Letter

Proof of the equivalence of the symplectic forms derived from the canonical and the covariant phase space formalisms

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We prove that, for any theory defined over a space-time with boundary, the symplectic form derived in the covariant phase space is equivalent to the one derived from the canonical formalism.

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I. INTRODUCTION

In physics, we can identify, roughly speaking, two somewhat disjoint frameworks to deal with a field theory: the canonical and the covariant. The former breaks all the objects of the theory into spatial and normal components. The main advantages are that we gain a dynamical perspective and some universal structures like the symplectic form of a cotangent bundle. This provides a way to approach the problem numerically [1] (essential in the study of gravitational waves and LIGO-Virgo-KAGRA observations) and a starting point for the Hamiltonian quantization [2–5]. A price to pay is the apparent loss of some symmetries.

The covariant approach, on the other hand, considers fields over the whole space-time [6]. Some noteworthy advantages are that symmetries are explicit, the study of null infinity is easier, there are methods to compute conserved quantities, and higher-derivative theories are treated on equal footing as first-order ones [7–9]. All this has important consequences in effective and perturbation theory, both of which are relevant to the of study string theories, edge modes, corner and Bondi-Metzner-Sachs algebras, or the analysis of consistent deformations [10–19]. However, one drawback is that there are no known canonical structures on the spaces involved. In particular, to have a symplectic structure, one has to rely on the covariant phase space (CPS) formalism and fix a local action.

The question that arises naturally is if both approaches are equivalent. It is possible to prove that in many aspects they are. However, the equivalence of the symplectic forms was unknown in general. The problem has been studied in concrete relevant examples, like GR without boundaries [20,21]. Also in [22], an analytical prove of the symplectic equivalence is provided over the reduced phase space (using Poisson brackets) for first-order theories with no boundaries. It is important to mention that adding boundaries complicates the matter a great deal even in concrete examples. In that sense, our paper [23] was a breakthrough because it provided a way to map any theory with boundary to another one without boundary in the CPS formalism. Although this approach would simplify the upcoming computations, we will not use it here as it requires the introduction of a lot of notation and definitions. However, in [23], we also provided a geometric language that bridges between the mathematical formalism (∞-jets framework) and the standard physics notation. We will see in this paper that this language turns out to be essential to prove, in full generality, the equivalence of both symplectic structures. As a by-product, the equivalence shows that our proposal of a CPS symplectic structure in manifolds with boundaries (introduced also in [23]) is the most natural one. Not only for its cohomological nature, as explained in that paper, but also because it is equivalent to the one coming from the canonical formalism.

II. THE GEOMETRIC ARENA

A. The spacetime

Consider a globally hyperbolic n-manifold M (up to diffeomorphism, $M=\mathbb{R}\times\Sigma$) with boundary $\partial M=\mathbb{R}\times\partial\Sigma$. We have the inclusions $\bar{\jmath}\colon \partial\Sigma\hookrightarrow\Sigma$, $\jmath\colon \partial M\hookrightarrow M$, and $\imath_t\colon \Sigma\hookrightarrow\{t\}\times\Sigma\subset M$. As usual, we have the exterior derivative d, the wedge product $\wedge$, the interior derivative $\iota_{\vec{V}}$, and the Lie derivative $\mathcal{L}_{\vec{V}}$.

In this setting, it is crucial to keep track of the orientations. We orient $\mathbb{R}$ with the standard volume form $\mathrm{d}t$ and Σ with some volume form vol_Σ [which, up to easy to handle technicalities, can be equally understood as an $(n-1)$-form on Σ or on M]. We then orient M with

$$\mathrm{vol}_M \coloneqq \mathrm{d}t \wedge \mathrm{vol}_\Sigma. \tag{1}$$

Boundaries are oriented so that Stokes' theorem holds

$$\int_M \mathrm{d}\alpha = \int_{\partial M} \jmath^* \alpha \qquad \int_\Sigma \mathrm{d}\beta = \int_{\partial\Sigma} \bar{\jmath}^* \beta. \tag{2}$$

For that, take any metric γ_{ij} on Σ and denote ν^i the unit vector field γ-normal to $\partial\Sigma$. Consider the adapted metric $g_{\alpha\beta}$ on $\mathbb{R}\times\Sigma$ such that $g^{\alpha\beta} = -\partial_t^\alpha \partial_t^\beta + (\iota_t)_i^\alpha (\iota_t)_j^\beta \gamma^{ij}$, then the unit vector field g-normal to ∂M is (up to pushforward) ν^i, i.e., $\mathcal{V}^\alpha \coloneqq (\iota_t)_i^\alpha \nu^i$. Finally, we define

$$\mathrm{vol}_{\partial\Sigma} \coloneqq \iota_{\vec{\nu}} \mathrm{vol}_\Sigma \qquad \mathrm{vol}_{\partial M} \coloneqq \iota_{\vec{\mathcal{V}}} \mathrm{vol}_M. \tag{3}$$

It is important to notice that

$$\mathrm{vol}_{\partial M} = \iota_{\vec{\mathcal{V}}}(\mathrm{d}t \wedge \mathrm{vol}_\Sigma) = -\mathrm{d}t \wedge \mathrm{vol}_{\partial\Sigma}. \tag{4}$$

B. The space of fields

Let $\mathcal{F}$ be a space of tensor fields (of any tensorial character) over M. This space is ∞-dimensional and nonlinear in general. Although it can rigorously be described with the ∞-jets formalism, for our purposes, it is enough to think of $\mathcal{F}$ as a standard smooth manifold with the usual operators such as the exterior derivative $\mathbb{d}$, the wedge product $\mathbin{\wedge\!\!\!\wedge}$, the interior derivative $\mathbb{i}_{\mathbb{V}}$, or the Lie derivative $\mathbb{L}_{\mathbb{V}}$. Here, $\mathbb{V}$ is a vector field of $\mathcal{F}$ (see [23] for a careful discussion). Of course, $\mathcal{F}$ may consist of different types of tensor fields; hence, $\mathcal{F} = \mathcal{F}^1 \times \cdots \times \mathcal{F}^N$ with the fields labeled as $(\phi^I)_{I=1\cdots N} \in \mathcal{F}$.

III. THE COVARIANT PHASE SPACE FORMALISM IN A NUTSHELL

Roughly speaking, the CPS method studies the space of solutions of a theory over the whole space-time. The equations of motion are not dynamic but, rather, give conditions for the fields to be solutions. We devote this section to summarizing how to define a presymplectic form canonically associated with a local action. For a detailed discussion and some applications, see [23–26].

Consider a local action $\mathbb{S}\colon \mathcal{F} \to \mathbb{R}$ given by

$$\mathbb{S} = \int_M L - \int_{\partial M} \bar{\ell}, \tag{5}$$

where $(L, \bar{\ell}) \in \Omega^{(n,0)}(M\times\mathcal{F}) \times \Omega^{(n-1,0)}(\partial M \times \mathcal{F})$, known as CPS bulk and boundary Lagrangians, are top-forms on $(M, \partial M)$ and 0-forms on $\mathcal{F}$ (they are bigraded forms). We assume that they are locally constructed, i.e., when evaluating $(L(\phi), \bar{\ell}(\phi)) \in \Omega^n(M) \times \Omega^{n-1}(\partial M)$ at $p \in M$, they only depend on p, $\phi(p)$, and finitely many of its derivatives at p.

It is a standard result [23,27] that the $(n,1)$-form $\mathbb{d}L$ can be split as

$$\mathbb{d}L = E_I \mathbin{\wedge\!\!\!\wedge} \mathbb{d}\phi^I + \mathrm{d}\Theta, \tag{6}$$

for some $(n,0)$-forms E_I (Euler-Lagrange forms), and some $(n-1,\ 1)$-form Θ (bulk symplectic potential, uniquely defined up to an exact form). In practice, this is achieved by using Leibniz's rule to remove all the derivatives from $\mathbb{d}\phi^I$. Taking the $\mathbb{d}$-exterior derivative of $\mathbb{S}$, using (6), and Stokes' theorem, we have

$$\mathbb{dS} = \int_M E^I \mathbin{\wedge\!\!\!\wedge} \mathbb{d}\phi^I - \int_{\partial M} (\mathbb{d}\bar{\ell} - \jmath^*\Theta). \tag{7}$$

We want to split the term of the boundary integral as in (6), but this is not always possible. We need to impose this condition by hand forcing it to be "splittable" as well, in which case we say that $\mathbb{S}$ defines a good variational principle. More specifically, we impose that

$$\mathbb{d}\bar{\ell} - \jmath^*\Theta = \bar{b}_I \mathbin{\wedge\!\!\!\wedge} \mathbb{d}\phi^I - \mathrm{d}\overline{\theta}, \tag{8}$$

for some $(n-1,\ 0)$-forms $\bar{b}_I$ (boundary Euler-Lagrange forms) and some $(n-2,\ 1)$-form $\bar{\theta}$ (boundary symplectic potential, uniquely defined up to an exact form). With these ingredients, we define the space of solutions,

$$\mathrm{Sol}(\mathbb{S}) = \{\phi \in \mathcal{F}/(E^I(\phi), \bar{b}^I(\phi)) = (0,0)\} \overset{\mathbb{j}_\mathbb{S}}{\hookrightarrow} \mathcal{F},$$

and the symplectic structure associated with an embedding $\iota\colon \Sigma \hookrightarrow M$,

$$\mathbf{\Omega}_\mathbb{S}^\iota = \int_\Sigma \mathbb{d}\iota^*\Theta - \int_{\partial\Sigma} \mathbb{d}\iota^*\bar{\theta}. \tag{9}$$

$\mathbf{\Omega}_\mathbb{S}^\iota$ is independent of the choice of Lagrangians (as long as they define the same $\mathbb{S}$) and of the symplectic potentials chosen in (6) and (8). Moreover, if we denote $\mathbf{\Omega}_\mathbb{S} = \mathbb{j}_\mathbb{S}^* \mathbf{\Omega}_\mathbb{S}^\iota$, the pullback of the symplectic form to $\mathrm{Sol}(\mathbb{S})$, then it can be proved that $\mathbf{\Omega}_\mathbb{S}$ does not depend on the embedding either. Thus, we have constructed a presymplectic form on $\mathrm{Sol}(\mathbb{S})$ canonically associated with $\mathbb{S}$.

IV. THE CANONICAL FORMALISM IN A NUTSHELL

The canonical (CAN) formalism deals, roughly speaking, with "instant fields" that evolve according to some dynamical equations. By evolving specific initial data, we obtain a curve in the space of "instantaneous fields," which corresponds (up to integrability issues) to a solution over the whole space-time. We devote this section to summarize the results necessary for the present work. For now, we will focus on first-order theories and delay the generalization to higher order ones to Sec. VI.

Consider the space $\mathcal{Q} = \mathcal{Q}^1 \times \cdots \times \mathcal{Q}^M$ of fields over Σ and its tangent bundle $T\mathcal{Q}$ (its standard geometric operators like $\mathbb{d}$ or $\curlywedge$ are denoted as the ones of $\mathcal{F}$). It consists of elements of the form $(q^1, \ldots, q^M;\, v^1, \ldots, v^M)$. Consider a local CAN Lagrangian-action $\mathcal{L}\colon\, T\mathcal{Q} \to \mathbb{R}$ given by

$$\mathcal{L} = \int_\Sigma \mathrm{L} - \int_{\partial\Sigma} l, \tag{10}$$

where $(\mathrm{L}, l) \in \Omega^{(n-1,0)}(\Sigma \times T\mathcal{Q}) \times \Omega^{(n-2,0)}(\partial\Sigma \times T\mathcal{Q})$ are some (locally constructed and possibly time dependent) CAN Lagrangians. $\mathcal{L}$ is historically called Lagrangian, but it plays a role similar to the action (5) although not entirely equal since the time integration is missing. Now we define the CAN action $\mathcal{S}\colon\, \mathcal{C}^\infty(\mathbb{R}, \mathcal{Q}) \to \mathbb{R}$,

$$\mathcal{S}(q) = \int_\mathbb{R} \mathcal{L}(q(t), \dot{q}(t))\mathrm{d}t. \tag{11}$$

The analogues to Eqs. (6) and (8) are

$$\begin{aligned} \mathbb{d}\mathrm{L} &= A_I^{(0)} \curlywedge \mathbb{d}q^I + A_I^{(1)} \curlywedge \mathbb{d}v^I + \mathrm{d}\tilde{\Theta} \\ \mathbb{d}l - \bar{\jmath}^*\tilde{\Theta} &= B_I^{(0)} \curlywedge \mathbb{d}q^I + B_I^{(1)} \curlywedge \mathbb{d}v^I - \mathrm{d}\tilde{\theta}. \end{aligned} \tag{12}$$

Taking the $\mathbb{d}$-exterior derivative of (11), using Stokes' theorem, and integrating by parts with respect to time (notice that in $\mathcal{S}$ every v^I is replaced by $\dot{q}^I$), we obtain

$$\mathbb{d}\mathcal{S} = \int_\mathbb{R} \mathrm{d}t \left(\int_\Sigma (A_I^{(0)} - \dot{A}_I^{(1)}) \curlywedge \mathbb{d}q^I - \int_{\partial\Sigma} (B_I^{(0)} - \dot{B}_I^{(1)}) \curlywedge \mathbb{d}q^I \right).$$

$(A_I^{(0)} - \dot{A}_I^{(1)},\ B_I^{(0)} - \dot{B}_I^{(1)})$ are the bulk and boundary dynamical equations (if some of them do not involve time derivatives, we obtain bulk or boundary constraints).

Once we have the Lagrangian formalism, we proceed to introduce the Hamiltonian formalism which, instead of living in the tangent bundle $T\mathcal{Q}$, lives on the cotangent bundle $T^*\mathcal{Q}$. The advantage of the latter is its canonical symplectic form, which plays an essential role on the Hamiltonian formulation. Indeed, denoting the elements of $T^*\mathcal{Q}$ as $(q; p) = (q^1, \ldots, q^M; p_1, \ldots, p_M)$, the canonical symplectic structure is given by

$$\mathbf{\Omega}_{T^*\mathcal{Q}} \coloneqq \mathbb{d}q^I \curlywedge \mathbb{d}p_I. \tag{13}$$

The usual pairing between position and momenta applies when evaluated over fields. In order to go from the Lagrangian (tangent bundle) to the Hamiltonian (cotangent bundle), we use the fiber derivative $F\mathcal{L}\colon\, T\mathcal{Q} \to T^*\mathcal{Q}$. For each $(q; v) \in T_q\mathcal{Q}$, we define $F\mathcal{L}_{(q;v)} \in T_q^*\mathcal{Q}$ as

$$F\mathcal{L}_{(q;v)}(q; w) = \frac{\mathrm{d}}{\mathrm{d}\tau}\bigg|_{\tau=0} \mathcal{L}(q; v_\tau), \tag{14}$$

where v_τ is a curve in $T_q\mathcal{Q}$ with $v_0 = v$ and $\frac{\mathrm{d}}{\mathrm{d}\tau}\big|_0 v_\tau = w$. In general, this map is not surjective, and the relevant symplectic structure is actually the one induced on its image (the primary constraint submanifold), i.e., induced by the inclusion $\mathbb{j}_\mathcal{L}\colon\, F\mathcal{L}(T\mathcal{Q}) \hookrightarrow T^*\mathcal{Q}$. Since over $F\mathcal{L}(T\mathcal{Q})$ we have $p = F\mathcal{L}_{(q;v)}$, we schematically obtain

$$\mathbf{\Omega}_\mathcal{L} \coloneqq \mathbb{j}_\mathcal{L}^* \mathbf{\Omega}_{T^*\mathcal{Q}} = \mathbb{d}q^I \curlywedge \mathbb{d}\mathbb{j}_\mathcal{L}^* p_I = \mathbb{d}q^I \curlywedge \mathbb{d}(F\mathcal{L}_{(q;v^I)}), \tag{15}$$

where by v^I we mean $(0, \ldots, 0, v^I, 0, \ldots, 0)$. In order to give a concrete sense to (15), we rely on the geometric language introduced before. The key observation is that the momenta can be rewritten as

$$\begin{aligned} p_I(q; w^I) &\coloneqq \mathbb{L}_{(0,w^I)}\mathcal{L} = \mathbb{i}_{(0,w^I)}\mathbb{d}\mathcal{L} \\ &= \int_\Sigma \mathbb{i}_{(0,w^I)}\mathbb{d}\mathrm{L} - \int_{\partial\Sigma} \mathbb{i}_{(0,w^I)}\mathbb{d}l. \end{aligned} \tag{16}$$

The Lie derivative acts following Cartan's rule while the interior derivative acts as follows: $\mathbb{i}_{(\alpha^I,\beta^J)}\mathbb{d}q^K = \delta_I^K \alpha^I$ and $\mathbb{i}_{(\alpha^I,\beta^J)}\mathbb{d}v^K = \delta_J^K \beta^J$. Using (16), Eq. (12), and Stokes' theorem leads to (removing the argument)

$$p_I(\cdot) = \int_\Sigma A_I^{(1)} \curlywedge \cdot - \int_{\partial\Sigma} B_I^{(1)} \curlywedge \cdot$$

Taking its $\mathbb{d}$-exterior derivative and using the $p - q$ pairing finally allows us to rewrite (15) as

$$\mathbf{\Omega}_\mathcal{L} = \int_\Sigma \mathbb{d}A_I^{(1)} \curlywedge \mathbb{d}q^I - \int_{\partial\Sigma} \mathbb{d}B_I^{(1)} \curlywedge \mathbb{d}q^I. \tag{17}$$

V. PROVING THE EQUIVALENCE FOR FIRST ORDER THEORIES

In the previous sections, we have seen that, on the one hand, we can derive a symplectic structure $\mathbf{\Omega}_\mathbb{S}$ canonically associated with a CPS action $\mathbb{S}$ (defined over the space of fields of M). On the other hand, from a CAN Lagrangian-action $\mathcal{L}$ defined over some $T\mathcal{Q}$, we get the symplectic structure $\mathbf{\Omega}_\mathcal{L}$ canonically associated with $\mathcal{L}$. It is well known that we can go from the CPS formalism to the CAN one by performing a $(1,\, n-1)$ decomposition (there are many equivalent ways of doing it). Therefore, for a given theory we end up with two symplectic structures canonically associated with it. The issue of understanding the relation between them has been a long-lasting open question that we answer now.

A. CANonicalizing the Lagrangian

In order to go from the CPS Lagrangians $(L, \bar{\ell})$ to the CAN Lagrangians (L, l), we need Fubini's theorem.

Theorem: Let A and B be manifolds oriented with vol_A and vol_B, respectively. We orient $A \times B$ with $\mathrm{vol}_A \wedge \mathrm{vol}_B$ and consider an integrable function $f\colon A \times B \to \mathbb{R}$, then

$$\int_{A\times B} f \mathrm{vol}_A \wedge \mathrm{vol}_B = \int_A \left(\int_B f \mathrm{vol}_B \right) \mathrm{vol}_A.$$

Let us now use the (1, $n-1$) decomposition induced from $M = \mathbb{R} \times \Sigma$ to decompose the CPS Lagrangians. In general, given $\alpha \in \Omega^k(M)$, we have $\alpha = \mathrm{d}t \wedge \alpha_\perp + \alpha^\top$ with $\alpha_\perp \coloneqq \iota_{\partial_t} \alpha$ and $\alpha^\top \coloneqq \iota_{\partial_t}(\mathrm{d}t \wedge \alpha)$. Since the CPS Lagrangians are top-forms, we simply have $L = \mathrm{d}t \wedge L_\perp$ and $\overline{\ell} = \mathrm{d}t \wedge \overline{\ell}_\perp$. Besides, there exist F, f such that

$$\mathrm{d}t \wedge L_\perp = L = F \mathrm{vol}_M \overset{(1)}{=} \mathrm{d}t \wedge (F \mathrm{vol}_\Sigma) \to F = \frac{L_\perp}{\mathrm{vol}_\Sigma}$$

$$\mathrm{d}t \wedge \overline{\ell}_\perp = \overline{\ell} = f \mathrm{vol}_{\partial M} \overset{(4)}{=} (-\mathrm{d}t) \wedge (f \mathrm{vol}_{\partial\Sigma}) \quad \to f = \frac{-\bar{\ell}_\perp}{\mathrm{vol}_{\partial\Sigma}},$$

where $\frac{\omega}{\mathrm{vol}}$ denotes the function that relates the top-form ω with the volume form vol. Plugging these expressions of $(L, \bar{\ell})$ into (5) and using Fubini's theorem, we get

$$\mathbb{S} = \int_{\mathbb{R}} \left(\int_\Sigma L_\perp - \int_{\partial\Sigma} (-\bar{\ell}_\perp) \right) \mathrm{d}t.$$

A minus sign appears in the boundary because, from (4), we have $(A, \mathrm{vol}_A) = (\mathbb{R}, -\mathrm{d}t)$ in Fubini's theorem.

All this allows us to identify L with $\iota_{\partial_t} L$ and l with $-\iota_{\partial_t} \bar{\ell}$. To formalize this identification, we perform the (1, $n-1$) decomposition on the fields $(\phi^I)_I$ and their derivatives. Since the theory is of first order, we end up with some tangential fields $(\hat{q}^J)_J$ and their velocities $\hat{v}^J \coloneqq \mathcal{L}_{\partial_t} \hat{q}^J$. We now express $(\iota_{\partial_t} L, -\iota_{\partial_t} \bar{\ell})$ in terms of $(\hat{q}^J; \hat{v}^J)_J$ and pull everything back to Σ [where the positions and velocities are denoted as $(q^J; v^J)_J$] to obtain (L, l). Plugging them into Eq. (10) leads to a CAN Lagrangian-action associated with $\mathbb{S}$ that we denote $\mathcal{L}_{\mathbb{S}}$.

B. CANonicalizing the symplectic potentials

Now we want to rewrite the symplectic potentials derived from the CPS formalism in a way suitable for the comparison with the CAN formalism. For that, we compute

$$\begin{aligned} E_I \wedge\!\!\!\wedge \mathbb{d}\phi^I + \mathrm{d}\Theta = \mathbb{d}L = \mathrm{d}t \wedge \mathbb{d}\iota_{\partial_t} L &\overset{\dagger}{=} \\ &= \mathrm{d}t \wedge (\hat{A}_I^{(0)} \wedge\!\!\!\wedge \mathbb{d}\hat{q}^I + \hat{A}_I^{(1)} \wedge\!\!\!\wedge \mathbb{d}\mathcal{L}_{\partial_t} \hat{q}^I + \mathrm{d}\hat{\Theta}) \\ &= \mathrm{d}t \wedge (\hat{A}_I^{(0)} - \mathcal{L}_{\partial_t} \hat{A}_I^{(1)}) \wedge\!\!\!\wedge \mathbb{d}\hat{q}^I \\ &\quad + \mathrm{d}(\iota_{\partial_t}(\mathrm{d}t \wedge \hat{A}_I^{(1)} \wedge\!\!\!\wedge \mathbb{d}\hat{q}^I) - \mathrm{d}t \wedge \hat{\Theta}). \end{aligned}$$

In the † equality, we have "lifted" (12) from Σ to M (the corresponding objects are denoted with a hat) taking into account that, although additional terms might seem to appear, they have a dt in them so they vanish in our previous computation. We know from [23,28,29], that if $r < n$ and $s > 0$, then any d-closed (r, s)-form is also d-exact. So there exists some $Z \in \Omega^{(n-2,1)}(M)$ such that

$$\Theta = \iota_{\partial_t}(\mathrm{d}t \wedge \hat{A}_I^{(1)} \wedge\!\!\!\wedge \mathbb{d}\hat{q}^I) - \mathrm{d}t \wedge \hat{\Theta} + \mathrm{d}Z. \tag{18}$$

Since $\jmath^*$ and ι_{∂_t} commute (∂_t is tangent to the boundary), if we compute $\jmath^*\Theta$, the first term vanishes. Hence,

$$\begin{aligned} b_I \wedge\!\!\!\wedge \mathbb{d}\phi^I - \mathrm{d}\bar{\theta} = \mathbb{d}\bar{\ell} - \jmath^*\Theta &\overset{(18)}{=} \\ &= -\mathrm{d}t \wedge (\mathbb{d}(-\iota_{\partial_t} \bar{\ell}) - \jmath^*\hat{\Theta}) - \mathrm{d}\jmath^* Z \overset{(12)}{=} \\ &= -\mathrm{d}t \wedge (\hat{B}_I^{(0)} - \mathcal{L}_{\partial_t} \hat{B}_I^{(1)}) \wedge\!\!\!\wedge \mathbb{d}\hat{q}^I \\ &\quad - \mathrm{d}(\iota_{\partial_t}(\mathrm{d}t \wedge \hat{B}_I^{(1)} \wedge\!\!\!\wedge \mathbb{d}\hat{q}^I) - \mathrm{d}t \wedge \hat{\theta} + \jmath^* Z). \end{aligned}$$

Thus, there exists some $\bar{z} \in \Omega^{(n-3,1)}(\partial M)$ such that

$$\bar{\theta} = \iota_{\partial_t}(\mathrm{d}t \wedge \hat{B}_I^{(1)} \wedge\!\!\!\wedge \mathbb{d}\hat{q}^I) - \mathrm{d}t \wedge \hat{\theta} + \jmath^* Z - \mathrm{d}\bar{z}. \tag{19}$$

Finally, we rewrite the CPS symplectic potentials in a more suitable way,

$$\begin{aligned} \Theta &= \hat{A}_I^{(1)} \wedge\!\!\!\wedge \mathbb{d}\hat{q}^I + \mathrm{d}t \wedge (\cdots) + \mathrm{d}Z \\ \bar{\theta} &= \hat{B}_I^{(1)} \wedge\!\!\!\wedge \mathbb{d}\hat{q}^I + \mathrm{d}t \wedge (\cdots) - \mathrm{d}t \wedge \hat{\theta} + \jmath^* Z - \mathrm{d}\bar{z}. \end{aligned} \tag{20}$$

C. CANonicalizing the symplectic form

Since the symplectic structure over Sol($\mathbb{S}$) does not depend on the embedding, we consider ι_{t_0} (where $\mathrm{d}t = 0$, so in particular, $\iota_{t_0}^* \hat{A}_I = A_I$ and $\iota_{t_0}^* \hat{B}_I = B_I$), and we finally obtain the desired equivalence,

$$\mathbf{\Omega}_{\mathbb{S}} \underset{(20)}{\overset{(9)}{=}} \int_\Sigma \mathbb{d}A_I^{(1)} \wedge\!\!\!\wedge \mathbb{d}q^I - \int_{\partial\Sigma} \mathbb{d}B_I^{(1)} \wedge\!\!\!\wedge \mathbb{d}q^I \overset{(17)}{=} \mathbf{\Omega}_{\mathcal{L}_{\mathbb{S}}}. \tag{21}$$

VI. PROVING THE EQUIVALENCE FOR HIGHER ORDER THEORIES

In Sec. V A, we obtained the first-order CAN Lagrangians from the CPS ones by breaking $(\phi^I)_I$ into

$(\hat{q}^J, \mathcal{L}_{\partial_t}\hat{q}^J)_J$. However, for general theories, higher order "velocities" $\{(\mathcal{L}_{\partial_t})^\mu \hat{q}^J\}_{\mu=1\cdots K}$ will appear. While the CPS formalism does not change, the CAN one changes drastically. We devote this section to briefly summarizing how to prove the equivalence for higher order theories.

A. The higher order canonical formalism in a nutshell

For a detailed description, see [30]. For our purposes, we only need to generalize some of the equations of Sec. V A. First, Eq. (12) has to be changed to

$$\begin{aligned} \mathbb{d}\mathrm{L} &= \sum_\mu A_I^{(\mu)} \mathbin{\wedge\!\!\!\wedge} \mathbb{d}q^I_{(\mu)} + \mathbb{d}\tilde{\Theta} \\ \mathbb{d}l - \bar{\jmath}^*\tilde{\Theta} &= \sum_\mu B_I^{(\mu)} \mathbin{\wedge\!\!\!\wedge} \mathbb{d}q^I_{(\mu)} - \mathbb{d}\tilde{\theta}, \end{aligned} \tag{22}$$

where $q^I_{(0)}$ are the positions, $q^I_{(1)}$ their velocities, $q^I_{(2)}$ their accelerations, and so on up to $\mu = K$. Thus, we are working on the K-th tangent bundle $T^K\mathcal{Q}$. The Hamiltonian formalism takes place in $T^*(T^{K-1}\mathcal{Q})$, where we have $\{q^I_{(\mu)}\}_{\mu=0\cdots K-1}$ and their momenta $\{p_I^{(\mu)}\}_{\mu=0\cdots K-1}$. The canonical symplectic structure generalizing (13) is

$$\mathbf{\Omega}_{T^*(T^{K-1}\mathcal{Q})} \coloneqq \sum_{\mu=0}^{K-1} \mathbb{d}q^I_{(\mu)} \mathbin{\wedge\!\!\!\wedge} \mathbb{d}p_I^{(\mu)}. \tag{23}$$

In order to go from the Lagrangian to the Hamiltonian formulation, we have to generalize the fiber derivative. For our purposes, it is enough to generalize (16). The standard way is

$$\begin{aligned} p_I^{(K)}(q; w^I) &\coloneqq \mathbb{L}_{(0,w^I)}\mathcal{L} \\ p_I^{(\mu)}(q; w^I) &\coloneqq \mathbb{L}_{(0,w^I)}\mathcal{L} - \frac{\mathrm{d}}{\mathrm{d}t} p_I^{(\mu+1)}(q; w^I), \end{aligned}$$

for $\mu = 1, \ldots, K-1$. It is not hard to prove by (backwards) induction that

$$p_I^{(\mu)}(\cdot) = \sum_{k=\mu}^{K} \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^{k-\mu} \left\{ \int_\Sigma A_I^{(k)} \mathbin{\wedge\!\!\!\wedge} \cdot - \int_{\partial\Sigma} B_I^{(k)} \mathbin{\wedge\!\!\!\wedge} \cdot \right\}.$$

Taking the $\mathbb{d}$-exterior derivative, plugging the result into (23), and performing some standard manipulations with double finite sums, we obtain

$$\begin{aligned} \mathbf{\Omega}_\mathcal{L} = \sum_{k=1}^{K}\sum_{\mu=0}^{k-1} \Bigg\{ &\int_\Sigma \mathbb{d}\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^{k-1-\mu} A_I^{(k)} \mathbin{\wedge\!\!\!\wedge} \mathbb{d}q^I_{(\mu)} \\ &- \int_{\partial\Sigma} \mathbb{d}\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^{k-1-\mu} B_I^{(k)} \mathbin{\wedge\!\!\!\wedge} \mathbb{d}q^I_{(\mu)} \Bigg\}. \end{aligned} \tag{24}$$

Taking $K = 1$ leads, as expected, to (17).

B. Equivalence for higher order theories

Reasoning by induction, using computations similar to those of Sec. V B, and Eq. (22) [where $q^I_{(\mu)}$ has to be replaced by $(\mathcal{L}_{\partial_t})^\mu \hat{q}^I$ since we are in the CPS formalism], we obtain

$$\begin{aligned} \mathbb{d}L &= \mathrm{d}t \wedge \sum_\mu (-\mathcal{L}_{\partial_t})^\mu \hat{A}_I^{(\mu)} \wedge \mathbb{d}\hat{q}^I + \mathrm{d}\Theta \\ \mathbb{d}\bar{\ell} - \jmath^*\Theta &= -\mathrm{d}t \wedge \sum_\mu (-\mathcal{L}_{\partial_t})^\mu \hat{B}_I^{(\mu)} \wedge \mathbb{d}\hat{q}^I - \mathrm{d}\bar{\theta}, \end{aligned} \tag{25}$$

where (up to exact forms as in (20))

$$\begin{aligned} \Theta &= \sum_{k=1}^{K}\sum_{\mu=0}^{k-1} (-\mathcal{L}_{\partial_t})^{k-1-\mu} \hat{A}_I^{(k)} \wedge \mathbb{d}(\mathcal{L}_{\partial_t})^\mu \hat{q}^I - \mathrm{d}t \wedge (\cdots) \\ \bar{\theta} &= \sum_{k=1}^{K}\sum_{\mu=0}^{k-1} (-\mathcal{L}_{\partial_t})^{k-1-\mu} \hat{B}_I^{(k)} \wedge \mathbb{d}(\mathcal{L}_{\partial_t})^\mu \hat{q}^I - \mathrm{d}t \wedge (\cdots). \end{aligned}$$

Plugging them into (9), considering an embedding $\imath_{t_0}$, and taking into account that in CAN $(\mathcal{L}_{\partial_t})^\mu \hat{q}^I$ is identified with $q^I_{(\mu)}$, we obtain the general equivalence,

$$\mathbf{\Omega}_\mathbb{S} \overset{(24)}{=} \mathbf{\Omega}_{\mathcal{L}_\mathbb{S}}.$$

VII. CONCLUSIONS AND COMMENTS

We have proved the equivalence of the symplectic form induced by the Hamiltonian formalism and the one derived in the covariant phase space. The equivalence, which has been an open question for several decades, holds for theories of any order and even when boundaries are present. The proof relies strongly on the geometric formalism introduced in [23], where we also checked the equivalence in some concrete examples. Finally, this work also proves that the symplectic form introduced in [23] for manifolds with boundaries is the natural one.

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