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QUALITATIVE ANALYSIS OF A COOPERATIVE REACTION-DIFFUSION SYSTEM IN A SPATIOTEMPORALLY DEGENERATE ENVIRONMENT*

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Abstract. In this paper, we are concerned with the cooperative system in which $\partial_t u - \Delta u = \mu u + \alpha(x,t)v - a(x,t)u^p$ and $\partial_t v - \Delta v = \mu v + \beta(x,t)u - b(x,t)v^q$ in $\Omega \times (0,\infty)$; $(\partial_\nu u, \partial_\nu v) = (0,0)$ on $\partial\Omega \times (0,\infty)$; and $(u(x,0), v(x,0)) = (u_0(x), v_0(x)) > (0,0)$ in Ω, where $p, q > 1$, $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded smooth domain, $\alpha, \beta > 0$ and $a, b \geq 0$ are smooth functions that are T-periodic in t, and μ is a varying parameter. The unknown functions $u(x,t)$ and $v(x,t)$ represent the densities of two cooperative species. We study the long-time behavior of (u,v) in the case that a and b vanish on some subdomains of $\Omega \times [0,T]$. Our results show that, compared to the nondegenerate case where $a, b > 0$ on $\overline{\Omega} \times [0,T]$, such a spatiotemporal degeneracy can induce a fundamental change to the dynamics of the cooperative system.

Key words. cooperative reaction-diffusion system, spatiotemporal degeneracy, positive periodic solutions, principal eigenvalue, dynamical behavior

AMS subject classifications. 35K40, 35K50, 35K57, 35K65

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1. Introduction.

In this paper, we analyze a class of cooperative reaction-diffusion systems of the form

$$\begin{cases} \partial_t u - \operatorname{div}(d_1(x,t)\nabla u) = \mu_1(x,t)u + \alpha(x,t)v - a(x,t)u^p \\ \partial_t v - \operatorname{div}(d_2(x,t)\nabla v) = \mu_2(x,t)v + \beta(x,t)u - b(x,t)v^q \end{cases} \text{in } \Omega \times (0,\infty), \\ (\partial_\nu u, \partial_\nu v) = (0,0) \quad \text{on } \partial\Omega \times (0,\infty), \\ (u(x,0), v(x,0)) = (u_0(x), v_0(x)) > (0,0) \quad \text{in } \Omega. \tag{1.1}$$

This problem may be used to describe the evolution of two cooperative species, with densities $u(x,t)$ and $v(x,t)$, respectively, that are randomly dispersing in a habitat Ω, whose mutual cooperative effects are measured by the positive functions α and β. The function pair $(u_0(x), v_0(x))$ stands for the initial density of the populations, and μ_1, μ_2 are the intrinsic growth rates of the species. The nonnegative functions a and b measure the strengths of intraspecific competition of the species, while the positive functions d_1 and d_2 represent the diffusion rates of the species. Moreover, the Neumann boundary condition means that the species are enclosed in Ω, with no

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population flux across the boundary $\partial\Omega$. Note that if the region Ω is surrounded by a lethal environment, then the homogeneous Dirichlet boundary conditions should be used instead. Throughout this paper, we assume that $p, q > 1$ and Ω is a bounded smooth domain in $\mathbb{R}^N$ $(N \geq 2)$.

In the case that all the coefficient functions in (1.1) are positive constants, and homogeneous Dirichlet boundary conditions are used, this system was investigated in [18, 19]. In [20] and [3], such a system was considered when all the coefficients are functions of x only (independent of t), with $a^{-1}(0) = b^{-1}(0) \neq \emptyset$. It was shown in [19] that in the situation considered there, qualitatively the system behaves like the classical scalar logistic equation, and in the case of [20, 3], it follows from [3] that the system behaves like the time-autonomous degenerate scalar logistic equation (as described in [8, 13]).

Since the natural environment is typically periodic in time, it is reasonable to assume that the coefficient functions in (1.1) are time-dependent with a common period T in t. In such a situation, in the absence of a second species, namely (1.1) with $v \equiv 0$, the problem reduces to a periodic-parabolic logistic problem, which was considered in [9]. It was shown in [9] that the degenerate periodic-parabolic logistic problem may behave in a drastically different way from the time-autonomous degenerate logistic problem as described in [8, 12].

The main purpose of this paper is to investigate the degenerate system (1.1), and to reveal some fundamental changes of behavior from those obtained in [3]. As remarked in [9], a good understanding of the model in the degenerate cases is important in order to determine the scope of changes of dynamical behaviors that a heterogeneous environment may cause to ecological systems. For example, based on the results of this paper, it is possible to use a perturbation approach as in [10] to reveal sophisticated patterns of the solutions to the corresponding standard cooperative system in a spatiotemporal environment which is close to the degenerate case considered here.

In order to make the main points of this paper more transparent, we consider only a special case of (1.1) by setting $d_1 = d_2 \equiv 1$ and $\mu_1 = \mu_2 \equiv \mu$, with μ regarded as a bifurcation parameter. The general case can be handled largely by the same method, but with more involved notation, etc. Therefore the system to be investigated in detail in this paper is given by

$$(1.2) \qquad \begin{cases} \partial_t u - \Delta u = \mu u + \alpha(x,t)v - a(x,t)u^p & \\ \partial_t v - \Delta v = \mu v + \beta(x,t)u - b(x,t)v^q & \text{in } \Omega \times (0,\infty), \\ (\partial_\nu u, \partial_\nu v) = (0,0) & \text{on } \partial\Omega \times (0,\infty), \\ (u(x,0), v(x,0)) = (u_0(x), v_0(x)) > (0,0) & \text{in } \Omega. \end{cases}$$

Here $\alpha(x,t)$, $\beta(x,t)$, $a(x,t)$, and $b(x,t)$ are T-periodic in time, i.e.,

$$\gamma(x, t+T) = \gamma(x,t), \quad \text{with } \gamma \in \{\alpha, \beta, a, b\}.$$

The habitat Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 2$, with $\partial\Omega$ of class $C^{2+\theta}$ for some $\theta \in (0,1)$, $\mu \in \mathbb{R}$, ν is the outward unit normal vector of $\partial\Omega$, and $\gamma \in C^{\theta,\theta/2}(\overline{\Omega} \times \mathbb{R})$ for all $\gamma \in \{\alpha, \beta, a, b\}$. By $(u_0(x), v_0(x)) > (0,0)$, we mean that both u_0 and v_0 are nonnegative and not identically zero.

We always assume that α, $\beta > 0$ and a, $b \geq 0$. If a and b are positive, then it is easy to show that (1.2) behaves like the scalar periodic-parabolic logistic problem as described in [17]. More precisely, let μ_0 be the principal eigenvalue of the cooperative periodic eigenvalue problem

$$\text{(1.3)}\qquad \begin{cases} \partial_t\Phi - \Delta\Phi - \alpha(x,t)\Psi = \mu_0\Phi \\ \partial_t\Psi - \Delta\Psi - \beta(x,t)\Phi = \mu_0\Psi \end{cases} \text{in } \Omega\times[0,T], \quad \begin{cases} (\partial_\nu\Phi, \partial_\nu\Psi) = (0,0) & \text{on } \partial\Omega\times[0,T], \\ (\Phi(x,0),\Psi(x,0)) = (\Phi(x,T),\Psi(x,T)) & \text{in } \Omega. \end{cases}$$

Then the nonlinear periodic cooperative problem

$$\text{(1.4)}\qquad \begin{cases} \partial_t u - \Delta u = \mu u + \alpha(x,t)v - a(x,t)u^p \\ \partial_t v - \Delta v = \mu v + \beta(x,t)u - b(x,t)v^q \end{cases} \text{in } \Omega\times\mathbb{R}, \quad \begin{cases} (\partial_\nu u, \partial_\nu v) = (0,0) & \text{on } \partial\Omega\times\mathbb{R}, \\ (u(x,0),v(x,0)) = (u(x,T),v(x,T)) & \text{in } \Omega \end{cases}$$

has a unique positive solution (u_μ, v_μ) if and only if $\mu \in (\mu_0,\infty)$. Moreover, the unique solution of (1.2) converges to $(0,0)$ as $t\to\infty$ if $\mu \le \mu_0$, while for $\mu > \mu_0$, $(u(x,t),v(x,t))$ converges to (u_μ, v_μ) as $t\to\infty$.

In this paper, we will focus on the degenerate case that both $a^{-1}(0)$ and $b^{-1}(0)$ are nonempty, but not identical. Specifically, we assume that

$$\{(x,t)\in\overline{\Omega}\times[0,T] : a(x,t)=0\} = (\overline{\Omega}\times[0,T^*]) \cup (\overline{\Omega^a_0}\times[T^*,T]),$$
$$\{(x,t)\in\overline{\Omega}\times[0,T] : b(x,t)=0\} = (\overline{\Omega}\times[0,T^*]) \cup (\overline{\Omega^b_0}\times[T^*,T]),$$

where $T^*\in(0,T)$ and Ω^a_0, Ω^b_0 are nonempty, open, connected sets satisfying

$$\overline{\Omega^a_0}\subset\Omega, \quad \overline{\Omega^b_0}\subset\Omega \quad \text{and} \quad \overline{\Omega^a_0}\cap\overline{\Omega^b_0}=\emptyset.$$

We also assume that $\partial\Omega^a_0$ and $\partial\Omega^b_0$ are smooth.

These assumptions are ecologically reasonable. Mathematically they allow us to avoid certain excessive technicalities. For example, if $(\partial\Omega^a_0\cup\partial\Omega^b_0)\cap\partial\Omega\neq\emptyset$, then considerable technical difficulties arise in the mathematical analysis, since singularly mixed boundary value problems have to be considered in the limit. Moreover, when $\overline{\Omega^a_0}\cap\overline{\Omega^b_0}\neq\emptyset$, additional techniques are required in the analysis. To keep the paper at a reasonable length, we have refrained from discussing these cases here.

Obviously,

$$\Sigma^a_+ := \{(x,t)\in\overline{\Omega}\times[0,T] : a(x,t)>0\} = (\Omega^a_+\cup\partial\Omega)\times(T^*,T),$$
$$\Sigma^b_+ := \{(x,t)\in\overline{\Omega}\times[0,T] : b(x,t)>0\} = (\Omega^b_+\cup\partial\Omega)\times(T^*,T),$$

with

$$\Omega^a_+ := \Omega\setminus\overline{\Omega^a_0}, \qquad \Omega^b_+ := \Omega\setminus\overline{\Omega^b_0}.$$

For convenience, we also use the notation

$$\Sigma^a_0 := (\Omega\times[0,T^*])\cup(\Omega^a_0\times[T^*,T]),$$
$$\Sigma^b_0 := (\Omega\times[0,T^*])\cup(\Omega^b_0\times[T^*,T]).$$

Clearly,

$$\{(x,t)\in\overline{\Omega}\times[0,T] : a(x,t)=0\} = \overline{\Sigma^a_0}, \quad \{(x,t)\in\overline{\Omega}\times[0,T] : b(x,t)=0\} = \overline{\Sigma^b_0}.$$

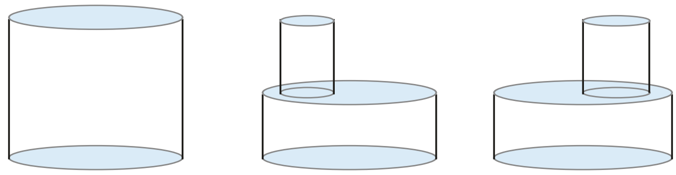

FIG. 1. *Illustrative graphs of* $\Omega \times [0,T]$ *(left),* Σ_0^a *(middle), and* Σ_0^b *(right).*

See Figure 1 for the illustrative graphs of $\Omega \times [0,T], \Sigma_0^a$, and Σ_0^b.

A simple special case in which the above conditions are satisfied is given by

$$c_1 p_a(x)q(t) \le a(x,t) \le c_2 p_a(x)q(t), \qquad c_1 p_b(x)q(t) \le b(x,t) \le c_2 p_b(x)q(t),$$

where c_1, c_2 are positive constants, $p_a(x), p_b(x)$ are nonnegative functions satisfying

$$p_a^{-1}(0) = \overline{\Omega_0^a}, \qquad p_b^{-1}(0) = \overline{\Omega_0^b},$$

and $q(t)$ is a nonnegative T-periodic function satisfying $q^{-1}(0) \cap [0,T] = [0,T^*]$.

Clearly, our assumptions on a and b allow much more general situations to happen. We remark that the results in [9, 10] also hold under more general conditions along the lines of this paper.

Under the above assumptions, we first consider the periodic-parabolic cooperative system (1.4), proving the following result.

THEOREM 1.1. *Problem* (1.4) *has a unique positive solution* (u_μ, v_μ) *if and only if* $\mu \in (\mu_0, \mu_\infty)$*, where* μ_∞ *is the principal eigenvalue of the periodic eigenvalue problem*

$$\text{(1.5)} \qquad \begin{cases} \partial_t \Phi - \Delta \Phi - \alpha(x,t)\chi_{\Omega\times[0,T^*]}\Psi = \mu_\infty \Phi & \text{in } \Sigma_0^a, \\ \partial_t \Psi - \Delta \Psi - \beta(x,t)\chi_{\Omega\times[0,T^*]}\Phi = \mu_\infty \Psi & \text{in } \Sigma_0^b, \\ \partial_\nu \Phi = \partial_\nu \Psi = 0 & \text{on } \partial\Omega \times (0,T^*], \\ \Phi = 0 & \text{on } \partial\Omega_0^a \times (T^*,T], \\ \Psi = 0 & \text{on } \partial\Omega_0^b \times (T^*,T], \\ \Phi(x,0) = \Phi(x,T) & \text{in } \Omega_0^a, \\ \Psi(x,0) = \Psi(x,T) & \text{in } \Omega_0^b, \end{cases}$$

where χ_ω *denotes the characteristic function of a set* ω*.*

We remark that the existence of the principal eigenvalue μ_∞ of (1.5), the regularity of the associated eigenfunction (Φ, Ψ), as well as upper and lower bounds of μ_∞, will be proved in detail in section 2 below.

Second, we study the long-time behavior of the unique solution (u,v) of (1.2), and our result reads as follows.

THEOREM 1.2. *Let* (u,v) *be the unique solution of* (1.2)*. Then the following assertions hold:*

(a) *If* $\mu \le \mu_0$*, then* $\lim_{t\to\infty}(u(x,t),v(x,t)) = (0,0)$ *uniformly in* $\overline{\Omega}$*.*

(b) *If* $\mu \in (\mu_0,\mu_\infty)$*, then* $\lim_{t\to\infty} |(u(x,t),v(x,t)) - (u_\mu(x,t),v_\mu(x,t))| = 0$ *uniformly in* $\overline{\Omega}$*.*

(c) *If* $\mu \ge \mu_\infty$*, then*

$$\lim_{n\to\infty} \big(u(x,t+nT), v(x,t+nT)\big) = (\infty,\infty)$$

uniformly in compact subsets of $(\overline{\Omega} \times (0,T^*]) \cup ((\overline{\Omega_0^a} \cup \overline{\Omega_0^b}) \times [0,T])$*, and*

$$\lim_{n\to\infty} \big(u(x,t+nT), v(x,t+nT)\big) = \Big(\underline{U}_\mu(x,t), \underline{V}_\mu(x,t)\Big)$$

uniformly in every compact subset of $(\overline{\Omega}\setminus(\overline{\Omega_0^a}\cup\overline{\Omega_0^b}))\times(T^*,T)$, *where* $(\underline{U}_\mu,\underline{V}_\mu)$ *is the minimal positive solution of*

$$(1.6)\quad \begin{cases} \partial_t u - \Delta u = \mu u + \alpha(x,t)v - a(x,t)u^p \\ \partial_t v - \Delta v = \mu v + \beta(x,t)u - b(x,t)v^q \end{cases} \textit{in } (\overline{\Omega}\setminus(\overline{\Omega_0^a}\cup\overline{\Omega_0^b}))\times(T^*,T),$$
$$\begin{cases} (\partial_\nu u,\partial_\nu v) = (0,0) & \textit{on } \partial\Omega\times(T^*,T), \\ (u,v) = (\infty,\infty) & \textit{on } (\partial\Omega_0^a\cup\partial\Omega_0^b)\times(T^*,T), \\ (u(x,T^*),v(x,T^*)) = (\infty,\infty) & \textit{in } \overline{\Omega}\setminus(\Omega_0^a\cup\Omega_0^b). \end{cases}$$

In Theorem 1.2(c), by $u=\infty$ on $(\partial\Omega_0^a\cup\partial\Omega_0^b)\times(T^*,T)$, we mean that

$$u(x,t)\to\infty \ \text{ as } d(x,\partial\Omega_0^a\cup\partial\Omega_0^b)\to 0 \ \text{ uniformly for } t\in(T^*,T),$$

and by $u(x,T^*)=\infty$ in $\overline{\Omega}\setminus(\Omega_0^a\cup\Omega_0^b)$ we mean that

$$u(x,t)\to\infty \ \text{ as } t \text{ decreases to } T^* \text{ uniformly for } x\in\overline{\Omega}\setminus(\Omega_0^a\cup\Omega_0^b).$$

The same interpretation applies to v. The existence of a minimal positive solution to (1.6) will follow from a more general result proved in section 3. As in [11], under further conditions on a and b at $t=T$, we can obtain a better understanding of the behavior of $(\underline{U}_\mu,\underline{V}_\mu)$ (and hence that of $(u(x,t+nT),v(x,t+nT)$ in $\overline{\Omega}\setminus(\overline{\Omega_0^a}\cup\overline{\Omega_0^b})$ at $t=T$ as $n\to\infty$) in part (c) of Theorem 1.2.

Compared with the nondegenerate case of a, $b>0$ on $\overline{\Omega}\times[0,T]$, we see from Theorem 1.2 that the dynamical behavior of (1.2) is fundamentally changed when degeneracy occurs. Moreover, the change is significantly different from the time-autonomous degenerate case described in [3]. The long-time dynamical behavior of the cooperative system obtained in this paper resembles that of the single species case considered in [9], but the mathematical techniques here are very different. Indeed, the system here is much harder to treat, and several new ideas and techniques are introduced in this paper to overcome a number of highly nontrivial difficulties associated with the system. Additional techniques are required to handle the other ecologically natural case $\Omega_0^a\cap\Omega_0^b\neq\emptyset$, which we have left for future work. We believe that the ideas and techniques developed in this paper should be useful for future investigations on this and related systems.

2. Linear eigenvalue problems. Our main aim in this section is to analyze the asymptotic behavior of the following linear periodic–parabolic eigenvalue problem:

$$(2.1)\quad \begin{cases} \partial_t\phi - \Delta\phi + \lambda a(x,t)\phi - \alpha(x,t)\psi = \mu\phi \\ \partial_t\psi - \Delta\psi + \lambda b(x,t)\psi - \beta(x,t)\phi = \mu\psi \end{cases} \text{in } \Omega\times\mathbb{R},$$
$$\begin{cases} (\partial_\nu\phi,\partial_\nu\psi) = (0,0) & \text{on } \partial\Omega\times\mathbb{R}, \\ (\phi(x,t),\psi(x,t)) = (\phi(x,t+T),\psi(x,t+T)) & \text{in } \Omega\times\mathbb{R}, \end{cases}$$

as the nonnegative parameter λ goes to infinity. We will show that the limits of the principal eigenvalue and the corresponding principal eigenfunction of (2.1) satisfy (1.5). Our results in this section will become crucial in the study of the qualitative properties of the nonlinear problem (1.2).

If (2.1) is time-autonomous, i.e., the coefficients are independent of time t, such an asymptotic limit (as $\lambda\to\infty$) was obtained in [2, 5, 15, 1, 6] under various conditions of the coefficient functions.

2.1. Basic properties of the principal eigenvalue of (2.1). Problem (2.1) can be formulated as an abstract eigenvalue problem,

$$\mathcal{L}_\lambda \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \mu \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \tag{2.2}$$

in the space

$$X_0 := \{(\phi, \psi) \in (C^{\theta,\theta/2}(\overline{\Omega} \times \mathbb{R}))^2 \,:\, \phi,\ \psi \text{ are } T\text{-periodic in } t\},$$

where $\mathcal{L}_\lambda$ is the operator defined by

$$\mathcal{L}_\lambda := \begin{pmatrix} \partial_t - \Delta + \lambda a(x,t) & -\alpha \\ -\beta & \partial_t - \Delta + \lambda b(x,t) \end{pmatrix},$$

assuming Neumann boundary conditions and T-periodicity, and with the domain of the operator $\operatorname{dom}(\mathcal{L}_\lambda) = X_1$ defined by

$$X_1 = \{(\phi,\psi) \in (C^{2+\theta,1+\theta/2}(\overline{\Omega}\times\mathbb{R}))^2 : \partial_\nu\phi = \partial_\nu\psi = 0 \text{ on } \partial\Omega\times\mathbb{R},\ \phi,\psi \text{ are } T\text{-periodic in } t\}.$$

One may argue as in [4] to show that, for fixed λ and large constant $m > 0$, the inverse operator $(\mathcal{L}_\lambda + mI)^{-1}$ from X_1 to X_1 is compact and strongly positive. Hence it follows from the Krein–Rutman theorem that there exists a unique $r > 0$ and a unique (subject to constant multiples) positive function pair $(\phi_\lambda, \psi_\lambda) \in X_1$ such that $(\mathcal{L}_\lambda + mI)^{-1}(\phi_\lambda, \psi_\lambda) = r(\phi_\lambda, \psi_\lambda)$. It follows that there is a unique value $\mu = \mu_1(\lambda)$ such that (2.2), and equivalently (2.1), has a unique positive solution $(\phi_\lambda, \psi_\lambda)$ (subject to a constant multiple). Such a value μ is known as the *principal eigenvalue* of (2.1) in Ω (under Neumann boundary conditions), and we denote it by

$$\mu_1[\mathcal{L}_\lambda, \Omega] := \mu_1(\lambda).$$

The corresponding positive eigenfunction pair $(\phi_\lambda, \psi_\lambda) \in X_1$ is called the *principal eigenfunction*. When Dirichlet boundary conditions are used, this kind of eigenvalue problem was discussed in detail in [4].

For convenience and later use, we denote by

$$\sigma_1[\mathcal{L}(V_1, V_2), \Omega] \quad \text{and} \quad \mu_1[\mathcal{L}(V_1, V_2), \Omega]$$

the principal eigenvalue of the operator

$$\mathcal{L}(V_1, V_2) := \begin{pmatrix} \partial_t - \Delta + V_1 & -\alpha \\ -\beta & \partial_t - \Delta + V_2 \end{pmatrix}$$

for any $(V_1, V_2) \in X_0$ under Dirichlet and Neumann boundary conditions, respectively. Moreover, the following properties are easily deduced from [4]:

(1) $\sigma_1[\mathcal{L}(V_1, V_2), \Omega] > \mu_1[\mathcal{L}(V_1, V_2), \Omega]$.

(2) Monotonicity with respect to the potentials,

$$\sigma_1[\mathcal{L}(V_1, V_2), \Omega] > \sigma_1[\mathcal{L}[U_1, U_2), \Omega],$$
$$\mu_1[\mathcal{L}(V_1, V_2), \Omega] > \mu_1[\mathcal{L}(U_1, U_2), \Omega],$$

if $V_1 \geq U_1$, $V_2 \geq U_2$ and $(V_1, V_2) \neq (U_1, U_2)$.

(3) Monotonicity with respect to the domain for Dirichlet boundary conditions,

$$\sigma_1[\mathcal{L}(V_1, V_2), \Omega_0] \geq \sigma_1[\mathcal{L}[V_1, V_2), \Omega],$$

if $\Omega_0 \subset \Omega$.

LEMMA 2.1. *The principal eigenvalue* $\mu_1(\lambda)$ *of* (2.1) *is continuous and strictly increasing. Moreover, the limit* $\mu_\infty := \lim_{\lambda\to\infty} \mu_1(\lambda)$ *satisfies*

$$\mu_0 := \mu_1(0) < \mu_\infty \leq \min\{\sigma_1^a,\ \sigma_1^b\}, \tag{2.3}$$

where σ_1^j, $j \in \{a, b\}$ *is the principal eigenvalue of the elliptic problem*

$$-\Delta\varphi = \sigma\varphi \quad \text{in } \Omega_0^j, \qquad \varphi = 0 \quad \text{on } \partial\Omega_0^j.$$

Proof. Since a, b are nonnegative and not identically zero, the monotonicity of $\mu_1(\lambda)$ is clear, and we also have $\mu_0 = \mu_1(0) < \mu_1(\lambda) < \mu_\infty$ for all $\lambda > 0$.

To prove the continuity, for any given λ, take a sequence $\{\lambda_n\}$ such that $\lambda_n \to \lambda$. Then consider the sequence of operators

$$\mathcal{L}_{\lambda_n} := \begin{pmatrix} \partial_t - \Delta + \lambda_n a & -\alpha \\ -\beta & \partial_t - \Delta + \lambda_n b \end{pmatrix}.$$

For any $\varepsilon > 0$, there exists an integer $n_0 = n_0(\varepsilon)$ such that, for every $n \geq n_0$,

$$\lambda a - \varepsilon \leq \lambda_n a \leq \lambda a + \varepsilon, \qquad \lambda b - \varepsilon \leq \lambda_n b \leq \lambda b + \varepsilon.$$

Thus, we find that, for any $n \geq n_0$,

$$\mu_1[\mathcal{L}_\lambda - \varepsilon\, \text{diag}\,\{1,1\}; \Omega] \leq \mu_1[\mathcal{L}_{\lambda_n}; \Omega] \leq \mu_1[\mathcal{L}_\lambda + \varepsilon\, \text{diag}\,\{1,1\}; \Omega]$$

or, equivalently,

$$\mu_1[\mathcal{L}_\lambda; \Omega] - \varepsilon \leq \mu_1[\mathcal{L}_{\lambda_n}; \Omega] \leq \mu_1[\mathcal{L}_\lambda; \Omega] + \varepsilon, \quad n \geq n_0.$$

This proves the continuity of $\mu_1(\lambda)$.

Furthermore, we have

$$\mu_1(\lambda) \leq \sigma_1[\mathcal{L}(\lambda a, \lambda b), \Omega] \leq \sigma_1[\mathcal{L}(\lambda a, \lambda b), \Omega_0^a].$$

Let (ϕ_0, ψ_0) be the principal eigenfunction corresponding to $\sigma^0 := \sigma_1[\mathcal{L}(\lambda a, \lambda b), \Omega_0^a]$. Then we have

$$\partial_t\phi_0 - \Delta\phi_0 \geq \sigma^0\phi_0, \quad \phi_0(x,t) = \phi_0(x, t+T) \quad \text{in } \Omega_0^a \times \mathbb{R}.$$

Multiplying the above inequality by a principal eigenfunction corresponding to σ_1^a and then integrating over $\Omega_0^a \times (0, T)$ by parts, it follows immediately, using the periodicity of ϕ_0, that $\sigma^0 \leq \sigma_1^a$. Therefore $\mu_1(\lambda) \leq \sigma_1^a$ for all $\lambda \in \mathbb{R}$, and so $\mu_\infty \leq \sigma_1^a$. Similarly, we have $\mu_\infty \leq \sigma_1^b$. ∎

2.2. Characterization of $\boldsymbol{\mu_\infty}$. In this subsection, we study the qualitative properties of μ_∞ as the principal eigenvalue of (1.5), as well as its associated principal eigenfunction. Our first result in this direction is the following.

THEOREM 2.2. *Let μ_∞ be given as above. Then there exists a function pair (Φ, Ψ) such that*

$$
\begin{aligned}
&\Phi \in C^{2+\theta,1+\theta/2}\left((\overline{\Omega} \times (0, T^*]) \cup (\overline{\Omega_0^a} \times (T^*, T])\right),\\
&\Psi \in C^{2+\theta,1+\theta/2}\left((\overline{\Omega} \times (0, T^*]) \cup (\overline{\Omega_0^b} \times (T^*, T])\right),\\
&\Phi \in C^{1+\theta,(1+\theta)/2}\left((\overline{\Omega_0^a} \times [T^*, T]) \setminus (\partial\Omega_0^a \times \{T^*\})\right),\\
&\Psi \in C^{1+\theta,(1+\theta)/2}\left((\overline{\Omega_0^b} \times [T^*, T]) \setminus (\partial\Omega_0^b \times \{T^*\})\right),\\
&\Phi = 0 \text{ in } \overline{\Omega_+^a} \times (T^*, T],\\
&\Psi = 0 \text{ in } \overline{\Omega_+^b} \times (T^*, T],\\
&\Phi > 0 \text{ in } (\overline{\Omega} \times (0, T^*]) \cup (\Omega_0^a \times (T^*, T]),\\
&\Psi > 0 \text{ in } (\overline{\Omega} \times (0, T^*]) \cup (\Omega_0^b \times (T^*, T]),
\end{aligned}
\tag{2.4}
$$

and (Φ, Ψ, μ_∞) *satisfies* (1.5).

Proof. Let $\{(\phi_\lambda, \psi_\lambda)\}$ be the principal eigenfunction corresponding to $\mu_1(\lambda)$, normalized by

$$\max_{\overline{\Omega}\times[0,T]} (\phi_\lambda + \psi_\lambda) = 1, \tag{2.5}$$

and satisfying $\phi_\lambda,\ \psi_\lambda > 0$. Since $0 \le \phi_\lambda,\ \psi_\lambda \le 1$ in $\overline{\Omega} \times [0, T]$, we can find a sequence $\{\lambda_n\}$ with $\lambda_n \to \infty$ as $n \to \infty$ such that $\{(\phi_{\lambda_n}, \psi_{\lambda_n})\}$ satisfies

$$
\begin{cases}
\left.\begin{aligned}\partial_t\phi_{\lambda_n} - \Delta\phi_{\lambda_n} + \lambda_n a(x,t)\phi_{\lambda_n} - \alpha(x,t)\psi_{\lambda_n} = \mu_1(\lambda_n)\phi_{\lambda_n}\\ \partial_t\psi_{\lambda_n} - \Delta\psi_{\lambda_n} + \lambda_n b(x,t)\psi_{\lambda_n} - \beta(x,t)\phi_{\lambda_n} = \mu_1(\lambda_n)\psi_{\lambda_n}\end{aligned}\right. & \text{in } \Omega \times \mathbb{R},\\
(\partial_\nu\phi_{\lambda_n}, \partial_\nu\psi_{\lambda_n}) = (0,0) & \text{on } \partial\Omega \times \mathbb{R},\\
(\phi_{\lambda_n}(x,0), \psi_{\lambda_n}(x,0)) = (\phi_{\lambda_n}(x,T), \psi_{\lambda_n}(x,T)) & \text{in } \Omega,
\end{cases}
\tag{2.6}
$$

and

$$(\phi_{\lambda_n}, \psi_{\lambda_n}) \longrightarrow (\Phi, \Psi) \text{ weakly in } L^2(\Omega \times (0,T)) \times L^2(\Omega \times (0,T)) \text{ as } n \to \infty,$$

for some $\Phi,\ \Psi \in L^2(\Omega \times (0, T))$ with $0 \le \Phi,\ \Psi \le 1$ almost everywhere (a.e.) in $\Omega \times (0, T)$. To avoid excessive notation, from now on we will write $\{(\phi_n, \psi_n)\}$ instead of $\{(\phi_{\lambda_n}, \psi_{\lambda_n})\}$.

Step 1. In this step, we prove

$$(\Phi, \Psi) \neq (0, 0) \text{ in } \Omega \times (0, T). \tag{2.7}$$

To do so we argue by contradiction. Suppose that

$$(\phi_n, \psi_n) \longrightarrow (0, 0) \text{ weakly in } L^2(\Omega \times (0,T)) \times L^2(\Omega \times (0,T)) \text{ as } n \to \infty.$$

For any fixed $n \ge 1$, let us consider the auxiliary problem

$$
\begin{cases}
\left.\begin{aligned}\partial_t\varphi - \Delta\varphi + \varphi = (\mu_\infty + 1)\phi_n + \alpha(x,t)\psi_n\\ \partial_t\eta - \Delta\eta + \eta = (\mu_\infty + 1)\psi_n + \beta(x,t)\phi_n\end{aligned}\right. & \text{in } \Omega \times (0,\infty),\\
(\partial_\nu\varphi, \partial_\nu\eta) = (0,0) & \text{on } \partial\Omega \times (0,\infty),\\
(\varphi(x,0), \eta(x,0)) = (1,1) & \text{in } \Omega.
\end{cases}
\tag{2.8}
$$

For each $n \geq 1$, problem (2.8) admits a unique solution, denoted by

$$(\varphi_n, \eta_n) \in C^{2+\theta, 1+\theta/2}(\overline{\Omega} \times (0, \infty)) \times C^{2+\theta, 1+\theta/2}(\overline{\Omega} \times (0, \infty)).$$

Moreover, by the parabolic comparison principle one can easily see that

$$\begin{array}{ll} \phi_n(x,t) \leq \varphi_n(x,t) \leq C_0 \\ \psi_n(x,t) \leq \eta_n(x,t) \leq C_0 \end{array} \quad \text{in } \Omega \times (0,\infty) \ \text{ for all large } n, \tag{2.9}$$

where

$$C_0 = |\mu_\infty| + 1 + \max_{\overline{\Omega} \times [0,T]} \{\alpha,\ \beta\}.$$

On the other hand, following the same argument as in step 1 of the proof of Theorem 3.3 of [9], we can conclude that

$$(\varphi_n, \eta_n) \to (e^{-t}, e^{-t}) \ \text{ in } C^{1,1/2}(\overline{\Omega} \times [0, \hat{T}]) \times C^{1,1/2}(\overline{\Omega} \times [0, \hat{T}])$$

for any $0 < \hat{T} < \infty$. Using (2.9), it therefore follows that, for any integer $k \geq 1$,

$$\begin{aligned} \max\nolimits_{\overline{\Omega} \times [0,T]}(\phi_n(x,t) + \psi_n(x,t)) &= \max\nolimits_{\overline{\Omega} \times [0,T]}(\phi_n(x, t+kT) + \psi_n(x, t+kT)) \\ &\leq \max\nolimits_{\overline{\Omega} \times [0,T]}(\varphi_n(x, t+kT) + \eta_n(x, t+kT)) \\ &\leq 4e^{-kT} \end{aligned}$$

for all large n. By sending $k \to \infty$ we immediately have

$$\lim_{n \to \infty} \max_{\overline{\Omega} \times [0,T]} (\phi_n(x,t) + \psi_n(x,t)) = 0.$$

This is a contradiction with (2.5). So (2.7) holds.

Step 2. (Φ, Ψ) in the range $(x,t) \in \overline{\Omega} \times (0, T^*]$.

In this range, since $a(x,t) = b(x,t) = 0$, (ϕ_n, ψ_n) is the unique solution of

$$\left\{ \begin{array}{ll} \partial_t \varphi - \Delta \varphi = \alpha(x,t)\psi_n + \mu_1(\lambda_n)\phi_n & \multirow{2}{*}{in $\Omega \times (0, T^*]$,} \\ \partial_t \eta - \Delta \eta = \beta(x,t)\phi_n + \mu_1(\lambda_n)\psi_n & \\ (\partial_\nu \varphi, \partial_\nu \eta) = (0,0) & \text{on } \partial\Omega \times (0, T^*], \\ (\varphi(x,0), \eta(x,0)) = (\phi_n(x,0), \psi_n(x,0)) & \text{in } \Omega. \end{array} \right. \tag{2.10}$$

Note that the right-hand side of the equations in (2.10) possesses an L^∞ bound that is independent of n. We use standard parabolic L^p-estimates and conclude that, for any $r > 1$ and $0 < \epsilon_0 < T^*$,

$$||(\phi_n, \psi_n)||_{X_r} \leq C_{\epsilon_0}$$

for some constant $C_{\epsilon_0} > 0$, independent of n, where

$$X_r = W_r^{2,1}(\Omega \times (\epsilon_0, T^*)) \times W_r^{2,1}(\Omega \times (\epsilon_0, T^*)).$$

By taking r large enough and applying the embedding theorems, we see that

$$||(\phi_n, \psi_n)||_Y \leq C = C(\epsilon_0),$$

with

$$Y = C^{1+\theta, (1+\theta)/2}(\overline{\Omega} \times [\epsilon_0, T^*]) \times C^{1+\theta, (1+\theta)/2}(\overline{\Omega} \times [\epsilon_0, T^*]).$$

Thus, using a standard diagonal argument we can pass to a further subsequence such that

$$(\phi_n, \psi_n) \longrightarrow (\hat{\phi}, \hat{\psi}) \text{ in } C^{1,1/2}(\overline{\Omega} \times [\epsilon_0, T^*]) \times C^{1,1/2}(\overline{\Omega} \times [\epsilon_0, T^*]) \text{ as } n \to \infty$$

for every given $\epsilon_0 \in (0, T^*)$. Necessarily $(\hat{\phi}, \hat{\psi}) = (\Phi, \Psi)$. Thus, (Φ, Ψ) solves

$$\text{(2.11)} \qquad \begin{cases} \partial_t \Phi - \Delta\Phi = \alpha(x,t)\Psi + \mu_\infty \Phi & \text{in } \Omega \times (0, T^*], \\ \partial_t \Psi - \Delta\Psi = \beta(x,t)\Phi + \mu_\infty \Psi & \\ (\partial_\nu \Phi, \partial_\nu \Psi) = (0,0) & \text{on } \partial\Omega \times (0, T^*]. \end{cases}$$

By standard parabolic regularity we see that $(\Phi, \Psi) \in Z$, and (Φ, Ψ) satisfies (2.11) in the classical sense, where

$$Z = C^{2+\theta,1+\theta/2}(\overline{\Omega} \times (0, T^*]) \times C^{2+\theta,1+\theta/2}(\overline{\Omega} \times (0, T^*]).$$

Step 3. (Φ, Ψ) in the range $\overline{\Omega} \times (T^*, T]$.

We choose ζ to be a smooth T-periodic function on $\overline{\Omega} \times \mathbb{R}$ with $\zeta = 0$ near $\partial\Omega \times \mathbb{R}$. Multiplying the first equation in (2.6) by ζ and then integrating the resulting equation over $\Omega \times (0, T)$ by parts, we have

$$\int_0^T \int_\Omega \{-\phi_n \zeta_t - \phi_n \Delta\zeta + \lambda_n a(x,t)\phi_n \zeta - \alpha(x,t)\psi_n \zeta\} = \mu_1(\lambda_n) \int_0^T \int_\Omega \phi_n \zeta.$$

Dividing the above identity by λ_n and then letting $n \to \infty$, combined with (2.3) and (2.5), we obtain

$$\int_0^T \int_\Omega a(x,t)\Phi(x,t)\zeta(x,t) = 0.$$

By the arbitrariness of ζ, it is necessary that

$$a(x,t)\Phi(x,t) = 0 \quad \text{a.e. in } \Omega \times (0, T).$$

In view of $a(x,t) > 0$ in $\Omega_+^a \times (T^*, T)$, it follows that

$$\text{(2.12)} \qquad \Phi(x,t) = 0 \quad \text{a.e. in } \Omega_+^a \times (T^*, T).$$

Arguing in a similar manner, from the second equation of (2.6) we can prove

$$\text{(2.13)} \qquad \Psi(x,t) = 0 \quad \text{a.e. in } \Omega_+^b \times (T^*, T).$$

We now restrict ourselves to $\Omega_0^a \times \mathbb{R}$. In this range, ϕ_n satisfies

$$\text{(2.14)} \qquad \begin{cases} \partial_t \phi_n - \Delta\phi_n - \alpha(x,t)\psi_n = \mu_1(\lambda_n)\phi_n & \text{in } \Omega_0^a \times \mathbb{R}, \\ \phi_n(x,0) = \phi_n(x,T) & \text{in } \Omega_0^a. \end{cases}$$

Due to (2.3) and $0 \le \phi_n$, $\psi_n \le 1$, by standard parabolic interior estimates (see, e.g., [14] or [16]) and embedding theorems, for any compact subset $K \subset \Omega_0^a \times \mathbb{R}$ and any $r > 1$, there exists a positive constant $C = C_K$ which is independent of n such that

$$\|\phi_n(x,t)\|_{W_r^{2,1}(K)} \le C.$$

Therefore, by passing to a subsequence of $\{\phi_n\}$ and a diagonal argument, we may assume that

$$\phi_n \to \Phi \quad \text{in } C^{1+\theta,(1+\theta)/2}_{\text{loc}}(\Omega_0^a \times \mathbb{R}),$$

and Φ satisfies, in $W_r^{2,1}$ and, hence, in the classical sense,

$$\partial_t \Phi - \Delta \Phi = \mu_\infty \Phi \quad \text{in } \Omega_0^a \times (T^*, T).$$

Here we have used (2.13) and $\Omega_0^a \subset \Omega_+^b$.

In the following, we will determine the boundary condition satisfied by $\Phi|_{\Omega_0^a \times (T^*,T]}$ over $\partial\Omega_0^a \times (T^*, T]$. Multiplying the first equation in (2.6) by ϕ_n and then integrating over $\Omega \times [0, T]$, we deduce

$$\int_0^T \int_\Omega |\nabla \phi_n|^2 \le \mu_1(\lambda_n) \int_0^T \int_\Omega \phi_n^2 + \int_0^T \int_\Omega \alpha \phi_n \psi_n \le (|\mu_\infty| + \|\alpha\|_\infty) T |\Omega| \text{ for all large } n.$$

Hence, it follows that

$$\int_0^T \int_\Omega |\nabla \phi_n|^2 + \int_0^T \int_\Omega \phi_n^2 \le (1 + |\mu_\infty| + \|\alpha\|_\infty) T |\Omega| \text{ for all large } n. \tag{2.15}$$

That is, $\{\phi_n\}$ is a bounded set in the Hilbert space $W_2^{1,0}(\Omega \times [0, T])$ with inner product

$$(u, v) = \int_0^T \int_\Omega \nabla u \cdot \nabla v + \int_0^T \int_\Omega uv.$$

Hence by passing to a subsequence if necessary, we have

$$\phi_n \to \Phi \quad \text{weakly in } W_2^{1,0}(\Omega \times (0, T)).$$

Thus $\Phi \in W_2^{1,0}(\Omega \times (0, T))$, and so for a.e. $t \in [0, T]$, $\Phi(\cdot, t) \in H^1(\Omega)$. Moreover, due to (2.12), for a.e. $t \in (T^*, T]$, $\Phi(\cdot, t) = 0$ over $\Omega \setminus \Omega_0^a$. Thanks to the smoothness of Ω_0^a, it then follows that

$$\Phi(\cdot, t)|_{\Omega_0^a} \in H_0^1(\Omega_0^a) \quad \text{for a.e.} \quad t \in (T^*, T].$$

As a result, our analysis shows that Φ is the unique weak solution of

$$\begin{cases} \partial_t \phi - \Delta \phi - \mu_\infty \phi = 0 & \text{in } \Omega_0^a \times (T^*, T], \\ \phi = 0 & \text{on } \partial\Omega_0^a \times (T^*, T], \\ \phi(x, T^*) = \Phi(x, T^*) & \text{in } \Omega_0^a. \end{cases} \tag{2.16}$$

By standard regularity theory (see, for instance, [16]), $\Phi \in C^{2+\theta,1+\theta/2}(\overline{\Omega_0^a} \times (T^*, T])$.

Similarly, using the second equation in (2.6), we can show that $\Psi \in C^{2+\theta,1+\theta/2}(\overline{\Omega_0^b} \times (T^*, T])$ solves

$$\begin{cases} \partial_t \psi - \Delta \psi - \mu_\infty \psi = 0 & \text{in } \Omega_0^b \times (T^*, T], \\ \psi = 0 & \text{on } \partial\Omega_0^b \times (T^*, T], \\ \psi(x, T^*) = \Psi(x, T^*) & \text{in } \Omega_0^b, \end{cases} \tag{2.17}$$

and passing to a subsequence,

$$\psi_n \to \Psi \text{ in } C^{(1+\theta),(1+\theta)/2}_{\text{loc}}(\Omega_0^b \times \mathbb{R}).$$

Step 4. $\phi_n \to 0$ uniformly on any compact subset of $\overline{\Omega_+^a} \times (T^*, T]$, and $\psi_n \to 0$ uniformly on any compact subset of $\overline{\Omega_+^b} \times (T^*, T]$.

We will prove this conclusion in three substeps. We first prove a weaker conclusion that $\phi_n \to 0$ uniformly in any compact subset of $(\Omega_+^a \cup \partial\Omega) \times (T^*, T)$, and $\psi_n \to 0$ weakly uniformly in any compact subset of $(\Omega_+^b \cup \partial\Omega) \times (T^*, T)$.

For sufficiently small $\epsilon > 0$, we denote

$$\Omega_{+,\epsilon}^a = \{x \in \overline{\Omega_+^a} : \ d(x, \partial\Omega_+^a \setminus \partial\Omega) > \epsilon\}.$$

We note that $\Omega_{+,\epsilon}^a$ is nonempty and smooth for all small $\epsilon > 0$. Clearly, $\partial\Omega \subset \Omega_{+,\epsilon}^a$. Furthermore, there exists $\sigma = \sigma(\epsilon) > 0$ such that $a(x,t) \geq \sigma$ for $(x,t) \in \Omega_{+,\epsilon}^a \times [T^* + \epsilon, T - \epsilon]$. Therefore, we have

$$\partial_t \phi_n - \Delta\phi_n + \lambda_n \sigma \phi_n \leq |\mu_\infty| + \|\alpha\|_\infty := C_0 \quad \text{in } \Omega_{+,\epsilon}^a \times [T^* + \epsilon, T - \epsilon] \tag{2.18}$$

for all large n.

We now take a function $\ell \in C^2([T^* + \epsilon, T - \epsilon])$ such that $\ell(T^* + \epsilon) = 0$, $\ell(t) = 1$ for $t \in [T^* + 2\epsilon, T - \epsilon]$ and $0 \leq \ell(t) \leq 1$ for $t \in [T^* + \epsilon, T - \epsilon]$. Then, for each $n \geq 1$, we define

$$h_n(x,t) = \ell(t)\phi_n(x,t).$$

By simple calculation, using (2.18), we find that, for all large n, h_n satisfies

$$\begin{cases} \partial_t h_n - \Delta h_n + \lambda_n \sigma h_n \leq C_1 & \text{in } \Omega_{+,\epsilon}^a \times (T^* + \epsilon, T - \epsilon], \\ \partial_\nu h_n = 0 & \text{on } \partial\Omega \times (T^* + \epsilon, T - \epsilon], \\ h_n \leq 1 & \text{on } (\partial\Omega_{+,\epsilon}^a \setminus \partial\Omega) \times (T^* + \epsilon, T - \epsilon], \\ h_n(x, T^* + \epsilon) = 0 & \text{in } \Omega_{+,\epsilon}^a, \end{cases} \tag{2.19}$$

where $C_1 = C_1(\epsilon) =: C_0 + \max_{[T^*+\epsilon, T-\epsilon]} |\ell'(t)| > 0$. Consequently, a simple comparison consideration shows that the unique solution u_n of the elliptic problem

$$\begin{cases} -\Delta u_n + \lambda_n \sigma u_n = C_1 & \text{in } \Omega_{+,\epsilon}^a, \\ \partial_\nu u_n = 0 & \text{on } \partial\Omega, \\ u_n = 1 & \text{on } (\partial\Omega_{+,\epsilon}^a \setminus \partial\Omega) \end{cases} \tag{2.20}$$

satisfies

$$h_n(x,t) \leq u_n(x) \quad \text{for } (x,t) \in \Omega_{+,\epsilon}^a \times [T^* + \epsilon, T - \epsilon].$$

In particular, as $\ell(t) = 1$ for $t \in [T^* + 2\epsilon, T - \epsilon]$, we have

$$\phi_n(x,t) \leq u_n(x) \quad \text{for } (x,t) \in \Omega_{+,\epsilon}^a \times [T^* + 2\epsilon, T - \epsilon].$$

We will show that $u_n \to 0$ uniformly on $\overline{\Omega_{+,2\epsilon}^a}$, and clearly our weaker conclusion on ϕ_n will follow from this claim due to the definition of $\Omega_{+,\epsilon}^a$. We choose $\overline{v} \in C^2(\overline{\Omega_{+,\epsilon}^a})$ such that

$$\begin{cases} \overline{v} = 0 & \text{on } \overline{\Omega_{+,2\epsilon}^a}, \\ \overline{v} = 1 & \text{on a small neighborhood of } \partial\Omega_{+,\epsilon}^a \setminus \partial\Omega, \\ 0 \leq \overline{v} \leq 1 & \text{on } \overline{\Omega_{+,\epsilon}^a}. \end{cases}$$

It is easy to check that, for all large n, $\overline{v}_n := \lambda_n^{-1/2} + \overline{v}$ is a supersolution to (2.20) and 0 is a subsolution. So the uniqueness of solutions to (2.20) ensures $u_n \le \overline{v}_n$ in $\overline{\Omega^a_{+,\epsilon}}$. Hence, we have

$$u_n(x,t) \le \lambda_n^{-\frac{1}{2}} \to 0 \quad \text{in } \overline{\Omega^a_{+,2\epsilon}}, \quad \text{as } n \to \infty,$$

which is what we wanted. This proves our weaker conclusion on ϕ_n. The same argument carries over to prove the weaker conclusion for ψ_n.

Second, we prove that on any compact subset of $(\overline{\Omega} \setminus (\Omega^a_0 \cup \Omega^b_0)) \times (T^*, T]$, $(\phi_n, \psi_n) \to (0,0)$ uniformly.

Recall that $(\phi_n, \psi_n) \to (\Phi, \Psi)$ weakly in $L^2(\Omega \times [0,T]) \times L^2(\Omega \times [0,T])$, $\Phi = 0$ over $\Omega^a_+ \times (T^*, T]$, and $\Psi = 0$ over $\Omega^b_+ \times (T^*, T]$. We now define

$$\xi_n(t) = \int_{\Omega^a_+} \phi_n(x,t)dx, \qquad \vartheta_n(t) = \int_{\Omega^b_+} \psi_n(x,t)dx.$$

Then $(\xi_n, \vartheta_n) \to (0,0)$ in $L^1([T^*, T])$. Hence $(\xi_n, \vartheta_n) \to (0,0)$ a.e. in $[T^*, T]$. Thus we can find a sequence t_k decreasing to T^* such that $(\xi_n(t_k), \vartheta_n(t_k)) \to (0,0)$ as $n \to \infty$ for each $k \ge 1$. So we have

$$0 \le \int_{\Omega^a_+} \phi_n(x,t_k)^2 dx \le \int_{\Omega^a_+} \phi_n(x,t_k)dx \to 0$$

and

$$0 \le \int_{\Omega^b_+} \psi_n(x,t_k)^2 dx \le \int_{\Omega^b_+} \psi_n(x,t_k)dx \to 0,$$

as $n \to \infty$ for each $k \ge 1$.

In view of what we have proved in the last part of Step 3, for any given small $\delta > 0$ and $k \ge 1$, we can find $\sigma > 0$ small such that, for all large n,

$$0 \le \Phi < \delta \text{ in } (\overline{\Omega} \setminus \Omega^a_\sigma) \times [t_k, T], \qquad 0 < \phi_n < \delta \text{ on } \partial\Omega^a_\sigma \times [t_k, T],$$

and

$$0 \le \Psi < \delta \text{ in } (\overline{\Omega} \setminus \Omega^b_\sigma) \times [t_k, T], \qquad 0 < \psi_n < \delta \text{ on } \partial\Omega^b_\sigma \times [t_k, T],$$

where

$$\Omega^a_\sigma = \{x \in \Omega^a_0 : \ d(x, \partial\Omega^a_0) > \sigma\} \ \text{ and } \ \Omega^b_\sigma = \{x \in \Omega^b_0 : \ d(x, \partial\Omega^b_0) > \sigma\}.$$

We now take a sequence $\{\epsilon_m\}$ which strictly decreases to 0 as $m \to \infty$. Since $\partial\Omega^b_\sigma \subset \Omega^a_+$ and $\partial\Omega^a_\sigma \subset \Omega^b_+$, by the weaker conclusions proved above for ϕ_n and ψ_n, for any given small $\delta > 0$, $k \ge 1$, and $m \ge 1$, we can find a large $n(m)$ with the sequence $\{n(m)\}$ converging to infinity as $m \to \infty$ such that

$$\phi_{n(m)}(x,t) < \delta \text{ on } \partial\Omega^b_\sigma \times [t_k, T - \epsilon_m], \qquad \psi_{n(m)}(x,t) < \delta \text{ on } \partial\Omega^a_\sigma \times [t_k, T - \epsilon_m]$$

for each $n(m)$.

Let $\xi_{n(m)}(t)$ be a smooth nondecreasing function such that

$$\xi_{n(m)}(t) = \delta \text{ for } t \in [t_k, T - \epsilon_{m-1}], \qquad \xi_{n(m)}(t) = 1 \text{ for } t \in [T - \epsilon_m, T].$$

We then consider the problem

$$(2.21)\quad \begin{cases} \partial_t g - \Delta g = \mu_\infty \phi_{n(m)} + \|\alpha\|_\infty \psi_{n(m)} & \text{in } [\Omega \setminus (\overline{\Omega^a_\sigma} \cup \overline{\Omega^b_\sigma})] \times (t_k, T], \\ \partial_\nu g = 0 & \text{on } \partial\Omega \times (t_k, T], \\ g = \xi_{n(m)} & \text{on } (\partial\Omega^a_\sigma \cup \partial\Omega^b_\sigma) \times (t_k, T], \\ g(x, t_k) = \phi_{n(m)}(x, t_k) & \text{in } \Omega \setminus (\overline{\Omega^a_\sigma} \cup \overline{\Omega^b_\sigma}). \end{cases}$$

Let $g_{n(m)}$ denote the unique solution of (2.21). Then a simple comparison consideration gives that, for all large $n(m)$,

$$\phi_{n(m)} \le g_{n(m)} \quad \text{in } \ \Omega \setminus (\overline{\Omega^a_\sigma} \cup \overline{\Omega^b_\sigma}) \times (t_k, T].$$

Moreover, by using L^p-estimates away from $\{t = t_k\}$ and $\{t = T\}$, and using a diagonal process, we find that, as $m \to \infty$, by passing to a subsequence,

$$g_{n(m)} \to g^* \quad \text{in} \quad C^{1+\theta,(1+\theta)/2}(\overline{\Omega} \setminus (\Omega^a_\sigma \cup \Omega^b_\sigma)) \times [\tau, \tilde{T}]) \ \text{ for all } [\tau, \tilde{T}] \subset (t_k, T),$$

and $g = g^*$ is a weak solution of

$$(2.22)\quad \begin{cases} \partial_t g - \Delta g = \mu_\infty \Phi + \|\alpha\|_\infty \Psi & \text{in } \Omega \setminus (\overline{\Omega^a_\sigma} \cup \overline{\Omega^b_\sigma}) \times (t_k, T], \\ \partial_\nu g = 0 & \text{on } \partial\Omega \times (t_k, T], \\ g = \delta & \text{on } (\partial\Omega^a_\sigma \cup \partial\Omega^b_\sigma) \times (t_k, T], \\ g(x, t_k) = g_0(x) & \text{in } \Omega \setminus (\overline{\Omega^a_\sigma} \cup \overline{\Omega^b_\sigma}), \end{cases}$$

where $g_0(x) = 0$ in $\Omega \setminus \overline{\Omega^a_0}$, and $g_0(x) = \Phi(x, t_k)$ for $x \in \Omega^a_0 \setminus \Omega^a_\sigma$. Thus $g_0 \le \delta$ in $\Omega \setminus (\Omega^a_\sigma \cup \Omega^b_\sigma)$. Hence, we find that

$$\hat{g}(x, t) = [(|\mu_\infty| + \|\alpha\|_\infty)t + 1]\delta$$

is a supersolution of (2.22). It follows that

$$g^*(x, t) \le C\delta \quad \text{in } \overline{\Omega} \setminus (\Omega^a_\sigma \cup \Omega^b_\sigma) \times [t_k, T],$$

with $C = (|\mu_\infty| + \|\alpha\|_\infty)T + 1$.

Applying interior parabolic estimates to (2.21) we find that there exists C_0 independent of $n(m)$ such that $\{g_{n(m)}\}$ over $\overline{\Omega} \setminus (\Omega^a_{\sigma/2} \cup \Omega^b_{\sigma/2}) \times [t_{k-1}, T]$ has a uniform bound for its $W^{2,1}_r$ norm for any $r > 1$. It follows that $g_{n(m)} \to g^*$ in the $C^{(1+\theta),(1+\theta)/2}$ norm over this set. As a result, for all large $n(m)$,

$$\phi_{n(m)} \le g_{n(m)} \le g^* + \delta \le (C + 1)\delta \quad \text{in } \overline{\Omega} \setminus (\Omega^a_{\sigma/2} \cup \Omega^b_{\sigma/2}) \times [t_{k-1}, T].$$

Obviously, this implies $\phi_{n(m)} \to 0$ uniformly in $\overline{\Omega} \setminus (\Omega^a_0 \cup \Omega^b_0) \times [t_{k-1}, T]$ as $m \to \infty$ for each $k \ge 2$. Since $t_k \to T^*$, this implies that a subsequence of $\{\phi_n\}$, and hence $\{\phi_n\}$ itself, converges to 0 uniformly on any compact subset of $\overline{\Omega} \setminus (\Omega^a_0 \cup \Omega^b_0) \times (T^*, T]$. The conclusion for ψ_n can be proved similarly.

Third, to complete the proof of Step 4, it remains to show that $\phi_n \to 0$ uniformly on any compact subset of $\overline{\Omega^b_0} \times (T^*, T]$, and $\psi_n \to 0$ uniformly on any compact subset of $\overline{\Omega^a_0} \times (T^*, T]$.

Again we give only the proof for ϕ_n, since that for ψ_n is similar. For any given small $\epsilon > 0$, the above proved conclusion tells us that there exists $n_0 = n_0(\epsilon)$ large such that

$$\phi_n \le \epsilon \text{ on } \partial\Omega^b_0 \times [T - \epsilon, T] \text{ for } n \ge n_0.$$

Since $\overline{\Omega_0^b} \subset \Omega_+^a$, our earlier weaker conclusion infers that, by enlarging n_0 if necessary,

$$\phi_n(x,t) \le \epsilon \text{ for } (x,t) \in \overline{\Omega_0^b} \times [T^* + \epsilon, T - \epsilon],\ n \ge n_0. \tag{2.23}$$

It follows that, for $n \ge n_0$,

$$\begin{cases} \partial_t \phi_n - \Delta \phi_n \le |\mu_\infty| + \|\alpha\|_\infty & \text{in } \Omega_0^b \times [T-\epsilon, T], \\ \phi_n \le \epsilon & \text{on } \partial\Omega_0^b \times [T-\epsilon, T], \\ \phi_n \le \epsilon & \text{in } \Omega_0^b \times \{T-\epsilon\}. \end{cases} \tag{2.24}$$

A simple comparison consideration applied to (2.24) then gives

$$\phi_n(x,t) \le \epsilon + (|\mu_\infty| + \|\alpha\|_\infty)(t - T + \epsilon) \ \text{ in } \overline{\Omega_0^b} \times [T-\epsilon, T]$$

for all $n \ge n_0$. Therefore, we get

$$\phi_n \le \epsilon(1 + |\mu_\infty| + \|\alpha\|_\infty) \ \text{ in } \overline{\Omega_0^b} \times [T-\epsilon, T] \ \text{ for all } n \ge n_0.$$

This, combined with (2.23), indicates that

$$\phi_n \le \epsilon(1 + |\mu_\infty| + \|\alpha\|_\infty) \ \text{ in } \overline{\Omega_0^b} \times [T^* + \epsilon, T] \ \text{ for all } \ n \ge n_0,$$

which implies that $\phi_n \to 0$ uniformly on any compact subset of $\overline{\Omega_0^b} \times (T^*, T]$. The proof of Step 4 is thus complete.

Step 5. Summary and positivity of (Φ, Ψ).

According to the above analysis, we have proved that, by passing to a subsequence,

- over $\overline{\Omega} \times (0, T^*]$, $(\phi_n, \psi_n) \to (\Phi, \Psi)$ locally in the $C^{2+\theta, 1+\theta/2}$ norm;
- over $\Omega_0^a \times \mathbb{R}$, $\phi_n \to \Phi$ locally in the $C^{1+\theta, (1+\theta)/2}$ norm;
- over $\Omega_0^b \times \mathbb{R}$, $\psi_n \to \Psi$ locally in the $C^{1+\theta, (1+\theta)/2}$ norm;
- over $\overline{\Omega_+^a} \times (T^*, T]$, $\phi_n \to 0 = \Phi$ locally uniformly;
- over $\overline{\Omega_+^b} \times (T^*, T]$, $\psi_n \to 0 = \Psi$ locally uniformly;
- $\Phi \in C^{2+\theta, 1+\theta/2}(\overline{\Omega_0^a} \times (T^*, T])$, $\Phi = 0$ on $\partial\Omega_0^a \times (T^*, T]$;
- $\Psi \in C^{2+\theta, 1+\theta/2}(\overline{\Omega_0^b} \times (T^*, T])$, $\Psi = 0$ on $\partial\Omega_0^b \times (T^*, T]$.

The above properties and the argument in Step 4 also tell us that, in the $C(\overline{\Omega})$ norm,

$$\phi_n(\cdot, 0) = \phi_n(\cdot, T) \to \Phi(\cdot, T) = \Phi(\cdot, 0) \ \text{ and } \ \psi_n(\cdot, 0) = \psi_n(\cdot, T) \to \Psi(\cdot, T) = \Psi(\cdot, 0).$$

In addition, from (2.11), it follows that $(\phi, \psi) = (\Phi, \Psi)$ is the unique classical solution of the problem

$$\begin{cases} \begin{array}{l} \partial_t \phi - \Delta \phi = \alpha(x,t)\psi + \mu_\infty \phi \\ \partial_t \psi - \Delta \psi = \beta(x,t)\phi + \mu_\infty \psi \end{array} & \text{in } \Omega \times (0, T^*], \\ (\partial_\nu \phi, \partial_\nu \psi) = (0,0) & \text{on } \partial\Omega \times (0, T^*], \\ (\phi(x,0), \psi(x,0)) = (\Phi(x,0), \Psi(x,0)) & \text{in } \Omega. \end{cases} \tag{2.25}$$

From (2.16) and (2.17), it further follows that $(\phi, \psi) = (\Phi, \Psi)$ is the unique weak solution of

$$\begin{cases} \partial_t \phi - \Delta \phi = \mu_\infty \phi & \text{in } \Omega_0^a \times (T^*, T], \\ \partial_t \psi - \Delta \psi = \mu_\infty \psi & \text{in } \Omega_0^b \times (T^*, T], \\ \phi = 0 & \text{on } \partial\Omega_0^a \times (T^*, T], \\ \psi = 0 & \text{on } \partial\Omega_0^b \times (T^*, T], \\ \phi(x, T^*) = \Phi(x, T^*) & \text{in } \Omega_0^a, \\ \psi(x, T^*) = \Psi(x, T^*) & \text{in } \Omega_0^b. \end{cases} \tag{2.26}$$

We now use the strong maximum principle to show that

$$\Phi > 0 \quad \text{in } \{\overline{\Omega} \times (0, T^*]\} \cup \{\Omega_0^a \times (T^*, T]\}$$

and

$$\Psi > 0 \quad \text{in } \{\overline{\Omega} \times (0, T^*]\} \cup \{\Omega_0^b \times (T^*, T]\}.$$

We first claim $\Phi(\cdot, 0) + \Psi(\cdot, 0) \not\equiv 0$ in Ω. Otherwise $\Phi(\cdot, 0) = \Psi(\cdot, 0) \equiv 0$, and so $(\phi, \psi) = (0, 0)$ is the unique solution of (2.25). In particular, this implies $(\Phi(\cdot, T^*), \Psi(\cdot, T^*)) = (0, 0)$. Hence, the unique solution of (2.26) must be $(\phi, \psi) = (\Phi, \Psi) = (0, 0)$.

Furthermore, we have already shown that $\Phi = 0$ over $(\Omega \setminus \Omega_0^a) \times (T^*, T]$ and $\Psi = 0$ over $(\Omega \setminus \Omega_0^b) \times (T^*, T]$. Hence $(\Phi, \Psi) \equiv (0, 0)$ over $\Omega \times [0, T]$, contradicting our conclusion in Step 1 that $(\Phi, \Psi) \not\equiv (0, 0)$, which therefore verifies $\Phi(\cdot, 0) + \Psi(\cdot, 0) \not\equiv 0$ in Ω.

Notice that (2.25) is a cooperative system and $(0, 0)$ is a strict subsolution. So the well-known parabolic comparison principle for cooperative systems infers that $\Phi(x, t) > 0$ and $\Psi(x, t) > 0$ for $(x, t) \in \overline{\Omega} \times (0, T^*]$. In particular, $\Phi(x, T^*) > 0$ and $\Psi(x, T^*) > 0$ on $\overline{\Omega}$. Thus, applying the parabolic comparison principle to the decoupled system (2.26), we can conclude $\Phi > 0$ in $\Omega_0^a \times (T^*, T]$ and $\Psi > 0$ in $\Omega_0^b \times (T^*, T]$. Our analysis also indicates that $\Phi(x, t)$ has a jumping discontinuity across $(\overline{\Omega} \setminus \Omega_0^a) \times \{T^*\}$ and $\Psi(x, t)$ has a jumping discontinuity across $(\overline{\Omega} \setminus \Omega_0^b) \times \{T^*\}$.

From Step 1 through to Step 5, we see that (Φ, Ψ) satisfies (2.4) and (1.5). The proof of Theorem 2.2 is now complete. $\square$

Consider the eigenvalue problem

$$(2.27) \qquad \begin{cases} \partial_t \phi - \Delta \phi - \alpha(x, t) \chi_{\Omega \times [0, T^*]} \psi = \mu \phi & \text{in } \Sigma_0^a, \\ \partial_t \psi - \Delta \psi - \beta(x, t) \chi_{\Omega \times [0, T^*]} \phi = \mu \psi & \text{in } \Sigma_0^b, \\ \partial_\nu \phi = \partial_\nu \psi = 0 & \text{on } \partial\Omega \times (0, T^*], \\ \phi = 0 & \text{on } \partial\Omega_0^a \times (T^*, T], \\ \psi = 0 & \text{on } \partial\Omega_0^b \times (T^*, T], \\ \phi(x, 0) = \phi(x, T) & \text{in } \Omega_0^a, \\ \psi(x, 0) = \psi(x, T) & \text{in } \Omega_0^b. \end{cases}$$

THEOREM 2.3. *The eigenvalue problem* (2.27) *admits a principal eigenvalue* $\mu = \mu_\infty > 0$ *which corresponds to a positive eigenfunction* (ϕ, ψ) *satisfying* (2.4). *Conversely, if* (2.27) *has a solution* (ϕ, ψ) *satisfying* (2.4), *then necessarily* $\mu = \mu_\infty$, *and* (ϕ, ψ) *is unique up to a constant multiple.*

Proof. For any given $(g_0, h_0) \in E := C_0^1(\overline{\Omega_0^a}) \times C_0^1(\overline{\Omega_0^b})$, we extend g_0 and h_0 by 0 to $\overline{\Omega}$ and denote the extended function pair by $(\tilde{g}, \tilde{h})$. Thus, $(\tilde{g}, \tilde{h}) \in C(\overline{\Omega}) \times C(\overline{\Omega})$. Let (g, h) be the unique solution of the problem

$$(2.28) \qquad \begin{cases} \left.\begin{array}{l} \partial_t \phi - \Delta \phi - \alpha(x, t) \psi = 0 \\ \partial_t \psi - \Delta \psi - \beta(x, t) \phi = 0 \end{array}\right. & \text{in } \Omega \times (0, T^*], \\ (\partial_\nu \phi, \partial_\nu \psi) = (0, 0) & \text{on } \partial\Omega \times (0, T^*], \\ \phi(x, 0) = \tilde{g} & \text{in } \Omega, \\ \psi(x, 0) = \tilde{h} & \text{in } \Omega. \end{cases}$$

Clearly, g, $h \in C^{2+\theta,1+\theta/2}(\overline{\Omega}\times(0,T^*])\cap C(\overline{\Omega}\times[0,T^*])$. We then consider the problem

$$(2.29)\qquad \begin{cases} \partial_t\phi - \Delta\phi = 0 & \text{in } \Omega_0^a \times (T^*,T], \\ \partial_t\psi - \Delta\psi = 0 & \text{in } \Omega_0^b \times (T^*,T], \\ \phi = 0 & \text{on } \partial\Omega_0^a \times (T^*,T], \\ \psi = 0 & \text{on } \partial\Omega_0^b \times (T^*,T], \\ \phi(x,T^*) = g(x,T^*) & \text{in } \Omega_0^a, \\ \psi(x,T^*) = h(x,T^*) & \text{in } \Omega_0^b. \end{cases}$$

By the well-known existence result for parabolic equations, we know that this problem has a unique solution $(w,z) \in C^\theta((T^*,T],E_1)\cap C^{1+\theta}((T^*,T],E_0)$, where $E_0 = L^r(\Omega_0^a)\times L^r(\Omega_0^b)$ and $E_1 = W_0^{2,r}(\Omega_0^a)\times W_0^{2,r}(\Omega_0^b)$, $r > 1$. We may choose r large enough such that $W_0^{2,r}(\Omega_0^a)\times W_0^{2,r}(\Omega_0^b)$ embeds compactly into E.

With (g_0,h_0) and (w,z) as above, we define the operator $\mathcal{K} : E \to E$ by

$$\mathcal{K}(g_0,h_0) = (w,z)(\cdot,T).$$

It is easily seen that $\mathcal{K}$ is a linear operator. We show next that $\mathcal{K}$ is compact. Suppose that $\{(g_n,h_n)\}$ is a bounded sequence in E. Then there exists $C > 0$ such that $-C \le \tilde{g}_n$, $\tilde{h}_n \le C$ in Ω, where $\tilde{g}_n$, $\tilde{h}_n$ are the extended functions as before. Denote by (g_n,h_n) the unique solution of (2.28), with $(\tilde{g},\tilde{h})$ replaced by $(\tilde{g}_n,\tilde{h}_n)$. Clearly,

$$\begin{aligned}&\left(-Ce^{t\max_{\overline{\Omega}\times[0,T]}(\alpha+\beta)}, -Ce^{t\max_{\overline{\Omega}\times[0,T]}(\alpha+\beta)}\right)\\ \text{and } &\left(Ce^{t\max_{\overline{\Omega}\times[0,T]}(\alpha+\beta)}, Ce^{t\max_{\overline{\Omega}\times[0,T]}(\alpha+\beta)}\right)\end{aligned}$$

are a pair of sub-supersolutions of (2.28); then a comparison consideration gives

$$-Ce^{t\max_{\overline{\Omega}\times[0,T]}(\alpha+\beta)} \le g_n,\ h_n \le Ce^{t\max_{\overline{\Omega}\times[0,T]}(\alpha+\beta)} \quad \text{in } \Omega\times[0,T^*].$$

In particular, we have

$$-Ce^{T^*\max_{\overline{\Omega}\times[0,T]}(\alpha+\beta)} \le g_n(x,T^*) \le Ce^{T^*\max_{\overline{\Omega}\times[0,T]}(\alpha+\beta)} \quad \text{in } \Omega_0^a$$

and

$$-Ce^{T^*\max_{\overline{\Omega}\times[0,T]}(\alpha+\beta)} \le h_n(x,T^*) \le Ce^{T^*\max_{\overline{\Omega}\times[0,T]}(\alpha+\beta)} \quad \text{in } \Omega_0^b.$$

Again, a simple comparison consideration, as applied to (2.29), yields

$$-Ce^{T^*\max_{\overline{\Omega}\times[0,T]}(\alpha+\beta)} \le w_n \le Ce^{T^*\max_{\overline{\Omega}\times[0,T]}(\alpha+\beta)} \quad \text{in } \Omega_0^a\times(T^*,T]$$

and

$$-Ce^{T^*\max_{\overline{\Omega}\times[0,T]}(\alpha+\beta)} \le z_n \le Ce^{T^*\max_{\overline{\Omega}\times[0,T]}(\alpha+\beta)} \quad \text{in } \Omega_0^b\times(T^*,T],$$

where (w_n,z_n) is the solution of (2.29), with $(g(x,T^*),h(x,T^*))$ replaced by $(g_n(x,T^*), h_n(x,T^*))$. We may now apply the standard parabolic L^p-estimates to the equation satisfied by (w_n,z_n) to conclude that, for any $r > 1$ and $\tau \in (T^*,T)$, there exists $C_0 > 0$ such that

$$\|w_n\|_{W_r^{2,1}(\Omega_0^a\times[\tau,T])} + \|z_n\|_{W_r^{2,1}(\Omega_0^b\times[\tau,T])} \le C_0 \quad \text{for all } n \ge 1.$$

Moreover, by the Sobolev embedding result in [14] (Lemma II 3.3) we deduce

$$\|w_n\|_{C^{1+\theta,(1+\theta)/2}(\overline{\Omega_0^a}\times[\tau,T])} + \|z_n\|_{C^{1+\theta,(1+\theta)/2}(\overline{\Omega_0^b}\times[\tau,T])} \le C$$

for some constant C and all $n \ge 1$. In particular, we have

$$\|w_n(\cdot,T)\|_{C^{1+\theta}(\overline{\Omega_0^a})} + \|z_n(\cdot,T)\|_{C^{1+\theta}(\overline{\Omega_0^b})} \le C.$$

Therefore, it follows from the compact embedding theorem that $\{(w_n(\cdot,T), z_n(\cdot,T))\}$ has a convergent subsequence in E. This proves the compactness of $\mathcal{K}$.

Let P denote the cone of nonnegative functions in E and P^o denote the interior of P. It is easily seen that P is reproducing, namely, $E = P - P$. We show that

$$\mathcal{K} \text{ is strongly positive, i.e., } \mathcal{K}(P \setminus \{(0,0)\}) \subset P^o.$$

Indeed, if $g, h \ge 0$ and $(g,h) \not\equiv (0,0)$ in E, then due to the comparison principle for cooperative systems we know that the unique solution (g,h) of (2.28) satisfies $g, h > 0$ in $\overline{\Omega} \times (0, T^*]$. Applying the comparison principle to (2.29) again, we find that the unique solution (w,z) of (2.29) satisfies

$$w > 0 \quad \text{in } \Omega_0^a \times (T^*, T] \quad \text{and} \quad z > 0 \quad \text{in } \Omega_0^b \times (T^*, T].$$

Moreover, thanks to the Hopf boundary lemma we also know that

$$\partial_{\nu_0^a} w < 0 \quad \text{on } \partial\Omega_0^a \times (T^*, T] \quad \text{and} \quad \partial_{\nu_0^b} z < 0 \quad \text{on } \partial\Omega_0^b \times (T^*, T],$$

where ν_0^a and ν_0^b, respectively, denote the unit outward normal of $\partial\Omega_0^a$ and $\partial\Omega_0^b$. In particular we have

$$w(x,T) > 0 \quad \text{in } \Omega_0^a, \quad \partial_{\nu_0^a} w(x,T) < 0 \quad \text{on } \partial\Omega_0^a$$

and

$$z(x,T) > 0 \quad \text{in } \Omega_0^b, \quad \partial_{\nu_0^b} z(x,T) < 0 \quad \text{on } \partial\Omega_0^b.$$

This implies that $(w(\cdot,T), z(\cdot,T)) \in P^o$. Hence $\mathcal{K}$ is strongly positive.

With the above properties for $\mathcal{K}$, we can now apply the Krein–Rutman theorem to conclude that the spectral radius $r(\mathcal{K})$ of $\mathcal{K}$ is positive and corresponds to an eigenvector $\varpi_0 \in P^o$. Moreover, if $\mathcal{K}\varpi_1 = r\varpi_1$ for some $\varpi_1 \in P^o$, then necessarily $r = r(\mathcal{K})$ and $\varpi_1 = c\varpi_0$ for some constant c.

Let us now see how $\mathcal{K}$ and $r(\mathcal{K})$ are related to the eigenvalue problem (2.27). Let $\varpi_0 \in P^o$ be an eigenvector of $\mathcal{K}$ corresponding to $r(\mathcal{K})$: $\mathcal{K}\varpi_0 = r(\mathcal{K})\varpi_0$. Let $U_0(t)$ be defined by

$$U_0(t) = (g(\cdot,t)|_{\overline{\Omega}},\ h(\cdot,t)|_{\overline{\Omega}}) \ \text{ for } t \in [0, T^*]$$

and

$$U_0(t) = (w(\cdot,t)|_{\overline{\Omega_0^a}},\ z(\cdot,t)|_{\overline{\Omega_0^b}}) \ \text{ for } t \in (T^*, T],$$

where (g,h) denotes the unique solution of (2.28) with ϖ_0 in place of (g_0, h_0), and (w,z) is the unique solution of (2.29).

By definition, $U_0(T) = \mathcal{K}\varpi_0 = r(\mathcal{K})\varpi_0$ in $\overline{\Omega_0^a} \times \overline{\Omega_0^b}$. Denote $(\hat{\Phi}(\cdot,t), \hat{\Psi}(\cdot,t)) = e^{\mu t}U_0(t)$ and $\mu = -\frac{1}{T}\ln r(\mathcal{K})$. Then, a simple analysis shows that $((\hat{\Phi}, \hat{\Psi}), \mu)$ satisfies (2.4) and (2.27).

Conversely, if (2.27) has a solution (Φ, Ψ) satisfying (2.4), then let

$$r_0 = e^{-\mu T} \quad \text{and} \quad (\phi, \psi) = e^{-\mu t}(\Phi, \Psi).$$

We easily see (ϕ, ψ) satisfies (2.28) with $(\tilde{g}(x,0), \tilde{h}(x,0))$ replaced by $(\phi(x,0), \psi(x,0))$, and it satisfies (2.29) with $(g(x,T^*), h(x,T^*))$ replaced by $(\phi(x,T^*), \psi(x,T^*))$. Moreover,

$$\begin{aligned}\mathcal{K}(\phi(\cdot,0), \psi(\cdot,0)) &= (\phi(\cdot,T), \psi(\cdot,T)) = e^{-\mu T}(\Phi(\cdot,T), \Psi(\cdot,T)) \\ &= r_0(\Phi(\cdot,0), \Psi(\cdot,0)) = r_0(\phi(\cdot,0), \psi(\cdot,0)).\end{aligned}$$

We also note that $(\phi(\cdot,0)|_{\overline{\Omega_0^a}}, \psi(\cdot,0)|_{\overline{\Omega_0^b}}) = (\Phi(\cdot,T), \Psi(\cdot,T)) \in P^o$ and it satisfies the equation $\mathcal{K}\varpi = r_0\varpi$ in E. Thus, due to the Krein–Rutman theorem, we necessarily have $r_0 = r(\mathcal{K})$ and $(\phi(\cdot,0)|_{\overline{\Omega_0^a}}, \psi(\cdot,0)|_{\overline{\Omega_0^b}}) = c\varpi_0$ for some constant c. It then follows that $(\Phi, \Psi) = c(\hat{\Phi}, \hat{\Psi})$. The proof is now complete. ◻

In Lemma 2.1 we have obtained an explicit upper bound for the principal eigenvalue μ_∞ of (2.27); we now deduce an explicit lower bound for μ_∞.

THEOREM 2.4. *There holds*

$$\mu_\infty \geq \Big(1 - \frac{T^*}{T}\Big)\min\{\sigma_1^a,\ \sigma_1^b\} - \max_{\overline{\Omega}\times[0,T]}\{\alpha,\ \beta\}.$$

Proof. Let $\mu_1(\lambda)$ be the principal eigenvalue in (2.1). Denote $\gamma(x,t) = \min\{a(x,t), b(x,t)\}$ on $\overline{\Omega}\times[0,T]$. Then, γ is a continuous and T-periodic function on $\overline{\Omega}\times\mathbb{R}$, $\gamma = 0$ on $(\overline{\Omega_0^a} \cup \overline{\Omega_0^b}) \times [0,T]$, and $\gamma > 0$ in $(\overline{\Omega} \setminus (\overline{\Omega_0^a} \cup \overline{\Omega_0^b})) \times [0,T]$. Then, by the monotonicity of the principal eigenvalue on the weight functions, we have

$$\mu_1(\lambda) \geq \overline{\mu}_1(\lambda), \tag{2.30}$$

where $\overline{\mu}_1(\lambda)$ is the principal eigenvalue of the problem

$$\begin{cases} \left.\begin{aligned}&\partial_t\phi - \Delta\phi + \lambda\gamma(x,t)\phi - \alpha(x,t)\psi = \mu\phi \\ &\partial_t\psi - \Delta\psi + \lambda\gamma(x,t)\psi - \beta(x,t)\phi = \mu\psi\end{aligned}\right. & \text{in } \Omega\times\mathbb{R}, \\ (\partial_\nu\phi, \partial_\nu\psi) = (0,0) & \text{on } \partial\Omega\times\mathbb{R}, \\ (\phi(x,0), \psi(x,0)) = (\phi(x,T), \psi(x,T)) & \text{in } \Omega. \end{cases} \tag{2.31}$$

Let $(\tilde{\phi}, \tilde{\psi})$ be a principal eigenfunction corresponding to $\overline{\mu}_1(\lambda)$. It is easy to check that $\rho = \tilde{\phi} + \tilde{\psi}$ satisfies

$$\begin{cases} \partial_t\rho - \Delta\rho + \lambda\gamma(x,t)\rho \leq (\max_{\overline{\Omega}\times[0,T]}\{\alpha, \beta\} + \overline{\mu}_1(\lambda))\rho & \text{in } \Omega\times\mathbb{R}, \\ \partial_\nu\rho = 0 & \text{on } \partial\Omega\times\mathbb{R}, \\ \rho(x,0) = \rho(x,T) & \text{in } \Omega. \end{cases} \tag{2.32}$$

We now denote by $\underline{\mu}_1(\lambda)$ the principal eigenvalue of

$$\begin{cases} \partial_t w - \Delta w + \lambda\gamma(x,t)w = \mu w & \text{in } \Omega\times\mathbb{R}, \\ \partial_\nu w = 0 & \text{on } \partial\Omega\times\mathbb{R}, \\ w(x,0) = w(x,T) & \text{in } \Omega. \end{cases} \tag{2.33}$$

It is well known (see, for instance, [17]) that $\underline{\mu}_1(\lambda)$ is also the principal eigenvalue of the adjoint problem to (2.33):

$$\begin{cases} -\partial_t w - \Delta w + \lambda\gamma(x,t)w = \mu w & \text{in } \Omega\times\mathbb{R}, \\ \partial_\nu w = 0 & \text{on } \partial\Omega\times\mathbb{R}, \\ w(x,0) = w(x,T) & \text{in } \Omega. \end{cases} \tag{2.34}$$

Take a principal eigenfunction $\underline{w}$ associated to $\underline{\mu}_1(\lambda)$ in (2.34). We then multiply the inequality in (2.32) by $\underline{w}$ and integrate over $\Omega\times(0,T)$ to obtain

$$\max_{\overline{\Omega}\times[0,T]}\{\alpha,\beta\} + \overline{\mu}_1(\lambda) \geq \underline{\mu}_1(\lambda). \tag{2.35}$$

On the other hand, in view of the choice of γ, from Proposition 3.5 of [10] it follows that

$$\lim_{\lambda\to\infty}\underline{\mu}_1(\lambda) \geq (1-T^*/T)\min\{\sigma_1^a,\ \sigma_1^b\}.$$

Therefore, this, combined with (2.30) and (2.35), implies

$$\mu_\infty = \lim_{\lambda\to\infty}\mu_1(\lambda) \geq \lim_{\lambda\to\infty}\overline{\mu}_1(\lambda) \geq (1-T^*/T)\min\{\sigma_1^a,\ \sigma_1^b\} - \max_{\overline{\Omega}\times[0,T]}\{\alpha,\ \beta\},$$

as we wanted. ∎

3. Nonlinear cooperative problems.

3.1. Existence and global stability of positive periodic solutions.

In this subsection, we investigate positive solutions of the periodic-parabolic system (1.4). As in [9] for the scalar logistic equation, we can obtain the necessary and sufficient condition for the existence of positive solutions of (1.4) as well as the global attractivity of such a unique positive periodic solution for the corresponding initial-boundary value problem (1.2). Precisely, we have the following.

THEOREM 3.1. *Problem* (1.4) *possesses a unique positive solution* (u_μ, v_μ) *if and only if*

$$\mu_0 < \mu < \mu_\infty. \tag{3.1}$$

Moreover, if (3.1) *holds, the unique solution of* (1.2) *satisfies*

$$\lim_{t\to\infty}\left|\big(u(x,t),v(x,t)\big) - \big(u_\mu(x,t),v_\mu(x,t)\big)\right| = 0 \quad \textit{uniformly for } x\in\overline{\Omega},$$

and if $\mu\leq\mu_0$, *then*

$$\lim_{t\to\infty}(u(x,t),v(x,t)) = (0,0) \quad \textit{uniformly for } x\in\overline{\Omega}.$$

Proof. Assume that (u^*,v^*) is a positive solution to (1.4). We set

$$m = \max_{\overline{\Omega}\times[0,T]}\{u^*(x,t)+v^*(x,t)\}+1 \quad\text{and}\quad r^* = \max\{p,q\} > 1.$$

Due to the uniqueness of the principal eigenvalues, it follows from (1.4) that

$$\mu = \mu_1[\mathcal{L}(a(u^*)^{p-1}, b(v^*)^{q-1}), \Omega].$$

Hence, in view of the monotonicity properties stated in section 2, we have

$$\mu_0 = \mu_1[\mathcal{L}(0,0), \Omega] < \mu$$

and

$$\mu = \mu_1[\mathcal{L}(a(u^*)^{p-1}, b(v^*)^{q-1}), \Omega] < \mu_1[\mathcal{L}(am^{r^*-1}, bm^{r^*-1}), \Omega] < \mu_\infty.$$

This shows that (3.1) is a necessary condition for (1.4) to admit a positive solution.

We now assume that (3.1) holds. We use a simple super-subsolution method to show that (1.4) has a positive solution. Indeed, we choose

$$(\overline{u}, \overline{v}) = (k\overline{\phi}, k\overline{\psi}),$$

where $(\overline{\phi}, \overline{\psi})$ is a principal eigenfunction corresponding to the principal eigenvalue $\mu_1[\mathcal{L}(\lambda a, \lambda b), \Omega]$. By the definition of μ_∞ and our assumption of $\mu < \mu_\infty$, we can take $\lambda > 0$ sufficiently large such that

$$\mu < \mu_1[\mathcal{L}(\lambda a, \lambda b), \Omega],$$

and we then choose $k > 0$ large enough so that

$$(k\overline{\phi})^{p-1} \geq \lambda \quad \text{and} \quad (k\overline{\psi})^{q-1} \geq \lambda \quad \text{on } \overline{\Omega} \times [0, T].$$

Then, it is easy to check that $(\overline{u}, \overline{v})$ is a positive supersolution of (1.4). On the other hand, we set

$$(\underline{u}, \underline{v}) = (\epsilon\underline{\phi}, \epsilon\underline{\psi}),$$

where $(\underline{\phi}, \underline{\psi})$ is a principal eigenfunction corresponding to the principal eigenvalue μ_0. Thus, if one chooses $\epsilon > 0$ small enough such that

$$\max_{\overline{\Omega}\times[0,T]} (a\underline{\phi}^{p-1})\epsilon^{p-1} < \mu - \mu_0 \quad \text{and} \quad \max_{\overline{\Omega}\times[0,T]} (b\underline{\psi}^{q-1})\epsilon^{q-1} < \mu - \mu_0,$$

then $(\underline{u}, \underline{v})$ is a subsolution of (1.4). Hence, a standard iteration consideration implies that (1.4) has a positive solution.

Furthermore, when condition (3.1) is satisfied, a simple modification of the arguments shown in [3] can be used to show that (1.4) has a unique positive solution, denoted by (u_μ, v_μ), which attracts all positive solutions of (1.4) in the sense that

$$\lim_{n\to\infty} (u(x, t+nT), v(x, t+nT)) = (u_\mu(x,t), v_\mu(x,t)) \quad \text{uniformly on } \overline{\Omega} \times [0, T],$$

and if $\mu \leq \mu_0$, $(0,0)$ is the unique nonnegative solution of (1.4), which is a global attractor of (1.2) in the sense that

$$\lim_{t\to\infty} (u(x,t), v(x,t)) = (0,0) \quad \text{uniformly on } \overline{\Omega}. \qquad \square$$

Clearly Theorem 1.1 and parts (a) and (b) of Theorem 1.2 follow directly from Theorem 3.1 and the results in section 2.

3.2. Asymptotic behavior of the positive solution of (1.4) when $\mu \to \mu_\infty$. In this subsection, we discuss the asymptotic behavior of the positive solution (u_μ, v_μ) of (1.4) as $\mu \to \mu_\infty$. First, by a simple comparison and sub-supersolution argument we can easily see that (u_μ, v_μ) is strictly increasing in μ for $\mu \in (\mu_0, \mu_\infty)$ in the sense of $u_{\mu_1} < u_{\mu_2}$ and $v_{\mu_1} < v_{\mu_2}$ on $\overline{\Omega} \times [0,T]$ for any $\mu_0 < \mu_1 < \mu_2 < \mu_\infty$.

We begin with the following lemma.

LEMMA 3.2. *For $j \in \{a, b\}$, denote $\Omega_*^j = \Omega \setminus \overline{\Omega_0^j}$. Let $m \in C^2(\overline{\Omega_*^j} \times [0,T])$ be a given positive T-periodic function. Then, for any $k \in (-\infty, \infty)$ and $r > 1$, the periodic-parabolic problem*

$$
(3.2) \qquad \begin{cases} \partial_t w - \Delta w = kw - j(x,t)w^r & \text{in } \Omega_*^j \times [0,T], \\ \partial_\nu w = 0 & \text{on } \partial\Omega \times [0,T], \\ w = m(x,t) & \text{on } \partial\Omega_0^j \times [0,T], \\ w(x,0) = w(x,T) & \text{in } \Omega_*^j \end{cases}
$$

has a unique solution $w_k^m \in C^{2,1}(\overline{\Omega_*^j} \times [0,T])$. Moreover $w_k^m > 0$ on $\overline{\Omega_*^j} \times [0,T]$, and w_k^m is a strictly increasing function with respect to $m(x,t)$ and k in the sense that $w_k^{m_1} > w_k^{m_2}$ in $\Omega_*^j \times [0,T]$ if $m_1 \geq, \not\equiv m_2$ on $\partial\Omega_0^j \times [0,T]$, and $w_{k_1}^m > w_{k_2}^m$ in $\Omega_*^j \times [0,T]$ if $k_1 > k_2$.

We remark that Lemma 3.2 is the same as Lemma 4.2 of [9] except that the condition imposed on a and b in this paper is less restrictive; the proof of Lemma 3.2 is similar to that of Lemma 4.2 of [9] (with some obvious modifications).

LEMMA 3.3. *Let (u_μ, v_μ) be the unique positive solution of (1.4) for $\mu \in (\mu_0, \mu_\infty)$. Then, as $\mu \to \mu_\infty$, u_μ and v_μ, respectively, converge to ∞ uniformly on every compact subset of $(\overline{\Omega} \times (0, T^*]) \cup (\overline{\Omega_0^a} \times [0,T])$ and $(\overline{\Omega} \times (0, T^*]) \cup (\overline{\Omega_0^b} \times [0,T])$.*

Proof. By the monotonicity of (u_μ, v_μ) with respect to μ, we need only prove the desired result along a sequence $\mu_n \to \mu_\infty$. Since $\mu_1[\mathcal{L}(\lambda a, \lambda b), \Omega] \to \mu_\infty$ as $\lambda \to \infty$, we take

$$\mu_n := \mu_1[\mathcal{L}(\lambda_n a, \lambda_n b), \Omega] \to \mu_\infty \quad \text{with } \lambda_n \to \infty, \quad \text{as } n \to \infty.$$

For simplicity, we denote (u_{μ_n}, v_{μ_n}) by (u_n, v_n). For fixed μ_n, we also let (ϕ_n, ψ_n) be the principal eigenfunction associated with the eigenvalue μ_n, with $\max_{\overline{\Omega} \times [0,T]}(\phi_n + \psi_n) = 1$.

A simple computation shows that

$$(\underline{u}, \underline{v}) = (\lambda_n^{\frac{1}{r^*-1}} \phi_n, \lambda_n^{\frac{1}{r^*-1}} \psi_n), \qquad (\overline{u}, \overline{v}) = (M_n \phi_n, M_n \psi_n),$$

form a pair of sub-supersolutions of (1.4), where $r^* = \max\{p, q\} > 1$ and M_n is chosen to be so large that

$$(M_n \phi_n)^{p-1} \geq \lambda_n, \quad (M_n \psi_n)^{q-1} \geq \lambda_n \quad \text{on } \overline{\Omega} \times [0,T],$$

which also implies

$$(\underline{u}, \underline{v}) \leq (\overline{u}, \overline{v}) \quad \text{on } \overline{\Omega} \times [0,T].$$

Thus, due to the uniqueness of (u_n, v_n) it follows that

$$\lambda_n^{\frac{1}{r^*-1}} \phi_n \leq u_n \leq M_n \phi_n, \quad \lambda_n^{\frac{1}{r^*-1}} \psi_n \leq v_n \leq M_n \psi_n \quad \text{on } \overline{\Omega} \times [0,T].$$

On the other hand, from Theorem 2.2 and its proof, for any compact subsets K_1 and K_2 satisfying

$$K_1 \subset (\overline{\Omega} \times (0, T^*]) \cup (\Omega_0^a \times \mathbb{R}), \qquad K_2 \subset (\overline{\Omega} \times (0, T^*]) \cup (\Omega_0^b \times \mathbb{R}),$$

we have that

$$(\phi_n, \psi_n) \to (\Phi, \Psi) \ \text{ in } C^{2,1}(K_1) \times C^{2,1}(K_2), \quad \text{as } n \to \infty,$$

where (Φ, Ψ) satisfies (1.5) and (2.4) with $\Phi > 0$ in K_1 and $\Psi > 0$ and K_2. Hence,

$$u_n(x,t) \geq \lambda_n^{\frac{1}{r^*-1}} \phi_n(x,t) \to \infty \quad \text{uniformly in } K_1$$

and

$$v_n(x,t) \geq \lambda_n^{\frac{1}{r^*-1}} \psi_n(x,t) \to \infty \quad \text{uniformly in } K_2.$$

To conclude the proof, it suffices to show that

$$u_n(x,t) \to \infty \ \text{ uniformly in } \overline{\Omega_0^a} \times [0,T], \quad \text{as } n \to \infty \tag{3.3}$$

and

$$v_n(x,t) \to \infty \ \text{ uniformly in } \overline{\Omega_0^b} \times [0,T], \quad \text{as } n \to \infty. \tag{3.4}$$

In what follows, we prove only (3.3) since (3.4) can be verified similarly. We first observe that u_n satisfies

$$\partial_t u_n - \Delta u_n = \mu_n u_n + \alpha(x,t) v_n \ \text{ in } \Omega_0^a \times [0,T],$$

with $u_n(x,t) > 0$ on $\overline{\Omega_0^a} \times [0,T]$, and we have already proved that

$$u_n(x,t) \to \infty \ \text{ uniformly on any compact subset of } \Omega_0^a \times [0,T], \quad \text{as } n \to \infty,$$

and

$$u_n(x,T^*) \to \infty \ \text{ uniformly in } \overline{\Omega_0^a}, \quad \text{as } n \to \infty.$$

We need only handle the case of $\mu_\infty > 0$; if $\mu_\infty \leq 0$, the proof is similar by considering the equation satisfied by $z_n(x,t) = e^{(1-\mu_\infty)\,t} u_n(x,t)$. Thus, we may assume that $\mu_n > 0$ for all $n \geq 1$, and so

$$\partial_t u_n - \Delta u_n = \mu_n u_n + \alpha(x,t) v_n \geq 0 \ \text{ in } \Omega_0^a \times [0,T].$$

Hence, by the parabolic maximum principle, it is enough to prove

$$u_n(x_n,t_n) = \min_{\partial\Omega_0^a \times \mathbb{R}} u_n(x,t) \to \infty \text{ as } n \to \infty, \tag{3.5}$$

where (x_n, t_n) on $\partial\Omega_0^a \times [T^*, T+T^*]$.

To verify (3.5) we proceed by contradiction. Suppose that (3.5) does not hold and so, by passing to a subsequence, we may assume that

$$u_n(x_n,t_n) \leq C^* \ \text{ for all } n \geq 1$$

for some positive constant C^*. Due to the smoothness of $\partial\Omega_0^a$, we can find a small $R > 0$ such that for any $x \in \partial\Omega_0^a$, there exists a ball $B_{x,R}$ of radius R such that $B_{x,R} \subset \overline{\Omega_0^a}$ and $\overline{B_{x,R}} \cap \partial\Omega_0^a = \{x\}$.

To produce a contradiction, we first claim that there is a constant $\delta > 0$ and a sequence of constants c_n satisfying $c_n \to \infty$ such that

$$u_n(x_n, t_n) + c_n\omega(x) \le u_n(x, t) \text{ if } \frac{R}{2} \le |x - y_n| \le R,\ T^* \le t \le T + T^*, \tag{3.6}$$

where $\omega(x) = e^{-\delta|x-y_n|^2} - e^{-\delta R^2}$, and y_n is the center of the ball $B_{x_n,R}$.

A simple computation gives

$$\Delta\omega + \mu_n\omega = (4\delta^2|x - y_n|^2 - 2N\delta + \mu_n)e^{-\delta|x-y_n|^2} - \mu_n e^{-\delta R^2}.$$

Thus, we can take a large $\delta > 0$ such that

$$\Delta\omega + \mu_n\omega \ge (4\delta^2|x - y_n|^2 - 2N\delta)e^{-\delta|x-y_n|^2} \ge 0 \quad \text{for all } x \in B_{x_n,R}\backslash B_{R/2}(y_n),$$

where $B_{R/2}(y_n) = \{x \in R^N : |x - y_n| < R/2\}$.

We now choose a compact set $K \subset\subset \Omega_0^a$ such that $K \supset \cup_{n=1}^{\infty} B_{R/2}(y_n)$. By what has already been proved, $u_n(x, t) \to \infty$ uniformly in $K \times \mathbb{R}$, and hence there is a sequence c_n with $c_n \to \infty$ such that

$$u_n(x_n, t_n)+c_n(e^{-\delta R^2/4}-e^{-\delta R^2}) \le u_n(x, t) \text{ for all } x \in \overline{B_{R/2}}(y_n) \subset K,\ t \in [T^*, T+T^*].$$

We may further require that

$$u_n(x_n, t_n) + c_n(e^{-\delta R^2/4} - e^{-\delta R^2}) \le u_n(x, T^*) \text{ for all } x \in \overline{\Omega_0^a}.$$

Then, as $u_n(x, t) \ge u_n(x_n, t_n)$ on $\overline{\Omega_0^a} \times [T^*, T + T^*]$, we find that $u_n(x, t)$ is a supersolution of the problem

$$\begin{cases} \partial_t w - \Delta w = \mu_n w & \text{in } B_{x_n,R}\backslash\overline{B_{R/2}}(y_n) \times [T^*, T + T^*], \\ w = u_n(x_n, t_n) & \text{on } \partial B_{x_n,R} \times [T^*, T + T^*], \\ w = u_n(x_n, t_n) + c_n(e^{-\delta R^2/4} - e^{-\delta R^2}) & \text{on } \partial B_{R/2}(y_n) \times [T^*, T + T^*], \\ w(x, T^*) = u_n(x, T^*) & \text{in } \{R/2 < |x - y_n| < R\}. \end{cases} \tag{3.7}$$

One also sees that $u_n(x_n, t_n) + c_n\omega(x)$ is a subsolution to (3.7). The comparison principle for parabolic equations then yields (3.6). Consequently, as $n \to \infty$, we find

$$\partial_{\nu_n} u_n|_{(x_n,t_n)} \ge c_n\partial_{\nu_n}\omega|_{x_n} = 2c_n\delta Re^{-\delta R^2} \to \infty, \tag{3.8}$$

where $\nu_n = (y_n - x_n)/|y_n - x_n|$.

On the other hand, by Lemma 3.2, for any $n \ge 1$, the T-periodic problem

$$\begin{cases} \partial_t w - \Delta w = \mu_n w - a(x, t)w^p & \text{in } \Omega\backslash\overline{\Omega_0^a} \times [0, T], \\ \partial_\nu w = 0 & \text{on } \partial\Omega \times [0, T], \\ w = u_n(x_n, t_n) & \text{on } \partial\Omega_0^a \times [0, T], \\ w(x, 0) = w(x, T) & \text{in } \Omega\backslash\overline{\Omega_0^a} \end{cases} \tag{3.9}$$

admits a unique positive solution, denoted by v_n. It is easily seen that u_n is a supersolution of (3.9). Due to Lemma 3.2, we have $v_n \le u_n$ on $\overline{\Omega}\backslash\Omega_0^a \times [0,T]$. If we replace μ_n by μ_∞ and replace $u_n(x_n,t_n)$ by its upper bound C^* in (3.9), we obtain a unique positive solution of (3.9), denoted by U_0. By Lemma 3.2 again, $v_n \le U_0$ on $\overline{\Omega}\backslash\Omega_0^a \times [0,T]$. In particular, $\|v_n\|_{L^\infty(\overline{\Omega}\backslash\Omega_0^a\times[0,T])}$ has a bound independent of n. Thus, the L^p-estimates and Sobolev embedding theorem imply that $\{v_n\}$ is bounded in $C^{1+\theta,\theta/2}(\overline{\Omega}\backslash\Omega_0^a\times[0,T])$, and so $\|\nabla v_n(x_n,t_n)\|_{L^\infty(\overline{\Omega}\backslash\Omega_0^a\times[0,T])} \le C_0$ for some $C_0 > 0$. Since

$$v_n(x,t) \le u_n(x,t) \text{ for all } (x,t) \in \overline{\Omega}\backslash\Omega_0^a \times [T^*, T+T^*] \quad \text{and} \quad u_n(x_n,t_n) = v_n(x_n,t_n),$$

we conclude that

$$\partial_{\nu_n} u_n|_{(x_n,t_n)} \le \partial_{\nu_n} v_n|_{(x_n,t_n)} \le C_0. \tag{3.10}$$

Clearly (3.8) and (3.10) contradict each other. This then implies that (3.3) is true. The proof is now complete. ∎

To better understand the asymptotic behavior of (u_μ, v_μ) as $\mu \to \mu_\infty$, we need the following lemma.

LEMMA 3.4. *Given two smooth bounded domains D_0, $D \subset \mathbb{R}^N$ with $\overline{D}_0 \subset\subset D$. Let $\{c_n\}_{n=1}^\infty$ be a given positive sequence with $c_n \to \infty$ as $n \to \infty$, and let φ_n be the unique solution of*

$$-\Delta\varphi + c_n\varphi = c_n\chi_{\overline{D}_0} \quad \text{in } D, \qquad \varphi = 0 \quad \text{on } \partial D. \tag{3.11}$$

Then, for all $n \ge 1$, we have

$$\max_{\overline{D}} \varphi_n \le 1, \quad \min_{\overline{D}_0} \varphi_n \ge \delta_0$$

for some constant $\delta_0 > 0$.

Proof. The existence and uniqueness of φ_n is obvious, and $\varphi_n > 0$ in D. Since 0 and 1 are a pair of sub-supersolutions to (3.11), the uniqueness ensures $\max_{\overline{D}} \varphi_n \le 1$.

To show $\min_{\overline{D}_0} \varphi_n \ge \delta_0 > 0$, we will argue indirectly. By passing to a subsequence, we may assume that

$$\varphi_n(x_n) = \min_{\overline{D}_0} \varphi_n \to 0 \quad \text{as } n \to \infty, \tag{3.12}$$

where $x_n \in \overline{D}_0$. We may also assume that $x_n \to x^* \in \overline{D}_0$ as $n \to \infty$. Then, we set $x = x_n + \frac{1}{\sqrt{c_n}}y$, namely, $y = \sqrt{c_n}(x - x_n)$, and define

$$\tilde{\varphi}_n(y) = \varphi_n\left(\frac{1}{\sqrt{c_n}}y + x_n\right).$$

Clearly, $\tilde{\varphi}_n$ solves

$$-\Delta\tilde{\varphi}_n = \chi_{\overline{D}_0^n} - \tilde{\varphi}_n \quad \text{in } D^n, \quad \tilde{\varphi}_n = 0 \quad \text{on } \partial D^n,$$

where

$$D_0^n = \left\{y : \frac{1}{\sqrt{c_n}}y + x_n \in D_0\right\}, \quad D^n = \left\{y : \frac{1}{\sqrt{c_n}}y + x_n \in D\right\}.$$

Moreover, $0 < \tilde{\varphi}_n \leq 1$ and $\tilde{\varphi}_n(0) = \varphi_n(x_n) \to 0$ as $n \to \infty$ due to (3.12).

If $x^* \in D_0$, then we notice that $\chi_{\overline{D_0^n}} \to 1$ locally uniformly in $\mathbb{R}^N$. By the standard elliptic interior L^p-estimates, for a given ball $B \subset \mathbb{R}^N$ and $r > 1$, we can conclude that $\|\tilde{\varphi}_n\|_{W^{2,r}} \leq C_0$ for some positive constant C_0, independent of n. Hence, using the Sobolev embedding theorem, a diagonal argument gives that, by passing to a subsequence if necessary, $\tilde{\varphi}_n \to \tilde{\varphi}$ in $C^1(B)$, for any $B \subset \mathbb{R}^N$. Additionally, $\tilde{\varphi}$ satisfies

$$-\Delta \tilde{\varphi} = 1 - \tilde{\varphi} \ \text{ in } \mathbb{R}^N, \quad 0 \leq \tilde{\varphi} \leq 1, \quad \tilde{\varphi}(0) = 0 = \min_{\mathbb{R}^N} \tilde{\varphi}. \tag{3.13}$$

Standard elliptic regularity theory implies that $\tilde{\varphi} \in C^2(\mathbb{R}^N)$. Thus, $-\Delta \tilde{\varphi}(0) \leq 0$, which gives $1 = 1 - \tilde{\varphi}(0) \leq 0$, a contradiction.

If $x^* \in \partial D_0$, we assume that $x_n^* \in \partial D_0$ is the closest point on ∂D_0 to x_n. By a translation and rotation of the coordinate system, we may also assume that $x^* = 0$ and the outward unit normal vector of ∂D_0 at 0 is the x^1-axis, where we write $x = (x^1, x^2, \ldots, x^N)$ and $x_n = (x_n^1, x_n^2, \ldots, x_n^N)$. Passing to a subsequence if necessary, we may assume that

$$\xi = \lim_{n \to \infty} \sqrt{c_n}\, |x_n - x_n^*| \quad \text{for some } \ \xi \in [0, +\infty].$$

If $\xi = +\infty$, it is easy to see that, subject to a subsequence, $\tilde{\varphi} = \lim_{n \to \infty} \tilde{\varphi}_n$ satisfies (3.13), and we obtain a contradiction as before.

We now assume that $+\infty > \xi \geq 0$. In this case, one can see that, up to a subsequence, $\tilde{\varphi} = \lim_{n \to \infty} \tilde{\varphi}_n$ satisfies

$$-\Delta \tilde{\varphi} = \chi_{\{y^1 \leq \xi\}} - \tilde{\varphi} \ \text{ in } \mathbb{R}^N, \quad 0 \leq \tilde{\varphi} \leq 1, \quad \tilde{\varphi}(0) = 0 = \min_{\mathbb{R}^N} \tilde{\varphi}.$$

Furthermore, the elliptic regularity theory guarantees $\tilde{\varphi} \in C^1(\mathbb{R}^N) \cap C^2(\{y^1 < \xi\})$. If $\xi > 0$, then 0 is an interior point of $\{y^1 \leq \xi\}$ and we can obtain a contradiction as before by using $-\Delta \tilde{\varphi}(0) \leq 0$.

If $\xi = 0$, then 0 lies on the boundary of $\{y^1 \leq \xi\} = \{y^1 \leq 0\}$. We must have $\tilde{\varphi}(y) > 0$ for $y \in \{y^1 < 0\}$ since otherwise we reach a contradiction as before using the fact that $-\Delta \tilde{\varphi}(y) \leq 0$ whenever $\tilde{\varphi}(y) = 0$ and y is an interior point. We now consider $\tilde{\varphi}$ on the half space $\{y^1 < 0\}$, where $\tilde{\varphi}$ satisfies

$$-\Delta \tilde{\varphi} + \tilde{\varphi} = 1 > 0.$$

Since $0 = \tilde{\varphi}(0) < \tilde{\varphi}(y)$ for $y \in \{y^1 < 0\}$, by the Hopf boundary lemma, we deduce $\partial_{y^1} \tilde{\varphi}(0) < 0$. On the other hand, $0 = \tilde{\varphi}(0) = \min_{\mathbb{R}^N} \tilde{\varphi}$ and $\tilde{\varphi} \in C^1(\mathbb{R}^N)$. So we have $\nabla \tilde{\varphi}(0) = 0$, which therefore implies $\partial_{y^1} \tilde{\varphi}(0) = 0$. Again, we have a contradiction.

Since every possibility leads to a contradiction, we have proved $\min_{\overline{D}_0} \varphi_n \geq \delta_0$ for some positive constant δ_0. $\square$

Theorem 3.5. *Let (u_μ, v_μ) be the unique positive solution of (1.4) for $\mu \in (\mu_0, \mu_\infty)$. Then, as $\mu \to \mu_\infty$,*

$$(u_\mu, v_\mu) \to (\infty, \infty) \quad \textit{uniformly on compact subsets of } \ (\overline{\Omega} \times (0, T^*]) \cup ((\overline{\Omega_0^a} \cup \overline{\Omega_0^b}) \times [0, T])$$

and

$$(u_\mu, v_\mu) \to (\underline{U}_{\mu_\infty}, \underline{V}_{\mu_\infty}) \ \textit{uniformly on compact subsets of } \ (\overline{\Omega} \setminus (\overline{\Omega_0^a} \cup \overline{\Omega_0^b})) \times (T^*, T),$$

where $(\underline{U}_{\mu_\infty}, \underline{V}_{\mu_\infty})$ *is the minimal positive solution of*

(3.14)
$$\begin{cases} \begin{array}{l} \partial_t u - \Delta u = \mu_\infty u + \alpha(x,t)v - a(x,t)u^p \\ \partial_t v - \Delta v = \mu_\infty v + \beta(x,t)u - b(x,t)v^q \end{array} & \text{in } (\overline{\Omega} \setminus (\overline{\Omega_0^a} \cup \overline{\Omega_0^b})) \times (T^*, T), \\ (\partial_\nu u, \partial_\nu v) = (0,0) & \text{on } \partial\Omega \times (T^*, T), \\ (u,v) = (\infty, \infty) & \text{on } (\partial\Omega_0^a \cup \partial\Omega_0^b) \times (T^*, T), \\ (u(x,T^*), v(x,T^*)) = (\infty,\infty) & \text{in } \overline{\Omega} \setminus (\Omega_0^a \cup \Omega_0^b). \end{cases}$$

Proof. As before, since (u_μ, v_μ) is increasing in $\mu \in (\mu_0, \mu_\infty)$, we need only consider the limit of $(u_n, v_n) := (u_{\mu_n}, v_{\mu_n})$ along an increasing sequence $\{\mu_n\}$ which converges to μ_∞ as $n \to \infty$. For clarity, we divide our proof into several steps.

Step 1. We show that for every compact subset K of $(\overline{\Omega} \setminus (\overline{\Omega_0^a} \cup \overline{\Omega_0^b})) \times (T^*, T)$,

$$\{u_n\} \text{ and } \{v_n\} \text{ are both uniformly bounded in } K. \tag{3.15}$$

For given small $\epsilon > 0$, we denote

$$\Omega_\epsilon = \{x \in \overline{\Omega} \setminus (\overline{\Omega_0^a} \cup \overline{\Omega_0^b}) : \; d(x, \overline{\Omega_0^a} \cup \overline{\Omega_0^b}) > \epsilon\}.$$

Then, for all small ϵ, Ω_ϵ is nonempty and smooth. Since $a(x,t)$, $b(x,t) > 0$ in $\overline{\Omega}_\epsilon \times (T^*, T)$, we may assume that

$$a(x,t),\, b(x,t) \geq c_\epsilon \quad \text{on } \overline{\Omega}_\epsilon \times [T^* + \epsilon, T - \epsilon]$$

for some positive constant c_ϵ. We also assume that

$$\alpha(x,t) + \beta(x,t) \leq m_0 \quad \text{on } \overline{\Omega} \times [0,T]$$

for some constant $m_0 > 0$, and denote $r_* = \min\{p,q\} > 1$. We may require $\mu_\infty + m_0 \geq c_\epsilon$.

For fixed $n \geq 1$, we consider the problem

$$\begin{cases} \begin{array}{l} \partial_t u - \Delta u = \mu_\infty u + m_0 v - c_\epsilon u^p \\ \partial_t v - \Delta v = \mu_\infty v + m_0 u - c_\epsilon v^q \end{array} & \text{in } \Omega_\epsilon \times (T^* + \epsilon, T - \epsilon], \\ (\partial_\nu u, \partial_\nu v) = (0,0) & \text{on } \partial\Omega \times (T^* + \epsilon, T - \epsilon], \\ u = u_n & \text{on } (\partial\Omega_\epsilon \setminus \partial\Omega) \times (T^* + \epsilon, T - \epsilon], \\ v = v_n & \text{on } (\partial\Omega_\epsilon \setminus \partial\Omega) \times (T^* + \epsilon, T - \epsilon], \\ u(x, T^* + \epsilon) = u_n(x, T^* + \epsilon) & \text{in } \Omega_\epsilon, \\ v(x, T^* + \epsilon) = v_n(x, T^* + \epsilon) & \text{in } \Omega_\epsilon. \end{cases} \tag{3.16}$$

Clearly, (3.16) has a classical solution, which is unique. In addition, (u_n, v_n) is a subsolution of (3.16).

In what follows, we are going to find a supersolution of (3.16). To this aim, let us consider the following two auxiliary problems:

$$w_t = (\mu_\infty + m_0)w - c_\epsilon w^{r_*}, \quad t > T^* + \epsilon; \quad w(T^* + \epsilon) = \infty, \tag{3.17}$$

and

$$\begin{cases} -\Delta z = (\mu_\infty + m_0)z - c_\epsilon z^{r_*} \quad \text{in } \Omega_\epsilon, \\ \partial_\nu z = 0 \;\; \text{on } \partial\Omega, \quad z = \infty \;\; \text{on } \partial\Omega_\epsilon \setminus \partial\Omega. \end{cases} \tag{3.18}$$

The unique solution $w(t)$ of (3.17) can be explicitly written as

$$w(t)=\Big(\frac{\mu_\infty+m_0}{c_\epsilon}\Big)^{\frac{1}{r_*-1}}e^{(\mu_\infty+m_0)t}\big[e^{(\mu_\infty+m_0)(r_*-1)t}-e^{(\mu_\infty+m_0)(r_*-1)(T^*+\epsilon)}\big]^{\frac{1}{1-r_*}},\ \ t>T^*+\epsilon.$$

From [7, 8], we also know that (3.18) admits a unique positive solution, denoted by $z(x)$. Clearly, 1 is a subsolution to (3.17) and (3.18) due to $\mu_\infty+m_0\geq c_\epsilon$, and so it follows from a comparison analysis that $w(t)\geq 1$ in $[T^*+\epsilon,T]$ and $z(x)\geq 1$ on $\overline{\Omega}_\epsilon$.

According to the definitions of $w(t)$ and $z(x)$, for fixed n, it is easy to see that

$$w(t)+z(x)>u_n(x,t)\quad\text{and}\quad w(t)+z(x)>v_n(x,t)\quad\text{in } (\partial\Omega_\epsilon\setminus\partial\Omega)\times(T^*+\epsilon,T-\epsilon)$$

and

$$w(T^*+\epsilon)>u_n(x,T^*+\epsilon)\quad\text{and}\quad w(T^*+\epsilon)>v_n(x,T^*+\epsilon)\quad\text{on } \overline{\Omega}_\epsilon.$$

Furthermore, using the fact that $w(t)$, $z(x)\geq 1$, and p, $q\geq r_*>1$, one can easily check that $(w(t)+z(x),w(t)+z(x))$ satisfies the required differential inequalities for a supersolution of (3.16). Hence, for any $n\geq 1$, by the comparison principle for cooperative parabolic systems, we conclude that

$$u_n(x,t),\ v_n(x,t)\leq w(t)+z(x)\quad\text{on } \overline{\Omega}_\epsilon\times[T^*+\epsilon,T-\epsilon].$$

In view of the fact that for fixed small $\epsilon>0$, $w(t)+z(x)$ is bounded on $\overline{\Omega}_{2\epsilon}\times[T^*+2\epsilon,T-\epsilon]$, we can find a positive constant C_0 such that

$$u_n,\ v_n\leq C_0\quad\text{on } \overline{\Omega}_{2\epsilon}\times[T^*+2\epsilon,T-\epsilon]\ \text{ for all } n\geq 1.$$

Due to the arbitrariness of ϵ and the choice of Ω_ϵ, this clearly implies (3.15).

Step 2. We now show that

$$u_n\to\infty\ \text{ uniformly in } \overline{\Omega_0^b}\times[T^*,T],\ \text{ as } n\to\infty, \tag{3.19}$$

and

$$v_n\to\infty\ \text{ uniformly in } \overline{\Omega_0^a}\times[T^*,T],\ \text{ as } n\to\infty. \tag{3.20}$$

We prove only (3.19) since (3.20) can be verified similarly.

According to Lemma 3.3, for any given small $\epsilon>0$ and a small neighborhood $\widetilde{\Omega}_0^b\subset\Omega$ of $\overline{\Omega_0^b}$, we can find a sequence $\{\sigma_n\}$ satisfying $\sigma_n\to\infty$ as $n\to\infty$ such that u_n satisfies

$$\begin{aligned}(u_n)_t-\Delta u_n&=\mu_n u_n+\alpha(x,t)v_n-a(x,t)u_n^p\\&\geq\mu_n u_n+\sigma_n\chi_{(\overline{\Omega_0^b}\times[T^*-\epsilon,T])}-a_0u_n^p,\end{aligned} \tag{3.21}$$

on $\widetilde{\Omega}_0^b\times[T^*-\epsilon,T]$, for some constant $a_0>0$. Moreover, we have $u_n(x,T^*-\epsilon)\geq\tilde{\sigma}_n$ on $\widetilde{\Omega}_0^b$ with $\tilde{\sigma}_n\to\infty$ as $n\to\infty$.

To simplify notation, we denote $\Omega_0^b=D_0$ and $\widetilde{\Omega}_0^b=D$. So $\overline{D}_0\subset D$, and D_0, D are smooth. We need only consider the case of $\mu_\infty>0$; if $\mu_\infty\leq 0$, the proof is similar by considering the equation satisfied by $z_n(x,t)=e^{(1-\mu_\infty)t}u_n(x,t)$. Furthermore, we also replace σ_n and $\tilde{\sigma}_n$ by $\min\{\sigma_n,\tilde{\sigma}_n\}$, which is still denoted by σ_n. Thus, from (3.21), for all large n, we have

$$\begin{cases}(u_n)_t-\Delta u_n\geq\sigma_n\chi_{(\overline{D}_0\times[T^*-\epsilon,T])}-a_0u_n^p & \text{in } D\times[T^*-\epsilon,T],\\ u_n(x,T^*-\epsilon)\geq\sigma_n & \text{in } \overline{D},\end{cases} \tag{3.22}$$

with $\sigma_n \to \infty$ as $n \to \infty$.

We next construct a lower bound for u_n. For this purpose, in Lemma 3.4 we take D_0, D as above, and the sequence $\{c_n\}$ satisfying $c_n \to \infty$ as $n \to \infty$ will be chosen later. For $\eta > 0$ to be determined, we define

$$w_n(x) = \eta c_n^{\frac{1}{p-1}} \varphi_n(x),$$

where c_n and φ_n are given as in Lemma 3.4. Then, a simple calculation gives

$$\begin{aligned} -\Delta w_n &= \eta c_n^{\frac{p}{p-1}} \chi_{\overline{D}_0} - \eta c_n^{\frac{p}{p-1}} \varphi_n \\ &\le \eta c_n^{\frac{p}{p-1}} \chi_{\overline{D}_0} - a_0 w_n^p \qquad \text{in } D, \end{aligned} \tag{3.23}$$

provided that

$$\eta c_n^{\frac{p}{p-1}} \varphi_n \ge a_0 w_n^p. \tag{3.24}$$

In view of $w_n^p = \eta^p c_n^{\frac{p}{p-1}} \varphi_n^p$ and $\varphi_n \le 1$, it is easy to check that (3.24) holds by choosing $\eta = a_0^{\frac{1}{p-1}}$. Moreover, if we take

$$c_n = a_0^{-\frac{1}{p}} \sigma_n^{\frac{p-1}{p}},$$

then $c_n \to \infty$ as $n \to \infty$ and w_n satisfies

$$-\Delta w_n \le \sigma_n \chi_{\overline{D}_0} - a_0 w_n^p \ \text{ in } D, \quad w_n = 0 \ \text{ on } \partial D. \tag{3.25}$$

Thanks to the choice of η, c_n, and the fact of $\varphi_n \le 1$, we have $w_n \le (a_0\sigma_n)^{\frac{1}{p}}$ on $\overline{D}$. Thus, for all large n, $w_n \le \sigma_n$ on $\overline{D}$. This, together with (3.25) and (3.22), allows us to use the comparison principle to conclude that $w_n \le u_n$ in $\overline{D} \times [T^* - \epsilon, T]$ for all large n. In particular, over $\overline{D}_0 \times [T^* - \epsilon, T]$, we have

$$u_n \ge w_n \ge \eta c_n^{\frac{1}{p-1}} \varphi_n(x) \ge (a_0\sigma_n)^{\frac{1}{p}} \delta_0 \to \infty \quad \text{as } n \to \infty,$$

where $\delta_0 > 0$ is given in Lemma 3.4. Clearly, this implies (3.19).

Step 3. Completion of the proof.

By Lemma 3.3 and the conclusions proved in Step 2, we find that, as $n \to \infty$,

$(u_n, v_n) \to (\infty, \infty)$ uniformly in compact subsets of $(\overline{\Omega} \times (0, T^*]) \cup ((\overline{\Omega_0^a} \cup \overline{\Omega_0^b}) \times [0, T])$.

Thanks to (3.15), a standard regularity argument concludes that

$(u_n, v_n) \to (U_{\mu_\infty}, V_{\mu_\infty})$ uniformly on any compact subset of $(\overline{\Omega} \backslash (\overline{\Omega_0^a} \cup \overline{\Omega_0^b})) \times (T^*, T)$,

as $n \to \infty$, where $(U_{\mu_\infty}, V_{\mu_\infty})$ satisfies the first two equations of (3.14), and $\partial_\nu U_{\mu_\infty} = \partial_\nu V_{\mu_\infty} = 0$ on $\partial\Omega \times (T^*, T)$.

We next show that

$$\lim_{t \downarrow T^*} (U_{\mu_\infty}, V_{\mu_\infty}) = (\infty, \infty) \ \text{ uniformly for } x \in \overline{\Omega} \setminus (\Omega_0^a \cup \Omega_0^b), \tag{3.26}$$

$$\lim_{d(x,\, \Omega_0^a \cup \Omega_0^b) \to 0} (U_{\mu_\infty}, V_{\mu_\infty}) = (\infty, \infty) \ \text{ uniformly for } t \in [T^*, T). \tag{3.27}$$

Since (u_n, v_n) increases to $(U_{\mu_\infty}, V_{\mu_\infty})$ as $n \to \infty$, we have $U_{\mu_\infty} > u_k$ and $V_{\mu_\infty} > v_k$ for all $k \geq 1$. In (3.26), we verify only $\lim_{t \downarrow T^*} U_{\mu_\infty} = \infty$ uniformly for $x \in \overline{\Omega} \setminus (\Omega_0^a \cup \Omega_0^b)$, since the other assertion can be proved similarly. Suppose that this is not true. Thus, there exist sequences $x_n \in \overline{\Omega} \setminus (\Omega_0^a \cup \Omega_0^b)$ and t_n decreasing to T^* such that $U_{\mu_\infty}(x_n, t_n) \leq M$ for all $n \geq 1$ and some constant $M > 0$. So we have

$$u_k(x_n, t_n) \leq M \quad \text{for all } n \geq 1, \text{ for all } k \geq 1. \tag{3.28}$$

On the other hand, by Lemma 3.3, we know that $u_k(x_n, T^*) \to \infty$ as $k \to \infty$ uniformly in $n \geq 1$. Thus there exists k_0 large such that $u_{k_0}(x_n, T^*) \geq 3M$ for all $n \geq 1$. Since the function $u_{k_0}(x, t)$ is uniformly continuous in its variables, and $t_n \to T^*$, we deduce $|u_{k_0}(x_n, t_n) - u_{k_0}(x_n, T^*)| \to 0$ as $n \to \infty$. Thus for all large n,

$$u_{k_0}(x_n, t_n) \geq u_{k_0}(x_n, T^*) - M \geq 2M,$$

which is in contradiction to (3.28). This proves (3.26). The proof of (3.27) is similar, where we use Lemma 3.3, (3.19), and (3.20).

The above analysis shows that $(U_{\mu_\infty}, V_{\mu_\infty})$ is a solution to (3.14). It remains to show that $(U_{\mu_\infty}, V_{\mu_\infty})$ is the minimal positive solution of (3.14). Let (U, V) be any positive solution of (3.14). Then applying the comparison principle for cooperative parabolic systems, we easily see that

$$(u_n, v_n) < (U, V) \quad \text{in } (\Omega \setminus (\overline{\Omega_0^a} \cup \overline{\Omega_0^b})) \times (T^*, T).$$

Letting $n \to \infty$ we deduce $(U_{\mu_\infty}, V_{\mu_\infty}) \leq (U, V)$. Hence $(U_{\mu_\infty}, V_{\mu_\infty})$ is the minimal positive solution of (3.14). ∎

3.3. Long-time behavior of the positive solution of (1.2) when $\mu \geq \mu_\infty$. As preparation, we first consider a more general version of (3.14); namely, for any given $\mu \in (-\infty, \infty)$, we study the problem

$$\begin{cases} \partial_t u - \Delta u = \mu u + \alpha(x,t)v - a(x,t)u^p \\ \partial_t v - \Delta v = \mu v + \beta(x,t)u - b(x,t)v^q \end{cases} \text{in } (\overline{\Omega} \setminus (\overline{\Omega_0^a} \cup \overline{\Omega_0^b})) \times (T^*, T),$$
$$\begin{cases} (\partial_\nu u, \partial_\nu v) = (0,0) & \text{on } \partial\Omega \times (T^*, T), \\ (u, v) = (\infty, \infty) & \text{on } (\partial\Omega_0^a \cup \partial\Omega_0^b) \times (T^*, T), \\ (u(x, T^*), v(x, T^*)) = (\infty, \infty) & \text{in } \overline{\Omega} \setminus (\Omega_0^a \cup \Omega_0^b). \end{cases} \tag{3.29}$$

LEMMA 3.6. *For any $\mu \in (-\infty, \infty)$, problem (3.29) has a minimal positive solution $(\underline{U}_\mu, \underline{V}_\mu)$ and a maximal positive solution $(\overline{U}_\mu, \overline{V}_\mu)$ in the sense that any positive solution (U, V) of (3.29) satisfies*

$$\underline{U}_\mu \leq U \leq \overline{U}_\mu, \quad \underline{V}_\mu \leq V \leq \overline{V}_\mu \quad \text{in } (\overline{\Omega} \setminus (\overline{\Omega_0^a} \cup \overline{\Omega_0^b})) \times (T^*, T).$$

Proof. Given small $\varepsilon \geq 0$, define

$$\Omega^\varepsilon := \{x \in \Omega : \, d(x, \Omega_0^a \cup \Omega_0^b) \leq \varepsilon\}.$$

Then, for each integer $n \geq 1$, let us consider the initial-boundary value problem

$$\begin{cases} \partial_t u - \Delta u = \mu u + \alpha(x,t)v - a(x,t)u^p \\ \partial_t v - \Delta v = \mu v + \beta(x,t)u - b(x,t)v^q \end{cases} \text{in } \Omega \setminus \Omega^\varepsilon \times (T^* + \varepsilon, T),$$
$$\begin{cases} (\partial_\nu u, \partial_\nu v) = (0,0) & \text{on } \partial\Omega \times (T^* + \varepsilon, T), \\ (u, v) = (n, n) & \text{on } \partial\Omega^\varepsilon \times (T^* + \varepsilon, T), \\ (u(x, T^* + \varepsilon), v(x, T^* + \varepsilon)) = (n, n) & \text{in } \overline{\Omega} \setminus \Omega^\varepsilon. \end{cases} \tag{3.30}$$

A standard analysis shows that (3.30) admits a unique positive solution, which is denoted by (u_n, v_n). Moreover, it is easy to show by a comparison argument that (u_n, v_n) increases to $(U_\varepsilon, V_\varepsilon)$ as $n \to \infty$, where $(U_\varepsilon, V_\varepsilon)$ stands for the minimal positive solution of

$$\begin{cases} \left.\begin{array}{l} \partial_t u - \Delta u = \mu u + \alpha(x,t)v - a(x,t)u^p \\ \partial_t v - \Delta v = \mu v + \beta(x,t)u - b(x,t)v^q \end{array}\right\} & \text{in } \Omega \setminus \Omega^\varepsilon \times (T^* + \varepsilon, T), \\ (\partial_\nu u, \partial_\nu v) = (0,0) & \text{on } \partial\Omega \times (T^* + \varepsilon, T), \\ (u,v) = (\infty, \infty) & \text{on } \partial\Omega^\varepsilon \times (T^* + \varepsilon, T), \\ (u(x, T^*+\varepsilon), v(x, T^*+\varepsilon)) = (\infty,\infty) & \text{in } \overline{\Omega} \setminus \Omega^\varepsilon. \end{cases}$$

Taking $\varepsilon = 0$ we find that (U_0, V_0) is the minimal positive solution of (3.29). Furthermore, using the parabolic comparison principle for cooperative systems we easily deduce that

$$U_{\varepsilon_1} \geq U_{\varepsilon_2} \geq U_0, \quad V_{\varepsilon_1} \geq V_{\varepsilon_2} \geq V_0 \quad \text{when } \varepsilon_1 > \varepsilon_2 > 0.$$

Thus, we can find a decreasing sequence ε_n with $\varepsilon_n \to 0$ such that

$$(U_{\varepsilon_n}, V_{\varepsilon_n}) \to (\overline{U}, \overline{V}),$$

where $(\overline{U}, \overline{V})$ is a positive solution of (3.29). Let (U, V) be any positive solution of (3.29). Then, applying the parabolic comparison principle, we deduce that

$$U_{\varepsilon_n} > U, \quad V_{\varepsilon_n} > V \quad \text{for every } n.$$

Passing to the limit $n \to \infty$ yields

$$\overline{U} \geq U, \quad \overline{V} \geq V,$$

which indicates that $(\overline{U}, \overline{V})$ is the maximal positive solution of (3.29), and this completes the proof. ∎

We are now ready to complete the proof of part (c) of Theorem 1.2.

THEOREM 3.7. *Assume that* $\mu \geq \mu_\infty$, $u_0, v_0 \in C(\overline{\Omega})$, *and* $u_0 \geq 0$ $v_0 \geq 0$, *with* $u_0 \not\equiv 0$ *and* $v_0 \not\equiv 0$. *Then, the unique solution* (u, v) *of problem* (1.2) *satisfies*

$$\lim_{n\to\infty} (u(x, t+nT), v(x, t+nT)) = (\infty, \infty)$$

uniformly on compact subsets of $(\overline{\Omega} \times (0, T^*]) \cup ((\overline{\Omega_0^a} \cup \overline{\Omega_0^b}) \times [0, T])$, *and*

$$\lim_{n\to\infty} (u(x, t+nT), v(t+nT)) = (\underline{U}_\mu(x,t), \underline{V}_\mu(x,t))$$

uniformly on every compact subset of $(\overline{\Omega} \setminus (\overline{\Omega_0^a} \cup \overline{\Omega_0^b})) \times (T^*, T)$.

Proof. For any given $\varepsilon > 0$, let us denote by $(u^\varepsilon, v^\varepsilon)$ the unique solution of the initial-boundary value problem

$$\text{(3.31)} \qquad \begin{cases} \left.\begin{array}{l} \partial_t u - \Delta u = (\mu_\infty - \varepsilon)u + \alpha(x,t)v - a(x,t)u^p \\ \partial_t v - \Delta v = \beta(x,t)u + (\mu_\infty - \varepsilon)v - b(x,t)v^q \end{array}\right\} & \text{in } \Omega \times (0, \infty), \\ (\partial_\nu u, \partial_\nu v) = (0,0) & \text{on } \partial\Omega \times (0,\infty), \\ (u(x,0), v(x,0)) = (u_0(x), v_0(x)) & \text{in } \Omega. \end{cases}$$

Since $\mu \geq \mu_\infty$, it is clear that (u, v) is a supersolution of (3.31). So we have

$$\text{(3.32)} \qquad u^\varepsilon(x,t) \leq u(x,t) \quad \text{and} \quad v^\varepsilon(x,t) \leq v(x,t) \quad \text{on } \overline{\Omega} \times [0, \infty).$$

Furthermore, let $(u_{\mu_\infty-\varepsilon}, v_{\mu_\infty-\varepsilon})$ be the unique positive solution of

$$\begin{cases} \begin{array}{l} \partial_t u - \Delta u = (\mu_\infty - \varepsilon)u + \alpha(x,t)v - a(x,t)u^p \\ \partial_t v - \Delta v = \beta(x,t)u + (\mu_\infty - \varepsilon)v - b(x,t)v^q \end{array} & \text{in } \Omega \times \mathbb{R}, \\ (\partial_\nu u, \partial_\nu v) = (0,0) & \text{on } \partial\Omega \times \mathbb{R}, \\ (u(x,t), v(x,t)) = (u(x,t+T), v(x,t+T)) & \text{in } \Omega \times \mathbb{R}. \end{cases}$$

It follows from Theorem 3.1 that, as $t \to \infty$,

$$(u^\varepsilon(x,t), v^\varepsilon(x,t)) - (u_{\mu_\infty-\varepsilon}(x,t), v_{\mu_\infty-\varepsilon}(x,t)) \to (0,0) \text{ uniformly on } \overline{\Omega} \times [0,T]. \tag{3.33}$$

Thus, due to (3.32) we obtain

$$\liminf_{n\to\infty} u(x,t+nT) \geq u_{\mu_\infty-\varepsilon}(x,t), \qquad \liminf_{n\to\infty} v(x,t+nT) \geq v_{\mu_\infty-\varepsilon}(x,t),$$

uniformly on $\overline{\Omega} \times [0,T]$, for every small $\varepsilon > 0$. Letting $\varepsilon \to 0$ and using Theorem 3.5, we conclude that

$$\lim_{n\to\infty} (u(x,t+nT), v(x,t+nT)) = (\infty, \infty),$$

uniformly on compact subsets of $(\overline{\Omega} \times (0,T^*]) \cup ((\overline{\Omega_0^a} \cup \overline{\Omega_0^b}) \times [0,T])$, and

$$\liminf_{n\to\infty} u(x,t+nT) \geq \underline{U}_{\mu_\infty}(x,t), \qquad \liminf_{n\to\infty} v(x,t+nT) \geq \underline{V}_{\mu_\infty}(x,t) \tag{3.34}$$

locally uniformly in $(\overline{\Omega} \setminus (\overline{\Omega_0^a} \cup \overline{\Omega_0^b})) \times (T^*, T)$. Furthermore, by the parabolic comparison principle, we easily see that, for every $n \geq 1$,

$$u(x,t+nT) \leq \underline{U}_\mu(x,t), \quad v(x,t+nT) \leq \underline{V}_\mu(x,t) \text{ in } (\overline{\Omega} \setminus (\overline{\Omega_0^a} \cup \overline{\Omega_0^b})) \times (T^*, T). \tag{3.35}$$

Hence, if we denote

$$(\tilde{u}_n(x,t), \tilde{v}_n(x,t)) := (u(x,t+nT), v(x,t+nT)),$$

then we can apply the standard parabolic regularity theory, together with (3.34), to deduce that, subject to a subsequence,

$$(\tilde{u}_n, \tilde{v}_n) \to (\tilde{U}_\mu, \tilde{V}_\mu) \text{ locally uniformly in } (\overline{\Omega} \setminus (\overline{\Omega_0^a} \cup \overline{\Omega_0^b})) \times (T^*, T) \text{ as } n \to \infty,$$

where $(\tilde{U}_\mu, \tilde{V}_\mu)$ is a positive solution of (3.29). Moreover,

$$\tilde{U}_\mu \geq \underline{U}_\mu \text{ and } \tilde{V}_\mu \geq \underline{V}_\mu.$$

From this, combined with (3.35), we find that

$$(\tilde{U}_\mu, \tilde{V}_\mu) = (\underline{U}_\mu, \underline{V}_\mu),$$

and therefore,

$$\lim_{n\to\infty} (u(x,t+nT), v(x,t+nT)) = (\underline{U}_\mu(x,t), \underline{V}_\mu(x,t))$$

locally uniformly in $(\overline{\Omega} \setminus (\overline{\Omega_0^a} \cup \overline{\Omega_0^b})) \times (T^*, T)$. ∎

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