaq, we observe a large difference between the QLIKEs when forecasts are made using DC and the QLIKEs calculated when forecasts are made using the BA approach. The former is significantly smaller except for the period before February 2022. For the AARSV model, we even find that the pattern of QLIKEs is different. DC shows QLIKEs that decrease over the out-of-sample period.

Finally, for the SP500, we also find that QLIKEs tend to decrease over time for the two models when we use the DC estimation method compared to the BA approach.

See Appendix B for figures B.9 and B.10 showing the rolling window MSEs and QLIKEs for 10-day volatility forecasts. The MSEs show that the DC and BA estimation methods provide fairly similar values for this measure, with the MSEs for the Nasdaq and SP500 model volatility forecasts tending to be slightly larger. This is especially true for the SP500 and TAARSV models. The predictability of the models and the patterns for the 10-day forecasts for DC and BA are closer than those for the 1-day forecasts.

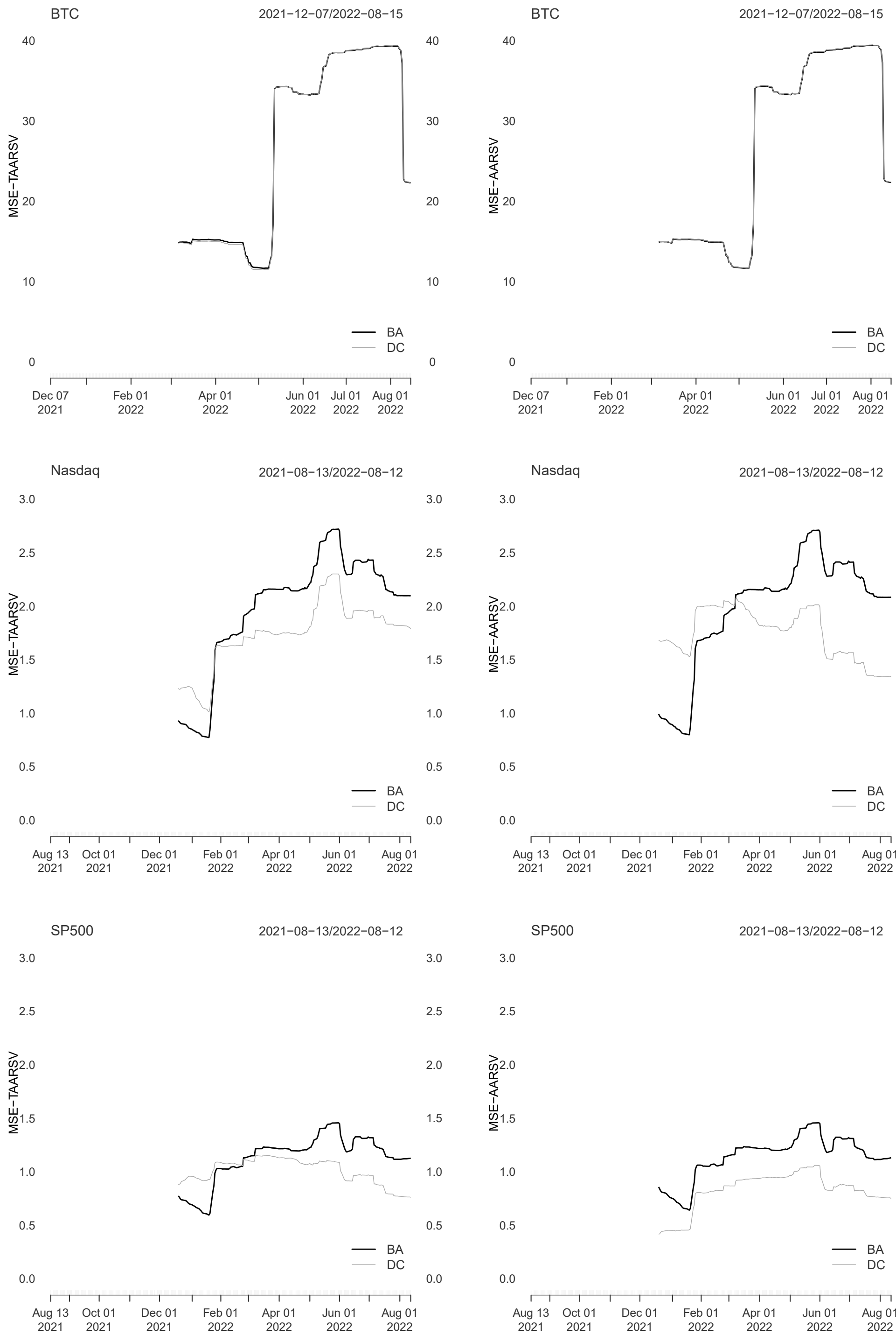


Figure 6. Rolling window MSE for 1-day volatility forecasts obtained from the TAARSV and AARSV models using either the Bayesian approach or DC.

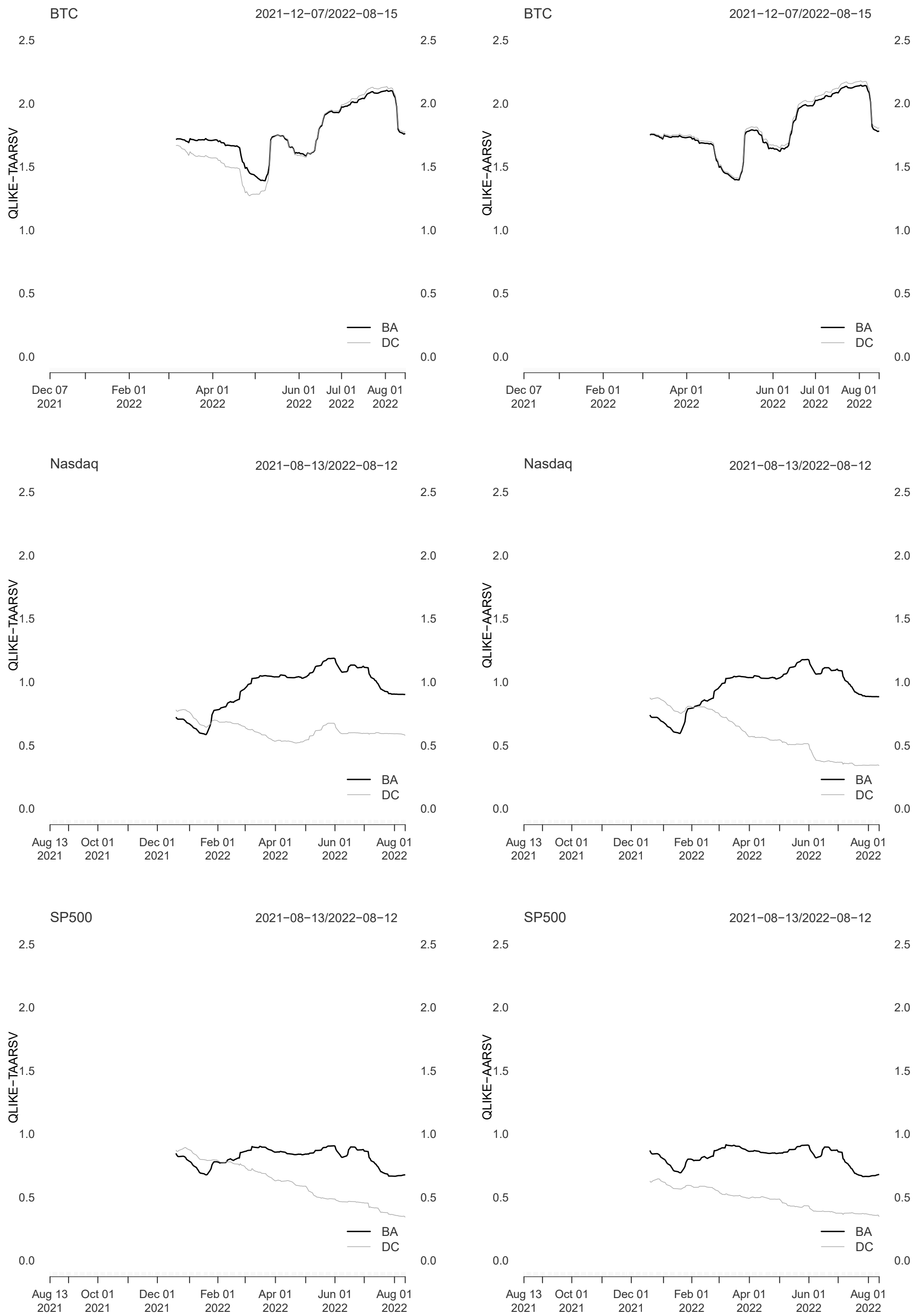


Figure 7. Rolling window QLIKE for 1-day volatility forecasts obtained from the TAARSV and AARSV models using either the Bayesian approach or DC.

4.1.2. Model confidence set–unconditional superior predictive ability

The predictive performance of the models is evaluated using the Model Confidence Set (MCS) procedure proposed by Hansen et al. (2011) and programmed in R by Catania and Bernardi (2015). The procedure of Hansen et al. (2011) consists of a series of statistical tests to create a set of models called "Superior Set Model" (SSM). The models in the SSM have statistically equal predictive ability and are ranked according to the value of a loss function. Test statistics are calculated for each loss function and evaluate the point predictions.

Table 4 shows the SSM of the prediction models and estimation methods studied. The loss functions are the squared error (SE) and the QLIKE.[4] We use two confidence levels (95% and 80%) to construct the SSM.

In the 1-day forecasts, TAARSV-DC always ranks first in BTC returns, and AARSV-BA and AARSV-DC are always excluded from the SSM unless we use the loss function QLIKE and the confidence level 95%. TAARSV-DC and TAARSV-BA appear to have similar predictive abilities.

With respect to Nasdaq returns, we find that both the TAARSV-DC and AARSV-DC models are always included in the SSM and rank first or second depending on the loss function. Moreover, in terms of predictive ability, the DC estimation method appears to be superior to the Bayesian approach for this financial time series.

On the other hand, the results for the 10-day volatility forecasts, the loss function SE and the confidence level 95% show that TAARSV-DC and AARSV-BA are the best models and estimation methods in terms of volatility predictability for BTC returns, while TAARSV-BA and AARSV perform better for the Nasdaq and the SP500. Both models and methods have the same predictive ability. For the other confidence levels and loss functions, AARSV appears to be the model that provides the most accurate volatility forecasts. BA also appears to be the preferred estimation method, except for the SP500.

Table 4
Model confidence set results

The table reports the rankings of volatility forecasters with different loss functions. – means that the model does not belong to the SSM. The statistical tests are done at 95% and 80% confidence levels. We use 5000 bootstrap samples.

Series	Models	Loss functions							
		1-day-ahead				**10-days-ahead**			
		95%		**80%**		**95%**		**80%**	
		SE	QLIKE	SE	QLIKE	SE	QLIKE	SE	QLIKE
BTC	TAARSV–BA	2	2	2	2	-	-	-	-
	TAARSV–DC	1	1	1	1	2	-	-	-
	AARSV–BA	-	3	-	-	1	1	1	1
	AARSV–DC	-	-	-	-	-	-	-	-
	p-value	0.644	0.069	0.640	0.067	0.075	0.032	0.074	0.041
Nasdaq	TAARSV–BA	3	-	-	-	3	-	-	-
	TAARSV–DC	1	2	1	2	-	-	-	-
	AARSV–BA	4	-	-	-	1	1	1	1
	AARSV–DC	2	1	2	1	2	-	-	-
	p-value	0.095	0.795	0.640	0.794	0.117	0.000	0.000	0.000
SP500	TAARSV–BA	-	-	-	-	3	-	-	-
	TAARSV–DC	-	-	-	-	4	-	-	-
	AARSV–BA	-	-	-	-	2	2	2	2
	AARSV–DC	1	1	1	1	1	1	1	1
	p-value	0.002	0.000	0.002	0.000	0.070	0.223	0.283	0.233

[4] The SE is defined as $(RV_{t+1} - \widehat{\sigma_{t+1}})^2$, while the QLIKE is defined as $qlike \equiv \frac{RV_{t+1}}{\widehat{\sigma_{t+1}}} - log\left(\frac{RV_{t+1}}{\widehat{\sigma_{t+1}}}\right) - 1$.

4.1.3. Conditional superior predictive ability

Li et al. (2022) use Conditional Superior Predictive Ability (CSPA) tests to evaluate the performance of predictive models. CSPA tests are advantageous because they examine the performance of models conditional on a state-dependent variable (in this case, RV). The null hypothesis states that the conditional expected loss of the benchmark model does not exceed that of the competing models for all conditional states where the smallest value of its loss function is statistically smallest. Rejection of this hypothesis indicates that the alternative models perform better than the benchmark model in certain states.

Two types of conditional tests are possible: "one-versus-one" in which each model is compared with an alternative for all pairs of models, and "one-versus-all", in which each model is compared with all other competing models; see Li et al. (2022).

Tables 5 and 6 show the results of CSPA tests for 1- and 10-day volatility forecasts using two loss functions and a 95% confidence level. The results for a confidence level of 80% can be found in Appendix B. A value of 1 means the null hypothesis is rejected, while a value of 0 means it is not rejected. The columns indicate the models used for the comparisons.

The CSPA tests (one-versus-all) for the BTC volatility forecasts show that the TAARSV model weakly dominates its competitors estimated using the DC or the Bayesian approach for the loss functions SE and QLIKE. This result is confirmed by the one-versus-one CSPA tests, with the TAARSV model least likely to be rejected when both loss functions are used.

For the Nasdaq volatility forecasts, the one-versus-all CSPA tests show that no model weakly dominates its competitors. However, the CSPA one-versus-one tests and the two loss functions show that the AARSV-BA model is most likely to be rejected.

For the SP500 volatility forecasts, the CSPA tests (one-versus-all) also show that no model weakly dominates its competitors. Nevertheless, both TAARSV-DC and AARSV-BA are most likely to be rejected using the CSPA tests and the two loss functions.

With a confidence level of 95%, loss functions SE and QLIKE, and predictions for 10 days ahead, the TAARSV model is the best model for predicting RV. No model weakly dominates the competitors for the other series.

With a confidence level of 80% and a QLIKE loss function, the results for BTC are slightly different, with TAARSV-DC and AARSV-BA weakly dominating the others; see Table B.7. The conclusion for the 95% confidence level also applies here for the Nasdaq and SP500.

Figure 8 shows the estimated difference between the conditional expected loss functions along with the upper confidence level of 95%; see Li et al. (2022). The loss function used to represent Figure 8 is the QLIKE. For example, the "diff loss" is defined as the difference between the QLIKE of the TAARSV-DC and the AARSV-DC.

For 1-day volatility forecasts and BTC, TAARSV-DC performs better at most values of RV, with AARSV-DC having some advantage at very low values of RV.

For 10-day volatility forecasts, the results are similar. The performance of TAARSV-DC is better than that of AARSV-BA for most values of RV and is highly significant at values above 6 for Nasdaq volatility forecasts.

For SP500 and 1-day volatility forecasts, the performance of TAARSV-BA is better than that of AARSV-BA for most RV values and is highly significant at values above 2. For the 10-day volatility forecasts, the opposite is true, with the performance of AARSV-BA better than that of TAARSV-BA for values above 2.

Table 5: Conditional superior predictive ability for excess returns using the SE loss function

		TAARSV–BA	TAARSV–DC	AARSV–BA	AARSV–DC
			1-day-ahead		
Panel A: One-versus-one CSPA tests against different competing models and estimation methods					
BTC	TAARSV–BA	-	0	1	1
	TAARSV–DC	0	-	1	1
	AARSV–BA	0	0	-	1
	AARSV–DC	0	0	0	-
Panel B: One-versus-all CSPA tests against all competing models and estimation methods					
		0	0	1	1
Panel A: One-versus-one CSPA tests against different competing models and estimation methods					
Nasdaq	TAARSV–BA	-	1	1	1
	TAARSV–DC	1	-	1	0
	AARSV–BA	0	1	-	1
	AARSV–DC	1	0	1	-
Panel B: One-versus-all CSPA tests against all competing models and estimation					
		1	1	1	1
Panel A: One-versus-one CSPA tests against different competing models and estimation methods					
SP500	TAARSV–BA	-	1	1	1
	TAARSV–DC	1	-	1	0
	AARSV–BA	0	1	-	1
	AARSV–DC	1	1	1	-
Panel B: One-versus-all CSPA tests against all competing models and estimation					
		1	1	1	1
			10-days-ahead		
Panel A: One-versus-one CSPA tests against different competing models and estimation methods					
BTC	TAARSV–BA	-	0	0	1
	TAARSV–DC	1	-	0	1
	AARSV–BA	1	0	-	1
	AARSV–DC	1	0	0	-
Panel B: One-versus-all CSPA tests against all competing models and estimation methods					
		1	0	0	1
Panel A: One-versus-one CSPA tests against different competing models and estimation methods					
Nasdaq	TAARSV–BA	-	1	1	1
	TAARSV–DC	0	-	1	1
	AARSV–BA	1	1	-	1
	AARSV–DC	1	1	0	-
Panel B: One-versus-all CSPA tests against all competing models and estimation					
		1	1	1	1
Panel A: One-versus-one CSPA tests against different competing models and estimation methods					
SP500	TAARSV–BA	-	0	1	1
	TAARSV–DC	1	-	1	1
	AARSV–BA	1	1	-	0
	AARSV–DC	1	1	1	-
Panel B: One-versus-all CSPA tests against all competing models and estimation					
		1	1	1	1

Panel A (resp. B) reports a value of 1, for which the one-versus-one (resp. one-versus-all), if the CSPA null hypothesis is rejected at 5% significance level. Each column corresponds to a different AARSV.

Table 6: Conditional superior predicative ability for excess returns using the QLIKE loss function

		TAARSV–BA	TAARSV–DC	AARSV–BA	AARSV–DC
			1-day-ahead		
Panel A: One-versus-one CSPA tests against different competing models and estimation methods					
BTC	TAARSV–BA	-	0	1	1
	TAARSV–DC	0	-	0	1
	AARSV–BA	0	0	-	1
	AARSV–DC	0	0	0	-
Panel B: One-versus-all CSPA tests against all competing models and estimation methods					
		0	0	1	1
Panel A: One-versus-one CSPA tests against different competing models and estimation methods					
Nasdaq	TAARSV–BA	-	1	1	1
	TAARSV–DC	1	-	1	0
	AARSV–BA	0	1	-	1
	AARSV–DC	1	0	1	-
Panel B: One-versus-all CSPA tests against all competing models and estimation					
		1	1	1	1
Panel A: One-versus-one CSPA tests against different competing models and estimation methods					
SP500	TAARSV–BA	-	1	1	1
	TAARSV–DC	1	-	1	0
	AARSV–BA	0	1	-	1
	AARSV–DC	1	1	1	-
Panel B: One-versus-all CSPA tests against all competing models and estimation					
		1	1	1	1
			10-days-ahead		
Panel A: One-versus-one CSPA tests against different competing models and estimation methods					
BTC	TAARSV–BA	-	0	0	1
	TAARSV–DC	1	-	0	1
	AARSV–BA	1	0	-	1
	AARSV–DC	1	0	0	-
Panel B: One-versus-all CSPA tests against all competing models and estimation methods					
		1	0	0	1
Panel A: One-versus-one CSPA tests against different competing models and estimation methods					
Nasdaq	TAARSV–BA	-	1	1	1
	TAARSV–DC	0	-	1	1
	AARSV–BA	1	1	-	1
	AARSV–DC	1	1	0	-
Panel B: One-versus-all CSPA tests against all competing models and estimation					
		1	1	1	1
Panel A: One-versus-one CSPA tests against different competing models and estimation methods					
SP500	TAARSV–BA	-	1	1	1
	TAARSV–DC	0	-	1	1
	AARSV–BA	1	1	-	0
	AARSV–DC	1	1	1	-
Panel B: One-versus-all CSPA tests against all competing models and estimation					
		1	1	1	1

Panel A (resp. B) reports a value of 1, for which the one-versus-one (resp. one-versus-all), if the CSPA null hypothesis is rejected at 5% significance level. Each column corresponds to a different AARSV.

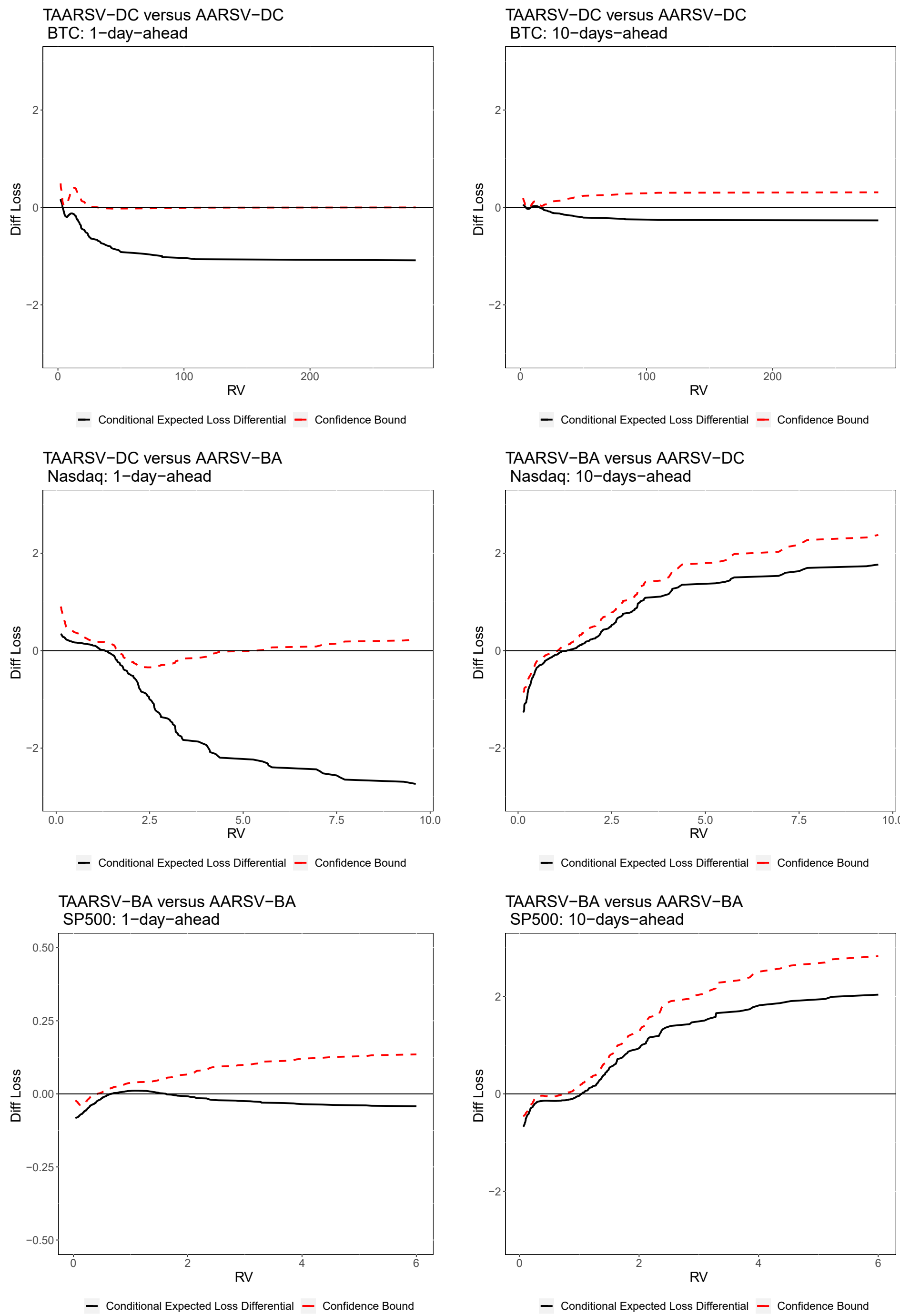


Figure 8. Predicting RV: one-versus-one CSPA tests. Loss function QLIKE.

4.2. Results' discussion

All in all, and taking into account the 1-day volatility forecasts, the predictive performance of the methods seems clear for the three series of returns. The BTC returns have higher volatility, skewness, and kurtosis than those of the Nasdaq and SP500, and the model that fits this data better and predicts volatility better is the TAARSV, regardless of the estimation method used.

For Nasdaq returns, both TAARSV and AARSV appear to have similar predictive abilities, but the DC estimation method appears to be preferable. For the SP500, the Hansen et al. (2011)

method supports the AARSV model estimated with DC, while the CSPA tests (one-versus-one) reject the TAARSV-BA model and the AARSV-DC model is less likely.

For 10-day volatility forecasts, CSPA also confirms that the TAARSV-DC model weakly dominates the benchmark model and the BA estimation method. Although there is no model and estimation method for the Nasdaq and SP500 that weakly dominate the others, the AARSV model appears to be preferable for forecasting the 10-day volatility of the Nasdaq and SP500.

The main conclusion is that the new model seems to be better suited to forecast the volatility of return series that have high volatility, kurtosis, and skewness, as is the case with BTC. This model also works well for the 1-day volatility forecasts of the Nasdaq and the SP500 according to the CSPA test (one-versus-one). In addition, DC seems to be the estimation method that often provides more accurate volatility forecasts.

5. Conclusion

In this paper, we propose a new stochastic volatility model that captures the asymmetric response of volatility with different magnitudes of this response. This model is a nesting of well-known stochastic volatility models. We explore two estimation methods, data cloning and a Bayesian approach, to determine which method is best suited for the proposed model, and estimate a benchmark model for comparison purposes.

The new model is able to generate higher leverage and propagation effects than a basic asymmetric stochastic volatility model and is best able to predict volatility when return series have high volatility, kurtosis, and skewness, as in the case of Bitcoin.

For the traditional return series and according to the unconditional and conditional superior predictive tests, the model often outperforms the basic asymmetric stochastic volatility model for 1-day volatility forecasts. Performance was evaluated during a particularly volatile period that included Russia's invasion of Ukraine and a wave of COVID contagion.

The simulations and empirical application show that data cloning is quite reliable in estimating the parameters of the models and that its advantage over the Bayesian approach is even greater for small sample sizes. It also leads to smaller standard error estimates and is often chosen when evaluating forecasts when unconditional and conditional tests of superior predictive ability are used for 1-day volatility forecasts. For 10-day volatility forecasts, the new model for Bitcoin remains the preferred specification.

Appendix A. Proving Theorem 1

Proof Theorem 1. *Consider the stationary process y_t defined by system (1) with $|\phi| < 1$. If ϵ_t follows a normal distribution and $\gamma_{int} = \gamma$, then the leverage effect and the propagation till time s is given by*

$$v(1) = var[y_t]\left[2\Phi(-\gamma) - 1\right], \tag{A.1}$$

where $\Phi(.)$ is the accumulative normal distribution,

$$var[y_t] = \exp\left(\frac{h_t}{2}\right) = \exp\left(\mu + \frac{\gamma^2 + \sigma_\eta^2}{2(1-\phi^2)}\right),$$

and

$$v(s) = var[y_t]\left[2\Phi(-\gamma\phi^{(s-1)}) - 1\right] \quad \textit{for } s > 1, \tag{A.2}$$

respectively.

Proof

We have

$$var(y_t|y_{t-1} < 0) = E(y_t^2|y_{t-1} < 0).$$

As given the AARSV model,

$$E(y_t|y_{t-1} < 0) = E\left[\exp\left(\frac{h_t}{2}\right)\cdot \epsilon_t \middle| y_{t-1} < 0\right] =$$
$$E\left[\exp\left(\frac{h_t}{2}\right)\middle| y_{t-1} < 0\right]\cdot E(\epsilon_t|y_{t-1} < 0) = 0.$$

On the other hand, $var(y_t) = \exp\left(\mu + \frac{\sigma_h^2}{2}\right)$, where $\sigma_h^2 = \frac{\gamma^2+\sigma_\eta^2}{(1-\phi^2)}$.

Computing now the value of

$$E(y_t^2|y_{t-1} < 0) = E\left[\exp(h_t)\cdot\epsilon_t^2|y_{t-1} < 0\right] = E\left[\exp(h_t)|y_{t-1} < 0\right].$$

Replacing h_t by equation (2), assuming normality of ϵ_t and lognormal distribution of the exponential function, it turns out that

$$E(y_t^2|y_{t-1} < 0) = E\left[\exp(h_t)|\, y_{t-1} < 0\right] =$$
$$E\left[\exp\left(\mu + \sum_{i=1}^{\infty}\phi^{(i-1)}\gamma\epsilon_{t-i} + \sum_{i=1}^{\infty}\phi^{(i-1)}\eta_{t-i}\right)\middle| y_{t-1} < 0\right]$$
$$\exp\left(\mu + \sum_{i=1}^{\infty}\frac{\phi^{2(i-1)}\sigma_\eta^2}{2}\right)E\left[\gamma\epsilon_{t-1} + \sum_{i=2}^{\infty}\phi^{(i-1)}\gamma\epsilon_{t-i}\middle| y_{t-1} < 0\right] =$$

$$\exp\left(\mu + \sum_{i=1}^{\infty}\frac{\phi^{2(i-1)}\sigma_\eta^2}{2} + \sum_{i=2}^{\infty}\frac{\phi^{2(i-1)}\gamma^2}{2}\right)\cdot E\left[\exp(\gamma\epsilon_{t-1})|y_{t-1} < 0\right]. \tag{A.3}$$

Given that $(\epsilon_{t-1}|y_{t-1} < 0)$ *is normal truncated distributed and* $\Phi(.)$ *is the accumulative normal distribution, then*

$$E\left[\exp(\gamma\epsilon_{t-1})|y_{t-1} < 0\right] = \int_{-\infty}^{0} \exp(\gamma\epsilon_{t-1}) \frac{\frac{1}{\sqrt{2\pi}}\exp(-\epsilon_{t-1}^2/2)}{\Phi(0)} d\epsilon_{t-1} =$$
$$\frac{2}{\sqrt{2\pi}} \int_{-\infty}^{0} \exp\left(\frac{-\epsilon_{t-1}^2}{2} + \gamma\epsilon_{t-1}\right) d\epsilon_{t-1} =$$
$$2\Phi(-\gamma)\exp\left(\frac{\gamma^2}{2}\right). \tag{A.4}$$

Replacing (A.4) in (A.3) and after some computations, we finally obtain that

$$E(y_t^2|y_{t-1} < 0) =$$
$$\exp\left(\mu + \sum_{i=1}^{\infty} \frac{\phi^{2(i-1)}\sigma_\eta^2}{2} + \sum_{i=2}^{\infty} \frac{\phi^{2(i-1)}\gamma^2}{2}\right) 2\Phi(-\gamma)\exp\left(\frac{\gamma^2}{2}\right) =$$
$$2\Phi(-\gamma)\exp\left(\mu + \sum_{i=1}^{\infty} \frac{\phi^{2(i-1)}\sigma_\eta^2}{2} + \sum_{i=2}^{\infty} \frac{\phi^{2(i-1)}\gamma^2}{2} + \frac{\gamma^2}{2}\right), \tag{A.5}$$

and it is obtained from (A.3) and (A.4) without truncation (conditioning to $y_{t-1} < 0$*):*

$$var(y_t) = E(y_t^2) =$$
$$\exp\left(\mu + \sum_{i=1}^{\infty} \frac{\phi^{2(i-1)}\sigma_\eta^2}{2} + \sum_{i=2}^{\infty} \frac{\phi^{2(i-1)}\gamma^2}{2}\right) \cdot E\left[\exp(\gamma\epsilon_{t-1})\right] =$$
$$\exp\left(\mu + \sum_{i=1}^{\infty} \frac{\phi^{2(i-1)}\sigma_\eta^2}{2} + \sum_{i=2}^{\infty} \frac{\phi^{2(i-1)}\gamma^2}{2}\right) \cdot \exp\left(\frac{\gamma^2}{2}\right) =$$
$$\exp\left(\mu + \sum_{i=1}^{\infty} \frac{\phi^{2(i-1)}\sigma_\eta^2}{2} + \sum_{i=2}^{\infty} \frac{\phi^{2(i-1)}\gamma^2}{2} + \frac{\gamma^2}{2}\right).$$

Then substituting in (A.5):

$$E(y_t^2|y_{t-1} < 0) = 2\Phi(-\gamma)var(y_t).$$

Therefore,

$$v(1) = var(y_t|y_{t-1} < 0) - var(y_t) = var(y_t)[2\Phi(-\gamma) - 1].$$

Proceeding the same way for $v(s)$*, we obtain as a general expression for the propagation effect*

$$v(s) = var[y_t]\left[2\Phi(-\gamma\phi^{(s-1)}) - 1\right] \quad \textit{for } s > 1.$$

Appendix B. Tables and figures

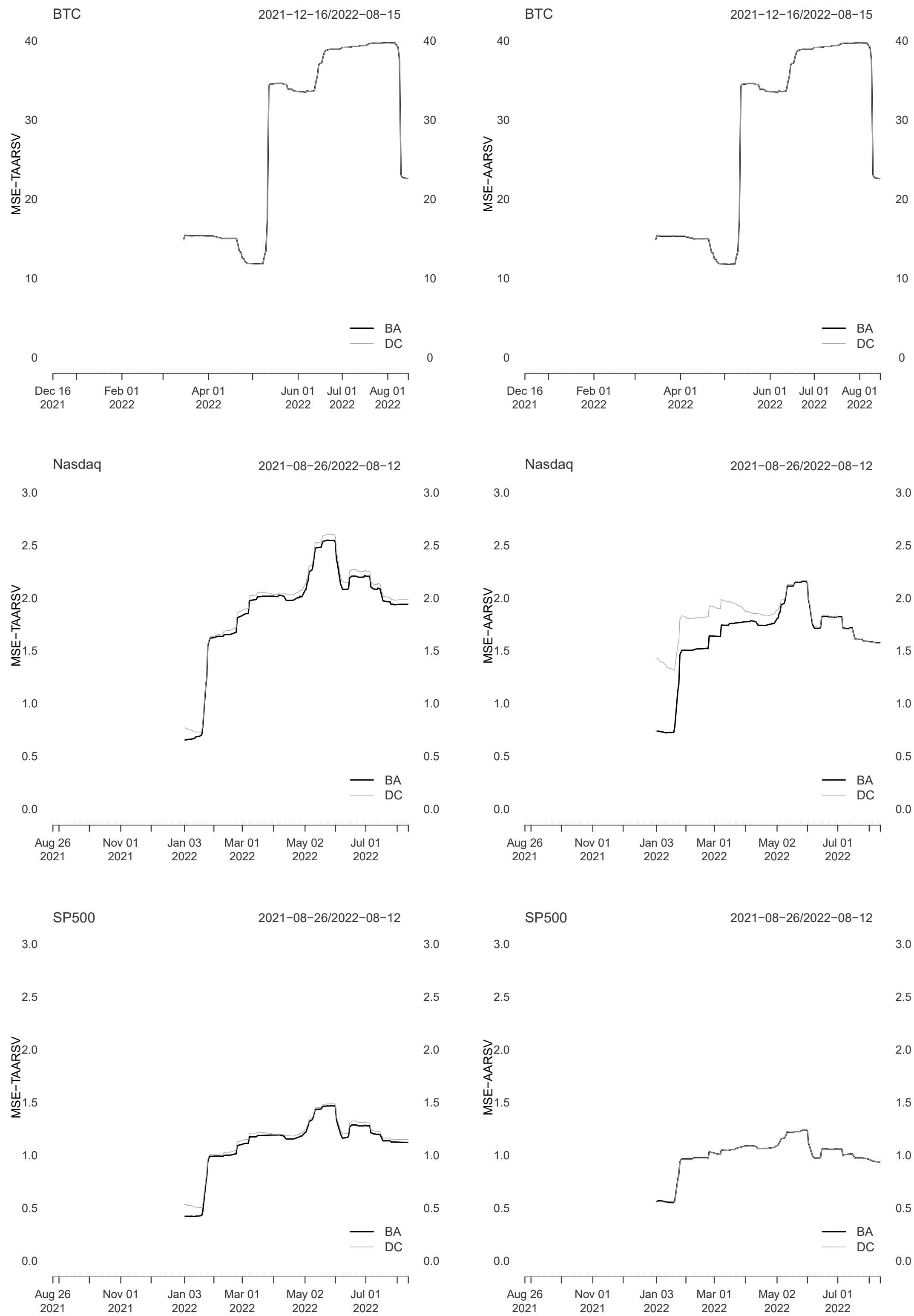


Figure B.9. Rolling window MSE (10-day volatility forecasts) for the TAARSV and AARSV models using either the Bayesian approach or DC.

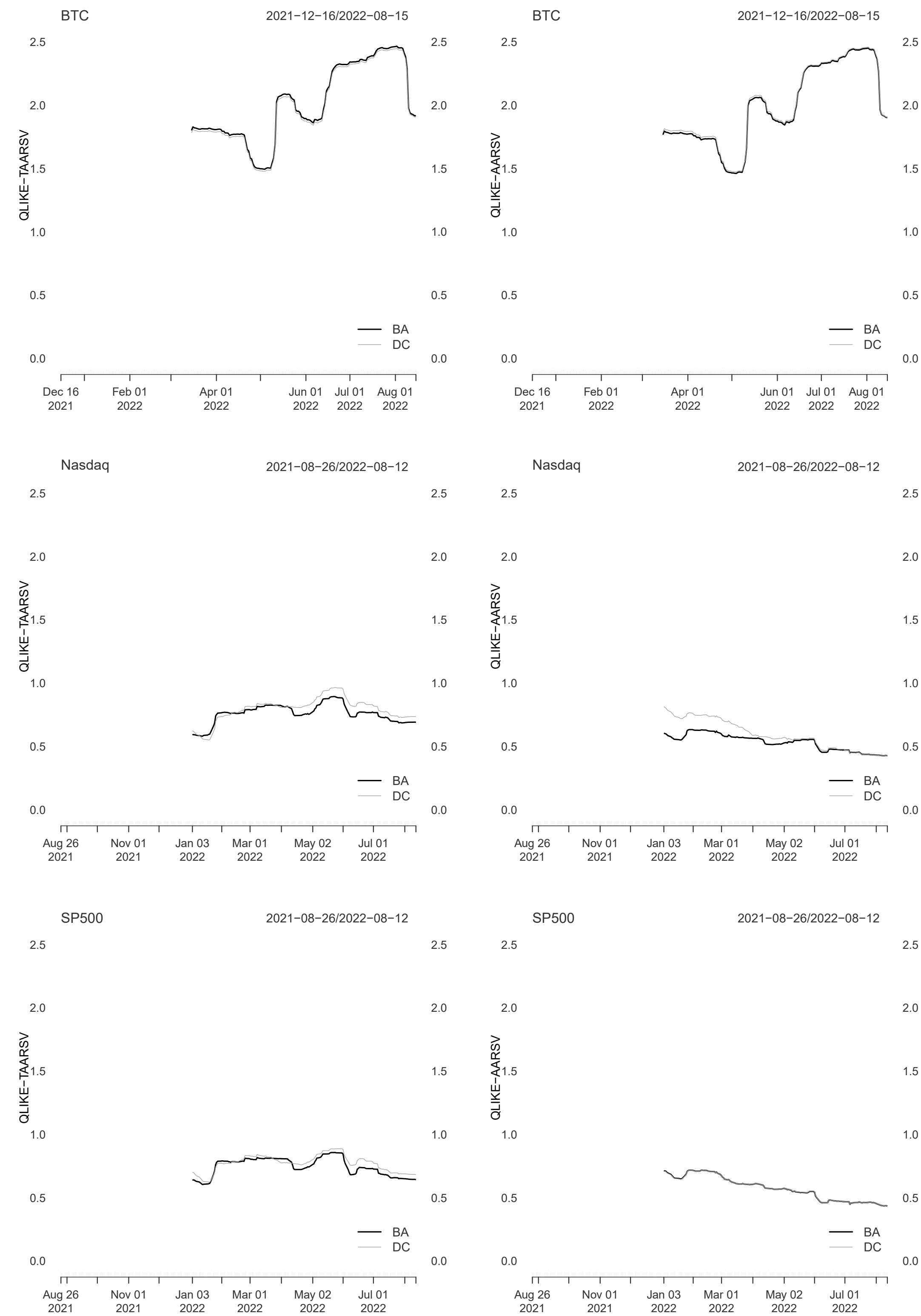


Figure B.10. Rolling window QLIKE (10-day volatility forecasts) for the TAARSV and AARSV models using either the Bayesian approach or DC.

Table B.7: Conditional superior predicative ability for excess returns using the QLIKE loss function.

		TAARSV–BA	TAARSV–DC	AARSV–BA	AARSV–DC
		1-day-ahead			
Panel A: One-versus-one CSPA tests against different competing models and estimation methods					
BTC	TAARSV–BA	-	0	1	1
	TAARSV–DC	0	-	1	1
	AARSV–BA	1	0	-	1
	AARSV–DC	1	0	1	-
Panel B: One-versus-all CSPA tests against all competing models and estimation methods					
		1	0	1	1
Panel A: One-versus-one CSPA tests against different competing models and estimation methods					
Nasdaq	TAARSV–BA	-	1	1	1
	TAARSV–DC	1	-	1	0
	AARSV–BA	0	1	-	1
	AARSV–DC	1	0	1	-
Panel B: One-versus-all CSPA tests against all competing models and estimation					
		1	1	1	1
Panel A: One-versus-one CSPA tests against different competing models and estimation methods					
SP500	TAARSV–BA	-	1	1	1
	TAARSV–DC	1	-	1	0
	AARSV–BA	0	1	-	1
	AARSV–DC	1	1	1	-
Panel B: One-versus-all CSPA tests against all competing models and estimation					
		1	1	1	1
		10-days-ahead			
Panel A: One-versus-one CSPA tests against different competing models and estimation methods					
BTC	TAARSV–BA	-	0	0	0
	TAARSV–DC	1	-	0	0
	AARSV–BA	1	1	-	1
	AARSV–DC	1	0	0	-
Panel B: One-versus-all CSPA tests against all competing models and estimation methods					
		1	0	0	1
Panel A: One-versus-one CSPA tests against different competing models and estimation methods					
Nasdaq	TAARSV–BA	-	1	1	1
	TAARSV–DC	0	-	1	1
	AARSV–BA	1	1	-	1
	AARSV–DC	1	1	0	-
Panel B: One-versus-all CSPA tests against all competing models and estimation					
		1	1	1	1
Panel A: One-versus-one CSPA tests against different competing models and estimation methods					
SP500	TAARSV–BA	-	1	1	1
	TAARSV–DC	0	-	1	1
	AARSV–BA	1	1	-	0
	AARSV–DC	1	1	1	-
Panel B: One-versus-all CSPA tests against all competing models and estimation					
		1	1	1	1

Panel A (resp. B) reports a value of 1, for which the one-versus-one (resp. one-versus-all), if the CSPA null hypothesis is rejected at 20% significance level. Each column corresponds to a different AARSV. Confidence level 80%.

Acknowledgements

Helena Veiga gratefully acknowledges financial support from Agencia Estatal de Investigación PID2019-108079GB-C21/AIE/10.13039/501100011033 and PID2021-122919NB-I00, and from Fundação para a Ciência e a Tecnologia, grant UIDB/00315/2020. Juan Miguel Marín gratefully acknowledges financial support from the Agencia Estatal de Investigación under project PID2020-115598RB-I00.

Disclosure statement

No potential conflicts of interest were reported by the authors.

References

Bandi, F. M. and R. Renò (2012). Time-varying leverage effects. *Journal of Econometrics 169*(1), 94–113.

Bermudez, P. d. Z., J. M. Marín, and H. Veiga (2020). Data cloning estimation for asymmetric stochastic volatility models. *Econometric Reviews 39*(10), 1057–1074.

Bollerslev, T., J. Litvinova, and G. Tauchen (2006, May). Leverage and volatility feedback effects in high-frequency data. *Journal of Financial Econometrics 4*(3), 353–384.

Breidt, F. J. (1996). A threshold autoregressive stochastic volatility model. In *VI Latin American Congress of Probability and Mathematical Statistics (CLAPEM), Valparaiso, Chile*. Citeseer.

Bretó, C. (2014). On idiosyncratic stochasticity of financial leverage effects. *Statistics & Probability Letters 91*, 20–26.

Catania, L. (2020). The leverage effect and propagation. *Available at SSRN: https://ssrn.com/abstract=3578656 or http://dx.doi.org/10.2139/ssrn.3578656*.

Catania, L. (2022). A stochastic volatility model with a general leverage specification. *Journal of Business & Economic Statistics 40*(2), 1–23.

Catania, L. and M. Bernardi (2015). *MCS: Model Confidence Set Procedure*. R package version 0.1.1.

Hansen, P. R., A. Lunde, and J. M. Nason (2011). The model confidence set. *Econometrica 79*, 453–497.

Harvey, A. C. and N. Shephard (1996). Estimation of an asymmetric stochastic volatility model for asset returns. *Journal of Business & Economic Statistics 14*(4), 429–34.

Lele, S. R., B. Dennis, and F. Lutscher (2007). Data cloning: Easy maximum likelihood estimation for complex ecological models using Bayesian Markov chain Monte Carlo methods. *Ecology Letters 10*(7), 551–563.

Lele, S. R., K. Nadeem, and B. Schmuland (2010). Estimability and likelihood inference for generalized linear mixed models using data cloning. *Journal of the American Statistical Association 105*(492), 1617–1625.

Li, J., Z. Liao, and R. Quaedvlieg (2022). Conditional superior predictive ability. *The Review of Economic Studies 89*, 843–875.

Mao, X., V. Czellar, E. Ruiz, and H. Veiga (2020). Asymmetric stochastic volatility models: Properties and particle filter-based simulated maximum likelihood estimation. *Econometrics and Statistics 13*, 84–105.

Meyer, R. and J. Yu (2000). BUGS for a Bayesian analysis of stochastic volatility models. *The Econometrics Journal 3*(2), 198–215.

Nguyen, H., T.-N. Nguyen, and M.-N. Tran (2023). A dynamic leverage stochastic volatility model. *Applied Economics Letters 30*(1), 97–102.

Patton, A. J. (2011). Volatility forecast comparison using imperfect volatility proxies. *Journal of Econometrics 160*(1), 246–256.

Plummer, M. (2003). JAGS: A program for analysis of Bayesian graphical models using Gibbs sampling.

So, M. K. P., W. K. Li, and K. Lam (2002). A threshold stochastic volatility model. *Journal of Forecasting 21*(7), 473–500.

Solymos, P. (2010). dclone: Data cloning in R. *The R Journal 2*(2), 29–37.

Solymos, P. (2016). *dcmle: Hierarchical Models Made Easy with Data Cloning*. R package version 0.3-1.

Wu, X. and X. Wang (2020). Forecasting volatility using realized stochastic volatility model with time-varying leverage effect. *Finance Research Letters 34*, 101271.

Yu, J. (2005). On leverage in a stochastic volatility model. *Journal of Econometrics 127*(2), 165–178.

Yu, J. (2012). A semiparametric stochastic volatility model. *Journal of Econometrics 167*(2), 473–482.