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Quadrature Formulae of Euler-Maclaurin Type Based on Generalized Euler Polynomials of Level m

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Dedicated to the memory of Professor Tom M. Apostol (1923 – 2016)

Abstract

This article deals with some properties -which are, to the best of our knowledge, new- of the generalized Euler polynomials of level m. These properties include a new recurrence relation satisfied by these polynomials and quadrature formulae of Euler-Maclaurin type based on them. Numerical examples are also given.

Keywords: Euler polynomials; generalized Euler polynomials of level m; Euler-Maclaurin quadrature formulae; quadrature formula.

1 Introduction

There exist standard quadrature formulae for numerically integrating different classes of real valued functions. When we consider the set $C^s[a, b]$ of all s-times continuously differentiable functions defined on $[a, b]$, then the so-called Euler-Maclaurin summation formula (also known as the composite

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trapezoidal rule) arises (cf. [1–3], and [4, Chap. 2, Sec. 3, p. 30]). More precisely, for a fixed $n \in \mathbb{N}$ and every $s \geq 1$, let $f \in C^s[0,n]$, then

$$\begin{aligned}\int_0^n f(t)\,dt =& \frac{f(0)+f(n)}{2} + \sum_{k=1}^{n-1} f(k) \\ &+ \sum_{r=1}^{\lfloor s/2 \rfloor} \left(f^{(2r-1)}(0) - f^{(2r-1)}(n)\right) \frac{B_{2r}}{(2r)!} + R_s(f),\end{aligned} \tag{1}$$

where the remainder term $R_s(f)$ can be written as

$$R_s(f) = \frac{(-1)^s}{s!} \int_0^n f^{(s)}(t) B_s(t - \lfloor t \rfloor)\,dt,$$

with $\lfloor t \rfloor$ the floor function, $B_s(t)$ and B_{2r} the s-th Bernoulli polynomial and the Bernoulli numbers for $1 \leq r \leq \lfloor s/2 \rfloor$, respectively, [1,3,5].

Or more generally, let $g \in C^s[a,b]$, for a fixed $n \in \mathbb{N}$ we set $h = \frac{b-a}{n}$, $x_i = a + ih$, $g_i = g(x_i)$, $i = 0, \ldots, n$, and $g_i^{(r)} = g^{(r)}(x_i)$, $r = 1, \ldots, s$. Then we have (cf. [3, Theorem 1]):

$$\begin{aligned}\int_a^b g(t)\,dt =& h\left(\frac{g(a)}{2} + g_1 + \cdots + g_{n-1} + \frac{g(b)}{2}\right) \\ &+ \sum_{r=1}^{\lfloor s/2 \rfloor} h^{2r} \left(g_0^{(2r-1)} - g_n^{(2r-1)}\right) \frac{B_{2r}}{(2r)!} + R_s(g),\end{aligned} \tag{2}$$

where

$$R_s(g) = \frac{(-h)^s}{s!} \int_a^b g^{(s)}(t) B_s\left(\frac{t-a}{h} - \left\lfloor \frac{t-a}{h} \right\rfloor\right) dt.$$

Thus, (2) may be viewed as an extension of the trapezoid rule by the inclusion of correction terms g_i, $g_0^{(2r-1)}$ and $g_n^{(2r-1)}$, $i = 1, \ldots, n-1$, $r = 1, \ldots, \lfloor s/2 \rfloor$.

It is well-known that neither Euler nor Maclaurin found the formulae with remainder (1) and (2), the first to do this was Poisson, in 1823. Eleven years later, Jacobi presented one of the earliest derivations of the Euler-Maclaurin summation formula [6]. Since then the formulae (1) and (2) have been derived in different ways (see e.g., [3] and the references thereof). An apart mention deserves the remarkable works of T.M. Apostol [1] and V. Lampret [3], which present nice and completely elementary derivations of the classical Euler-Maclaurin formula, respectively.

The quadrature formula (1) may be regarded as an extension of the trapezoidal rule and it can be useful for numerically integrating of periodic functions (cf., e.g., [2, Chap. 2, Sec. 2.9, p. 134], or [7–9]). The quadrature formula (2) refers to a fixed interval, which would be advantageous in certain situations: some types of integrals can be transformed to a form suitable for the trapezoidal rule, such transformations are known as exponential and double exponential quadrature rules [10].

Also, it is well-known that the Euler-Maclaurin summation formula is implemented in the Wolfram *Mathematica* as the function NSum with option Method $\rightarrow$ Integrate [3, 11]. The command NSum is used for Wolfram *Mathematica* to obtain a numerical evaluation of sums, it includes a certain number of terms explicitly, and then tries to estimate the contribution of the remaining ones. There are three approaches to estimating this contribution, one of such approaches uses the Euler-Maclaurin formula, and it is based on approximating the sum by an integral (cf. [11, pp. 269-270]).

There exists several earlier papers associated with generalizations, modifications and applications of the classical quadrature formula of Euler-Maclaurin (see for instance, [12–19]).

Recent and interesting works dealing with the Appell and Apostol type polynomials, their properties and applications in several areas as such as combinatorics, number theory, numerical analysis and partial differential equations, can be found by reviewing the current literature on this subject. For a broad information on new research trends about these classes of polynomials we strongly recommend to the interested reader see [20–28].

This paper provides quadrature formulae of Euler-Maclaurin type based on generalized Euler polynomials of level $m \in \mathbb{N}$. This class of polynomials can be seen as a generalization of the classical Euler polynomials and it constitutes a particular case of the so-called generalized Apostol-Euler polynomials and the extensions of generalized Apostol-type polynomials [29, 30], respectively. The interested reader may find recent literature which contains a large number of new and interesting properties involving these polynomials (see for instance, [29] and the references thereof).

The outline of the paper is as follows. In Section 2 some relevant properties of the generalized Euler polynomials of level m are given. In particular, we show a new recurrence formula for these polynomials which is compared with whose recently obtained in [31]. Section 3 contains the basic ideas in order to obtain quadrature formulae of Euler-Maclaurin type based on generalized Euler polynomials of level m (see Theorems 2 and 3). Finally, Section 4 is devoted to show some numerical examples. As usual, throughout this paper the convention $0^0 = 1$ will be adopted and an empty sum will be interpreted to be zero.

2 Some properties of the generalized Euler polynomials

For a fixed $m \in \mathbb{N}$, the generalized Euler polynomials of level m are defined by means of the following generating function [5].

$$\frac{2^m e^{xz}}{e^z + \sum_{l=0}^{m-1} \frac{z^l}{l!}} = \sum_{n=0}^{\infty} E_n^{[m-1]}(x) \frac{z^n}{n!}, \quad |z| < \pi. \tag{3}$$

And, the generalized Euler numbers of level m are defined by $E_n^{[m-1]} := E_n^{[m-1]}(0)$, for all $n \geq 0$. It is clear that if $m = 1$ in (3), then we obtain the classical Euler polynomials $E_n(x)$, and classical Euler numbers, respectively, i.e., $E_n(x) = E_n^{[0]}(x)$, and $\mathcal{E}_n = 2^n E_n^{[0]}\left(\frac{1}{2}\right) = 2^n E_n\left(\frac{1}{2}\right)$, respectively, for all $n \geq 0$.

The generalized Euler polynomials of level m and the generalized Euler numbers of level m can be seen as the analogous of the generalized Bernoulli polynomials of level m and the generalized Bernoulli numbers of level m, respectively. These last polynomials and numbers were introduced by Natalini and Bernardini in [32] as a generalization of the classical Bernoulli polynomials, and classical Bernoulli numbers, respectively.

For example, the first six generalized Euler polynomials of level $m = 3$ are:

$$\begin{aligned}
E_0^{[2]}(x) &= 4,\\
E_1^{[2]}(x) &= 4(x-1)\\
E_2^{[2]}(x) &= 4(x-1)^2,\\
E_3^{[2]}(x) &= 4x^3-12x^2+12x-2,\\
E_4^{[2]}(x) &= 4x^4-16x^3+24x^2-8x-10,\\
E_5^{[2]}(x) &= 4x^5-20x^4+40x^3-20x^2-50x+58.
\end{aligned}$$

The following theorem summarizes some properties of the generalized Euler polynomials of level m (cf. [29, 30]).

Theorem 1. *For a fixed $m \in \mathbb{N}$, let $\left\{E_n^{[m-1]}(x)\right\}_{n\geq 0}$ be the sequence of generalized Euler polynomials of level m. Then the following statements hold.*

(a) *Summation formulas (cf., e.g., [22]). For every $n \geq 0$,*

$$E_n^{[m-1]}(x+y) = \sum_{k=0}^{n} \binom{n}{k} y^k \, E_{n-k}^{[m-1]}(x) = \sum_{k=0}^{n} \binom{n}{k} E_k^{[m-1]}(y) \, x^{n-k}.$$

In particular,

$$E_n^{[m-1]}(x) = \sum_{k=0}^{n} \binom{n}{k} E_k^{[m-1]} \, x^{n-k}.$$

(b) *Differential relations (Appell polynomial sequences, cf.[33]). For $n, j \geq 0$ with $0 \leq j \leq n$, we have*

$$[E_n^{[m-1]}(x)]^{(j)} = \frac{n!}{(n-j)!} \, E_{n-j}^{[m-1]}(x). \tag{4}$$

(c) *Inversion formula. For every $n \geq 0$,*

$$2^m x^n = \sum_{k=0}^{n} \binom{n}{k} (1+a_{k,m}) E_{n-k}^{[m-1]}(x), \tag{5}$$

where

$$a_{k,m} = \begin{cases} 1, & 0 \leq k < m, \\ 0, & k \geq m. \end{cases}$$

(d) *Integral formulas.*

$$\int_{x_0}^{x_1} E_n^{[m-1]}(x)dx = \frac{1}{n+1}\left[E_{n+1}^{[m-1]}(x_1) - E_{n+1}^{[m-1]}(x_0)\right]$$
$$= \sum_{k=0}^{n} \frac{1}{n-k+1}\binom{n}{k} E_k^{[m-1]}((x_1)^{n-k+1} - (x_0)^{n-k+1}).$$

$$E_n^{[m-1]}(x) = n\int_0^x E_{n-1}^{[m-1]}(t)dt + E_n^{[m-1]}. \tag{6}$$

(e) *Recurrence relation. For any $m \geq 2$ and $n \geq 0$, the following recurrence relation for the generalized Euler polynomials of level m is satisfied.*

$$\begin{aligned} E_{n+1}^{[m-1]}(x) = {} & \left(2xE_n^{[m-2]} - E_n^{[m-1]}\right) \\ & + \frac{1}{2^{m-1}} \sum_{k=1}^{n} \left[\binom{n}{k}\left(2xE_{n-k}^{[m-2]} - E_{n-k}^{[m-1]}\right)\right. \\ & \left. -2\binom{n}{k-1} E_{n-k+1}^{[m-2]}\right] E_k^{[m-1]}(x). \end{aligned} \tag{7}$$

(f) *Differential equation. For any $m \geq 2$, the generalized Euler polynomials $E_n^{[m-1]}(x)$ satisfy the differential equation:*

$$\begin{aligned} 0 = {} & \left[\frac{2}{n!}\left(E_n^{[m-2]} - 1\right) + \frac{2xE_{n-1}^{[m-2]} - E_{n-1}^{[m-1]}}{(n-1)!}\right] y^{(n)} \\ & + \left[\frac{2}{(n-1)!}\left(E_{n-1}^{[m-2]} - 2\right) + \frac{2xE_{n-2}^{[m-2]} - E_{n-2}^{[m-1]}}{(n-2)!}\right] y^{(n-1)} \\ & + \cdots + \left[2^{m-1}(1-x) - n + 1 + E_2^{[m-2]}\right] y'' \\ & + \left[2^{m-1}(x-2) - 2n\right] y' - n2^{m-1}y. \end{aligned} \tag{8}$$

Proof. Since $(a), (b)$ and (d) are straightforward consequences of (3), and a suitable use of the Fundamental Theorem of Calculus, respectively, we shall omit their proof. So, we focus our efforts on the proof of $(c), (e)$ and (f).

By (3) and direct calculations, we have

$$\begin{aligned} 2^m e^{xz} &= \left[e^z + \sum_{l=0}^{m-1} \frac{z^l}{l!}\right] \left[\sum_{n=0}^{\infty} E_n^{[m-1]}(x) \frac{z^n}{n!}\right], \\ &= \left[\sum_{n=0}^{\infty} (1 + a_{n,m}) \frac{z^n}{n!}\right] \left[\sum_{n=0}^{\infty} E_n^{[m-1]}(x) \frac{z^n}{n!}\right], \end{aligned}$$

where

$$a_{n,m} = \begin{cases} 1, & 0 \leq n < m, \\ 0, & n \geq m. \end{cases}$$

Or equivalently,

$$2^m \sum_{n=0}^{\infty} x^n \frac{z^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} (1 + a_{k,m}) E_{n-k}^{[m-1]}(x) \right) \frac{z^n}{n!}. \tag{9}$$

Comparing the coefficients of z^n on both sides of (9), we get the desired result of the part (c).

In order to prove (e), we proceed as in the proof of [32, Lemma 3.2], making the corresponding modifications. For $m \geq 2$ and $n \geq 0$, let us consider the generating function

$$E^{[m-1]}(x, z) = \frac{2^m e^{xz}}{e^z + \sum_{l=0}^{m-1} \frac{z^l}{l!}}.$$

Then, differentiation of $E^{[m-1]}(x, z)$ with respect to z, yields

$$\begin{aligned} \frac{\partial}{\partial z} E^{[m-1]}(x, z) =& x E^{[m-1]}(x, z) - \frac{E^{[m-1]}(0, z) E^{[m-1]}(x, z)}{2} \left(\sum_{l=0}^{m-2} \frac{z^l}{l!} \right) \\ & \left(x - \frac{E^{[m-1]}(0, z)}{2 E^{[m-2]}(0, z)} \right) E^{[m-1]}(x, z). \end{aligned} \tag{10}$$

So, from differentiation with respect to z on the right hand side of (3) and (10), we can deduce that

$$\sum_{n=0}^{\infty} E_{n+1}^{[m-1]}(x) \frac{z^n}{n!} = \left(x - \frac{E^{[m-1]}(0, z)}{2 E^{[m-2]}(0, z)} \right) E^{[m-1]}(x, z). \tag{11}$$

Or equivalently,

$$2E^{[m-2]}(0,z)\sum_{n=0}^{\infty}E_{n+1}^{[m-1]}(x)\frac{z^n}{n!}=\left(2xE^{[m-2]}(0,z)-E^{[m-1]}(0,z)\right)E^{[m-1]}(x,z). \tag{12}$$

The left hand side of (12) coincides with the product of the following two series:

$$\left[\sum_{n=0}^{\infty}2E_n^{[m-2]}\frac{z^n}{n!}\right]\left[\sum_{n=0}^{\infty}E_{n+1}^{[m-1]}(x)\frac{z^n}{n!}\right]=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}2\binom{n}{k}E_{n-k}^{[m-2]}E_{k+1}^{[m-1]}(x)\right]\frac{z^n}{n!}, \tag{13}$$

and the right hand side of (12) coincides with the product of the following two series:

$$\begin{aligned}&\left[\sum_{n=0}^{\infty}\left(2xE_n^{[m-2]}-E_n^{[m-1]}\right)\frac{z^n}{n!}\right]\left[\sum_{n=0}^{\infty}E_n^{[m-1]}(x)\frac{z^n}{n!}\right]\\&=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}\left(2xE_{n-k}^{[m-2]}-E_{n-k}^{[m-1]}\right)E_k^{[m-1]}(x)\right]\frac{z^n}{n!}.\end{aligned} \tag{14}$$

Comparing the equations (13) and (14) (on the right hand side of each one), we get

$$2\sum_{k=0}^{n}\binom{n}{k}E_{n-k}^{[m-2]}E_{k+1}^{[m-1]}(x)=\sum_{k=0}^{n}\binom{n}{k}\left(2xE_{n-k}^{[m-2]}-E_{n-k}^{[m-1]}\right)E_k^{[m-1]}(x). \tag{15}$$

Then (7) immediately follows by a suitable rearrangement of the terms on both sides of (15).

Apart from minor changes, the proof of (f) relies on similar arguments to those from the proof of [32, Theorem 3.1]. Using (4) we can rewrite (15) as follows

$$2^{m-1}E_{n+1}^{[m-1]}(x)=\left[\sum_{k=0}^{n}\left(\frac{2xE_{n-k}^{[m-2]}-E_{n-k}^{[m-1]}}{(n-k)!}-\frac{2k}{(n-k+1)!}\right)D_x^{n-k}\right]E_n^{[m-1]}(x),$$

where $D_x^{n-k}:=\frac{d^{n-k}}{dx^{n-k}}$. So, the operator $\mathcal{D}_{n,m}^{+}$ given by

$$\mathcal{D}_{n,m}^{+}(f):=\frac{1}{2^{m-1}}\sum_{k=0}^{n}\left(\frac{2xE_{n-k}^{[m-2]}-E_{n-k}^{[m-1]}}{(n-k)!}-\frac{2k}{(n-k+1)!}\right)D_x^{n-k}(f),$$

satisfies

$$\mathcal{D}^{+}_{n,m} E^{[m-1]}_{n}(x) = E^{[m-1]}_{n+1}(x). \tag{16}$$

Now, applying the operator $\Delta^{-}_{n+1} := \frac{1}{n+1} D_x$ on both sides of (16), we have

$$\left(\Delta^{-}_{n+1} \mathcal{D}^{+}_{n,m}\right) E^{[m-1]}_{n}(x) = E^{[m-1]}_{n}(x).$$

This last equation and the use of the inversion formula (5) lead to the differential equation (8) with $E^{[m-1]}_{n}(x)$ as a polynomial solution. $\square$

Remark 1. *It is an easy consequence of (5) that for a fixed $m \geq 2$,*

$$E^{[m-1]}_{j}(x) = 2^{m-1}(x-1)^{j}, \quad \textit{whenever } 0 \leq j \leq m-1.$$

The particular case $m = 1$ in (5) reads as

$$\begin{aligned} 2x^n = & \sum_{k=0}^{n} \binom{n}{k} (1 + a_{k,1}) E_{n-k}(x) = \sum_{k=0}^{n} \binom{n}{k} (1 + a_{n-k,1}) E_k(x) \\ & \sum_{k=0}^{n} \binom{n}{k} (1 + \delta_{n-k,0}) E_k(x), \end{aligned} \tag{17}$$

where $\delta_{n-k,0}$ is the Kronecker delta. So, the expression (17) *is an equivalent form of the familiar expansion (cf., e.g., [34, p. 30])*

$$2x^n = E_n(x) + \sum_{k=0}^{n} \binom{n}{k} E_k(x), \quad n \geq 0.$$

Remark 2. *The following inversion formula was deduced in [31]*

$$x^n = \sum_{k=0}^{n} \binom{n}{k} \frac{k!}{(k+m)!} E^{[m-1]}_{n-k}(x), \quad n \geq 0. \tag{18}$$

We would like to note that (18) is wrong. In order to check that the generalized Euler polynomials of level m do not satisfy (18), it suffices to consider $m = 1$. Since $2^{m-1} = m!$ only when $m = 1$, then it is easy to check that the expression (18) is correct for $n = 0, 1$. However, when $n = 2$, the situation changes. Using (18) we obtain

$$E_2(x) = x^2 - x + \frac{1}{6}. \tag{19}$$

But, taking into account (5) or (17) we have

$$E_2(x) = x^2 - x,$$

this last polynomial is the classical Euler polynomial of degree 2. Also, it is possible to check that the polynomial in (19) does not satisfy the well-known relation (see, for instance [35, p. 804]):

$$E_n(x+1) + E_n(x) = 2x^n, \quad n = 0, 1, \ldots \tag{20}$$

Remark 3. *It is worthwhile to mention that (8) represents the analogue of [32, Eq. (3.1)] in the setting of the generalized Euler polynomials of level* $m \geq 2$*. As it was pointed out in the aforementioned paper, the Appell-type polynomials, satisfying a differential operator of finite order, can be considered as an exceptional case (cf. [36] for additional details about this assertion).*

3 The quadrature formulae of Euler-Maclaurin type

The integration by parts formula asserts that the following result holds.

Lemma 1. *Let* $s \geq 1$ *and* $f \in C^s[0,1]$*. For a fixed* $m \in \mathbb{N}$*, we have*

$$\int_0^1 f(t)dt = \frac{1}{2^{m-1}} \left[\sum_{k=1}^{s} A_k^{[m-1]}(f) + \frac{(-1)^s}{s!} \int_0^1 f^{(s)}(t) E_s^{[m-1]}(t)dt \right], \tag{21}$$

where

$$A_k^{[m-1]}(f) = \frac{(-1)^{k-1}}{k!} \left(f^{(k-1)}(1) E_k^{[m-1]}(1) - f^{(k-1)}(0) E_k^{[m-1]} \right), \; k = 1, \ldots, s.$$

Proof. Since the integral on the left-hand side of (21) can be expressed as follows,

$$\int_0^1 f(t)dt = \frac{1}{2^{m-1}} \int_0^1 f(t) E_0^{[m-1]}(t)dt,$$

it suffices to apply repeated integration by parts on the right-hand side of above equation, using a suitable form of (4) in each step. $\square$

Making the substitution $f(t) = E_{s+r}^{[m-1]}(t)$ into (21) and taking into account (4), (6) and some straightforward calculations, we can show that

$$\begin{aligned}\int_0^1 E_s^{[m-1]}(t)E_r^{[m-1]}(t)dt =& \frac{2^{m-1}(-1)^s s!r!}{(s+r+1)!}\left(E_{s+r+1}^{[m-1]}(1) - E_{s+r+1}^{[m-1]}\right) \\ &+ \frac{s!r!}{(s+r+1)!}\sum_{k=1}^{s} A_k^{[m-1]}, \quad s, r \geq 1,\end{aligned} \tag{22}$$

where

$$A_k^{[m-1]} = (-1)^{k+s}\binom{s+r+1}{k}\left(E_{s+r-k+1}^{[m-1]}(1)E_k^{[m-1]}(1) - E_{s+r-k+1}^{[m-1]}E_k^{[m-1]}\right),$$

for $k = 1, \ldots, s$.

The relation (22) is of independent interest. For instance, its combination with (5) allows us to connect with operational matrix methods based on generalized Euler polynomials of level m. Recently, in [23] the authors introduce an operational matrix method based on generalized Bernoulli polynomials of level m and analyze it in order to obtain numerical solutions of initial value problems. Their computational results demonstrate that such operational matrix method can lead to very ill-conditioned matrix equations.

Remark 4. *When $m = 1$, from the equation (20) and the symmetric relation for classical Euler polynomials $E_n(1-x) = (-1)^n E_n(x)$, it is possible to deduce that*

$$E_n(1) = \begin{cases} 0, & \text{if } n \text{ is even}, \\ -E_n, & \text{if } n \text{ is odd}. \end{cases} \tag{23}$$

This last relation yields the following particular forms of (21):

$$\int_0^1 f(t)dt = \frac{f(0)+f(1)}{2} - \sum_{k=1}^{\lfloor s/2 \rfloor}\left(f^{(2k)}(1) + f^{(2k)}(0)\right)\frac{E_{2k+1}}{(2k+1)!} - R_E(f),$$

being s an odd number.

$$\int_0^1 f(t)dt = \frac{f(0)+f(1)}{2} - \sum_{k=1}^{\lfloor s/2 \rfloor - 1}\left(f^{(2k)}(1) + f^{(2k)}(0)\right)\frac{E_{2k+1}}{(2k+1)!} + R_E(f), \tag{24}$$

being s an even number. In both cases, the term $R_E(f)$ is given by

$$R_E(f) = \frac{1}{s!}\int_0^1 f^{(s)}(t)E_s(t)dt.$$

Notice that (24) is a particular case of [15, Eq. (3.3)].

Remark 5. *For $s \geq 1$ and $f \in C^s[0,n]$ we can give the analogous of (1) depending on the parity of s as follows:*

If s is odd, we have

$$\begin{aligned}\int_0^n f(t)\,dt =& \frac{f(0)+f(n)}{2} + \sum_{j=1}^{n-1} f(j)\\ &- \sum_{j=0}^{n-1}\sum_{k=1}^{\lfloor s/2\rfloor}\left(f^{(2k)}(j+1)+f^{(2k)}(j)\right)\frac{E_{2k+1}}{(2k+1)!} - R_E(f,s).\end{aligned}$$

And if s is even, we have

$$\begin{aligned}\int_0^n f(t)\,dt =& \frac{f(0)+f(n)}{2} + \sum_{j=1}^{n-1} f(j)\\ &- \sum_{j=0}^{n-1}\sum_{k=1}^{\lfloor s/2\rfloor-1}\left(f^{(2k)}(j+1)+f^{(2k)}(j)\right)\frac{E_{2k+1}}{(2k+1)!} + R_E(f,s).\end{aligned}$$

In both cases the term $R_E(f,s)$ can be written as

$$R_E(f,s) = \frac{1}{s!}\int_0^n f^{(s)}(t)E_s(t-\lfloor t\rfloor)\,dt.$$

With these ideas in mind, we can connect Riemann sums and integrals and obtain the following result.

Theorem 2. *Let $s \geq 1$ and $f \in C^s[a,b]$. For a fixed $n \in \mathbb{N}$ let $x_j = a + jh$, $j = 0,1,\ldots,n$, where $h = \frac{b-a}{n}$, and $f_j = f(x_j)$, $f_j^{(k)} = f^{(k)}(x_j)$, $k = 1,2,\ldots,s$. Then, the following composite trapezoidal rules hold.*

If s is odd:

$$\begin{aligned}\int_a^b f(t)\,dt =& h\left(\frac{f(a)}{2} + f_1 + \cdots + f_{n-1} + \frac{f(b)}{2}\right)\\ &- \sum_{j=0}^{n-1}\sum_{k=1}^{\lfloor s/2\rfloor} h^{2k+1}\left(f_j^{(2k)} + f_{j+1}^{(2k)}\right)\frac{E_{2k+1}}{(2k+1)!} - \rho_E[f].\end{aligned} \tag{25}$$

If s is even:

$$\begin{aligned}\int_a^b f(t)\,dt =& h\left(\frac{f(a)}{2}+f_1+\cdots+f_{n-1}+\frac{f(b)}{2}\right)\\ &-\sum_{j=0}^{n-1}\sum_{k=1}^{\lfloor s/2\rfloor-1} h^{2k+1}\left(f_j^{(2k)}+f_{j+1}^{(2k)}\right)\frac{E_{2k+1}}{(2k+1)!}+\rho_E[f].\end{aligned} \tag{26}$$

In both cases the correction terms are expressed by means of the derivatives of f at the extrema, and term $\rho_E[f]$ is given by

$$\begin{aligned}\rho_E[f] =& \frac{h^s}{s!}\sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}} f^{(s)}(t)E_s\left(\frac{t-x_j}{h}\right)dt\\ =& \frac{h^s}{s!}\int_a^b f^{(s)}(t)E_s\left(\frac{t-a}{h}-\left\lfloor\frac{t-a}{h}\right\rfloor\right)dt.\end{aligned} \tag{27}$$

Proof. In order to prove (26), we proceed as in the proof of [15, Theorem 3.1], making the corresponding modifications. Put $f(t)=f(a+ux)=g(x)$, where $u=b-a$, $x=\frac{t-a}{u}$, and using (24) we obtain

$$\begin{aligned}\int_a^b f(t)\,dt =& u\int_0^1 g(x)\,dx = u\left(\frac{f(a)+f(b)}{2}\right)\\ &-\sum_{k=1}^{\lfloor s/2\rfloor-1} u^{2k+1}\left(f^{(2k)}(b)+f^{(2k)}(a)\right)\frac{E_{2k+1}}{(2k+1)!}\\ &+\frac{u^s}{s!}\int_a^b f^{(s)}(t)E_s\left(\frac{t-a}{u}\right)dt.\end{aligned} \tag{28}$$

Consider now the partition of the interval $[a,b]$ into n subintervals by means of the equidistant nodes $x_j=a+jh$, $j=0,1,\ldots,n$, where $h=\frac{b-a}{n}$, and $f_j = f(x_j)$, $f_j^{(k)} = f^{(k)}(x_j)$, $k = 1, 2, \ldots, s$. By (28) we have

$$\begin{aligned}\int_a^b f(t)\,dt =& \sum_{j=0}^{n-1} \int_{x_j}^{x_{j+1}} f(t)\,dt \\ & h\left(\frac{f(a)}{2} + f_1 + \cdots + f_{n-1} + \frac{f(b)}{2}\right) \\ & - \sum_{j=0}^{n-1} \sum_{k=1}^{\lfloor s/2\rfloor-1} h^{2k+1}\left(f_{j+1}^{(2k)} + f_j^{(2k)}\right)\frac{E_{2k+1}}{(2k+1)!} \\ & + \frac{h^s}{s!}\sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}} f^{(s)}(t) E_s\left(\frac{t-x_j}{h}\right) dt.\end{aligned} \tag{29}$$

Note that last summand on the right-hand side of (29) is equal to $\rho_E[f]$. In order to simplify the last summand on the right-hand side of (29), we only need to recall that

$$\begin{aligned}\int_{x_j}^{x_{j+1}} f^{(s)}(t) E_s\left(\frac{t-x_j}{h}\right) dt &= \int_{x_j}^{x_{j+1}} f^{(s)}(t) E_s\left(\frac{t-a}{h} - j\right) dt \\ &= \int_{x_j}^{x_{j+1}} f^{(s)}(t) E_s\left(\frac{t-a}{h} - \left\lfloor \frac{t-a}{h}\right\rfloor\right) dt.\end{aligned}$$

Consequently,

$$\begin{aligned}\rho_E[f] =& \frac{h^s}{s!}\sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}} f^{(s)}(t) E_s\left(\frac{t-a}{h} - \left\lfloor \frac{t-a}{h}\right\rfloor\right) dt \\ =& \frac{h^s}{s!}\int_a^b f^{(s)}(t) E_s\left(\frac{t-a}{h} - \left\lfloor \frac{t-a}{h}\right\rfloor\right) dt.\end{aligned} \tag{30}$$

Finally, substituting (30) into (29) we get (26). The proof of (25) is similar. □

Remark 6. *Notice that the relation (23) and the integration by parts do not allow the sum*

$$\sum_{j=0}^{n-1}\left(f_{j+1}^{(2k)} + f_j^{(2k)}\right)$$

satisfies the summation telescoping property. So, the second summand on the right-hand side of (26) cannot be simplified as the classical Euler-Maclaurin *formula (2). Consequently, the expression (26) is the correct form for [15, Equation (3.1)].*

Now, using (21) and proceeding as in the proof of Theorem 2, we obtain the following quadrature formulae of Euler-Maclaurin type based on generalized Euler polynomials of level $m \in \mathbb{N} \setminus \{1\}$.

Theorem 3. *Let $s \geq 1$, $f \in C^s[a,b]$ and $m \in \mathbb{N} \setminus \{1\}$. For a fixed $n \in \mathbb{N}$ let $x_j = a + jh$, $j = 0, 1, \ldots, n$, where $h = \frac{b-a}{n}$, and $f_j = f(x_j)$, $f_j^{(k)} = f^{(k)}(x_j)$, $k = 1, 2, \ldots, s$. Then, the following composite trapezoidal rules hold.*

$$\begin{aligned}\int_a^b f(t)\,dt =& \frac{1}{2^{m-1}} \sum_{j=0}^{n-1} \sum_{k=1}^{s} \frac{(-1)^{k-1}}{k!} h^k \\ & \left(f_{j+1}^{(k-1)} E_k^{[m-1]}(1) - f_j^{(k-1)} E_k^{[m-1]} \right) + \rho_E^{[m-1]}[f],\end{aligned} \tag{31}$$

where the remainder term $\rho_E^{[m-1]}[f]$ can be written as

$$\begin{aligned}\rho_E^{[m-1]}[f] =& \frac{(-h)^s}{2^{m-1}s!} \sum_{j=0}^{n-1} \int_{x_j}^{x_{j+1}} f^{(s)}(t) E_s^{[m-1]} \left(\frac{t - x_j}{h} \right) dt \\ =& \frac{(-h)^s}{2^{m-1}s!} \int_a^b f^{(s)}(t) E_s^{[m-1]} \left(\frac{t-a}{h} - \left\lfloor \frac{t-a}{h} \right\rfloor \right) dt.\end{aligned}$$

4 Numerical examples

In this section, four numerical examples are examined to illustrate the efficiency of the composite trapezoidal rules presented in Section 3. All of the numerical experiments are performed using MAPLE 15 and the approximations of definite integrals are given up to 30 decimal places. Also, we would like to point out that some examples of this section have been considered in [15, Section 4].

Example 1. *Consider the elementary integral*

$$I_1 := \int_0^1 \frac{1}{1+x} dx = \ln(2) \approx 0.693147180559945309417232121458 \ldots$$

For $s = 12$ and $n = 90$ the formula (26) yields the following numerical approximation of I_1:

$$I_1 \approx 0.693147180559945309417232121456\ldots$$

Notice that an absolute error for this approximation is less than 10^{-29}, and a relative error is less than 3×10^{-30}.

Also, for $s = 12$ the formula (24) yields the following numerical approximation of I_1:

$$I_1 \approx 0.6931471805599453094172321215\ldots$$

So, an absolute error for this approximation is less than 10^{-28}, and a relative error is less than 7×10^{-29}.

We strongly recommend that the reader compare these numerical evidences with the numerical approximation for I_1 presented in [15, Section 4].

By using (31) *with $m = 2$, $s = 12$ and $n = 90$, we find the following numerical approximation for I_1:*

$$I_1 \approx 0.693147180559945309417232121463\ldots$$

Hence, an absolute error for this approximation is less than 10^{-29}, and a relative error is less than 8×10^{-30}.

Finally, when we consider the formula (21) with $m = 2$ and $s = 12$, the corresponding numerical approximation for I_1 is

$$I_1 \approx 0.69314718055994530941723\ldots,$$

and an absolute error for this approximation is less than 10^{-23}, and a relative error is less than 4×10^{-24}.

MAPLE uses a sophisticated numerical integration routine with automatic error control to evaluate definite integrals that it cannot do analytically, for instance, definite integrals whose integrand does not have elementary anti-derivatives. The most common command of MAPLE for numerical integration is evalf($Int(f, x = a..b)$) where the integration command is expressed in inert form to avoid first invoking the symbolic integration routines [37,38].

The examples below, show approximations via quadrature formulae of Euler-Maclaurin type for definite integrals whose integrand does not have elementary anti-derivatives.

Example 2.

$$I_2 := \int_1^2 \frac{e^x}{x} dx \approx 3.05911653964595340791298419590\ldots \tag{32}$$

For $s = 14$ and $n = 90$, the formula (26) yields the following numerical approximation of I_2:

$$I_2 \approx 3.05911653964595340791298419590\ldots$$

In this case, our approximation coincides exactly with (32). If we take $n = 4$, then (26) yields the following approximation:

$$I_2 \approx 3.05911653964595340791298419589\ldots$$

An absolute error for this last approximation is less than 10^{-28}.

Notice that the authors of [15, Section 4] found an absolute error for their numerical approximation of I_2 which is less than 10^{-5}.

Example 3. *Let us consider the following integral.*

$$I_3 := \int_{-1}^1 e^{-x^2} dx \approx 1.49364826562485405079893487226\ldots \tag{33}$$

Table 1 shows some absolute errors for numerical approximations of (33) using (21), when different values of m and s are considered.

Tab. 1: Absolute errors for approximations of I_3.

Level: m	**Der. order:** s	**Polynomial degree:** n	**Abs. error**
1	6	90	2×10^{-29}
2	6	90	2×10^{-29}
3	8	90	10^{-29}
4	6	90	10^{-29}
5	5	90	10^{-29}

Example 4. *Let us consider the following integral.*

$$I_4 := \int_0^1 \cos(x^3) dx \approx 0.931704440591544226076926390685\ldots$$

Applying the quadrature formula (25) with $s = 5$ and $n = 90$, we get

$$I_4 \approx 0.930016201613048171856211810179\ldots$$

And an absolute error for this approximation is less than 2×10^{-3}.

While, by using (31) with $m = 5$, $s = 3$ and $n = 90$, we find the following numerical approximation for I_4:

$$I_4 \approx 0.931704440591544226076926390684\ldots$$

Hence, an absolute error for this approximation is less than 10^{-29}.

5 Conclusion

A composite trapezoidal rule based on generalized Euler polynomials of level $m \in \mathbb{N}$ has been presented in order to obtain numerical approximations of definite integrals. Such definite integrals possess an integrand regular enough. The comparative numerical evidence suggests that the Euler-Maclaurin type quadrature formula (31) produces smaller absolute errors than standard formulae (25) and (26).

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References

[1] T.M. Apostol. An elementary view of Euler's summation formula. *AM. Math. Monthly*, 106(5):409–418, 1999. doi: 10.2307/2589145.

[2] P.J. Davis and P. Rabinowitz. *Methods of Numerical Integration*. Academic Press, Inc., 1984. ISBN 978-0-12-206360-2.

[3] V. Lampret. The Euler-Maclaurin and Taylor formulas: twin, elementary derivations. *Math. Mag.*, 74(2):109–122, 2001. http://www.kmf.fgg.uni-lj.si/matematika/anal2/pred_vito/em.pdf.

[4] N.E. Nørlund. *Vorlesungen über differenzenrechnung*. Chelsea Publishing Co, 1954. ISBN 978-1114527232, Chelsea print from 1954 of the 1923 original.

[5] H.M. Srivastava and H.L. Manocha. *A Treatise on Generating Functions*, volume 176 of *Mathematics and its applications*. Ellis Horwood Ltd., 1984. ISBN 978-0470200100.

[6] C.G. Jacobi. De usu legitimo formulae summatoriae maclaurinianae. *Journal für die reine und angew. Math.*, 12:263–272, 1834. http://eudml.org/doc/146908.

[7] L.N. Trefethen and J.A.C. Weideman. The exponentially convergent trapezoidal rule. *SIAM Review*, 56(3):385–458, 2014. doi: 10.1137/130932132.

[8] J. Waldvogel. Towards a general error theory of the trapezoidal rule. *In W. Gautschi, G. Mastroianni, T.M. Rassias editors, Approximation and Computation: In Honor of Gradimir V. Milovanović, Springer Optim. Appl.*, 42(3):267–282, 2011. http://www.sam.math.ethz.ch/ joergw/Projects/nisJoerg.pdf.

[9] J.A.C. Weideman. Numerical integration of periodic functions: a few examples. *AM. Math. Monthly*, 109(3):21–36, 2002. doi: 10.2307/2695765.

[10] M. Mori and M. Sugihara. The double-exponential transformation in numerical analysis. *J. Comput. Appl. Math.*, 127(1-2):287–296, 2001. doi: 10.1016/S0377-0427(00)00501-X.

[11] Wolfram Research. *Mathematics and Algorithms*. Wolfram Mathematica Tutorial Collection. Wolfram Research, Inc., 2008. ISBN 978-1-57955-054-7, http://library.wolfram.com/infocenter/Books/8501/MathematicsAnd Algorithms.pdf.

[12] C. Belingeri and B. Germano. On Appell-type quadrature rules. *Georgian Math. J.*, 9(3):405–412, 2002. https://www.emis.de/journals/GMJ/vol9/v9n3-1.pdf.

[13] G. Bretti and C. Belingeri. An Euler-type quadrature rule derived by using Radon's method. *J. Concr. Appl. Math.*, 3(3):335–345, 2005. https://www.researchgate.net/profile/Gabriella_Bretti/publication/265317070_An_Euler-type_quadrature_rule_derived_by_using_Radon%27s_method/links/542bd1c10cf277d58e8a3b60/An-Euler-type-quadrature-rule-derived-by-using-Radons-method.pdf.

[14] G. Bretti, M.X. He, and P.E. Ricci. On quadrature rules associated with Appell polynomials. *Int. J. Appl. Math.*, 11(1):1–14, 2002. https://nsuworks.nova.edu/math_facarticles/170/.

[15] G. Bretti and P.E. Ricci. Euler polynomials and the related quadrature rule. *Georgian Math. J.*, 8(3):447–453, 2001. https://eudml.org/doc/49338.

[16] J. Franjić and J. Pečarić. Corrected Euler-Maclaurin's formulae. *J. Rend. Circ. Mat. Palermo*, 54(2):259–272, 2005. doi: 10.1007/BF02874640.

[17] Lj. Dedić, M. Matić, and J. Pečarić. On Euler trapezoid formulae. *Appl. Math. Comput.*, 123(1):37–62, 2001. doi: 10.1016/S0096-3003(00)00054-0.

[18] Lj. Dedić, M. Matić, and J. Pečarić. Some inequalities of Euler-Grüss type. *Computers & Mathematics with Applications*, 41(7-8):843–856, 2001. doi: 10.1016/S0898-1221(00)00324-2.

[19] Lj. Dedić, C.E.M. Pearce, and J. Pečarić. The euler formulae and convex functions. *Math. Inequal. Appl.*, 3(2):211–221, 2000. http://files.ele-math.com/abstracts/mia-03-24-abs.pdf.

[20] Y. He, S. Araci, and H.M. Srivastava. Some new formulas for the products of the Apostol type polynomials. *Adv. Difference Equ., Article ID 287*, pages 1–18, 2016. doi: 10.1186/s13662-016-1014-0.

[21] Y. He, S. Araci, H.M. Srivastava, and M. Acikgöz. Some new identities for the Apostol-Bernoulli polynomials and the Apostol-Genocchi polynomials. *Appl. Math. Comput.*, 262:31–41, 2015. doi: 10.1016/j.amc.2015.03.132.

[22] Á. Pintér and H.M. Srivastava. Addition theorems for the Appell polynomials and the associated classes of polynomial expansions. *Aequationes Math.*, 85(3):483–495, 2013. doi: 10.1007/s00010-012-0148-8.

[23] Y. Quintana, W. Ramírez, and A. Urieles. On an operational matrix method based on generalized Bernoulli polynomials of level m. *Calcolo*, 55(3):1–29, 2018. doi: 10.1007/s10092-018-0272-5.

[24] H.M. Srivastava, M.A. Boutiche, and M. Rahmani. A class of frobenius-type eulerian polynomials. *Rocky Mountain J. Math.*, 48(3):1003–1013, 2018. doi: 10.1216/RMJ-2018-48-3-1003.

[25] H.M. Srivastava, I. Kucukoğlu, and Y. Simsek. Partial differential equations for a new family of numbers and polynomials unifying the apostol-type numbers and the apostol-type polynomials. *J. Number Theory*, 181:117–146, 2017. doi: 10.1016/j.jnt.2017.05.008.

[26] H.M. Srivastava, M. Masjed-Jamei, and M. Reza Beyki. A parametric type of the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials. *Appl. Math. Inform. Sci.*, 12(5):907–916, 2018. doi: 10.18576/amis/120502.

[27] H.M. Srivastava, M.A. Özarslan, and C. Kaanuglu. Some generalized Lagrange-based Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials. *Russian J. Math. Phys.*, 20(1):110–120, 2013. doi: 10.1134/S106192081301010X.

[28] H.M. Srivastava, M.A. Özarslan, and B. Yilmaz. Some families of differential equations associated with the Hermite-based Appell polynomials and other classes of Hermite-based polynomials. *Filomat*, 28(4):695–708, 2014. http://journal.pmf.ni.ac.rs/filomat/index.php/filomat/article/view/201.

[29] P. Hernández-Llanos, Y. Quintana, and A. Urieles. About extensions of generalized Apostol-type polynomials. *Results. Math.*, 68(1-2):203–225, 2015. doi: 10.1007/s00025-014-0430-2.

[30] B. Kurt. Some relationships between the generalized Apostol-Bernoulli and Apostol-Euler polynomials. *Turk. J. Anal. Number Theory*, 1(1):54–58, 2013. doi: 10.12691/tjant-1-1-11.

[31] M. Pathan and A. Waseem. Some implicit summation formulas and symmetric identities for the generalized Hermite-Bernoulli polynomials. *Mediterr. J. Math.*, 12(3):679–695, 2015. doi: 10.1007/s00009-014-0423-0.

[32] P. Natalini and A. Bernardini. A generalization of the Bernoulli polynomials. *J. Appl. Math.*, 3(3):155–163, 2003. doi: 10.1155/S1110757X03204101.

[33] H.M. Srivastava. Some characterizations of Appel and q-Appell polynomials. *Ann. Mat. Pura Appl.*, 130(1):321–329, 1982. doi: 10.1007/BF01761501.

[34] W. Magnus, F. Oberhettinger, and R.P. Soni. *Formulas and Theorems for the Special Functions of Mathematical Physics*, volume 52 of *Grundlehren der mathematischen Wissenschaften.* Springer-Verlag Berlin Heidelberg, 1966. ISBN 978-3-662-11761-3.

[35] M. Abramowitz and I.A. Stegun. *Handbook of Mathematical Functions*, volume 55 of *National Bureau of Standards Applied Mathematics Series.* Dover Publications, 1970. 9th Edition, ISBN -486-61272-4.

[36] G. Dattoli, P.E. Ricci, and C. Cesarano. Differential equations for Appell type polynomials. *Fract. Calc. Appl. Anal.*, 5(1):69–75, 2002. https://www.researchgate.net/publication/267149646_Differential_equations_for_Appell_type_polynomials.

[37] K.O. Geddes and G.J. Fee. Hybrid symbolic-numeric integration in MAPLE. *Proc. ISSAC'92 Papers from the international symposium on Symbolic and algebraic computation*, pages 34–41, 1992. doi: 10.1145/143242.143262.

[38] K.O. Geddes and G. Labahn. Symbolic and numeric integration in MAPLE. *Proc. First Asian Technology Conference in Mathematics*, pages 377–386, 1995. https://cs.uwaterloo.ca/ glabahn/Papers/intbnew.ps.