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# Hessenberg–Sobolev Matrices and Favard Type Theorem

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**Abstract**

We study the relation between certain non-degenerate lower Hessenberg infinite matrices $\mathcal{G}$ and the existence of sequences of orthogonal polynomials with respect to Sobolev inner products. In other words, we extend the well-known Favard theorem for Sobolev orthogonality. We characterize the structure of the matrix $\mathcal{G}$ and the associated matrix of formal moments $\mathcal{M}_{\mathcal{G}}$ in terms of certain matrix operators.

**Keywords** Sobolev orthogonality · Orthogonal polynomials · Moment problem · Favard theorem · Hessenberg matrices · Hankel matrices

**Mathematics Subject Classification** 42C05 · 33C47 · 44A60 · 30E05 · 11B37 · 47B35

## 1 Introduction

Let $\mathcal{A} = (a_{i,j})_{i,j=0}^{\infty}$ be an infinite matrix of real numbers. For $m \in \mathbb{Z}$, we say that the entry $a_{i,j}$ lies in the $m$-diagonal if $j = i + m$. Obviously, the 0-diagonal is the usual main diagonal of $\mathcal{A}$. The matrix $\mathcal{A}$ is an *$m$-diagonal matrix* if all of its nonzero elements lie in its $m$-diagonal and *lower (upper) triangular matrix* if $a_{i,j} = 0$ whenever $j > i$ ($j < i$). The symbols $\mathcal{A}^{\mathsf{T}}$ and $[\mathcal{A}]_n$ denote the transposed matrix and the squared matrix of the first $n$ rows and columns of $\mathcal{A}$, respectively. $\mathcal{I}$ is called the

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*unit matrix*; its $(i, j)$th entry is $\delta_{i,j}$ where $\delta_{i,j} = 0$ if $i \neq j$ and $\delta_{i,i} = 1$. $\mathcal{A}$ is called *positive definite of infinite order* if $\det([\mathcal{A}]_n) > 0$ for all $n \geq 1$, where $\det([\mathcal{A}]_n)$ is the determinant of $[\mathcal{A}]_n$. If $\det([\mathcal{A}]_n) > 0$ for all $1 \leq n \leq k$ and $\det([\mathcal{A}]_n) = 0$ for all $n > k$, we say that $\mathcal{A}$ is a *positive definite matrix of order* $k$.

According to the definitions given in [5, Ch. II], if $\mathcal{A}$ and $\mathcal{B}$ are two infinite matrices such that $\mathcal{A} \cdot \mathcal{B} = \mathcal{I}$, then $\mathcal{B}$ is called a right-hand inverse of $\mathcal{A}$, denoted by $\mathcal{A}^{-1}$; and $\mathcal{A}$ is called a left-hand inverse of $\mathcal{B}$, denoted by ${}^{-1}\mathcal{B}$. The transposes of $\mathcal{A}^{-1}$ (${}^{-1}\mathcal{A}$) and $\mathcal{A}^m$ (the $m$th power of the matrix $\mathcal{A}$, with $m \in \mathbb{Z}_+$) are denoted by $\mathcal{A}^{-\mathsf{T}}$ (${}^{-\mathsf{T}}\mathcal{A}$) and $\mathcal{A}^{m\mathsf{T}}$, respectively. Moreover, $\mathcal{A}^{-m} = \left(\mathcal{A}^{-1}\right)^m$, where $m \in \mathbb{Z}_+$.

A difficulty of dealing with infinite matrices is that matrix products can be ill-defined (c.f. [4, (1)]). Nevertheless, in this paper we will only consider the product of infinite matrices $\mathcal{A} = (a_{i,j})_{i,j=0}^{\infty}$ and $\mathcal{B} = (b_{i,j})_{i,j=0}^{\infty}$ when $\mathcal{A}\mathcal{B} = (\sum_k a_{i,k} b_{k,j})_{i,j=0}^{\infty}$ is such that each sum ($i, j$-dependent) involves only a finite number of non-null summands (c.f. [4, Def. 1]).

We will denote by $\mathcal{U}$ the infinite matrix whose $(i, j)$th entry is $\delta_{i+1,j}$ for $i, j \in \mathbb{Z}_+$; i.e. the *upper (or backward) shift infinite matrix* given by the expression

$$\mathcal{U} = \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The matrix $\mathcal{U}^{\mathsf{T}}$ is called the *lower (or forward) shift infinite matrix* and it is easy to check that $\mathcal{U} \cdot \mathcal{U}^{\mathsf{T}} = \mathcal{I}$; i.e. $\mathcal{U}^{\mathsf{T}} = \mathcal{U}^{-1}$, the right-hand inverse of $\mathcal{U}$ (c.f. [9, Sec. 0.9]).

An infinite Hankel matrix is an infinite matrix in which each ascending skew-diagonal from left to right is constant. In other words, $\mathcal{H} = (h_{i,j})_{i,j=0}^{\infty}$ is a Hankel matrix if $h_{i,j+1} = h_{i+1,j}$ for all $i, j \in \mathbb{Z}_+$ or equivalently if

$$\mathcal{U}\mathcal{H} - \mathcal{H}\mathcal{U}^{-1} = \mathcal{O}, \tag{1}$$

where $\mathcal{O}$ denote the infinite null matrix. If $\{r_i\}_{i=0}^{\infty}$ is a sequence of real numbers, we denote $\mathcal{D}(r_i)$ the infinite diagonal matrix whose $i$th main diagonal entry is $r_i$, and by $\mathcal{H}[r_i]$ the associated Hankel matrix, defined as

$$\mathcal{H}[r_i] = \begin{pmatrix} r_0 & r_1 & r_2 & \cdots \\ r_1 & r_2 & r_3 & \cdots \\ r_2 & r_3 & r_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We say that a matrix $\mathcal{M} = (m_{i,j})_{i,j=0}^{\infty}$ is a *Hankel-Sobolev matrix* if there exists a sequence of Hankel matrices $\{\mathcal{H}_k\}_{k=0}^{\infty}$ such that

$$\mathcal{M} = \sum_{k=0}^{\infty} \left( \mathcal{U}^{-k} \, \mathcal{D}_k \, \mathcal{H}_k \, \mathcal{D}_k \, \mathcal{U}^k \right), \tag{2}$$

where $\mathcal{D}_0 = \mathcal{I}$ and $\mathcal{D}_k = \mathcal{D}\left(\frac{(k+i)!}{i!}\right)$ for each $k > 0$, e.g.

$$\mathcal{D}_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 2 & 0 & \cdots \\ 0 & 0 & 3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \ \mathcal{D}_2 = \begin{pmatrix} 2 & 0 & 0 & \cdots \\ 0 & 6 & 0 & \cdots \\ 0 & 0 & 12 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \text{ and } \mathcal{D}_3 = \begin{pmatrix} 6 & 0 & 0 & \cdots \\ 0 & 24 & 0 & \cdots \\ 0 & 0 & 60 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We say that a Hankel-Sobolev matrix $\mathcal{M}$ is of *index* $d \in \mathbb{Z}_+$ if $\mathcal{H}_d \neq \mathcal{O}$ and $\mathcal{H}_k = \mathcal{O}$ for all $k > d$. Otherwise, we will say that $\mathcal{M}$ is of *infinite index*.

Let $\mathcal{M}$ be a Hankel-Sobolev matrix of index $d \in \overline{\mathbb{Z}}_+ = \mathbb{Z}_+ \cup \{\infty\}$. If $\mathcal{H}_k \neq \mathcal{O}$ for all $k < d$, we say that $\mathcal{M}$ is *non-lacunary* and *lacunary* in any other case.

Hankel-Sobolev matrices appeared for the first time in [3, 16] in close connection with the moment problem for a Sobolev inner product. Some of the properties of this class of infinite matrices have also been studied in [6, 13, 14, 17, 19, 22].

Let $\mathbb{M}$ be the linear space of all infinite matrices of real numbers. For each $\eta \in \mathbb{Z}_+$ fixed, we denote by $\Phi(\cdot, \eta)$ the operator from $\mathbb{M}$ to itself given by the expression

$$\Phi(\mathcal{A}, \eta) := \sum_{\ell=0}^{\eta} (-1)^{\ell} \binom{\eta}{\ell} \mathcal{U}^{\eta-\ell} \mathcal{A} \mathcal{U}^{-\ell}, \tag{3}$$

where $\mathcal{A} \in \mathbb{M}$ and $\binom{\eta}{\ell}$ denote binomial coefficients. Obviously, $\Phi(\cdot, \eta)$ is a linear operator.

One of the main results of this work is the following intrinsic characterization of the Hankel-Sobolev matrices using the operator $\Phi(\cdot, \eta)$, which we will be prove in section 2.

**Theorem 1** *An infinite matrix $\mathcal{M}$ is a Hankel-Sobolev matrix of index $d \in \mathbb{Z}_+$, if and only if $\mathcal{M}$ is a symmetric matrix and*

$$\Phi(\mathcal{M}, 2d+1) = \mathcal{O} \quad \textit{and} \quad \Phi(\mathcal{M}, 2d) \neq \mathcal{O}. \tag{4}$$

*Moreover, for $k = 0, 1, \ldots, d$; the Hankel matrix $\mathcal{H}_{d-k}$ in (2) is given by*

$$\mathcal{H}_{d-k} = \frac{(-1)^{d-k}}{(2d-2k)!} \, \Phi(\mathcal{M}_{d-k}, 2d-2k), \tag{5}$$

*where $\mathcal{M}_d = \mathcal{M}$ and $\mathcal{M}_{d-k} = \mathcal{M}_{d-k+1} - \mathcal{U}^{-d-1+k} \mathcal{D}_{d+1-k} \mathcal{H}_{d+1-k} \mathcal{D}_{d+1-k} \mathcal{U}^{d+1-k}$ for $k = 1, 2, \ldots, d$.*

An infinite matrix $\mathcal{G} = (g_{i,j})_{i,j=0}^{\infty}$ is a *lower Hessenberg infinite matrix* if $g_{i,j} = 0$ whenever $j - i > 1$ and at least one entry of the 1-diagonal is different from zero, i.e.

$$\mathcal{G} = \begin{pmatrix} g_{0,0} & g_{0,1} & 0 & \cdots & 0 & \cdots \\ g_{1,0} & g_{1,1} & g_{1,2} & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \cdots \\ g_{n,0} & g_{n,1} & g_{n,2} & \cdots & g_{n,n+1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{6}$$

Let us denote by $\mathbb{H}$ the set of all lower Hessenberg infinite matrices. Additionally, if $\mathcal{G} \in \mathbb{H}$ and all the entries in the 1-diagonal are equal to 1, we say that $\mathcal{G}$ is *monic*. If all the entries of the 1-diagonal of $\mathcal{G}$ are nonzero, we say that $\mathcal{G}$ is a *non-degenerate lower Hessenberg infinite matrix* (for brevity hereafter referred as *non-degenerate Hessenberg matrix*). An upper Hessenberg matrix is a matrix whose transpose is a lower Hessenberg matrix.

For each $\eta \in \mathbb{Z}_+$ fixed, we denote by $\Psi(\cdot, \eta)$ the operator from $\mathbb{H}$ to $\mathbb{M}$ given by the expression

$$\Psi(\mathcal{B}, \eta) = \sum_{k=0}^{\eta} (-1)^k \binom{\eta}{k} \mathcal{B}^k \, \mathcal{B}^{(\eta-k)\,\mathsf{T}}, \quad \mathcal{B} \in \mathbb{H}. \tag{7}$$

Theorem 9 establishes the relation between the operators (3) and (7).

Given a non-degenerate matrix $\mathcal{G} \in \mathbb{H}$, we can generate a sequence of polynomials $\{Q_n\}_{n=0}^{\infty}$ as follows. Assume that $Q_0(z) = t_{0,0} > 0$, then

$$\begin{aligned} g_{0,1} Q_1(x) =& x Q_0(x) - g_{0,0} Q_0(x), \\ g_{1,2} Q_2(x) =& x Q_1(x) - g_{1,1} Q_1(x) - g_{1,0} Q_0(x), \\ &\vdots \quad \vdots \quad \vdots \\ g_{n,n+1} Q_{n+1}(x) =& x Q_n(x) - \sum_{k=0}^{n} g_{n,k}\, Q_k(x), \\ &\vdots \quad \vdots \quad \vdots \end{aligned} \tag{8}$$

Hereafter, we will say that $\{Q_n\}$ is the *sequence of polynomials generated by* $\mathcal{G}$. As $\mathcal{G}$ is non-degenerate, $Q_n$ is a polynomial of degree $n$.

Let $\mathcal{T}$ be the lower triangular infinite matrix whose entries are the coefficients of the sequence of polynomials $\{Q_n\}$, i.e.

$$\mathcal{Q}(x) = \mathcal{T}\,\mathcal{P}(x) \quad \text{where} \quad \mathcal{T} = \begin{pmatrix} t_{0,0} & 0 & \cdots & 0 & \cdots \\ t_{1,0} & t_{1,1} & \cdots & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t_{n,0} & t_{n,1} & \cdots & t_{n,n} & \cdots \\ \vdots & \vdots & \dots & \vdots & \ddots \end{pmatrix}, \tag{9}$$

$\mathcal{Q}(x) = (Q_0(x), Q_1(x), \cdots, Q_n(x), \cdots)^{\mathsf{T}}$ and $\mathcal{P}(x) = (1, x, \cdots, x^n, \cdots)^{\mathsf{T}}$.

As $\mathcal{G}$ is non-degenerate, $t_{i,i} = t_{0,0}\left(\prod_{k=0}^{i-1} g_{k,k+1}\right)^{-1} \neq 0$ for all $i \geq 1$. Therefore, there exists a unique lower triangular infinite matrix $\mathcal{T}^{-1}$ such that $\mathcal{T} \cdot \mathcal{T}^{-1} = \mathcal{I}$ (c.f. [5, (2.1.I)]), i.e. $\mathcal{T}$ has a unique right-hand inverse. Furthermore, in this case $\mathcal{T}^{-1}$ is also a left-hand inverse of $\mathcal{T}$ and it is its only two-sided inverse (c.f. [5, Remark (a) pp. 22]),

$$\mathcal{T}^{-1} = \begin{pmatrix} \tau_{0,0} & 0 & \cdots & 0 & \cdots \\ \tau_{1,0} & \tau_{1,1} & \cdots & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \tau_{n,0} & \tau_{n,1} & \cdots & \tau_{n,n} & \cdots \\ \vdots & \vdots & \dots & \vdots & \ddots \end{pmatrix}, \quad \text{where} \quad \tau_{i,i} = t_{i,i}^{-1}. \tag{10}$$

We will denote by $\mathcal{M}_{\mathcal{G}}$ the *matrix of formal moments associated with* $\mathcal{G}$ (a non-degenerate Hessenberg matrix) defined by

$$\mathcal{M}_{\mathcal{G}} = \left(m_{i,j}\right)_{i,j=0}^{\infty} = \mathcal{T}^{-1}\,\mathcal{T}^{-\mathsf{T}}. \tag{11}$$

We say that a non-degenerate Hessenberg matrix $\mathcal{G}$ is a *Hessenberg–Sobolev matrix of index* $d \in \overline{\mathbb{Z}}_+$ if its associated matrix of formal moments is a Hankel-Sobolev matrix of index $d$. In the following theorem, we give a characterization of these matrices.

**Theorem 2** *A non-degenerate Hessenberg matrix $\mathcal{G}$ is a Hessenberg–Sobolev matrix of index $d \in \mathbb{Z}_+$, if and only if*

$$\Psi(\mathcal{G}, 2d+1) = \mathcal{O} \quad \textit{and} \quad \Psi(\mathcal{G}, 2d) \neq \mathcal{O}.$$

The proof of this theorem is an immediate consequence of Theorem 1 and Theorem 9 (stated in section 3).

Let $\mathbb{P}$ be the linear space of polynomials with real coefficients, $\mathcal{G}$ be a non-degenerate Hessenberg matrix and $\mathcal{M}_{\mathcal{G}}$ its associated matrix of formal moments. If $p(x) = \sum_{i=0}^{n_1} a_i x^i$ and $q(x) = \sum_{j=0}^{n_2} b_j x^j$ are two polynomials in $\mathbb{P}$ of degree $n_1$ and $n_2$, respectively. Then, the bilinear form

$$\langle p, q\rangle_{\mathcal{G}} = (a_0, \cdots, a_{n_1}, 0, \cdots)\mathcal{M}_{\mathcal{G}}(b_0, \cdots, b_{n_2}, 0, \cdots)^{\mathsf{T}} = \sum_{i=0}^{n_1}\sum_{j=0}^{n_2} a_i\, m_{i,j}\, b_j; \tag{12}$$

defines an inner product on $\mathbb{P}$ and $\|\cdot\|_{\mathcal{G}} = \sqrt{\langle\cdot,\cdot\rangle_{\mathcal{G}}}$ is the norm induced by (12) on $\mathbb{P}$ (c.f. Theorem 7).

Let $d \in \mathbb{Z}_+$ and $\vec{\boldsymbol{\mu}}_d = (\mu_0, \mu_1, \ldots, \mu_d)$ be a vector of $d+1$ measures, we write $\vec{\boldsymbol{\mu}}_d \in \mathfrak{M}_d(\mathbb{R})$ if for each $k$ $(0 \leq k \leq d)$ the measure $\mu_k$ is a non-negative finite Borel measure with support $\Delta_k \subset \mathbb{R}$, $\mathbb{P} \subset \mathbf{L}_1(\mu_k)$, $\mu_0$ is positive and $\Delta_0$ contains infinite points. If $d = \infty$ and $\vec{\boldsymbol{\mu}}_\infty$ is a sequence of measures that satisfy the above conditions, we write $\vec{\boldsymbol{\mu}}_\infty \in \mathfrak{M}_\infty(\mathbb{R})$.

For $d \in \overline{\mathbb{Z}}_+$ and $\vec{\boldsymbol{\mu}}_d \in \mathfrak{M}_d(\mathbb{R})$, we define on $\mathbb{P}$ the Sobolev inner product

$$\langle f, g\rangle_{\vec{\boldsymbol{\mu}}_d} = \sum_{k=0}^{d} \int f^{(k)}(x) g^{(k)}(x) d\mu_k(x) = \sum_{k=0}^{d} \langle f^{(k)}, g^{(k)}\rangle_{\mu_k}, \tag{13}$$

where $f, g \in \mathbb{P}$ and $f^{(k)}$ denote the $k$th derivative of $f$. The symbol $\|\cdot\|_{\vec{\boldsymbol{\mu}}_d} = \sqrt{\langle\cdot,\cdot\rangle_{\vec{\boldsymbol{\mu}}_d}}$ denotes the Sobolev norm associated with (13). Note that although $d$ is usually considered a non-negative integer (c.f. [12]), the case $d = \infty$ has sense on $\mathbb{P}$. If all the measures $\mu_k$ involved in (13) are positive, we say that the Sobolev inner product is *non-lacunary* and *lacunary* in any other case.

Taking into account the nature of the support of the measures involved in (13), we have the following three cases:

*Continuous case*. The measures $\mu_0, \cdots, \mu_d$ are supported on infinite subsets.

*Discrete case*. The support of the measure $\mu_0$ is an infinite subset and the measures $\mu_1, \cdots, \mu_d$ are supported on finite subsets.

*Discrete-continuous case*. The support of the measure $\mu_d$ is an infinite subset and the measures $\mu_0, \cdots, \mu_{d-1}$ are supported on finite subsets.

The notion of Sobolev moment and several related topics were firstly introduced in [3]. The $(n,k)$-*moment associated with the inner product* (13) is defined as $s_{n,k} = \langle x^n, x^k\rangle_{\vec{\boldsymbol{\mu}}_d}$ $(n, k \geq 0)$, provided the integral exists. In [3], it was proved that the infinite matrix of moments $\mathcal{S}$ with entries $s_{n,k}$, $(n, k \geq 0)$ is a Hankel-Sobolev matrix (c.f. [3] and Sect. 2.1 of this paper). Furthermore, if $Q_n$ is the sequence of orthonormal polynomials with respect to (13) with leading coefficient $c_n > 0$, then the infinite matrix $\mathcal{G}_{\vec{\boldsymbol{\mu}}_d}$ with entries $g_{i,j} = \langle x Q_i, Q_j\rangle_{\vec{\boldsymbol{\mu}}_d}$ is a non-degenerate Hessenberg matrix. In this case, the sequence of orthonormal polynomials $Q_n$ is the sequence of polynomials generated by $\mathcal{G}_{\vec{\boldsymbol{\mu}}_d}$.

The following theorem gives a characterization of the non-degenerate Hessenberg matrices whose sequence of generated polynomials is orthogonal with respect to a Sobolev inner product as (13).

**Theorem 3** (Favard type theorem for continuous case) *Let $\mathcal{G}$ be a non-degenerate Hessenberg matrix. Then, there exists $d \in \mathbb{Z}_+$ and $\vec{\boldsymbol{\mu}}_d \in \mathfrak{M}_d(\mathbb{R})$ such that $\langle p, q\rangle_{\vec{\boldsymbol{\mu}}_d} = \langle p, q\rangle_{\mathcal{G}}$ if and only if*

1. *$\mathcal{G}$ is a Hessenberg–Sobolev matrix of index $d \in \mathbb{Z}_+$.*
2. *For each $k = 0, 1, \ldots, d$; the Hankel matrix $\mathcal{H}_{d-k}$ defined by* (5)*, is a positive definite matrix of infinite order.*

The Favard type theorems for the cases discrete and the discrete-continuous are Theorems 11 and 12, respectively. Some basic aspects about the classical moment problem and the Sobolev moment problem are revisited in Sect. 2.1.

In Sect. 2, we proceed with the study of the properties of the matrix operator $\Phi(\cdot, \eta)$, the Hankel-Sobolev matrices and the proof of Theorem 1. We revisit the Sobolev moment problem in Sect. 2.1. The third section is devoted to study the properties of the bilinear form (12) and the nexus between the operators $\Phi(\cdot, \eta)$ and $\Psi(\cdot, \eta)$. In the last section, we prove the extension of the Favard Theorem for Sobolev orthogonality stated in Theorem 3.

## 2 Hankel-Sobolev Matrices

First of all, we need to prove that the notion of a Hankel-Sobolev matrix introduced in (2) is well defined.

**Proposition 2.1** *Let $\mathcal{M}$ be a Hankel-Sobolev matrix, then the decomposition of $\mathcal{M}$ established in (2) is unique.*

***Proof*** We first recall that for each $k \in \mathbb{Z}_+$, $\mathcal{D}_k$ is a diagonal matrix with positive entries in the main diagonal. Furthermore, if $\mathcal{A}$ is an infinite matrix and $k \in \mathbb{Z}_+$ is fixed, the matrix $\left(\mathcal{U}^{-k}\, \mathcal{A}\, \mathcal{U}^k\right)$ is obtained adding to $\mathcal{A}$ the first $k$ rows and columns of zeros.

Suppose there are two sequences of Hankel matrices, $\{\mathcal{H}_k\}_{k=0}^{\infty}$ and $\left\{\widehat{\mathcal{H}}_k\right\}_{k=0}^{\infty}$, such that

$$\mathcal{M} = \sum_{k=0}^{\infty} \left(\mathcal{U}^{-k}\, \mathcal{D}_k\, \mathcal{H}_k\, \mathcal{D}_k\, \mathcal{U}^k\right) \quad \text{and} \quad \mathcal{M} = \sum_{k=0}^{\infty} \left(\mathcal{U}^{-k}\, \mathcal{D}_k\, \widehat{\mathcal{H}}_k\, \mathcal{D}_k\, \mathcal{U}^k\right).$$

Therefore,

$$\sum_{k=0}^{\infty} \left(\mathcal{U}^{-k}\, \mathcal{D}_k\, \left(\mathcal{H}_k - \widehat{\mathcal{H}}_k\right)\, \mathcal{D}_k\, \mathcal{U}^k\right) = \mathcal{O}.$$

Hence, for each $k \in \mathbb{Z}_+$ fixed, the matrix $\left(\mathcal{H}_k - \widehat{\mathcal{H}}_k\right)$ is a Hankel matrix whose first row has all its entries equal to zero, i.e. $\mathcal{H}_k = \widehat{\mathcal{H}}_k$, which completes the proof. □

Obviously, the matrix operator $\Phi(\cdot, \eta)$ defined in (3) is linear. Before proving Theorem 1, we need to study some other properties of this operator and some auxiliary results.

**Proposition 2.2** (Recurrence) *Let $\eta \in \mathbb{Z}_+$ fixed and $\mathcal{A} \in \mathbb{M}$, then*

$$\Phi(\mathcal{A}, \eta + 1) = \mathcal{U}\, \Phi(\mathcal{A}, \eta) - \Phi(\mathcal{A}, \eta)\, \mathcal{U}^{-1}. \tag{14}$$

***Proof***

$$
\begin{aligned}
\Phi(\mathcal{A},\eta+1) &= \mathcal{U}^{\eta+1}\mathcal{A} + \left(\sum_{\ell=1}^{\eta}(-1)^{\ell}\binom{\eta+1}{\ell}\mathcal{U}^{\eta+1-\ell}\mathcal{A}\mathcal{U}^{-\ell}\right) + (-1)^{\eta+1}\mathcal{A}\mathcal{U}^{-\eta-1} \\
&= \mathcal{U}^{\eta+1}\mathcal{A} + \left(\sum_{\ell=1}^{\eta}(-1)^{\ell}\left(\binom{\eta}{\ell-1}+\binom{\eta}{\ell}\right)\mathcal{U}^{\eta+1-\ell}\mathcal{A}\mathcal{U}^{-\ell}\right) \\
&\quad + (-1)^{\eta+1}\mathcal{A}\mathcal{U}^{-\eta-1} \\
&= \sum_{\ell=0}^{\eta}(-1)^{\ell}\binom{\eta}{\ell}\mathcal{U}^{\eta+1-\ell}\mathcal{A}\mathcal{U}^{-\ell} + \sum_{\ell=1}^{\eta+1}(-1)^{\ell}\binom{\eta}{\ell-1}\mathcal{U}^{\eta+1-\ell}\mathcal{A}\mathcal{U}^{-\ell} \\
&= \mathcal{U}\left(\sum_{\ell=0}^{\eta}(-1)^{\ell}\binom{\eta}{\ell}\mathcal{U}^{\eta-\ell}\mathcal{A}\mathcal{U}^{-\ell}\right) - \left(\sum_{\ell=0}^{\eta}(-1)^{\ell}\binom{\eta}{\ell}\mathcal{U}^{\eta-\ell}\mathcal{A}\mathcal{U}^{-\ell}\right)\mathcal{U}^{-1} \\
&= \mathcal{U}\,\Phi(\mathcal{A},\eta) - \Phi(\mathcal{A},\eta)\mathcal{U}^{-1}
\end{aligned}
$$

□

The following proposition is an immediate consequence of Proposition 2.2 and (1).

**Proposition 2.3** *If for a matrix $\mathcal{A} \in \mathbb{M}$, there exists $\eta \in \mathbb{Z}_+$ such that $\Phi(\mathcal{A},\eta) = \mathcal{O}$, then*

(a) $\Phi(\mathcal{A},\eta_1) = \mathcal{O}$ *for all $\eta_1 \geq \eta$.*
(b) *For all $c \in \mathbb{R}$ and $\eta \geq 1$, the matrix $c\,\Phi(\mathcal{A},\eta-1)$ is a Hankel matrix.*

**Proposition 2.4** *Assume that $\mathcal{A} \in \mathbb{M}$ is a symmetric matrix, then $\Phi(\mathcal{A},\eta)$ is a symmetric (antisymmetric) matrix if and only if $\eta$ is an even (odd) integer number.*

***Proof***

$$
\begin{aligned}
(\Phi(\mathcal{A},\eta))^{\mathsf{T}} &= \sum_{\ell=0}^{\eta}(-1)^{\ell}\binom{\eta}{\ell}\left(\mathcal{U}^{\eta-\ell}\mathcal{A}\mathcal{U}^{-\ell}\right)^{\mathsf{T}} = \sum_{\ell=0}^{\eta}(-1)^{\ell}\binom{\eta}{\eta-\ell}\mathcal{U}^{\ell}\mathcal{A}\mathcal{U}^{\ell-\eta} \\
&= \sum_{\ell=0}^{\eta}(-1)^{\eta-\ell}\binom{\eta}{\ell}\mathcal{U}^{\eta-\ell}\mathcal{A}\mathcal{U}^{-\ell} = (-1)^{\eta}\sum_{\ell=0}^{\eta}(-1)^{\ell}\binom{\eta}{\ell}\mathcal{U}^{\eta-\ell}\mathcal{A}\mathcal{U}^{-\ell} \\
&=(-1)^{\eta}\Phi(\mathcal{A},\eta).
\end{aligned}
$$

□

**Theorem 4** *Let $d \in \overline{\mathbb{Z}}_+$ and $\mathcal{M}$ be a Hankel-Sobolev matrix of index $d$, as in (2). Denote*

$$
\mathcal{M}_{\eta} = \sum_{k=0}^{\eta}\mathcal{U}^{-k}\mathcal{D}_k\mathcal{H}_k\mathcal{D}_k\mathcal{U}^{k}, \quad 0 \leq \eta \leq d.
$$

*Then,*

(a) $\Phi(\mathcal{M}_\eta, 2\eta+1) = \mathcal{O}$.

(b) $\mathcal{H}_\eta = \dfrac{(-1)^\eta}{(2\eta)!}\Phi(\mathcal{M}_\eta, 2\eta)$.

Before proving the previous theorem, we need the next two lemmas. The first one is a version of the famous Euler's finite difference theorem (c.f. [18, Sec. 6.1]).

**Lemma 2.1** *Let $f(z)$ be a complex polynomial of degree $n$ and leading coefficient $a_n \in \mathbb{C}$. Then, for all $\nu \in \mathbb{Z}_+$*

$$\sum_{\ell=0}^{\nu}(-1)^\ell\binom{\nu}{\ell}f(\ell) = \begin{cases} 0, & \text{if } 0 \le n < \nu, \\ (-1)^\nu\ \nu!\ a_\nu, & \text{if } n = \nu. \end{cases}$$

**Lemma 2.2** *Let $\mathcal{A} = [a_{i,j}]$ be an infinite symmetric matrix, whose $(i,j)$ entry is $a_{i,j}$. Then, for all $\eta, \nu \in \mathbb{Z}_+$, the matrix $\left(\mathcal{U}^\eta \mathcal{A}\mathcal{U}^{-\nu}\right)$ is symmetric.*

***Proof*** Given a sequence of double indexes $\{a_{i,j}\}$ and two functions $f,\ g$ on $\mathbb{Z}_+$, we denote by $\mathcal{A} = [a_{f(i),g(j)}]$ the corresponding infinite matrix, whose $(i,j)$ entry is $a_{f(i),g(j)}$. Therefore, as $a_{i,j} = a_{j,i}$ we get

$$\begin{aligned}\mathcal{U}^\eta \mathcal{A}\mathcal{U}^{-\nu} &= \left[a_{\eta+i,\nu+j}\right] = \left[a_{\nu+j,\eta+i}\right] = \mathcal{U}^\nu \mathcal{A}\mathcal{U}^{-\eta} \\ &= \left(\mathcal{U}^\eta \mathcal{A}\mathcal{U}^{-\nu}\right)^{\mathsf{T}};\ i,j = 1,2,\ldots.\end{aligned}$$

□

***Proof of Theorem 4*** Let $0 \le k \le d$ fixed and $\mathcal{R}_k = \mathcal{U}^{-k}\mathcal{D}_k\mathcal{H}_k\mathcal{D}_k\mathcal{U}^k$. Then, $\mathcal{M}_\eta = \sum_{k=0}^{\eta}\mathcal{R}_k$ and from linearity $\Phi(\mathcal{M}_\eta, 2\eta+1) = \sum_{k=0}^{\eta}\Phi\left(\mathcal{R}_k, 2\eta+1\right)$.

$$\begin{aligned}\Phi(\mathcal{R}_k, 2\eta+1) &= \sum_{\ell=0}^{2\eta+1}(-1)^\ell\binom{2\eta+1}{\ell}\mathcal{U}^{2\eta+1-\ell}\mathcal{R}_k\mathcal{U}^{-\ell} \\ &= \sum_{\ell=0}^{\eta}(-1)^\ell\binom{2\eta+1}{\ell}\mathcal{U}^{2\eta+1-\ell}\mathcal{R}_k\mathcal{U}^{-\ell} \\ &\quad + \sum_{\ell=\eta+1}^{2\eta+1}(-1)^\ell\binom{2\eta+1}{\ell}\mathcal{U}^{2\eta+1-\ell}\mathcal{R}_k\mathcal{U}^{-\ell} \\ &= \sum_{\ell=0}^{\eta}(-1)^\ell\binom{2\eta+1}{\ell}\mathcal{U}^{2\eta+1-\ell}\mathcal{R}_k\mathcal{U}^{-\ell} \\ &\quad + \sum_{\ell=\eta+1}^{2\eta+1}(-1)^\ell\binom{2\eta+1}{2\eta+1-\ell}\mathcal{U}^{2\eta+1-\ell}\mathcal{R}_k\mathcal{U}^{-\ell}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\ell=0}^{\eta} (-1)^{\ell} \binom{2\eta+1}{\ell} \mathcal{U}^{2\eta+1-\ell} \mathcal{R}_k \mathcal{U}^{-\ell} \\
&\quad - \sum_{\ell=0}^{\eta} (-1)^{\ell} \binom{2\eta+1}{\ell} \mathcal{U}^{\ell} \mathcal{R}_k \mathcal{U}^{\ell-2\eta-1} \\
&= \sum_{\ell=0}^{\eta} (-1)^{\ell} \binom{2\eta+1}{\ell} \left( \mathcal{U}^{2\eta+1-\ell} \mathcal{R}_k \mathcal{U}^{-\ell} - \mathcal{U}^{\ell} \mathcal{R}_k \mathcal{U}^{\ell-2\eta-1} \right).
\end{aligned} \tag{15}
$$

Clearly, $\mathcal{R}_k$ is a symmetric matrix. Therefore, from Lemma 2.2, $\mathcal{U}^{2\eta+1-\ell} \mathcal{R}_k \mathcal{U}^{-\ell}$ is a symmetric matrix too and

$$
\mathcal{U}^{2\eta+1-\ell} \mathcal{R}_k \mathcal{U}^{-\ell} = \left( \mathcal{U}^{2\eta+1-\ell} \mathcal{R}_k \mathcal{U}^{-\ell} \right)^{\mathsf{T}} = \mathcal{U}^{\ell} \mathcal{R}_k \mathcal{U}^{\ell-2\eta-1}. \tag{16}
$$

Combining (15)–(16), we get (a) in Theorem 4.

From (a) and Proposition 2.2, it follows that

$$
\begin{aligned}
\Phi(\mathcal{M}_\eta, 2\eta) =& \Phi(\mathcal{R}_\eta, 2\eta) + \sum_{k=0}^{\eta-1} \Phi(\mathcal{R}_k, 2\eta) \\
=& \Phi(\mathcal{R}_\eta, 2\eta) + \mathcal{U} \sum_{k=0}^{\eta-1} \Phi(\mathcal{R}_k, 2\eta-1) - \sum_{k=0}^{\eta-1} \Phi(\mathcal{R}_k, 2\eta-1) \mathcal{U}^{-1} \\
=& \Phi(\mathcal{R}_\eta, 2\eta) + \mathcal{U}\, \Phi(\mathcal{M}_{\eta-1}, 2\eta-1) - \Phi(\mathcal{M}_{\eta-1}, 2\eta-1) \mathcal{U}^{-1} \\
=& \Phi(\mathcal{R}_\eta, 2\eta).
\end{aligned}
$$

$$
\Phi(\mathcal{R}_\eta, 2\eta) = \sum_{\ell=0}^{2\eta} (-1)^{\ell} \binom{2\eta}{\ell} \mathcal{U}^{2\eta-\ell} \mathcal{R}_\eta \mathcal{U}^{-\ell} = \sum_{\ell=0}^{2\eta} (-1)^{\ell} \binom{2\eta}{\ell} \mathcal{U}^{\eta-\ell} \mathcal{D}_\eta \mathcal{H}_\eta \mathcal{D}_\eta \mathcal{U}^{\eta-\ell}.
$$

Let $\{m_{k,i}\}$ be the sequence of real numbers such that $m_{k,i+j-2}$ is the $(i, j)$th entry of the infinite Hankel matrix $\mathcal{H}_k$, where $i, j = 1, 2, \ldots$ In that case, we write $\mathcal{H}_k = \left[ m_{k,i+j-2} \right]$. Thus,

$$
\begin{aligned}
\mathcal{D}_\eta \mathcal{H}_\eta \mathcal{D}_\eta &= \left[ \frac{(\eta+i-1)!}{(i-1)!} \frac{(\eta+j-1)!}{(j-1)!} m_{\eta,i+j-2} \right] \\
&= (\eta!)^2 \left[ \binom{\eta+i-1}{\eta} \binom{\eta+j-1}{\eta} m_{\eta,i+j-2} \right],
\end{aligned}
$$

and therefore

$$
\Phi(\mathcal{R}_\eta, 2\eta) = (\eta!)^2 \left[ \left( \sum_{\ell=0}^{2\eta} (-1)^{\ell} \binom{2\eta}{\ell} \binom{\eta+i-1+(\eta-\ell)}{\eta} \right.\right.
$$

$$\left.\binom{\eta+j-1-(\eta-\ell)}{\eta}\right)\right)\ m_{\eta,i+j-2}\Bigg]$$

$$=(\eta!)^2\left[\left(\sum_{\ell=0}^{2\eta}(-1)^{\ell}\binom{2\eta}{\ell}\binom{2\eta+i-1-\ell}{\eta}\right.\right.$$

$$\left.\left.\binom{j-1+\ell}{\eta}\right)\right)\ m_{\eta,i+j-2}\Bigg]. \tag{17}$$

Clearly $f(\ell)=\binom{2\eta+i-1-\ell}{\eta}\binom{j-1+\ell}{\eta}$ is a polynomial of degree $2\eta$ in $\ell$ and leading coefficient $\frac{(-1)^{\eta}}{(\eta!)^2}$. By Lemma 2.1, we deduce that

$$\sum_{\ell=0}^{2\eta}(-1)^{\ell}\binom{2\eta}{\ell}\binom{2\eta+i-1-\ell}{\eta}\binom{j-1+\ell}{\eta}=(-1)^{\eta}\binom{2\eta}{\eta}. \tag{18}$$

Hence, from (17)–(18) we get $\Phi(\mathcal{R}_\eta,2\eta)=(-1)^{\eta}(2\eta)![m_{\eta,i+j-2}]=(-1)^{\eta}\ (2\eta)!\,\mathcal{H}_\eta$ and (b). ◻

We will assume that $\mathcal{A}$ is an infinite symmetric matrix because this is obviously a necessary condition for (2) to take place since the Hankel matrices $\mathcal{H}_k$ are symmetric.

**Theorem 5** *Let $\mathcal{A}$ be an infinite symmetric matrix, $\eta\in\mathbb{Z}_+$ (fixed) such that $\Phi(\mathcal{A},2\eta+1)=\mathcal{O}$. Then,*

$$\Phi(\mathcal{A}_\eta,2\eta-1)=\mathcal{O}, \tag{19}$$

*where $\mathcal{A}_\eta=\mathcal{A}-\mathcal{R}_\eta$ and $\mathcal{R}_\eta=\dfrac{(-1)^{\eta}}{(2\eta)!}\mathcal{U}^{-\eta}\mathcal{D}_\eta\Phi(\mathcal{A},2\eta)\mathcal{D}_\eta\mathcal{U}^{\eta}$.*

***Proof*** If $\mathcal{A}_\eta=\mathcal{O}$, the theorem is obvious. Assume that $\mathcal{A}_\eta\neq\mathcal{O}$, from Theorem 4, we get $\Phi(\mathcal{R}_\eta,2\eta)=\frac{(-1)^{\eta}}{(2\eta)!}\Phi(\mathcal{A},2\eta)$, i.e. $\Phi(\mathcal{A}_\eta,2\eta)=\mathcal{O}$. According to the recurrence formula (14), we have $\mathcal{U}\,\Phi(\mathcal{A}_\eta,2\eta-1)=\Phi(\mathcal{A}_\eta,2\eta-1)\mathcal{U}^{-1}$ which is equivalent to stating that $\Phi(\mathcal{A}_\eta,2\eta-1)$ is a Hankel matrix and therefore it is a symmetric matrix.

On the other hand, $\mathcal{A}_\eta$ is a symmetric matrix since it is the difference of two symmetric matrices. Hence, from Proposition 2.4 we get that $\Phi(\mathcal{A}_\eta,2\eta-1)$ is antisymmetric which establishes (19). ◻

***Proof of Theorem 1*** From Theorem 4, a Hankel-Sobolev matrix of index $d\in\mathbb{Z}_+$ satisfies the conditions (4) and each Hankel matrix $\mathcal{H}_k$ holds (5), which establishes the first implication of the theorem.

For the converse, assume that $\mathcal{M}$ is a symmetric infinite matrix and there exits $d\in\mathbb{Z}_+$ such that the conditions (4) are satisfied. From Proposition 2.3, $H_d=\frac{(-1)^d}{(2d)!}\Phi(\mathcal{M},2d)\neq\mathcal{O}$ is a Hankel matrix.

Denote $\mathcal{M}_{d-1}=\mathcal{M}_d-\mathcal{R}_d$, where $\mathcal{M}_d=\mathcal{M}$ and $\mathcal{R}_d=\mathcal{U}^{-d}\mathcal{D}_d\mathcal{H}_d\mathcal{D}_d\mathcal{U}^d$. From (19) and Proposition 2.3, $H_{d-1}=\frac{(-1)^{d-1}}{(2d-2)!}\Phi(\mathcal{M}_{d-1},2d-2)$ is a Hankel matrix.

Let $\mathcal{M}_{d-k} = \mathcal{M}_{d+1-k} - \mathcal{R}_{d+1-k}$ and $\mathcal{R}_{d+1-k} = \mathcal{U}^{-d-1+k}\mathcal{D}_{d+1-k}\mathcal{H}_{d+1-k}\mathcal{D}_{d+1-k}\mathcal{U}^{d+1-k}$. Repeating the previous argument, we get that $H_{d-k} = \frac{(-1)^{d-k}}{(2d-2k)!}\Phi(\mathcal{M}_{d-k}, 2d-2k)$ is a Hankel matrix for $k = 2, \ldots, d$.

By construction, it is clear that

$$\mathcal{M} = \sum_{k=0}^{d} \mathcal{R}_{d-k} = \sum_{k=0}^{d} \mathcal{U}^{k-d}\mathcal{D}_{d-k}\mathcal{H}_{d-k}\mathcal{D}_{d-k}\mathcal{U}^{d-k},$$

i.e. $\mathcal{M}$ is a Hankel-Sobolev matrix and the proof is complete. □

### 2.1 The Sobolev Moment Problem

Let $\mu$ be a finite positive Borel measure supported on the real line and $\mathbf{L}_2(\mu)$ be the usual Hilbert space of square integrable functions with respect to $\mu$ with the inner product

$$\langle f, g\rangle_\mu = \int_{\mathbb{R}} f(x)g(x)d\mu(x), \quad \text{for all } f, g \in \mathbf{L}_2(\mu). \tag{20}$$

The $n$th moment associated with the inner product (20) (or the measure $\mu$) is defined as $m_n = \langle x^n, 1\rangle_\mu$ $(n \geq 0)$, provided the integral exists. The Hankel matrix $\mathcal{H}[m_n]$ is called the *matrix of moments* associated with $\mu$.

The classical moment problem consists in solving the following question: given an arbitrary sequence of real numbers $\{m_n\}_{n\geq 0}$ (or equivalently the associated Hankel matrix $\mathcal{H}[m_n]$) and a closed subset $\Delta \subset \mathbb{R}$, find a positive Borel measure $\mu$ supported on $\Delta$, whose $n$th moment is $m_n$, i.e.

$$m_n = \int_{\Delta} x^n d\mu(x), \quad \text{for all } n \geq 0.$$

It is said that the moment problem $(\mathcal{H}; \Delta)$ is *definite*, if it has at least one solution and *determinate* if the solution is unique. There are three named *classical moment problems*: the *Hamburger moment problem* when the support of $\mu$ is on the whole real line, the *Stieltjes moment problem* if $\Delta = [0, \infty)$, and the *Hausdorff moment problem* for a bounded interval $\Delta$ (without loss of generality, $\Delta = [0, 1]$).

As H. J. Landau write in the introduction of [11, p.1]: *"The moment problem is a classical question in analysis, remarkable not only for its own elegance, but also for the extraordinary range of subjects, theoretical and applied, which it has illuminated"*. For more details on the classical moment problem, the reader is referred to [2, 8, 11, 20, 21] and for historical aspects to [10] or [8, Sec. 2.4].

Without restriction of generality, we now turn our attention to the Hamburger moment problem referring to the following lemma as a necessary and sufficient condition for the problem of moments to be defined and determined.

**Lemma 2.3** ([21, Th. 1.2]) *Let $\{m_n\}_{n=0}^{\infty}$ be a sequence of real numbers and denote by $\mathcal{H} = \mathcal{H}[m_n]$ the associated Hankel matrix. Then,*

1. *The Hamburger moment problem* $(\mathcal{H};\mathbb{R})$ *has a solution, whose support is not reducible to a finite set of points, if and only if* $\mathcal{H}$ *is a matrix positive definite of infinite order (i.e.* $\det([\mathcal{H}]_n) > 0$ *for all* $n \geq 1$*).*
2. *The Hamburger moment problem* $(\mathcal{H};\mathbb{R})$ *has a solution, whose support consists of precisely k distinct points, if and only if* $\mathcal{H}$ *is a matrix positive definite of order k (i.e.* $\det([\mathcal{H}]_n) > 0$ *for all* $1 \leq n \leq k$*, and* $\det([\mathcal{H}]_n) = 0$ *for all* $n > k$*). The moment problem is determined in this case.*

The analogous results for the moment problem of Stieltjes $(\mathcal{H};\mathbb{R}_+)$ or the moment problem of Hausdorff $(\mathcal{H};[0,1])$ are [21, Th. 1.3] and [21, Th. 1.5], respectively. Other equivalent formulations of these results can be seen in [15, Ch. 1, Sec. 7] or [21, Sec. 3.2].

The $(n,k)$-moment associated with the inner product (13) is defined as $m_{n,k} = \langle x^n, x^k\rangle_{\vec{\mu}}$ $(n,k \geq 0)$, provided the integral exists. In the sequel, the values $\langle x^n, x^k\rangle_{\vec{\mu}}$ are called *S-moments*. Here, instead of a sequence of moments, we have the infinite matrix of moments $\mathcal{M}$ with entries $m_{n,k} = \langle x^n, x^k\rangle_{\vec{\mu}}$, $(n,k \geq 0)$.

Now, the Sobolev moment problem (or $\mathcal{S}$-moment problem) consists of solving the following question: given an infinite matrix $\mathcal{M} = (m_{i,j})_{i,j=0}^{\infty}$ and $d+1$ subsets $\Delta_k \subset \mathbb{R}$ $(0 \leq k \leq d)$, find a set of $d+1$ measures $\{\mu_0, \mu_1, \dots, \mu_d\}$, where $\mu_d \neq 0$ and supp $(\mu_k) \subset \Delta_k$, such that $m_{i,j} = \langle x^i, x^j\rangle_{\vec{\mu}}$ for $i, j = 0, 1, \dots$ As in the standard case, the problem is considered *definite* if it has at least one solution, and *determinate* if this solution is unique. There are three conventional cases of $\mathcal{S}$-moment problems: the *Hamburger* $\mathcal{S}$*-moment problem* when $\Delta_0 = \cdots = \Delta_d = \mathbb{R}$; the *Stieltjes* $\mathcal{S}$*-moment problem* if $\Delta_0 = \cdots = \Delta_d = [0,\infty)$, and the *Hausdorff* $\mathcal{S}$*-moment problem* for $\Delta_0 = \cdots = \Delta_d = [0,1]$. Nonetheless, other combinations of the sets $\Delta_k \subset \mathbb{R}$ $(0 \leq k \leq d)$ are possible too. An equivalent formulation of the Sobolev moment problem is made for the special case $d = \infty$.

The following result was proved in [3, Th. 1]. In virtue of Theorem 1, we can now reformulate it in the following equivalent form.

**Theorem 6** *Given an infinite symmetric matrix* $\mathcal{M} = (m_{i,j})_{i,j=0}^{\infty}$ *and* $d+1$ *subset* $\Delta_k \subset \mathbb{R}$ $(0 \leq k \leq d \in \mathbb{Z}_+)$*, the* $\mathcal{S}$*-moment problem is definite (or determinate) if and only if* $\Phi(\mathcal{M}, 2d+1) = \mathcal{O}$*,* $\Phi(\mathcal{M}, 2d) \neq \mathcal{O}$ *and for each* $k = 0, 1, \dots, d$ *the Hankel matrix* $\mathcal{H}_k$ *(defined in (5)) is such that the classical moment problem* $(\mathcal{H}_k;\Delta_k)$ *is definite (or determinate).*

Although [3] is devoted to the study of the case in which $d$ is finite and the measures involved are supported on subset of the real line, there are no difficulties in extending these results when $d = \infty$ or if the measures are supported on the unit circle, as confirmed by the authors of [13, 14]. The $\mathcal{S}$-moments problem for discrete Sobolev-type inner products was studied in [22].

## 3 Hessenberg–Sobolev Matrices

From the definition of the matrix of formal moments $\mathcal{M}$ in (11), we have two immediate consequences.

**Proposition 3.1** *Let $\mathcal{G}$ be a non-degenerate Hessenberg matrix and $\mathcal{M}_{\mathcal{G}}$ be its associated matrix of formal moments. Then, $\mathcal{M}_{\mathcal{G}}$ is a symmetric and positive definite infinite matrix.*

***Proof*** Obviously, $\mathcal{M}_{\mathcal{G}}^{\mathsf{T}} = \left(\mathcal{T}^{-1}\,\mathcal{T}^{-\mathsf{T}}\right)^{\mathsf{T}} = \mathcal{T}^{-1}\,\mathcal{T}^{-\mathsf{T}} = \mathcal{M}_{\mathcal{G}}$. Moreover, for all $n \geq 1$,

$$\det\left([\mathcal{M}_{\mathcal{G}}]_k\right) = \det\left([\mathcal{T}]_k^{-1}\right)\det\left([\mathcal{T}]_k^{-\mathsf{T}}\right) = \tau_{0,0}^2 \cdot \tau_{1,1}^2 \cdot \ldots \cdot \tau_{k-1,k-1}^2 > 0,$$

where $\tau_{i,i}$ is the $(i,i)$-entry of $\mathcal{T}^{-1}$. □

The following theorem clarifies the relation between the sequence of polynomials generated by $\mathcal{G}$ and the matrix of formal moments.

**Theorem 7** *Let $\mathcal{G}$ be a non-degenerate Hessenberg matrix and $\mathcal{M}_{\mathcal{G}} = \left(m_{i,j}\right)$ be its associated matrix of formal moments.*

1. *If $p(x) = \sum_{i=0}^{n_1} a_i x^i$ and $q(x) = \sum_{j=0}^{n_2} b_j x^j$ are two polynomials in $\mathbb{P}$ of degree $n_1$ and $n_2$, respectively. Then, the bilinear form (12) defines an inner product on $\mathbb{P}$ and $\|\cdot\|_{\mathcal{G}} = \sqrt{\langle\cdot,\cdot\rangle_{\mathcal{G}}}$ is the norm induced by (12) on $\mathbb{P}$.*
2. *Let $m_{i,j}$ be the $(i,j)$th entry of $\mathcal{M}_{\mathcal{G}}$, as in (11), then $m_{i,j} = \langle x^i, x^j\rangle_{\mathcal{G}}$.*
3. *$\{Q_n\}$, the sequence of polynomials generated by $\mathcal{G}$, is the sequence of orthonormal polynomials with respect to the inner product (12).*
4. *$g_{j,k} = \langle xQ_j, Q_k\rangle_{\mathcal{G}}$, where $g_{j,k}$ is the $(j,k)$-entry of the matrix $\mathcal{G}$ given in (6), with $j \geq 0$ and $0 \leq k \leq j+1$.*

***Proof*** From Proposition 3.1, the statement 1) is straightforward. The assertion 2) follows from (12).

Let $\mathcal{E}_j$ be the infinite column-vector whose $i$-entry is $\delta_{i,j}$, where $i, j \in \mathbb{Z}_+$. Denote by $Q_n(x) = \sum_{i=0}^{n} t_{n,i}\, x^i$ the $n$th polynomial generated by $\mathcal{G}$, as in (9). Then, for $j = 0, \ldots, n$

$$\begin{aligned}\langle Q_n, x^j\rangle_{\mathcal{G}} =&(t_{n,0}, \cdots, t_{n,n}, 0, \cdots)\; \mathcal{M}_{\mathcal{G}}\, \mathcal{E}_j = \mathcal{E}_n^{\mathsf{T}}\mathcal{T}\mathcal{M}_{\mathcal{G}}\mathcal{E}_j = \mathcal{E}_n^{\mathsf{T}}\mathcal{T}\,\mathcal{T}^{-1}\,\mathcal{T}^{-\mathsf{T}}\,\mathcal{E}_j\\ =&\mathcal{E}_n^{\mathsf{T}}\,\mathcal{T}^{-\mathsf{T}}\,\mathcal{E}_j = \tau_{n,n}\,\mathcal{E}_n^{\mathsf{T}}\,\mathcal{E}_j = \tau_{n,n}\,\delta_{n,j};\end{aligned}$$

where $\tau_{n,n} \neq 0$. Furthermore,

$$\begin{aligned}\langle Q_n, Q_n\rangle_{\mathcal{G}} =&(t_{n,0}, \cdots, t_{n,n}, 0, \cdots)\; \mathcal{M}_{\mathcal{G}}\; (t_{n,0}, \cdots, t_{n,n}, 0, \cdots)^{\mathsf{T}}\\ =&\mathcal{E}_n^{\mathsf{T}}\,\mathcal{T}\,\mathcal{M}_{\mathcal{G}}\,\mathcal{T}^{\mathsf{T}}\,\mathcal{E}_n = \mathcal{E}_n^{\mathsf{T}}\,\mathcal{T}\,\mathcal{T}^{-1}\,\mathcal{T}^{-\mathsf{T}}\,\mathcal{T}^{\mathsf{T}}\,\mathcal{E}_n = 1.\end{aligned}$$

Hence, $Q_n$ is the $n$th orthonormal polynomial with respect to (12). The fourth assertion is straightforward from (8) and the orthogonality. □

***Remark 3.1*** From (8)–(9), we have that the leading coefficient of $Q_n$ is

$$t_{n,n} = t_{0,0}\left(\prod_{k=0}^{n-1} g_{k,k+1}\right)^{-1} \neq 0; \quad \text{for all } n \geq 1.$$

Therefore, the corresponding $n$th-monic polynomial orthogonal with respect to $\langle \cdot, \cdot \rangle_{\mathcal{G}}$ is $q_n = \tau_{n,n} Q_n$ and $\|q_n\|_{\mathcal{G}} = \tau_{n,n} = t_{n,n}^{-1}$ as in (10).

**Theorem 8** *The matrices $\mathcal{G}$ and $\mathcal{M}_{\mathcal{G}}$ are closely related by the expression*

$$\mathcal{G} = \mathcal{T}\mathcal{U}\mathcal{M}_{\mathcal{G}}\mathcal{T}^T. \tag{21}$$

***Proof*** From (9) $Q_n(x) = \sum_{i=0}^{n} t_{n,i} x^i$, therefore

$$g_{k,\ell} = \langle x Q_k, Q_\ell \rangle_{\mathcal{G}} = \sum_{i=0}^{k}\sum_{j=0}^{\ell} t_{k,i}\, t_{\ell,j}\, \langle x^{i+1}, x^j \rangle_{\mathcal{G}} = \sum_{i=0}^{k}\sum_{j=0}^{\ell} t_{k,i}\, t_{\ell,j}\, m_{i+1,j}$$

which is the $(k, \ell)$ entry of matrix $\mathcal{T}\mathcal{U}\mathcal{M}_{\mathcal{G}}\mathcal{T}^{\mathsf{T}}$ and (21) is proved. □

**Theorem 9** *Let $\mathcal{G} \in \mathbb{M}$ be a non-degenerate Hessenberg matrix, $\mathcal{M}_{\mathcal{G}} \in \mathbb{M}$ be its matrix of formal moments associated and $\eta \in \mathbb{Z}_+$ fixed. Then*

$$\Psi(\mathcal{G}, \eta) = (-1)^n\, \mathcal{T}\, \Phi(\mathcal{M}_{\mathcal{G}}, \eta)\, \mathcal{T}^T.$$

***Proof*** From (11) and (21), we get that $\mathcal{G} = \mathcal{T}\mathcal{U}\mathcal{T}^{-1}$. Therefore, for each $k \in \mathbb{Z}_+$ we obtain

$$\mathcal{G}^k = \mathcal{T}\mathcal{U}^k\mathcal{T}^{-1} \quad \text{and} \quad \mathcal{G}^{k\mathsf{T}} = \mathcal{T}^{-\mathsf{T}}\mathcal{U}^{-k}\mathcal{T}^{\mathsf{T}}. \tag{22}$$

Now, from (3), (7), (11) and (22) it follows that

$$\begin{aligned}\Psi(\mathcal{G}, \eta) =& \sum_{k=0}^{\eta}(-1)^k \binom{\eta}{k} \mathcal{G}^k\, \mathcal{G}^{(\eta-k)\,\mathsf{T}} = \sum_{k=0}^{\eta}(-1)^k \binom{\eta}{k} \mathcal{T}\,\mathcal{U}^k\, \mathcal{M}_{\mathcal{G}}\, \mathcal{U}^{k-\eta}\, \mathcal{T}^{\mathsf{T}} \\ =& \mathcal{T}\left((-1)^n \sum_{\ell=0}^{\eta}(-1)^\ell \binom{\eta}{\ell} \mathcal{U}^{\eta-\ell}\, \mathcal{M}_{\mathcal{G}}\, \mathcal{U}^{-\ell}\right)\mathcal{T}^{\mathsf{T}}.\end{aligned}$$

□

## 4 Favard Type Theorems

One of the main problems in the general theory of orthogonal polynomials is to characterize the non-degenerate Hessenberg matrices, for which there exists a non-discrete positive measure $\mu$ supported on the real line such that the inner product (12) can be represented as

$$\langle p, q \rangle_{\mathcal{G}} = \langle p, q \rangle_\mu := \int p\, q\, d\mu. \tag{23}$$

The aforementioned characterization is the well-known *Favard Theorem* (c.f. [7] or [1] for an overview of this theorem and its extensions), that we revisit according to the view-point of this paper.

**Theorem 10** (Favard theorem) *Let $\mathcal{G}$ be a non-degenerate Hessenberg matrix and $\langle\cdot,\cdot\rangle_{\mathcal{G}}$ be the inner product on $\mathbb{P}$ defined by (12). Then, there exists a non-discrete positive measure $\mu$ such that $\langle p, q\rangle_{\mathcal{G}} = \langle p, q\rangle_{\mu}$ for all $p, q \in \mathbb{P}$ if and only if $\Psi(\mathcal{G}, 1) = \mathcal{O}$.*

***Proof*** Assume that there exists a non-discrete positive measure $\mu$ such that $\langle p, q\rangle_{\mathcal{G}} = \langle p, q\rangle_{\mu}$ for all $p, q \in \mathbb{P}$, where $\mathcal{G}$ is a non-degenerate Hessenberg matrix. From the orthogonality of the generated polynomials $Q_n$ (Theorem 7) and the fact that the operator of multiplication by the variable is symmetric with respect to $\langle\cdot,\cdot\rangle_{\mu}$ ($\langle xp, q\rangle_{\mu} = \langle p, xq\rangle_{\mu}$), it is clear that $\mathcal{G}$ is a symmetric tridiagonal matrix, which is equivalent to $\Psi(\mathcal{G}, 1) = \mathcal{O}$.

On the other hand, if $\mathcal{G}$ is a non-degenerate Hessenberg matrix such that $\Psi(\mathcal{G}, 1) = \mathcal{O}$, we get that $\mathcal{G}$ a symmetric Hessenberg matrix or equivalently a non-degenerate tridiagonal matrix. From Theorem 9,

$$\mathcal{O} = \Phi(\mathcal{M}_{\mathcal{G}}, 1) = \mathcal{U}\mathcal{M}_{\mathcal{G}} - \mathcal{M}_{\mathcal{G}}\mathcal{U}^{-1},$$

i.e. $\mathcal{M}_{\mathcal{G}}$ is a Hankel matrix, which from Proposition 3.1 is positive definite. From Lemma 2.3, the proof is complete. □

Obviously, under the assumptions of Theorems 7 and 10, the sequence $\{Q_n\}$ of polynomials generated by $\mathcal{G}$ is the sequence of orthogonal polynomials with respect to the measure $\mu$ (i.e. with respect to the inner product (23)).

***Example 4.1*** The sequence of polynomials $\{1, x, x^2, \ldots, x^n, \ldots\}$ is generated by the non-degenerated Hessenberg matrix

$$\mathcal{G} = \begin{pmatrix} 0 & 1 & 0 & \ldots \\ 0 & 0 & 1 & \ldots \\ 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

hence from Theorem 7, the sequence is orthonormal with respect to the inner product (12). As $\mathcal{G}$ is a non-symmetric matrix, $\Psi(\mathcal{G}, 1) \neq \mathcal{O}$. Then, from Theorem 10 there does not exist a non-discrete positive measure $\mu$, such that $\langle p, q\rangle_{\mathcal{G}} = \langle p, q\rangle_{\mu}$ for all $p, q \in \mathbb{P}$.

***Proof of Theorem 3*** Let $p(x) = \sum_{i=0}^{n_1} a_i x^i$ and $q(x) = \sum_{j=0}^{n_2} b_j x^j$ be polynomials in $\mathbb{P}$ of degree $n_1$ and $n_2$, respectively. Then, from the Sobolev inner product (13) we have the representation

$$\langle p, q\rangle_{\vec{\boldsymbol{\mu}}_d} = (a_0, \cdots, a_{n_1}, 0, \cdots)\mathcal{S}(b_0, \cdots, b_{n_2}, 0, \cdots)^{\mathsf{T}} = \sum_{i=0}^{n_1}\sum_{j=0}^{n_2} a_i\, s_{i,j}\, b_j\ , \quad (24)$$

where $\mathcal{S} = (s_{i,j})_{i,j=0}^{\infty}$ with $s_{i,j} = \langle x^i, x^j\rangle_{\vec{\boldsymbol{\mu}}_d}$, is a Hankel-Sobolev matrix of index $d$. If $\mathcal{G}$ is a Hessenberg–Sobolev matrix of index $d \in \mathbb{Z}_+$, from (12) and (24), to prove that $\langle p,q\rangle_{\vec{\mu}_d}=\langle p,q\rangle_{\mathcal{G}}$ is equivalent to prove that $\mathcal{S}=\mathcal{M}_{\mathcal{G}}$, where $\mathcal{M}_{\mathcal{G}}$ is the matrix of formal moments associated with $\mathcal{G}$.

Let $\mathcal{G}$ be a non-degenerate Hessenberg matrix. Assume that there exists $d\in\mathbb{Z}_+$ and $\vec{\mu}_d=(\mu_0,\mu_1,\dots,\mu_d)\in\mathfrak{M}_d(\mathbb{R})$ (continuous case), such that $\mathcal{S}=\mathcal{M}_{\mathcal{G}}$. From Theorem 6, $\mathcal{S}$ is a Hankel-Sobolev matrix. Therefore, combining Theorem 1 and Theorem 9, we get that $\Psi(\mathcal{G},2d+1)=\mathcal{O}$ and $\Psi(\mathcal{G},2d)\neq\mathcal{O}$. Furthermore, each matrix $\mathcal{H}_k$ defined by (5), is the moment matrix of the measure $\mu_k$, which is a non-negative finite Borel measure whose support is an infinite subset. Hence, from Lemma 2.3 we have that $\mathcal{H}_k$ s a positive definite matrix of infinite order.

Reciprocally, let $\mathcal{G}$ be a non-degenerate Hessenberg matrix satisfying 1 and 2. From Theorems 2 and 9, we conclude that $\mathcal{M}_{\mathcal{G}}$, the matrix of formal moments associated with $\mathcal{G}$, is a Hankel-Sobolev matrix of index $d$, i.e. there exist Hankel matrices $\mathcal{H}_k$, $k=0,\ 1,\dots,\ d$, such that

$$\mathcal{M}_{\mathcal{G}}=\sum_{k=0}^{d}\left(\mathcal{U}^{-k}\ \mathcal{D}_k\ \mathcal{H}_k\ \mathcal{D}_k\ \mathcal{U}^{k}\right).$$

From Theorem 6 and Lemma 2.3, the $\mathcal{S}$-moment problem for $\mathcal{M}_{\mathcal{G}}$ is defined. Let $\mu_{d-k}$ be a solution of the problem of moments with respect to $\mathcal{H}_{d-k}$ for each $k=0,\ 1,\dots,\ d$. If $\langle p,q\rangle_{\vec{\mu}_d}$ is as in (13), from Proposition 2.1 we get $\mathcal{S}=\mathcal{M}_{\mathcal{G}}$. □

The following result may be proved in much the same way as Theorem 3, using the appropriate assertions of Lemma 2.3 for the case of measures supported on finite subsets.

**Theorem 11** (Favard type theorem for discrete case) *Let $\mathcal{G}$ be a non-degenerate Hessenberg matrix. Then, there exists $d\in\mathbb{Z}_+$ and $\vec{\mu}_d\in\mathfrak{M}_d(\mathbb{R})$ such that $\langle p,q\rangle_{\vec{\mu}_d}=\langle p,q\rangle_{\mathcal{G}}$ if and only if*

1. *$\mathcal{G}$ is a Hessenberg–Sobolev matrix of index $d\in\mathbb{Z}_+$.*
2. *The Hankel matrix $\mathcal{H}_0$ defined by (5) is a positive definite matrix of infinite order and for each $k=0,\ 1,\dots,\ d-1$; the matrix $\mathcal{H}_{d-k}$ is a positive definite matrix of order $m_k\in\mathbb{Z}_+$.*

The previous theorem is a refinement of [7, Lemma 3].

**Theorem 12** (Favard type theorem for discrete-continuous case) *Let $\mathcal{G}$ be a non-degenerate Hessenberg matrix. Then, there exists $d\in\mathbb{Z}_+$ and $\vec{\mu}_d\in\mathfrak{M}_d(\mathbb{R})$ such that $\langle p,q\rangle_{\vec{\mu}_d}=\langle p,q\rangle_{\mathcal{G}}$ if and only if*

1. *$\mathcal{G}$ is a Hessenberg–Sobolev matrix of index $d\in\mathbb{Z}_+$.*
2. *The Hankel matrix $\mathcal{H}_d$ defined by (5), is a positive definite matrix of infinite order and for each $k=1,\ 2\dots,\ d$; the matrix $\mathcal{H}_{d-k}$ is a positive definite matrix of order $m_k\in\mathbb{Z}_+$.*

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## Declarations

**Conflict of interest** All authors declare that they have no conflicts of interest.



## References

1. Álvarez-Nordase, R., Marcellán, F.: On the Favard theorem and its extensions. J. Comput. Appl. Math. **127**, 231–254 (2001)
2. Akhiezer, N.I.: The Classical Moment Problem and Some Related Questions in Analysis. Oliver and Boyd, Edinburgh (1965)
3. Barrios-Rolanía, D., López-Lagomasino, G., Pijeira-cabrera, H.: The moment problem for a Sobolev inner product. J. Approx. Theory **100**, 364–380 (1999)
4. Cantero, M.J., Marcellán, F., Moral, L., Velázquez, L.: Darboux transformations for CMV matrices. Adv. Math. **298**, 122–206 (2016)
5. Cooke, R.G.: Infinite Matrices and Sequence Spaces. Macmillan and Co., London (1950)
6. Díaz-Millán, R.: Characterization of linear functionals associated with bilinear forms of Sobolev type. Rev. Colomb. Mat. **42**, 85–98 (2008). ((**in Spanish**))
7. Durán, A.J.: A generalization of Favard's Theorem for polynomials satisfying a recurrence relation. J. Approx. Theory **74**, 83–09 (1993)
8. Golub, G.H., Meurant, G.: Matrices, Moments and Quadrature with Applications. Princeton Univ. Press, Princeton, NJ (2010)
9. Horn, R.A., Johnson, C.R.: Matrix Analysis. Cambridge Univ. Press, Cambridge (2013)
10. Kjeldsen, T.H.: The early history of the moment problem. Historia Math. **20**, 19–44 (1993)
11. Landau , H.J.: (Ed.) Moments in Mathematics, Proc. Symp. Appl. Math., Vol. 37, Amer. Math. Soc., Providence, RI, (1987)
12. Marcellán, F., Xu, Y.: On Sobolev orthogonal polynomials. Expo. Math. **33**, 308–352 (2015)
13. Marcellán, F., Szafraniec, F.H.: The Sobolev-type moment problem. Proc. Amer. Math. Soc. **128**, 2309–2317 (2000)
14. Marcellán, F., Szafraniec, F.H.: A matrix algorithm towards solving the moment problem of Sobolev type. Linear Algebra Appl. **331**, 155–164 (2001)
15. Peller, V.: Hankel Operators and Their Applications. Springer-Verlag, New York (2003)
16. Pijeira-Cabrera, H.: Theory of moments and asymptotic properties of Sobolev orthogonal polynomials, Doctoral Dissertation, Universidad Carlos III de Madrid, (1998) (in Spanish), https://doi.org/10.13140/RG.2.1.4382.9920
17. Pijeira-Cabrera, H., Quintana-Mato, Y., Rodríguez-García, J.M.: Sobolev formal orthogonality on algebraic curves and extensions of Favard theorem. Jaen J. Approx. **3**, 193–207 (2011)
18. Quaintance, J., Gould, H.W.: Combinatorial Identities for Stirling Numbers. The unpublished notes of H. W. Gould, World Scientific. Singapore, (2016)
19. Robert, L., Santiago, L.: On a class of Sobolev scalar products in the polynomials. J. Approx. Theory **125**, 169–189 (2003)
20. Schmüdgen, K.: The Moment Problem, Graduate Texts in Mathematics, vol. 27. Springer, Cham (2017)
21. Shohat, J.A., Tamarkin, J.D.: The Problem of Moments, Mathematical Surveys, vol. I. Amer. Math. Soc, Providence, RI (1963)

22. Zagorodnyuk, S.M.: On the moment problem of discrete Sobolev type. Ukr. Math. Bull. **2**, 351–367 (2005)