Publ. Mat. **67** (2023), 277–410
DOI: 10.5565/PUBLMAT6712307

COERCIVITY FOR TRAVELLING WAVES IN THE GROSS–PITAEVSKII EQUATION IN $\mathbb{R}^2$ FOR SMALL SPEED

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Abstract: In a previous paper, we constructed a smooth branch of travelling waves for the 2-dimensional Gross–Pitaevskii equation. Here, we continue the study of this branch. We show some coercivity results, and we deduce from them the kernel of the linearized operator, a spectral stability result, as well as a uniqueness result in the energy space. In particular, our result proves the nondegeneracy of these travelling waves, which is a key step in their classification and for the construction of multi-travelling waves.

2020 Mathematics Subject Classification: 35C07, 35Q55, 35A02, 35B35.

Key words: travelling waves, coercivity, local uniqueness.

1. Introduction and statement of the results

We consider the Gross–Pitaevskii equation

$$0 = (\mathrm{GP})(\mathfrak{u}) := i\partial_t \mathfrak{u} + \Delta \mathfrak{u} - (|\mathfrak{u}|^2 - 1)\mathfrak{u}$$

in dimension 2 for $\mathfrak{u}\colon \mathbb{R}_t \times \mathbb{R}^2_x \to \mathbb{C}$. The Gross–Pitaevskii equation is a physical model for Bose–Einstein condensates [**8**], [**17**], and is associated with the Ginzburg–Landau energy

$$E(v) := \frac{1}{2}\int_{\mathbb{R}^2} |\nabla v|^2 + \frac{1}{4}\int_{\mathbb{R}^2} (1 - |v|^2)^2.$$

The condition at infinity for (GP) will be

$$|\mathfrak{u}| \to 1 \quad \text{as} \quad |x| \to +\infty.$$

The equation (GP) has some well-known stationary solutions of infinite energy called vortices, which are solutions of (GP) of degrees $n \in \mathbb{Z}^*$ (see [**2**]):

$$V_n(x) = \rho_n(r) e^{in\theta},$$

where $x = re^{i\theta}$, solving

$$\begin{cases} \Delta V_n - (|V_n|^2 - 1)V_n = 0, \\ |V_n| \to 1 \text{ as } |x| \to \infty. \end{cases}$$

Amongst other properties, V_1 and V_{-1} have exactly one zero ($\rho_n(r) = 0$ only if $r = 0$), and we call it the centre of the vortex. Since the equation is invariant by translation, we can define a vortex by its degree and its centre (the only point where its value is zero).

We are interested here in travelling wave solutions of (GP):

$$\mathfrak{u}(t, x) = v(x_1, x_2 + ct),$$

where $x = (x_1, x_2)$ and $c > 0$ is the speed of the travelling wave, which moves along the direction $-\overrightarrow{e_2}$. The equation on v is then

$$0 = (\mathrm{TW}_c)(v) := -ic\partial_{x_2} v - \Delta v - (1 - |v|^2)v.$$

We use the following notations throughout this paper. We denote, for functions $f, g \in L^2_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{C})$ such that $\mathfrak{Re}(f\bar{g}) \in L^1(\mathbb{R}^2, \mathbb{C})$, the quantity

$$\langle f, g \rangle := \int_{\mathbb{R}^2} \mathfrak{Re}(f\bar{g}),$$

even if $f, g \notin L^2(\mathbb{R}^2, \mathbb{C})$. We also use the notation $B(x, r)$ to define the closed ball in $\mathbb{R}^2$ of centre $x \in \mathbb{R}^2$ and radius $r > 0$ for the Euclidean norm. We define between two vectors $X = (X_1, X_2) \in \mathbb{R}^2$, $Y = (Y_1, Y_2) \in \mathbb{C}^2$ the complex quantity

$$X \cdot Y := X_1 Y_1 + X_2 Y_2.$$

Finally, we use the notation $o^{\nu}_{c\to 0}(1)$ to describe a quantity that goes to 0 when $c \to 0$ for a fixed value of ν.

1.1. Branch of travelling waves at small speed. In the previous paper [**4**], we constructed solutions of (TW_c) for small values of c as a perturbation of two well-separated vortices (the distance between their centres is large when c is small). We showed the following result.

Theorem 1.1 ([**4**, Theorem 1.1])**.** *There exists $c_0 > 0$ a small constant such that for any $0 < c \leqslant c_0$, there exists a solution of (TW_c) of the form*

$$Q_c = V_1(\cdot - d_c\overrightarrow{e_1})V_{-1}(\cdot + d_c\overrightarrow{e_1}) + \Gamma_c,$$

where $d_c = \frac{1+o_{c\to 0}(1)}{c}$ is a continuous function of c. This solution has finite energy ($E(Q_c) < +\infty$) and $Q_c \to 1$ at infinity.

Furthermore, for all $+\infty \geqslant p > 2$, there exists $c_0(p) > 0$ such that if $c \leqslant c_0(p)$, for the norm

$$\|h\|_p := \|h\|_{L^p(\mathbb{R}^2)} + \|\nabla h\|_{L^{p-1}(\mathbb{R}^2)}$$

of the space $X_p := \{f \in L^p(\mathbb{R}^2),\ \nabla f \in L^{p-1}(\mathbb{R}^2)\}$, one has

$$\|\Gamma_c\|_p = o_{c\to 0}(1).$$

In addition,

$$c \mapsto Q_c - 1 \in C^1(]0, c_0(p)[, X_p),$$

with the estimate

$$\left\| \partial_c Q_c + \left(\frac{1 + o_{c\to 0}(1)}{c^2} \right) \partial_d (V_1(\cdot - d\vec{e_1}) V_{-1}(\cdot + d\vec{e_1}))_{|d=d_c} \right\|_p = o_{c\to 0}\left(\frac{1}{c^2} \right).$$

The main idea of the proof of Theorem 1.1 is to use perturbative methods around a quasi-solution $V_1(\cdot - d\vec{e_1})V_{-1}(\cdot + d\vec{e_1})$, get Γ_c by a fixed point theorem and the value of d_c by the cancellation of a Lagrange multiplier. With an implicit function theorem, we can show that this construction gives us a C^1 branch with respect to the speed c. In [**4**], we showed additional and more precise estimates on Q_c and $\partial_c Q_c$ in some weighted L^∞ norms that will be useful in the proof of the next results (they will be recalled later on). Still in [**4**], we wrote the perturbation Γ_{c,d_c} to make the dependence on c and d_c clearer, but it is no longer needed here, and we will only write Γ_c.

With this solution Q_c, we can construct travelling waves of any small speed, i.e. solutions of

$$(\mathrm{TW}_{\vec{c}})(v) := i\vec{c}\cdot\nabla v - \Delta v - (1 - |v|^2)v$$

for any $\vec{c} \in \mathbb{R}^2$ of small modulus. For $\vec{c} = |\vec{c}|e^{i(\theta_{\vec{c}} - \pi/2)} \in \mathbb{R}^2$, $|\vec{c}| \leqslant c_0$, we have that

$$Q_{\vec{c}} := Q_{|\vec{c}|} \circ R_{-\theta_{\vec{c}}} \tag{1.1}$$

is a solution of $(\mathrm{TW}_{\vec{c}})$, with R_α being the rotation of angle α and $Q_{|\vec{c}|}$ defined in Theorem 1.1. Furthermore, the equation is invariant by translation and by changing the phase. Thus, we have a family of solutions of (GP) depending on five real parameters, $\vec{c} \in \mathbb{R}^2$, $|\vec{c}| \leqslant c_0$, $X \in \mathbb{R}^2$, and $\gamma \in \mathbb{R}$:

$$Q_{\vec{c}}(\cdot - X - t\vec{c})e^{i\gamma}.$$

We note that, for a vortex of degree ± 1, the family of solutions has three parameters (the two translations and the phase): $V_{\pm 1}(\cdot - X)e^{i\gamma}$ is a solution of (GP) for $X \in \mathbb{R}^2$, $\gamma \in \mathbb{R}$. In particular, between a travelling wave and the two vortices that compose it, we lose a parameter (since the phase is global). This is one of the difficulties that will appear when we study the stability of this branch.

First, we give additional results on this branch of travelling waves: we will study the position of its zeros, its energy and momentum, as well as some particular values appearing in the linearization. The (additive) linearized operator around Q_c is

$$L_{Q_c}(\varphi) := -\Delta\varphi - ic\partial_{x_2}\varphi - (1 - |Q_c|^2)\varphi + 2\mathfrak{Re}(\overline{Q_c}\varphi)Q_c.$$

We want to define and use four particular directions for the linearized operator around Q_c, which are

$$\partial_{x_1} Q_c, \quad \partial_{x_2} Q_c,$$

related to the translations (i.e. related to the parameter $X \in \mathbb{R}^2$ in the family of travelling waves), and

$$\partial_c Q_c, \quad \partial_{c^\perp} Q_c,$$

related to the variation of speed (i.e. related to the parameter $\vec{c} \in \mathbb{R}^2$), if we change respectively its modulus or its direction. The functions $\partial_{x_1} Q_c$, $\partial_{x_2} Q_c$, and $\partial_c Q_c$ are defined in Theorem 1.1, and we will show that

$$\partial_{c^\perp} Q_c(x) := \partial_\alpha (Q_c \circ R_{-\alpha})_{|\alpha=0} = -x^\perp \cdot \nabla Q_c(x),$$

with $x^\perp = (-x_2, x_1)$ (see Lemma 2.7). We infer the following properties.

Proposition 1.2. *There exists $c_0 > 0$ such that, for $0 < c \leqslant c_0$, the momentum $\vec{P}(Q_c) = (P_1(Q_c), P_2(Q_c))$ of Q_c from Theorem 1.1, defined by*

$$P_1(Q_c) := \frac{1}{2}\langle i\partial_{x_1} Q_c, Q_c - 1\rangle,$$
$$P_2(Q_c) := \frac{1}{2}\langle i\partial_{x_2} Q_c, Q_c - 1\rangle,$$

verifies $c \mapsto \vec{P}(Q_c) \in C^1(]0, c_0[, \mathbb{R}^2)$,

$$P_1(Q_c) = \partial_c P_1(Q_c) = 0,$$
$$P_2(Q_c) = \frac{2\pi + o_{c\to 0}(1)}{c},$$

and

$$\partial_c P_2(Q_c) = \frac{-2\pi + o_{c\to 0}(1)}{c^2}.$$

Furthermore, the energy satisfies $c \mapsto E(Q_c) \in C^1(]0, c_0[, \mathbb{R})$, and

$$E(Q_c) = (2\pi + o_{c\to 0}(1)) \ln\left(\frac{1}{c}\right).$$

Additionally, $\mathfrak{Re}(L_{Q_c}(A)\bar{A}) \in L^1(\mathbb{R}^2, \mathbb{R})$ for $A \in \{\partial_{x_1} Q_c, \partial_{x_2} Q_c, \partial_c Q_c, \partial_{c^\perp} Q_c\}$, and

$$\langle L_{Q_c}(\partial_{x_1} Q_c), \partial_{x_1} Q_c\rangle = \langle L_{Q_c}(\partial_{x_2} Q_c), \partial_{x_2} Q_c\rangle = 0,$$
$$\langle L_{Q_c}(\partial_c Q_c), \partial_c Q_c\rangle = \partial_c P_2(Q_c) = \frac{-2\pi + o_{c\to 0}(1)}{c^2},$$
$$\langle L_{Q_c}(\partial_{c^\perp} Q_c), \partial_{c^\perp} Q_c\rangle = c P_2(Q_c) = 2\pi + o_{c\to 0}(1),$$

and

$$\partial_c E(Q_c) = c\partial_c P_2(Q_c) = \frac{-2\pi + o_{c\to 0}(1)}{c}.$$

Finally, the function Q_c has exactly two zeros. Their positions are $\pm \widetilde{d_c}\overrightarrow{e_1}$, with

$$|d_c - \tilde{d}_c| = o_{c\to 0}(1),$$

where d_c is defined in Theorem 1.1.

The momentum has a generalized definition for finite energy functions (see [**16**] in $3d$ and [**3**]). For travelling waves going to 1 at infinity, it is equal to the quantity defined in Proposition 1.2. The proof of Proposition 1.2 is done in Section 2.

The equality $\langle L_{Q_c}(\partial_c Q_c), \partial_c Q_c\rangle = \partial_c P_2(Q_c)$ is a general property for Hamiltonian systems; see [**12**]. The equality $\partial_c E(Q_c) = c\partial_c P_2(Q_c)$ was conjectured and formally shown in [**14**], provided we have a smooth branch $c \mapsto Q_c$, which is precisely shown in Theorem 1.1. We note that the energy $E(Q_c)$ is of the same order as the energy of the travelling waves constructed in [**1**], which also exhibit two vortices at a distance of order $\frac{1}{c}$. We believe that both constructions give the same branch, and that this branch minimizes globally the energy at fixed momentum. However, we were not able to show even a local minimization result of the energy for Q_c defined in Theorem 1.1.

In the limit $c \to 0$, the four directions $(\partial_{x_1} Q_c, \partial_{x_2} Q_c, c^2\partial_c Q_c, c\partial_{c^\perp} Q_c)$ are going to zeros of the quadratic form (while being of size of order one), and we see here the splitting for small values of c. In particular, two directions give zero ($\partial_{x_1} Q_c$ and $\partial_{x_2} Q_c$), one becomes positive ($\partial_{c^\perp} Q_c$), and one negative ($\partial_c Q_c$).

1.2. Coercivity results. One of the main ideas is to reduce the problem of the coercivity of a travelling wave to the coercivity of vortices. We will first state such a result for vortices (Proposition 1.3) before the results on the travelling waves (see in particular Theorem 1.5).

1.2.1. Coercivity in the case of one vortex. A coercivity result for one vortex of degree ± 1 is already known; see [**5**], and in particular equation (2.42) there. We consider both vortices of degrees $+1$ and -1 here at the same time, since $V_1 = \overline{V_{-1}}$. Here, we present a slight variation of the results in [**5**] that will be useful for the coercivity of the travelling waves. We recall from [**5**] the quadratic form around V_1:

$$B_{V_1}(\varphi) = \int_{\mathbb{R}^2} |\nabla\varphi|^2 - (1 - |V_1|^2)|\varphi|^2 + 2\mathfrak{Re}^2(\overline{V_1}\varphi),$$

for functions in the energy space

$$H_{V_1} = \left\{ \varphi \in H^1_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{C}),\ \|\varphi\|^2_{H_{V_1}} := \int_{\mathbb{R}^2} |\nabla\varphi|^2 + (1 - |V_1|^2)|\varphi|^2 + \mathfrak{Re}^2(\overline{V_1}\varphi) < +\infty \right\}.$$

As the family of vortices has three parameters, we expect a coercivity result under three orthogonality conditions. The three associated directions are $\partial_{x_1} V_1$, $\partial_{x_2} V_1$ (for the translations), and iV_1 (for the phase).

Proposition 1.3. *There exist $K > 0$, $R > 5$, such that, if the following three orthogonality conditions are satisfied for $\varphi = V_1\psi \in C_c^\infty(\mathbb{R}^2\backslash\{0\}, \mathbb{C})$,*

$$\int_{B(0,R)} \mathfrak{Re}(\partial_{x_1} V_1 \overline{V_1\psi}) = \int_{B(0,R)} \mathfrak{Re}(\partial_{x_2} V_1 \overline{V_1\psi}) = \int_{B(0,R)\backslash B(0,R/2)} \mathfrak{Im}(\psi) = 0,$$

then

$$B_{V_1}(\varphi) \geqslant K\left(\int_{B(0,10)} |\nabla\varphi|^2 + |\varphi|^2 + \int_{\mathbb{R}^2\backslash B(0,5)} |\nabla\psi|^2|V_1|^2 + \mathfrak{Re}^2(\psi)|V_1|^4 + \frac{|\psi|^2}{r^2\ln^2(r)}\right).$$

The same result holds if we replace V_1 by V_{-1}. We note that the coercivity norm is not $\|\cdot\|_{H_{V_1}}$, but is weaker (the decay in position is stronger), and this is due to the fact that $iV_1 \notin H_{V_1}$. That is why this result is stated for a compactly supported function. The fact that the support of φ avoids 0 is technical at this point.

Proposition 1.3 is shown in Subsection 4.2. The proofs there are mostly slight variations or improvements of proofs given in [**5**].

1.2.2. Coercivity and kernel in the energy space. The main part of this section consists of coercivity results for the family of travelling waves constructed in Theorem 1.1. We will show them on Q_c defined in Theorem 1.1, and with (1.1), they extend to all speed values $\vec{c}$ of small norm. We recall the linearized operator around Q_c:

$$L_{Q_c}(\varphi) = -\Delta\varphi - ic\partial_{x_2}\varphi - (1 - |Q_c|^2)\varphi + 2\mathfrak{Re}(\overline{Q_c}\varphi)Q_c.$$

The natural associated energy space is

$$H_{Q_c} := \{\varphi \in H^1_{\mathrm{loc}}(\mathbb{R}^2),\ \|\varphi\|_{H_{Q_c}} < +\infty\},$$

where

$$\|\varphi\|^2_{H_{Q_c}} := \int_{\mathbb{R}^2} |\nabla\varphi|^2 + |1 - |Q_c|^2||\varphi|^2 + \mathfrak{Re}^2(\overline{Q_c}\varphi).$$

First, there are difficulties in the definition of the quadratic form for $\varphi \in H_{Q_c}$, because of the transport term. A natural definition for the associated quadratic form for $\varphi \in H_{Q_c}$ could be

$$(1.2) \qquad \int_{\mathbb{R}^2} |\nabla\varphi|^2 - (1 - |Q_c|^2)|\varphi|^2 + 2\mathfrak{Re}^2(\overline{Q_c}\varphi) - \mathfrak{Re}(ic\partial_{x_2}\varphi\bar{\varphi}).$$

Unfortunately the last term is not well defined for $\varphi \in H_{Q_c}$, because we lack a control on $\mathfrak{Im}(\overline{Q_c}\varphi)$ in $L^2(\mathbb{R}^2)$ in $\|\cdot\|_{H_{Q_c}}$; see [**16**]. We can resolve this issue by decomposing this term and doing an integration by parts, but the proof of the integration by parts cannot be done if we only suppose $\varphi \in H_{Q_c}$ (see Section 3 for more details). We therefore define the quadratic form with the integration by parts already done. Take a smooth cutoff function η such that $\eta(x) = 0$ on $B(\pm\widetilde{d_c}\vec{e_1}, 1)$, $\eta(x) = 1$

on $\mathbb{R}^2 \backslash B(\pm\widetilde{d_c}\overrightarrow{e_1}, 2)$, where $\pm\widetilde{d_c}\overrightarrow{e_1}$ are the zeros of Q_c. We define, for $\varphi = Q_c\psi \in H_{Q_c}$,

$$\begin{aligned}(1.3)\quad B_{Q_c}(\varphi) := \int_{\mathbb{R}^2} &|\nabla\varphi|^2 - (1-|Q_c|^2)|\varphi|^2 + 2\mathfrak{Re}^2(\overline{Q_c}\varphi) \\ &- c\int_{\mathbb{R}^2}(1-\eta)\mathfrak{Re}(i\partial_{x_2}\varphi\bar{\varphi}) - c\int_{\mathbb{R}^2}\eta\mathfrak{Re}(i\partial_{x_2}Q_c\overline{Q_c})|\psi|^2 \\ &+ 2c\int_{\mathbb{R}^2}\eta\mathfrak{Re}\psi\mathfrak{Im}(\partial_{x_2}\psi)|Q_c|^2 + c\int_{\mathbb{R}^2}\partial_{x_2}\eta\mathfrak{Re}\psi\mathfrak{Im}\psi|Q_c|^2 \\ &+ c\int_{\mathbb{R}^2}\eta\mathfrak{Re}\psi\mathfrak{Im}\psi\partial_{x_2}(|Q_c|^2).\end{aligned}$$

See Subsection 3.3 for the details of the computation. For functions $\varphi \in H^1(\mathbb{R}^2)$ for instance, both quadratic forms (1.2) and (1.3) are well defined and are equal (see Lemma 5.7). We will show that B_{Q_c} is well defined for $\varphi \in H_{Q_c}$ (see Lemma 3.3), and that for $A \in \{\partial_{x_1}Q_c, \partial_{x_2}Q_c, \partial_c Q_c, \partial_{c^\perp}Q_c\}$, $B_{Q_c}(A) = \langle L_{Q_c}(A), A\rangle$.

From Proposition 1.2, we know that Q_c has only two zeros. We will write the quadratic form B_{Q_c} around the zeros of Q_c (for a function $\varphi = Q_c\psi \in H_{Q_c}$) as the quadratic form for one vortex (computed in Proposition 1.3), up to some small error. As we want to avoid adding an orthogonality condition on the phase, we change the coercivity norm to a weaker seminorm that avoids iQ_c, the direction connected to the shift of phase.

We will therefore infer a coercivity result under four orthogonality conditions near the zeros of Q_c (two for each zero). Then we shall show that, far from the zeros of Q_c, the coercivity holds, without any additional orthogonality conditions.

Proposition 1.4. *There exist $c_0, R > 0$ such that, for $0 < c \leqslant c_0$, if one defines $\tilde{V}_{\pm 1}$ to be the vortices centred around $\pm\widetilde{d_c}\overrightarrow{e_1}$ ($\widetilde{d_c}$ is defined in Proposition 1.2), there exists $K > 0$ such that for $\varphi = Q_c\psi \in H_{Q_c}$, $0 < c < c_0$, if the four orthogonality conditions*

$$\int_{B(\tilde{d}_c\overrightarrow{e_1},R)} \mathfrak{Re}(\partial_{x_1}\tilde{V}_1\overline{\tilde{V}_1\psi}) = \int_{B(\tilde{d}_c\overrightarrow{e_1},R)} \mathfrak{Re}(\partial_{x_2}\widetilde{V_1}\overline{\widetilde{V_1}\psi}) = 0,$$

$$\int_{B(-\tilde{d}_c\overrightarrow{e_1},R)} \mathfrak{Re}(\partial_{x_1}\tilde{V}_{-1}\overline{\tilde{V}_{-1}\psi}) = \int_{B(-\tilde{d}_c\overrightarrow{e_1},R)} \mathfrak{Re}(\partial_{x_2}\tilde{V}_{-1}\overline{\tilde{V}_{-1}\psi}) = 0$$

are satisfied, then, for

$$\|\varphi\|_{\mathcal{C}}^2 := \int_{\mathbb{R}^2} |\nabla\psi|^2|Q_c|^4 + \mathfrak{Re}^2(\psi)|Q_c|^4,$$

the following coercivity result holds:

$$B_{Q_c}(\varphi) \geqslant K\|\varphi\|_{\mathcal{C}}^2.$$

We will check that $\|\varphi\|_{\mathcal{C}}$ is well defined for $\varphi \in H_{Q_c}$ (see Section 3). Proposition 1.4 is proven in Subsection 4.4.

We point out that $\varphi = Q_c\psi \mapsto \|\varphi\|_{\mathcal{C}}$ is not a norm but a seminorm since $\int_{\mathbb{R}^2} |\nabla\psi|^2|Q_c|^4 + \mathfrak{Re}^2(\psi)|Q_c|^4 = 0$ implies only that $\varphi = \lambda i Q_c$ for some $\lambda \in \mathbb{R}$, and iQ_c is the direction connected to the shift of phase. Note also that in this proposition $\varphi = Q_c\psi$ but the orthogonality conditions are on $\widetilde{V_1}\psi$. This is a consequence of Proposition 1.3 and the fact that the coercivity is shown with a seminorm.

Now, we want to change the orthogonality conditions in Proposition 1.4 to quantities linked to the parameters $\vec{c}$ and X of the travelling waves, that is, $\partial_{x_1}Q_c$, $\partial_{x_2}Q_c$, $\partial_c Q_c$, and $\partial_{c^\perp}Q_c$. We can show that for $\varphi = Q_c\psi \in H_{Q_c}$, for instance

$$\left|\int_{B(\tilde{d}_c\vec{e_1},R)} \mathfrak{Re}(\partial_{x_1}\tilde{V}_1\overline{\widetilde{V_1}\psi})\right| \leqslant K\|\varphi\|_{\mathcal{C}},$$

but such an estimate might not hold for $\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)}\partial_{x_1}Q_c\overline{Q_c\psi}$ (because of the lack of control on $\mathfrak{Im}(\psi)$ in $L^2(\mathbb{R}^2)$ in the coercivity norm $\|\cdot\|_{\mathcal{C}}$). It is therefore difficult to have a local orthogonality condition directly on $\partial_{x_1}Q_c$ for instance.

To solve this issue, we shall use the harmonic decomposition around $\pm\tilde{d}_c\vec{e_1}$. For the constructed travelling wave Q_c, two distances play a particular role: d_c (defined in Theorem 1.1) and $\widetilde{d_c}$ (defined in Proposition 1.2 and connected to the position of the zeros of Q_c). In particular, we define the following polar coordinates for $x \in \mathbb{R}^2$:

$$\begin{aligned} re^{i\theta} &:= x \in \mathbb{R}^2, \\ r_{\pm1}e^{i\theta_{\pm1}} &:= x - (\pm d_c)\vec{e_1} \in \mathbb{R}^2, \\ \tilde{r}_{\pm1}e^{i\tilde{\theta}_{\pm1}} &:= x - (\pm\widetilde{d_c})\vec{e_1} \in \mathbb{R}^2. \end{aligned}$$

We will also use $\tilde{r} := \min(r_1, r_{-1})$ and $\check{r} := \min(\widetilde{r_1}, \tilde{r}_{-1})$. For a function ψ such that $Q_c\psi \in H^1_{\mathrm{loc}}(\mathbb{R}^2)$ and $j \in \mathbb{Z}$, we define its j-harmonic around $\pm\widetilde{d_c}\vec{e_1}$ by the radial function around $\pm\widetilde{d_c}\vec{e_1}$:

$$\psi^{j,\pm1}(\tilde{r}_{\pm1}) := \frac{1}{2\pi}\int_0^{2\pi} \psi(\tilde{r}_{\pm1}e^{i\tilde{\theta}_{\pm1}})e^{-ij\tilde{\theta}_{\pm1}}\,d\tilde{\theta}_{\pm1}.$$

Summing over the Fourier modes leads to

$$\psi(x) = \sum_{j\in\mathbb{Z}} \psi^{j,\pm1}(\tilde{r}_{\pm1})e^{ij\tilde{\theta}_{\pm1}}$$

and we define, to simplify the notations later on, the function $\psi^{\neq0}$, by

$$\psi^{\neq0}(x) := \psi(x) - \psi^{0,1}(\tilde{r}_1)$$

in the right half-plane, and

$$\psi^{\neq0}(x) := \psi(x) - \psi^{0,-1}(\tilde{r}_{-1})$$

in the left half-plane. This notation will only be used far from the line $\{x_1 = 0\}$. We now state the main coercivity result.

Theorem 1.5. *There exist $c_0, K, \beta_0 > 0$ such that, for $R > 0$ defined in Proposition 1.4, for any $0 < \beta < \beta_0$, there exist $c_0(\beta), K(\beta) > 0$ such that, for $c < c_0(\beta)$, if $\varphi = Q_c\psi \in H_{Q_c}$ satisfies the following three orthogonality conditions:*

$$\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)} \partial_{x_1}Q_c\overline{Q_c\psi^{\neq 0}} = \mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)} \partial_{x_2}Q_c\overline{Q_c\psi^{\neq 0}} = 0,$$

$$\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)} \partial_c Q_c\overline{Q_c\psi^{\neq 0}} = 0,$$

then

$$B_{Q_c}(\varphi) \geqslant K(\beta)c^{2+\beta}\|\varphi\|_{\mathcal{C}}^2,$$

with

$$\|\varphi\|_{\mathcal{C}}^2 = \int_{\mathbb{R}^2} |\nabla\psi|^2|Q_c|^4 + \mathfrak{Re}^2(\psi)|Q_c|^4.$$

If $\varphi = Q_c\psi$ also satisfies the fourth orthogonality condition (with $0 < c < c_0$)

$$\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)} \partial_{c^\perp} Q_c\overline{Q_c\psi^{\neq 0}} = 0,$$

then

$$B_{Q_c}(\varphi) \geqslant K\|\varphi\|_{\mathcal{C}}^2.$$

Theorem 1.5 shows that, under four orthogonality conditions, we have a coercivity result in a weaker norm $\|\cdot\|_{\mathcal{C}}$, instead of $\|\cdot\|_{H_{Q_c}}$ with a constant independent of c, and with only three orthogonality conditions, we have the coercivity but the constant is a $O^{\beta}_{c\to 0}(c^{2+\beta})$. This is because, of the four particular directions of the linearized operator, $\partial_{x_1}Q_c$, $\partial_{x_2}Q_c$ are in its kernel, $\partial_c Q_c$ is a small negative direction, and $\partial_{c^\perp}Q_c$ is a small positive direction (see Proposition 1.2). Concerning the orthogonality conditions, we note that, for $\varphi = Q_c\psi \in H_{Q_c}$,

$$\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)} \partial_{x_1}Q_c\overline{Q_c\psi^{\neq 0}}$$

is close to

$$\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)} \partial_{x_1}Q_c\overline{Q_c\psi}$$

(we have $\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)} \partial_{x_1}Q_c\overline{Q_c\psi^{0,1}} = o_{c\to 0}(1)\|\varphi\|_{H_{Q_c}}$ for instance), but the first quantity can be controlled by $\|\varphi\|_{\mathcal{C}}$, and the second cannot be.

Theorem 1.5 is a consequence of Proposition 1.4, and is shown in Section 5. From this result, we can also deduce the kernel of the linearized operator in H_{Q_c}.

Corollary 1.6. *There exists $c_0 > 0$ such that, for $0 < c < c_0$, Q_c defined in Theorem 1.1, for $\varphi \in H_{Q_c}$, the following properties are equivalent:*

(i) $L_{Q_c}(\varphi) = 0$ *in* $H^{-1}(\mathbb{R}^2)$, *that is,* $\forall \varphi^* \in H^1(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} \mathfrak{Re}(\nabla\varphi \cdot \nabla\overline{\varphi^*}) - (1 - |Q_c|^2)\mathfrak{Re}(\varphi\overline{\varphi^*}) + 2\mathfrak{Re}(\overline{Q_c}\varphi)\mathfrak{Re}(\overline{Q_c}\varphi^*) \\ - \mathfrak{Re}(ic\partial_{x_2}\varphi\overline{\varphi^*}) = 0.$$

(ii) $\varphi \in \mathrm{Span}_{\mathbb{R}}(\partial_{x_1}Q_c, \partial_{x_2}Q_c)$.

This corollary is proven in Subsection 5.5. This nondegeneracy result is, to our knowledge, the first one on this type of model. It is a building block in the analysis of the dynamical stability of travelling waves and the construction of multi-travelling waves. Here, the travelling wave is not radial, nor has a simple profile, which means that we cannot use classical techniques for radial ground states for instance (see [**19**]).

1.2.3. Spectral stability in $H^1(\mathbb{R}^2)$. In this subsection, we give some results on the spectrum of $L_{Q_c} \colon H^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$. In particular, we are interested in negative eigenvalues of the linearized operator. We can show that $H^1(\mathbb{R}^2) \subset H_{Q_c}$ and prove the following corollary of Theorem 1.5.

Corollary 1.7. *There exists $c_0 > 0$ such that, for $0 < c \leqslant c_0$, Q_c defined in Theorem 1.1, if $\varphi \in H^1(\mathbb{R}^2)$ satisfies*

$$\langle \varphi, i\partial_{x_2}Q_c \rangle = 0,$$

then

$$B_{Q_c}(\varphi) \geqslant 0.$$

We can show that $L_{Q_c}(\partial_c Q_c) = i\partial_{x_2}Q_c \in L^2(\mathbb{R}^2)$, and thus $\varphi \overline{i\partial_{x_2}Q_c} \in L^1(\mathbb{R}^2)$ for $\varphi \in H^1(\mathbb{R}^2)$. This result shows that we expect only one negative direction for the linearized operator, and it should also hold in H_{Q_c}. For $\varphi \in H^1(\mathbb{R}^2)$, we have that $B_{Q_c}(\varphi)$ is equal to the expression (1.2).

Now, we define $\mathfrak{G}$ to be the collection of subspaces $S \subset H^1(\mathbb{R}^2)$ such that $B_{Q_c}(\varphi) < 0$ for all $\varphi \neq 0$, $\varphi \in S$, and we define

$$n^-(L_{Q_c}) := \max\{\dim S, S \in \mathfrak{G}\}.$$

Proposition 1.8. *There exists $c_0 > 0$ such that, for $0 < c < c_0$, for Q_c defined in Theorem 1.1,*

$$n^-(L_{Q_c}) = 1.$$

Furthermore, $L_{Q_c} \colon H^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ has exactly one negative eigenvalue with eigenvector in $L^2(\mathbb{R}^2)$.

With this result, Theorem 1.1 and Proposition 1.2, we have met all the conditions to show the spectral stability of the travelling wave:

Theorem 1.9 (Theorem 11.8(i) of [**15**]). *For $0 < c_1 < c_2$ and $c \mapsto U_c$ a C^1 branch of solutions of $(\mathrm{TW}_c)(U_c) = 0$ on $]c_1, c_2[$ with finite energy, for $c_* \in]c_1, c_2[$, under the following conditions:*

(i) *for all $c \in]c_1, c_2[$, $\mathfrak{Re}(U_c - 1) \in H^1(\mathbb{R}^2)$, $\mathfrak{Im}(\nabla U_c) \in L^2(\mathbb{R}^2)$, $|U_c| \to 1$ at infinity, and $\|U_c\|_{C^1(\mathbb{R}^2)} < +\infty$,*

(ii) *$n^-(L_{U_{c_*}}) \leqslant 1$,*

(iii) *$\partial_c P_2(U_c)_{|c=c_*} < 0$,*

then U_{c_*} is spectrally stable. That is, it is not an exponentially unstable solution of the linearized equation in $\dot{H}^1(\mathbb{R}^2, \mathbb{C})$.

Corollary 1.10. *There exists $c_0 > 0$ such that, for any $0 < c < c_0$, the function Q_c defined in Theorem 1.1 is spectrally stable in the sense of Theorem 1.9.*

The notion of spectral stability of [**15**] is the following: for any $u_0 \in H^1(\mathbb{R}^2, \mathbb{C})$, the solution to the problem

$$\begin{cases} i\partial_t u = L_{Q_c}(u), \\ u(t = 0) = u_0 \end{cases}$$

satisfies that, for all $\lambda > 0$,

$$\left(\int_{\mathbb{R}^2} |\nabla u|^2(t)\, dx \right) e^{-\lambda t} \to 0$$

when $t \to \infty$. The result of [**15**] is a little stronger: the norm that does not grow exponentially in time is better than the one on $\dot{H}^1(\mathbb{R}^2, \mathbb{C})$, but weaker than the one on $H^1(\mathbb{R}^2, \mathbb{C})$, and is not explicit.

1.3. Generalization to a larger energy space and use of the phase. There are two main difficulties with the phase. The first one, as previously stated, is that we lose a parameter when passing from two vortices to a travelling wave. The second one is that for the direction linked to the phase shift, namely iQ_c, we have $iQ_c \notin H_{Q_c}$ (and even for one vortex, $iV_1 \notin H_{V_1}$). This will be an obstacle when we modulate the phase for the local uniqueness result. Therefore, we define here a space larger than H_{Q_c}.

1.3.1. Definition and properties of the space $H^{\exp}_{Q_c}$. We define the space $H^{\exp}_{Q_c}$, the expanded energy space, by

$$H^{\exp}_{Q_c} := \{\varphi \in H^1_{\mathrm{loc}}(\mathbb{R}^2),\ \|\varphi\|_{H^{\exp}_{Q_c}} < +\infty\},$$

with the norm, for $\varphi = Q_c\psi \in H^1_{\mathrm{loc}}(\mathbb{R}^2)$,

$$\|\varphi\|^2_{H^{\exp}_{Q_c}} := \|\varphi\|^2_{H^1(\{\tilde{r} \leqslant 10\})} + \int_{\{\tilde{r} \geqslant 5\}} |\nabla\psi|^2 + \mathfrak{Re}^2(\psi) + \frac{|\psi|^2}{\tilde{r}^2 \ln^2(\tilde{r})},$$

where $\tilde{r} = \min(\widetilde{r_1}, \tilde{r}_{-1})$, the minimum of the distance to the zeros of Q_c. It is easy to check that there exists $K > 0$ independent of c such that, for $\varphi = Q_c\psi \in H^{\exp}_{Q_c}$,

$$\frac{1}{K}\|\varphi\|^2_{H^1(\{5\leqslant\tilde{r}\leqslant 10\})} \leqslant \int_{\{5\leqslant\tilde{r}\leqslant 10\}} |\nabla\psi|^2 + \mathfrak{Re}^2(\psi) + \frac{|\psi|^2}{\tilde{r}^2\ln(\tilde{r})^2} \leqslant K\|\varphi\|^2_{H^1(\{5\leqslant\tilde{r}\leqslant 10\})}.$$

We will show that $H_{Q_c} \subset H^{\exp}_{Q_c}$ and $iQ_c \in H^{\exp}_{Q_c}$, whereas $iQ_c \notin H_{Q_c}$. This space will appear in the proof of the local uniqueness (Theorem 1.14 below). The main difficulty is that $B_{Q_c}(\varphi)$ is not well defined for $\varphi \in H^{\exp}_{Q_c}$ because, for instance, of the term $(1-|Q_c|^2)|\varphi|^2$ integrated at infinity. If we write the linearized operator multiplicatively, for $\varphi = Q_c\psi$ (using $(\mathrm{TW}_c)(Q_c) = 0$),

$$Q_c L'_{Q_c}(\psi) := L_{Q_c}(\varphi) = Q_c\left(-ic\partial_{x_2}\psi - \Delta\psi - 2\frac{\nabla Q_c}{Q_c}\cdot\nabla\psi + 2\mathfrak{Re}(\psi)|Q_c|^2\right),$$

then there will be no problem at infinity for $\varphi \in H^{\exp}_{Q_c}$ for the associated quadratic form (in ψ), but there are instead some integrability issues near the zeros of Q_c. We take as before a smooth cutoff function η such that $\eta(x) = 0$ on $B(\pm\widetilde{d_c}\overrightarrow{e_1}, 1)$, $\eta(x) = 1$ on $\mathbb{R}^2\backslash B(\pm\widetilde{d_c}\overrightarrow{e_1}, 2)$, where $\pm\widetilde{d_c}\overrightarrow{e_1}$ are the zeros of Q_c. The natural linear operator for which we want to consider the quadratic form is then

$$L^{\exp}_{Q_c}(\varphi) := (1-\eta)L_{Q_c}(\varphi) + \eta Q_c L'_{Q_c}(\psi),$$

and we therefore define, for $\varphi = Q_c\psi \in H^{\exp}_{Q_c}$,

$$\begin{aligned} B^{\exp}_{Q_c}(\varphi) := &\int_{\mathbb{R}^2}(1-\eta)(|\nabla\varphi|^2 - \mathfrak{Re}(ic\partial_{x_2}\varphi\bar{\varphi}) - (1-|Q_c|^2)|\varphi|^2 + 2\mathfrak{Re}^2(\overline{Q_c}\varphi)) \\ &- \int_{\mathbb{R}^2}\nabla\eta\cdot(\mathfrak{Re}(\nabla Q_c\overline{Q_c})|\psi|^2 - 2\mathfrak{Im}(\nabla Q_c\overline{Q_c})\mathfrak{Re}(\psi)\mathfrak{Im}(\psi)) \\ &+ \int_{\mathbb{R}^2} c\partial_{x_2}\eta\mathfrak{Re}(\psi)\mathfrak{Im}(\psi)|Q_c|^2 \\ &+ \int_{\mathbb{R}^2}\eta(|\nabla\psi|^2|Q_c|^2 + 2\mathfrak{Re}^2(\psi)|Q_c|^4) \\ &+ \int_{\mathbb{R}^2}\eta(4\mathfrak{Im}(\nabla Q_c\overline{Q_c})\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi) + 2c|Q_c|^2\mathfrak{Im}(\partial_{x_2}\psi)\mathfrak{Re}(\psi)). \end{aligned} \tag{1.4}$$

This quantity is independent of the choice of η.

We will show that $B^{\exp}_{Q_c}(\varphi)$ is well defined for $\varphi \in H^{\exp}_{Q_c}$ and that, if $\varphi \in H_{Q_c} \subset H^{\exp}_{Q_c}$, then $B^{\exp}_{Q_c}(\varphi) = B_{Q_c}(\varphi)$. Writing the quadratic form $B^{\exp}_{Q_c}$ is a way to enlarge the space of possible perturbations to add in particular the remaining zero of the linearized operator. We infer the following result.

Proposition 1.11. *There exist $c_0, K, R, \beta_0 > 0$ such that, for any $0 < \beta < \beta_0$, there exist $c_0(\beta), K(\beta) > 0$ such that, for $0 < c < c_0(\beta)$, if $\varphi = Q_c\psi \in H^{\exp}_{Q_c}$ satisfies the following three orthogonality conditions:*

$$\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)} \partial_{x_1}Q_c\overline{Q_c\psi^{\neq 0}} = \mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)} \partial_{x_2}Q_c\overline{Q_c\psi^{\neq 0}} = 0,$$

$$\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)} \partial_c Q_c\overline{Q_c\psi^{\neq 0}} = 0,$$

then

$$B^{\exp}_{Q_c}(\varphi) \geqslant K(\beta)c^{2+\beta}\|\varphi\|^2_{\mathcal{C}},$$

with

$$\|\varphi\|^2_{\mathcal{C}} = \int_{\mathbb{R}^2} |\nabla\psi|^2|Q_c|^4 + \mathfrak{Re}^2(\psi)|Q_c|^4.$$

If $\varphi = Q_c\psi$ also satisfies the fourth orthogonality condition (with $0 < c < c_0$)

$$\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)} \partial_{c^\perp} Q_c\overline{Q_c\psi^{\neq 0}} = 0,$$

then

$$B^{\exp}_{Q_c}(\varphi) \geqslant K\|\varphi\|^2_{\mathcal{C}}.$$

Furthermore, for $\varphi \in H^{\exp}_{Q_c}$, the following properties are equivalent:

(i) *$L_{Q_c}(\varphi) = 0$ in $H^{-1}(\mathbb{R}^2)$, that is, $\forall\varphi^* \in H^1(\mathbb{R}^2)$,*

$$\int_{\mathbb{R}^2} \mathfrak{Re}(\nabla\varphi\cdot\nabla\overline{\varphi^*}) - (1-|Q_c|^2)\mathfrak{Re}(\varphi\overline{\varphi^*}) + 2\mathfrak{Re}(\overline{Q_c}\varphi)\mathfrak{Re}(\overline{Q_c}\varphi^*) - \mathfrak{Re}(ic\partial_{x_2}\varphi\overline{\varphi^*}) = 0.$$

(ii) *$\varphi \in \mathrm{Span}_{\mathbb{R}}(iQ_c, \partial_{x_1}Q_c, \partial_{x_2}Q_c)$.*

Proposition 1.11 is proven in Subsection 6.1. The additional direction in the kernel comes from the invariance of phase ($L_{Q_c}(iQ_c) = 0$). The main difficulty, compared to Theorem 1.5, is to show that the considered quantities are well defined with only $\varphi \in H^{\exp}_{Q_c}$, and that we can conclude by density in this bigger space.

1.3.2. Coercivity results with an orthogonality condition on the phase. The main problem with adding a local orthogonality condition on iQ_c is to choose where to put it. Indeed, we want this condition near both zeros of Q_c, or else the coercivity constant will depend on the distance between the vortices, which itself depends on c.

The first option is to let the coercivity constant depend on c. In that case, we can also remove the orthogonality condition on $\partial_{c^\perp}Q_c$, the small positive direction. We infer the following result.

Proposition 1.12. *There exist universal constants $K_1, c_0 > 0$ such that, with $R > 0$ defined in Proposition 1.4, for $0 < c < c_0$, for the function Q_c defined in Theorem 1.1, there exists $K_2(c) > 0$ depending on c such that, if $\varphi = Q_c\psi \in H^{\exp}_{Q_c}$ satisfies the following four orthogonality conditions:*

$$\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)} \partial_{x_1}Q_c\overline{Q_c\psi^{\neq 0}} = \mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)} \partial_{x_2}Q_c\overline{Q_c\psi^{\neq 0}} = 0,$$

$$\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)} \partial_c Q_c\overline{Q_c\psi^{\neq 0}} = \mathfrak{Re}\int_{B(0,R)} i\psi = 0,$$

then

$$K_1\|\varphi\|^2_{H^{\exp}_{Q_c}} \geqslant B^{\exp}_{Q_c}(\varphi) \geqslant K_2(c)\|\varphi\|^2_{H^{\exp}_{Q_c}}.$$

Here, the orthogonality condition on iQ_c is around 0, between the two vortices, but it can be chosen near one of the vortices for instance, and the result still holds.

The second possibility is to work with symmetric perturbations, since the orthogonality condition can then be at both the zeros of Q_c. We then study the space

$$H^{\exp,s}_{Q_c} := \{\varphi \in H^{\exp}_{Q_c},\ \forall x = (x_1, x_2) \in \mathbb{R}^2,\ \varphi(x_1, x_2) = \varphi(-x_1, x_2)\}.$$

We show that, under three orthogonality conditions, the quadratic form is equivalent to the norm on $H^{\exp}_{Q_c}$.

Theorem 1.13. *There exist $R, K, c_0 > 0$ such that, for $0 < c \leqslant c_0$, Q_c defined in Theorem 1.1, if a function $\varphi \in H^{\exp,s}_{Q_c}$ satisfies the three orthogonality conditions:*

$$\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)} \partial_c Q_c\bar{\varphi} = \mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)} \partial_{x_2}Q_c\bar{\varphi} = 0,$$

$$\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)} iQ_c\bar{\varphi} = 0,$$

then

$$\frac{1}{K}\|\varphi\|^2_{H^{\exp}_{Q_c}} \geqslant B^{\exp}_{Q_c}(\varphi) \geqslant K\|\varphi\|^2_{H^{\exp}_{Q_c}}.$$

We note that here the orthogonality conditions to $\partial_{x_1}Q_c$ and $\partial_{c^\perp}Q_c$ are freely given by the symmetry. We also do not need to remove the 0-harmonic near the zeros of Q_c.

If we remove the symmetry, and if we add the two orthogonality conditions related to $\partial_{x_1}Q_c$ and $\partial_{c^\perp}Q_c$, it is not clear that we can get a similar result (with a coercivity constant independent of c). The main difficulty would come from the phase, because we would have one orthogonality condition on it, but we would like two, one on each vortex.

Proposition 1.12 and Theorem 1.13 hold if we replace $B^{\exp}_{Q_c}$ by B_{Q_c} for $\varphi = Q_c\psi \in H_{Q_c}$ with the symmetry, but the coercivity norm will still be $\|\cdot\|_{H^{\exp}_{Q_c}}$.

1.4. Local uniqueness result. With Propositions 1.11 and 1.12, we can modulate the five parameters $(\vec{c}, X, \gamma)$ of the travelling wave, and these coercivity results will be enough to show the following theorem.

Theorem 1.14. *There exist constants $K, c_0, \varepsilon_0, \mu_0 > 0$ such that, for $0 < c < c_0$, Q_c defined in Theorem 1.1, there exists $R_c > 0$ depending on c such that, for any $\lambda > R_c$, if a function $Z \in C^2(\mathbb{R}^2, \mathbb{C})$ satisfies, for some small constant $\varepsilon(c,\lambda) > 0$, depending on c and λ,*

(i) $(\mathrm{TW}_c)(Z) = 0$,
(ii) $E(Z) < +\infty$,
(iii) $\|Z - Q_c\|_{C^1(\mathbb{R}^2\setminus B(0,\lambda))} \leqslant \mu_0$,
(iv) $\|Z - Q_c\|_{H^{\exp}_{Q_c}} \leqslant \varepsilon(c,\lambda)$,

then, there exists $X \in \mathbb{R}^2$ such that $|X| \leqslant K\|Z - Q_c\|_{H^{\exp}_{Q_c}}$, and

$$Z = Q_c(\cdot - X).$$

The conditions $E(Z) < +\infty$ and $\|Z - Q_c\|_{H^{\exp}_{Q_c}} \leqslant \varepsilon(c,\lambda)$ imply that the travelling wave $Z \to 1$ at infinity, and therefore $Z = Q_c e^{i\gamma}$ with $\gamma \in \mathbb{R}$, $\gamma \neq 0$, is excluded. The fact that $\varepsilon(c,\lambda)$ depends on c comes in part from the constant of coercivity in Proposition 1.12, which itself depends on c. The condition that $\|Z - Q_c\|_{C^1(\mathbb{R}^2\setminus B(0,\lambda))} \leqslant \mu_0$ outside of $B(0,\lambda)$ is mainly technical. We believe that this condition is automatically satisfied with the other ones (with λ depending only on c), but we were not able to show it.

To the best of our knowledge, this is the first result of local uniqueness for travelling waves in (GP). It does not suppose any symmetries on Z, and therefore shows that we cannot bifurcate from this branch, even to nonsymmetric travelling waves.

We believe that, at least in the symmetric case, Theorem 1.14 should hold for $\|Z - Q_c\|_{H^{\exp}_{Q_c}} \leqslant \varepsilon$ with $\varepsilon > 0$ independent of c and λ. We also note that the condition $\|Z - Q_c\|_{H^{\exp}_{Q_c}} \leqslant \varepsilon(c,\lambda)$ is weaker than $\|Z - Q_c\|_{H_{Q_c}} \leqslant \varepsilon(c,\lambda)$, and thus we can state a result in H_{Q_c}.

1.5. Plan of the proofs. Section 2 is devoted to the proof of Proposition 1.2. We start by giving some estimates on the branch of travelling waves in Subsection 2.1. We then show the equivalents when $c \to 0$ for the energy and momentum, as well as the relations between them and some specific values of the quadratic form in Subsection 2.2. Finally, in Subsection 2.3, we study the travelling wave near its zeros.

In Section 3, we infer some properties of the space H_{Q_c}. First, we explain why we cannot have a coercivity result in the energy norm in Subsection 3.1, and we show the well-posedness of several quantities in Subsections 3.2 and 3.3. A density argument is given in Subsection 3.4 that will be needed for the proof of Proposition 1.4.

Section 4 is devoted to the proofs of Propositions 1.3 and 1.4. We start by writing the quadratic form for test functions in a particular form (Subsection 4.1), and we then show Propositions 1.3 and 1.4 respectively in Subsections 4.2 and 4.4. To show Proposition 1.4, we use Proposition 1.3 and the fact that we know well the travelling wave near its zeros from Subsection 2.3.

The next part, Section 5, is devoted to the proof of Theorem 1.5 and its corollaries. We show the coercivity under four orthogonality conditions by showing that we can modify the initial function by a small amount to have the four orthogonality conditions of Proposition 1.4, and that the error committed is small in the coercivity norm. We then focus on the corollaries of Theorem 1.5 in Subsection 5.5. We show the composition of the kernel of L_{Q_c} (Corollary 1.6), and the results in $H^1(\mathbb{R}^2)$: Corollary 1.7, Proposition 1.8, and Corollary 1.10.

The penultimate Section 6 is devoted to the proofs of Propositions 1.11 and 1.12 and Theorem 1.13. In Subsection 6.1, we study the space $H^{\exp}_{Q_c}$; in particular, we give a density argument that allows us to finish the proof of Proposition 1.11. Then, in Subsection 6.2, we compute how the additional orthogonality condition improves the coercivity norm, both in the symmetric and nonsymmetric case, and we can then show Proposition 1.12 and Theorem 1.13.

Section 7 is devoted to the proof of Theorem 1.14. Here we use classical methods for the proof of local uniqueness, by modulating the five parameters of the family and using a coercivity result. One of the main points is to write the problem additively near the zeros of Q_c and multiplicatively far from them. The reason for that is that we do not know the link between the speed and the position of the zeros of a travelling wave in general, and we therefore cannot write a perturbation multiplicatively in the whole space. Because of that, here we require an orthogonality condition on the phase, and we cannot avoid it, as we did for instance in the proof of Proposition 1.4 by choosing correctly the position of the vortices.

We will use many cutoffs in the proofs. As a rule of thumb, a function written as η, χ, or $\tilde{\chi}$ will be smooth and have value 1 at infinity and 0 in some compact domain. The function η itself is reserved for B_{Q_c} and $B_{Q_c}^{\exp}$ (see equations (1.3) and (1.4)).

Acknowledgments. The authors would like to thank Pierre Raphaël for helpful discussions. E. Pacherie is supported by the ERC-2014-CoG 646650 SingWave.

2. Properties of the branch of travelling waves

This section is devoted to the proof of Proposition 1.2. In Subsection 2.1, we recall some estimates on Q_c defined in Theorem 1.1 from previous works ([**2**], [**4**], [**9**], and [**13**]). In Subsection 2.2, we compute some equalities and equivalents when $c \to 0$ on the energy, momentum and the four particular directions ($\partial_{x_1} Q_c$, $\partial_{x_2} Q_c$, $\partial_c Q_c$ and $\partial_{c^\perp} Q_c$). Finally, the properties of the zeros of Q_c are studied in Subsection 2.3.

2.1. Decay estimates.

2.1.1. Estimates on vortices. We recall that vortices are stationary solutions of (GP) of degrees $n \in \mathbb{Z}^*$ (see [**2**]):

$$V_n(x) = \rho_n(r) e^{in\theta},$$

where $x = re^{i\theta}$, solving

$$\begin{cases} \Delta V_n - (|V_n|^2 - 1)V_n = 0, \\ |V_n| \to 1 \text{ as} |x| \to \infty. \end{cases}$$

Here we regroup estimates on quantities involving vortices. We start with estimates on $V_{\pm 1}$.

Lemma 2.1 ([**2**] and [**13**]). *A vortex centred around* 0, $V_1(x) = \rho_1(r) e^{i\theta}$, *verifies* $V_1(0) = 0$, *and there exist constants* $K, \kappa > 0$ *such that*

$$\forall r > 0,\, 0 < \rho_1(r) < 1,\, \rho_1(r) \sim_{r \to 0} \kappa r, \rho_1'(r) \sim_{r \to 0} \kappa,$$

$$\rho_1'(r) > 0;\, \rho_1'(r) = O_{r \to \infty}\left(\frac{1}{r^3}\right),\, |\rho_1''(r)| + |\rho_1'''(r)| \leqslant K,$$

$$1 - |V_1(x)| = \frac{1}{2r^2} + O_{r \to \infty}\left(\frac{1}{r^3}\right),$$

$$|\nabla V_1| \leqslant \frac{K}{1+r},\, |\nabla^2 V_1| \leqslant \frac{K}{(1+r)^2},$$

and

$$\nabla V_1(x) = iV_1(x)\frac{x^\perp}{r^2} + O_{r \to \infty}\left(\frac{1}{r^3}\right),$$

where $x^\perp = (-x_2, x_1)$*,* $x = re^{i\theta} \in \mathbb{R}^2$*. Furthermore, similar properties hold for* V_{-1}*, since*

$$V_{-1}(x) = \overline{V_1(x)}.$$

We also define, as in [**4**],

$$V(\cdot) := V_1(\cdot - d_c\overrightarrow{e_1})V_{-1}(\cdot + d_c\overrightarrow{e_1})$$

and

$$\partial_d V(\cdot) := \partial_d(V_1(\cdot - d\overrightarrow{e_1})V_{-1}(\cdot + d\overrightarrow{e_1}))_{|d=d_c}.$$

We will also estimate

$$\partial_d^2 V := \partial_d^2(V_1(\cdot - d\overrightarrow{e_1})V_{-1}(\cdot + d\overrightarrow{e_1}))_{|d=d_c}.$$

The function $V(x) = V_1(x - d_c\overrightarrow{e_1})V_{-1}(x + d_c\overrightarrow{e_1})$ is close to $V_1(x - d_c\overrightarrow{e_1})$ in $B(d_c\overrightarrow{e_1}, 2d_c^{1/2})$, since, from Lemma 2.1 and [**2**], we have, uniformly in $B(d_c\overrightarrow{e_1}, 2d_c^{1/2})$,

$$V_{-1}(\cdot + d_c\overrightarrow{e_1}) = 1 + O_{c\to 0}(c^{1/2}) \tag{2.1}$$

and

$$|\nabla V_{-1}(\cdot + d_c\overrightarrow{e_1})| \leqslant \frac{o_{c\to 0}(c^{1/2})}{(1+\widetilde{r_1})}. \tag{2.2}$$

We recall that $B(d_c\overrightarrow{e_1}, 2d_c^{1/2})$ is near the vortex of degree $+1$ of Q_c and that $\tilde{r} = \min(r_1, r_{-1})$, with $r_{\pm 1} = |x \mp d_c\overrightarrow{e_1}|$.

2.1.2. Estimates on Q_c from [**4**]**.** We recall, for the function Q_c defined in Theorem 1.1, that

$$\forall (x_1, x_2) \in \mathbb{R}^2,\ Q_c(x_1, x_2) = \overline{Q_c(x_1, -x_2)} = Q_c(-x_1, x_2). \tag{2.3}$$

In particular, $\partial_c Q_c$ enjoys the same symmetries, since (2.3) holds for any $c > 0$ small enough. We recall that $Q_c \in C^\infty(\mathbb{R}^2, \mathbb{C})$ by standard elliptic regularity arguments.

Finally, we recall some estimates on Q_c and its derivatives, coming from Lemma 3.8 of [**4**]. We denote $\tilde{r} = \min(r_1, r_{-1})$, the minimum of the distances to $d_c\vec{e}_1$ and $-d_c\vec{e}_1$, and we recall that $V(x) = V_1(x - d_c\overrightarrow{e_1})V_{-1}(x + d_c\overrightarrow{e_1})$.

We write $Q_c = V + \Gamma_c$ or $Q_c = (1-\eta)V\Psi_c + \eta V e^{\Psi_c}$, where $\Gamma_c = (1-\eta)V\Psi_c + \eta V(e^{\Psi_c} - 1)$ (see equation (3.4) of [**4**]). There exists $K > 0$ and, for any $0 < \sigma < 1$, there exists $K(\sigma) > 0$ such that

$$|\Gamma_c| \leqslant \frac{K(\sigma)c^{1-\sigma}}{(1+\tilde{r})^{\sigma}}, \tag{2.4}$$

$$|\nabla\Gamma_c| \leqslant \frac{K(\sigma)c^{1-\sigma}}{(1+\tilde{r})^{1+\sigma}}, \tag{2.5}$$

$$|1 - |Q_c|| \leqslant \frac{K(\sigma)}{(1+\tilde{r})^{1+\sigma}}, \tag{2.6}$$

$$|Q_c - V| \leqslant \frac{K(\sigma)c^{1-\sigma}}{(1+\tilde{r})^{\sigma}}, \tag{2.7}$$

$$||Q_c|^2 - |V|^2| \leqslant \frac{K(\sigma)c^{1-\sigma}}{(1+\tilde{r})^{1+\sigma}}, \tag{2.8}$$

$$|\mathfrak{Re}(\nabla Q_c \overline{Q_c})| \leqslant \frac{K(\sigma)}{(1+\tilde{r})^{2+\sigma}}, \tag{2.9}$$

$$|\mathfrak{Im}(\nabla Q_c \overline{Q_c})| \leqslant \frac{K}{1+\tilde{r}}, \tag{2.10}$$

and for $0 < \sigma < \sigma' < 1$, there exists $K(\sigma, \sigma') > 0$ such that

$$|D^2\mathfrak{Im}(\Psi_c)| + |\nabla\mathfrak{Re}(\Psi_c)| + |\nabla^2\mathfrak{Re}(\Psi_c)| \leqslant \frac{K(\sigma,\sigma')c^{1-\sigma'}}{(1+\tilde{r})^{2+\sigma}}. \tag{2.11}$$

From Lemma 2.1, with Theorem 1.1, we deduce in particular that, for c small enough, there exist universal constants $K_1, K_2 > 0$ such that on $\mathbb{R}^2 \backslash B(\pm d_c \overrightarrow{e_1}, 1)$ we have

$$K_1 \leqslant |Q_c| \leqslant K_2. \tag{2.12}$$

To these estimates, we add two additional lemmas. We write

$$\begin{aligned}\|\psi\|_{\sigma,d_c} := &\|V\psi\|_{C^1(\{\tilde{r}\leqslant 3\})} + \|\tilde{r}^{1+\sigma}\mathfrak{Re}(\psi)\|_{L^\infty(\{\tilde{r}\geqslant 2\})} + \|\tilde{r}^{2+\sigma}\nabla\mathfrak{Re}(\psi)\|_{L^\infty(\{\tilde{r}\geqslant 2\})}\\ &+ \|\tilde{r}^{\sigma}\mathfrak{Im}(\psi)\|_{L^\infty(\{\tilde{r}\geqslant 2\})} + \|\tilde{r}^{1+\sigma}\nabla\mathfrak{Im}(\psi)\|_{L^\infty(\{\tilde{r}\geqslant 2\})},\end{aligned}$$

where $\tilde{r} = \min(r_1, r_{-1})$, with

$$r_{\pm 1} = |x \mp d_c \overrightarrow{e_1}|, \tag{2.13}$$

and with d_c defined in Theorem 1.1. The first lemma is about Q_c and the second one about $\partial_c Q_c$.

Lemma 2.2. *For any $0 < \sigma < 1$, there exist $c_0(\sigma), K(\sigma) > 0$ such that, for $0 < c < c_0(\sigma)$ and Q_c defined in Theorem 1.1, if*

$$\Gamma_c = Q_c - V,$$

then

$$\left\| \frac{\Gamma_c}{V} \right\|_{\sigma, d_c} \leqslant K(\sigma) c^{1-\sigma}.$$

Proof: This estimate is a consequence of

$$\Gamma_c = (1-\eta) V \Psi_c + \eta V (e^{\Psi_c} - 1)$$

and equation (3.10) of [**4**]. □

Lemma 2.3 (Lemma 4.6 of [**4**])**.** *There exists $1 > \beta_0 > 0$ such that, for all $0 < \sigma < \beta_0 < \sigma' < 1$, there exists $c_0(\sigma, \sigma') > 0$ such that for any $0 < c < c_0(\sigma, \sigma')$, Q_c defined in Theorem 1.1, $c \mapsto Q_c$ is a C^1 function from $]0, c_0(\sigma, \sigma')[$ to $C^1(\mathbb{R}^2, \mathbb{C})$, and*

$$\left\| \frac{\partial_c Q_c}{V} + \left(\frac{1 + o^{\sigma,\sigma'}_{c\to 0}(c^{1-\sigma'})}{c^2} \right) \frac{\partial_d V_{|d=d_c}}{V} \right\|_{\sigma, d_c} = o^{\sigma,\sigma'}_{c\to 0}\left(\frac{c^{1-\sigma'}}{c^2} \right).$$

These results are technical, but quite precise. They give both a decay in position and the size in c of the error term. The statement of Lemma 4.6 of [**4**] has $o_{c\to 0}(1)$ and $o_{c\to 0}\left(\frac{1}{c^2}\right)$ instead of respectively $o_{c\to 0}(c^{1-\sigma'})$ and $o_{c\to 0}\left(\frac{c^{1-\sigma'}}{c^2}\right)$, but its proof gives this better estimate (given that σ' is close enough to 1). We recall that $o^{\sigma,\sigma'}_{c\to 0}(1)$ is a quantity going to 0 when $c \to 0$ at fixed σ, σ'. We recall that $\partial_c \nabla Q_c = \nabla \partial_c Q_c$. We conclude this subsection with a link between the $\|\cdot\|_\sigma$ norms and $\|\cdot\|_{H_{Q_c}}$. We recall

$$\|\varphi\|^2_{H_{Q_c}} = \int_{\mathbb{R}^2} |\nabla \varphi|^2 + |1 - |Q_c|^2| |\varphi|^2 + \mathfrak{Re}^2(\overline{Q_c} \varphi).$$

Lemma 2.4. *There exists a universal constant $K > 0$ (independent of c) such that, for Q_c defined in Theorem 1.1,*

$$\|h\|_{H_{Q_c}} \leqslant K \left\| \frac{h}{V} \right\|_{3/4, d_c}.$$

The value $\sigma = 3/4$ is arbitrary here; this estimate holds for $\sigma \in \left]\frac{1}{2}, 1\right[$.

Proof: We compute, using Lemma 2.1, that

$$\begin{aligned} \int_{\mathbb{R}^2} |\nabla h|^2 &\leqslant K \left\| \frac{h}{V} \right\|^2_{3/4, d_c} + \int_{\{\tilde r \geqslant 1\}} \left| \nabla \left(\frac{h}{V} V \right) \right|^2 \\ &\leqslant K \left\| \frac{h}{V} \right\|^2_{3/4, d_c} + 2 \int_{\{\tilde r \geqslant 1\}} \left| \nabla \left(\frac{h}{V} \right) \right|^2 + |\nabla V|^2 \frac{|h|^2}{|V|^2}. \end{aligned}$$

With Lemma 2.1 and the definition of $\|\cdot\|_{3/4,d_c}$, we check that

$$2\int_{\{\tilde r\geqslant 1\}}\left|\nabla\left(\frac{h}{V}\right)\right|^2+|\nabla V|^2\frac{|h|^2}{|V|^2}\leqslant K\left\|\frac{h}{V}\right\|^2_{3/4,d_c}\int_{\{\tilde r\geqslant 1\}}\frac{1}{(1+\tilde r)^{3+1/2}}\leqslant K\left\|\frac{h}{V}\right\|^2_{3/4,d_c}.$$

Indeed, we have the estimate

$$\int_{\{\tilde r\geqslant 1\}}\frac{1}{(1+\tilde r)^{3+1/2}}\leqslant 2\int_{\{r\geqslant 1\}}\frac{1}{(1+r)^{3+1/2}}\leqslant K.$$

Furthermore, from equation (2.6) with $\sigma=1/2$, we have the estimate

$$\int_{\mathbb{R}^2}|1-|Q_c|^2||h|^2\leqslant K\left\|\frac{h}{V}\right\|^2_{3/4,d_c}\int_{\mathbb{R}^2}\frac{1}{(1+\tilde r)^{9/4}}\leqslant K\left\|\frac{h}{V}\right\|^2_{3/4,d_c}.$$

Finally, we compute

$$\int_{\mathbb{R}^2}\mathfrak{Re}^2(\overline{Q_c}h)\leqslant K\left\|\frac{h}{V}\right\|^2_{3/4,d_c}+\int_{\{\tilde r\geqslant 1\}}\mathfrak{Re}^2(\overline{Q_c}h),$$

and

$$\begin{aligned}\int_{\{\tilde r\geqslant 1\}}\mathfrak{Re}^2(\overline{Q_c}h)&=\int_{\{\tilde r\geqslant 1\}}\mathfrak{Re}^2\left(V\overline{Q_c}\frac{h}{V}\right)\\&\leqslant 2\int_{\{\tilde r\geqslant 1\}}\mathfrak{Re}^2\left(\frac{h}{V}\right)\mathfrak{Re}^2(V\overline{Q_c})+\mathfrak{Im}^2\left(\frac{h}{V}\right)\mathfrak{Im}^2(V\overline{Q_c}).\end{aligned}$$

With the definition of $\|\cdot\|_{3/4,d_c}$, Lemmas 2.1 and 2.2, we check that

$$\begin{aligned}\int_{\{\tilde r\geqslant 1\}}\mathfrak{Re}^2\left(\frac{h}{V}\right)\mathfrak{Re}^2(V\overline{Q_c})&\leqslant K\int_{\{\tilde r\geqslant 1\}}\mathfrak{Re}^2\left(\frac{h}{V}\right)\\&\leqslant K\left\|\frac{h}{V}\right\|^2_{3/4,d_c}\int_{\{\tilde r\geqslant 1\}}\frac{1}{(1+\tilde r)^{3+1/2}}\leqslant K\left\|\frac{h}{V}\right\|^2_{3/4,d_c}.\end{aligned}$$

With Lemma 2.2 with $\sigma=1/2$, we check that, since $\mathfrak{Im}^2(V\overline{Q_c})=\mathfrak{Im}^2(V\overline{V+\Gamma_c})=\mathfrak{Im}^2(V\bar\Gamma_c)$, we have

$$\int_{\{\tilde r\geqslant 1\}}\mathfrak{Im}^2\left(\frac{h}{V}\right)\mathfrak{Im}^2(V\overline{Q_c})\leqslant K\left\|\frac{h}{V}\right\|^2_{3/4,d_c}\int_{\{\tilde r\geqslant 1\}}\frac{1}{(1+\tilde r)^{2+1/2}}\leqslant K\left\|\frac{h}{V}\right\|^2_{3/4,d_c}.$$

Combining these estimates, we end the proof of this lemma. □

2.1.3. Faraway estimates on Q_c. Since $E(Q_c)<+\infty$ thanks to Theorem 1.1, from Theorem 7 of [**9**], we have the following result.

Theorem 2.5 ([**9**, Theorem 7]). *There exists a constant $C(c) > 0$ (depending on c) such that, for Q_c defined in Theorem 1.1,*

$$|1 - |Q_c|^2| \leqslant \frac{C(c)}{(1+r)^2},$$

$$|1 - Q_c| \leqslant \frac{C(c)}{1+r},$$

$$|\nabla Q_c| \leqslant \frac{C(c)}{(1+r)^2},$$

and

$$|\nabla |Q_c|| \leqslant \frac{C(c)}{(1+r)^3}.$$

Furthermore, such estimates hold for any travelling wave with finite energy (but then the constant $C(c)$ also depends on the travelling wave, and not only on its speed).

This result is crucial to show that some terms are well defined, since it gives better decay estimates in position than the estimates in Subsection 2.1.2 (but with no smallness in c). Note that $1 - |Q_c|^2$ is not necessarily positive. In fact it is not at infinity (see [**10**]). In particular, the estimate

$$|1 - |Q_c|^2| \geqslant \frac{C(c)}{1+r^2}$$

does not hold because of the possibility of $|Q_c| = 1$. This happens, but only for few directions and it can be offset. We show the following sufficient result, which is needed to show that some quantities we will use are well defined. Furthermore, in these estimates, the constant depends on c, and thus cannot be used in error estimates (since the smallness of the errors there will depend on c).

Lemma 2.6. *There exists $c_0 > 0$ such that, for $0 < c < c_0$, there exists $C(c) > 0$ such that for $\varphi \in H_{Q_c}$ and the function Q_c defined in Theorem 1.1,*

$$\int_{\mathbb{R}^2} \frac{|\varphi|^2}{(1+|x|)^2}\, dx \leqslant C(c) \left(\int_{\mathbb{R}^2} |\nabla \varphi|^2 + |1 - |Q_c|^2||\varphi|^2 \right).$$

See Appendix A.1 for the proof of this result.

2.2. Construction and properties of the four particular directions.

2.2.1. Definitions. The four directions we want to study here are $\partial_{x_1} Q_c$, $\partial_{x_2} Q_c$, $\partial_c Q_c$, and $\partial_{c^\perp} Q_c$. The first two are derivatives of Q_c with respect to the position, the third one is the derivative of Q_c with respect to the speed, and we have its first order term in Theorem 1.1.

The fourth direction is defined in Lemma 2.7 below. The directions $\partial_{x_1} Q_c$ and $\partial_{x_2} Q_c$ correspond to the translations of the travelling wave, $\partial_c Q_c$ and $\partial_{c^\perp} Q_c$ to changes respectively in the modulus and direction of its speed. These directions will also appear in the orthogonality conditions for some of the coercivity results.

Lemma 2.7. *Take $\vec{c} \in \mathbb{R}^2$ such that $|\vec{c}| < c_0$ for c_0 defined in Theorem 1.1. Define α such that $\vec{c} = |\vec{c}| R_\alpha(-\vec{e}_2)$, where $R_\theta \colon \mathbb{R}^2 \to \mathbb{R}^2$ is the rotation of angle θ. Then, $Q_{\vec{c}} := Q_{|\vec{c}|} \circ R_{-\alpha}$ solves*

$$\begin{cases} (\mathrm{TW}_{\vec{c}})(v) = i\vec{c} \cdot \nabla v - \Delta v - (1 - |v|^2)v = 0, \\ |v| \to 1 \text{ as } |x| \to +\infty, \end{cases}$$

where $Q_{|\vec{c}|}$ is the solution of $(\mathrm{TW}_{|\vec{c}|})$ in Theorem 1.1. In particular, $Q_{\vec{c}}$ is a C^1 function of α and

$$\partial_\alpha Q_{\vec{c}}(x) = -R_{-\alpha}(x^\perp) \cdot \nabla Q_{|\vec{c}|}(R_{-\alpha}(x)).$$

Furthermore, at $\alpha = 0$, the quantity

$$\partial_{c^\perp} Q_c := (\partial_\alpha Q_{\vec{c}})_{|\alpha=0}$$

satisfies

$$\partial_{c^\perp} Q_c(x) = -x^\perp \cdot \nabla Q_c(x),$$

is in $C^\infty(\mathbb{R}^2, \mathbb{C})$, and

$$L_{Q_c}(\partial_{c^\perp} Q_c) = -ic\partial_{x_1} Q_c.$$

Proof: Since the Laplacian operator is invariant by rotation, it is easy to check that $Q_{|\vec{c}|} \circ R_{-\alpha}$ solves $(\mathrm{TW}_{\vec{c}})(Q_{|\vec{c}|} \circ R_{-\alpha}) = 0$. The function $\theta \mapsto R_\theta$ is C^1, hence $(\alpha, x) \mapsto Q_{\vec{c}}(x)$ is a C^1 function, and we compute

$$(\partial_\alpha Q_{\vec{c}})(x) = \partial_\alpha (Q_{|\vec{c}|} \circ R_{-\alpha})(x) = \partial_\alpha (R_{-\alpha}(x)) \cdot \nabla Q_{|\vec{c}|}(R_{-\alpha}(x)).$$

We note that

$$\partial_\alpha (R_{-\alpha}(x)) = -R_{-\alpha}(x^\perp),$$

where $x^\perp = (-x_2, x_1)$, hence

$$\partial_\alpha Q_{\vec{c}}(x) = -R_{-\alpha}(x^\perp) \cdot \nabla Q_{|\vec{c}|}(R_{-\alpha}(x)).$$

In particular, for $\alpha = 0$,

$$\partial_\alpha Q_{\vec{c}}(x)_{|\alpha=0} = -x^\perp \cdot \nabla Q_c(x).$$

We recall that $Q_{\vec{c}}$ solves

$$i\vec{c} \cdot \nabla Q_{\vec{c}} - \Delta Q_{\vec{c}} - (1 - |Q_{\vec{c}}|^2) Q_{\vec{c}} = 0,$$

and when we differentiate this equation with respect to α (with $|\vec{c}| = c$), we have

$$-i\partial_\alpha \vec{c} \cdot (\nabla Q_{\vec{c}}) + L_{Q_{\vec{c}}}(\partial_\alpha Q_{\vec{c}}) = 0.$$

At $\alpha = 0$, $Q_{\vec{c}} = Q_c$, $\partial_\alpha \vec{c} = -c\vec{e}_1$, and $\partial_\alpha Q_{\vec{c}|\alpha=0} = \partial_{c^\perp} Q_c$, therefore

$$L_{Q_c}(\partial_{c^\perp} Q_c) = -ic\partial_{x_1} Q_c. \qquad \square$$

2.2.2. Estimates on the four directions. We shall now show that the functions $\partial_{x_1}Q_c$, $\partial_{x_2}Q_c$, $\partial_c Q_c$, and $\partial_{c^\perp}Q_c$ are in the energy space and we will also compute their values through the linearized operator around Q_c, namely

$$L_{Q_c}(\varphi) = -\Delta\varphi - ic\partial_{x_2}\varphi - (1-|Q_c|^2)\varphi + 2\mathfrak{Re}(\overline{Q_c}\varphi)Q_c.$$

Lemma 2.8. *There exists $c_0 > 0$ such that, for $0 < c < c_0$, Q_c defined in Theorem 1.1, we have*

$$\partial_{x_1}Q_c, \partial_{x_2}Q_c, \partial_c Q_c, \partial_{c^\perp}Q_c \in H_{Q_c},$$

and

$$\begin{aligned} L_{Q_c}(\partial_{x_1}Q_c) = L_{Q_c}(\partial_{x_2}Q_c) &= 0,\\ L_{Q_c}(\partial_c Q_c) &= i\partial_{x_2}Q_c,\\ L_{Q_c}(\partial_{c^\perp}Q_c) &= -ic\partial_{x_1}Q_c. \end{aligned}$$

We could check that we also have $\partial_{x_1}Q_c, \partial_{x_2}Q_c \in H^1(\mathbb{R}^2)$ (see [**10**]), but we expect that $\partial_c Q_c, \partial_{c^\perp}Q_c \notin L^2(\mathbb{R}^2)$. For $\partial_{c^\perp}Q_c$, this can be shown with Lemma 2.7 and [**10**].

Proof: We have defined

$$\|\varphi\|^2_{H_{Q_c}} = \int_{\mathbb{R}^2} |\nabla\varphi|^2 + |1-|Q_c|^2||\varphi|^2 + \mathfrak{Re}^2(\overline{Q_c}\varphi).$$

For any of the four functions, since they are in $C^\infty(\mathbb{R}^2,\mathbb{C})$, the only possible problem for the integrability is at infinity.

Step 1. We have $\partial_{x_1}Q_c, \partial_{x_2}Q_c \in H_{Q_c}$.

From Lemma 2.1 and equation (2.11) (for $1 > \sigma' > \sigma = 3/4$), we have

$$\int_{\mathbb{R}^2} |\nabla\partial_{x_1}Q_c|^2 + \int_{\mathbb{R}^2} |\nabla\partial_{x_2}Q_c|^2 \leqslant \int_{\mathbb{R}^2} \frac{K(c,\sigma')}{(1+r)^{7/2}} < +\infty.$$

From Theorem 2.5, we have

$$\int_{\mathbb{R}^2} |1-|Q_c|^2||\nabla Q_c|^2 + \mathfrak{Re}^2(\overline{Q_c}\nabla Q_c) \leqslant \int_{\mathbb{R}^2} \frac{K(c)}{(1+r)^4} < +\infty,$$

hence $\partial_{x_1}Q_c, \partial_{x_2}Q_c \in H_{Q_c}$.

Step 2. We have $\partial_c Q_c \in H_{Q_c}$.

From Lemmas 2.3 and 2.4, we have that for $\sigma > 0$ small enough

$$\partial_c Q_c + \frac{1+o^{\sigma}_{c\to 0}(c^\sigma)}{c^2}\partial_d V_{|d=d_c} \in H_{Q_c},$$

therefore we just have to check that $\partial_d V_{|d=d_c} \in H_{Q_c}$, which is a direct consequence of Lemma 2.6 of [**4**].

Step 3. We have $\partial_{c^\perp} Q_c \in H_{Q_c}$.

From Lemma 2.7, we have $\partial_{c^\perp} Q_c = -x^\perp \cdot \nabla Q_c$. With Theorem 2.5, Lemma 2.1, and equation (2.11), we check that

$$\int_{\mathbb{R}^2} |\nabla \partial_{c^\perp} Q_c|^2 + |(1-|Q_c|^2)||\partial_{c^\perp} Q_c|^2 < +\infty.$$

Now, from Lemma 2.1 and equation (2.6) (with $\sigma = 1/2$), we have

$$\int_{\mathbb{R}^2} \mathfrak{Re}^2(\overline{Q_c}\partial_{c^\perp} Q_c) \leqslant K \int_{\mathbb{R}^2} (1+r^2)\mathfrak{Re}^2(\nabla Q_c \overline{Q_c}) \leqslant K(c) \int_{\mathbb{R}^2} \frac{1}{(1+r)^3} < +\infty,$$

thus $\partial_{c^\perp} Q_c \in H_{Q_c}$.

Step 4. Computation of the linearized operator on $\partial_{x_1} Q_c$, $\partial_{x_2} Q_c$, $\partial_c Q_c$, $\partial_{c^\perp} Q_c$.

For the values in the linearized operator, since

$$-ic\partial_{x_2} Q_c - \Delta Q_c - (1-|Q_c|^2)Q_c = (\mathrm{TW}_c)(Q_c) = 0,$$

by differentiating it with respect to x_1 and x_2, we have

$$L_{Q_c}(\partial_{x_1} Q_c) = L_{Q_c}(\partial_{x_2} Q_c) = 0.$$

By differentiating it with respect to c, we have (we recall that $\partial_c Q_c \in C^\infty(\mathbb{R}^2, \mathbb{C})$)

$$-i\partial_{x_2} Q_c + L_{Q_c}(\partial_c Q_c) = 0.$$

Finally, the quantity $L_{Q_c}(\partial_{c^\perp} Q_c)$ is given by Lemma 2.7. $\square$

The next two lemmas are additional estimates on the four directions that will be useful later on. They estimate in particular the dependence on c of $\|\cdot\|_{\mathcal{C}}$ on these four directions.

Lemma 2.9. *There exists $K > 0$ a universal constant, independent of c, such that, for Q_c defined in Theorem 1.1,*

$$\|\partial_{x_1} Q_c\|_{\mathcal{C}} + \|\partial_{x_2} Q_c\|_{\mathcal{C}} + \|c^2 \partial_c Q_c\|_{\mathcal{C}} \leqslant K.$$

Furthermore, for any $1 > \beta > 0$,

$$\|c\partial_{c^\perp} Q_c\|_{\mathcal{C}} = o^{\beta}_{c\to 0}(c^{-\beta}).$$

Proof: We have defined, for $\varphi = Q_c\psi \in H_{Q_c}$,

$$\|\varphi\|^2_{\mathcal{C}} = \int_{\mathbb{R}^2} |\nabla\psi|^2|Q_c|^4 + \mathfrak{Re}^2(\psi)|Q_c|^4.$$

We recall that, since $\varphi = Q_c\psi$,

$$\int_{\mathbb{R}^2} |\nabla\psi|^2|Q_c|^4 = \int_{\mathbb{R}^2} |\nabla\varphi - \nabla Q_c\psi|^2|Q_c|^2 \leqslant K \int_{\mathbb{R}^2} |\nabla\varphi|^2|Q_c|^2 + |\nabla Q_c|^2|\varphi|^2. \tag{2.14}$$

Step 1. We have $\|\partial_{x_1} Q_c\|_{\mathcal{C}} + \|\partial_{x_2} Q_c\|_{\mathcal{C}} \leqslant K$.

From Lemmas 2.1 and 2.2 and equations (2.9) to (2.11), we have that, for $\tilde{r} = \min(r_1, r_{-1})$,

$$|\nabla Q_c| \leqslant \frac{K}{(1+\tilde{r})} \quad \text{and} \quad |\nabla^2 Q_c| \leqslant \frac{K}{(1+\tilde{r})^2}.$$

Therefore,

$$\int_{\mathbb{R}^2} |\nabla(\partial_{x_1} Q_c)|^2 |Q_c|^2 + |\nabla(\partial_{x_2} Q_c)|^2 |Q_c|^2 \leqslant K,$$

and we also have

$$\int_{\mathbb{R}^2} |\nabla Q_c|^2 |\nabla Q_c|^2 \leqslant K,$$

thus, with equation (2.14),

$$\int_{\mathbb{R}^2} \left|\nabla\left(\frac{\partial_{x_1} Q_c}{Q_c}\right)\right|^2 |Q_c|^4 + \int_{\mathbb{R}^2} \left|\nabla\left(\frac{\partial_{x_2} Q_c}{Q_c}\right)\right|^2 |Q_c|^4 \leqslant K.$$

By equation (2.9) (for $\sigma = 1/4$), we have

$$\int_{\mathbb{R}^2} \mathfrak{Re}^2\left(\frac{\nabla Q_c}{Q_c}\right)|Q_c|^4 \leqslant K \int_{\mathbb{R}^2} \mathfrak{Re}^2(\nabla Q_c \overline{Q_c}) \leqslant K \int_{\mathbb{R}^2} \frac{1}{(1+\tilde{r})^{5/2}} \leqslant K.$$

We conclude that $\|\partial_{x_1} Q_c\|_{\mathcal{C}} + \|\partial_{x_2} Q_c\|_{\mathcal{C}} \leqslant K$.

Step 2. We have $\|c^2 \partial_c Q_c\|_{\mathcal{C}} \leqslant K$.

From Lemma 2.3, we have, writing $c^2\partial_c Q_c = (1+o_{c\to 0}(1))\partial_d V_{|d=d_c} + h$, that $\left\|\frac{h}{V}\right\|_{\sigma,d_c} = o_{c\to 0}(1)$. In particular, if we show that $\|\partial_d V_{|d=d_c}\|_{\mathcal{C}} \leqslant K$ and $\|h\|_{\mathcal{C}} \leqslant K$, then $\|c^2\partial_c Q_c\|_{\mathcal{C}} \leqslant K$. With Lemma 2.6 of [**4**], we check directly that

$$\int_{\mathbb{R}^2} |\nabla \partial_d V_{|d=d_c}|^2 + \frac{|\partial_d V_{|d=d_c}|^2}{(1+\tilde{r})^{3/2}} + \mathfrak{Re}^2(V\partial_d V_{|d=d_c}) \leqslant K.$$

In particular, with (2.14), this implies that

$$\int_{\mathbb{R}^2} \left|\nabla\left(\frac{\partial_d V_{|d=d_c}}{Q_c}\right)\right|^2 |Q_c|^4 \leqslant K$$

and we estimate

$$\int_{\mathbb{R}^2} \mathfrak{Re}^2\left(\frac{\partial_d V_{|d=d_c}}{Q_c}\right)|Q_c|^4 \leqslant K \int_{\mathbb{R}^2} \mathfrak{Re}^2(\bar{V}\partial_d V_{|d=d_c}) + |V - Q_c|^2 |\partial_d V_{|d=d_c}|^2 \leqslant K$$

with the same arguments and equation (2.7). Similarly,

$$\int_{\mathbb{R}^2} \left|\nabla \frac{\partial_d V_{|d=d_c}}{Q_c}\right|^2 |Q_c|^4 \leqslant 2\int_{\mathbb{R}^2} |\nabla \partial_d V_{|d=d_c}|^2 |Q_c|^2 + |\nabla Q_c \partial_d V_{|d=d_c}|^2 \leqslant K,$$

therefore $\|\partial_d V_{|d=d_c}\|_{\mathcal{C}} \leqslant K$. We now have to estimate $\|h\|_{\mathcal{C}}$. The computations are similar, since we check easily that

$$\int_{\mathbb{R}^2} |\nabla h|^2 + |\nabla Q_c|^2 |h|^2 \leqslant K \left\| \frac{h}{V} \right\|^2_{3/4,d_c}$$

and

$$\int_{\mathbb{R}^2} \mathfrak{Re}^2(\bar{Q}_c h) \leqslant K \int_{\mathbb{R}^2} \mathfrak{Re}^2(\bar{V} h) + |V - Q_c|^2 |h|^2 \leqslant K \left\| \frac{h}{V} \right\|^2_{3/4,d_c}.$$

Step 3. We have $\|c\partial_{c^\perp} Q_c\|_{\mathcal{C}} = o^\beta_{c\to 0}(c^{-\beta})$.

By definition, $c\partial_{c^\perp} Q_c = -cx^\perp \cdot \nabla Q_c(x)$, and we check by triangular inequality that $c|x^\perp| \leqslant K(1+\tilde{r})$ since $\tilde{r} = \min(|x - \widetilde{d_c}\overrightarrow{e_1}|, |x + \widetilde{d_c}\overrightarrow{e_1}|)$ and $c\tilde{d_c} \to 1$. Therefore,

$$\begin{aligned}\int_{\mathbb{R}^2} |\nabla(c\partial_{c^\perp} Q_c)|^2 &\leqslant c^2 \int_{\mathbb{R}^2} |\nabla Q_c|^2 + \int_{\mathbb{R}^2} (c|x^\perp|)^2 |\nabla^2 Q_c|^2 \\ &\leqslant K\left(1 + \int_{\mathbb{R}^2} |\nabla^2 Q_c|^2 (1+\tilde{r})^2\right).\end{aligned}$$

We have $|\nabla^2 Q_c| \leqslant |\nabla^2 V| + |\nabla^2 \Gamma_c|$, and with equation (2.11), we check that $\int_{\mathbb{R}^2} |\nabla^2 \Gamma_c|^2 (1+\tilde{r})^2 \leqslant K$. With computations similar to the ones of Lemma 2.3 of [**4**] and Lemma 2.1, we can show that

$$|\nabla^2 V| \leqslant \frac{K}{(1+\tilde{r})^2} \quad \text{and} \quad |\nabla^2 V| \leqslant \frac{K}{c(1+\tilde{r})^3},$$

therefore, for any $1 > \beta > 0$,

$$|\nabla^2 V| \leqslant \frac{Kc^{-\beta}}{(1+\tilde{r})^{2+\beta}},$$

and thus, by (2.14),

$$\int_{\mathbb{R}^2} \left|\nabla\left(\frac{c\partial_{c^\perp} Q_c}{Q_c}\right)\right|^2 |Q_c|^4 \leqslant K \int_{\mathbb{R}^2} |\nabla c\partial_{c^\perp} Q_c|^2 |Q_c|^2 + |\nabla Q_c|^2 |c\partial_{c^\perp} Q_c|^2 \leqslant K(\beta) c^{-2\beta}.$$

Furthermore, by equations (2.9) (for $\sigma = 1/2$) and (2.12), we have

$$\int_{\mathbb{R}^2} \mathfrak{Re}^2\left(\frac{cx^\perp \cdot \nabla Q_c(x)}{Q_c}\right) |Q_c|^4 \leqslant K \int_{\mathbb{R}^2} (1+\tilde{r})^2 \mathfrak{Re}^2(\nabla Q_c \overline{Q_c}) \leqslant K \int_{\mathbb{R}^2} \frac{1}{(1+\tilde{r})^3} \leqslant K.$$

We conclude that $\|c\partial_{c^\perp} Q_c\|_{\mathcal{C}} = o^\beta_{c\to 0}(c^{-\beta})$. □

2.2.3. Link with the energy and momentum and computations of equivalents. In this subsection, we compute the value of the four previous particular directions $\partial_{x_1}Q_c$, $\partial_{x_2}Q_c$, $\partial_c Q_c$, $\partial_{c^\perp}Q_c$ on the quadratic form. In particular, we shall show that one of them is negative.

Lemma 2.10. *There exists* $c_0 > 0$ *such that for* $0 < c < c_0$*, and for* Q_c *defined in Theorem 1.1, for* $A \in \{\partial_{x_1}Q_c, \partial_{x_2}Q_c, \partial_c Q_c, \partial_{c^\perp}Q_c\}$, $\mathfrak{Re}(L_{Q_c}(A)\bar{A}) \in L^1(\mathbb{R}^2)$*, and*

$$\langle L_{Q_c}(\partial_{x_1}Q_c), \partial_{x_1}Q_c\rangle = \langle L_{Q_c}(\partial_{x_2}Q_c), \partial_{x_2}Q_c\rangle = 0,$$
$$\langle L_{Q_c}(\partial_c Q_c), \partial_c Q_c\rangle = \frac{-2\pi + o_{c\to 0}(1)}{c^2},$$
$$\langle L_{Q_c}(\partial_{c^\perp} Q_c), \partial_{c^\perp} Q_c\rangle = 2\pi + o_{c\to 0}(1).$$

Proof: For $A \in \{\partial_{x_1}Q_c, \partial_{x_2}Q_c, \partial_c Q_c, \partial_{c^\perp}Q_c\}$, we recall from Lemma 2.8 that $A \in H_{Q_c}$. To show that $\mathfrak{Re}(L_{Q_c}(A)\bar{A}) \in L^1(\mathbb{R}^2)$, we need to show that

$$-\mathfrak{Re}(\Delta A\bar{A}) - \mathfrak{Re}(ic\partial_{x_2}A\bar{A}) - (1-|Q_c|^2)|A|^2 + 2\mathfrak{Re}^2(\overline{Q_c}A) \in L^1(\mathbb{R}^2).$$

For that, we check that, for some $\sigma > 1/2$,

$$\begin{aligned}&\|(1+r)^\sigma A\|_{L^\infty(\mathbb{R}^2)} + \|(1+r)^{1+\sigma}(|\nabla A| + |\mathfrak{Re}(A)|)\|_{L^\infty(\mathbb{R}^2)}\\ &\quad + \|(1+r)^{2+\sigma}\mathfrak{Im}(\Delta A)\|_{L^\infty(\mathbb{R}^2)} + \|(1+r)^{1+\sigma}\mathfrak{Re}(\Delta A)\|_{L^\infty(\mathbb{R}^2)} < +\infty.\end{aligned} \tag{2.15}$$

For $\partial_{x_1}Q_c$ and $\partial_{x_2}Q_c$, this follows from Theorem 2.5, and also, since $L_{Q_c}(\partial_{x_{1,2}}Q_c) = 0$, from

$$\Delta(\partial_{x_{1,2}}Q_c) = -ic\partial^2_{x_2 x_{1,2}}Q_c - (1-|Q_c|^2)\partial_{x_{1,2}}Q_c + 2\mathfrak{Re}(\overline{Q_c}\partial_{x_{1,2}}Q_c)Q_c,$$

which allows us to estimate $\Delta(\partial_{x_{1,2}}Q_c)$ with Theorem 2.5, Lemma 2.1, and equation (2.11) for any $\sigma > 1/2$.

Now, for $\partial_c Q_c$, the estimates not on its Laplacian are a consequence of Lemma 2.3, Theorem 2.5, and Lemma 2.6 of [**4**]. Then, from Lemma 2.8, we have $L_{Q_c}(\partial_c Q_c) = i\partial_{x_2}Q_c$, thus

$$\Delta(\partial_c Q_c) = -i\partial_{x_2}Q_c - ic\partial_{x_2}\partial_c Q_c - (1-|Q_c|^2)\partial_c Q_c + 2\mathfrak{Re}(\overline{Q_c}\partial_c Q_c)Q_c.$$

By Theorem 2.5 and Lemma 2.3, we have, for any $\sigma > 1/2$,

$$|(1-|Q_c|^2)\partial_c Q_c| + |2\mathfrak{Re}(\overline{Q_c}\partial_c Q_c)Q_c| \leqslant \frac{K(c,\sigma)}{(1+r)^{2+\sigma}},$$
$$|\partial_{x_2}Q_c| + |\partial_{x_2}\partial_c Q_c| \leqslant \frac{K(c,\sigma)}{(1+r)^{1+\sigma}},$$

and

$$|\mathfrak{Re}(\partial_{x_2}Q_c)| + |\mathfrak{Re}(\partial_{x_2}\partial_c Q_c)| \leqslant \frac{K(c,\sigma)}{(1+r)^{2+\sigma}},$$

which is enough to show the estimates for $\partial_c Q_c$.

Finally, from Lemma 2.7 we recall that

$$\partial_{c^\perp} Q_c = -x^\perp \cdot \nabla Q_c(x)$$

and

$$L_{Q_c}(\partial_{c^\perp} Q_c) = -ic\partial_{x_1} Q_c.$$

Similarly, the estimates not on its Laplacian follow from Theorem 2.5, Lemmas 2.1 and 2.2, and equation (2.11). We also have

$$\Delta(\partial_{c^\perp} Q_c) = ic\partial_{x_1} Q_c - ic\partial_{x_2}\partial_{c^\perp} Q_c - (1-|Q_c|^2)\partial_{c^\perp} Q_c + 2\mathfrak{Re}(\overline{Q_c}\partial_{c^\perp} Q_c)Q_c,$$

and with the same previous estimates, we conclude that $\partial_{c^\perp} Q_c$ satisfies the required estimates. With the definition $\|\cdot\|_{H_{Q_c}}$, we check that the last two terms are in $L^1(\mathbb{R}^2)$, and for the first two, the integrands are in $L^1(\mathbb{R}^2, \mathbb{R})$ by estimates in Subsection 2.1.1 and (2.15).

Step 1. We have $\langle L_{Q_c}(\partial_{x_1} Q_c), \partial_{x_1} Q_c\rangle = \langle L_{Q_c}(\partial_{x_2} Q_c), \partial_{x_2} Q_c\rangle = 0$.

From Lemma 2.8, we have $L_{Q_c}(\partial_{x_1} Q_c) = L_{Q_c}(\partial_{x_2} Q_c) = 0$, hence

$$\langle L_{Q_c}(\partial_{x_1} Q_c), \partial_{x_1} Q_c\rangle = \langle L_{Q_c}(\partial_{x_2} Q_c), \partial_{x_2} Q_c\rangle = 0.$$

Step 2. We have $\langle L_{Q_c}(\partial_c Q_c), \partial_c Q_c\rangle = \frac{-2\pi + o_{c\to 0}(1)}{c^2}$.

From Lemma 2.8, we have

$$L_{Q_c}(\partial_c Q_c) = i\partial_{x_2} Q_c,$$

therefore

$$\langle L_{Q_c}(\partial_c Q_c), \partial_c Q_c\rangle = \langle i\partial_{x_2} Q_c, \partial_c Q_c\rangle. \tag{2.16}$$

From Lemma 2.3, we can write $\partial_c Q_c = -\big(\frac{1+o_{c\to 0}(1)}{c^2}\big)\partial_d V_{|d=d_c} + h$ with $\big\|\frac{h}{V}\big\|_{\sigma,d_c} = o_{c\to 0}\big(\frac{1}{c^2}\big)$. Similarly, from Lemma 2.2, we write $Q_c = V + \Gamma_c$ with $\big\|\frac{\Gamma_c}{V}\big\|_{\sigma,d_c} = o_{c\to 0}(1)$, and we compute

$$\begin{aligned}\langle L_{Q_c}(\partial_c Q_c), \partial_c Q_c\rangle = &\left\langle i\partial_{x_2} V, -\left(\frac{1+o_{c\to 0}(1)}{c^2}\right)\partial_d V_{|d=d_c}\right\rangle + \langle i\partial_{x_2} V, h\rangle\\ &+\left\langle i\partial_{x_2}\Gamma_c, -\left(\frac{1+o_{c\to 0}(1)}{c^2}\right)\partial_d V_{|d=d_c}\right\rangle + \langle i\partial_{x_2}\Gamma_c, h\rangle.\end{aligned} \tag{2.17}$$

By symmetry in x_1 of V, we compute

$$\langle i\partial_{x_2} V, \partial_d V_{|d=d_c}\rangle = -2\langle i\partial_{x_2} V_1 V_{-1}, \partial_{x_1} V_1 V_{-1}\rangle + 2\langle i\partial_{x_2} V_1 V_{-1}, \partial_{x_1} V_{-1} V_1\rangle.$$

In equation (2.25) of [**4**], we computed

$$\langle i\partial_{x_2} V_1 V_{-1}, \partial_{x_1} V_1 V_{-1}\rangle = -\pi + o_{c\to 0}(1).$$

Furthermore,

$$|\langle i\partial_{x_2}V_1V_{-1},\partial_{x_1}V_{-1}V_1\rangle| = \left|\int_{\mathbb{R}^2}\mathfrak{Re}(i\partial_{x_2}V_1\overline{V_1\partial_{x_1}V_{-1}}V_{-1})\right|$$
$$\leqslant \left|\int_{\mathbb{R}^2}\mathfrak{Re}(\partial_{x_2}V_1\overline{V_1})\mathfrak{Im}(\overline{\partial_{x_1}V_{-1}}V_{-1})\right| + \left|\int_{\mathbb{R}^2}\mathfrak{Im}(\partial_{x_2}V_1\overline{V_1})\mathfrak{Re}(\overline{\partial_{x_1}V_{-1}}V_{-1})\right|.$$

From Lemma 2.1, we have the estimates

$$|\mathfrak{Re}(\partial_{x_2}V_{-1}\overline{V_{-1}})| \leqslant \frac{K}{(1+r_{-1})^3} \quad \text{and} \quad |\mathfrak{Re}(\overline{\partial_{x_1}V_1}V_1)| \leqslant \frac{K}{(1+r_1)^3},$$

as well as

$$|\mathfrak{Im}(\partial_{x_2}V_{-1}\overline{V_{-1}})| \leqslant \frac{K}{1+r_{-1}} \quad \text{and} \quad |\mathfrak{Im}(\overline{\partial_{x_1}V_1}V_1)| \leqslant \frac{K}{1+r_1}.$$

We deduce, in the right half-plane, where $r_{-1}\geqslant d_c$, that $|\mathfrak{Im}(\nabla V_{-1}\overline{V_{-1}})| = o_{c\to 0}(1)$ and thus

$$\left|\int_{\{x_1\geqslant 0\}}\mathfrak{Re}(\partial_{x_2}V_1\overline{V_1})\mathfrak{Im}(\overline{\partial_{x_1}V_{-1}}V_{-1})\right| \leqslant o_{c\to 0}(1)\int_{\{x_1\geqslant 0\}}\frac{1}{(1+r_1)^3} = o_{c\to 0}(1).$$

In the left half-plane, we have $\frac{1}{1+r_1} \leqslant \frac{K}{1+r_{-1}}$ and $\frac{1}{1+r_1} = o_{c\to 0}(1)$, therefore

$$\left|\int_{\{x_1\leqslant 0\}}\mathfrak{Re}(\partial_{x_2}V_1\overline{V_1})\mathfrak{Im}(\overline{\partial_{x_1}V_{-1}}V_{-1})\right| \leqslant o_{c\to 0}(1)\int_{\{x_1\leqslant 0\}}\frac{1}{(1+r_{-1})^3} = o_{c\to 0}(1).$$

We therefore have

$$\left|\int_{\mathbb{R}^2}\mathfrak{Re}(\partial_{x_2}V_1\overline{V_1})\mathfrak{Im}(\overline{\partial_{x_1}V_{-1}}V_{-1})\right| = o_{c\to 0}(1),$$

and by similar estimates,

$$\left|\int_{\mathbb{R}^2}\mathfrak{Im}(\partial_{x_2}V_1\overline{V_1})\mathfrak{Re}(\overline{\partial_{x_1}V_{-1}}V_{-1})\right| = o_{c\to 0}(1).$$

We can thus conclude that $\langle i\partial_{x_2}V_1V_{-1},\partial_{x_1}V_{-1}V_1\rangle = o_{c\to 0}(1)$. Therefore,

$$\left(\frac{1+o_{c\to 0}(1)}{c^2}\right)\langle i\partial_{x_2}V, -\partial_d V_{|d=d_c}\rangle = \frac{-2\pi}{c^2} + o\left(\frac{1}{c^2}\right). \tag{2.18}$$

Now, we estimate

$$|\langle i\partial_{x_2}V,h\rangle| = \left|\int_{\mathbb{R}^2}\mathfrak{Re}(i\partial_{x_2}V\bar{h})\right| \leqslant o_{c\to 0}(1) + \left|\int_{\{\tilde{r}\geqslant 1\}}\mathfrak{Re}(i\partial_{x_2}V\bar{h})\right|$$
$$\leqslant o_{c\to 0}(1) + \left|\int_{\{\tilde{r}\geqslant 1\}}\mathfrak{Re}\left(i\partial_{x_2}V\bar{V}\overline{\left(\frac{h}{V}\right)}\right)\right|$$

because $\|h\|_{L^\infty} = o_{c\to 0}(1)$ and $|\partial_{x_2} V|$ is bounded near $\tilde{d}_c$ by a universal constant. Furthermore,

$$\left|\int_{\{\tilde{r}\geqslant 1\}} \mathfrak{Re}\left(i\partial_{x_2} V\bar{V}\overline{\left(\frac{h}{V}\right)}\right)\right| \leqslant \left|\int_{\{\tilde{r}\geqslant 1\}} \mathfrak{Re}(\partial_{x_2} V\bar{V})\mathfrak{Im}\left(\frac{h}{V}\right)\right| + \left|\int_{\{\tilde{r}\geqslant 1\}} \mathfrak{Im}(\partial_{x_2} V\bar{V})\mathfrak{Re}\left(\frac{h}{V}\right)\right|.$$

From Lemmas 2.1 and 2.3 (taking $\sigma = 1/2$), we have

$$\left|\int_{\{\tilde{r}\geqslant 1\}} \mathfrak{Re}(\partial_{x_2} V\bar{V})\mathfrak{Im}\left(\frac{h}{V}\right)\right| \leqslant K \left\|\frac{h}{V}\right\|_{1/2,d_c} \int_{\{\tilde{r}\geqslant 1\}} \frac{1}{(1+\tilde{r})^{3+1/2}} = o_{c\to 0}\left(\frac{1}{c^2}\right)$$

and

$$\left|\int_{\{\tilde{r}\geqslant 1\}} \mathfrak{Im}(\partial_{x_2} V\bar{V})\mathfrak{Re}\left(\frac{h}{V}\right)\right| \leqslant K \left\|\frac{h}{V}\right\|_{1/2,d_c} \int_{\{\tilde{r}\geqslant 1\}} \frac{1}{(1+\tilde{r})^{2+1/2}} = o_{c\to 0}\left(\frac{1}{c^2}\right),$$

therefore

$$|\langle i\partial_{x_2} V, h\rangle| = o_{c\to 0}\left(\frac{1}{c^2}\right).$$

Now, by Lemmas 2.1 and 2.2 (taking $\sigma = 1/2$), we have

$$\left(\frac{1+o_{c\to 0}(1)}{c^2}\right)|\langle i\partial_{x_2}\Gamma_c, \partial_d V_{|d=d_c}\rangle| \leqslant \frac{K}{c^2}\left\|\frac{\Gamma_c}{V}\right\|_{1/2,d_c} \int_{\mathbb{R}^2} \frac{1}{(1+\tilde{r})^{2+1/2}} = o_{c\to 0}\left(\frac{1}{c^2}\right).$$

Finally, by Lemmas 2.2 and 2.3, we check easily that

$$|\langle i\partial_{x_2}\Gamma_c, h\rangle| \leqslant K \left\|\frac{\Gamma_c}{V}\right\|_{3/4,d_c} \left\|\frac{h}{V}\right\|_{1/2,d_c} \int_{\mathbb{R}^2} \frac{1}{(1+\tilde{r})^{2+1/4}} = o_{c\to 0}\left(\frac{1}{c^2}\right). \tag{2.19}$$

Combining (2.18) to (2.19) with (2.17), we conclude that

$$\langle L_{Q_c}(\partial_c Q_c), \partial_c Q_c\rangle = \frac{-2\pi + o_{c\to 0}(1)}{c^2}.$$

Step 3. We have $\langle L_{Q_c}(\partial_{c^\perp} Q_c), \partial_{c^\perp} Q_c\rangle = 2\pi + o_{c\to 0}(1)$.

From Lemma 2.8, we have $L_{Q_c}(\partial_{c^\perp} Q_c) = -ic\partial_{x_1} Q_c$ and from Lemma 2.7, we have $\partial_{c^\perp} Q_c = -x^\perp \cdot \nabla Q_c$. Therefore,

$$\langle L_{Q_c}(\partial_{c^\perp} Q_c), \partial_{c^\perp} Q_c\rangle = c\langle i\partial_{x_1} Q_c, x^\perp \cdot \nabla Q_c\rangle.$$

We have

$$\langle i\partial_{x_1} Q_c, -x_2\partial_{x_1} Q_c\rangle = -\int_{\mathbb{R}^2} \mathfrak{Re}(ix_2|\partial_{x_1} Q_c|^2) = 0,$$

hence

$$\langle L_{Q_c}(\partial_{c^\perp} Q_c), \partial_{c^\perp} Q_c\rangle = c\langle i\partial_{x_1} Q_c, x_1\partial_{x_2} Q_c\rangle. \tag{2.20}$$

From Lemma 2.2, we write $Q_c = V + \Gamma_c$ with $\left\| \frac{\Gamma_c}{V} \right\|_{\sigma, d_c} \leqslant K(\sigma) c^{1-\sigma}$ for any $0 < \sigma < 1$, and we compute

$$\langle i\partial_{x_1} Q_c, x_1 \partial_{x_2} Q_c \rangle = \langle i\partial_{x_1} V, x_1 \partial_{x_2} V \rangle + \langle i\partial_{x_1} V, x_1 \partial_{x_2} \Gamma_c \rangle + \langle i\partial_{x_1} \Gamma_c, x_1 \partial_{x_2} V \rangle + \langle i\partial_{x_1} \Gamma_c, x_1 \partial_{x_2} \Gamma_c \rangle.$$

We write $x_1 = d_c + y_1$, therefore

$$\langle i\partial_{x_1} V, x_1 \partial_{x_2} V \rangle = d_c \langle i\partial_{x_1} V, \partial_{x_2} V \rangle + \langle i\partial_{x_1} V, y_1 \partial_{x_2} V \rangle.$$

We have

$$\langle i\partial_{x_1} V, \partial_{x_2} V \rangle = \langle i\partial_{x_1} V_1 V_{-1}, \partial_{x_2} V_1 V_{-1} \rangle + \langle i\partial_{x_1} V_{-1} V_1, \partial_{x_2} V_{-1} V_1 \rangle + \langle i\partial_{x_1} V_1 V_{-1}, \partial_{x_2} V_{-1} V_1 \rangle + \langle i\partial_{x_1} V_{-1} V_1, \partial_{x_2} V_1 V_{-1} \rangle,$$

and, from the previous step and by symmetry, we have

$$\langle i\partial_{x_1} V_1 V_{-1}, \partial_{x_2} V_1 V_{-1} \rangle = \langle i\partial_{x_1} V_{-1} V_1, \partial_{x_2} V_{-1} V_1 \rangle = \pi + o_{c \to 0}(1)$$

and

$$|\langle i\partial_{x_1} V_1 V_{-1}, \partial_{x_2} V_{-1} V_1 \rangle| + |\langle i\partial_{x_1} V_{-1} V_1, \partial_{x_2} V_1 V_{-1} \rangle| = o_{c \to 0}(1),$$

thus

$$\langle i\partial_{x_1} V, \partial_{x_2} V \rangle = 2\pi + o_{c \to 0}(1).$$

With $V_{\pm 1}$ centred around $\pm d_c \overrightarrow{e_1}$, we write $V = V_1 V_{-1}$ and we compute

$$\langle i\partial_{x_1} V, y_1 \partial_{x_2} V \rangle = \int_{\mathbb{R}^2} \mathfrak{Re}(iy_1 \partial_{x_1} V_1 \overline{\partial_{x_2} V_1} |V_{-1}|^2 + iy_1 \partial_{x_1} V_{-1} \overline{\partial_{x_2} V_{-1}} |V_1|^2) + \int_{\mathbb{R}^2} \mathfrak{Re}(iy_1 \partial_{x_1} V_1 \overline{V_1 \overline{V_{-1}} \partial_{x_2} V_{-1}} + iy_1 \partial_{x_1} V_{-1} \overline{V_{-1} \overline{V_1} \partial_{x_2} V_1}).$$

By decomposition in polar coordinates, with the notation of (2.13) and Lemma 2.1, we compute

$$\int_{\mathbb{R}^2} \mathfrak{Re}(iy_1 \partial_{x_1} V_1 \overline{\partial_{x_2} V_1} |V_{-1}|^2) = \int_0^{+\infty} \int_0^{2\pi} |V_{-1}|^2 \rho_1(r_1) \rho_1'(r_1) \cos(\theta_1) r_1 \, dr_1 \, d\theta_1.$$

By integration in polar coordinates, we check that

$$\int_0^{+\infty} \int_0^{2\pi} \rho_1(r_1) \rho_1'(r_1) \cos(\theta_1) r_1 \, dr_1 \, d\theta_1 = 0,$$

hence

$$\int_{\mathbb{R}^2} \mathfrak{Re}(iy_1 \partial_{x_1} V_1 \overline{\partial_{x_2} V_1} |V_{-1}|^2) = \int_{\mathbb{R}^2} (1 - |V_{-1}|^2) \mathfrak{Re}(iy_1 \partial_{x_1} V_1 \overline{\partial_{x_2} V_1}).$$

In particular, since, from Lemma 2.1, we have

$$(1-|V_{-1}|^2) \leqslant \frac{K}{(1+r_{-1})^2}$$

and

$$|\rho_1'(r_1)| \leqslant \frac{K}{(1+r_1)^3},$$

we can deduce that

$$\int_{\mathbb{R}^2} \mathfrak{Re}(iy_1\partial_{x_1}V_1\overline{\partial_{x_2}V_1}|V_{-1}|^2) = o_{c\to 0}(1)$$

and, similarly,

$$\int_{\mathbb{R}^2} \mathfrak{Re}(iy_1\partial_{x_1}V_{-1}\overline{\partial_{x_2}V_{-1}}|V_1|^2) = o_{c\to 0}(1).$$

Therefore, we conclude that

$$\langle i\partial_{x_1}V, x_1\partial_{x_2}V\rangle = (2\pi + o_{c\to 0}(1))\tilde{d}_c = \frac{2\pi + o_{c\to 0}(1)}{c}.$$

Now, we want to show that

$$|\langle i\partial_{x_1}V, x_1\partial_{x_2}\Gamma_c\rangle| + |\langle i\partial_{x_1}\Gamma_c, x_1\partial_{x_2}V\rangle| + |\langle i\partial_{x_1}\Gamma_c, x_1\partial_{x_2}\Gamma_c\rangle| = o_{c\to 0}\left(\frac{1}{c}\right),$$

which is enough to end the proof of this step.

By triangular inequality, we have $|x_1| \leqslant \frac{K(1+\tilde{r})}{c}$, and with Lemmas 2.1 and 2.2 (for $\sigma = 1/2$), we estimate

$$\begin{aligned}
|\langle i\partial_{x_1}V, x_1\partial_{x_2}\Gamma_c\rangle| &= \left|\int_{\mathbb{R}^2} x_1\mathfrak{Re}(\partial_{x_1}V\bar{V})\mathfrak{Im}(\overline{\partial_{x_2}\Gamma_c\bar{V}})\right| \\
&\quad + \left|\int_{\mathbb{R}^2} x_1\mathfrak{Im}(\partial_{x_1}V\bar{V})\mathfrak{Re}(\overline{\partial_{x_2}\Gamma_c\bar{V}})\right| \\
&\leqslant \frac{K}{c}\left(\int_{\mathbb{R}^2} \frac{(1+\tilde{r})}{(1+\tilde{r})^3} \times \frac{c^{1/2}}{(1+\tilde{r})^{3/2}} + \frac{(1+\tilde{r})}{(1+\tilde{r})} \times \frac{c^{1/2}}{(1+\tilde{r})^{5/2}}\right) \\
&= o_{c\to 0}\left(\frac{1}{c}\right).
\end{aligned}$$

Similarly, we check with the same computations that $|\langle i\partial_{x_1}\Gamma_c, x_1\partial_{x_2}V\rangle| = o_{c\to 0}\left(\frac{1}{c}\right)$.

Finally, using Lemma 2.2 (for $\sigma = 1/4$), we estimate

$$|\langle i\partial_{x_1}\Gamma_c, x_1\partial_{x_2}\Gamma_c\rangle| \leqslant Kc^{3/2}\|x_1\|_{L^\infty(\{\tilde{r}\leqslant 1\})} + K\left|\int_{\{\tilde{r}\geqslant 1\}} \mathfrak{Re}\left(ix_1\frac{\partial_{x_1}\Gamma_c}{V}\overline{\frac{\partial_{x_2}\Gamma_c}{V}}\right)\right|.$$

We have $\|x_1\|_{L^\infty(\{\tilde r\leqslant 1\})} \leqslant \frac{K}{c}$. Moreover, we infer

$$\left|\int_{\{\tilde r\geqslant 1\}} \mathfrak{Re}\left(ix_1\frac{\partial_{x_1}\Gamma_c}{V}\overline{\frac{\partial_{x_2}\Gamma_c}{V}}\right)\right| \leqslant \int_{\{\tilde r\geqslant 1\}}|x_1|\left|\mathfrak{Re}\left(\frac{\partial_{x_1}\Gamma_c}{V}\right)\mathfrak{Im}\left(\frac{\partial_{x_2}\Gamma_c}{V}\right)\right| + \int_{\{\tilde r\geqslant 1\}}|x_1|\left|\mathfrak{Im}\left(\frac{\partial_{x_1}\Gamma_c}{V}\right)\mathfrak{Re}\left(\frac{\partial_{x_2}\Gamma_c}{V}\right)\right|,$$

and, with Lemma 2.2 (for $\sigma = 1/4$), we have

$$\left|\int_{\{\tilde r\geqslant 1\}} \mathfrak{Re}\left(ix_1\frac{\partial_{x_1}\Gamma_c}{V}\overline{\frac{\partial_{x_2}\Gamma_c}{V}}\right)\right| \leqslant K\int_{\{\tilde r\geqslant 1\}}|x_1|\frac{c^{3/2}}{(1+\tilde r)^{3+1/2}} = o_{c\to 0}(1),$$

since $\frac{|x_1|c}{(1+\tilde r)} \leqslant K$ by triangular inequality. We conclude that

$$\langle i\partial_{x_1}\Gamma_c, x_1\partial_{x_2}\Gamma_c\rangle = o_{c\to 0}(1),$$

which, together with the previous estimates, shows that

$$\langle L_{Q_c}(\partial_{c^\perp}Q_c), \partial_{c^\perp}Q_c\rangle = 2\pi + o_{c\to 0}(1). \qquad \square$$

These quantities are connected to the energy and momentum. This is shown in the next lemma.

Lemma 2.11. *There exists $c_0 > 0$ such that for $0 < c < c_0$, Q_c defined in Theorem 1.1, we have*

$$P_1(Q_c) = \partial_c P_1(Q_c) = 0,$$

$$P_2(Q_c) = \frac{1}{c}B_{Q_c}(\partial_{c^\perp}Q_c) = \frac{2\pi + o_{c\to 0}(1)}{c},$$

and

$$\partial_c P_2(Q_c) = B_{Q_c}(\partial_c Q_c) = \frac{-2\pi + o_{c\to 0}(1)}{c^2}.$$

Furthermore,

$$\partial_c E(Q_c) = c\partial_c P_2(Q_c),$$

and

$$E(Q_c) = (2\pi + o_{c\to 0}(1))\ln\left(\frac{1}{c}\right).$$

Proof: We have

$$P_1(Q_c) = \frac{1}{2}\langle i\partial_{x_1}Q_c, Q_c - 1\rangle,$$

by the symmetries (2.3), $\partial_{x_1}Q_c$ is odd in x_1 and $Q_c - 1$ is even. Therefore,

$$P_1(Q_c) = \partial_c P_1(Q_c) = 0.$$

We have

$$P_2(Q_c) = \frac{1}{2}\langle i\partial_{x_2}Q_c, Q_c - 1\rangle,$$

and from Lemma 2.10 and (2.20), we have

$$2\pi + o_{c\to 0}(1) = B_{Q_c}(\partial_{c^\perp}Q_c) = c\langle i\partial_{x_1}Q_c, x_1\partial_{x_2}Q_c\rangle.$$

By integration by parts (which can be done thanks to Theorem 2.5, Lemma 2.1, and equation (2.11)), we compute

$$\langle i\partial_{x_1} Q_c, x_1 \partial_{x_2} Q_c\rangle = -\langle i(Q_c - 1), \partial_{x_2} Q_c\rangle - \langle i(Q_c - 1), x_1\partial_{x_1x_2} Q_c\rangle,$$

and

$$\langle i(Q_c - 1), x_1\partial_{x_1x_2} Q_c\rangle = -\langle i\partial_{x_2} Q_c, x_1\partial_{x_1} Q_c\rangle = \langle i\partial_{x_1} Q_c, x_1\partial_{x_2} Q_c\rangle.$$

Therefore,

$$P_2(Q_c) = \frac{1}{2}\langle i\partial_{x_1} Q_c, x_1\partial_{x_2} Q_c\rangle = \frac{1}{c} B_{Q_c}(\partial_{c^\perp} Q_c) = \frac{2\pi + o_{c\to 0}(1)}{c}.$$

We have $P_2(Q_c) = \frac{1}{2}\int_{\mathbb{R}^2} \mathfrak{Re}(i\partial_{x_2} Q_c(\overline{Q_c} - 1))$, and we check, with Lemmas 2.2 and 2.3, that

$$|\partial_c\partial_{x_2} Q_c(\overline{Q_c} - 1)| + |\partial_{x_2} Q_c \partial_c \overline{Q_c}| \leqslant \frac{K}{(1+\tilde{r})^{5/2}},$$

and is therefore dominated by an integrable function independent of $c \in]c_1, c_2[$ given that $c_1, c_2 > 0$ are small enough. We deduce that $c \mapsto P_2(Q_c) \in C^1(]0, c_0[, \mathbb{R})$ for some small $c_0 > 0$ and that, by integration by parts,

$$2\partial_c P_2(Q_c) = \langle i\partial_{x_2}\partial_c Q_c, Q_c - 1\rangle + \langle i\partial_{x_2} Q_c, \partial_c Q_c\rangle = 2\langle i\partial_{x_2} Q_c, \partial_c Q_c\rangle,$$

and, from Lemma 2.10 and equation (2.16), we have

$$\langle i\partial_{x_2} Q_c, \partial_c Q_c\rangle = B_{Q_c}(\partial_c Q_c) = \frac{-2\pi + o_{c\to 0}(1)}{c^2},$$

therefore

$$\partial_c P_2(Q_c) = \frac{-2\pi + o_{c\to 0}(1)}{c^2}.$$

We recall that

$$E(Q_c) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla Q_c|^2 + \frac{1}{4}\int_{\mathbb{R}^2} (1 - |Q_c|^2)^2.$$

We check, with Lemmas 2.2 and 2.3, that

$$|\partial_c \nabla Q_c \cdot \overline{\nabla Q_c}| + |\partial_c(|Q_c|^2)(1 - |Q_c|^2)| \leqslant \frac{K}{(1+\tilde{r})^{5/2}}$$

and is therefore dominated by an integrable function independent of $c \in]c_1, c_2[$ given that $c_1, c_2 > 0$ are small enough. We deduce that $c \mapsto E(Q_c) \in C^1(]0, c_0[, \mathbb{R})$ for some small $c_0 > 0$ and that

$$\partial_c \left(\frac{1}{2}\int_{\mathbb{R}^2} |\nabla Q_c|^2\right) = \frac{1}{2}\int_{\mathbb{R}^2} \mathfrak{Re}(\nabla Q_c \overline{\nabla \partial_c Q_c}) + \mathfrak{Re}(\nabla\partial_c Q_c \overline{\nabla Q_c}).$$

We check, with Theorem 2.5 and $(\mathrm{TW}_c)(Q_c) = 0$, that we can do the integration by parts, which yields

$$\partial_c\left(\frac{1}{2}\int_{\mathbb{R}^2}|\nabla Q_c|^2\right) = \langle -\Delta Q_c, \partial_c Q_c\rangle.$$

We check similarly that

$$\partial_c\left(\frac{1}{4}\int_{\mathbb{R}^2}(1-|Q_c|^2)^2\right) = -\int_{\mathbb{R}^2}(1-|Q_c|^2)\mathfrak{Re}(\partial_c Q_c\overline{Q_c}),$$

hence

$$\partial_c\left(\frac{1}{4}\int_{\mathbb{R}^2}(1-|Q_c|^2)^2\right) = \langle -(1-|Q_c|^2)Q_c, \partial_c Q_c\rangle.$$

Now, since $-ic\partial_{x_2}Q_c - \Delta Q_c - (1-|Q_c|^2)Q_c = 0$, we have

$$\partial_c E(Q_c) = \langle -\Delta Q_c - (1-|Q_c|^2)Q_c, \partial_c Q_c\rangle = c\langle -i\partial_{x_2}Q_c, \partial_c Q_c\rangle.$$

Now, since $P_2(Q_c) = \frac{1}{2}\langle i\partial_{x_2}Q_c, Q_c - 1\rangle$, we have

$$\partial_c P_2(Q_c) = \frac{1}{2}(\langle i\partial_{x_2}\partial_c Q_c, Q_c - 1\rangle + \langle i\partial_{x_2}Q_c, \partial_c Q_c\rangle).$$

By integrations by parts, we compute

$$\partial_c P_2(Q_c) = \langle -i\partial_{x_2}Q_c, \partial_c Q_c\rangle.$$

We deduce that $\partial_c E(Q_c) = c\partial_c P_2(Q_c)$, and in particular, we deduce that

$$\partial_c E(Q_c) = \frac{-2\pi + o_{c\to 0}(1)}{c}.$$

By integration (from some fixed $c_0 > c > 0$), we check that $E(Q_c) = (2\pi + o_{c\to 0}(1))\ln\left(\frac{1}{c}\right)$. □

We conclude this subsection with an estimate on Q_c connected to the energy that will be useful later on.

Lemma 2.12. *There exists $K > 0$, a universal constant independent of c, such that, if c is small enough, for Q_c defined in Theorem 1.1,*

$$\int_{\mathbb{R}^2}\frac{|\mathfrak{Im}(\nabla Q_c\overline{Q_c})|^2}{|Q_c|^2} \leqslant K\ln\left(\frac{1}{c}\right).$$

Proof: We recall that $r_{\pm 1} = |x \mp d_c\overrightarrow{e_1}|$. Since ∇Q_c is bounded near the zeros of Q_c (by Lemmas 2.1 and 2.2), and $|Q_c| \geqslant K$ on $\mathbb{R}^2\backslash B(\pm\widetilde{d_c}\overrightarrow{e_1}, 1)$ by (2.12), we have

$$\int_{\mathbb{R}^2}\frac{|\mathfrak{Im}(\nabla Q_c\overline{Q_c})|^2}{|Q_c|^2} \leqslant K\left(1 + \int_{\{\tilde{r}\geqslant 1\}}|\mathfrak{Im}(\nabla Q_c\overline{Q_c})|^2\right).$$

Now, by (2.12), Lemma 2.11, and the definition of the energy,

$$\int_{\{\tilde{r}\geqslant 1\}} |\mathfrak{Im}(\nabla Q_c \overline{Q_c})|^2 \leqslant \int_{\{\tilde{r}\geqslant 1\}} |\nabla Q_c|^2 |Q_c|^2 \leqslant K \int_{\mathbb{R}^2} |\nabla Q_c|^2 \leqslant K E(Q_c) \leqslant K \ln\left(\frac{1}{c}\right).$$

□

We could check that this estimate is optimal with respect to its growth in c when $c \to 0$.

2.3. Zeros of Q_c. In this subsection, we show that Q_c has only two zeros and we compute estimates on Q_c around them. In a bounded domain, a general result about the zeros of solutions to the Ginzburg–Landau problem is already known; see [**18**].

Lemma 2.13. *For $c > 0$ small enough, the function Q_c defined in Theorem 1.1 has exactly two zeros. Their positions are $\pm \widetilde{d}_c \overrightarrow{e_1}$, and, for any $0 < \sigma < 1$,*

$$|d_c - \widetilde{d}_c| = o^{\sigma}_{c\to 0}(c^{1-\sigma}),$$

where d_c is defined in Theorem 1.1.

The notation $o^{\sigma}_{c\to 0}(1)$ denotes a quantity going to 0 when $c \to 0$ at fixed σ. Combining Lemmas 2.10, 2.11, and 2.13, we end the proof of Proposition 1.2.

Proof: From (2.3), we know that Q_c enjoys the symmetry $Q_c(x_1, x_2) = Q_c(-x_1, x_2)$ for $(x_1, x_2) \in \mathbb{R}^2$, hence we look at zeros only in the right half-plane. From Theorem 1.1, we have $Q_c = V_1(\cdot - d_c \overrightarrow{e_1}) V_{-1}(\cdot + d_c \overrightarrow{e_1}) + \Gamma_c$ with $\|\Gamma_c\|_{L^\infty(\mathbb{R}^2)} + \|\nabla \Gamma_c\|_{L^\infty(\mathbb{R}^2)} = o_{c\to 0}(1)$. In the right half-plane and outside of $B(d_c \overrightarrow{e_1}, \Lambda)$ for any $\Lambda > 0$, by Lemma 2.1, we estimate

$$|Q_c| \geqslant |V_1(\cdot - d_c \overrightarrow{e_1}) V_{-1}(\cdot + d_c \overrightarrow{e_1})| - o_{c\to 0}(1) \geqslant K(\Lambda) > 0$$

if c is small enough (depending on Λ). Now, we consider the smooth function $F\colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}$ defined by

$$F(\mu, z) := (V_1(\cdot - d_c \overrightarrow{e_1}) V_{-1}(\cdot + d_c \overrightarrow{e_1}) + \mu \Gamma_c(\cdot))(z + d_c \vec{e}_1).$$

We have $F(0,0) = V_1(0) V_{-1}(2 d_c \overrightarrow{e_1}) = 0$ by Lemma 2.1 and $F(1, z) = Q_c(z + d_c \vec{e}_1)$. For $|\mu| \leqslant 1$ and $|z| \leqslant 1$, since $\|\nabla \Gamma_c\|_{L^\infty(\mathbb{R}^2)} = o^{\sigma}_{c\to 0}(c^{1-\sigma})$ by equation (2.5), with Lemma 2.1 and equation (2.1), we check that

$$|d_z F_{(\mu,z)}(\xi) - \nabla V_1(z) \cdot \xi| = o_{c\to 0}(1) |\xi| \tag{2.21}$$

uniformly in $\mu \in [0, 1]$.

Now, from Lemma 2.1, we estimate (for $x = re^{i\theta} \neq 0 \in \mathbb{R}^2$)

$$\begin{aligned}\partial_{x_1} V_1(x) &= \left(\cos(\theta)\rho'(r) - \frac{i}{r}\sin(\theta)\rho(r)\right)e^{i\theta}\\ &= \kappa(\cos(\theta) - i\sin(\theta))e^{i\theta} + o_{r\to 0}(1)\\ &= \kappa + o_{r\to 0}(1),\end{aligned}$$

and thus, by continuity, $\partial_{x_1} V_1(0) = \kappa > 0$. Similarly, we check that $\partial_{x_2} V_1(0) = -i\kappa$, and therefore,

$$\nabla V_1(z) = \kappa \begin{pmatrix} 1 \\ -i \end{pmatrix} + o_{|z|\to 0}(1).$$

Identifying $\mathbb{C}$ with $\mathbb{R}^2$ canonically, we deduce that the Jacobian determinant of F in z, $J(F)$, satisfies

$$J(F)(\mu, z) = J(V_1)(z) + o_{c\to 0}(1) = -\kappa^2 + o_{c\to 0}(1) + o_{|z|\to 0}(1) \neq 0,$$

given that c and $|z|$ are small enough. By the implicit function theorem, there exists $\mu_0 > 0$ such that, for $|\mu| \leqslant \mu_0$, there exists a unique value $z(\mu)$ in a neighbourhood of 0 such that $F(\mu, z(\mu)) = 0$, and since $\partial_\mu F(\mu, z) = \Gamma_c(d_c\vec{e}_1 + z) = o^{\sigma}_{c\to 0}(c^{1-\sigma})$ uniformly in z (by (2.4)), it satisfies additionally $z(\mu) = o^{\sigma}_{c\to 0}(c^{1-\sigma})$.

Now, let us show that we can take $\mu_0 = 1$. Indeed, if we define $\mu_0 = \sup\{\nu > 0,\ \mu \to z(\mu) \in C^1([0,\nu], \mathbb{R}^2)\} > 0$ and we have $\mu_0 < 1$, since $\mu \to z(\mu) \in C^1([0,\mu_0], \mathbb{R}^2)$ with $|d_\mu z|(\mu) = o^{\sigma}_{c\to 0}(c^{1-\sigma})$ uniformly in $[0, \mu_0]$, it can be continuously extended to μ_0 with $F(\mu_0, z(\mu_0)) = 0$ and $z(\mu_0) = o^{\sigma}_{c\to 0}(c^{1-\sigma})$. Then, by the implicit function theorem at $(\mu_0, z(\mu_0))$ (since $\mu_0 < 1$ with equation (2.21)), it can be extended above μ_0, which is in contradiction with the definition of μ_0.

Since $F(1, \cdot) = Q_c(\cdot + d_c\vec{e}_1)$, we have shown that there exists $z \in \mathbb{R}^2$ with $|z| = o^{\sigma}_{c\to 0}(c^{1-\sigma})$ such that $Q_c(z + d_c\vec{e}_1) = 0$. Now, for c small enough and $|\xi| \leqslant 1$, we have

$$\begin{aligned}\nabla(Q_c(\xi + z + d_c\vec{e}_1)) &= \nabla V_1(z) + o_{c\to 0}(1) + o_{|\xi|\to 0}(1)\\ &= \kappa\begin{pmatrix} 1 \\ -i\end{pmatrix} + o_{c\to 0}(1) + o_{|\xi|\to 0}(1).\end{aligned}$$

We deduce, with $Q_c(\zeta + z + d_c\vec{e}_1) = \int_0^{|\zeta|} \nabla Q_c\big(\xi\frac{\zeta}{|\zeta|} + z + d_c\vec{e}_1\big)\cdot\frac{\zeta}{|\zeta|}\,d\xi$, that

$$\left|Q_c(\zeta + z + d_c\vec{e}_1) - \zeta\cdot\begin{pmatrix} 1 \\ -i\end{pmatrix}\kappa\right| = o_{|\zeta|\to 0}(|\zeta|) + o_{c\to 0}(1)|\zeta|.$$

Therefore, Q_c has no other zeros in $B(z + d_c\vec{e}_1, \Lambda)$ for some $\Lambda > 0$ independent of c. Therefore, since for c small enough, $|Q_c| > K(\Lambda) > 0$ outside of $B(z + d_c\vec{e}_1, \Lambda)$ in the right half-plane, Q_c has only one zero in the right half-plane.

By the symmetry $Q_c(x_1, x_2) = \overline{Q_c(x_1, -x_2)}$ (see (2.3)), z must be colinear to $\overrightarrow{e_1}$, therefore we define $\widetilde{d_c} \in \mathbb{R}$ by $\widetilde{d_c}\overrightarrow{e_1} := z + d_c\vec{e}_1$, and we conclude that, since $|z| = o^{\sigma}_{c\to 0}(c^{1-\sigma})$,

$$|d_c - \widetilde{d_c}| = o^{\sigma}_{c\to 0}(c^{1-\sigma}). \qquad \square$$

We define the vortices around the zeros of Q_c by

$$\tilde{V}_{\pm 1}(x) := V_{\pm 1}(x \mp \widetilde{d_c}\overrightarrow{e_1}),$$

and we will use the already defined polar coordinates around $\pm\widetilde{d_c}\overrightarrow{e_1}$ of $x \in \mathbb{R}^2$, namely

$$\tilde{r}_{\pm 1} = |x \mp \widetilde{d_c}\overrightarrow{e_1}|, \quad \tilde{\theta}_{\pm 1} = \arg(x \mp \widetilde{d_c}\overrightarrow{e_1}).$$

One of the idea of the proof is to understand how Q_c is close, multiplicatively, to vortices $\tilde{V}_{\pm 1}$ centred at its zeros, since by construction it is close to a vortex centred around $\pm d_c\overrightarrow{e_1}$, which is itself close to $\pm\widetilde{d_c}\overrightarrow{e_1}$. In particular, Lemma 2.15 below will show that the ratio $\left|\frac{Q_c}{\tilde{V}_1}\right|$ is bounded and close to 1 near $\widetilde{d_c}\overrightarrow{e_1}$.

In Lemma 2.14 to follow, we compute the additive perturbation between derivatives of Q_c and a vortex $\tilde{V}_{\pm 1}$ centred around one of its zeros. In Lemma 2.15, we compute the multiplicative perturbation. All along, we work in $B(\widetilde{d_c}\overrightarrow{e_1}, \widetilde{d_c}^{1/2})$, the size of the ball $\widetilde{d_c}^{1/2}$ being arbitrary (any quantity that both goes to infinity when $c \to 0$ and is a $o_{c\to 0}(\widetilde{d_c})$ should work). We recall that $\tilde{r}_{\pm 1} = |x \mp \widetilde{d_c}\overrightarrow{e_1}|$.

Lemma 2.14. *Uniformly in $B(\widetilde{d_c}\overrightarrow{e_1}, \widetilde{d_c}^{1/2})$, for Q_c defined in Theorem 1.1, one has*

$$|Q_c - \widetilde{V_1}| = o_{c\to 0}(1),$$

$$|\nabla Q_c - \nabla\widetilde{V_1}| \leqslant \frac{o_{c\to 0}(1)}{1 + \tilde{r}_1},$$

and

$$|\nabla^2 Q_c - \nabla^2\widetilde{V_1}| \leqslant \frac{o_{c\to 0}(1)}{1 + \tilde{r}_1}.$$

See Appendix A.2 for the proof of this result.

Lemma 2.15. *In $B(\widetilde{d_c}\overrightarrow{e_1}, \widetilde{d_c}^{1/2})$, for Q_c defined in Theorem 1.1, we have*

$$\left|\frac{Q_c}{\widetilde{V_1}} - 1\right| = o_{c\to 0}(c^{1/10}).$$

In particular,

$$\left|\frac{Q_c}{\widetilde{V_1}}\right| = 1 + o_{c\to 0}(c^{1/10}).$$

The power $1/10$ is arbitrary, but enough here for the upcoming estimates.

Proof: We recall that both Q_c and $\tilde{V}_1$ are C^∞ since they are solutions of elliptic equations. We have that $Q_c(\widetilde{d_c}\overrightarrow{e_1}) = 0$ by Lemma 2.13, thus, for $x \in \mathbb{R}^2$, by Taylor expansion, for $|x| \leqslant 1$,

$$Q_c(x + \widetilde{d_c}\overrightarrow{e_1}) = x \cdot \nabla Q_c(\widetilde{d_c}\overrightarrow{e_1}) + O_{|x|\to 0}(|x|^2).$$

From Theorem 1.1, we have $Q_c = V_1(\cdot - d_c\overrightarrow{e_1})V_{-1}(\cdot + d_c\overrightarrow{e_1}) + \Gamma_c$, therefore, with $V_{\pm 1}$ being centred around $\pm d_c\overrightarrow{e_1}$ for the rest of the proof,

$$\nabla Q_c(\widetilde{d_c}\overrightarrow{e_1}) = \nabla V_1(\widetilde{d_c}\overrightarrow{e_1})V_{-1}(\widetilde{d_c}\overrightarrow{e_1}) + V_1(\widetilde{d_c}\overrightarrow{e_1})\nabla V_{-1}(\widetilde{d_c}\overrightarrow{e_1}) + \nabla\Gamma_c(\widetilde{d_c}\overrightarrow{e_1}).$$

We have $V_1(\widetilde{d_c}\overrightarrow{e_1})\nabla V_{-1}(\widetilde{d_c}\overrightarrow{e_1}) + \nabla\Gamma_c(\widetilde{d_c}\overrightarrow{e_1}) = o_{c\to 0}(c^{1/2})$ by Theorem 1.1, Lemma 2.1, and (2.2). Furthermore, by (2.1) and Lemmas 2.1 and 2.13,

$$\nabla V_1(\widetilde{d_c}\overrightarrow{e_1})V_{-1}(\widetilde{d_c}\overrightarrow{e_1}) = \nabla V_1(\widetilde{d_c}\overrightarrow{e_1}) + o_{c\to 0}(c^{1/4}).$$

We deduce that

$$Q_c(x + \widetilde{d_c}\overrightarrow{e_1}) = x \cdot (\nabla V_1(d_c\overrightarrow{e_1}) + o_{c\to 0}(c^{1/4})) + O_{x\to 0}(|x|^2).$$

We also have $\widetilde{V_1}(x + \widetilde{d_c}\overrightarrow{e_1}) = x \cdot \nabla\widetilde{V_1}(\widetilde{d_c}\overrightarrow{e_1}) + O_{x\to 0}(|x|^2)$ (since $\widetilde{V_1}(\widetilde{d_c}\overrightarrow{e_1}) = 0$) and $\nabla V_1(d_c\overrightarrow{e_1}) = \nabla\widetilde{V_1}(\widetilde{d_c}\overrightarrow{e_1})$, hence

$$Q_c(x + \widetilde{d_c}\overrightarrow{e_1}) = \widetilde{V_1}(x + \widetilde{d_c}\overrightarrow{e_1}) + x \cdot o_{c\to 0}(c^{1/4}) + O_{|x|\to 0}(|x|^2).$$

Now, by Lemma 2.1, there exists $K > 0$ such that, in $B(\widetilde{d_c}\overrightarrow{e_1}, c^{1/4})$ for c small enough, $|\widetilde{V_1}(x + \widetilde{d_c}\overrightarrow{e_1})| \geqslant K|x|$. We deduce that

$$\begin{aligned}\left|\frac{Q_c}{\widetilde{V_1}} - 1\right| &\leqslant \frac{|x| o_{c\to 0}(c^{1/4})}{|\widetilde{V_1}(x + \widetilde{d_c}\overrightarrow{e_1})|} + \frac{O_{|x|\to 0}(|x|^2)}{|\widetilde{V_1}(x + \widetilde{d_c}\overrightarrow{e_1})|} \\ &\leqslant o_{c\to 0}(c^{1/4}) + O_{|x|\to 0}(|x|) \\ &\leqslant o_{c\to 0}(c^{1/5}).\end{aligned}$$

Outside of $B(\widetilde{d_c}\overrightarrow{e_1}, c^{1/4})$ and in $B(\widetilde{d_c}\overrightarrow{e_1}, \widetilde{d_c}^{1/2})$, we have $|\widetilde{V_1}| \geqslant Kc^{1/4}$ by Lemma 2.1, and

$$Q_c = V_1 + O_{c\to 0}(c^{1/2})$$

by Theorem 1.1 and equations (2.7) and (2.1). We deduce

$$\left|\frac{Q_c}{\widetilde{V_1}} - 1\right|(x) = \left|\frac{V_1 + O_{c\to 0}(c^{1/2})}{\widetilde{V_1}} - 1\right|(x) = \left|\frac{V_1(x)}{\widetilde{V_1}(x)} - 1\right| + o_{c\to 0}(c^{1/10}).$$

Furthermore, by Lemma 2.13 (for $\sigma = 1/2$), we have

$$\begin{aligned}\left|\frac{V_1(x)}{\widetilde{V_1}(x)} - 1\right| &= \left|\frac{\widetilde{V_1}(x) + O_{|d_c - \tilde{d}_c|\to 0}(|d_c - \widetilde{d_c}|)}{\widetilde{V_1}(x)} - 1\right| \\ &= \frac{O_{|d_c - \tilde{d}_c|\to 0}(|d_c - \widetilde{d_c}|)}{c^{1/4}} = o_{c\to 0}(c^{1/10}).\end{aligned}$$

We conclude that $\left|\frac{Q_c}{\tilde{V}_1} - 1\right| = o_{c\to 0}(c^{1/10})$ in $B(\widetilde{d_c}\overrightarrow{e_1}, \widetilde{d_c}^{1/2})$. □

By the symmetries of Q_c (see (2.3)), the result of Lemma 2.15 holds if we replace $\overrightarrow{e_1}$ by $-\overrightarrow{e_1}$ and $\widetilde{V_1}$ by $\tilde{V}_{-1}$.

We conclude this section with the proof that in $B(\pm\tilde{d}_c\overrightarrow{e_1}, \tilde{d}_c^{1/2})$, we have, for $\psi \in C_c^\infty(\mathbb{R}^2 \backslash \{\pm\tilde{d}_c\overrightarrow{e_1}\}, \mathbb{C})$,

$$\int_0^{2\pi} |\psi^{\neq 0}|^2 \, d\tilde{\theta}_{\pm 1} \leqslant \tilde{r}_{\pm 1}^2 \int_0^{2\pi} |\nabla\psi|^2 \, d\tilde{\theta}_{\pm 1}. \tag{2.22}$$

We recall that the function $\psi^{\neq 0}$ is defined by

$$\psi^{\neq 0}(x) = \psi(x) - \psi^{0,1}(\tilde{r}_1)$$

in the right half-plane, and

$$\psi^{\neq 0}(x) = \psi(x) - \psi^{0,-1}(\tilde{r}_{-1})$$

in the left half-plane.

To prove (2.22), it is enough to show that, for $\psi \in C_c^\infty(\mathbb{R}^2 \backslash \{0\}, \mathbb{C})$, we have, with $x = re^{i\theta}$,

$$\int_0^{2\pi} \left| \psi - \int_0^{2\pi} \psi \, d\gamma \right|^2 d\theta \leqslant r^2 \int_0^{2\pi} |\nabla\psi|^2 \, d\theta.$$

This is a Poincaré inequality. By decomposition into harmonics and Parseval's equality, we have

$$\begin{aligned}
\int_0^{2\pi} \left| \psi - \int_0^{2\pi} \psi(\gamma) \, d\gamma \right|^2 d\theta &= \int_0^{2\pi} \left| \sum_{n\in\mathbb{Z}^*} \psi_n(r) e^{in\theta} \right|^2 d\theta \\
&= \int_0^{2\pi} \sum_{n\in\mathbb{Z}^*} |\psi_n(r)|^2 \, d\theta,
\end{aligned}$$

and

$$\begin{aligned}
\int_0^{2\pi} |\nabla\psi|^2 \, d\theta &\geqslant \int_0^{2\pi} \frac{1}{r^2} |\partial_\theta \psi|^2 \, d\theta \\
&\geqslant \int_0^{2\pi} \left| \sum_{n\in\mathbb{Z}^*} i \frac{n\psi_n(r)}{r} e^{in\theta} \right|^2 d\theta \\
&\geqslant \frac{1}{r^2} \int_0^{2\pi} \sum_{n\in\mathbb{Z}^*} n^2 |\psi_n(r)|^2 \, d\theta \\
&\geqslant \frac{1}{r^2} \int_0^{2\pi} \sum_{n\in\mathbb{Z}^*} |\psi_n(r)|^2 \, d\theta.
\end{aligned}$$

This concludes the proof of (2.22). With $|Q_c(x\pm\tilde{d}_c\overrightarrow{e_1})| = O_{\tilde{r}_{\pm1}\to 0}(\tilde{r}_{\pm1})$ and (2.22), we have, for $\tilde{r}_{\pm1} \leqslant R$,

$$
\begin{aligned}
\int_0^{2\pi} |Q_c|^2 |\psi^{\neq 0}|^2 \, d\tilde{\theta}_{\pm1} &\leqslant K \int_0^{2\pi} \tilde{r}_{\pm1}^2 |\psi^{\neq 0}|^2 \, d\tilde{\theta}_{\pm1} \\
&\leqslant K \int_0^{2\pi} \tilde{r}_{\pm1}^4 |\nabla\psi|^2 \, d\tilde{\theta}_{\pm1} \\
&\leqslant K(R) \int_0^{2\pi} |Q_c|^4 |\nabla\psi|^2 \, d\tilde{\theta}_{\pm1}.
\end{aligned}
\tag{2.23}
$$

This result will be useful to estimate the quantities in the orthogonality conditions.

3. Estimates in H_{Q_c}

We give several estimates for functions in H_{Q_c}. They will in particular allow us to use a density argument to show Proposition 1.4 once it is shown for test functions in Section 4. We will also explain why a coercivity result with the energy norm $\|\cdot\|_{H_{Q_c}}$ is impossible with any number of local orthogonality conditions, and show that the quadratic form and the coercivity norm are well defined for functions in H_{Q_c}.

3.1. Comparison of the energy and coercivity norms. In the introduction, we have defined the quadratic form by

$$
\begin{aligned}
B_{Q_c}(\varphi) = &\int_{\mathbb{R}^2} |\nabla\varphi|^2 - (1-|Q_c|^2)|\varphi|^2 + 2\mathfrak{Re}^2(\overline{Q_c}\varphi) \\
&-c\int_{\mathbb{R}^2} (1-\eta)\mathfrak{Re}(i\partial_{x_2}\varphi\bar{\varphi}) - c\int_{\mathbb{R}^2} \eta\mathfrak{Re}(i\partial_{x_2}Q_c\overline{Q_c})|\psi|^2 \\
&+2c\int_{\mathbb{R}^2} \eta\mathfrak{Re}\psi\mathfrak{Im}\partial_{x_2}\psi|Q_c|^2 + c\int_{\mathbb{R}^2} \partial_{x_2}\eta\mathfrak{Re}\psi\mathfrak{Im}\psi|Q_c|^2 \\
&+c\int_{\mathbb{R}^2} \eta\mathfrak{Re}\psi\mathfrak{Im}\psi\partial_{x_2}(|Q_c|^2)
\end{aligned}
$$

(see (1.3)). We will show in Lemma 3.3 below that this quantity is well defined for $\varphi \in H_{Q_c}$. As we have seen, the natural energy space H_{Q_c} is given by the norm

$$
\|\varphi\|_{H_{Q_c}}^2 = \int_{\mathbb{R}^2} |\nabla\varphi|^2 + |1-|Q_c|^2||\varphi|^2 + \mathfrak{Re}^2(\overline{Q_c}\varphi).
$$

We could expect to replace Theorem 1.5 by a result of the form: up to some local orthogonality conditions, for $\varphi \in H_{Q_c}$ we have

$$
B_{Q_c}(\varphi) \geqslant K(c)\|\varphi\|_{H_{Q_c}}^2.
$$

However such a result cannot hold. This is because of a formal zero of L_{Q_c} which is not in the space H_{Q_c}: iQ_c (which comes from the phase invariance of the equation). We have $L_{Q_c}(iQ_c) = 0$ and $iQ_c \notin H_{Q_c}$ because

$$(1 - |Q_c|^2)|iQ_c|^2$$

is not integrable at infinity (see [**10**], where it is shown that this quantity decays like $1/r^2$). We can then create functions in H_{Q_c} getting close to iQ_c, for instance

$$f_R = \eta_R iQ_c,$$

where η_R is a C^∞ real function with value 1 if $R_0 < |x| < R$ and value 0 if $|x| < R_0 - 1$ or $|x| > 2R$. In that case, when $R \to +\infty$, $\|f_R\|_{H_{Q_c}} \to +\infty$ and $B_{Q_c}(f_R) \to C$ a constant independent of R, making the inequality $B_{Q_c}(\varphi) \geqslant K\|\varphi\|^2_{H_{Q_c}}$ impossible (and the local orthogonality conditions are verified for R_0 large enough since $f_R = 0$ on $B(0, R_0 - 1)$). That is why we get the result in a weaker norm in Proposition 1.12: we will only get for $\varphi \in H_{Q_c}$, up to some local orthogonality conditions,

$$B_{Q_c}(\varphi) \geqslant K(c)\|\varphi\|^2_{H^{\exp}_{Q_c}},$$

where $\|\cdot\|_{H^{\exp}_{Q_c}}$ is defined in Subsection 1.3.1. In particular, $\|\cdot\|_{H^{\exp}_{Q_c}}$ is not equivalent to $\|\cdot\|_{H_{Q_c}}$.

3.2. The coercivity norm and other quantities are well defined in $\boldsymbol{H_{Q_c}}$. We have defined the energy space H_{Q_c} by the norm

$$\|\varphi\|^2_{H_{Q_c}} = \int_{\mathbb{R}^2} |\nabla\varphi|^2 + |1 - |Q_c|^2||\varphi|^2 + \mathfrak{Re}^2(\overline{Q_c}\varphi).$$

By Lemma 2.6, we have that, for $\varphi \in H_{Q_c}$,

$$\int_{\mathbb{R}^2} \frac{|\varphi|^2}{(1+|x|)^2}\,dx \leqslant C(c)\|\varphi\|^2_{H_{Q_c}}. \tag{3.1}$$

The goal of this subsection is to show that, for $\varphi \in H_{Q_c}$, $\|\varphi\|_{\mathcal{C}}$, and $B_{Q_c}(\varphi)$, as well as the quantities in the orthogonality conditions of Proposition 1.4 and Theorem 1.5, are well defined. This is done in Lemmas 3.1 to 3.3.

Lemma 3.1. *There exists $c_0 > 0$ such that for $0 < c \leqslant c_0$, there exists $C(c) > 0$ such that, for Q_c defined in Theorem 1.1 and for any $\varphi = Q_c\psi \in H_{Q_c}$,*

$$\|\varphi\|^2_{\mathcal{C}} = \int_{\mathbb{R}^2} |\nabla\psi|^2|Q_c|^4 + \mathfrak{Re}^2(\psi)|Q_c|^4 \leqslant C(c)\|\varphi\|^2_{H_{Q_c}}.$$

Proof: We estimate for $\varphi = Q_c\psi \in H_{Q_c}$, using equations (2.12) and (3.1) and $|\nabla Q_c| \leqslant \frac{C(c)}{(1+r)^2}$ from Theorem 2.5, that

$$\begin{aligned}\int_{\mathbb{R}^2} |\nabla\psi|^2|Q_c|^4 &= \int_{\mathbb{R}^2} |\nabla\varphi - \nabla Q_c\psi|^2|Q_c|^2 \\ &\leqslant K\int_{\mathbb{R}^2} |\nabla\varphi|^2|Q_c|^2 + |\nabla Q_c|^2|Q_c\psi|^2 \\ &\leqslant K(c)\int_{\mathbb{R}^2} |\nabla\varphi|^2 + \frac{|\varphi|^2}{(1+r)^4} \\ &\leqslant K(c)\|\varphi\|^2_{H_{Q_c}}.\end{aligned}$$

Similarly, for $\varphi = Q_c\psi$,

$$\int_{\mathbb{R}^2} \mathfrak{Re}^2(\psi)|Q_c|^4 = \int_{\mathbb{R}^2} \mathfrak{Re}^2(\overline{Q_c}\varphi) \leqslant \|\varphi\|^2_{H_{Q_c}}.$$

We conclude that

$$\int_{\mathbb{R}^2} |\nabla\psi|^2|Q_c|^4 + \mathfrak{Re}^2(\psi)|Q_c|^4 \leqslant C(c)\|\varphi\|^2_{H_{Q_c}}. \tag{3.2}$$

□

We conclude this subsection with the proof that the quantities in the orthogonality conditions are well defined for $\varphi \in H_{Q_c}$.

Lemma 3.2. *There exists $K > 0$ and, for c small enough, there exists $K(c) > 0$ such that, for Q_c defined in Theorem 1.1 and $\varphi = Q_c\psi \in H_{Q_c}$, $0 < R < \tilde{d}_c^{1/2}$, we have*

$$\int_{B(\pm\tilde{d}_c\vec{e_1},R)} |\mathfrak{Re}(\partial_{x_1}\tilde{V}_{\pm 1}\overline{\tilde{V}_{\pm 1}\psi})| + \int_{B(\pm\tilde{d}_c\vec{e_1},R)} |\mathfrak{Re}(\partial_{x_2}\tilde{V}_{\pm 1}\overline{\tilde{V}_{\pm 1}\psi})| \leqslant K(c)\|\varphi\|_{H_{Q_c}},$$

$$\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)} |\mathfrak{Re}(\partial_{x_{1,2}}Q_c\overline{Q_c\psi^{\neq 0}})| \leqslant K(c)\|\varphi\|_{H_{Q_c}},$$

$$\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)} |\mathfrak{Re}(\partial_c Q_c\overline{Q_c\psi^{\neq 0}})| \leqslant K(c)\|\varphi\|_{H_{Q_c}},$$

and

$$\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)} |\mathfrak{Re}(-x^{\perp}\cdot\nabla Q_c\overline{Q_c\psi^{\neq 0}})| \leqslant K(c)\|\varphi\|_{H_{Q_c}}.$$

We recall that $\psi^{\neq 0}(x) = \psi(x) - \psi^{0,1}(\tilde{r}_1)$ in the right half-plane and $\psi^{\neq 0}(x) = \psi(x) - \psi^{0,-1}(\tilde{r}_{-1})$ in the left half-plane, with $\tilde{r}_{\pm 1} = |x \mp \widetilde{d_c}\vec{e_1}|$ and $\psi^{0,\pm 1}(\tilde{r}_{\pm 1})$ the 0-harmonic of ψ around $\pm\widetilde{d_c}\vec{e_1}$.

Proof: From Lemma 2.15, we have, for $\varphi = Q_c\psi \in H_{Q_c}$,

$$|\tilde{V}_{\pm 1}\psi| = |\varphi| \times \left|\frac{\tilde{V}_{\pm 1}}{Q_c}\right| \leqslant 2|\varphi|$$

given that c is small enough. We deduce by Cauchy–Schwarz and Lemmas 2.1 and 2.6 that

$$\begin{aligned}\int_{B(\pm\tilde{d}_c\vec{e_1},R)} |\mathfrak{Re}(\partial_{x_1}\tilde{V}_{\pm 1}\overline{\tilde{V}_{\pm 1}\psi})| &\leqslant 2\int_{B(\pm\tilde{d}_c\vec{e_1},R)} |\partial_{x_1}\tilde{V}_{\pm 1}| \times |\varphi| \\ &\leqslant K(c)\|\varphi\|_{H^1(B(\pm\tilde{d}_c\vec{e_1},R))} \\ &\leqslant K(c)\|\varphi\|_{H_{Q_c}},\end{aligned}$$

and similarly $\int_{B(\pm\tilde{d}_c\vec{e_1},R)} |\mathfrak{Re}(\partial_{x_2}\tilde{V}_{\pm 1}\overline{\tilde{V}_{\pm 1}\psi})| \leqslant K(c)\|\varphi\|_{H_{Q_c}}$.

By Cauchy–Schwarz, equation (3.2), and Theorem 1.1 (for $p = +\infty$), we conclude that

$$\begin{aligned}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)} |\mathfrak{Re}(\partial_c Q_c\overline{Q_c\psi^{\neq 0}})| &\leqslant K(c)\sqrt{\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)} |\nabla\psi|^2|Q_c|^4} \\ &\leqslant K(c)\|\varphi\|_{H_{Q_c}}.\end{aligned}$$

We can estimate the other terms similarly. □

3.3. On the definition of B_{Q_c}. We start by explaining how to get $B_{Q_c}(\varphi)$ from the "natural" quadratic form

$$\int_{\mathbb{R}^2} |\nabla\varphi|^2 - (1 - |Q_c|^2)|\varphi|^2 + 2\mathfrak{Re}^2(\overline{Q_c}\varphi) - \mathfrak{Re}(ic\partial_{x_2}\varphi\bar{\varphi}).$$

For the first three terms of this quantity, it is obvious that they are well defined for $\varphi \in H_{Q_c}$, but the term $-\mathfrak{Re}(ic\partial_{x_2}\varphi\bar{\varphi})$ is not clearly integrable.

Take a smooth cutoff function η such that $\eta(x) = 0$ on $B(\pm\widetilde{d_c}\vec{e_1}, 1)$, $\eta(x) = 1$ on $\mathbb{R}^2 \backslash B(\pm\widetilde{d_c}\vec{e_1}, 2)$. Then, taking for now $\varphi = Q_c\psi \in C_c^\infty(\mathbb{R}^2)$,

$$\mathfrak{Re}(i\partial_{x_2}\varphi\bar{\varphi}) = \eta\mathfrak{Re}(i\partial_{x_2}\varphi\bar{\varphi}) + (1-\eta)\mathfrak{Re}(i\partial_{x_2}\varphi\bar{\varphi}),$$

and writing $\varphi = Q_c\psi$,

$$\begin{aligned}\eta\mathfrak{Re}(i\partial_{x_2}\varphi\bar{\varphi}) &= \eta\mathfrak{Re}(i\partial_{x_2}Q_c\overline{Q_c})|\psi|^2 + \eta\mathfrak{Re}(i\partial_{x_2}\psi\bar{\psi})|Q_c|^2 \\ &= \eta\mathfrak{Re}(i\partial_{x_2}Q_c\overline{Q_c})|\psi|^2 - \eta\mathfrak{Re}\psi\mathfrak{Im}\partial_{x_2}\psi|Q_c|^2 + \eta\mathfrak{Re}\partial_{x_2}\psi\mathfrak{Im}\psi|Q_c|^2.\end{aligned}$$

Furthermore,

$$\eta\mathfrak{Re}\partial_{x_2}\psi\mathfrak{Im}\psi|Q_c|^2 = \partial_{x_2}(\eta\mathfrak{Re}\psi\mathfrak{Im}\psi|Q_c|^2) - \partial_{x_2}\eta\mathfrak{Re}\psi\mathfrak{Im}\psi|Q_c|^2$$
$$-\eta\mathfrak{Re}\psi\mathfrak{Im}\partial_{x_2}\psi|Q_c|^2 - \eta\mathfrak{Re}\psi\mathfrak{Im}\psi\partial_{x_2}(|Q_c|^2),$$

thus we can write

$$\begin{aligned}\int_{\mathbb{R}^2}\mathfrak{Re}(i\partial_{x_2}\varphi\bar{\varphi}) = {} & \int_{\mathbb{R}^2}\partial_{x_2}(\eta\mathfrak{Re}\psi\mathfrak{Im}\psi|Q_c|^2)\\ & +\int_{\mathbb{R}^2}(1-\eta)\mathfrak{Re}(i\partial_{x_2}\varphi\bar{\varphi}) + \int_{\mathbb{R}^2}\eta\mathfrak{Re}(i\partial_{x_2}Q_c\overline{Q_c})|\psi|^2\\ & -2\int_{\mathbb{R}^2}\eta\mathfrak{Re}\psi\mathfrak{Im}\partial_{x_2}\psi|Q_c|^2 - \int_{\mathbb{R}^2}\partial_{x_2}\eta\mathfrak{Re}\psi\mathfrak{Im}\psi|Q_c|^2\\ & -\int_{\mathbb{R}^2}\eta\mathfrak{Re}\psi\mathfrak{Im}\psi\partial_{x_2}(|Q_c|^2).\end{aligned}$$

The only difficulty here is that the first integral is not well defined for $\varphi \in H_{Q_c}$, but it is the integral of a derivative. This is why we defined instead the quadratic form

$$\begin{aligned}B_{Q_c}(\varphi) = {} & \int_{\mathbb{R}^2}|\nabla\varphi|^2 - (1-|Q_c|^2)|\varphi|^2 + 2\mathfrak{Re}^2(\overline{Q_c}\varphi)\\ & -c\int_{\mathbb{R}^2}(1-\eta)\mathfrak{Re}(i\partial_{x_2}\varphi\bar{\varphi}) - c\int_{\mathbb{R}^2}\eta\mathfrak{Re}(i\partial_{x_2}Q_c\overline{Q_c})|\psi|^2\\ & +2c\int_{\mathbb{R}^2}\eta\mathfrak{Re}\psi\mathfrak{Im}\partial_{x_2}\psi|Q_c|^2 + c\int_{\mathbb{R}^2}\partial_{x_2}\eta\mathfrak{Re}\psi\mathfrak{Im}\psi|Q_c|^2\\ & +c\int_{\mathbb{R}^2}\eta\mathfrak{Re}\psi\mathfrak{Im}\psi\partial_{x_2}(|Q_c|^2).\end{aligned}$$

It is easy to check that this quantity is independent of the choice of η. We will show in Lemma 3.3 that this quantity is well defined for $\varphi \in H_{Q_c}$. By adding some conditions on φ, for instance if $\varphi \in H^1(\mathbb{R}^2)$, we can show that $\int_{\mathbb{R}^2}\partial_{x_2}(\eta\mathfrak{Re}\psi\mathfrak{Im}\psi|Q_c|^2)$ is well defined and is 0. In these cases, we therefore have

$$B_{Q_c}(\varphi) = \int_{\mathbb{R}^2}|\nabla\varphi|^2 - (1-|Q_c|^2)|\varphi|^2 + 2\mathfrak{Re}^2(\overline{Q_c}\varphi) - \mathfrak{Re}(ic\partial_{x_2}\varphi\bar{\varphi}).$$

This is a classical situation for Schrödinger equations with nonzero limit at infinity (see [**3**] or [**16**]): the quadratic form is defined up to a term which is a derivative of some function in some L^p space.

Lemma 3.3. *There exists $c_0 > 0$ such that, for $0 < c \leqslant c_0$, Q_c defined in Theorem 1.1, there exists a constant $C(c) > 0$ such that, for $\varphi = Q_c\psi \in H_{Q_c}$ and η a smooth cutoff function such that $\eta(x) = 0$ on $B(\pm\widetilde{d_c}\overrightarrow{e_1}, 1)$, $\eta(x) = 1$ on $\mathbb{R}^2\backslash B(\pm\widetilde{d_c}\overrightarrow{e_1}, 2)$, we have*

$$\int_{\mathbb{R}^2} |(1-\eta)\mathfrak{Re}(i\partial_{x_2}\varphi\bar{\varphi})| + \int_{\mathbb{R}^2} |\eta\mathfrak{Re}(i\partial_{x_2}Q_c\overline{Q_c})|\psi|^2| + \int_{\mathbb{R}^2} |\eta\mathfrak{Re}\psi\mathfrak{Im}(\partial_{x_2}\psi)|Q_c|^2| + \int_{\mathbb{R}^2} |\partial_{x_2}\eta\mathfrak{Re}\psi\mathfrak{Im}\psi|Q_c|^2| + \int_{\mathbb{R}^2} |\eta\mathfrak{Re}\psi\mathfrak{Im}\psi\partial_{x_2}(|Q_c|^2)| \leqslant C(c)\|\varphi\|^2_{H_{Q_c}}.$$

Proof: Since $|1-|Q_c|^2| \geqslant K > 0$ on $B(\pm\widetilde{d_c}\overrightarrow{e_1}, 2)$ for c small enough by Lemma 2.1 and Theorem 1.1, we estimate

$$\int_{\mathbb{R}^2} |(1-\eta)\mathfrak{Re}(ic\partial_{x_2}\varphi\bar{\varphi})| \leqslant C(c)\int_{B(\tilde{d}_c\overrightarrow{e_1},2)\cup B(-\tilde{d}_c\overrightarrow{e_1},2)} |1-|Q_c|^2||\varphi||\partial_{x_2}\varphi| \leqslant C(c)\|\varphi\|^2_{H_{Q_c}}.$$

Furthermore, by (2.12) and Lemma 2.6,

$$\int_{\mathbb{R}^2} |\eta\mathfrak{Re}(ic\partial_{x_2}Q_c\overline{Q_c})|\psi|^2| \leqslant C(c)\int_{\mathbb{R}^2} \eta|\nabla Q_c||\psi|^2 \leqslant C(c)\int_{\mathbb{R}^2} \eta|\nabla Q_c||\varphi|^2 \leqslant C(c)\|\varphi\|^2_{H_{Q_c}}$$

since $|\nabla Q_c| \leqslant \frac{C(c)}{(1+r)^2}$ from Theorem 2.5. By Cauchy–Schwarz, equation (2.12), and Lemma 3.1,

$$\int_{\mathbb{R}^2} |\eta\mathfrak{Re}\psi\mathfrak{Im}\partial_{x_2}\psi|Q_c|^2| \leqslant K\sqrt{\int_{\mathbb{R}^2} \eta\mathfrak{Re}^2(\psi)\int_{\mathbb{R}^2} \eta|\nabla\psi|^2} \leqslant C(c)\|\varphi\|^2_{H_{Q_c}}.$$

Now, still by equation (2.12) and Lemma 3.1, since $\partial_{x_2}\eta$ is supported in $B(\pm\widetilde{d_c}\overrightarrow{e_1}, 2)\backslash B(\pm\widetilde{d_c}\overrightarrow{e_1}, 1)$,

$$\int_{\mathbb{R}^2} |\partial_{x_2}\eta\mathfrak{Re}\psi\mathfrak{Im}\psi|Q_c|^2| \leqslant K\|\varphi\|^2_{H_{Q_c}}.$$

Finally, since $|\nabla Q_c| \leqslant \frac{C(c)}{(1+r)^2}$ by Theorem 2.5, by Cauchy–Schwarz and Lemma 2.6,

$$\int_{\mathbb{R}^2} |\eta\mathfrak{Re}\psi\mathfrak{Im}\psi\partial_{x_2}(|Q_c|^2)| \leqslant C(c)\sqrt{\int_{\mathbb{R}^2} \eta\mathfrak{Re}^2(\psi)\int_{\mathbb{R}^2} \eta\frac{\mathfrak{Im}^2\psi}{(1+r)^4}} \leqslant C(c)\|\varphi\|^2_{H_{Q_c}}. \square$$

3.4. Density of test functions in H_{Q_c}. We shall prove coercivity with test functions that are 0 in a neighbourhood of the zeros of Q_c. This will allow us to divide by Q_c in several computations. Here we give a density result to show that it is not a problem to remove a neighbourhood of the zeros of Q_c for test functions.

Lemma 3.4. *$C_c^\infty(\mathbb{R}^2\backslash\{\widetilde{d_c}\overrightarrow{e_1}, -\widetilde{d_c}\overrightarrow{e_1}\}, \mathbb{C})$ is dense in H_{Q_c} for the norm $\|\cdot\|_{H_{Q_c}}$.*

This result uses similar arguments to those used in [**5**] for the density in H_{V_1}. See Appendix B.1 for a proof of it.

4. Coercivity results in H_{Q_c}

This section is devoted to the proofs of Propositions 1.3 and 1.4. Here, we will do most of the computations with test functions, that is, functions in $C_c^\infty(\mathbb{R}^2\backslash\{\tilde{d}_c\vec{e}_1, -\tilde{d}_c\vec{e}_1\}, \mathbb{C})$. This will allow us to do many computations, including dividing by Q_c in some quantities.

4.1. Expression of the quadratic forms. We recall that η is a smooth cutoff function such that $\eta(x) = 0$ on $B(\pm\widetilde{d_c}\overrightarrow{e_1}, 1)$, $\eta(x) = 1$ on $\mathbb{R}^2\backslash(B(\widetilde{d_c}\overrightarrow{e_1}, 2)\cup B(-\widetilde{d_c}\overrightarrow{e_1}, 2))$, where $\pm\widetilde{d_c}\overrightarrow{e_1}$ are the zeros of Q_c. Furthermore, from [**5**], we recall the quadratic form around a vortex V_1:

$$B_{V_1}(\varphi) = \int_{\mathbb{R}^2} |\nabla\varphi|^2 - (1-|V_1|^2)|\varphi|^2 + 2\mathfrak{Re}^2(\overline{V_1}\varphi).$$

We want to write the quadratic form around V_1 and Q_c in a special form. For the one around Q_c, it will be of the form $B_{Q_c}^{\exp}$, defined in (1.4).

Lemma 4.1. *For $\varphi = Q_c\psi \in C_c^\infty(\mathbb{R}^2\backslash\{\widetilde{d_c}\overrightarrow{e_1}, -\widetilde{d_c}\overrightarrow{e_1}\}, \mathbb{C})$, we have*

$$\langle L_{Q_c}(\varphi), \varphi\rangle = B_{Q_c}^{\exp}(\varphi),$$

where $B_{Q_c}^{\exp}(\varphi)$ is defined in (1.4). Furthermore, for $\varphi = V_1\psi \in C_c^\infty(\mathbb{R}^2\backslash \{0\}, \mathbb{C})$, where V_1 is centred at 0, and $\tilde{\eta}$ a smooth radial cutoff function with value 0 in $B(0,1)$, and value 1 outside of $B(0,2)$,

$$\begin{aligned} B_{V_1}(\varphi) = &\int_{\mathbb{R}^2} (1-\tilde{\eta})(|\nabla\varphi|^2 - (1-|V_1|^2)|\varphi|^2 + 2\mathfrak{Re}^2(\overline{V_1}\varphi)) \\ &- \int_{\mathbb{R}^2} \nabla\tilde{\eta}\cdot(\mathfrak{Re}(\nabla V_1\overline{V_1})|\psi|^2 - 2\mathfrak{Im}(\nabla V_1\overline{V_1})\mathfrak{Re}(\psi)\mathfrak{Im}(\psi)) \\ &+ \int_{\mathbb{R}^2} \tilde{\eta}(|\nabla\psi|^2|V_1|^2 + 2\mathfrak{Re}^2(\psi)|V_1|^4 + 4\mathfrak{Im}(\nabla V_1\overline{V_1})\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi)). \end{aligned}$$

See Appendix B.2 for the proof of this result.

4.2. A coercivity result for the quadratic form around one vortex. This subsection is devoted to the proof of Proposition 1.3, and a localized version of it (see Lemma 4.2).

4.2.1. Coercivity for test functions.

Proof of Proposition 1.3: We recall the result from [**5**]; see Lemma 3.1 and equation (2.42) there. If $\varphi = V_1\psi \in C_c^\infty(\mathbb{R}^2\backslash\{0\}, \mathbb{C})$ with the two orthogonality conditions

$$\int_{B(0,R)} \mathfrak{Re}(\partial_{x_1} V_1 \bar{\varphi}) = \int_{B(0,R)} \mathfrak{Re}(\partial_{x_1} V_1 \bar{\varphi}) = 0,$$

then, writing $\psi^0(x) = \frac{1}{2\pi}\int_0^{2\pi} \psi(|x|\cos(\theta), |x|\sin(\theta))\, d\theta$, the 0-harmonic around 0 of ψ, and $\psi^{\neq 0} = \psi - \psi^0$, then

$$B_{V_1}(\varphi) \geqslant K \int_{\mathbb{R}^2} |\nabla(V_1\psi^{\neq 0})|^2 + |\nabla\psi^0|^2|V_1|^2 + \frac{|V_1\psi^{\neq 0}|^2}{(1+r)^2} + \mathfrak{Re}^2(\psi)|V_1|^4.$$

We recall from Lemma 2.1 that there exists $K_1 > 0$ such that, for all $r > 0$, $K_1 \leqslant \frac{|V_1|}{r} \leqslant \frac{1}{K_1}$, and that $|V_1|$ is a radial function around 0. Therefore, by Hardy inequality in dimension 4,

$$\int_{B(0,1)} |\psi^0|^2 \leqslant K \left(\int_{B(0,2)} |\nabla\psi^0|^2|V_1|^2 + \int_{B(0,2)\backslash B(0,1)} |\psi^0|^2 \right).$$

By Poincaré inéquality, using $\int_{B(0,R)\backslash B(0,R/2)} \mathfrak{Im}(\psi) = 0$ and $|V_1|^2 \geqslant K$ outside of $B(0,1)$, we have

$$\int_{B(0,10)\backslash B(0,1)} |\psi^0|^2 \leqslant K \left(\int_{B(0,R)} |\nabla\psi^0|^2|V_1|^2 + \mathfrak{Re}^2(\psi^0)|V_1|^4 \right).$$

Here, the constant $K > 0$ depends on $R > 0$, but we consider R as a universal constant. We deduce that

$$\begin{aligned}\int_{B(0,10)} |\varphi|^2 &\leqslant \int_{B(0,10)} |V_1\psi|^2 \\ &\leqslant K\left(\int_{B(0,10)} |V_1\psi^0|^2 + \int_{B(0,10)} |V_1\psi^{\neq 0}|^2 \right) \\ &\leqslant K\left(\int_{\mathbb{R}^2} |\nabla(V_1\psi^{\neq 0})|^2 + |\nabla\psi^0|^2|V_1|^2 + \frac{|V_1\psi^{\neq 0}|^2}{(1+r)^2} + \mathfrak{Re}^2(\psi)|V_1|^4 \right).\end{aligned}$$

Similarly,

$$\begin{aligned}\int_{B(0,10)}|\nabla\varphi|^2 &\leqslant \int_{B(0,10)}|\nabla(V_1(\psi^0+\psi^{\neq 0}))|^2\\ &\leqslant K\left(\int_{B(0,10)}|\nabla(V_1\psi^0)|^2+\int_{B(0,10)}|\nabla(V_1\psi^{\neq 0})|^2\right)\\ &\leqslant K\left(\int_{B(0,10)}|\nabla\psi^0|^2|V_1|^2+|\psi^0|^2|\nabla V_1|^2+\int_{B(0,10)}|\nabla(V_1\psi^{\neq 0})|^2\right)\\ &\leqslant K\left(\int_{\mathbb{R}^2}|\nabla(V_1\psi^{\neq 0})|^2+|\nabla\psi^0|^2|V_1|^2+\frac{|V_1\psi^{\neq 0}|^2}{(1+r)^2}+\mathfrak{Re}^2(\psi)|V_1|^4\right).\end{aligned}$$

Finally, outside of $B(0,5)$, we have, by Lemma 2.1, that

$$\int_{\mathbb{R}^2\setminus B(0,5)}|\nabla\psi|^2 \leqslant K\int_{\mathbb{R}^2\setminus B(0,5)}|\nabla\psi|^2|V_1|^2.$$

Let us show that

$$\int_{\mathbb{R}^2\setminus B(0,5)}\frac{|\psi|^2}{r^2\ln^2(r)} \leqslant K\left(\int_{\mathbb{R}^2\setminus B(0,5)}|\nabla\psi|^2+\int_{B(0,10)\setminus B(0,5)}|\psi|^2\right).$$

This is a Hardy-type inequality, and it would conclude the proof of this proposition. Note that for the harmonics other than zeros, this is a direct consequence of

$$\int_{\mathbb{R}^2\setminus B(0,5)}\frac{|\psi^{\neq 0}|^2}{r^2} \leqslant \int_{\mathbb{R}^2\setminus B(0,5)}|\nabla\psi|^2.$$

We therefore suppose that ψ is a radial compactly supported function. We define χ a smooth radial cutoff function with $\chi(r)=0$ if $r\leqslant 4$ and $\chi(r)=1$ if $r\geqslant 5$. Then, by Cauchy–Schwarz,

$$\begin{aligned}\left|\int_{\mathbb{R}^2}\frac{\chi(r)|\psi|^2}{r^2\ln^2(r)}\right| &= \left|-\int_0^{+\infty}\chi(r)|\psi|^2(r)\partial_r\left(\frac{1}{\ln(r)}\right)dr\right|\\ &= \left|\int_0^{+\infty}\partial_r(\chi|\psi|^2)(r)\frac{dr}{\ln(r)}\right|\\ &\leqslant K\left(\int_{B(0,10)\setminus B(0,5)}|\psi|^2+\int_5^{+\infty}|\psi|(r)\partial_r|\psi|(r)\frac{dr}{\ln(r)}\right)\\ &\leqslant K\left(\int_{B(0,10)\setminus B(0,5)}|\psi|^2+\sqrt{\int_{\mathbb{R}^2\setminus B(0,5)}\frac{|\psi|^2}{r^2\ln^2(r)}\int_{\mathbb{R}^2\setminus B(0,5)}|\nabla\psi|^2}\right).\end{aligned}$$

The proof is complete. □

4.2.2. Localization of the coercivity for one vortex. Now, we want to localize the coercivity result. We define, for $D > 10$, $\varphi = V_1\psi \in H_{V_1}$,

$$B_{V_1}^{\mathrm{loc}_D}(\varphi) := \int_{B(0,D)} (1-\tilde{\eta})(|\nabla\varphi|^2 - (1-|V_1|^2)|\varphi|^2 + 2\mathfrak{Re}^2(\overline{V_1}\varphi))$$
$$-\int_{B(0,D)} \nabla\tilde{\eta}\cdot(\mathfrak{Re}(\nabla V_1\overline{V_1})|\psi|^2 - 2\mathfrak{Im}(\nabla V_1\overline{V_1})\mathfrak{Re}(\psi)\mathfrak{Im}(\psi))$$
$$+\int_{B(0,D)} \tilde{\eta}(|\nabla\psi|^2|V_1|^2 + 2\mathfrak{Re}^2(\psi)|V_1|^4 + 4\mathfrak{Im}(\nabla V_1\overline{V_1})\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi)),$$

where $\tilde{\eta}$ is a smooth radial cutoff function such that $\tilde{\eta}(x) = 0$ on $B(0,1)$, $\tilde{\eta}(x) = 1$ on $\mathbb{R}^2\backslash B(0,2)$.

Lemma 4.2. *There exist $K, R, D_0 > 0$ with $D_0 > R$, such that, for $D > D_0$ and $\varphi = V_1\psi \in C_c^\infty(\mathbb{R}^2\backslash\{0\},\mathbb{C})$, if the following three orthogonality conditions*

$$\int_{B(0,R)} \mathfrak{Re}(\partial_{x_1}V_1\bar{\varphi}) = \int_{B(0,R)} \mathfrak{Re}(\partial_{x_2}V_1\bar{\varphi}) = \int_{B(0,R)\backslash B(0,R/2)} \mathfrak{Im}(\psi) = 0$$

are satisfied, then

$$B_{V_1}^{\mathrm{loc}_D}(\varphi) \geqslant K\left(\int_{B(0,10)} |\nabla\varphi|^2 + |\varphi|^2 \right.$$
$$\left. + \int_{B(0,D)\backslash B(0,5)} |\nabla\psi|^2|V_1|^2 + \mathfrak{Re}^2(\psi)|V_1|^4 + \frac{|\psi|^2}{r^2\ln^2(r)}\right).$$

Proof: We decompose ψ into harmonics $j \in \mathbb{N}$, $l \in \{1,2\}$, with the same decomposition as in (2.5) of [**5**]. This decomposition is adapted to the quadratic form $B_{V_1}^{\mathrm{loc}_D}$ (see equation (2.4) of [**5**]), which also holds if the integral is only on $B(0,D)$.

For $j = 0$, the proof is identical. For $j \geqslant 2$, $l \in \{1,2\}$, from equation (2.38) of [**5**] (which holds on $B(0,D)$ as the inequality is pointwise), the proof holds if it does for $j = 1$, $l \in \{1,2\}$.

We therefore focus on the case $j = l = 1$. We write $\psi = \psi_1(r)\cos(\theta) + i\psi_2(r)\sin(\theta)$, with $\psi_1, \psi_2 \in C_c^\infty(\mathbb{R}^{+*},\mathbb{R})$. The other possibility ($l = 2$) is $\psi = \psi_1(r)i\cos(\theta) + \psi_2(r)\sin(\theta)$, which is done similarly. We will show a more general result, that is, for any $\varphi = V_1\psi \in C_c^\infty(\mathbb{R}^2\backslash\{0\},\mathbb{C})$ satisfying the orthogonality conditions,

$$B_{V_1}^{\mathrm{loc}_D}(V_1\psi^{\neq 0}) \geqslant K\left(\int_{B(0,10)} |\nabla(V_1\psi^{\neq 0})|^2 + |V_1\psi^{\neq 0}|^2 \right.$$
$$\left. + \int_{B(0,D)\backslash B(0,5)} |\nabla\psi^{\neq 0}|^2|V_1|^2 + \mathfrak{Re}^2(\psi^{\neq 0})|V_1|^4 + \frac{|\psi^{\neq 0}|^2}{r^2}\right).$$

With the previous remark, it is enough to conclude the proof of this lemma. In the rest of the proof, to simplify the notation, we write ψ instead of $\psi^{\neq 0}$, but it still has no 0-harmonic.

We note that, for $D > R_0 > 2$,

$$\begin{aligned}
&\int_{B(0,D)\setminus B(0,R_0)} |\nabla\psi|^2|V_1|^2 + 2\mathfrak{Re}^2(\psi)|V_1|^4 + 4\mathfrak{Im}(\nabla V_1\overline{V_1})\cdot\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi)\\
&\geqslant \int_{B(0,D)\setminus B(0,R_0)} |\nabla\psi|^2|V_1|^2 + 2\mathfrak{Re}^2(\psi)|V_1|^4 - \frac{K|V_1|^2}{R_0}|\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi)| \qquad (4.1)\\
&\geqslant \frac{1}{2}\int_{B(0,D)\setminus B(0,R_0)} |\nabla\psi|^2|V_1|^2 + 2\mathfrak{Re}^2(\psi)|V_1|^4
\end{aligned}$$

if R_0 is large enough. We therefore take $R_0 > R$ large enough such that (4.1) holds. For $\frac{D}{2} > \lambda > R_0$, we define χ_λ a smooth cutoff function such that $\chi_\lambda(r) = 1$ if $r \leqslant \lambda$, $\chi_\lambda = 0$ if $r \geqslant 2\lambda$, and $|\chi_\lambda'| \leqslant \frac{K}{\lambda}$. In particular, since $R_0 > 2$, we have $\mathrm{Supp}(\chi_\lambda') \subset \mathrm{Supp}(\tilde\eta)$ and $\mathrm{Supp}(1-\tilde\eta) \subset \mathrm{Supp}(\chi_\lambda)$. This implies that

$$\begin{aligned}
&\int_{B(0,D)} (1-\tilde\eta)(|\nabla\varphi|^2 - (1-|V_1|^2)|\varphi|^2 + 2\mathfrak{Re}^2(\overline{V_1}\varphi))\\
&\qquad = \int_{B(0,D)} (1-\tilde\eta)(|\nabla(\chi_\lambda\varphi)|^2 - (1-|V_1|^2)|\chi_\lambda\varphi|^2 + 2\mathfrak{Re}^2(\overline{V_1}\chi_\lambda\varphi))
\end{aligned}$$

and

$$\begin{aligned}
&\int_{B(0,D)} \nabla\tilde\eta\cdot(\mathfrak{Re}(\nabla V_1\overline{V_1})|\psi|^2 - 2\mathfrak{Im}(\nabla V_1\overline{V_1})\mathfrak{Re}(\psi)\mathfrak{Im}(\psi))\\
&\qquad = \int_{B(0,D)} \nabla\tilde\eta\cdot(\mathfrak{Re}(\nabla V_1\overline{V_1})|\chi_\lambda\psi|^2 - 2\mathfrak{Im}(\nabla V_1\overline{V_1})\mathfrak{Re}(\chi_\lambda\psi)\mathfrak{Im}(\chi_\lambda\psi)).
\end{aligned}$$

Now, we decompose

$$\begin{aligned}
&\int_{B(0,D)} \tilde\eta(|\nabla\psi|^2|V_1|^2 + 2\mathfrak{Re}^2(\psi)|V_1|^4 + 4\mathfrak{Im}(\nabla V_1\overline{V_1})\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi))\\
&= \int_{B(0,D)} (1-\chi_\lambda^2)\tilde\eta(|\nabla\psi|^2|V_1|^2 + 2\mathfrak{Re}^2(\psi)|V_1|^4 + 4\mathfrak{Im}(\nabla V_1\overline{V_1})\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi))\\
&\quad + \int_{B(0,D)} \chi_\lambda^2\tilde\eta(|\nabla\psi|^2|V_1|^2 + 2\mathfrak{Re}^2(\psi)|V_1|^4 + 4\mathfrak{Im}(\nabla V_1\overline{V_1})\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi)),
\end{aligned}$$

and by equation (4.1),

$$\begin{aligned}
&\int_{B(0,D)} (1-\chi_\lambda^2)\tilde\eta(|\nabla\psi|^2|V_1|^2 + 2\mathfrak{Re}^2(\psi)|V_1|^4 + 4\mathfrak{Im}(\nabla V_1\overline{V_1})\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi))\\
&\qquad\qquad \geqslant K\int_{B(0,D)} (1-\chi_\lambda^2)|\nabla\psi|^2|V_1|^2 + 2\mathfrak{Re}^2(\psi)|V_1|^4.
\end{aligned}$$

Furthermore,

$$\begin{aligned}\int_{B(0,D)} & \chi_\lambda^2\tilde{\eta}(|\nabla\psi|^2|V_1|^2 + 2\mathfrak{Re}^2(\psi)|V_1|^4 + 4\mathfrak{Im}(\nabla V_1\overline{V_1})\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi)) \\ &= \int_{B(0,D)} \tilde{\eta}(|\nabla(\chi_\lambda\psi)|^2|V_1|^2 + 2\mathfrak{Re}^2(\chi_\lambda\psi)|V_1|^4 \\ &\qquad + 4\mathfrak{Im}(\nabla V_1\overline{V_1})\mathfrak{Im}(\nabla(\chi_\lambda\psi))\mathfrak{Re}(\chi_\lambda\psi)) \\ &\quad - \int_{B(0,D)} \tilde{\eta}(((|\nabla(\chi_\lambda\psi) - \nabla\chi_\lambda\psi|^2 - |\nabla(\chi_\lambda\psi)|^2)|V_1|^2 \\ &\qquad - 4\mathfrak{Im}(\nabla V_1\overline{V_1})\cdot\nabla\chi_\lambda\mathfrak{Im}(\psi)\mathfrak{Re}(\chi_\lambda\psi)),\end{aligned}$$

and thus

$$\begin{aligned}B_{V_1}^{\mathrm{loc}_D}(V_1\psi) \geqslant B_{V_1}^{\mathrm{loc}_D}(V_1\chi_\lambda\psi) &+ K\int_{B(0,D)}(1-\chi_\lambda^2)|\nabla\psi|^2|V_1|^2 + 2\mathfrak{Re}^2(\psi)|V_1|^4 \\ &- \int_{B(0,D)} \tilde{\eta}(((|\nabla(\chi_\lambda\psi) - \nabla\chi_\lambda\psi|^2 - |\nabla(\chi_\lambda\psi)|^2)|V_1|^2 \\ &\qquad - 4\mathfrak{Im}(\nabla V_1\overline{V_1})\cdot\nabla\chi_\lambda\mathfrak{Im}(\psi)\mathfrak{Re}(\chi_\lambda\psi)).\end{aligned}$$

Since $V_1\chi_\lambda\psi \in C_c^\infty(B(0,D))$, we have $B_{V_1}^{\mathrm{loc}_D}(V_1\chi_\lambda\psi) = B_{V_1}(V_1\chi_\lambda\psi)$, and since $\chi_\lambda = 1$ in $B(0,R)$ and $V_1\psi$ satisfied the orthogonality conditions, so does $V_1\chi_\lambda\psi$. By Proposition 1.3, we deduce that

$$\begin{aligned}B_{V_1}^{\mathrm{loc}_D}(V_1\chi_\lambda\psi) \geqslant K\int_{B(0,10)} & |\nabla(V_1\chi_\lambda\psi)|^2 + |V_1\chi_\lambda\psi|^2 \\ &+ K\int_{B(0,D)\setminus B(0,5)} |\nabla(\chi_\lambda\psi)|^2|V_1|^2 + \mathfrak{Re}^2(\chi_\lambda\psi)|V_1|^4 + \frac{|\chi_\lambda\psi|^2}{r^2\ln^2(r)}.\end{aligned}$$

Now, noting that

$$|\nabla(\chi_\lambda\psi)|^2|V_1|^2 \geqslant K_1|\nabla\psi|^2\chi_\lambda^2|V_1|^2 - K_2|\nabla\chi_\lambda|^2|\psi|^2|V_1|^2,$$

and since $\chi_\lambda = 1$ in $B(0,10)$, we deduce that

$$\begin{aligned}B_{V_1}^{\mathrm{loc}_D}(V_1\psi) \geqslant K&\left(\int_{B(0,10)}|\nabla\varphi|^2 + |\varphi|^2 + \int_{B(0,D)\setminus B(0,5)}|\nabla\psi|^2|V_1|^2 + \mathfrak{Re}^2(\psi)|V_1|^4\right) \\ &- K\int_{B(0,D)} \tilde{\eta}(|(|\nabla(\chi_\lambda\psi) - \nabla\chi_\lambda\psi|^2 - |\nabla(\chi_\lambda\psi)|^2)||V_1|^2 \\ &\qquad + |\mathfrak{Im}(\nabla V_1\overline{V_1})\cdot\nabla\chi_\lambda\mathfrak{Im}(\psi)\mathfrak{Re}(\chi_\lambda\psi)|) \\ &- K\int_{B(0,D)\setminus B(0,5)} |\nabla\chi_\lambda|^2|\psi|^2|V_1|^2.\end{aligned} \tag{4.2}$$

Since $\nabla\chi_\lambda$ is supported in $B(0,2\lambda)\backslash B(0,\lambda)$ with $|\nabla\chi_\lambda| \leqslant \frac{K}{\lambda}$, we have

$$\int_{B(0,D)\backslash B(0,5)} |\nabla\chi_\lambda|^2|\psi|^2|V_1|^2 \leqslant K\int_{B(0,2\lambda)\backslash B(0,\lambda)} \frac{|\psi|^2}{(1+r)^2},$$

and by Cauchy–Schwarz we have that

$$\int_{B(0,D)} \tilde{\eta}|\mathfrak{Im}(\nabla V_1\overline{V_1})\cdot\nabla\chi_\lambda\mathfrak{Im}(\psi)\mathfrak{Re}(\chi_\lambda\psi)| \leqslant K\sqrt{\int_{B(0,2\lambda)\backslash B(0,\lambda)} \frac{|\psi|^2}{(1+r)^2}\int_{B(0,D)\backslash B(0,5)}\mathfrak{Re}^2(\psi)}$$

and

$$\int_{B(0,D)} \tilde{\eta}(|(|\nabla(\chi_\lambda\psi) - \nabla\chi_\lambda\psi|^2 - |\nabla(\chi_\lambda\psi)|^2)||V_1|^2) \leqslant K\left(\sqrt{\int_{B(0,2\lambda)\backslash B(0,\lambda)}\frac{|\psi|^2}{(1+r)^2}\int_{B(0,D)\backslash B(0,5)}|\nabla\psi|^2|V_1|^2}+\int_{B(0,2\lambda)\backslash B(0,\lambda)}\frac{|\psi|^2}{(1+r)^2}\right).$$

Since ψ has no 0-harmonics we have that

$$\int_{B(0,D)\backslash B(0,5)} \frac{|\psi|^2}{(1+r)^2} \leqslant K\int_{B(0,D)\backslash B(0,5)} |\nabla\psi|^2|V_1|^2.$$

We infer that there exists $D_0 > R_0$ a large constant such that, for $D > D_0$, for all $\varphi = V_1\psi \in C_c^\infty(\mathbb{R}^2\backslash\{0\},\mathbb{C})$, there exists $\lambda \in \left[R_0, \frac{D_0}{2}\right]$ such that

$$\int_{B(0,2\lambda)\backslash B(0,\lambda)} \frac{|\psi|^2}{(1+r)^2} \leqslant \varepsilon\int_{B(0,D)\backslash B(0,5)} |\nabla\psi|^2|V_1|^2 \tag{4.3}$$

for some small fixed constant $\varepsilon > 0$. Indeed, if this does not hold, then $\int_{B(0,D)\backslash B(0,5)} |\nabla\psi|^2|V_1|^2 \neq 0$ and

$$\begin{aligned}\int_{B(0,D)\backslash B(0,5)} \frac{|\psi|^2}{(1+r)^2} &\geqslant \int_{R_0}^{D_0} \frac{|\psi|^2}{(1+r)^2} r\,dr \\ &\geqslant \sum_{n=0}^{\left\lfloor\log_2\left(\frac{D_0}{2R_0}\right)\right\rfloor-2} \int_{2^nR_0}^{2^{n+1}R_0} \frac{|\psi|^2}{(1+r)^2} r\,dr \\ &\geqslant \sum_{n=0}^{\left\lfloor\log_2\left(\frac{D_0}{2R_0}\right)\right\rfloor-2} \varepsilon\int_{B(0,D)\backslash B(0,5)} |\nabla\psi|^2|V_1|^2 \\ &\geqslant \varepsilon\left(\left\lfloor\log_2\left(\frac{D_0}{2R_0}\right)\right\rfloor - 1\right)\int_{B(0,D)\backslash B(0,5)} |\nabla\psi|^2|V_1|^2 \\ &> \frac{1}{K}\int_{B(0,D)\backslash B(0,5)} |\nabla\psi|^2|V_1|^2\end{aligned}$$

for D_0 large enough. Taking $\varepsilon > 0$ small enough, with equations (4.2) to (4.3), we conclude the proof of this lemma. □

A consequence of Lemma 4.2 is that, for a function $\varphi = V_1\psi \in C_c^\infty(\mathbb{R}^2\backslash\{0\}, \mathbb{C})$ satisfying the three orthogonality conditions in Lemma 4.2 and $D > D_0$, then

$$B_{V_1}^{\mathrm{loc}_D}(\varphi) \geqslant K(D)\|\varphi\|^2_{H^1(B(0,D))}. \tag{4.4}$$

4.3. Coercivity for a travelling wave near its zeros. We recall from Lemma 4.1 that, for $\varphi \in C_c^\infty(\mathbb{R}^2\backslash\{\tilde{d}_c\vec{e}_1, -\tilde{d}_c\vec{e}_1\}, \mathbb{C})$, we have

$$\begin{aligned}\langle L_{Q_c}(\varphi), \varphi\rangle = &\int_{\mathbb{R}^2}(1-\eta)(|\nabla\varphi|^2 - \mathfrak{Re}(ic\partial_{x_2}\varphi\bar{\varphi}) - (1-|Q_c|^2)|\varphi|^2 + 2\mathfrak{Re}^2(\overline{Q_c}\varphi))\\ &-\int_{\mathbb{R}^2}\nabla\eta\cdot(\mathfrak{Re}(\nabla Q_c\overline{Q_c})|\psi|^2 - 2\mathfrak{Im}(\nabla Q_c\overline{Q_c})\mathfrak{Re}(\psi)\mathfrak{Im}(\psi))\\ &+\int_{\mathbb{R}^2} c\partial_{x_2}\eta\mathfrak{Re}(\psi)\mathfrak{Im}(\psi)|Q_c|^2\\ &+\int_{\mathbb{R}^2}\eta(|\nabla\psi|^2|Q_c|^2 + 2\mathfrak{Re}^2(\psi)|Q_c|^4)\\ &+\int_{\mathbb{R}^2}\eta(4\mathfrak{Im}(\nabla Q_c\overline{Q_c})\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi) + 2c|Q_c|^2\mathfrak{Im}(\partial_{x_2}\psi)\mathfrak{Re}(\psi)).\end{aligned}$$

For $D > D_0$ ($D_0 > 0$ being defined in Lemma 4.2), we define, with $\varphi = Q_c\psi$,

$$\begin{aligned}B_{Q_c}^{\mathrm{loc}_{\pm1,D}}(\varphi) := &\int_{B(\pm\tilde{d}_c\vec{e_1},D)}(1-\eta)(|\nabla\varphi|^2 - \mathfrak{Re}(ic\partial_{x_2}\varphi\bar{\varphi})\\ &\qquad\qquad - (1-|Q_c|^2)|\varphi|^2 + 2\mathfrak{Re}^2(\overline{Q_c}\varphi))\\ &-\int_{B(\pm\tilde{d}_c\vec{e_1},D)}\nabla\eta\cdot(\mathfrak{Re}(\nabla Q_c\overline{Q_c})|\psi|^2 - 2\mathfrak{Im}(\nabla Q_c\overline{Q_c})\mathfrak{Re}(\psi)\mathfrak{Im}(\psi))\\ &+\int_{B(\pm\tilde{d}_c\vec{e_1},D)} c\partial_{x_2}\eta\mathfrak{Re}(\psi)\mathfrak{Im}(\psi)|Q_c|^2\\ &+\int_{B(\pm\tilde{d}_c\vec{e_1},D)}\eta(|\nabla\psi|^2|Q_c|^2 + 2\mathfrak{Re}^2(\psi)|Q_c|^4)\\ &+\int_{B(\pm\tilde{d}_c\vec{e_1},D)}\eta(4\mathfrak{Im}(\nabla Q_c\overline{Q_c})\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi)\\ &\qquad\qquad + 2c|Q_c|^2\mathfrak{Im}(\partial_{x_2}\psi)\mathfrak{Re}(\psi)).\end{aligned}$$

We infer that this quantity is close enough to $B^{\mathrm{loc}_D}_{\tilde V_{\pm 1}}(\varphi)$ for coercivity to hold, with $\tilde V_{\pm 1}$ being centred at $\pm\tilde d_c \overrightarrow{e_1}$, the zero of Q_c in the right half-plane.

Lemma 4.3. *There exist $R, D_0 > 0$ with $D_0 > R$, such that, for $D > D_0$, $0 < c < c_0(D)$, and $\varphi = Q_c\psi \in C^\infty_c(\mathbb{R}^2\backslash\{\tilde d_c \overrightarrow{e_1}\}, \mathbb{C})$, if the following three orthogonality conditions*

$$\int_{B(\tilde d_c \overrightarrow{e_1}, R)} \mathfrak{Re}(\partial_{x_1}\tilde V_1 \bar\varphi) = \int_{B(\tilde d_c \overrightarrow{e_1}, R)} \mathfrak{Re}(\partial_{x_2}\tilde V_1 \bar\varphi) = \int_{B(\tilde d_c \overrightarrow{e_1}, R)\backslash B(\tilde d_c \overrightarrow{e_1}, R/2)} \mathfrak{Im}(\psi) = 0$$

are satisfied, then

$$B^{\mathrm{loc}_{1,D}}_{Q_c}(\varphi) \geqslant K(D)\|\varphi\|^2_{H^1(B(\tilde d_c \overrightarrow{e_1}, D))}.$$

Proof: First, note that we write $\varphi = Q_c\psi$ and not $\varphi = \tilde V_1\psi$, as we did in the proof of Proposition 1.3. Hence, to apply Lemma 4.2, the third orthogonality condition becomes

$$\int_{B(\tilde d_c \overrightarrow{e_1}, R)\backslash B(\tilde d_c \overrightarrow{e_1}, R/2)} \mathfrak{Im}\left(\psi\frac{Q_c}{\tilde V_1}\right) = 0.$$

With Lemma 2.15, we check that

$$\begin{aligned}\left|\int_{B(\tilde d_c \overrightarrow{e_1}, R)\backslash B(\tilde d_c \overrightarrow{e_1}, R/2)} \mathfrak{Im}\left(\psi\frac{Q_c}{\tilde V_1}\right)\right| &\leqslant \left|\int_{B(\tilde d_c \overrightarrow{e_1}, R)\backslash B(\tilde d_c \overrightarrow{e_1}, R/2)} \mathfrak{Im}(\psi)\right| \\ &\quad + o_{c\to 0}(1)\|\psi\|_{L^2(B(\tilde d_c \overrightarrow{e_1}, R)\backslash B(\tilde d_c \overrightarrow{e_1}, R/2))} \\ &\leqslant \left|\int_{B(\tilde d_c \overrightarrow{e_1}, R)\backslash B(\tilde d_c \overrightarrow{e_1}, R/2)} \mathfrak{Im}(\psi)\right| \\ &\quad + o^D_{c\to 0}(1)\|\varphi\|_{H^1(B(\tilde d_c \overrightarrow{e_1}, D))},\end{aligned}$$

therefore, by a standard coercivity argument, we can change this orthogonality condition, given that c is small enough (depending on D). With equation (4.4), it is therefore enough to show that

$$|B^{\mathrm{loc}_D}_{Q_c}(\varphi) - B^{\mathrm{loc}_D}_{\tilde V_1}(\varphi)| \leqslant o^D_{c\to 0}(1)\|\varphi\|^2_{H^1(B(\tilde d_c \overrightarrow{e_1}, D))}$$

to complete the proof. Thus, for $\varphi = Q_c\psi \in C_c^\infty(\mathbb{R}^2\backslash\{\tilde{d}_c\overrightarrow{e_1}\}, \mathbb{C})$, writing $\varphi = V_1\left(\frac{Q_c}{V_1}\psi\right)$ in $B_{\tilde{V}_1}^{\mathrm{loc}_D}(\varphi)$, we have

$$
\begin{aligned}
&B_{Q_c}^{\mathrm{loc}_{1,D}}(\varphi) - B_{\tilde{V}_1}^{\mathrm{loc}_D}(\varphi)\\
&\quad= \int_{B(\tilde{d}_c\overrightarrow{e_1},D)} -\mathfrak{Re}(ic\partial_{x_2}\varphi\bar{\varphi}) + (|Q_c|^2 - |\tilde{V}_1|^2)|\varphi|^2 + 2(\mathfrak{Re}^2(\overline{Q_c}\varphi) - \mathfrak{Re}^2(\overline{\tilde{V}_1}\varphi))\\
&\qquad - \int_{B(\tilde{d}_c\overrightarrow{e_1},D)} \nabla\eta\cdot(\mathfrak{Re}(\nabla Q_c\overline{Q_c})|\psi|^2 - 2\mathfrak{Im}(\nabla Q_c\overline{Q_c})\mathfrak{Re}(\psi)\mathfrak{Im}(\psi))\\
&\qquad + \int_{B(\tilde{d}_c\overrightarrow{e_1},D)} \nabla\eta\cdot\left(\mathfrak{Re}(\nabla\tilde{V}_1\overline{\tilde{V}_1})\left|\frac{Q_c}{\tilde{V}_1}\psi\right|^2 - 2\mathfrak{Im}(\nabla\tilde{V}_1\overline{\tilde{V}_1})\mathfrak{Re}\left(\frac{Q_c}{\tilde{V}_1}\psi\right)\mathfrak{Im}\left(\frac{Q_c}{\tilde{V}_1}\psi\right)\right)\\
&\qquad + \int_{B(\tilde{d}_c\overrightarrow{e_1},D)} c\partial_{x_2}\eta\mathfrak{Re}(\psi)\mathfrak{Im}(\psi)|Q_c|^2\\
&\qquad + \int_{B(\tilde{d}_c\overrightarrow{e_1},D)} \eta(|\nabla\psi|^2|Q_c|^2 + 2\mathfrak{Re}^2(\psi)|Q_c|^4)\\
&\qquad - \int_{B(\tilde{d}_c\overrightarrow{e_1},D)} \eta\left(\left|\nabla\left(\frac{Q_c}{\tilde{V}_1}\psi\right)\right|^2|Q_c|^2 + 2\mathfrak{Re}^2\left(\frac{Q_c}{\tilde{V}_1}\psi\right)|Q_c|^4\right)\\
&\qquad + \int_{B(\tilde{d}_c\overrightarrow{e_1},D)} \eta(4\mathfrak{Im}(\nabla Q_c\overline{Q_c})\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi) + 2c|Q_c|^2\mathfrak{Im}(\partial_{x_2}\psi)\mathfrak{Re}(\psi))\\
&\qquad - \int_{B(\tilde{d}_c\overrightarrow{e_1},D)} \eta\left(4\mathfrak{Im}(\nabla Q_c\overline{Q_c})\mathfrak{Im}\left(\nabla\left(\frac{Q_c}{\tilde{V}_1}\psi\right)\right)\mathfrak{Re}\left(\frac{Q_c}{\tilde{V}_1}\psi\right)\right).
\end{aligned}
$$

With Theorem 1.1 (for $p = +\infty$) and Cauchy–Schwarz we check easily that

$$
\begin{aligned}
\int_{B(\tilde{d}_c\overrightarrow{e_1},D)} &|\mathfrak{Re}(ic\partial_{x_2}\varphi\bar{\varphi})| + ||Q_c|^2 - |\tilde{V}_1|^2||\varphi|^2 + 2|\mathfrak{Re}^2(\overline{Q_c}\varphi) - \mathfrak{Re}^2(\overline{\tilde{V}_1}\varphi)|\\
&\leqslant o^D_{c\to 0}(1)\|\varphi\|^2_{H^1(B(\tilde{d}_c\overrightarrow{e_1},D))}.
\end{aligned}
$$

Since $\nabla\eta$ is supported in $B(\tilde{d}_c\overrightarrow{e_1},2)\backslash B(\tilde{d}_c\overrightarrow{e_1},1)$, still with Theorem 1.1 (for $p = +\infty$), we check that

$$
\begin{aligned}
&\int_{B(\tilde{d}_c\overrightarrow{e_1},D)} \left|\nabla\eta\cdot\mathfrak{Re}(\nabla Q_c\overline{Q_c})|\psi|^2 - \nabla\eta\mathfrak{Re}(\nabla\tilde{V}_1\overline{\tilde{V}_1})\left|\frac{Q_c}{\tilde{V}_1}\psi\right|^2\right|\\
&\quad\leqslant K\int_{B(\tilde{d}_c\overrightarrow{e_1},D)} \left|\nabla\eta\cdot\mathfrak{Re}(\nabla Q_c\overline{Q_c})|\varphi|^2 - \nabla\eta\mathfrak{Re}(\nabla\tilde{V}_1\overline{\tilde{V}_1})\left|\frac{Q_c}{\tilde{V}_1}\varphi\right|^2\right|\\
&\quad\leqslant \left\|\nabla\eta\cdot\mathfrak{Re}(\nabla Q_c\overline{Q_c}) - \nabla\eta\mathfrak{Re}(\nabla\tilde{V}_1\overline{\tilde{V}_1})\left|\frac{Q_c}{\tilde{V}_1}\right|^2\right\|_{L^\infty((\tilde{d}_c\overrightarrow{e_1},D))}\|\varphi\|^2_{H^1(B(\tilde{d}_c\overrightarrow{e_1},D))}\\
&\quad\leqslant o^D_{c\to 0}(1)\|\varphi\|^2_{H^1(B(\tilde{d}_c\overrightarrow{e_1},D))}.
\end{aligned}
$$

We check similarly that the same estimate holds for all the remaining error terms, using the fact that η is supported in $\mathbb{R}^2\backslash B(\tilde{d}_c\overrightarrow{e_1},1)$. □

Note that, by a density argument (see the proof of Lemma 3.4), Lemma 4.3 holds for any $\varphi \in H^1(B(0,D))$. Now, we want to remove the orthogonality condition on the phase. For that, we have to change the coercivity norm.

Lemma 4.4. *There exist $R, D_0 > 0$ with $D_0 > R$, such that, for $D > D_0$, $0 < c < c_0(D)$, and $\varphi = Q_c\psi \in C_c^\infty(\mathbb{R}^2 \backslash \{\tilde{d}_c\overrightarrow{e_1}\}, \mathbb{C})$, if the following two orthogonality conditions*

$$\int_{B(\tilde{d}_c\overrightarrow{e_1},R)} \mathfrak{Re}(\partial_{x_1}\tilde{V}_1\overline{\tilde{V}_1\psi}) = \int_{B(\tilde{d}_c\overrightarrow{e_1},R)} \mathfrak{Re}(\partial_{x_2}\tilde{V}_1\overline{\tilde{V}_1\psi}) = 0$$

are satisfied, then

$$B_{Q_c}^{\mathrm{loc}_1,D}(\varphi) \geqslant K(D)\int_{B(\tilde{d}_c\overrightarrow{e_1},D)} |\nabla\psi|^2|Q_c|^4 + \mathfrak{Re}^2(\psi)|Q_c|^4.$$

Proof: Take a function $\varphi \in H^1(B(0,D))$ that satisfies the orthogonality conditions

$$\begin{aligned}\int_{B(\tilde{d}_c\overrightarrow{e_1},R)} \mathfrak{Re}(\partial_{x_1}\tilde{V}_1\overline{\tilde{V}_1\psi}) &= \int_{B(\tilde{d}_c\overrightarrow{e_1},R)} \mathfrak{Re}(\partial_{x_2}\tilde{V}_1\overline{\tilde{V}_1\psi}) \\ &= \int_{B(\tilde{d}_c\overrightarrow{e_1},R)\backslash B(\tilde{d}_c\overrightarrow{e_1},R/2)} \mathfrak{Im}(\psi) = 0,\end{aligned}$$

and let us show that $B_{Q_c}^{\mathrm{loc}_1,D}(\varphi) \geqslant K\|\varphi\|^2_{H^1(B(\tilde{d}_c\overrightarrow{e_1},D))}$. Take $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \mathbb{R}$ and we define

$$\tilde{\varphi} = \varphi - \varepsilon_1\partial_{x_1}Q_c - \varepsilon_2\partial_{x_2}Q_c - \varepsilon_3 iQ_c.$$

We have, for $\varphi = Q_c\psi$, by Theorem 1.1 (for $p = +\infty$) and Lemma 2.15,

$$\begin{aligned}&\left|\int_{B(\tilde{d}_c\overrightarrow{e_1},R)} \mathfrak{Re}(\partial_{x_1}\tilde{V}_1\overline{\tilde{V}_1\psi}) - \int_{B(\tilde{d}_c\overrightarrow{e_1},R)} \mathfrak{Re}(\partial_{x_1}Q_c\overline{Q_c\psi})\right| \\ &\qquad\leqslant \left|\int_{B(\tilde{d}_c\overrightarrow{e_1},R)} \mathfrak{Re}\left(\partial_{x_1}\tilde{V}_1\frac{\tilde{V}_1}{Q_c}\bar{\varphi} - \partial_{x_1}Q_c\bar{\varphi}\right)\right| \\ &\qquad\leqslant K\left\|\partial_{x_1}\tilde{V}_1\frac{\tilde{V}_1}{Q_c} - \partial_{x_1}Q_c\right\|_{L^\infty(B(\tilde{d}_c\overrightarrow{e_1},R))}\|\varphi\|_{H^1(B(\tilde{d}_c\overrightarrow{e_1},D))} \\ &\qquad\leqslant o^D_{c\to 0}(1)\|\varphi\|_{H^1(B(\tilde{d}_c\overrightarrow{e_1},D))}.\end{aligned}$$

Similar estimates hold for $\int_{B(\tilde{d}_c\overrightarrow{e_1},R)} \mathfrak{Re}(\partial_{x_2}\tilde{V}_1\overline{\tilde{V}_1\psi})$. By the implicit function theorem, we check that there exist $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \mathbb{R}$ with $|\varepsilon_1| + |\varepsilon_2| + |\varepsilon_3| \leqslant o_{c\to 0}(1)\|\varphi\|_{H^1(B(\tilde{d}_c\overrightarrow{e_1},D))}$ such that $\tilde{\varphi}$ satisfies the three orthogonality conditions of Lemma 4.3. We deduce that, since (by Theorem 1.1 for $p=+\infty$)

$$\|\partial_{x_1} Q_c\|_{H^1(B(\tilde{d}_c\overrightarrow{e_1},D))} + \|\partial_{x_2} Q_c\|_{H^1(B(\tilde{d}_c\overrightarrow{e_1},D))} + \|iQ_c\|_{H^1(B(\tilde{d}_c\overrightarrow{e_1},D))} \leqslant K(D),$$

$$\begin{aligned} B^{\mathrm{loc}_1,D}_{Q_c}(\varphi) &\geqslant B^{\mathrm{loc}_1,D}_{Q_c}(\tilde{\varphi}) - o^D_{c\to 0}(1)\|\varphi\|^2_{H^1(B(\tilde{d}_c\overrightarrow{e_1},D))} \\ &\geqslant K(D)\|\tilde{\varphi}\|^2_{H^1(B(\tilde{d}_c\overrightarrow{e_1},D))} - o^D_{c\to 0}(1)\|\varphi\|^2_{H^1(B(\tilde{d}_c\overrightarrow{e_1},D))} \\ &\geqslant K(D)\|\varphi\|^2_{H^1(B(\tilde{d}_c\overrightarrow{e_1},D))} - o^D_{c\to 0}(1)\|\varphi\|^2_{H^1(B(\tilde{d}_c\overrightarrow{e_1},D))} \\ &\geqslant K(D)\|\varphi\|^2_{H^1(B(\tilde{d}_c\overrightarrow{e_1},D))}, \end{aligned}$$

given that c is small enough (depending on D). For $\varphi = Q_c\psi$, we infer that

$$\int_{B(\tilde{d}_c\overrightarrow{e_1},D)} |\nabla\psi|^2|Q_c|^4 + \mathfrak{Re}^2(\psi)|Q_c|^4 \leqslant K(D)\|\varphi\|^2_{H^1(B(\tilde{d}_c\overrightarrow{e_1},D))}.$$

Indeed, we have

$$\int_{B(\tilde{d}_c\overrightarrow{e_1},D)} \mathfrak{Re}^2(\psi)|Q_c|^4 \leqslant K\int_{B(\tilde{d}_c\overrightarrow{e_1},D)} \mathfrak{Re}^2(\varphi) \leqslant K\|\varphi\|^2_{H^1(B(\tilde{d}_c\overrightarrow{e_1},D))},$$

and

$$\begin{aligned} \int_{B(\tilde{d}_c\overrightarrow{e_1},D)} |\nabla\psi|^2|Q_c|^4 &= \int_{B(\tilde{d}_c\overrightarrow{e_1},D)} |\nabla\varphi - \nabla Q_c\psi|^2|Q_c|^2 \\ &\leqslant K\left(\int_{B(\tilde{d}_c\overrightarrow{e_1},D)} |\nabla\varphi|^2 + \int_{B(\tilde{d}_c\overrightarrow{e_1},D)} |\nabla Q_c\psi|^2|Q_c|^2\right) \\ &\leqslant K\left(\int_{B(\tilde{d}_c\overrightarrow{e_1},D)} |\nabla\varphi|^2 + \int_{B(\tilde{d}_c\overrightarrow{e_1},D)} |\varphi|^2\right). \end{aligned}$$

We deduce that, under the three orthogonality conditions, for $\varphi = Q_c\psi$,

$$\begin{aligned} \int_{B(\tilde{d}_c\overrightarrow{e_1},R)} \mathfrak{Re}(\partial_{x_1}\tilde{V}_1\overline{\tilde{V}_1\psi}) &= \int_{B(\tilde{d}_c\overrightarrow{e_1},R)} \mathfrak{Re}(\partial_{x_2}\tilde{V}_1\overline{\tilde{V}_1\psi}) \\ &= \int_{B(\tilde{d}_c\overrightarrow{e_1},R)\setminus B(\tilde{d}_c\overrightarrow{e_1},R/2)} \mathfrak{Im}(\psi) = 0, \end{aligned}$$

then

$$B^{\mathrm{loc}_1,D}_{Q_c}(\varphi) \geqslant K(D)\int_{B(\tilde{d}_c\overrightarrow{e_1},D)} |\nabla\psi|^2|Q_c|^4 + \mathfrak{Re}^2(\psi)|Q_c|^4.$$

Now, let us show that for any $\lambda\in\mathbb{R}$, $\varphi\in H^1(B(\tilde{d}_c\overrightarrow{e_1},D))$,

$$B^{\mathrm{loc}_1,D}_{Q_c}(\varphi - i\lambda Q_c) = B^{\mathrm{loc}_1,D}_{Q_c}(\varphi).$$

For $\varphi \in C_c^\infty(\mathbb{R}^2, \mathbb{C})$, we have $L_{Q_c}(\varphi - i\lambda Q_c) = L_{Q_c}(\varphi) \in C_c^\infty(\mathbb{R}^2, \mathbb{C})$, thus $\langle L_{Q_c}(\varphi - i\lambda Q_c), \varphi - i\lambda Q_c\rangle$ is well defined, and

$$\begin{aligned}\langle L_{Q_c}(\varphi - i\lambda Q_c), \varphi - i\lambda Q_c\rangle &= \langle L_{Q_c}(\varphi), \varphi - i\lambda Q_c\rangle \\ &= \langle \varphi, L_{Q_c}(\varphi - i\lambda Q_c)\rangle = \langle L_{Q_c}(\varphi), \varphi\rangle.\end{aligned}$$

With computations similar to that of the proof of Lemma 4.1 and by density, using $\nabla(\psi - i\lambda) = \nabla\psi$ and $\mathfrak{Re}(\psi - i\lambda) = \mathfrak{Re}(\psi)$, we deduce that $B_{Q_c}^{\mathrm{loc}_{1,D}}(\varphi - i\lambda Q_c) = B_{Q_c}^{\mathrm{loc}_{1,D}}(\varphi)$.

Now, for $\lambda \in \mathbb{R}$, $\tilde{\varphi} = \varphi - i\lambda Q_c$, $\tilde{\psi} = \psi - i\lambda$, $\tilde{\varphi} = Q_c\tilde{\psi}$, we have $B_{Q_c}^{\mathrm{loc}_{1,D}}(\varphi) = B_{Q_c}^{\mathrm{loc}_{1,D}}(\tilde{\varphi})$,

$$\int_{B(\tilde{d}_c\vec{e_1}, D)} |\nabla\psi|^2|Q_c|^4 + \mathfrak{Re}^2(\psi)|Q_c|^4 = \int_{B(\tilde{d}_c\vec{e_1}, D)} |\nabla\tilde{\psi}|^2|Q_c|^4 + \mathfrak{Re}^2(\tilde{\psi})|Q_c|^4$$

and

$$\int_{B(\tilde{d}_c\vec{e_1}, R)} \mathfrak{Re}(\nabla\tilde{V}_1\overline{\tilde{V}_1\psi}) = \int_{B(\tilde{d}_c\vec{e_1}, R)} \mathfrak{Re}(\nabla\tilde{V}_1\overline{\tilde{V}_1\tilde{\psi}}).$$

For this last equality, it comes from the fact that $\int_{B(\tilde{d}_c\vec{e_1}, R)} \mathfrak{Re}(i\nabla\tilde{V}_1\overline{\tilde{V}_1}) = 0$, since $\mathfrak{Re}(i\nabla\tilde{V}_1\overline{\tilde{V}_1})$ has no 0-harmonic (see Lemma 2.1). We also check that

$$\int_{B(\tilde{d}_c\vec{e_1}, R)\setminus B(\tilde{d}_c\vec{e_1}, R/2)} \mathfrak{Im}(\psi) = \int_{B(\tilde{d}_c\vec{e_1}, R)\setminus B(\tilde{d}_c\vec{e_1}, R/2)} \mathfrak{Im}(\tilde{\psi}) + K\lambda$$

for a universal constant $K > 0$. Therefore, choosing $\lambda \in \mathbb{R}$ such that $\int_{B(\tilde{d}_c\vec{e_1}, R)\setminus B(\tilde{d}_c\vec{e_1}, R/2)} \mathfrak{Im}(\tilde{\psi}) = 0$, we have, for a function $\varphi = Q_c\psi$ that satisfies

$$\int_{B(\tilde{d}_c\vec{e_1}, R)} \mathfrak{Re}(\partial_{x_1}\tilde{V}_1\overline{\tilde{V}_1\psi}) = \int_{B(\tilde{d}_c\vec{e_1}, R)} \mathfrak{Re}(\partial_{x_2}\tilde{V}_1\overline{\tilde{V}_1\psi}) = 0,$$

that

$$\begin{aligned}B_{Q_c}^{\mathrm{loc}_{1,D}}(\varphi) &= B_{Q_c}^{\mathrm{loc}_{1,D}}(\tilde{\varphi}) \\ &\geqslant \int_{B(\tilde{d}_c\vec{e_1}, D)} |\nabla\tilde{\psi}|^2|Q_c|^4 + \mathfrak{Re}^2(\tilde{\psi})|Q_c|^4 \\ &= \int_{B(\tilde{d}_c\vec{e_1}, D)} |\nabla\psi|^2|Q_c|^4 + \mathfrak{Re}^2(\psi)|Q_c|^4.\end{aligned}$$

This concludes the proof of this lemma. □

4.4. Proof of Proposition 1.4. From Lemma 4.1, we have, for $\varphi = Q_c\psi \in C_c^\infty(\mathbb{R}^2 \backslash \{\tilde{d}_c\overrightarrow{e_1}, -\tilde{d}_c\overrightarrow{e_1}\}, \mathbb{C})$ that

$$\begin{aligned} B_{Q_c}(\varphi) = & \int_{\mathbb{R}^2} (1-\eta)(|\nabla\varphi|^2 - \mathfrak{Re}(ic\partial_{x_2}\varphi\bar{\varphi}) - (1-|Q_c|^2)|\varphi|^2 + 2\mathfrak{Re}^2(\overline{Q_c}\varphi)) \\ & - \int_{\mathbb{R}^2} \nabla\eta\cdot(\mathfrak{Re}(\nabla Q_c\overline{Q_c})|\psi|^2 - 2\mathfrak{Im}(\nabla Q_c\overline{Q_c})\mathfrak{Re}(\psi)\mathfrak{Im}(\psi)) \\ & + \int_{\mathbb{R}^2} c\partial_{x_2}\eta\mathfrak{Re}(\psi)\mathfrak{Im}(\psi)|Q_c|^2 \\ & + \int_{\mathbb{R}^2} \eta(|\nabla\psi|^2|Q_c|^2 + 2\mathfrak{Re}^2(\psi)|Q_c|^4) \\ & + \int_{\mathbb{R}^2} \eta(4\mathfrak{Im}(\nabla Q_c\overline{Q_c})\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi) + 2c|Q_c|^2\mathfrak{Im}(\partial_{x_2}\psi)\mathfrak{Re}(\psi)). \end{aligned}$$

We decompose the integral in three domains, $B(\pm\tilde{d}_c\overrightarrow{e_1}, D)$ (which yield $B_{Q_c}^{\mathrm{loc}_{\pm1,D}}(\varphi)$) and $\mathbb{R}^2\backslash(B(\tilde{d}_c\overrightarrow{e_1}, D)\cup B(-\tilde{d}_c\overrightarrow{e_1}, D))$ for some $D > D_0 > 0$, where D_0 is defined in Lemma 4.3.

Then, with the four orthogonality conditions and Lemma 4.3, we check that

$$B_{Q_c}^{\mathrm{loc}_{1,D}}(\varphi) \geqslant K(D)\int_{B(\tilde{d}_c\overrightarrow{e_1}, D)} |\nabla\psi|^2|Q_c|^4 + \mathfrak{Re}^2(\psi)|Q_c|^4,$$

and, by symmetry of the problem around $B(\pm\tilde{d}_c\overrightarrow{e_1}, D)$, since $Q_c = -V_{-1}(\cdot + \tilde{d}_c\vec{e}_1) + o_{c\to0}(1)$ in $L^\infty(B(-\tilde{d}_c\overrightarrow{e_1}, D))$, and checking that multiplying the vortex by -1 does not change the result, that

$$B_{Q_c}^{\mathrm{loc}_{-1,D}}(\varphi) \geqslant K(D)\int_{B(-\tilde{d}_c\overrightarrow{e_1}, D)} |\nabla\psi|^2|Q_c|^4 + \mathfrak{Re}^2(\psi)|Q_c|^4.$$

Furthermore, there exist $K_1, K_2 > 0$, universal constants, such that, outside of $B(\widetilde{d_c}\overrightarrow{e_1}, 1)\cup B(-\widetilde{d_c}\overrightarrow{e_1}, 1)$ for c small enough, we have

$$K_1 \geqslant |Q_c|^2 \geqslant K_2$$

by (2.12). We also have

$$|\mathfrak{Im}(\nabla Q_c\overline{Q_c})| \leqslant K\left(\frac{1}{(1+\tilde{r}_1)} + \frac{1}{(1+\tilde{r}_{-1})}\right)$$

by (2.10). With these estimates and by Cauchy–Schwarz, for $D > D_0$,

$$\begin{aligned} & \int_{\mathbb{R}^2\backslash(B(\tilde{d}_c\overrightarrow{e_1}, D)\cup B(-\tilde{d}_c\overrightarrow{e_1}, D))} 2c|Q_c|^2\mathfrak{Im}(\partial_{x_2}\psi)\mathfrak{Re}(\psi) \\ & \qquad \geqslant -Kc\int_{\mathbb{R}^2\backslash(B(\tilde{d}_c\overrightarrow{e_1}, D)\cup B(-\tilde{d}_c\overrightarrow{e_1}, D))} |\nabla\psi|^2|Q_c|^4 + \mathfrak{Re}^2(\psi)|Q_c|^4, \end{aligned}$$

and

$$\begin{aligned} & \int_{\mathbb{R}^2\backslash(B(\tilde{d}_c\overrightarrow{e_1}, D)\cup B(-\tilde{d}_c\overrightarrow{e_1}, D))} 4\mathfrak{Im}(\nabla Q_c\overline{Q_c})\cdot\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi) \\ & \qquad \geqslant \frac{-K}{(1+D)}\int_{\mathbb{R}^2\backslash(B(\tilde{d}_c\overrightarrow{e_1}, D)\cup B(-\tilde{d}_c\overrightarrow{e_1}, D))} |\nabla\psi|^2|Q_c|^4 + \mathfrak{Re}^2(\psi)|Q_c|^4. \end{aligned}$$

Therefore, taking $D > D_0$ large enough (independently of c or c_0, $D \geqslant 10K+1$) and c small enough ($c \leqslant \frac{10}{K}$), we have

$$\int_{\mathbb{R}^2\backslash(B(\tilde{d}_c\vec{e_1},D)\cup B(-\tilde{d}_c\vec{e_1},D))} |\nabla\psi|^2|Q_c|^2 + 2\mathfrak{Re}^2(\psi)|Q_c|^4 + \int_{\mathbb{R}^2\backslash(B(\tilde{d}_c\vec{e_1},D)\cup B(-\tilde{d}_c\vec{e_1},D))} 4\mathfrak{Im}(\nabla Q_c\overline{Q_c})\cdot\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi) + 2c|Q_c|^2\mathfrak{Im}(\partial_{x_2}\psi)\mathfrak{Re}(\psi)$$

$$\geqslant K\int_{\mathbb{R}^2\backslash(B(\tilde{d}_c\vec{e_1},D)\cup B(-\tilde{d}_c\vec{e_1},D))} |\nabla\psi|^2|Q_c|^4 + \mathfrak{Re}^2(\psi)|Q_c|^4.$$

We deduce that, for $\varphi = Q_c\psi \in C_c^\infty(\mathbb{R}^2\backslash\{\widetilde{d_c}\vec{e_1}, -\widetilde{d_c}\vec{e_1}\}, \mathbb{C})$,

$$B_{Q_c}(\varphi) \geqslant K\|\varphi\|_{\mathcal{C}}^2$$

if

$$\int_{B(\tilde{d}_c\vec{e_1},R)} \mathfrak{Re}(\partial_{x_1}\tilde{V}_1\overline{\widetilde{V_1}\psi}) = \int_{B(\tilde{d}_c\vec{e_1},R)} \mathfrak{Re}(\partial_{x_2}\widetilde{V_1}\overline{\widetilde{V_1}\psi}) = 0,$$
$$\int_{B(-\tilde{d}_c\vec{e_1},R)} \mathfrak{Re}(\partial_{x_1}\tilde{V}_{-1}\overline{\tilde{V}_{-1}\psi}) = \int_{B(-\tilde{d}_c\vec{e_1},R)} \mathfrak{Re}(\partial_{x_2}\tilde{V}_{-1}\overline{\tilde{V}_{-1}\psi}) = 0.$$

We argue by density to show this result in H_{Q_c}. From Lemma 3.1, we know that $\|\cdot\|_{\mathcal{C}}$ is continuous with respect to $\|\cdot\|_{H_{Q_c}}$. Furthermore, we recall from Lemma 3.2 that

$$\int_{B(\tilde{d}_c\vec{e_1},R)} |\mathfrak{Re}(\partial_{x_1}\widetilde{V_1}\overline{\widetilde{V_1}\psi})| \leqslant K(c)\|\varphi\|_{H_{Q_c}},$$

and similar estimates hold for

$$\int_{B(\tilde{d}_c\vec{e_1},R)} \mathfrak{Re}(\partial_{x_2}\widetilde{V_1}\overline{\widetilde{V_1}\psi}), \int_{B(-\widetilde{d_c}\vec{e_1},R)} \mathfrak{Re}(\partial_{x_1}\widetilde{V}_{-1}\overline{\widetilde{V}_{-1}\psi})$$

and

$$\int_{B(-\tilde{d}_c\vec{e_1},R)} \mathfrak{Re}(\partial_{x_2}\tilde{V}_{-1}\overline{\tilde{V}_{-1}\psi}).$$

In particular, we check that these quantities are continuous for the norm $\|\cdot\|_{H_{Q_c}}$, and that we can pass to the limit by density in these quantities by Lemma 3.4.

We are left with the passage to the limit for the quadratic form. For $\varphi \in H_{Q_c}$, we recall from (1.3) that

$$\begin{aligned} B_{Q_c}(\varphi) = \int_{\mathbb{R}^2} & |\nabla\varphi|^2 - (1-|Q_c|^2)|\varphi|^2 + 2\mathfrak{Re}^2(\overline{Q_c}\varphi) \\ & + c\int_{\mathbb{R}^2} (1-\eta)\mathfrak{Re}(i\partial_{x_2}\varphi\bar{\varphi}) + c\int_{\mathbb{R}^2} \eta\mathfrak{Re}(i\partial_{x_2}Q_c\overline{Q_c})|\psi|^2 \\ & - 2c\int_{\mathbb{R}^2} \eta\mathfrak{Re}\psi\mathfrak{Im}\partial_{x_2}\psi|Q_c|^2 - c\int_{\mathbb{R}^2} \partial_{x_2}\eta\mathfrak{Re}\psi\mathfrak{Im}\psi|Q_c|^2 \\ & - c\int_{\mathbb{R}^2} \eta\mathfrak{Re}\psi\mathfrak{Im}\psi\partial_{x_2}(|Q_c|^2). \end{aligned}$$

Following the proof of Lemma 3.3, we check easily that, for $\varphi_1 = Q_c\psi_1$, $\varphi_2 = Q_c\psi_2 \in H_{Q_c}$, we have

$$\begin{aligned} \int_{\mathbb{R}^2} & |\nabla\varphi_1\overline{\nabla\varphi_2}| + |(1-|Q_c|^2)\varphi_1\overline{\varphi_2}| + |\mathfrak{Re}(\overline{Q_c}\varphi_1)\mathfrak{Re}(\overline{Q_c}\varphi_2)| \\ & + \int_{\mathbb{R}^2} (1-\eta)|\mathfrak{Re}(i\partial_{x_2}\varphi_1\overline{\varphi_2})| + \int_{\mathbb{R}^2} \eta|\mathfrak{Re}(i\partial_{x_2}Q_c\overline{Q_c})||\psi_1\psi_2| \\ & + \int_{\mathbb{R}^2} \eta|\mathfrak{Re}\psi_1\mathfrak{Im}\partial_{x_2}\psi_2||Q_c|^2 + \int_{\mathbb{R}^2} |\partial_{x_2}\eta\mathfrak{Re}\psi_1\mathfrak{Im}\psi_2||Q_c|^2 \\ & + \int_{\mathbb{R}^2} \eta|\mathfrak{Re}\psi_1\mathfrak{Im}\psi_2\partial_{x_2}(|Q_c|^2)| \\ & \leqslant K(c)\|\varphi_1\|_{H_{Q_c}}\|\varphi_2\|_{H_{Q_c}}, \end{aligned}$$

and thus we can pass to the limit in B_{Q_c} by Lemma 3.4. This concludes the proof of Proposition 1.4. □

5. Proof of Theorem 1.5 and its corollaries

5.1. Link between the sets of orthogonality conditions. The first goal of this subsection is to show that the four particular directions $(\partial_{x_1}Q_c, \partial_{x_2}Q_c, c^2\partial_c Q_c, c\partial_{c^\perp}Q_c)$ are almost orthogonal to each other near the zeros of Q_c, and that they can replace the four orthogonality conditions of Proposition 1.4. This is computed in the following lemma.

Lemma 5.1. *For $R > 0$ given by Proposition 1.4, there exist $K_1, K_2 > 0$, two constants independent of c, such that, for Q_c defined in Theorem 1.1,*

$$\begin{aligned} K_1 \leqslant & \int_{B(\pm\tilde{d}_c\vec{e_1},R)} |\partial_{x_1}Q_c|^2 + \int_{B(\pm\tilde{d}_c\vec{e_1},R)} |\partial_{x_2}Q_c|^2 \\ & + \int_{B(\pm\tilde{d}_c\vec{e_1},R)} |c^2\partial_c Q_c|^2 + \int_{B(\pm\widetilde{d_c}\vec{e_1},R)} |c\partial_{c^\perp}Q_c|^2 \leqslant K_2. \end{aligned}$$

Furthermore, for $A, B \in \{\partial_{x_1} Q_c, \partial_{x_2} Q_c, c^2 \partial_c Q_c, c\partial_{c^\perp} Q_c\}$, $A \neq B$, *we have that, for* $1 > \beta_0 > 0$ *a small constant,*

$$\int_{B(\tilde{d}_c \overrightarrow{e_1}, R) \cup B(-\tilde{d}_c \overrightarrow{e_1}, R)} \mathfrak{Re}(A\bar{B}) = o_{c\to 0}(c^{\beta_0}).$$

Proof: From Lemma 2.2, we have, in $B(\pm\widetilde{d_c}\overrightarrow{e_1}, R)$, that (for $0 < \sigma = 1 - \beta_0 < 1$)

$$Q_c(x) = V_1(x - d_c\overrightarrow{e_1})V_{-1}(x + d_c\overrightarrow{e_1}) + o_{c\to 0}(c^{\beta_0})$$

and

$$\nabla Q_c(x) = \nabla(V_1(x - d_c\overrightarrow{e_1})V_{-1}(x + d_c\overrightarrow{e_1})) + o_{c\to 0}(c^{\beta_0}).$$

In this proof a $o_{c\to 0}(c^{\beta_0})$ may depend on R, but we consider R as a universal constant. From Lemmas 2.1 and 2.13 and equation (2.7), we show that, by the mean value theorem, in $B(\pm\widetilde{d_c}\overrightarrow{e_1}, R)$,

$$Q_c = V_1 V_{-1} + o_{c\to 0}(c^{\beta_0}) = V_{\pm 1} + o_{c\to 0}(c^{\beta_0}) = \tilde{V}_{\pm 1} + o_{c\to 0}(c^{\beta_0})$$

and, similarly,

$$\nabla Q_c = \nabla \tilde{V}_{\pm 1} + o_{c\to 0}(c^{\beta_0}).$$

Thus, in $B(\pm\widetilde{d_c}\overrightarrow{e_1}, R)$, we have

$$\partial_{x_1} Q_c = \partial_{x_1} \tilde{V}_{\pm 1} + o_{c\to 0}(c^{\beta_0}) \tag{5.1}$$

and

$$\partial_{x_2} Q_c = \partial_{x_2} \tilde{V}_{\pm 1} + o_{c\to 0}(c^{\beta_0}). \tag{5.2}$$

Furthermore, by Lemma 2.3, we have in particular that in $B(\pm\widetilde{d_c}\overrightarrow{e_1}, R)$,

$$c^2 \partial_c Q_c = (1 + o_{c\to 0}(c^{\beta_0}))\partial_d(V_1(x - d\overrightarrow{e_1})V_{-1}(x + d\overrightarrow{e_1}))_{|d=d_c} + o_{c\to 0}(c^{\beta_0}).$$

Thus, in $B(\pm\widetilde{d_c}\overrightarrow{e_1}, R)$, with Lemmas 2.1 and 2.13, we estimate

$$c^2 \partial_c Q_c = \mp \partial_{x_1} \tilde{V}_{\pm 1} + o_{c\to 0}(c^{\beta_0}). \tag{5.3}$$

Finally, from Lemma 2.7, we have

$$c\partial_{c^\perp} Q_c = -cx^\perp \cdot \nabla Q_c$$

with $x^\perp = (-x_2, x_1)$. In $B(\pm\widetilde{d_c}\overrightarrow{e_1}, R)$, we have, since $c\widetilde{d_c} = 1 + o_{c\to 0}(c^{\beta_0})$ and using Lemma 2.13,

$$cx^\perp = \mp\overrightarrow{e_2} + o_{c\to 0}(c^{\beta_0}).$$

Therefore, in $B(\pm\widetilde{d_c}\overrightarrow{e_1}, R)$, we have

$$c\partial_{c^\perp} Q_c = \pm \partial_{x_2} \tilde{V}_{\pm 1} + o_{c\to 0}(c^{\beta_0}). \tag{5.4}$$

Now, from Lemma 2.1, we have

$$K_1 \leqslant \int_{B(\pm\tilde{d}_c\overrightarrow{e_1}, R)} |\partial_{x_1} \tilde{V}_{\pm 1}|^2 + \int_{B(\pm\tilde{d}_c\overrightarrow{e_1}, R)} |\partial_{x_2} \tilde{V}_{\pm 1}|^2 \leqslant K_2 \tag{5.5}$$

for universal constant $K_1, K_2 > 0$ (depending only on R). By a change of variable, we have, writing $\tilde{V}_{\pm 1} = \rho(\tilde{r}_{\pm 1})e^{i\tilde{\theta}_{\pm 1}}$ (with the notations of Lemma 2.1),

$$\partial_{x_1}\tilde{V}_{\pm 1} = \left(\cos(\tilde{\theta}_{\pm 1})\frac{\rho'(\tilde{r}_{\pm 1})}{\rho(\tilde{r}_{\pm 1})} - \frac{\pm i}{\tilde{r}_{\pm 1}}\sin(\tilde{\theta}_{\pm 1})\right)\tilde{V}_{\pm 1} \tag{5.6}$$

and

$$\partial_{x_2}\tilde{V}_{\pm 1} = \left(\sin(\tilde{\theta}_{\pm 1})\frac{\rho'(\tilde{r}_{\pm 1})}{\rho(\tilde{r}_{\pm 1})} + \frac{\pm i}{\tilde{r}_{\pm 1}}\cos(\tilde{\theta}_{\pm 1})\right)\tilde{V}_{\pm 1}.$$

Since

$$\mathfrak{Re}(\partial_{x_1}\tilde{V}_{\pm 1}\overline{\partial_{x_2}\tilde{V}_{\pm 1}}) = 2\cos(\tilde{\theta}_{\pm 1})\sin(\tilde{\theta}_{\pm 1})\frac{\rho'(\tilde{r}_{\pm 1})}{\tilde{r}_{\pm 1}\rho(\tilde{r}_{\pm 1})}|\tilde{V}_{\pm 1}|^2,$$

by integration in polar coordinates, we have

$$\int_{B(\pm\tilde{d}_c\overrightarrow{e_1},R)} \mathfrak{Re}(\partial_{x_1}\tilde{V}_{\pm 1}\overline{\partial_{x_2}\tilde{V}_{\pm 1}}) = 0. \tag{5.7}$$

Combining (5.1) to (5.4) with (5.5) and (5.7), we can do every estimate stated in the lemma. □

With (5.1) to (5.4), we check that these four directions are close to the ones in the orthogonality conditions of Proposition 1.4. This will appear in the proof of Lemma 5.5. Now, we give a way to develop the quadratic form for some particular functions.

Lemma 5.2. *For* $\varphi \in C_c^\infty(\mathbb{R}^2\backslash\{\widetilde{d_c}\overrightarrow{e_1}, -\widetilde{d_c}\overrightarrow{e_1}\}, \mathbb{C})$ *and* $A \in \operatorname{Span}\{\partial_{x_1}Q_c, \partial_{x_2}Q_c, \partial_c Q_c, \partial_{c^\perp}Q_c\}$*, we have*

$$\langle L_{Q_c}(\varphi + A), \varphi + A\rangle = \langle L_{Q_c}(\varphi), \varphi\rangle + \langle 2L_{Q_c}(A), \varphi\rangle + \langle L_{Q_c}(A), A\rangle.$$

Furthermore, $\langle L_{Q_c}(\varphi + A), \varphi + A\rangle = B_{Q_c}(\varphi + A)$ *and* $\langle L_{Q_c}(A), A\rangle = B_{Q_c}(A)$.

Proof: Since $\varphi \in C_c^\infty(\mathbb{R}^2\backslash\{\widetilde{d_c}\overrightarrow{e_1}, -\widetilde{d_c}\overrightarrow{e_1}\}, \mathbb{C})$, it is enough to check that $\mathfrak{Re}(L_{Q_c}(A)\bar{A}) \in L^1(\mathbb{R}^2, \mathbb{R})$ for $A \in \operatorname{Span}\{\partial_{x_1}Q_c, \partial_{x_2}Q_c, \partial_c Q_c, \partial_{c^\perp}Q_c\}$ to show that

$$\langle L_{Q_c}(\varphi + A), \varphi + A\rangle = \langle L_{Q_c}(\varphi), \varphi\rangle + \langle 2L_{Q_c}(A), \varphi\rangle + \langle L_{Q_c}(A), A\rangle.$$

From Lemma 2.8, we have, for $A = \mu_1\partial_{x_1}Q_c + \mu_2\partial_{x_2}Q_c + \mu_3\partial_c Q_c + \mu_4\partial_{c^\perp}Q_c$, that

$$L_{Q_c}(A) = \mu_3 i\partial_{x_2}Q_c - \mu_4 i\partial_{x_1}Q_c.$$

Now, with (2.15) (that holds also for A by linearity) and (2.9), (2.10), we check easily that $\mathfrak{Re}(L_{Q_c}(A)\bar{A}) \in L^1(\mathbb{R}^2, \mathbb{R})$.

Now, from Subsection 3.3, to show that for $\Phi = Q_c\Psi \in H_{Q_c} \cap C^2(\mathbb{R}^2,\mathbb{C})$, we have $\langle L_{Q_c}(\Phi),\Phi\rangle = B_{Q_c}(\Phi)$, it is enough to show that the integral $\int_{\mathbb{R}^2} \partial_{x_2}(\eta \mathfrak{Re}\Psi \mathfrak{Im}\Psi |Q_c|^2)$ is well defined and is 0. For $\Phi = A$ or $\Phi = \varphi + A$, this is a consequence of (2.15), Lemma 2.15, and $\varphi \in C_c^\infty(\mathbb{R}^2 \backslash \{\widetilde{d_c}\overrightarrow{e_1}, -\widetilde{d_c}\overrightarrow{e_1}\},\mathbb{C})$. □

5.2. Some useful elliptic estimates. We want to improve slightly the coercivity norm near the zeros of Q_c. This is done in the following lemma. The improvement is in the exponent of the weight in front of f^2.

Lemma 5.3. *There exists a universal constant $K > 0$ such that, for any $D > 2$, for V_1 centred at 0, and any function $f \in C_c^\infty(\mathbb{R}^2\backslash\{0\},\mathbb{R})$, we have*

$$\int_{B(0,D)} f^2|V_1|^3\,dx \leqslant K \int_{B(0,D)} |\nabla f|^2|V_1|^4 + f^2|V_1|^4\,dx.$$

In particular, this implies that, for $\psi \in C_c^\infty(\mathbb{R}^2\backslash\{0\},\mathbb{C})$,

$$\int_{B(0,D)} \mathfrak{Re}^2(\psi)|V_1|^3\,dx \leqslant K \int_{B(0,D)} |\nabla \psi|^2|V_1|^4 + \mathfrak{Re}^2(\psi)|V_1|^4\,dx.$$

This lemma, with Lemmas 2.15 and 3.4, implies that, for $\varphi = Q_c\psi \in H_{Q_c}$,

$$\int_{\mathbb{R}^2} \mathfrak{Re}^2(\psi)|Q_c|^3 \leqslant K\|\varphi\|_{\mathcal{C}}^2.$$

Proof: Since $|V_1| \geqslant K > 0$ outside of $B(0,1)$, we take χ a smooth radial non-negative cutoff with value 0 in $B(0,1)$ and value 1 outside $B(0,3/2)$. We have

$$\int_{B(0,D)} \chi f^2|V_1|^3\,dx \leqslant K \int_{B(0,D)} \chi f^2|V_1|^4\,dx \leqslant K \int_{B(0,D)} f^2|V_1|^4\,dx.$$

In $B(0,2)$, from Lemma 2.1, there exist $K_1, K_2 > 0$ such that $K_1 \geqslant \frac{|V_1|}{r} \geqslant K_2$, and thus

$$\int_{B(0,D)} (1-\chi) f^2|V_1|^3\,dx \leqslant K \left(\int_0^{2\pi} \int_0^2 (1-\chi(r)) f^2(x) r^4\,dr\right) d\theta.$$

For $g \in C_c^\infty(\mathbb{R}\backslash\{0\},\mathbb{R})$, we have

$$\begin{aligned}\int_0^2 (1-\chi(r))g^2(r)r^4\,dr &= \frac{-1}{5}\int_0^2 \partial_r((1-\chi)g^2)r^5\,dr \\ &= \frac{-2}{5}\int_0^2 (1-\chi(r))\partial_r g(r) g(r) r^5\,dr + \frac{1}{4}\int_0^2 \chi'(r)g^2(r)r^5\,dr,\end{aligned}$$

and since $\chi'(r) \neq 0$ only for $r \in [1,2]$, we have

$$\int_0^2 |\chi'(r)|g^2(r)r^5\,dr \leqslant K \int_0^2 g^2(r)r^4\,dr,$$

and, by Cauchy–Schwarz,

$$\int_0^2 (1-\chi(r))|\partial_r g(r) g(r)| r^5\, dr \leqslant \sqrt{\int_0^2 (\partial_r g)^2 r^5\, dr \int_0^2 g^2(r) r^5\, dr}.$$

We deduce that

$$\int_0^2 (1-\chi(r)) g^2(r) r^4\, dr \leqslant K\left(\int_0^2 (\partial_r g)^2 r^5\, dr + \int_0^2 g^2(r) r^5\, dr\right),$$

and taking, for any $\theta \in [0, 2\pi]$, $g(r) = f(r\cos(\theta), r\sin(\theta))$, and since $r \leqslant K|V_1|$ in $B(0,2)$ (by Lemma 2.1), by integration with respect to θ, we conclude that

$$\int_{B(0,D)} (1-\chi) f^2 |V_1|^3\, dx \leqslant K \int_{B(0,D)} |\nabla f|^2 |V_1|^4 + f^2 |V_1|^4\, dx,$$

which ends the proof of this lemma. □

We estimate here some quantities with the coercivity norm. These computations will be useful later on.

Lemma 5.4. *There exists $K > 0$, a universal constant independent of c, such that, if c is small enough, for Q_c defined in Theorem 1.1, for $\varphi = Q_c \psi \in C_c^\infty(\mathbb{R}^2 \backslash \{\widetilde{d_c}\overrightarrow{e_1}, -\widetilde{d_c}\overrightarrow{e_1}\}, \mathbb{C})$, we have*

$$\left|\int_{\mathbb{R}^2} \mathfrak{Re}(\psi)\mathfrak{Im}(\nabla Q_c \overline{Q_c})\right| \leqslant K \ln\left(\frac{1}{c}\right) \|\varphi\|_{\mathcal{C}}$$

and

$$\left|\int_{\mathbb{R}^2} \mathfrak{Im}(\psi)\mathfrak{Re}(\nabla Q_c \overline{Q_c})\right| \leqslant K \|\varphi\|_{\mathcal{C}}.$$

Proof: By Cauchy–Schwarz and Lemmas 2.12 (with a slight modification near the zeros of Q_c that does not change the result) and 5.3,

$$\begin{aligned}\left|\int_{\mathbb{R}^2} \mathfrak{Re}(\psi)\mathfrak{Im}(\nabla Q_c \overline{Q_c})\right| &\leqslant \sqrt{\int_{\mathbb{R}^2} \mathfrak{Re}^2(\psi)|Q_c|^3 \int_{\mathbb{R}^2} \frac{|\mathfrak{Im}(\nabla Q_c \overline{Q_c})|^2}{|Q_c|^3}} \\ &\leqslant K \ln\left(\frac{1}{c}\right)\sqrt{\int_{\mathbb{R}^2} \mathfrak{Re}^2(\psi)|Q_c|^3} \\ &\leqslant K \ln\left(\frac{1}{c}\right) \|\varphi\|_{\mathcal{C}}.\end{aligned}$$

We now focus on the second estimate. We take χ a smooth function with value 1 outside of $\{\tilde{r} \geqslant 2\}$ and 0 inside $\{\tilde{r} \leqslant 1\}$, and that is radial around $\pm\tilde{d}_c\overrightarrow{e_1}$ in $B(\pm\tilde{d}_c\overrightarrow{e_1}, 2)$. We note that

$$\mathfrak{Re}(\nabla Q_c \overline{Q_c}) = \frac{1}{2}\nabla(|Q_c|^2) = \frac{1}{2}\nabla(\chi(|Q_c|^2 - 1) + (1-\chi)|Q_c|^2) + \frac{1}{2}\nabla\chi,$$

thus, by integration by parts, we have

$$\begin{aligned}\int_{\mathbb{R}^2}\mathfrak{Im}(\psi)\mathfrak{Re}(\nabla Q_c\overline{Q_c}) &= \frac{1}{2}\int_{\mathbb{R}^2}\mathfrak{Im}(\psi)\nabla(\chi(|Q_c|^2-1)+(1-\chi)|Q_c|^2)\\ &\quad+\frac{1}{2}\int_{\mathbb{R}^2}\nabla\chi\mathfrak{Im}(\psi)\\ &= \frac{-1}{2}\int_{\mathbb{R}^2}\mathfrak{Im}(\nabla\psi)\chi(|Q_c|^2-1)-\frac{1}{2}\int_{\mathbb{R}^2}\mathfrak{Im}(\nabla\psi)(1-\chi)|Q_c|^2\\ &\quad+\frac{1}{2}\int_{\mathbb{R}^2}\nabla\chi\mathfrak{Im}(\psi),\end{aligned}$$

and, since χ is radial around $\pm\tilde{d}_c\overrightarrow{e_1}$ in $B(\pm\tilde{d}_c\overrightarrow{e_1},2)$,

$$\int_{\mathbb{R}^2}\mathfrak{Im}(\psi)\nabla\chi=\int_{B(\tilde{d}_c\overrightarrow{e_1},2)\cup B(-\tilde{d}_c\overrightarrow{e_1},2)}\mathfrak{Im}(\psi^{\neq 0})\nabla\chi.$$

Since $\nabla\chi$ is supported in

$$(B(\tilde{d}_c\overrightarrow{e_1},2)\cup B(-\tilde{d}_c\overrightarrow{e_1},2))\backslash(B(\tilde{d}_c\overrightarrow{e_1},1)\cup B(-\tilde{d}_c\overrightarrow{e_1},1)),$$

by equations (2.12) and (2.22) and Cauchy–Schwarz,

$$\left|\int_{B(\tilde{d}_c\overrightarrow{e_1},2)\cup B(-\tilde{d}_c\overrightarrow{e_1},2)}\mathfrak{Im}(\psi^{\neq 0})\nabla\chi\right|\leqslant K\sqrt{\int_{\mathbb{R}^2}|\nabla\psi|^2|Q_c|^4}.$$

Now, by Cauchy–Schwarz, we check that

$$\left|\int_{\mathbb{R}^2}\mathfrak{Im}(\nabla\psi)(1-\chi)|Q_c|^2\right|\leqslant K\sqrt{\int_{\mathbb{R}^2}|\nabla\psi|^2|Q_c|^4\int_{\mathbb{R}^2}(1-\chi)^2}\leqslant K\sqrt{\int_{\mathbb{R}^2}|\nabla\psi|^2|Q_c|^4}.$$

Furthermore, we check that (χ being supported in $\{\tilde{r}\geqslant 1\}$)

$$\begin{aligned}\left|\int_{\mathbb{R}^2}\mathfrak{Im}(\nabla\psi)\chi(|Q_c|^2-1)\right| &\leqslant \sqrt{\int_{\mathbb{R}^2}|\nabla\psi|^2\chi^2\int_{\mathbb{R}^2}(|Q_c|^2-1)^2}\\ &\leqslant K\sqrt{\int_{\mathbb{R}^2}|\nabla\psi|^2|Q_c|^4}.\end{aligned}$$

Indeed, we have, from equation (2.6) (for $\sigma=1/2$), that

$$||Q_c|^2-1|\leqslant\frac{K}{(1+\tilde{r})^{3/2}},$$

which is enough to show that

$$\int_{\mathbb{R}^2}(|Q_c|^2-1)^2\leqslant K.$$

Combining these estimates, we conclude the proof of

$$\left|\int_{\mathbb{R}^2}\mathfrak{Im}(\psi)\mathfrak{Re}(\nabla Q_c\overline{Q_c})\right|\leqslant K\sqrt{\int_{\mathbb{R}^2}|\nabla\psi|^2|Q_c|^4}\leqslant K\|\varphi\|_{\mathcal{C}}.\qquad\square$$

5.3. Coercivity result under four orthogonality conditions. The next result is the first part of Theorem 1.5, and the second part (for coercivity under three orthogonality conditions) is done in Lemma 5.6 below. We recall that, in $B(\pm\tilde{d}_c\overrightarrow{e_1}, R)$, we have $\psi^{\neq 0}(x) = \psi(x) - \psi^{0,\pm 1}(\tilde{r}_{\pm 1})$ with $\psi^{0,\pm 1}(\tilde{r}_{\pm 1})$ the 0-harmonic centred around $\pm\tilde{d}_c\overrightarrow{e_1}$ of ψ.

Lemma 5.5. *There exist $R, K, c_0 > 0$ such that, for $0 < c \leqslant c_0$ and $\varphi = Q_c\psi \in H_{Q_c}$, Q_c defined in Theorem 1.1, if*

$$\mathfrak{Re}\int_{B(\tilde{d}_c\overrightarrow{e_1},R)\cup B(-\tilde{d}_c\overrightarrow{e_1},R)} \partial_{x_1}Q_c\overline{Q_c\psi^{\neq 0}} = \mathfrak{Re}\int_{B(\tilde{d}_c\overrightarrow{e_1},R)\cup B(-\tilde{d}_c\overrightarrow{e_1},R)} \partial_{x_2}Q_c\overline{Q_c\psi^{\neq 0}} = 0,$$

$$\mathfrak{Re}\int_{B(\tilde{d}_c\overrightarrow{e_1},R)\cup B(-\tilde{d}_c\overrightarrow{e_1},R)} \partial_c Q_c\overline{Q_c\psi^{\neq 0}} = \mathfrak{Re}\int_{B(\tilde{d}_c\overrightarrow{e_1},R)\cup B(-\tilde{d}_c\overrightarrow{e_1},R)} \partial_{c^\perp}Q_c\overline{Q_c\psi^{\neq 0}} = 0,$$

then

$$B_{Q_c}(\varphi) \geqslant K\|\varphi\|_{\mathcal{C}}^2.$$

Proof: For $\varphi = Q_c\psi \in C_c^\infty(\mathbb{R}^2\backslash\{\widetilde{d_c}\overrightarrow{e_1}, -\widetilde{d_c}\overrightarrow{e_1}\}, \mathbb{C})$, we take ε_1, ε_2, ε_3, ε_4 four real parameters and we define

$$\psi^* := \psi + \varepsilon_1\frac{\partial_{x_1}Q_c}{Q_c} + \varepsilon_2\frac{c^2\partial_c Q_c}{Q_c} + \varepsilon_3\frac{\partial_{x_2}Q_c}{Q_c} + \varepsilon_4\frac{c\partial_{c^\perp}Q_c}{Q_c}.$$

Since, by Lemma 2.8, $\partial_{x_1}Q_c, \partial_{x_2}Q_c, \partial_c Q_c, \partial_{c^\perp}Q_c \in H_{Q_c}$, we deduce that $Q_c\psi^* \in H_{Q_c}$. Furthermore, we have

$$\begin{aligned}\int_{B(\tilde{d}_c\overrightarrow{e_1},R)} \mathfrak{Re}(\partial_{x_1}\widetilde{V_1}\overline{\widetilde{V_1}\psi^*}) = {} & \int_{B(\tilde{d}_c\overrightarrow{e_1},R)} \mathfrak{Re}(\partial_{x_1}\widetilde{V_1}\overline{\widetilde{V_1}\psi}) \\ & +\varepsilon_1\int_{B(\tilde{d}_c\overrightarrow{e_1},R)} \mathfrak{Re}\left(\partial_{x_1}\widetilde{V_1}\overline{\partial_{x_1}Q_c\frac{\widetilde{V_1}}{Q_c}}\right) \\ & +\varepsilon_2\int_{B(\tilde{d}_c\overrightarrow{e_1},R)} \mathfrak{Re}\left(\partial_{x_1}\widetilde{V_1}\overline{c^2\partial_c Q_c\frac{\widetilde{V_1}}{Q_c}}\right) \\ & +\varepsilon_3\int_{B(\tilde{d}_c\overrightarrow{e_1},R)} \mathfrak{Re}\left(\partial_{x_1}\widetilde{V_1}\overline{\partial_{x_2}Q_c\frac{\widetilde{V_1}}{Q_c}}\right) \\ & +\varepsilon_4\int_{B(\tilde{d}_c\overrightarrow{e_1},R)} \mathfrak{Re}\left(\partial_{x_1}\widetilde{V_1}\overline{c\partial_{c^\perp}Q_c\frac{\widetilde{V_1}}{Q_c}}\right).\end{aligned}$$

From (5.6), we compute

$$\partial_{x_1}\widetilde{V_1}\overline{\widetilde{V_1}} = \left(\cos(\tilde{\theta}_1)\frac{\rho'(\tilde{r}_1)}{\rho(\tilde{r}_1)} - \frac{i}{\tilde{r}_1}\sin(\tilde{\theta}_1)\right)|\widetilde{V_1}|^2,$$

and in particular, it has no 0-harmonic (since $|\widetilde{V_1}|^2$ is radial). Therefore,

$$\begin{aligned}\int_{B(\tilde{d}_c\vec{e_1},R)} \mathfrak{Re}(\partial_{x_1}\widetilde{V_1}\overline{\widetilde{V_1}\psi}) &= \int_{B(\tilde{d}_c\vec{e_1},R)} \mathfrak{Re}(\partial_{x_1}\widetilde{V_1}\overline{\widetilde{V_1}\psi^{\neq 0}})\\ &= \int_{B(\tilde{d}_c\vec{e_1},R)} \mathfrak{Re}(\partial_{x_1}Q_c\overline{Q_c\psi^{\neq 0}})\\ &\quad + \int_{B(\tilde{d}_c\vec{e_1},R)} \mathfrak{Re}((\partial_{x_1}\widetilde{V_1}\overline{\widetilde{V_1}} - \partial_{x_1}Q_c\overline{Q_c})\psi^{\neq 0}).\end{aligned}$$

By Cauchy–Schwarz and equation (2.23),

$$\begin{aligned}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)} |Q_c\psi^{\neq 0}|^2 &\leqslant K\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)} |Q_c|^4|\nabla\psi|^2\\ &\leqslant K\|\varphi\|^2_{\mathcal{C}}.\end{aligned} \tag{5.8}$$

Here, K depends on R, but we consider R as a universal constant. We note, by equations (5.1), (5.3), and (5.8), that

$$\begin{aligned}&\frac{1}{2}\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)} (\partial_{x_1}Q_c - c^2\partial_c Q_c)\overline{Q_c\psi^{\neq 0}}\\ &\qquad = \int_{B(\tilde{d}_c\vec{e_1},R)} \mathfrak{Re}(\partial_{x_1}Q_c\overline{Q_c\psi^{\neq 0}}) + o_{c\to 0}(c^{\beta_0})K\|\varphi\|^2_{\mathcal{C}},\end{aligned}$$

where $\beta_0 > 0$ is a small constant. We suppose that

$$\begin{aligned}&\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)} \partial_{x_1}Q_c\overline{Q_c\psi^{\neq 0}}\\ &\qquad = \mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)} \partial_c Q_c\overline{Q_c\psi^{\neq 0}} = 0,\end{aligned}$$

therefore

$$\int_{B(\tilde{d}_c\vec{e_1},R)} \mathfrak{Re}(\partial_{x_1}Q_c\overline{Q_c\psi^{\neq 0}}) = o_{c\to 0}(c^{\beta_0})K\|\varphi\|^2_{\mathcal{C}}.$$

Furthermore, by equations (2.7), (2.23), and (5.1), Lemma 2.15, and Cauchy–Schwarz,

$$\begin{aligned}\left|\int_{B(\tilde{d}_c\vec{e_1},R)} \mathfrak{Re}((\partial_{x_1}\widetilde{V_1}\overline{\widetilde{V_1}} - \partial_{x_1}Q_c\overline{Q_c})\psi^{\neq 0})\right| &\leqslant o_{c\to 0}(c^{\beta_0})\sqrt{\int_{B(\tilde{d}_c\vec{e_1},R)} |\psi^{\neq 0}|^2|Q_c|^2}\\ &\leqslant o_{c\to 0}(c^{\beta_0})K\|\varphi\|_{\mathcal{C}}.\end{aligned}$$

Now, from Lemma 2.15 and equation (5.1), we estimate

$$\int_{B(\tilde{d}_c\vec{e_1},R)} \mathfrak{Re}\left(\partial_{x_1}\widetilde{V_1}\overline{\partial_{x_1}Q_c\frac{\widetilde{V_1}}{Q_c}}\right) = \int_{B(\tilde{d}_c\vec{e_1},R)} |\partial_{x_1}\widetilde{V_1}|^2 + o_{c\to 0}(1).$$

With (5.2), we check

$$\int_{B(\tilde{d}_c\vec{e_1},R)} \mathfrak{Re}\left(\partial_{x_1}\widetilde{V_1}\overline{\partial_{x_2}Q_c\frac{\widetilde{V_1}}{Q_c}}\right) = o_{c\to 0}(1).$$

Similarly, by (5.3) and Lemma 2.15, we have

$$\int_{B(\tilde{d}_c\vec{e_1},R)} \mathfrak{Re}\left(\partial_{x_1}\widetilde{V_1}\overline{c^2\partial_c Q_c\frac{\widetilde{V_1}}{Q_c}}\right) = -\int_{B(\tilde{d}_c\vec{e_1},R)} |\partial_{x_1}\widetilde{V_1}|^2 + o_{c\to 0}(1)$$

and by (5.4), we have

$$\int_{B(\tilde{d}_c\vec{e_1},R)} \mathfrak{Re}\left(\partial_{x_1}\widetilde{V_1}\overline{c\partial_{c^\perp}Q_c\frac{\widetilde{V_1}}{Q_c}}\right) = o_{c\to 0}(1).$$

Thus, with (5.5) we deduce that, writing

$$K(R) = \int_{B(0,R)} |\partial_{x_1}V_1(x)|^2\,dx,$$

since

$$\begin{aligned} K(R) &= \int_{B(\tilde{d}_c\vec{e_1},R)} |\partial_{x_1}\widetilde{V_1}|^2 = \int_{B(-\tilde{d}_c\vec{e_1},R)} |\partial_{x_1}\tilde{V}_{-1}|^2 \\ &= \int_{B(\tilde{d}_c\vec{e_1},R)} |\partial_{x_2}\widetilde{V_1}|^2 = \int_{B(-\tilde{d}_c\vec{e_1},R)} |\partial_{x_2}\tilde{V}_{-1}|^2, \end{aligned}$$

we have

$$\begin{aligned} \int_{B(\tilde{d}_c\vec{e_1},R)} \mathfrak{Re}(\partial_{x_1}\widetilde{V_1}\overline{\widetilde{V_1}\psi^*}) &= (\varepsilon_1-\varepsilon_2)K(R) \\ &\quad + o_{c\to 0}(1)(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4) + o_{c\to 0}(c^{\beta_0})K\|\varphi\|_{\mathcal{C}}. \end{aligned}$$

Similarly, we can do the same computation for all the orthogonality conditions, and we have the system

$$\begin{pmatrix} \int_{B(\tilde{d}_c\vec{e_1},R)} \mathfrak{Re}(\partial_{x_1}\widetilde{V_1}\overline{\widetilde{V_1}\psi^*}) \\ \int_{B(-\tilde{d}_c\vec{e_1},R)} \mathfrak{Re}(\partial_{x_1}\tilde{V}_{-1}\overline{\tilde{V}_{-1}\psi^*}) \\ \int_{B(\tilde{d}_c\vec{e_1},R)} \mathfrak{Re}(\partial_{x_2}\widetilde{V_1}\overline{\widetilde{V_1}\psi^*}) \\ \int_{B(-\tilde{d}_c\vec{e_1},R)} \mathfrak{Re}(\partial_{x_2}\tilde{V}_{-1}\overline{\tilde{V}_{-1}\psi^*}) \end{pmatrix} = \left(K(R)\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} + o_{c\to 0}(1)\right)\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \end{pmatrix}$$
$$+ o_{c\to 0}(c^{\beta_0})K\|\varphi\|_{\mathcal{C}}.$$

Therefore, since the matrix is invertible and $K(R) > 0$, for c small enough, we can find $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in \mathbb{R}$ such that

$$|\varepsilon_1| + |\varepsilon_2| + |\varepsilon_3| + |\varepsilon_4| \leqslant o_{c\to 0}(c^{\beta_0}) K \|\varphi\|_{\mathcal{C}} \tag{5.9}$$

and

$$\int_{B(\tilde{d}_c\vec{e_1},R)} \mathfrak{Re}(\partial_{x_1}\widetilde{V_1}\overline{\widetilde{V_1}\psi^*}) = \int_{B(\tilde{d}_c\vec{e_1},R)} \mathfrak{Re}(\partial_{x_2}\widetilde{V_1}\overline{\widetilde{V_1}\psi^*}) = 0,$$

$$\int_{B(-\tilde{d}_c\vec{e_1},R)} \mathfrak{Re}(\partial_{x_1}\tilde{V}_{-1}\overline{\tilde{V}_{-1}\psi^*}) = \int_{B(-\tilde{d}_c\vec{e_1},R)} \mathfrak{Re}(\partial_{x_2}\tilde{V}_{-1}\overline{\tilde{V}_{-1}\psi^*}) = 0.$$

Therefore, by Proposition 1.4, since $Q_c\psi^* \in H_{Q_c}$, we have

$$B_{Q_c}(Q_c\psi^*) \geqslant K\|Q_c\psi^*\|^2_{\mathcal{C}}.$$

From Lemma 2.9, we have

$$\|\partial_{x_1}Q_c\|_{\mathcal{C}} + \|\partial_{x_2}Q_c\|_{\mathcal{C}} + \|c^2\partial_c Q_c\|_{\mathcal{C}} + c^{\beta_0/2}\|c\partial_{c^\perp}Q_c\|_{\mathcal{C}} \leqslant K(\beta_0),$$

hence, since $Q_c(\psi^* - \psi) = \varepsilon_1\partial_{x_1}Q_c + \varepsilon_2 c^2\partial_c Q_c + \varepsilon_3\partial_{x_2}Q_c + \varepsilon_4 c\partial_{c^\perp}Q_c$,

$$\begin{aligned}\|Q_c\psi\|^2_{\mathcal{C}} &\leqslant \|Q_c\psi^*\|^2_{\mathcal{C}} + \|Q_c(\psi - \psi^*)\|^2_{\mathcal{C}}\\ &\leqslant \|Q_c\psi^*\|^2_{\mathcal{C}} + K(\beta_0)(|\varepsilon_1| + |\varepsilon_2| + |\varepsilon_3| + c^{-\beta_0/2}|\varepsilon_4|)^2,\end{aligned}$$

therefore, for c small enough, by (5.9), we have

$$\|Q_c\psi^*\|^2_{\mathcal{C}} \geqslant K\|Q_c\psi\|^2_{\mathcal{C}}$$

and

$$B_{Q_c}(Q_c\psi^*) \geqslant K\|Q_c\psi\|^2_{\mathcal{C}}.$$

Finally, we compute, since $Q_c(\psi-\psi^*) = \varepsilon_1\partial_{x_1}Q_c + \varepsilon_2 c^2\partial_c Q_c + \varepsilon_3\partial_{x_2}Q_c + \varepsilon_4 c\partial_{c^\perp}Q_c$, by Lemma 5.2, that

$$B_{Q_c}(\varphi) = B_{Q_c}(Q_c\psi^*) + B_{Q_c}(Q_c(\psi - \psi^*)) + 2\langle Q_c\psi^*, L_{Q_c}(Q_c(\psi - \psi^*))\rangle.$$

Furthermore, we compute, still by Lemma 5.2,

$$\langle Q_c\psi^*, L_{Q_c}(Q_c(\psi - \psi^*))\rangle = -B_{Q_c}(Q_c(\psi - \psi^*)) + \langle Q_c\psi, L_{Q_c}(Q_c(\psi - \psi^*))\rangle,$$

therefore

$$\begin{aligned}B_{Q_c}(\varphi) &= B_{Q_c}(Q_c\psi^*) - B_{Q_c}(Q_c(\psi - \psi^*)) + 2\langle Q_c\psi, L_{Q_c}(Q_c(\psi - \psi^*))\rangle\\ &\geqslant K\|Q_c\psi\|^2_{\mathcal{C}} - B_{Q_c}(Q_c(\psi - \psi^*)) + 2\langle Q_c\psi, L_{Q_c}(Q_c(\psi - \psi^*))\rangle.\end{aligned}$$

We have

$$Q_c(\psi-\psi^*) = -(\varepsilon_1\partial_{x_1}Q_c + \varepsilon_2 c^2\partial_c Q_c + \varepsilon_3\partial_{x_2}Q_c + \varepsilon_4 c\partial_{c^\perp}Q_c),$$

and from Lemma 2.8, we have

$$L_{Q_c}(Q_c(\psi-\psi^*)) = -c^2\varepsilon_2 i\partial_{x_2}Q_c + c^2\varepsilon_4 i\partial_{x_1}Q_c.$$

We compute

$$\begin{aligned}B_{Q_c}(Q_c(\psi-\psi^*)) = \langle -(\varepsilon_1\partial_{x_1}Q_c + \varepsilon_2 c^2\partial_c Q_c + \varepsilon_3\partial_{x_2}Q_c + \varepsilon_4 c\partial_{c^\perp}Q_c),\\ - c^2\varepsilon_2 i\partial_{x_2}Q_c + c^2\varepsilon_4 i\partial_{x_1}Q_c\rangle,\end{aligned}$$

and with (2.3), we check that

$$B_{Q_c}(Q_c(\psi-\psi^*)) = \varepsilon_2^2c^4\langle L_{Q_c}(\partial_c Q_c), \partial_c Q_c\rangle - \varepsilon_4^2c^2\langle L_{Q_c}(\partial_{c^\perp}Q_c), \partial_{c^\perp}Q_c\rangle.$$

With Lemma 2.10 and equation (5.9), we estimate

$$|B_{Q_c}(Q_c(\psi-\psi^*))| \leqslant Kc^2(\varepsilon_2^2+\varepsilon_4^2) \leqslant o_{c\to 0}(1)\|Q_c\psi\|_{\mathcal{C}}^2.$$

Finally, we have

$$\langle Q_c\psi, L_{Q_c}(Q_c(\psi-\psi^*))\rangle = \langle Q_c\psi, -c^2\varepsilon_2 i\partial_{x_2}Q_c + c^2\varepsilon_4 i\partial_{x_1}Q_c\rangle.$$

We compute

$$c^2\langle Q_c\psi, i\nabla Q_c\rangle = c^2\int_{\mathbb{R}^2}\mathfrak{Im}(\psi)\mathfrak{Re}(\nabla Q_c\overline{Q_c}) - c^2\int_{\mathbb{R}^2}\mathfrak{Re}(\psi)\mathfrak{Im}(\nabla Q_c\overline{Q_c}),$$

and to finish the proof, we use

$$|c\langle Q_c\psi, i\nabla Q_c\rangle| \leqslant Kc\ln\left(\frac{1}{c}\right)\|Q_c\psi\|_{\mathcal{C}}$$

for a constant $K>0$ independent of c by Lemma 5.4, which is enough to show that

$$|\langle Q_c\psi, L_{Q_c}(Q_c(\psi-\psi^*))\rangle| \leqslant o_{c\to 0}(1)(|\varepsilon_2|+|\varepsilon_4|)\|Q_c\psi\|_{\mathcal{C}} \leqslant o_{c\to 0}(1)\|Q_c\psi\|_{\mathcal{C}}^2,$$

since $c\ln\left(\frac{1}{c}\right) = o_{c\to 0}(1)$. We have shown that, for $\varphi\in C_c^\infty(\mathbb{R}^2\backslash\{\widetilde{d_c}\overrightarrow{e_1}, -\widetilde{d_c}\overrightarrow{e_1}\},\mathbb{C})$

$$\begin{aligned}B_{Q_c}(\varphi) &\geqslant K\|Q_c\psi\|_{\mathcal{C}}^2 - B_{Q_c}(Q_c(\psi-\psi^*)) + 2\langle Q_c\psi, L_{Q_c}(Q_c(\psi-\psi^*))\rangle\\ &\geqslant (K-o_{c\to 0}(1))\|Q_c\psi\|_{\mathcal{C}}^2\\ &\geqslant \frac{K}{2}\|Q_c\psi\|_{\mathcal{C}}^2\end{aligned}$$

for c small enough. Now, by Lemma 3.4, we conclude by density as in the proof of Proposition 1.4. □

5.4. Coercivity under three orthogonality conditions.

Lemma 5.6. *There exist $R, K > 0$ such that, for $0 < \beta < \beta_0$, β_0 a small constant, there exist $c_0(\beta), K(\beta) > 0$ with, for $0 < c < c_0(\beta)$, Q_c defined in Theorem 1.1, $\varphi = Q_c\psi \in H_{Q_c}$, if*

$$\mathfrak{Re}\int_{B(\tilde{d}_c\overrightarrow{e_1},R)\cup B(-\tilde{d}_c\overrightarrow{e_1},R)} \partial_{x_1}Q_c\overline{Q_c\psi^{\neq 0}} = \mathfrak{Re}\int_{B(\tilde{d}_c\overrightarrow{e_1},R)\cup B(-\tilde{d}_c\overrightarrow{e_1},R)} \partial_{x_2}Q_c\overline{Q_c\psi^{\neq 0}} = 0,$$

$$\mathfrak{Re}\int_{B(\tilde{d}_c\overrightarrow{e_1},R)\cup B(-\tilde{d}_c\overrightarrow{e_1},R)} \partial_c Q_c\overline{Q_c\psi^{\neq 0}} = 0,$$

then

$$B_{Q_c}(\varphi) \geqslant K(\beta)c^{2+\beta}\|\varphi\|^2_{\mathcal{C}}.$$

Proof: As for the proof of Lemma 5.5, we show the result for $\varphi = Q_c\psi \in C^\infty_c(\mathbb{R}^2\backslash\{\widetilde{d_c}\overrightarrow{e_1}, -\widetilde{d_c}\overrightarrow{e_1}\}, \mathbb{C})$, and we conclude by density for $\varphi \in H_{Q_c}$.

For $\varphi = Q_c\psi \in C^\infty_c(\mathbb{R}^2\backslash\{\widetilde{d_c}\overrightarrow{e_1}, -\widetilde{d_c}\overrightarrow{e_1}\}, \mathbb{C})$, we take ε_1, ε_2, ε_3, ε_4 four real parameters and we define

$$\psi^* := \psi + \varepsilon_1\frac{\partial_{x_1}Q_c}{Q_c} + \varepsilon_2\frac{c^2\partial_c Q_c}{Q_c} + \varepsilon_3\frac{\partial_{x_2}Q_c}{Q_c} + \varepsilon_4\frac{c\partial_{c^\perp}Q_c}{Q_c}.$$

With the same computation as in the proof of Lemma 5.5, we check that $Q_c\psi^* \in H_{Q_c}$, and using similarly the estimates of Lemma 5.1, we can take $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in \mathbb{R}$ such that

$$|\varepsilon_1| + |\varepsilon_2| + |\varepsilon_3| = o_{c\to 0}(c^{\beta_0})\|\varphi\|_{\mathcal{C}},$$

$|\varepsilon_4| \leqslant K\|\varphi\|_{\mathcal{C}}$ and such that ψ^* satisfies the four orthogonality conditions of Lemma 5.5. The estimates on ε_4 are with a constant independent of c because $c\partial_{c^\perp}Q_c$ is of size independent of c in $B(\tilde{d}_c\overrightarrow{e_1}, R) \cup B(-\tilde{d}_c\overrightarrow{e_1}, R)$. Therefore,

$$B_{Q_c}(Q_c\psi^*) \geqslant K\|Q_c\psi^*\|^2_{\mathcal{C}}. \tag{5.10}$$

We write

$$T = \varepsilon_1\partial_{x_1}Q_c + \varepsilon_2 c^2\partial_c Q_c + \varepsilon_3\partial_{x_2}Q_c,$$

and we develop, by Lemma 5.2,

$$\begin{aligned} B_{Q_c}(Q_c\psi) = {} & B_{Q_c}(Q_c\psi^*) + c^2\varepsilon_4^2 B_{Q_c}(\partial_{c^\perp}Q_c) + B_{Q_c}(T) \\ & -2\langle Q_c\psi^*, c\varepsilon_4 L_{Q_c}(\partial_{c^\perp}Q_c)\rangle \\ & -2\langle Q_c\psi^*, L_{Q_c}(T)\rangle + 2c\varepsilon_4\langle L_{Q_c}(\partial_{c^\perp}Q_c), T\rangle. \end{aligned}$$

Using Lemmas 2.8 and 2.10, we compute

$$\begin{aligned} |B_{Q_c}(T)| &= |\langle L_{Q_c}(T), T\rangle| = |\langle L_{Q_c}(\varepsilon_2 c^2\partial_c Q_c), \varepsilon_2 c^2\partial_c Q_c\rangle| \\ &= \varepsilon_2^2 c^4|\langle L_{Q_c}(\partial_c Q_c), \partial_c Q_c\rangle| \leqslant K\varepsilon_2^2 c^2 = o_{c\to 0}(c^{2+2\beta_0})\|\varphi\|^2_{\mathcal{C}}. \end{aligned}$$

Now, we compute, by Lemma 2.8, that

$$\langle Q_c\psi^*, c\varepsilon_4 L_{Q_c}(\partial_{c^\perp}Q_c)\rangle = \varepsilon_4 c^2 \langle Q_c\psi^*, i\partial_{x_1}Q_c\rangle.$$

From Lemma 5.4, we have

$$|c\langle Q_c\psi^*, i\partial_{x_1}Q_c\rangle| \leqslant o_{c\to 0}(c^{1-\beta_0/2})\|\varphi^*\|_{\mathcal{C}},$$

therefore

$$|\langle Q_c\psi^*, c\varepsilon_4 L_{Q_c}(\partial_{c^\perp}Q_c)\rangle| \leqslant o_{c\to 0}(c^{1+\beta_0/2})\|\varphi^*\|_{\mathcal{C}}\|\varphi\|_{\mathcal{C}}.$$

Similarly, we compute

$$\langle Q_c\psi^*, L_{Q_c}(T)\rangle = \langle Q_c\psi^*, \varepsilon_2 c^2 L_{Q_c}(\partial_c Q_c)\rangle = \varepsilon_2 c^2\langle Q_c\psi^*, i\partial_{x_2}Q_c\rangle.$$

Still from Lemma 5.4, we have

$$|c\langle Q_c\psi^*, i\partial_{x_2}Q_c\rangle| \leqslant Kc\ln\left(\frac{1}{c}\right)\|\varphi^*\|_{\mathcal{C}},$$

therefore

$$|\langle Q_c\psi^*, L_{Q_c}(T)\rangle| \leqslant K|\varepsilon_2|c^2\ln\left(\frac{1}{c}\right)\|\varphi^*\|_{\mathcal{C}} \leqslant o_{c\to 0}(c^{1+\beta_0})\|\varphi^*\|_{\mathcal{C}}\|\varphi\|_{\mathcal{C}}.$$

Finally, we compute similarly that

$$c|\varepsilon_4\langle L_{Q_c}(\partial_{c^\perp}Q_c), T\rangle| = c|\varepsilon_4\langle ic\partial_{x_1}Q_c, T\rangle| = c^2|\varepsilon_4\langle i\partial_{x_1}Q_c, \varepsilon_2 c^2\partial_c Q_c + \varepsilon_3\partial_{x_2}Q_c\rangle|.$$

Using Lemma 5.4 for $\varphi = c^2\partial_c Q_c$ (with Lemma 3.4), we infer

$$|\langle i\partial_{x_1}Q_c, c^2\partial_c Q_c\rangle| \leqslant K\|c^2\partial_c Q_c\|_{\mathcal{C}},$$

and $\|c^2\partial_c Q_c\|_{\mathcal{C}} \leqslant K$ by Lemma 2.9. Furthermore, since $Q_c(-x_1,x_2) = Q_c(x_1,x_2)$, we have

$$\langle i\partial_{x_1}Q_c, \partial_{x_2}Q_c\rangle = 0.$$

We conclude that

$$|c\varepsilon_4\langle L_{Q_c}(\partial_{c^\perp}Q_c), T\rangle| \leqslant Kc^2|\varepsilon_4|(|\varepsilon_2|+|\varepsilon_3|) = o_{c\to 0}(c^{2+\beta_0/2})\|\varphi\|^2_{\mathcal{C}}. \tag{5.11}$$

Now, combining (5.10) to (5.11), and with $B_{Q_c}(\partial_{c^\perp}Q_c) = 2\pi + o_{c\to 0}(1)$ from Lemma 2.10, we have

$$B_{Q_c}(\varphi) \geqslant K\|\varphi^*\|^2_{\mathcal{C}} + K\varepsilon_4^2c^2 - o_{c\to 0}(c^{2+\beta_0/2})\|\varphi\|^2_{\mathcal{C}} - o_{c\to 0}(c^{1+\beta_0/2})\|\varphi^*\|_{\mathcal{C}}\|\varphi\|_{\mathcal{C}}.$$

Similarly as in the proof of Lemma 5.5, we have from Lemma 2.9 that, for any $\beta_0/2 > \beta > 0$,

$$\|\varphi\|_{\mathcal{C}}^2 \leqslant K\|\varphi^*\|_{\mathcal{C}}^2 + K(\beta)\varepsilon_4^2 c^{-\beta},$$

hence

$$\varepsilon_4^2 c^2 \geqslant K(\beta)c^{2+\beta}(\|\varphi\|_{\mathcal{C}}^2 - \|\varphi^*\|_{\mathcal{C}}^2),$$

therefore

$$\begin{aligned} B_{Q_c}(\varphi) &\geqslant K_1(\beta)(\|\varphi^*\|_{\mathcal{C}}^2 + c^{2+\beta}\|\varphi\|_{\mathcal{C}}^2) - K_2(\beta)c^{2+\beta}\|\varphi^*\|_{\mathcal{C}}^2 \\ &\quad - o_{c\to 0}(c^{2+\beta_0/2})\|\varphi\|_{\mathcal{C}}^2 - o_{c\to 0}(c^{1+\beta_0})\|\varphi^*\|_{\mathcal{C}}\|\varphi\|_{\mathcal{C}} \\ &\geqslant K(\beta)c^{2+\beta}\|\varphi\|_{\mathcal{C}}^2 \end{aligned}$$

for c small enough (depending on β). □

Lemmas 2.13, 5.5, and 5.6 together end the proof of Theorem 1.5. Note that in both Lemmas 5.5 and 5.6, we could replace the orthogonality condition $\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)} \partial_c Q_c \overline{Q_c\psi^{\neq 0}} = 0$ by

$$\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)} \partial_d(V_1(x-d\vec{e}_1)V_{-1}(x+d\vec{e}_1))_{|d=d_c}\overline{Q_c\psi^{\neq 0}}(x)\,dx = 0, \tag{5.12}$$

since, by Theorem 1.1 (for $p = +\infty$),

$$\|c^2\partial_c Q_c - \partial_d(V_1(x-d\vec{e}_1)V_{-1}(x+d\vec{e}_1))_{|d=d_c}\|_{C^1(B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R))} = o_{c\to 0}(1),$$

and thus this replacement creates an error term that can be estimated like the other ones in the proof of Lemma 5.5.

5.5. Proof of the corollaries of Theorem 1.5.

5.5.1. Proof of Corollary 1.6. We start with the proof that (i) implies (ii). We start by showing that, for $\varphi_0 \in C_c^\infty(\mathbb{R}^2, \mathbb{C})$,

$$B_{Q_c}(\varphi + \varphi_0) = B_{Q_c}(\varphi_0).$$

We take $\varphi_0 = Q_c\psi_0 \in C_c^\infty(\mathbb{R}^2, \mathbb{C})$ and, by integration by parts, from (i), we check that

$$\langle L_{Q_c}(\varphi_0), \varphi\rangle = 0.$$

Furthermore, we check (for $\varphi \in C_c^\infty(\mathbb{R}^2\backslash\{\widetilde{d}_c\vec{e_1}, -\widetilde{d}_c\vec{e_1}\}, \mathbb{C})$ and then by density for $\varphi \in H_{Q_c}$) that for $\varphi_0 \in C_c^\infty(\mathbb{R}^2, \mathbb{C})$,

$$B_{Q_c}(\varphi + \varphi_0) = B_{Q_c}(\varphi) + B_{Q_c}(\varphi_0) + 2\langle \varphi, L_{Q_c}(\varphi_0)\rangle,$$

hence

$$B_{Q_c}(\varphi + \varphi_0) = B_{Q_c}(\varphi) + B_{Q_c}(\varphi_0). \tag{5.13}$$

Similarly as in the proof of Proposition 1.4, we argue by density that this result holds for $\varphi_0 \in H_{Q_c}$. Now, taking $\varphi_0 = -\varphi$, we infer from (5.13) that $B_{Q_c}(\varphi) = 0$, thus, for $\varphi \in H_{Q_c}$,

$$B_{Q_c}(\varphi + \varphi_0) = B_{Q_c}(\varphi_0). \tag{5.14}$$

Now, similarly as in the proof of Lemma 5.5, we decompose $\varphi = Q_c\psi \in H_{Q_c}$ into

$$\varphi = \varphi^* + \varepsilon_1\partial_{x_1}Q_c + \varepsilon_2\partial_{x_2}Q_c + \varepsilon_3 c^2\partial_c Q_c$$

with

$$|\varepsilon_1| + |\varepsilon_2| + |\varepsilon_3| \leqslant K\|\varphi\|_{\mathcal{C}},$$

such that φ^* verifies the three orthogonality conditions of Lemma 5.6 (all the functions of Q_c considered in the orthogonality conditions are of size independent of c in $B(\tilde{d}_c\overrightarrow{e_1}, R) \cup B(-\tilde{d}_c\overrightarrow{e_1}, R)$). We write

$$A = \varepsilon_1\partial_{x_1}Q_c + \varepsilon_2\partial_{x_2}Q_c + \varepsilon_3 c^2\partial_c Q_c \in H_{Q_c}$$

by Lemma 2.8, and using (5.14), we have

$$B_{Q_c}(\varphi^*) = B_{Q_c}(\varphi - A) = B_{Q_c}(A).$$

From Lemma 5.6, we have $B_{Q_c}(\varphi^*) \geqslant Kc^{2+\beta_0/2}\|\varphi^*\|^2_{\mathcal{C}}$. Furthermore, from Lemmas 2.8 and 2.10,

$$B_{Q_c}(A) = \varepsilon_3^2 c^2 B_{Q_c}(\partial_c Q_c) = (-2\pi + o_{c\to 0}(1))\varepsilon_3^2 \leqslant 0.$$

We deduce that $\varepsilon_3 = 0$ and $\|\varphi^*\|_{\mathcal{C}} = 0$, hence $\varphi^* = i\mu Q_c$ for some $\mu \in \mathbb{R}$. Since $\varphi^* = \varphi - R \in H_{Q_c}$, we deduce that $\mu = 0$ (or else $\|\varphi^*\|^2_{H_{Q_c}} \geqslant \int_{\mathbb{R}^2} \frac{|\varphi^*|^2}{(1+\tilde{r})^2} = +\infty$). Therefore,

$$\varphi = \varepsilon_1\partial_{x_1}Q_c + \varepsilon_2\partial_{x_2}Q_c \in \mathrm{Span}_{\mathbb{R}}(\partial_{x_1}Q_c, \partial_{x_2}Q_c).$$

Finally, the fact that (ii) implies (i) is a consequence of Lemma 2.8. This concludes the proof of this lemma. □

5.5.2. Spectral stability. We have $H^1(\mathbb{R}^2) \subset H_{Q_c}$, therefore $B_{Q_c}(\varphi)$ is well defined for $\varphi \in H^1(\mathbb{R}^2)$. Furthermore, $i\partial_{x_2}Q_c \in L^2(\mathbb{R}^2)$ is a consequence of Theorem 2.5, and in particular this justifies that $\langle\varphi, i\partial_{x_2}Q_c\rangle$ is well defined for $\varphi \in H^1(\mathbb{R}^2)$. For $\varphi \in H^1(\mathbb{R}^2)$, there is no issue in the definition of the quadratic form, as shown in the following lemma.

Lemma 5.7. *There exists $c_0 > 0$ such that, for $0 < c < c_0$, Q_c defined in Theorem 1.1, if $\varphi \in H^1(\mathbb{R}^2)$, then*

$$B_{Q_c}(\varphi) = \int_{\mathbb{R}^2} |\nabla\varphi|^2 - \mathfrak{Re}(ic\partial_{x_2}\varphi\bar{\varphi}) - (1 - |Q_c|^2)|\varphi|^2 + 2\mathfrak{Re}^2(\overline{Q_c}\varphi).$$

Proof: We recall that $H^1(\mathbb{R}^2) \subset H_{Q_c}$ and, for $\varphi = Q_c\psi \in H^1(\mathbb{R}^2)$,

$$\begin{aligned} B_{Q_c}(\varphi) = \int_{\mathbb{R}^2} |\nabla\varphi|^2 - (1-|Q_c|^2)|\varphi|^2 + 2\mathfrak{Re}^2(\overline{Q_c}\varphi) \\ -c\int_{\mathbb{R}^2}(1-\eta)\mathfrak{Re}(i\partial_{x_2}\varphi\bar{\varphi}) - c\int_{\mathbb{R}^2}\eta\mathfrak{Re}i\partial_{x_2}Q_c\overline{Q_c}|\psi|^2 \\ +2c\int_{\mathbb{R}^2}\eta\mathfrak{Re}\psi\mathfrak{Im}\partial_{x_2}\psi|Q_c|^2 + c\int_{\mathbb{R}^2}\partial_{x_2}\eta\mathfrak{Re}\psi\mathfrak{Im}\psi|Q_c|^2 \\ +c\int_{\mathbb{R}^2}\eta\mathfrak{Re}\psi\mathfrak{Im}\psi\partial_{x_2}(|Q_c|^2). \end{aligned}$$

Since $\varphi \in H^1(\mathbb{R}^2)$, the integral $\int_{\mathbb{R}^2}\mathfrak{Re}(ic\partial_{x_2}\varphi\bar{\varphi})$ is well defined as the scalar product of two $L^2(\mathbb{R}^2)$ functions. Now, still because $\varphi = Q_c\psi \in H^1(\mathbb{R}^2)$, we can integrate by parts, and we check that

$$\begin{aligned} \int_{\mathbb{R}^2}\eta\mathfrak{Re}\psi\mathfrak{Im}\partial_{x_2}\psi|Q_c|^2 = -\int_{\mathbb{R}^2}\eta\mathfrak{Re}\partial_{x_2}\psi\mathfrak{Im}\psi|Q_c|^2 \\ -\int_{\mathbb{R}^2}\partial_{x_2}\eta\mathfrak{Re}\psi\mathfrak{Im}\psi|Q_c|^2 - \int_{\mathbb{R}^2}\eta\mathfrak{Re}\psi\mathfrak{Im}\psi\partial_{x_2}(|Q_c|^2). \end{aligned}$$

We conclude by expanding

$$\begin{aligned} \int_{\mathbb{R}^2}\eta\mathfrak{Re}(i\partial_{x_2}\varphi\bar{\varphi}) &= \int_{\mathbb{R}^2}\eta\mathfrak{Re}(i\partial_{x_2}Q_c\overline{Q_c})|\psi|^2 + \int_{\mathbb{R}^2}\eta\mathfrak{Re}(i\partial_{x_2}\psi\bar{\psi})|Q_c|^2 \\ &= \int_{\mathbb{R}^2}\eta\mathfrak{Re}(i\partial_{x_2}Q_c\overline{Q_c})|\psi|^2 + \int_{\mathbb{R}^2}\eta\mathfrak{Re}(\partial_{x_2}\psi)\mathfrak{Im}\psi|Q_c|^2 \\ &\quad + \int_{\mathbb{R}^2}\eta\mathfrak{Re}(\psi)\mathfrak{Im}\partial_{x_2}\psi|Q_c|^2. \qquad \square \end{aligned}$$

The rest of this subsection is devoted to the proofs of Corollary 1.7, Proposition 1.8, and Corollary 1.10.

Proof of Corollary 1.7: For $\varphi \in H^1(\mathbb{R}^2)$ such that $\langle\varphi, i\partial_{x_2}Q_c\rangle = 0$, we decompose it into

$$\varphi = \varphi^* + \varepsilon_1\partial_{x_1}Q_c + \varepsilon_2\partial_{x_2}Q_c + c^2\varepsilon_3\partial_c Q_c.$$

Similarly as in the proof of Lemma 5.5, we can find $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \mathbb{R}$ such that φ^* satisfies the three orthogonality conditions of Lemma 5.6, and thus (since $\varphi \in H^1(\mathbb{R}^2) \subset H_{Q_c}$, for $\beta = \beta_0/2$)

$$B_{Q_c}(\varphi^*) \geqslant Kc^{2+\beta_0/2}\|\varphi^*\|^2_{\mathcal{C}}.$$

Now, we compute, by Lemma 5.2 and with a density argument, that

$$B_{Q_c}(\varphi) = B_{Q_c}(\varphi^*) + 2\langle\varphi^*, L_{Q_c}(\varepsilon_1\partial_{x_1}Q_c + \varepsilon_2\partial_{x_2}Q_c + c^2\varepsilon_3\partial_c Q_c)\rangle + \varepsilon_3^2c^4B_{Q_c}(\partial_c Q_c).$$

We have from Lemma 2.8 that $L_{Q_c}(\varepsilon_1\partial_{x_1}Q_c + \varepsilon_2\partial_{x_2}Q_c + c^2\varepsilon_3\partial_c Q_c) = c^2\varepsilon_3 i\partial_{x_2}Q_c$, therefore

$$B_{Q_c}(\varphi) \geqslant Kc^{2+\beta_0/2}\|\varphi^*\|^2_{\mathcal{C}} + 2c^2\varepsilon_3\langle\varphi^*, i\partial_{x_2}Q_c\rangle + \varepsilon_3^2c^4B_{Q_c}(\partial_c Q_c).$$

Since $\langle\varphi, i\partial_{x_2}Q_c\rangle = 0$ and $\varphi = \varphi^* + \varepsilon_1\partial_{x_1}Q_c + \varepsilon_2\partial_{x_2}Q_c + c^2\varepsilon_3\partial_c Q_c$, we have

$$\langle\varphi^*, i\partial_{x_2}Q_c\rangle = -\langle\varepsilon_1\partial_{x_1}Q_c + \varepsilon_2\partial_{x_2}Q_c + c^2\varepsilon_3\partial_c Q_c, i\partial_{x_2}Q_c\rangle.$$

We have $\langle\varepsilon_1\partial_{x_1}Q_c, i\partial_{x_2}Q_c\rangle = 0$ since $\partial_{x_1}Q_c$ is odd in x_1 and $i\partial_{x_2}Q_c$ is even in x_1. Furthermore,

$$\langle\varepsilon_2\partial_{x_2}Q_c, i\partial_{x_2}Q_c\rangle = \varepsilon_2\int_{\mathbb{R}^2}\mathfrak{Re}(i|\partial_{x_2}Q_c|^2) = 0,$$

and, from Lemma 2.10, we have

$$B_{Q_c}(\partial_c Q_c) = \langle\partial_c Q_c, i\partial_{x_2}Q_c\rangle = \frac{-2\pi + o_{c\to 0}(1)}{c^2},$$

thus

$$\langle\varphi^*, L_{Q_c}(\varepsilon_1\partial_{x_1}Q_c + \varepsilon_2\partial_{x_2}Q_c + c^2\varepsilon_3\partial_c Q_c)\rangle = (2\pi + o_{c\to 0}(1))\varepsilon_3 B_{Q_c}(\partial_c Q_c),$$

and

$$\begin{aligned} B_{Q_c}(\varphi) &\geqslant Kc^{2+\beta_0/2}\|\varphi^*\|^2_{\mathcal{C}} - \varepsilon_3^2 c^4 B_{Q_c}(\partial_c Q_c) \\ &\geqslant Kc^{2+\beta_0/2}\|\varphi^*\|^2_{\mathcal{C}} + 2\pi\varepsilon_3^2 c^2(1 + o_{c\to 0}(1)) \geqslant 0 \end{aligned}$$

for c small enough. This also shows that if $\varphi \in H^1(\mathbb{R}^2)$, $B_{Q_c}(\varphi) = 0$, and $\langle\varphi, i\partial_{x_2}Q_c\rangle = 0$, then $\varphi \in \mathrm{Span}_{\mathbb{R}}\{\partial_{x_1}Q_c, \partial_{x_2}Q_c\}$. □

We can now finish the proof of Proposition 1.8.

Proof of Proposition 1.8: First, we have from Theorem 2.5 that $i\partial_{x_2}Q_c \in L^2(\mathbb{R}^2)$. Now, with Corollary 1.7, it is easy to check that $n^-(L_{Q_c}) \leqslant 1$. Indeed, if it is false, we can find $u, v \in H^1(\mathbb{R}^2)$ such that for all $\lambda, \mu \in \mathbb{R}$ with $(\lambda, \mu) \neq (0,0)$, $\lambda u + \mu v \neq 0$, and $B_{Q_c}(\lambda u + \mu v) < 0$. Then, we can take $(\lambda, \mu) \neq (0,0)$ such that

$$\langle\lambda u + \mu v, i\partial_{x_2}Q_c\rangle = 0,$$

which implies $B_{Q_c}(\lambda u + \mu v) \geqslant 0$ and is therefore a contradiction.

Let us show that L_{Q_c} has at least one negative eigenvalue (with eigenvector in $H^1(\mathbb{R}^2)$), which implies that $n^-(L_{Q_c}) = 1$ and that it is the only negative eigenvalue. We consider

$$\alpha_c := \inf_{\varphi\in H^1(\mathbb{R}^2),\, \|\varphi\|_{L^2(\mathbb{R}^2)}=1} B_{Q_c}(\varphi).$$

We recall, from Lemma 5.7, that (since $\varphi \in H^1(\mathbb{R}^2)$)

$$B_{Q_c}(\varphi) = \int_{\mathbb{R}^2}|\nabla\varphi|^2 - \mathfrak{Re}(ic\partial_{x_2}\varphi\bar{\varphi}) - (1 - |Q_c|^2)|\varphi|^2 + 2\mathfrak{Re}^2(\overline{Q_c}\varphi),$$

and if $\varphi \in H^1(\mathbb{R}^2)$ with $\|\varphi\|_{L^2(\mathbb{R}^2)} = 1$, we have, by Cauchy–Schwarz,

$$B_{Q_c}(\varphi) \geqslant \int_{\mathbb{R}^2}|\nabla\varphi|^2 - Kc\|\partial_{x_2}\varphi\|_{L^2(\mathbb{R}^2)} - K \geqslant -K(c).$$

In particular, this implies that $\alpha_c \neq -\infty$.

Now, assume that there exists no $\varphi \in C_c^\infty(\mathbb{R}^2,\mathbb{C})$ such that $B_{Q_c}(\varphi) < 0$. Then, for any $\varphi \in C_c^\infty(\mathbb{R}^2,\mathbb{C})$, we have $B_{Q_c}(\varphi) \geqslant 0$. Following the density argument at the end of the proof of Proposition 1.4, we have $B_{Q_c}(\varphi) \geqslant 0$ for all $\varphi \in H_{Q_c}$, and in particular $B_{Q_c}(\partial_c Q_c) \geqslant 0$ (we recall that $\partial_c Q_c \in H_{Q_c}$ but is not a priori in $H^1(\mathbb{R}^2)$), which is in contradiction with Lemma 2.10. Therefore, there exists $\varphi \in C_c^\infty(\mathbb{R}^2,\mathbb{C}) \subset H^1(\mathbb{R}^2)$ such that $B_{Q_c}(\varphi) < 0$, and in particular $B_{Q_c}\big(\frac{\varphi}{\|\varphi\|_{L^2(\mathbb{R}^2)}}\big) < 0$ and $\big\|\frac{\varphi}{\|\varphi\|_{L^2(\mathbb{R}^2)}}\big\|_{L^2(\mathbb{R}^2)} = 1$, hence $\alpha_c < 0$.

Note that we do not show that $\partial_c Q_c \in L^2(\mathbb{R}^2)$, and we believe this to be false. This estimate on α_c is the only time we need to work specifically with Q_c from Theorem 1.1. From now on, we can suppose that Q_c is a travelling wave with finite energy such that $\alpha_c < 0$.

To show that there exists at least one negative eigenvalue, it is enough to show that α_c is achieved for a function $\varphi \in H^1(\mathbb{R}^2)$. Let us take a minimizing sequence $\varphi_n \in H^1(\mathbb{R}^2)$ such that $\|\varphi_n\|_{L^2(\mathbb{R}^2)} = 1$ and $B_{Q_c}(\varphi_n) \to \alpha_c$. We have

$$\int_{\mathbb{R}^2} |\nabla\varphi_n|^2 = B_{Q_c}(\varphi_n) + \int_{\mathbb{R}^2} \mathfrak{Re}(ic\partial_{x_2}\varphi_n\overline{\varphi_n}) + (1-|Q_c|^2)|\varphi_n|^2 - 2\mathfrak{Re}^2(\overline{Q_c}\varphi_n),$$

therefore, by Cauchy–Schwarz,

$$\int_{\mathbb{R}^2} |\nabla\varphi_n|^2 \leqslant |\alpha_c| + Kc\|\nabla\varphi_n\|_{L^2(\mathbb{R}^2)} + K.$$

We deduce that, for c small enough,

$$\|\nabla\varphi_n\|^2_{L^2(\mathbb{R}^2)} - Kc\|\nabla\varphi_n\|_{L^2(\mathbb{R}^2)} \leqslant K(c),$$

hence $\|\nabla\varphi_n\|^2_{L^2(\mathbb{R}^2)}$ is bounded uniformly in n given that $c < c_0$ for some constant c_0 small enough. We deduce that φ_n is bounded in $H^1(\mathbb{R}^2)$, therefore, up to a subsequence, $\varphi_n \to \varphi$ weakly in $H^1(\mathbb{R}^2)$.

Now, we note that for any $\varphi \in H^1(\mathbb{R}^2)$, by integration by parts (see Lemma 5.7),

$$\begin{aligned}\int_{\mathbb{R}^2} -\mathfrak{Re}(ic\partial_{x_2}\varphi\bar{\varphi}) &= -c\int_{\mathbb{R}^2}\mathfrak{Re}(\partial_{x_2}\varphi)\mathfrak{Im}(\varphi) + c\int_{\mathbb{R}^2}\mathfrak{Re}(\varphi)\mathfrak{Im}(\partial_{x_2}\varphi)\\ &= 2c\int_{\mathbb{R}^2}\mathfrak{Re}(\varphi)\mathfrak{Im}(\partial_{x_2}\varphi).\end{aligned}$$

For $R > 0$, since $\varphi_n \to \varphi$ weakly in $H^1(\mathbb{R}^2)$, this implies that $\varphi_n \to \varphi$ strongly in $L^2(B(0,R))$ by the Rellich–Kondrachov theorem. In particular, we have

$$\int_{B(0,R)} \mathfrak{Re}(\varphi_n)\mathfrak{Im}(\partial_{x_2}\varphi_n) \to \int_{B(0,R)} \mathfrak{Re}(\varphi)\mathfrak{Im}(\partial_{x_2}\varphi),$$

since $\varphi_n \to \varphi$ strongly in $L^2(B(0,R))$ and $\partial_{x_2}\varphi_n \to \partial_{x_2}\varphi$ weakly in $L^2(B(0,R))$. We deduce that, up to a subsequence,

$$\int_{B(0,R)} |\nabla\varphi|^2 + 2c\mathfrak{Re}(\varphi)\mathfrak{Im}(\partial_{x_2}\varphi) - (1-|Q_c|^2)|\varphi|^2 + 2\mathfrak{Re}^2(\overline{Q_c}\varphi)$$
$$\leqslant \liminf_{n\to\infty}\int_{B(0,R)} |\nabla\varphi_n|^2 + 2c\mathfrak{Re}(\varphi_n)\mathfrak{Im}(\partial_{x_2}\varphi_n) - (1-|Q_c|^2)|\varphi_n|^2 + 2\mathfrak{Re}^2(\overline{Q_c}\varphi_n) + o^R_{n\to\infty}(1).$$

Furthermore, we have, by weak convergence,

$$\|\varphi\|_{H^1(\mathbb{R}^2)} \leqslant \liminf_{n\to\infty}\|\varphi_n\|_{H^1(\mathbb{R}^2)} \leqslant K(c),$$

therefore, we estimate

$$\int_{\mathbb{R}^2\setminus B(0,R)} |\nabla\varphi|^2 + 2c\mathfrak{Re}(\varphi)\mathfrak{Im}(\partial_{x_2}\varphi) - (1-|Q_c|^2)|\varphi|^2 + 2\mathfrak{Re}^2(\overline{Q_c}\varphi)$$
$$\leqslant K\|\varphi\|^2_{H^1(\mathbb{R}^2\setminus B(0,R))} = o_{R\to\infty}(1).$$

We deduce that

$$B_{Q_c}(\varphi) \leqslant \liminf_{n\to\infty}\int_{B(0,R)} |\nabla\varphi_n|^2 + 2c\mathfrak{Re}(\varphi_n)\mathfrak{Im}(\partial_{x_2}\varphi_n) - (1-|Q_c|^2)|\varphi_n|^2$$
$$+ 2\mathfrak{Re}^2(\overline{Q_c}\varphi_n) + o^R_{n\to\infty}(1) + o_{R\to\infty}(1).$$

Now, we have

$$\liminf_{n\to\infty}\int_{B(0,R)} |\nabla\varphi_n|^2 + 2c\mathfrak{Re}(\varphi_n)\mathfrak{Im}(\partial_{x_2}\varphi_n) - (1-|Q_c|^2)|\varphi_n|^2 + 2\mathfrak{Re}^2(\overline{Q_c}\varphi_n)$$
$$= \liminf_{n\to\infty} B_{Q_c}(\varphi_n) - \liminf_{n\to\infty}\int_{\mathbb{R}^2\setminus B(0,R)} |\nabla\varphi_n|^2 + 2c\mathfrak{Re}(\varphi_n)\mathfrak{Im}(\partial_{x_2}\varphi_n)$$
$$- (1-|Q_c|^2)|\varphi_n|^2 + 2\mathfrak{Re}^2(\overline{Q_c}\varphi_n)$$

and $B_{Q_c}(\varphi_n) \to \alpha_c$, therefore

$$B_{Q_c}(\varphi) \leqslant \alpha_c + o^R_{n\to\infty}(1) + o_{R\to\infty}(1)$$
$$- \liminf_{n\to\infty}\int_{\mathbb{R}^2\setminus B(0,R)} |\nabla\varphi_n|^2 + 2c\mathfrak{Re}(\varphi_n)\mathfrak{Im}(\partial_{x_2}\varphi_n)$$
$$- (1-|Q_c|^2)|\varphi_n|^2 + 2\mathfrak{Re}^2(\overline{Q_c}\varphi_n).$$

From Theorem 2.5, we have $(1-|Q_c|^2)(x) \to 0$ when $|x| \to \infty$, therefore, since $\|\varphi_n\|_{L^2(\mathbb{R}^2)} = 1$, we have by dominated convergence that

$$\int_{\mathbb{R}^2\setminus B(0,R)} (1-|Q_c|^2)|\varphi_n|^2 \leqslant o_{R\to\infty}(1).$$

Furthermore, we check easily that (since $(A+B)^2 \geqslant \frac{1}{2}A^2 - B^2$)

$$\int_{\mathbb{R}^2\setminus B(0,R)} \mathfrak{Re}^2(\overline{Q_c}\varphi_n) \geqslant \frac{1}{2}\int_{\mathbb{R}^2\setminus B(0,R)} \mathfrak{Re}^2(Q_c)\mathfrak{Re}^2(\varphi_n) - \int_{\mathbb{R}^2\setminus B(0,R)} \mathfrak{Im}^2(Q_c)\mathfrak{Im}^2(\varphi_n),$$

and from Theorem 2.5, $\mathfrak{Im}(Q_c)(x) \to 0$ and $\mathfrak{Re}(Q_c)(x) \to 1$ when $|x| \to \infty$, therefore, since $\|\varphi_n\|_{L^2(\mathbb{R}^2)} = 1$, by dominated convergence,

$$\int_{\mathbb{R}^2\setminus B(0,R)} 2\mathfrak{Re}^2(\overline{Q_c}\varphi_n) \geqslant \int_{\mathbb{R}^2\setminus B(0,R)} \mathfrak{Re}^2(\varphi_n) - o_{R\to\infty}(1).$$

We deduce that, since $c < \sqrt{2}$,

$$\begin{aligned}
B_{Q_c}(\varphi) &\leqslant \alpha_c + o^R_{n\to\infty}(1) + o_{R\to\infty}(1) \\
&\quad - \liminf_{n\to\infty}\left(\int_{\mathbb{R}^2\setminus B(0,R)} |\nabla\varphi_n|^2 + 2c\mathfrak{Re}(\varphi_n)\mathfrak{Im}(\partial_{x_2}\varphi_n) + \mathfrak{Re}^2(\varphi_n)\right) \\
&\leqslant \alpha_c + o^R_{n\to\infty}(1) + o_{R\to\infty}(1) \\
&\quad - \liminf_{n\to\infty}\left(\int_{\mathbb{R}^2\setminus B(0,R)} (|\nabla\varphi_n| + c\mathfrak{Re}(\varphi_n))^2 + (2-c^2)\mathfrak{Re}^2(\varphi_n)\right) \\
&\leqslant \alpha_c + o^R_{n\to\infty}(1) + o_{R\to\infty}(1).
\end{aligned}$$

Thus, by letting $n \to \infty$ and then $R \to \infty$,

$$B_{Q_c}(\varphi) \leqslant \alpha_c.$$

In particular, this implies that $\|\varphi\|_{L^2(\mathbb{R}^2)} \neq 0$, or else $B_{Q_c}(\varphi) = 0 \leqslant \alpha_c$, and we know that $\alpha_c < 0$. Furthermore, by weak convergence, we have $\|\varphi\|_{L^2(\mathbb{R}^2)} \leqslant 1$, and if it is not 1, then, since $\alpha_c < 0$,

$$B_{Q_c}\left(\frac{\varphi}{\|\varphi\|_{L^2(\mathbb{R}^2)}}\right) \leqslant \frac{\alpha_c}{\|\varphi\|^2_{L^2(\mathbb{R}^2)}} < \alpha_c,$$

which is in contradiction with the definition of α_c. Therefore $\|\varphi\|_{L^2(\mathbb{R}^2)} = 1$ and $B_{Q_c}(\varphi) = \alpha_c$. This concludes the proof of Proposition 1.8. □

Proof of Corollary 1.10: The hypotheses to have the spectral stability from Theorem 11.8 of [**15**] are:

(1) The curve of travelling waves is C^1 from $]0, c_0[$ to $C^1(\mathbb{R}^2, \mathbb{C})$ with respect to the speed. This is a consequence of Theorem 1.1. This is enough to legitimate the computations done in the proof of Theorem 11.8 of [**15**].

(2) $\mathfrak{Re}(Q_c) - 1 \in H^1(\mathbb{R}^2)$, $\nabla Q_c \in L^2(\mathbb{R}^2)$, $|Q_c| \to 1$ at infinity, and $\|Q_c\|_{C^1(\mathbb{R}^2)} \leqslant K$. These are consequences of Theorem 7 of [**11**].

(3) $n^-(L_{Q_c}) \leqslant 1$. This is a consequence of Proposition 1.8.

(4) $\partial_c P_2(Q_c) < 0$. This is a consequence of Proposition 1.2. □

6. Coercivity results with an orthogonality condition on the phase

This section is devoted to the proofs of Propositions 1.11 and 1.12 and Theorem 1.13.

6.1. Properties of the space $H^{\exp}_{Q_c}$. In this subsection, we look at the space $H^{\exp}_{Q_c}$. We recall the norm

$$\|\varphi\|^2_{H^{\exp}_{Q_c}} = \|\varphi\|^2_{H^1(\{\tilde{r}\leqslant 10\})} + \int_{\{\tilde{r}\geqslant 5\}} |\nabla\psi|^2 + \mathfrak{Re}^2(\psi) + \frac{|\psi|^2}{\tilde{r}^2 \ln(\tilde{r})^2}.$$

The quadratic form we look at is

$$\begin{aligned}
B^{\exp}_{Q_c}(\varphi) = & \int_{\mathbb{R}^2} \eta(|\nabla\varphi|^2 - \mathfrak{Re}(ic\partial_{x_2}\varphi\bar{\varphi}) - (1-|Q_c|^2)|\varphi|^2 + 2\mathfrak{Re}^2(\overline{Q_c}\varphi)) \\
& - \int_{\mathbb{R}^2} \nabla\eta \cdot (\mathfrak{Re}(\nabla Q_c \overline{Q_c})|\psi|^2 - 2\mathfrak{Im}(\nabla Q_c\overline{Q_c})\mathfrak{Re}(\psi)\mathfrak{Im}(\psi)) \\
& + \int_{\mathbb{R}^2} c\partial_{x_2}\eta |Q_c|^2 \mathfrak{Re}(\psi)\mathfrak{Im}(\psi) \\
& + \int_{\mathbb{R}^2} (1-\eta)(|\nabla\psi|^2|Q_c|^2 + 2\mathfrak{Re}^2(\psi)|Q_c|^4) \\
& + \int_{\mathbb{R}^2} (1-\eta)(4\mathfrak{Im}(\nabla Q_c\overline{Q_c})\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi) + 2c|Q_c|^2\mathfrak{Im}(\partial_{x_2}\psi)\mathfrak{Re}(\psi)).
\end{aligned}$$

We will show in Lemma 6.1 that $B^{\exp}_{Q_c}(\varphi)$ is well defined for $\varphi \in H^{\exp}_{Q_c}$. The main difference between B_{Q_c} and $B^{\exp}_{Q_c}$ is the space on which they are defined. In particular, we can check easily for instance that, for $\varphi \in C^\infty_c(\mathbb{R}^2)$ with support far from the zeros of Q_c, we have $B^{\exp}_{Q_c}(\varphi) = B_{Q_c}(\varphi)$, as the terms with the gradient of the cutoff are exactly the ones coming from the integrations by parts. We start with a lemma about the space $H^{\exp}_{Q_c}$.

Lemma 6.1. *The following properties of $H^{\exp}_{Q_c}$ hold:*

$$H_{Q_c} \subset H^{\exp}_{Q_c},$$

$$iQ_c \in H^{\exp}_{Q_c}.$$

Furthermore, there exists $K(c) > 0$ such that, for $\varphi \in H_{Q_c}^{\exp}$,

$$\|\varphi\|_{\mathcal{C}} \leqslant K\|\varphi\|_{H_{Q_c}^{\exp}},$$
$$\|\varphi\|_{H_{Q_c}^{\exp}} \leqslant K(c)\|\varphi\|_{H_{Q_c}}, \tag{6.1}$$

and the integrands of $B_{Q_c}^{\exp}(\varphi)$, defined in (1.4), are in $L^1(\mathbb{R}^2)$ for $\varphi \in H_{Q_c}^{\exp}$, and $B_{Q_c}^{\exp}$ does not depend on the choice of η. Finally, if $\varphi \in H_{Q_c} \subset H_{Q_c}^{\exp}$,

$$B_{Q_c}(\varphi) = B_{Q_c}^{\exp}(\varphi).$$

See Appendix B.3 for the proof of this result.

Now, we state some lemmas that were shown previously in H_{Q_c}, which we have to extend to $H_{Q_c}^{\exp}$ to replace some arguments that were used in the proof of Proposition 1.4 by the proofs of Propositions 1.11 and 1.12 and Theorem 1.13. We start with the density argument.

Lemma 6.2. *$C_c^\infty(\mathbb{R}^2\backslash\{\tilde{d}_c\vec{e}_1, -\tilde{d}_c\vec{e}_1\}, \mathbb{C})$ is dense in $H_{Q_c}^{\exp}$ for $\|\cdot\|_{H_{Q_c}^{\exp}}$.*

Proof: The proof is identical to that of Lemma 3.4, as we check easily that, for $\lambda > \frac{10}{c}$ large enough,

$$\|\varphi\|^2_{H^1(\{\tilde{r}\leqslant 10\})} + \int_{\{\tilde{r}\geqslant 5\}\cap B(0,\lambda)} |\nabla\psi|^2 + \mathfrak{Re}^2(\psi) + \frac{|\psi|^2}{\tilde{r}^2\ln(\tilde{r})^2} \leqslant K_1(\lambda, c)\|\varphi\|^2_{H^1(B(0,\lambda))}$$

and

$$\|\varphi\|^2_{H^1(\{\tilde{r}\leqslant 10\})} + \int_{\{\tilde{r}\geqslant 5\}\cap B(0,\lambda)} |\nabla\psi|^2 + \mathfrak{Re}^2(\psi) + \frac{|\psi|^2}{\tilde{r}^2\ln(\tilde{r})^2} \geqslant K_2(\lambda, c)\|\varphi\|^2_{H^1(B(0,\lambda))}.$$

□

We also want to decompose the quadratic form, but with a fifth possible direction: iQ_c.

Lemma 6.3. *For $A \in \operatorname{Span}\{\partial_{x_1}Q_c, \partial_{x_2}Q_c, \partial_c Q_c, \partial_{c^\perp}Q_c, iQ_c\}$ and $\varphi \in C_c^\infty(\mathbb{R}^2\backslash\{\widetilde{d_c}\overrightarrow{e_1}, -\widetilde{d_c}\overrightarrow{e_1}\}, \mathbb{C})$, we have*

$$\langle L_{Q_c}(\varphi + A), \varphi + A\rangle = \langle L_{Q_c}(\varphi), \varphi\rangle + \langle 2L_{Q_c}(A), \varphi\rangle + \langle L_{Q_c}(A), A\rangle.$$

Furthermore, $\langle L_{Q_c}(\varphi + A), \varphi + A\rangle = B_{Q_c}^{\exp}(\varphi + A)$, $L_{Q_c}(iQ_c) = 0$ and

$$\|\partial_{x_1}Q_c\|_{H_{Q_c}^{\exp}} + \|\partial_{x_2}Q_c\|_{H_{Q_c}^{\exp}} + \|c^2\partial_c Q_c\|_{H_{Q_c}^{\exp}} + c^{\beta_0/2}\|c\partial_{c^\perp}Q_c\|_{H_{Q_c}^{\exp}} + \|iQ_c\|_{H_{Q_c}^{\exp}} \leqslant K(\beta_0).$$

Proof: As for the proof of Lemma 5.2, we only have to show that $\mathfrak{Re}(L_{Q_c}(A)\bar{A}) \in L^1(\mathbb{R}^2)$ to show the first equality.

By simple computation (or by invariance of the phase), we check that $L_{Q_c}(iQ_c) = 0$. Writing $A = T + \varepsilon iQ_c$ for $\varepsilon \in \mathbb{R}$, $T \in \operatorname{Span}\{\partial_{x_1} Q_c, \partial_{x_2} Q_c, \partial_c Q_c, \partial_{c^\perp} Q_c\}$, we compute from Lemma 2.8 that

$$L_{Q_c}(A) = L_{Q_c}(T) \in \operatorname{Span}_{\mathbb{R}}(i\partial_{x_1} Q_c, i\partial_{x_2} Q_c),$$

thus

$$\mathfrak{Re}(L_{Q_c}(A)\bar{A}) = \mathfrak{Re}(L_{Q_c}(T)\overline{T + \varepsilon iQ_c}) = \mathfrak{Re}(L_{Q_c}(T)\bar{T}) + \varepsilon \mathfrak{Re}(L_{Q_c}(T)\overline{iQ_c}).$$

From the proof of Lemma 5.2, we have $\mathfrak{Re}(L_{Q_c}(T)\bar{T}) \in L^1(\mathbb{R}^2)$, and since $L_{Q_c}(T) \in \operatorname{Span}_{\mathbb{R}}(i\partial_{x_1} Q_c, i\partial_{x_2} Q_c)$, with Theorem 2.5, we have

$$|\mathfrak{Re}(L_{Q_c}(T)\overline{iQ_c})| \leqslant \frac{K(c)}{(1+r)^3} \in L^1(\mathbb{R}^2).$$

Let us check that, for $\varphi \in H^{\exp}_{Q_c}$, $B^{\exp}_{Q_c}(\varphi + \varepsilon iQ_c) = B^{\exp}_{Q_c}(\varphi)$ for $\varepsilon \in \mathbb{R}$.

We check, from (1.4), that, for $\varphi \in C^\infty_c(\mathbb{R}^2 \backslash \{\widetilde{d_c}\overrightarrow{e_1}, -\widetilde{d_c}\overrightarrow{e_1}\}, \mathbb{C})$, this equality holds by integration by parts and because $\mathfrak{Re}(\psi + i) = \mathfrak{Re}(\psi)$, $\mathfrak{Im}(\nabla(\psi + i)) = \mathfrak{Im}(\nabla \psi)$. We then argue by density, as in the proof of Proposition 1.4.

We deduce, from Lemmas 2.8 and 5.2, that for $\varphi \in C^\infty_c(\mathbb{R}^2 \backslash \{\widetilde{d_c}\overrightarrow{e_1}, -\widetilde{d_c}\overrightarrow{e_1}\}, \mathbb{C})$,

$$\begin{aligned} B^{\exp}_{Q_c}(\varphi + A) &= B^{\exp}_{Q_c}(\varphi + T) = B_{Q_c}(\varphi + T) \\ &= \langle L_{Q_c}(\varphi + T), \varphi + T\rangle = \langle L_{Q_c}(\varphi + A), \varphi + T\rangle \\ &= \langle L_{Q_c}(\varphi + A), \varphi + A\rangle - \langle L_{Q_c}(\varphi + A), \varepsilon iQ_c\rangle, \end{aligned}$$

and we check, with Lemma 2.8, that for some $v \in \mathbb{R}^2$ depending on A,

$$\begin{aligned} \langle L_{Q_c}(\varphi + A), \varepsilon iQ_c\rangle &= \langle L_{Q_c}(\varphi), \varepsilon iQ_c\rangle + \langle L_{Q_c}(P), \varepsilon iQ_c\rangle \\ &= \varepsilon\langle \varphi, L_{Q_c}(iQ_c)\rangle + \varepsilon v \cdot \int_{\mathbb{R}^2} \mathfrak{Re}(\nabla Q_c \overline{Q_c}) \\ &= 0. \end{aligned}$$

From Lemma 2.9, we have

$$\|\partial_{x_1} Q_c\|_{\mathcal{C}} + \|\partial_{x_2} Q_c\|_{\mathcal{C}} + \|c^2 \partial_c Q_c\|_{\mathcal{C}} + c^{\beta_0/2}\|c\partial_{c^\perp} Q_c\|_{\mathcal{C}} \leqslant K(\beta_0),$$

and with Lemmas 2.1 and 2.3 and equations (2.9), (2.10), and (2.11), we check with the definition of $\|\cdot\|_{H^{\exp}_{Q_c}}$ and $\|\cdot\|_{\mathcal{C}}$ that, for $A \in \{\partial_{x_1} Q_c, \partial_{x_2} Q_c, c^2 \partial_c Q_c, c^{1+\beta_0/2}\partial_{c^\perp} Q_c\}$,

$$\|A\|^2_{H^{\exp}_{Q_c}} \leqslant K\|A\|^2_{H^1(\{\tilde{r} \leqslant 10\})} + \|A\|^2_{\mathcal{C}} \leqslant K(\beta_0).$$

Finally, we check that

$$\|iQ_c\|^2_{H^{\exp}_{Q_c}} = \|iQ_c\|^2_{H^1(\{\tilde{r} \leqslant 10\})} + \int_{\{\tilde{r} \geqslant 5\}} |\nabla i|^2 + \mathfrak{Re}^2(i) + \frac{|i|^2}{\tilde{r}^2 \ln(\tilde{r})^2} \leqslant K. \quad \square$$

We can now end the proof of Proposition 1.11.

Proof of Proposition 1.11: From Theorem 1.5, for $\varphi \in C_c^\infty(\mathbb{R}^2 \backslash \{\widetilde{d_c}\overrightarrow{e_1}, -\widetilde{d_c}\overrightarrow{e_1}\}, \mathbb{C})$, under the four orthogonality conditions of Proposition 1.11, we have, by Lemma 6.1,

$$B_{Q_c}^{\exp}(\varphi) = B_{Q_c}(\varphi) = \langle L_{Q_c}(\varphi), \varphi \rangle \geqslant K \|\varphi\|_{\mathcal{C}}^2.$$

We then conclude by density, as in the proof of Proposition 1.4, using Lemma 6.2. The proof for the density in $B_{Q_c}^{\exp}$ is similar to the one for B_{Q_c} in the proof of Proposition 1.4. Coercivity under three orthogonality conditions can be shown similarly.

Then, for the computation of the kernel, the proof is identical to that of Corollary 1.6. With Lemma 6.1, we check easily that we can do the same computation simply by replacing $B_{Q_c}(\varphi)$ by $B_{Q_c}^{\exp}(\varphi)$. The only difference is at the end, when we have $\|\varphi^*\|_{\mathcal{C}} = 0$; this implies that $\varphi^* = \lambda i Q_c$ for some $\lambda \in \mathbb{R}$, and we cannot conclude that $\lambda = 0$, since we only have $\varphi^* \in H_{Q_c}^{\exp}$ instead of $\varphi^* \in H_{Q_c}$. This implies that

$$\varphi \in \mathrm{Span}_{\mathbb{R}}(\partial_{x_1} Q_c, \partial_{x_2} Q_c, i Q_c).$$

Using Lemmas 2.8 and 6.3, we check easily the implication from (ii) to (i). □

6.2. Change of the coercivity norm with an orthogonality condition on the phase. We now focus on the proofs of Proposition 1.12 and Theorem 1.13. In these results, we add an orthogonality condition on the phase. We start with a lemma giving the coercivity result but with the original orthogonality conditions on the vortices, adding the one on the phase.

Lemma 6.4. *For $\varphi = Q_c \psi \in H_{Q_c}^{\exp}$, if the following four orthogonality conditions are satisfied:*

$$\int_{B(\tilde{d}_c \overrightarrow{e_1}, R)} \mathfrak{Re}(\partial_{x_1} \tilde{V}_1 \overline{\tilde{V}_1 \psi}) = \int_{B(\tilde{d}_c \overrightarrow{e_1}, R)} \mathfrak{Re}(\partial_{x_2} \widetilde{V_1} \overline{\widetilde{V_1} \psi}) = 0,$$

$$\int_{B(-\tilde{d}_c \overrightarrow{e_1}, R)} \mathfrak{Re}(\partial_{x_1} \tilde{V}_{-1} \overline{\tilde{V}_{-1} \psi}) = \int_{B(-\tilde{d}_c \overrightarrow{e_1}, R)} \mathfrak{Re}(\partial_{x_2} \tilde{V}_{-1} \overline{\tilde{V}_{-1} \psi}) = 0,$$

then, if $\mathfrak{Re} \int_{B(0,R)} i\psi = 0$, we have (with $K(c) \leqslant 1$)

$$B_{Q_c}^{\exp}(\varphi) \geqslant K(c) \|\varphi\|_{H_{Q_c}^{\exp}}^2 + K \|\varphi\|_{\mathcal{C}}^2,$$

or if $\forall x \in \mathbb{R}$, $\varphi(x_1, x_2) = \varphi(-x_1, x_2)$ *and* $\mathfrak{Re} \int_{B(\tilde{d}_c \vec{e_1}, R) \cup B(-\tilde{d}_c \vec{e_1}, R)} iQ_c \bar{\varphi} = 0$, *then*

$$B^{\exp}_{Q_c}(\varphi) \geqslant K \|\varphi\|^2_{H^{\exp}_{Q_c}}.$$

Proof: Let us show these results for $\varphi = Q_c \psi \in C^\infty_c(\mathbb{R}^2 \backslash \{\tilde{d}_c \vec{e}_1, -\tilde{d}_c \vec{e}_1\}, \mathbb{C})$. We then conclude by density. We start with the nonsymmetric case.

By Lemma 4.4, for $\varphi = Q_c \psi \in C^\infty_c(\mathbb{R}^2 \backslash \{\tilde{d}_c \vec{e}_1, -\tilde{d}_c \vec{e}_1\}, \mathbb{C})$ such that

$$\int_{B(\tilde{d}_c \vec{e_1}, R)} \mathfrak{Re}(\partial_{x_1} \tilde{V}_1 \overline{\widetilde{V_1} \psi}) = \int_{B(\tilde{d}_c \vec{e_1}, R)} \mathfrak{Re}(\partial_{x_2} \widetilde{V_1} \overline{\widetilde{V_1} \psi}) = 0,$$

we have

$$B^{\mathrm{loc}_1, D}_{Q_c}(\varphi) \geqslant K(D) \int_{B(\tilde{d}_c \vec{e_1}, D)} |\nabla \psi|^2 |Q_c|^4 + \mathfrak{Re}^2(\psi) |Q_c|^4.$$

By Lemma 4.3, we infer, by a standard proof by contradiction (with the first two orthogonality conditions),

$$B^{\mathrm{loc}_1, D}_{Q_c}(\varphi) \geqslant K_1(D) \|\varphi\|^2_{H^1(B(\tilde{d}_c \vec{e_1}, D))} - K_2(D) \left(\int_{B(\tilde{d}_c \vec{e_1}, R) \backslash B(\tilde{d}_c \vec{e_1}, R/2)} \mathfrak{Im}(\psi) \right)^2.$$

We deduce, with Lemma 4.3, that for any small $\varepsilon > 0$,

$$\begin{aligned} B^{\mathrm{loc}_1, D}_{Q_c}(\varphi) \geqslant{} & K(D)(1-\varepsilon) \int_{B(\tilde{d}_c \vec{e_1}, D)} |\nabla \psi|^2 |Q_c|^4 + \mathfrak{Re}^2(\psi) |Q_c|^4 \\ & + K_1(D) \varepsilon \|\varphi\|^2_{H^1(B(\tilde{d}_c \vec{e_1}, D))} - K_2(D) \varepsilon \left(\int_{B(\tilde{d}_c \vec{e_1}, R) \backslash B(\tilde{d}_c \vec{e_1}, R/2)} \mathfrak{Im}(\psi) \right)^2. \end{aligned}$$

By Poincaré inequality, if $\mathfrak{Re} \int_{B(0,R)} i\psi = 0$, then

$$\begin{aligned} \int_{B(\tilde{d}_c \vec{e_1}, R) \backslash B(\tilde{d}_c \vec{e_1}, R/2)} \mathfrak{Im}(\psi) & \leqslant K(c) \sqrt{\int_{\mathbb{R}^2 \backslash (B(\tilde{d}_c \vec{e_1}, R/2) \cup B(-\tilde{d}_c \vec{e_1}, R/2))} |\nabla \psi|^2} \\ & \leqslant K(c) \sqrt{\int_{\mathbb{R}^2} |\nabla \psi|^2 |Q_c|^4}. \end{aligned}$$

Therefore, for any small $\mu > 0$, taking $\varepsilon > 0$ small enough (depending on c, D, and μ),

$$\begin{aligned} B^{\mathrm{loc}_1, D}_{Q_c}(\varphi) \geqslant{} & K(D) \int_{B(\tilde{d}_c \vec{e_1}, D)} |\nabla \psi|^2 |Q_c|^4 + \mathfrak{Re}^2(\psi) |Q_c|^4 \\ & + K_1(D, c, \mu) \|\varphi\|^2_{H^1(B(\tilde{d}_c \vec{e_1}, D))} - \mu \int_{\mathbb{R}^2} |\nabla \psi|^2 |Q_c|^4. \end{aligned}$$

With similar arguments, we have a similar result for $B_{Q_c}^{\mathrm{loc}_{-1},D}(\varphi)$. Now, as in the proof of Proposition 1.4, we have, taking $\mu>0$ small enough and $D>0$ large enough,

$$\begin{aligned}B_{Q_c}(\varphi) &\geqslant B_{Q_c}^{\mathrm{loc}_{1},D}(\varphi)+B_{Q_c}^{\mathrm{loc}_{-1},D}(\varphi)\\ &\quad +K\left(\int_{\mathbb{R}^2\setminus(B(\tilde{d}_c\vec{e_1},D)\cup B(-\tilde{d}_c\vec{e_1},D))}|\nabla\psi|^2|Q_c|^4+\mathfrak{Re}^2(\psi)|Q_c|^4\right)\\ &\geqslant K\int_{\mathbb{R}^2}|\nabla\psi|^2|Q_c|^4+\mathfrak{Re}^2(\psi)|Q_c|^4+K_1(c,\mu)\|\varphi\|^2_{H^1(B(\tilde{d}_c\vec{e_1},10))}\\ &\quad -\mu\int_{\mathbb{R}^2}|\nabla\psi|^2|Q_c|^4\\ &\geqslant K\|\varphi\|^2_{\mathcal{C}}+K(c)\|\varphi\|^2_{H^1(B(\tilde{d}_c\vec{e_1},10))}.\end{aligned}$$

Then, by the same Hardy-type inequality as in the proof of Proposition 1.4, we show that

$$\int_{\mathbb{R}^2}\frac{|\varphi|^2}{(1+\tilde{r})^2\ln^2(2+\tilde{r})}\leqslant K\left(\|\varphi\|^2_{H^1(B(\tilde{d}_c\vec{e_1},10))}+\int_{\mathbb{R}^2}|\nabla\psi|^2|Q_c|^4\right),$$

therefore

$$B_{Q_c}(\varphi)\geqslant K\|\varphi\|^2_{\mathcal{C}}+K(c)\|\varphi\|^2_{H^{\exp}_{Q_c}}.$$

In the symmetric case, the proof is identical, except that, by symmetry,

$$\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)}iQ_c\bar{\varphi}=0,$$

and we check by Poincaré inequality that for a function φ satisfying this orthogonality condition, $\varphi=Q_c\psi$,

$$\left|\int_{B(\tilde{d}_c\vec{e_1},R)\setminus B(\tilde{d}_c\vec{e_1},R/2)}\mathfrak{Im}(\psi)\right|\leqslant K\|\varphi\|_{H^1(B(\tilde{d}_c\vec{e_1},R))},$$

for a universal constant $K>0$. By a similar computation as previously, we conclude the proof of this lemma. □

We now have all the elements necessary to conclude the proof of Proposition 1.12.

Proof of Proposition 1.12: This proof follows the proof of Lemma 5.5. For $\varphi\in C_c^\infty(\mathbb{R}^2\setminus\{\tilde{d}_c\vec{e}_1,-\tilde{d}_c\vec{e}_1\},\mathbb{C})$ and five real-valued parameters ε_1, ε_2, ε_3, ε_4, ε_5 we define $\varphi^*=Q_c\psi^*$ by

$$\psi^*=\psi+\varepsilon_1\frac{\partial_{x_1}Q_c}{Q_c}+\varepsilon_2\frac{c^2\partial_cQ_c}{Q_c}+\varepsilon_3\frac{\partial_{x_2}Q_c}{Q_c}+\varepsilon_4\frac{c\partial_{c^\perp}Q_c}{Q_c}+\varepsilon_5 i.$$

With Lemma 6.3, we check that $\varphi^* \in H^{\exp}_{Q_c}$. Now, similarly as in the proof of Lemma 5.5, we check that

$$\begin{aligned}\int_{B(\tilde{d}_c\vec{e_1},R)} \mathfrak{Re}(\partial_{x_1}\widetilde{V_1}\overline{\widetilde{V_1}\psi^*}) &= \int_{B(\tilde{d}_c\vec{e_1},R)} \mathfrak{Re}(\partial_{x_1}\widetilde{V_1}\overline{\widetilde{V_1}\psi}) \\ &+\varepsilon_1 \int_{B(\tilde{d}_c\vec{e_1},R)} \mathfrak{Re}\left(\partial_{x_1}\widetilde{V_1}\overline{\partial_{x_1}Q_c\frac{\widetilde{V_1}}{Q_c}}\right) \\ &+\varepsilon_2 \int_{B(\tilde{d}_c\vec{e_1},R)} \mathfrak{Re}\left(\partial_{x_1}\widetilde{V_1}\overline{c^2\partial_c Q_c\frac{\widetilde{V_1}}{Q_c}}\right) \\ &+\varepsilon_3 \int_{B(\tilde{d}_c\vec{e_1},R)} \mathfrak{Re}\left(\partial_{x_1}\widetilde{V_1}\overline{\partial_{x_2}Q_c\frac{\widetilde{V_1}}{Q_c}}\right) \\ &+\varepsilon_4 \int_{B(\tilde{d}_c\vec{e_1},R)} \mathfrak{Re}\left(\partial_{x_1}\widetilde{V_1}\overline{c\partial_{c^\perp}Q_c\frac{\widetilde{V_1}}{Q_c}}\right) \\ &+\varepsilon_5 \int_{B(\tilde{d}_c\vec{e_1},R)} \mathfrak{Re}(\partial_{x_1}\widetilde{V_1}\overline{i\widetilde{V_1}}).\end{aligned}$$

Furthermore, with Lemma 2.1, we check that

$$\int_{B(\tilde{d}_c\vec{e_1},R)} \mathfrak{Re}(\partial_{x_1}V_1\overline{i\widetilde{V_1}}) = 0,$$

and the other terms are estimated as in the proof of Lemma 5.5. Similarly,

$$\begin{aligned}\int_{B(\tilde{d}_c\vec{e_1},R)} \mathfrak{Re}(\partial_{x_2}V_1\overline{i\widetilde{V_1}}) &= \int_{B(-\tilde{d}_c\vec{e_1},R)} \mathfrak{Re}(\partial_{x_1}V_{-1}\overline{i\widetilde{V_{-1}}}) \\ &= \int_{B(-\tilde{d}_c\vec{e_1},R)} \mathfrak{Re}(\partial_{x_2}V_{-1}\overline{i\widetilde{V_{-1}}}) = 0.\end{aligned}$$

We also check that, from (2.9) and (2.10) and Lemmas 2.3 and 2.7 that

$$\begin{aligned}&\left|\int_{B(0,R)} \mathfrak{Re}\left(i\frac{\partial_{x_1}Q_c}{Q_c}\right)\right| + \left|\int_{B(0,R)} \mathfrak{Re}\left(i\frac{\partial_{x_2}Q_c}{Q_c}\right)\right| \\ &\quad + \left|\int_{B(0,R)} \mathfrak{Re}\left(ic^2\frac{\partial_c Q_c}{Q_c}\right)\right| + \left|\int_{B(0,R)} \mathfrak{Re}\left(ci\frac{\partial_{c^\perp}Q_c}{Q_c}\right)\right| \\ &= o_{c\to 0}(1),\end{aligned}$$

and

$$\int_{B(0,R)} \mathfrak{Re}(i\times i) = -\pi R^2 < 0.$$

We deduce, as in the proof of Lemma 5.5, that

$$\begin{pmatrix} \int_{B(\tilde{d}_c\vec{e_1},R)} \mathfrak{Re}(\partial_{x_1}\widetilde{V_1}\overline{\widetilde{V_1}\psi^*}) \\ \int_{B(-\tilde{d}_c\vec{e_1},R)} \mathfrak{Re}(\partial_{x_1}\tilde{V}_{-1}\overline{\tilde{V}_{-1}\psi^*}) \\ \int_{B(\tilde{d}_c\vec{e_1},R)} \mathfrak{Re}(\partial_{x_2}\widetilde{V_1}\overline{\widetilde{V_1}\psi^*}) \\ \int_{B(-\tilde{d}_c\vec{e_1},R)} \mathfrak{Re}(\partial_{x_2}\tilde{V}_{-1}\overline{\tilde{V}_{-1}\psi^*}) \\ \mathfrak{Re}\int_{B(0,R)} i\psi = 0 \end{pmatrix}$$

$$= \left(\begin{pmatrix} K(R) & -K(R) & 0 & 0 & 0 \\ K(R) & K(R) & 0 & 0 & 0 \\ 0 & 0 & K(R) & -K(R) & 0 \\ 0 & 0 & K(R) & K(R) & 0 \\ 0 & 0 & 0 & 0 & -\pi R^2 \end{pmatrix} + o_{c\to 0}(1)\right)\begin{pmatrix} c\varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \end{pmatrix}$$
$$+ o_{c\to 0}(c^{\beta_0})K\|\varphi\|_{\mathcal{C}}.$$

Therefore, we can find $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5 \in \mathbb{R}$ such that

$$|\varepsilon_1| + |\varepsilon_2| + |\varepsilon_3| + |\varepsilon_4| + |\varepsilon_5| \leqslant o_{c\to 0}(c^{\beta_0})\|\varphi\|_{\mathcal{C}}$$

and φ^* satisfies the five orthogonality conditions of Lemma 6.4. Therefore,

$$B^{\exp}_{Q_c}(\varphi^*) \geqslant K(c)\|\varphi^*\|^2_{H^{\exp}_{Q_c}} + K\|\varphi^*\|^2_{\mathcal{C}}.$$

We continue as in the proof of Lemma 5.5, and with the same arguments, we have

$$B^{\exp}_{Q_c}(\varphi) \geqslant K(c)\|\varphi^*\|^2_{H^{\exp}_{Q_c}} + K\|\varphi\|^2_{\mathcal{C}}.$$

Now, by Lemma 6.3, we have

$$\begin{aligned} \|\varphi^*\|_{H^{\exp}_{Q_c}} &\geqslant \|\varphi\|_{H^{\exp}_{Q_c}} - \|\varepsilon_1\partial_{x_1}Q_c + \varepsilon_2 c^2\partial_c Q_c + \varepsilon_3\partial_{x_2}Q_c + \varepsilon_4 c\partial_{c^\perp}Q_c + \varepsilon_5 i\|_{H^{\exp}_{Q_c}} \\ &\geqslant \|\varphi\|_{H^{\exp}_{Q_c}} - o_{c\to 0}(c^{\beta_0/2})\|\varphi\|_{\mathcal{C}}, \end{aligned}$$

thus, since we can take $K(c) \leqslant 1$, we have

$$B^{\exp}_{Q_c}(\varphi) \geqslant K(c)\|\varphi\|^2_{H^{\exp}_{Q_c}}.$$

We conclude by density as in the proof of Proposition 1.4, thanks to Lemma 6.2. We are left with the proof of $B^{\exp}_{Q_c}(\varphi) \leqslant K\|\varphi\|^2_{H^{\exp}_{Q_c}}$. With regard to (1.4), the local terms can be estimated by $K\|\varphi\|^2_{H^1(\{\tilde{r}\leqslant 10\})} \leqslant K\|\varphi\|^2_{H^{\exp}_{Q_c}}$ and the terms at infinity, by Cauchy–Schwarz, can be estimated by $K\int_{\{\tilde{r}\geqslant 5\}} |\nabla\psi|^2 + \mathfrak{Re}^2(\psi) + \frac{|\psi|^2}{\tilde{r}^2\ln^2(\tilde{r})} \leqslant K\|\varphi\|^2_{H^{\exp}_{Q_c}}$. □

As it was done in equation (5.12), we can replace the orthogonality condition $\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)}\partial_c Q_c\overline{Q_c\psi^{\neq 0}}=0$ by

(6.2) $$\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)}\partial_d(V_1(x-d\vec{e}_1)V_{-1}(x+d\vec{e}_1))_{|d=d_c}\overline{Q_c\psi^{\neq 0}}(x)\,dx=0$$

in Propositions 1.11 and 1.12.

Proof of Theorem 1.13: This proof follows closely the proof of Proposition 1.12.

First, with Lemma 2.3 and the definition of $\partial_{c^\perp}Q_c$ in Lemma 2.7, we check that $\partial_{x_1}Q_c$ and $\partial_{c^\perp}Q_c$ are odd in x_1, and for $\varphi = Q_c\psi \in C_c^\infty(\mathbb{R}^2\backslash\{\tilde{d}_c\vec{e}_1,-\tilde{d}_c\vec{e}_1\},\mathbb{C})$ with $\forall(x_1,x_2)\in\mathbb{R}^2$, $\varphi(x_1,x_2)=\varphi(-x_1,x_2)$, we check that in $B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)$, $Q_c\psi^{\neq 0}$ is even in x_1. Therefore, these two orthogonality conditions are freely given.

We decompose as previously, for ε_1, ε_2, ε_3 three real-valued parameters,

$$\varphi=\varphi^*+\varepsilon_1 iQ_c+\varepsilon_2\partial_{x_2}Q_c+\varepsilon_3c^2\partial_cQ_c.$$

We suppose that

$$\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)}\partial_cQ_c\bar{\varphi}=\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)}\partial_{x_2}Q_c\bar{\varphi}=0,$$

$$\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)}iQ_c\bar{\varphi}=0,$$

and we show, as in the proof of Lemma 5.5, that we can find $\varepsilon_1,\varepsilon_2,\varepsilon_3\in\mathbb{R}$ such that

$$|\varepsilon_1|+|\varepsilon_2|+|\varepsilon_3|\leqslant o_{c\to 0}(c^{\beta_0})\|\varphi\|_{H^{\exp}_{Q_c}},$$

and φ^* satisfies the five orthogonality conditions of Lemma 6.4 (we recall that two of them are given by symmetry). Here, since we did not remove the 0-harmonics, the error is only controlled by $\|\varphi\|_{H^{\exp}_{Q_c}}$ instead of $\|\varphi\|_{\mathcal{C}}$. For instance, we have

$$\begin{aligned}\int_{B(\tilde{d}_c\vec{e_1},R)}|\mathfrak{Re}((\partial_{x_2}\widetilde{V_1}\overline{\widetilde{V_1}}-\partial_{x_2}Q_c\overline{Q_c})\psi)|&\leqslant o_{c\to 0}(1)\|Q_c\psi\|_{L^2(B(\tilde{d}_c\vec{e_1},R))}\\&=o_{c\to 0}(1)\|\varphi\|_{H^{\exp}_{Q_c}}.\end{aligned}$$

Now, from Lemma 6.4, since $\varphi^*\in H^{\exp}_{Q_c}$, we have

$$B^{\exp}_{Q_c}(\varphi^*)\geqslant K\|\varphi^*\|^2_{H^{\exp}_{Q_c}}.$$

We continue, as in the proof of Lemma 5.5, with $|\varepsilon_1|+|\varepsilon_2|+|\varepsilon_3| = o_{c\to 0}(1)\|\varphi\|_{H^{\exp}_{Q_c}}$ and Lemma 6.3. We show that

$$B^{\exp}_{Q_c}(\varphi)\geqslant K\|\varphi\|^2_{H^{\exp}_{Q_c}}.$$

We conclude the proof of Theorem 1.13 by density. □

7. Local uniqueness result

This section is devoted to the proof of Theorem 1.14. This proof will follow classical schemes for local uniqueness using coercivity. Here, we will use Propositions 1.11 and 1.12, with equation (6.2).

7.1. Construction of a perturbation. For a given $\vec{c'} \in \mathbb{R}^2$, $0 < |\vec{c'}| \leqslant c_0$ (c_0 defined in Theorem 1.1), $X \in \mathbb{R}^2$, and $\gamma \in \mathbb{R}$, we define, thanks to (1.1), the travelling wave

$$Q := Q_{\vec{c'}}(\cdot - X)e^{i\gamma}. \tag{7.1}$$

We define a smooth cutoff function η, with value 0 in $B(\pm\tilde{d}_c\overrightarrow{e_1}, R+1)$ ($R > 10$ is defined in Theorem 1.5), and 1 outside of $B(\tilde{d}_c\overrightarrow{e_1}, R+2) \cup B(-\tilde{d}_c\overrightarrow{e_1}, R+2)$. The first step is to define a function ψ such that

$$(1-\eta)Q\psi + \eta Q(e^{\psi} - 1) = Z - Q, \tag{7.2}$$

with $Q\psi$ satisfying the orthogonality conditions of Propositions 1.11 and 1.12. We start by showing that there exists a function ψ solution of (7.2). We denote $\delta^{|\cdot|}(c\overrightarrow{e_2}, \vec{c'}) := \left|(c\overrightarrow{e_2} - \vec{c'}) \cdot \frac{\vec{c'}}{|\vec{c'}|}\right|$ and $\delta^{\perp}(c\overrightarrow{e_2}, \vec{c'}) := \left|c\overrightarrow{e_2} \cdot \frac{\vec{c'}^{\perp}}{|\vec{c'}|}\right|$. At fixed c, these two quantities characterize $\vec{c'}$, since they are the coordinates of the vector $c\overrightarrow{e_2} - \vec{c'}$ in the basis $\left(\frac{\vec{c'}}{|\vec{c'}|}, \frac{\vec{c'}^{\perp}}{|\vec{c'}|}\right)$. We will use them as variables instead of $\vec{c'}$, this decomposition being well adapted to the problem.

Since both Z and $|Q|$ go to 1 at infinity, we have that such a function ψ is bounded at infinity. The perturbation here is chosen additively close to the zeros of the travelling wave, and multiplicatively at infinity. This seems to be a fit form for the perturbation, and we have already used it in the construction of Q_c.

Lemma 7.1. *There exists $c_0 > 0$ such that, for $0 < c < c_0$ and any $\Lambda > \frac{10}{c}$, with Z a function satisfying the hypothesis of Theorem 1.14 and Q defined by (7.1) with $\frac{c}{2} \leqslant |\vec{c'}| \leqslant 2c$, there exist $K, K(\Lambda) > 0$ such that*

$$\|Z-Q\|_{C^1(B(0,\Lambda))} \leqslant K(\Lambda)\|Z-Q_c\|_{H^{\exp}_{Q_c}} + K\left(|X| + \frac{\delta^{|\cdot|}(c\overrightarrow{e_2}, \vec{c'})}{c^2} + \frac{\delta^{\perp}(c\overrightarrow{e_2}, \vec{c'})}{c} + |\gamma|\right).$$

We will mainly use this result for $\Lambda = \lambda + 1$, $\lambda > 0$, defined in Theorem 1.14.

Proof: We recall that such a function Z is in $C^\infty(\mathbb{R}^2, \mathbb{C})$ by elliptic regularity.

We start with the estimate of $w := Q_c - Z$ in $B(0,\Lambda)$. Since both Z and Q_c solve (TW_c), we have

$$-\Delta w = (1-|Q_c|^2)Q_c - (1-|Z|^2)Z + ic\partial_{x_2} w.$$

From Theorem 8.8 of [**7**], $\Omega := B(0,\Lambda)$, $2\Omega = B(0,2\Lambda)$,

$$\|w\|_{H^2(\Omega)} \leqslant K(\Lambda)(\|w\|_{H^1(2\Omega)} + \|ic\partial_{x_2} w + (1-|Q_c|^2)Q_c - (1-|Z|^2)Z\|_{L^2(2\Omega)}).$$

We compute that

$$(1-|Q_c|^2)Q_c - (1-|Z|^2)Z = (Q_c - Z)(1-|Q_c|^2) + Z(|Q_c| - |Z|)(|Q_c| + |Z|).$$

From [**6**], we have that any travelling wave of finite energy is bounded in $L^\infty(\mathbb{R}^2)$ by a universal constant, i.e.

$$|Q_c| + |Z| \leqslant K,$$

therefore

$$|1-|Q_c|^2| + |Z|(|Q_c| + |Z|) \leqslant K$$

for a universal constant K. Thus,

$$\|(1-|Q_c|^2)Q_c - (1-|Z|^2)Z\|_{L^2(2\Omega)} \leqslant K\|w\|_{L^2(2\Omega)},$$

and we deduce, from Lemma 2.6, that

$$\|w\|_{H^2(\Omega)} \leqslant K(\Lambda)(\|w\|_{H^1(2\Omega)} + \|ic\partial_{x_2} w\|_{L^2(2\Omega)} + \|w\|_{L^2(2\Omega)}) \leqslant K(\Lambda)\|w\|_{H^{\exp}_{Q_c}}.$$

By standard elliptic arguments, we have that for every $k \geqslant 2$,

$$\|w\|_{W^{k,2}(\Omega)} \leqslant K(\Lambda,k)\|w\|_{H^{\exp}_{Q_c}}.$$

By Sobolev embeddings, we estimate

$$\|w\|_{C^1(\Omega)} \leqslant K(\Lambda)\|w\|_{W^{4,2}(\Omega)} \leqslant K(\Lambda)\|w\|_{H^{\exp}_{Q_c}}. \tag{7.3}$$

From (7.3), we have

$$\|Z-Q\|_{L^\infty(\Omega)} \leqslant \|Q-Q_c\|_{L^\infty(\Omega)} + \|w\|_{L^\infty(\Omega)} \leqslant \|Q-Q_c\|_{L^\infty(\mathbb{R}^2)} + K(\Lambda)\|w\|_{H^{\exp}_{Q_c}}.$$

We estimate

$$\begin{aligned}\|Q-Q_c\|_{L^\infty(\mathbb{R}^2)} &= \|Q_{\vec{c'}}(\cdot - X)e^{i\gamma} - Q_c\|_{L^\infty(\mathbb{R}^2)} \\ &\leqslant \|Q_{\vec{c'}}(\cdot - X)e^{i\gamma} - Q_{\vec{c'}}(\cdot - X)\|_{L^\infty(\mathbb{R}^2)} + \|Q_{\vec{c'}}(\cdot - X) - Q_{\vec{c'}}\|_{L^\infty(\mathbb{R}^2)} \\ &\quad + \|Q_{\vec{c'}} - Q_{|\vec{c'}|\vec{e}_2}\|_{L^\infty(\mathbb{R}^2)} + \|Q_{|\vec{c'}|\vec{e}_2} - Q_c\|_{L^\infty(\mathbb{R}^2)}.\end{aligned}$$

We check, with Theorem 1.1 and Lemma 2.7, that

$$\|\nabla Q\|_{L^\infty(\mathbb{R}^2)} + c^2\|\partial_c Q\|_{L^\infty(\mathbb{R}^2)} + c\|\partial_{c^\perp} Q\|_{L^\infty(\mathbb{R}^2)} + \|iQ\|_{L^\infty(\mathbb{R}^2)} \leqslant K,$$

and that it also holds for any travelling wave of the form $Q_{\vec{\varsigma}}(\cdot - Y)e^{i\beta}$ if $2c \geqslant |\vec{\varsigma}| \geqslant c/2$, $Y \in \mathbb{R}^2$, and $\beta \in \mathbb{R}$.

We check that $\|Q_{\vec{c'}}(\cdot - X)e^{i\gamma} - Q_{\vec{c'}}(\cdot - X)\|_{L^\infty(\mathbb{R}^2)} \leqslant |e^{i\gamma} - 1|\|Q_{\vec{c'}}(\cdot - X)\|_{L^\infty(\mathbb{R}^2)} \leqslant K|\gamma|$, and we estimate (by the mean value theorem)

$$\|Q_{\vec{c'}}(\cdot - X) - Q_{\vec{c'}}\|_{L^\infty(\mathbb{R}^2)} \leqslant K|X|\|\nabla Q_{\vec{c'}}\|_{L^\infty(\mathbb{R}^2)} \leqslant K|X|.$$

Similarly, we have

$$\|Q_{\vec{c'}} - Q_{|\vec{c'}|\vec{e}_2}\|_{L^\infty(\mathbb{R}^2)} \leqslant K\frac{\delta^{\perp}(c\overrightarrow{e_2}, \vec{c'}) + \delta^{|\cdot|}(c\overrightarrow{e_2}, \vec{c'})}{c}$$

and $\|Q_{|\vec{c'}|\vec{e}_2} - Q_c\|_{L^\infty(\mathbb{R}^2)} \leqslant K\frac{\delta^{|\cdot|}(c\overrightarrow{e_2}, \vec{c'})}{c^2}$. We deduce that (since $c \leqslant 1$)

$$\|Q - Q_c\|_{L^\infty(\mathbb{R}^2)} \leqslant K\left(|X| + \frac{\delta^{|\cdot|}(c\overrightarrow{e_2}, \vec{c'})}{c^2} + \frac{\delta^{\perp}(c\overrightarrow{e_2}, \vec{c'})}{c} + |\gamma|\right),$$

and thus

$$\|Z - Q\|_{L^\infty(B(0,\Lambda))} \leqslant K(\Lambda)\|Z - Q_c\|_{H^{\exp}_{Q_c}} + K\left(|X| + \frac{\delta^{|\cdot|}(c\overrightarrow{e_2}, \vec{c'})}{c^2} + \frac{\delta^{\perp}(c\overrightarrow{e_2}, \vec{c'})}{c} + |\gamma|\right).$$

Finally, from Lemmas 2.1, 2.2, and 2.3, $\partial_{c^\perp} Q_c = -x^\perp \cdot \nabla Q_c$ and equation (2.11), we have

$$\|\nabla\partial_{x_2} Q\|_{L^\infty(\mathbb{R}^2)} + c^2\|\nabla\partial_c Q\|_{L^\infty(\mathbb{R}^2)} + c\|\nabla\partial_{c^\perp} Q\|_{L^\infty(\mathbb{R}^2)} + \|i\nabla Q_c\|_{L^\infty(\mathbb{R}^2)} \leqslant K.$$

We deduce that

$$\|\nabla(Q - Q_c)\|_{L^\infty(\mathbb{R}^2)} \leqslant K\left(|X| + \frac{\delta^{|\cdot|}(c\overrightarrow{e_2}, \vec{c'})}{c^2} + \frac{\delta^{\perp}(c\overrightarrow{e_2}, \vec{c'})}{c} + |\gamma|\right),$$

and, by (7.3),

$$\|\nabla(Z - Q)\|_{L^\infty(B(0,\Lambda))} \leqslant K(\Lambda)\|Z - Q_c\|_{H^{\exp}_{Q_c}} + K\left(|X| + \frac{\delta^{|\cdot|}(c\overrightarrow{e_2}, \vec{c'})}{c^2} + \frac{\delta^{\perp}(c\overrightarrow{e_2}, \vec{c'})}{c} + |\gamma|\right). \qquad \square$$

Lemma 7.2. *There exists $\varepsilon_0(c) > 0$ small such that, for Z a function satisfying the hypothesis of Theorem 1.14 with*

$$|X| + \frac{\delta^{|\cdot|}(c\overrightarrow{e_2}, \vec{c'})}{c^2} + \frac{\delta^{\perp}(c\overrightarrow{e_2}, \vec{c'})}{c} + |\gamma| \leqslant \varepsilon_0(c),$$

there exists a function $Q\psi \in C^1(\mathbb{R}^2, \mathbb{C})$ such that (7.2) holds. Furthermore, for any $\Lambda > \frac{10}{c}$, there exist $K, K(\Lambda) > 0$ such that

$$\|Q\psi\|_{C^1(B(0,\Lambda))} \leqslant K(\Lambda)\|Z - Q_c\|_{H^{\exp}_{Q_c}} + K\left(|X| + \frac{\delta^{|\cdot|}(c\overrightarrow{e_2}, \vec{c'})}{c^2} + \frac{\delta^{\perp}(c\overrightarrow{e_2}, \vec{c'})}{c} + |\gamma|\right).$$

Proof: First, taking $\varepsilon_0(c)$ small enough (depending on c), we check that $\frac{c}{2} \leqslant |\vec{c'}| \leqslant 2c$.

We recall equation (7.2):

$$(1-\eta)Q\psi + \eta Q(e^{\psi} - 1) = Z - Q.$$

We write it in the form

$$\psi + \eta(e^{\psi} - 1 - \psi) = \frac{Z-Q}{Q},$$

and in $\{\eta = 0\}$, we therefore define

$$\psi = \frac{Z-Q}{Q}. \tag{7.4}$$

Now, we define the set $\Omega := B(0,\lambda+1)\backslash(B(d_c\vec{e}_1, R-1) \cup B(-d_c\vec{e}_1, R-1))$. In this set, we have that

$$\left\| \frac{Z-Q}{Q} \right\|_{C^1(\Omega)} \leqslant K\varepsilon_0(c) + K(\lambda)\|Z - Q_c\|_{H^{\exp}_{Q_c}}$$

by Lemma 7.1 and (2.12). Therefore, since $e^{\psi} - 1 - \psi$ is at least quadratic in $\psi \in C^1(\Omega, \mathbb{C})$, by a fixed point argument (on $H(\psi) := \frac{Z-Q}{Q} - \eta(e^{\psi} - 1 - \psi)$, which is a contraction on $\|\psi\|_{L^\infty(\{\eta \neq 0\})} < \mu$ for $\mu > 0$ small enough), we deduce that on Ω, given that ε_0 and $\|Z - Q_c\|_{H^{\exp}_{Q_c}}$ are small enough (depending on λ for $\|Z - Q_c\|_{H^{\exp}_{Q_c}}$), there exists a unique function $\psi \in C^1(\Omega, \mathbb{C})$ such that $\psi + \eta(e^{\psi} - 1 - \psi) = \frac{Z-Q}{Q}$ in Ω. By uniqueness, since we have a solution of the same problem on $\{\eta = 0\}$ which intersects Ω, we can construct $Q\psi \in C^1(B(0,\lambda+1), \mathbb{C})$ such that $\eta Q\psi + (1-\eta)Q(e^{\psi} - 1) = Z - Q$ in $B(0, \lambda+1)$.

Furthermore, here we use the hypothesis that, outside of $B(0,\lambda)$, $|Z - Q_c| \leqslant \mu_0$. We deduce that (taking $\mu_0 < \frac{1}{4}$) there exists $\delta > 0$ such that $|Z| > \delta$ outside of $B(0,\lambda)$. In particular, since λ can be taken large, we have that outside of $B(0,\lambda)$, $\eta = 1$. The equation on ψ then becomes

$$e^{\psi} = \frac{Z}{Q},$$

and by equation (2.12) and $|Z| > \delta$, we deduce that there exists a unique solution to this problem in $C^1(\mathbb{R}^2 \backslash B(0,\lambda), \mathbb{C})$ that is equal on $B(0,\lambda+1)\backslash B(0,\lambda)$ to the previously constructed function ψ.

Therefore, there exists $Q\psi \in C^1(\mathbb{R}^2, \mathbb{C})$ such that $(1-\eta)Q\psi + \eta Q(e^{\psi} - 1) = Z - Q$ in $\mathbb{R}^2$. Furthermore, we check that (by the fixed point argument), since $\{\eta \neq 1\} \subset B(0,\lambda)$,

$$\|\psi\|_{C^1(\{\eta\neq 1\})} \leqslant K \left\| \frac{Z-Q}{Q} \right\|_{C^1(\{\eta\neq 1\})} \leqslant K(\lambda)\|Z-Q_c\|_{H^{\exp}_{Q_c}} + K\left(|X| + \frac{\delta^{|\cdot|}(c\overrightarrow{e_2}, \vec{c'})}{c^2} + \frac{\delta^{\perp}(c\overrightarrow{e_2}, \vec{c'})}{c} + |\gamma|\right).$$

From equation (2.12) and Lemma 7.1, we have

$$\|Q\psi\|_{C^1(B(0,\Lambda))} \leqslant \|Z - Q\|_{C^1(B(0,\Lambda))} + K\|\psi\|_{C^1(\{\eta\neq 1\})} + K(\Lambda)\|Z-Q_c\|_{H^{\exp}_{Q_c}} \leqslant K(\Lambda)\|Z-Q_c\|_{H^{\exp}_{Q_c}} + K\left(|X| + \frac{\delta^{|\cdot|}(c\overrightarrow{e_2}, \vec{c'})}{c^2} + \frac{\delta^{\perp}(c\overrightarrow{e_2}, \vec{c'})}{c} + |\gamma|\right).$$

This concludes the proof of the lemma. □

Lemma 7.3. *The functions Q and ψ, defined respectively in* (7.1) *and Lemma 7.2, satisfy*

$$\varphi := Q\psi \in H^{\exp}_Q.$$

Furthermore, $\varphi \in C^2(\mathbb{R}^2, \mathbb{C})$ and there exists $K(\lambda, c, \|Z-Q_c\|_{H^{\exp}_{Q_c}}, \varepsilon_0, Z) > 0$ such that, in $\{\eta = 1\}$ (i.e. far from the vortices),

$$|\nabla\psi| + |\mathfrak{Re}(\psi)| + |\Delta\psi| \leqslant \frac{K(\lambda, c, \|Z-Q_c\|_{H^{\exp}_{Q_c}}, \varepsilon_0, Z)}{(1+r)^2},$$

$$|\nabla\mathfrak{Re}(\psi)| \leqslant \frac{K(\lambda, c, \|Z-Q_c\|_{H^{\exp}_{Q_c}}, \varepsilon_0, Z)}{(1+r)^3},$$

and

$$|\mathfrak{Im}(\psi + i\gamma)| \leqslant \frac{K(\lambda, c, \|Z-Q_c\|_{H^{\exp}_{Q_c}}, \varepsilon_0, Z)}{(1+r)}.$$

We note that here, since $\psi \nrightarrow 0$ at infinity (if $\gamma \neq 0$), we do not have $Q\psi \in H_Q$. This is one of the main reasons to introduce the space $H^{\exp}_Q$. See Appendix C.1 for the proof of this result.

Lemma 7.4. *The functions Q and ψ, defined respectively in* (7.1) *and Lemma 7.2, satisfy, with $\varphi = Q\psi$,*

$$\langle L^{\exp}_Q(\varphi), (\varphi + i\gamma Q)\rangle = B^{\exp}_Q(\varphi),$$

where $L^{\exp}_Q(\varphi) = (1-\eta)L_Q(\varphi) + \eta Q L'_Q(\psi)$, with

$$L'_Q(\psi) = -\Delta\psi - 2\frac{\nabla Q}{Q}\cdot\nabla\psi + i\vec{c}\cdot\nabla\psi + 2\mathfrak{Re}(\psi)|Q|^2.$$

Furthermore,

$$L_Q(\varphi) = QL'_Q(\psi).$$

See Appendix C.2 for the proof of this result.

The equality $\langle L_Q^{\exp}(\varphi), (\varphi + i\gamma Q)\rangle = B_Q^{\exp}(\varphi)$ is not obvious for functions $\varphi \in C^2(\mathbb{R}^2, \mathbb{C}) \cap H_{Q_c}^{\exp}$ (because of some integration by parts to justify) and we need to check that, for the particular function φ we have constructed, this result holds. We will use mainly the decay estimates of Lemma 7.3.

Morally, we are showing that, since $L_Q(i\gamma Q) = 0$, we can do the following computation: $\langle L_Q(\varphi), \varphi + i\gamma Q\rangle = \langle \varphi, L_Q(\varphi + i\gamma Q)\rangle = \langle \varphi, L_Q(\varphi)\rangle = B_Q(\varphi)$. The goal of this lemma is simply to check that, with the estimates of Lemma 7.3, the integrands are integrable and the integration by parts can be done to have $\langle L_Q^{\exp}(\varphi), (\varphi + i\gamma Q)\rangle = B_Q^{\exp}(\varphi)$.

7.2. Properties of the perturbation. We look for the equation satisfied by $\varphi = Q\psi$ in the next lemma.

Lemma 7.5. *The functions Q and ψ, defined respectively in* (7.1) *and Lemma 7.2, with $\varphi = Q\psi$, satisfy the equation*

$$L_Q(Q\psi) - i(c\overrightarrow{e_2} - \vec{c'}) \cdot H(\psi) + \mathrm{NL}_{\mathrm{loc}}(\psi) + F(\psi) = 0,$$

with L_Q the linearized operator around Q: $L_Q(\varphi) := -\Delta\varphi - i\vec{c} \cdot \nabla\varphi - (1 - |Q|^2)\varphi + 2\mathfrak{Re}(\bar{Q}\varphi)Q$,

$$S(\psi) := e^{2\mathfrak{Re}(\psi)} - 1 - 2\mathfrak{Re}(\psi),$$

$$F(\psi) := Q\eta(-\nabla\psi \cdot \nabla\psi + |Q|^2 S(\psi)),$$

$$H(\psi) := \nabla Q + \frac{\nabla(Q\psi)(1-\eta) + Q\nabla\psi\eta e^{\psi}}{(1-\eta) + \eta e^{\psi}},$$

and $\mathrm{NL}_{\mathrm{loc}}(\psi)$ is a sum of terms at least quadratic in ψ, localized in the area where $\eta \neq 1$. Furthermore,

$$|\langle \mathrm{NL}_{\mathrm{loc}}(\psi), Q(\psi + i\gamma)\rangle| \leqslant K(\|Q\psi\|_{C^1(\{\eta\neq 1\})} + |\gamma|)\|Q\psi\|^2_{H^1(\{\eta\neq 1\})}.$$

Note that here, the equation satisfied by φ has a "source" term, $i(c\overrightarrow{e_2} - \vec{c'}) \cdot H(\psi)$, coming from the fact that Z and Q_c might not have the same speed at this point. We will estimate it later on.

Proof: The function Z solves (TW_c), hence,

$$i(c\overrightarrow{e_2} - \vec{c'}) \cdot \nabla Z = -i\vec{c'} \cdot \nabla Z - \Delta Z - (1 - |Z|^2)Z.$$

From (7.2), we have

$$Z = Q + (1-\eta)Q\psi + \eta Q(e^{\psi} - 1).$$

We define

$$\zeta := 1 + \psi - e^{\psi}.$$

We replace $Z = Q+(1-\eta)Q\psi+\eta Q(e^{\psi}-1)$ in $-i\vec{c'}\cdot\nabla Z-\Delta Z-(1-|Z|^2)Z$ exactly as in the proof of Lemma 2.7 of [**4**], by simply changing V, Ψ, $c\vec{e}_2$, η respectively to Q, ψ, $\vec{c'}$, $1-\eta$. In particular, $E-ic\partial_{x_2}V$ becomes 0 (since $\mathrm{TW}_{\vec{c'}}(Q) = 0$). This computation yields

$$i(\overrightarrow{ce_2} - \vec{c'}) \cdot \nabla Z = ((1-\eta) + \eta e^{\psi})(L_Q(Q\psi) + \widetilde{\mathrm{NL}}_{\mathrm{loc}}(\psi) + F(\psi)).$$

Furthermore, we have that $((1-\eta) + \eta e^{\psi}) \neq 0$ by Lemma 7.2 and equation (C.2) (for the same reason as in the proof of Lemma 2.7 of [**4**]), and we compute (as in Lemma 2.7 of [**4**]) that

$$\frac{\eta e^{\psi}}{(1-\eta) + \eta e^{\psi}} = \eta + \eta(1-\eta)\left(\frac{e^{\psi} - 1}{(1-\eta) + \eta e^{\psi}}\right). \tag{7.5}$$

Furthermore, we have

$$\begin{aligned}\nabla Z &= \nabla Q - Q\nabla\eta\zeta + \nabla Q((1-\eta)\psi + \eta(e^{\psi} - 1)) + Q\nabla\psi((1-\eta) + \eta e^{\psi}) \\ &= \nabla Q(1 - \eta + \eta e^{\psi}) - Q\nabla\eta\zeta + \nabla(Q\psi)(1-\eta) + Q\nabla\psi\eta e^{\psi},\end{aligned}$$

hence

$$\frac{\nabla Z}{(1-\eta) + \eta e^{\psi}} = \nabla Q - \frac{Q\nabla\eta\zeta}{(1-\eta) + \eta e^{\psi}} + \frac{\nabla(Q\psi)(1-\eta) + Q\nabla\psi\eta e^{\psi}}{(1-\eta) + \eta e^{\psi}},$$

therefore, with $\mathrm{NL}_{\mathrm{loc}}(\psi) = \widetilde{\mathrm{NL}}_{\mathrm{loc}}(\psi) + i(\overrightarrow{ce_2} - \vec{c'}) \cdot \frac{-Q\nabla\eta\zeta}{(1-\eta)+\eta e^{\psi}}$, we have

$$L_Q(Q\psi) - i(\overrightarrow{ce_2} - \vec{c'}) \cdot H(\psi) + \mathrm{NL}_{\mathrm{loc}}(\psi) + F(\psi) = 0.$$

Finally, we check, similarly as in the proof of Lemma 2.7 of [**4**], that

$$|\langle \mathrm{NL}_{\mathrm{loc}}(\psi), Q(\psi + i\gamma)\rangle| \leqslant K(\|Q\psi\|_{C^1(\{\eta\neq 1\})} + |\gamma|) \int_{\mathbb{R}^2} |\mathrm{NL}_{\mathrm{loc}}(\psi)|,$$

hence

$$|\langle \mathrm{NL}_{\mathrm{loc}}(\psi), Q(\psi + i\gamma)\rangle| \leqslant K(\|Q\psi\|_{C^1(\{\eta\neq 1\})} + |\gamma|)\|Q\psi\|^2_{H^1(\{\eta\neq 1\})}. \qquad \square$$

Now, we want to choose the right parameters γ, $\vec{c'}$, X so that φ satisfies the orthogonality conditions of Propositions 1.11 and 1.12 (with equation (6.2)).

Lemma 7.6. *For the functions Q and ψ, defined respectively in* (7.1) *and Lemma 7.2, there exist $X, \vec{c'} \in \mathbb{R}^2$, $\gamma \in \mathbb{R}$ such that*

$$|X| + \frac{\delta^{|\cdot|}(c\overrightarrow{e_2}, \vec{c'})}{c^2} + \frac{\delta^{\perp}(c\overrightarrow{e_2}, \vec{c'})}{c} + |\gamma| \leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1),$$

and

$$\mathfrak{Re}\int_{B(\boldsymbol{d}_{\vec{c'},1},R)\cup B(\boldsymbol{d}_{\vec{c'},2},R)} \partial_{x_1} Q\overline{Q\psi^{\neq 0}} = \mathfrak{Re}\int_{B(\boldsymbol{d}_{\vec{c'},1},R)\cup B(\boldsymbol{d}_{\vec{c'},2},R)} \partial_{x_2} Q\overline{Q\psi^{\neq 0}} = 0,$$

$$\mathfrak{Re}\int_{B(\boldsymbol{d}_{\vec{c'},1},R)\cup B(\boldsymbol{d}_{\vec{c'},2},R)} \partial_{c^\perp} Q\overline{Q\psi^{\neq 0}} = 0,$$

$$\mathfrak{Re}\int_{B(\boldsymbol{d}_{\vec{c'},1},R)\cup B(\boldsymbol{d}_{\vec{c'},2},R)} \partial_{\boldsymbol{d}}\boldsymbol{V}\overline{Q\psi^{\neq 0}} = 0,$$

$$\mathfrak{Re}\int_{B((\boldsymbol{d}_{\vec{c'},1}+\boldsymbol{d}_{\vec{c'},2})/2,R)} i\psi = 0,$$

where $\boldsymbol{d}_{\vec{c'},1}$ and $\boldsymbol{d}_{\vec{c'},2}$ are the zeros of Q, $\boldsymbol{d}_{\vec{c'},1}$ being the closest one to $\tilde{d}_c\overrightarrow{e_1}$, and $\partial_{\boldsymbol{d}}\boldsymbol{V}$ is the first order of Q by Theorem 1.1 and (1.1).

See Appendix C.3 for the proof of this result.

Here, the notations for the harmonics are done for Q, and are therefore centred around $\boldsymbol{d}_{\vec{c'},1}$ or $\boldsymbol{d}_{\vec{c'},2}$. This means that $\psi^{\neq 0}(x) = \psi(x) - \psi^{\mathbf{0},1}(\boldsymbol{r}_1)$ with $\boldsymbol{r}_1 := |x - \boldsymbol{d}_{\vec{c'},1}|$, $x - \boldsymbol{d}_{\vec{c'},1} = \boldsymbol{r}_1 e^{i\boldsymbol{\theta}_1} \in \mathbb{R}^2$, and $\psi^{\mathbf{0},2}$ being the 0-harmonic of ψ around $\boldsymbol{d}_{\vec{c'},1}$ in $B(\boldsymbol{d}_{\vec{c'},1}, R)$, and $\psi^{\neq 0}(x) = \psi(x) - \psi^{\mathbf{0},2}(\boldsymbol{r}_2)$ with $\boldsymbol{r}_2 := |x - \boldsymbol{d}_{\vec{c'},2}|$ in $B(\boldsymbol{d}_{\vec{c'},2}, R)$ and $\psi^{\mathbf{0},1}$ being the 0-harmonic of ψ around $\boldsymbol{d}_{\vec{c'},2}$. We will denote $\psi^{\mathbf{0}}(x)$ the quantity equal to $\psi^{\mathbf{0},1}(\boldsymbol{r}_1)$ in the right half-plane and to $\psi^{\mathbf{0},2}(\boldsymbol{r}_2)$ in the left half-plane. Note that $\boldsymbol{d}_{\vec{c'},1} \in \mathbb{R}^2$, whereas $\tilde{d}_c \in \mathbb{R}$. We recall that, taking $\|Z - Q_c\|_{H^{\exp}_{Q_c}}$ small enough, we have $\frac{\delta^{|\cdot|}(c\overrightarrow{e_2},\vec{c'})}{c^2} \leqslant 1$, and in particular, for c small enough, this implies that $\frac{c}{2} \leqslant |\vec{c'}| \leqslant 2c$. We recall that $o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)$ is a quantity going to 0 when $\|Z - Q_c\|_{H^{\exp}_{Q_c}} \to 0$ at fixed λ and c.

7.3. End of the proof of Theorem 1.14. From Lemmas 7.3 and 7.6, we can find $\varphi = Q\psi \in H_Q^{\exp}$ such that

$$|X| + \frac{\delta^{|\cdot|}(c\overrightarrow{e_2}, \vec{c'})}{c^2} + \frac{\delta^{\perp}(c\overrightarrow{e_2}, \vec{c'})}{c} + |\gamma| \leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1), \tag{7.6}$$

and

$$\mathfrak{Re}\int_{B(\boldsymbol{d}_{\vec{c'},1},R)\cup B(\boldsymbol{d}_{\vec{c'},2},R)} \partial_{x_1} Q\overline{Q\psi^{\neq 0}} = \mathfrak{Re}\int_{B(\boldsymbol{d}_{\vec{c'},1},R)\cup B(\boldsymbol{d}_{\vec{c'},2},R)} \partial_{x_2} Q\overline{Q\psi^{\neq 0}} = 0,$$
$$\mathfrak{Re}\int_{B(\boldsymbol{d}_{\vec{c'},1},R)\cup B(\boldsymbol{d}_{\vec{c'},2},R)} \partial_{\boldsymbol{d}} \boldsymbol{V}\overline{Q\psi^{\neq 0}} = \mathfrak{Re}\int_{B(\boldsymbol{d}_{\vec{c'},1},R)\cup B(\boldsymbol{d}_{\vec{c'},2},R)} \partial_{c^\perp} Q\overline{Q\psi^{\neq 0}} = 0,$$
$$\mathfrak{Re}\int_{B((\boldsymbol{d}_{\vec{c'},1}+\boldsymbol{d}_{\vec{c'},2})/2,R)} i\psi = 0.$$

Now, from Lemma 7.5, ψ satisfies the equation

$$L_Q(Q\psi) - i(\vec{c'} - c\overrightarrow{e_2})\cdot H(\psi) + \mathrm{NL}_{\mathrm{loc}}(\psi) + F(\psi) = 0. \tag{7.7}$$

We note that

$$L_Q(Q\psi) = (1-\eta)L_Q(Q\psi) + \eta Q L'_Q(\psi),$$

and by Lemmas 7.3 and 7.4,

$$\langle (1-\eta)L_Q(Q\psi) + \eta Q L'_Q(\psi), Q(\psi + i\gamma)\rangle = B_Q^{\exp}(\varphi).$$

We deduce that

$$\begin{aligned} B_Q^{\exp}(\varphi) - \langle i(\vec{c'} - c\overrightarrow{e_2})\cdot H(\psi), Q(\psi+i\gamma)\rangle& \\ +\langle \mathrm{NL}_{\mathrm{loc}}(\psi), Q(\psi+i\gamma)\rangle + \langle F(\psi), Q(\psi+i\gamma)\rangle &= 0.\end{aligned} \tag{7.8}$$

Since $Q\psi \in H_Q^{\exp}$ by Lemma 7.3, with the orthogonality conditions satisfied (see Lemma 7.6), we can apply Propositions 1.11 and 1.12 with equation (6.2). We have

$$B_Q^{\exp}(\varphi) \geqslant K\|\varphi\|_{\mathcal{C}}^2 + K(c)\|\varphi\|^2_{H_Q^{\exp}}. \tag{7.9}$$

7.3.1. Better estimates on $\vec{c'} - c\overrightarrow{e_2}$. The term $i(\vec{c'} - c\overrightarrow{e_2})\cdot H(\psi)$ contains a "source" term, because Z and Q do not satisfy the same equation (since the travelling waves Z and Q may not have the same speed at this point). We want to show the following estimates:

$$\delta^{|\cdot|}(c\overrightarrow{e_2},\vec{c'}) \leqslant \left(Kc^2\ln\left(\frac{1}{c}\right) + o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\right)\|\varphi\|_{\mathcal{C}} + o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|_{H^{\exp}_{Q_c}} \tag{7.10}$$

and

$$\delta^{\perp}(c\overrightarrow{e_2},\vec{c'}) \leqslant \left(Kc^2\ln\left(\frac{1}{c}\right) + o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\right)\|\varphi\|_{\mathcal{C}} + o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|_{H^{\exp}_{Q_c}}. \tag{7.11}$$

This subsection is devoted to the proof of (7.10) and (7.11).

Step 1. We have the estimate (7.10).

We take the scalar product of (7.7) with $c^2\partial_c Q$, which yields

$$\langle i(\vec{c'}-c\overrightarrow{e_2})\cdot H(\psi), c^2\partial_c Q\rangle = \langle Q\psi, c^2 L_Q(\partial_c Q)\rangle + \langle \mathrm{NL}_{\mathrm{loc}}(\psi)+F(\psi), c^2\partial_c Q\rangle.$$

We check here, with the L^∞ estimates on ψ and its derivatives, as well as on $\partial_c Q$ (see Lemmas 2.3 and 7.3), that $\langle L_Q(Q\psi), c^2\partial_c Q\rangle$ is well defined and that all the integrations by parts can be done.

We recall that $H(\psi) = \nabla Q + \frac{\nabla(Q\psi)(1-\eta)+Q\nabla\psi\eta e^{\psi}}{(1-\eta)+\eta e^{\psi}}$, and we check with equation (7.5) that, since $1-\eta$ is compactly supported (in a domain with size independent of c, $\vec{c'}$), we have

$$\left|\left\langle i(\vec{c'}-c\overrightarrow{e_2})\cdot\frac{\nabla(Q\psi)(1-\eta)+Q\nabla\psi\eta e^{\psi}}{(1-\eta)+\eta e^{\psi}}, c^2\partial_c Q\right\rangle\right| \leqslant K|(\vec{c'}-c\overrightarrow{e_2})\cdot\langle \eta iQ\nabla\psi, c^2\partial_c Q\rangle| + K|\vec{c'}-c\overrightarrow{e_2}|\|\varphi\|_{H^{\exp}_{Q_c}}.$$

We compute with Lemma 2.3 that

$$\begin{aligned}
|\langle \eta iQ\nabla\psi, c^2\partial_c Q\rangle| &= \left|\int_{\mathbb{R}^2}\eta\mathfrak{Re}(\nabla\psi iQc^2\overline{\partial_c Q})\right| \\
&\leqslant \left|\int_{\mathbb{R}^2}\eta\mathfrak{Re}(\nabla\psi)\mathfrak{Im}(Qc^2\overline{\partial_c Q})\right| + \left|\int_{\mathbb{R}^2}\eta\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(Qc^2\overline{\partial_c Q})\right| \\
&\leqslant \left|\int_{\mathbb{R}^2}\eta\mathfrak{Re}(\psi)\nabla(\mathfrak{Im}(Qc^2\overline{\partial_c Q}))\right| + K\|\varphi\|_{H^{\exp}_{Q_c}} \\
&\quad + \|\varphi\|_{\mathcal{C}}\sqrt{\int_{\mathbb{R}^2}\eta\mathfrak{Re}^2(Qc^2\overline{\partial_c Q})}.
\end{aligned}$$

With Lemmas 2.2 and 2.3, we check that $\int_{\mathbb{R}^2}\eta\mathfrak{Re}^2(Qc^2\overline{\partial_c Q})\leqslant K$, and furthermore,

$$|\nabla(\mathfrak{Im}(Qc^2\overline{\partial_c Q}))|\leqslant c^2|\partial_c Q||\nabla Q|+Kc^2|\nabla\partial_c Q|$$

and with Lemma 2.3 (with $\sigma=1/2$), we check that

$$|\nabla(\mathfrak{Im}(Qc^2\overline{\partial_c Q}))|\leqslant\frac{K}{(1+\tilde{r})^{3/2}},$$

thus, by Cauchy–Schwarz,

$$\left|\int_{\mathbb{R}^2}\eta\mathfrak{Re}(\psi)\nabla(\mathfrak{Im}(Qc^2\overline{\partial_c Q}))\right|\leqslant K\|\varphi\|_{\mathcal{C}}.$$

Using $|\vec{c'}-c\overrightarrow{e_2}|\leqslant K(c)(\delta^{|\cdot|}(c\overrightarrow{e_2},\vec{c'})+\delta^{\perp}(c\overrightarrow{e_2},\vec{c'}))\leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)$ and $\|\varphi\|_{\mathcal{C}}\leqslant K\|\varphi\|_{H^{\exp}_{Q_c}}$, we deduce that

$$\left|\left\langle i(\vec{c'}-c\overrightarrow{e_2})\cdot\frac{(1-\eta)\nabla(Q\psi)+\eta e^{\psi}Q\nabla\psi}{(1-\eta)+\eta e^{\psi}},c^2\partial_c Q\right\rangle\right|\leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1).$$

Furthermore, we check that, by symmetry (see (2.3)),

$$\langle i(\vec{c'}-c\overrightarrow{e_2})\cdot\nabla_x Q,c^2\partial_c Q\rangle=\delta^{|\cdot|}(c\overrightarrow{e_2},\vec{c'})\left\langle i\frac{\vec{c'}}{|\vec{c'}|}\cdot\nabla Q,c^2\partial_c Q\right\rangle.$$

Furthermore, from Lemma 2.8, we have $L_Q(\partial_c Q)=i\nabla_{\vec{c'}}Q$, therefore, from Proposition 1.2,

$$\left\langle i\frac{\vec{c'}}{|\vec{c'}|}\cdot\nabla Q,c^2\partial_c Q\right\rangle=c^2B_Q(\partial_c Q)=-2\pi+o_{c\to 0}(1).$$

We deduce that

$$\delta^{|\cdot|}(c\overrightarrow{e_2},\vec{c'})\leqslant K|\langle Q\psi,c^2L_Q(\partial_c Q)\rangle+\langle\mathrm{NL}_{\mathrm{loc}}(\psi)+F(\psi),c^2\partial_c Q\rangle|$$
$$+o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|_{H^{\exp}_{Q}}.$$

Now, since $L_Q(\partial_c Q)=i\frac{\vec{c'}}{|\vec{c'}|}\cdot\nabla Q$, we check that

$$\langle Q\psi,c^2L_Q(\partial_c Q)\rangle=c^2\left\langle Q\psi,i\frac{\vec{c'}}{|\vec{c'}|}\cdot\nabla Q\right\rangle,$$

and

$$\left|\left\langle Q\psi,i\frac{\vec{c'}}{|\vec{c'}|}\cdot\nabla Q\right\rangle\right|\leqslant\left|\int_{\mathbb{R}^2}\mathfrak{Re}(\psi)\mathfrak{Im}\left(\frac{\vec{c'}}{|\vec{c'}|}\cdot\nabla Q\bar{Q}\right)\right|+\left|\int_{\mathbb{R}^2}\mathfrak{Im}(\psi)\mathfrak{Re}\left(\frac{\vec{c'}}{|\vec{c'}|}\cdot\nabla Q\bar{Q}\right)\right|.$$

From Lemma 5.4, we deduce that

$$|\langle Q\psi,c^2L_Q(\partial_c Q)\rangle|\leqslant Kc^2\ln\left(\frac{1}{c}\right)\|\varphi\|_{\mathcal{C}}.$$

Now, we check easily that, with Lemmas 7.1 and 7.5,

$$|\langle \mathrm{NL_{loc}}(\psi), c^2\partial_c Q\rangle| \leqslant K(c)\|\varphi\|_{H_Q^{\exp}}\|\varphi\|_{C^1(B(0,\lambda))} \leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|_{H_Q^{\exp}}.$$

To conclude the proof of estimate (7.10), we shall estimate

$$|\langle F(\psi), c^2\partial_c Q\rangle| \leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|_{H_Q^{\exp}} + (K\lambda_0 + o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1))\|\varphi\|_C,$$

with $F(\psi) = Q\eta(-\nabla\psi\cdot\nabla\psi + |Q|^2 S(\psi))$. First, we estimate, for $\Lambda > \lambda > \frac{10}{c}$, with Lemma 7.2,

$$\begin{aligned}
|\langle -Q\eta\nabla\psi\cdot\nabla\psi, c^2\partial_c Q\rangle| &= \left|\int_{\mathbb{R}^2}\eta\mathfrak{Re}(\nabla\psi\cdot\nabla\psi c^2\bar{Q}\partial_c Q)\right| \\
&\leqslant \int_{\mathbb{R}^2}\eta|\nabla\psi|^2|c^2\bar{Q}\partial_c Q| \\
&\leqslant K\|\nabla\psi\|_{L^\infty(B(0,\lambda)\cap\{\eta\neq 0\})}\sqrt{\int_{B(0,\lambda)}\eta|\nabla\psi|^2}\sqrt{\int_{B(0,\lambda)}\eta|c^2\bar{Q}\partial_c Q|^2} \\
&\quad + \|c^2\bar{Q}\partial_c Q\|_{L^\infty(\mathbb{R}^2\setminus B(0,\Lambda))}\int_{\mathbb{R}^2\setminus B(0,\lambda)}\eta|\nabla\psi|^2 \\
&\leqslant o^{\Lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|_{\mathcal{C}} + o_{\Lambda\to\infty}(1)\|\varphi\|_{\mathcal{C}},
\end{aligned}$$

since, by Lemma 2.3, $|c^2\bar{Q}\partial_c Q| \leqslant \frac{K}{(1+\tilde{r})^{1/2}}$. We deduce that

$$|\langle -Q\eta\nabla\psi\cdot\nabla\psi, c^2\partial_c Q\rangle| \leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|_{\mathcal{C}}.$$

Now, in $\{\eta = 1\}$, since $e^\psi = \frac{Z}{Q}$ and $1 - K\lambda_0 \leqslant \frac{|Z|}{|Q|} \leqslant 1 + K\mu_0$ (by our assumptions on Z), we have $|\mathfrak{Re}(\psi)| \leqslant K\mu_0$. We deduce, with Lemma 7.1, that in $\{\eta \neq 0\}$,

$$|\mathfrak{Re}(\psi)| \leqslant K\mu_0 + o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1).$$

With $S(\psi) = e^{2\mathfrak{Re}(\psi)} - 1 - 2\mathfrak{Re}(\psi)$, we check that, in $\eta \neq 0$, $|S(\psi)| \leqslant K|\mathfrak{Re}(\psi)|^2$ (given that μ_0 and $\|Z-Q_c\|_{H^{\exp}_{Q_c}}$ are small enough), and with similar computations as for $|\langle -Q\eta\nabla\psi\cdot\nabla\psi, c^2\partial_c Q\rangle|$, we conclude that

$$|\langle F(\psi), c^2\partial_c Q\rangle| \leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|_{\mathcal{C}}.$$

This concludes the proof of

$$\begin{aligned}
\delta^{|\cdot|}(c\overrightarrow{e_2}, \vec{c'}) &\leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|_{H_Q^{\exp}} \\
&\qquad + \left(Kc^2\ln\left(\frac{1}{c}\right) + o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\right)\|\varphi\|_{\mathcal{C}}.
\end{aligned}$$

Step 2. We have the estimate (7.11).

Now, we take the scalar product of (7.7) with $c\partial_{c^\perp}Q$:

$$\langle i(\vec{c'}-c\overrightarrow{e_2})\cdot H(\psi), c\partial_{c^\perp}Q\rangle = \langle Q\psi, cL_Q(\partial_{c^\perp}Q)\rangle + \langle \mathrm{NL}_{\mathrm{loc}}(\psi)+F(\psi), c\partial_{c^\perp}Q\rangle.$$

We check that

$$\left\langle i(\vec{c'}-c\overrightarrow{e_2})\cdot\frac{\nabla(Q\psi)(1-\eta)+Q\nabla\psi\eta e^{\psi}}{(1-\eta)+\eta e^{\psi}}, c\partial_{c^\perp}Q\right\rangle \leqslant K|(\vec{c'}-c\overrightarrow{e_2})\cdot\langle(1-\eta)iQ\nabla\psi, c\partial_{c^\perp}Q\rangle| + K|\vec{c'}-c\overrightarrow{e_2}|\|\varphi\|_{H^{\exp}_{Q_c}}$$

and

$$\begin{aligned}|\langle\eta iQ\nabla\psi, c\partial_{c^\perp}Q\rangle| &= \left|\int_{\mathbb{R}^2}\eta\mathfrak{Re}(\nabla\psi iQc\overline{\partial_{c^\perp}Q})\right|\\ &\leqslant \left|\int_{\mathbb{R}^2}\eta\mathfrak{Re}(\nabla\psi)\mathfrak{Im}(Qc\overline{\partial_{c^\perp}Q})\right| + \left|\int_{\mathbb{R}^2}\eta\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(Qc\overline{\partial_{c^\perp}Q})\right|\\ &\leqslant \left|\int_{\mathbb{R}^2}\eta\mathfrak{Re}(\psi)\nabla(\mathfrak{Im}(Qc\overline{\partial_{c^\perp}Q}))\right| + K\|\varphi\|_{H^{\exp}_{Q_c}}\\ &\quad+\|\varphi\|_{\mathcal{C}}\int_{\mathbb{R}^2}\eta\mathfrak{Re}^2(Qc\overline{\partial_{c^\perp}Q}).\end{aligned}$$

We check, with Lemmas 2.2 and 2.3, that

$$\int_{\mathbb{R}^2}\eta\mathfrak{Re}^2(Qc\overline{\partial_{c^\perp}Q})\leqslant K$$

and

$$|\nabla(\mathfrak{Im}(Q\overline{\partial_{c^\perp}Q}))|\leqslant|\nabla Q||\partial_{c^\perp}Q|+|\nabla\partial_{c^\perp}Q|\leqslant\frac{K(c)}{(1+r)^2},$$

therefore, as for the previous estimate,

$$\left|\left\langle i(\vec{c'}-c\overrightarrow{e_2})\cdot\frac{(1-\eta)\nabla(Q\psi)+\eta e^{\psi}Q\nabla\psi}{(1-\eta)+\eta e^{\psi}}, c\partial_{c^\perp}Q\right\rangle\right|\leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to0}(1)\|\varphi\|_{H^{\exp}_{Q}}.$$

We check that, by symmetry (see equation (2.3)),

$$\langle i(\vec{c'}-c\overrightarrow{e_2})\cdot\nabla Q, c\partial_{c^\perp}Q\rangle = \delta^\perp(c\overrightarrow{e_2},\vec{c'})\left\langle i\frac{\vec{c'}}{|\vec{c'}|}\cdot\nabla Q, c\partial_{c^\perp}Q\right\rangle.$$

Furthermore, from Lemma 2.8, we have $L_Q(\partial_{c^\perp}Q)=-ic\frac{\vec{c'}^\perp}{|\vec{c'}|}\cdot\nabla Q$, therefore, from Proposition 1.2,

$$c\left\langle i\frac{\vec{c'}^\perp}{|\vec{c'}|}\cdot\nabla Q, \partial_{c^\perp}Q\right\rangle = -B_Q(\partial_{c^\perp}Q) = -2\pi + o_{c\to0}(1).$$

We deduce that

$$\delta^{\perp}(c\overrightarrow{e_2}, \vec{c'}) \leqslant K|\langle Q\psi, cL_Q(\partial_{c^\perp} Q)\rangle + \langle \mathrm{NL}_{\mathrm{loc}}(\psi) + F(\psi), c\partial_{c^\perp} Q\rangle| \\ + o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|_{H^{\exp}_Q}.$$

As previously, we check that

$$|\langle \mathrm{NL}_{\mathrm{loc}}(\psi) + F(\psi), c\partial_{c^\perp} Q\rangle| \leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|_{H^{\exp}_Q} \\ + o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|_{\mathcal{C}}$$

and from Lemma 2.8, we have

$$|\langle Q\psi, L_Q(\partial_{c^\perp} Q)\rangle| = \left|\left\langle Q\psi, i\frac{\vec{c'}^{\perp}}{|\vec{c'}|}\cdot\nabla Q\right\rangle\right| \\ \leqslant \left|\int_{\mathbb{R}^2}\mathfrak{Re}(\psi)\mathfrak{Im}\left(\frac{\vec{c'}^{\perp}}{|\vec{c'}|}\cdot\nabla Q\bar{Q}\right)\right| + \left|\int_{\mathbb{R}^2}\mathfrak{Im}(\psi)\mathfrak{Re}\left(\frac{\vec{c'}^{\perp}}{|\vec{c'}|}\cdot\nabla Q\bar{Q}\right)\right|,$$

and with Lemma 5.4, we deduce that

$$c|\langle Q\psi, L_Q(\partial_{c^\perp} Q)\rangle| \leqslant Kc\ln\left(\frac{1}{c}\right)\|\varphi\|_{\mathcal{C}}.$$

We conclude that

$$\delta^{\perp}(c\overrightarrow{e_2}, \vec{c'}) \leqslant \left(Kc^2\ln\left(\frac{1}{c}\right) + o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\right)\|\varphi\|_{\mathcal{C}} \\ + o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|_{H^{\exp}_{Q_c}}.$$

7.3.2. Estimates on the remaining terms. Let us show in this subsection that

$$|\langle i(\vec{c'} - c\overrightarrow{e_2})\cdot H(\psi), Q(\psi+i\gamma)\rangle| + |\langle \mathrm{NL}_{\mathrm{loc}}(\psi), Q(\psi+i\gamma)\rangle| + |\langle F(\psi), Q(\psi+i\gamma)\rangle| \\ \leqslant (o_{c\to 0}(1) + o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1) + K\lambda_0)\|\varphi\|^2_{\mathcal{C}} + o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|^2_{H^{\exp}_Q}.$$

Step 1. Proof of $|\langle \mathrm{NL}_{\mathrm{loc}}(\psi), Q(\psi+i\gamma)\rangle| \leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|^2_{H^{\exp}_Q}$.

From Lemma 7.5, we have

$$|\langle \mathrm{NL}_{\mathrm{loc}}(\psi), Q(\psi+i\gamma)\rangle| \leqslant K(\|Q\psi\|_{C^1(\{\eta\neq 1\})} + |\gamma|)\|\varphi\|^2_{H^1(\{\eta\neq 1\})},$$

therefore, from Lemmas 7.2 and 7.6 and equation (7.6), we deduce

$$|\langle \mathrm{NL}_{\mathrm{loc}}(\psi), Q\psi\rangle| \leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|^2_{H^{\exp}_Q}.$$

Step 2. Proof of

$$|\langle i(\vec{c'} - c\overrightarrow{e_2}) \cdot H(\psi), Q(\psi + i\gamma)\rangle| \leqslant (o_{c\to 0}(1) + o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1))\|\varphi\|^2_{\mathcal{C}} + o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|^2_{H^{\exp}_Q}.$$

We separate the estimate into two parts. First, we look at $\langle i(\vec{c'} - c\overrightarrow{e_2}) \cdot H(\psi), Q\psi\rangle$. We recall that $H(\psi) = \nabla Q + \frac{(1-\eta)\nabla(Q\psi)+\eta e^{\psi} Q\nabla\psi}{(1-\eta)+\eta e^{\psi}}$, and, since $|\vec{c'} - c\overrightarrow{e_2}| \leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)$ and $1-\eta$ is compactly supported, we check easily that

$$\left|\left\langle i(\vec{c'} - c\overrightarrow{e_2}) \cdot \frac{(1-\eta)\nabla(Q\psi) + \eta e^{\psi} Q\nabla\psi}{(1-\eta) + \eta e^{\psi}}, Q\psi\right\rangle\right| \leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)(|\langle \eta i Q\nabla\psi, Q\psi\rangle| + K(c)\|\varphi\|^2_{H^{\exp}_Q}).$$

Furthermore, we check that

$$|\langle \eta i Q\nabla\psi, Q\psi\rangle| \leqslant \left|\int_{\mathbb{R}^2} \mathfrak{Re}(\psi)\mathfrak{Im}(\nabla\psi)|Q|^2\eta\right| + \left|\int_{\mathbb{R}^2} \mathfrak{Im}(\psi)\mathfrak{Re}(\nabla\psi)|Q|^2\eta\right|,$$

and by Cauchy–Scwharz $\left|\int_{\mathbb{R}^2} \mathfrak{Re}(\psi)\mathfrak{Im}(\nabla\psi)|Q|^2\eta\right| \leqslant K\|\varphi\|^2_{\mathcal{C}}$. Now, by integration by parts (using Lemma 7.3), we have

$$\left|\int_{\mathbb{R}^2} \mathfrak{Im}(\psi)\mathfrak{Re}(\nabla\psi)|Q|^2\eta\right| \leqslant \left|\int_{\mathbb{R}^2} \mathfrak{Re}(\psi)\mathfrak{Im}(\nabla\psi)|Q|^2\eta\right| + \left|\int_{\mathbb{R}^2} \mathfrak{Im}(\psi)\mathfrak{Re}(\psi)\nabla(|Q|^2)\eta\right| + \left|\int_{\mathbb{R}^2} \mathfrak{Im}(\psi)\mathfrak{Re}(\psi)|Q|^2\nabla\eta\right|,$$

and by Cauchy–Schwarz we check that

$$\left|\int_{\mathbb{R}^2} \mathfrak{Im}(\psi)\mathfrak{Re}(\nabla\psi)|Q|^2\eta\right| \leqslant K\|\varphi\|^2_{H^{\exp}_Q}.$$

We deduce that

$$\left|\left\langle i(\vec{c'} - c\overrightarrow{e_2}) \cdot \frac{(1-\eta)\nabla(Q\psi) + \eta e^{\psi} Q\nabla\psi}{(1-\eta) + \eta e^{\psi}}, Q\psi\right\rangle\right| \leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|^2_{H^{\exp}_Q}.$$

Finally, we write

$$|\langle i(\vec{c'} - c\vec{e_2}) \cdot \nabla Q, Q\psi\rangle| \leqslant \delta^{|\cdot|}(c\vec{e_2}, \vec{c'}) \left| \left\langle i\frac{\vec{c'}}{|\vec{c'}|} \cdot \nabla Q, Q\psi \right\rangle \right| + \delta^{\perp}(c\vec{e_2}, \vec{c'}) \left| \left\langle i\frac{\vec{c'}^{\perp}}{|\vec{c'}|} \cdot \nabla Q, Q\psi \right\rangle \right|.$$

With Lemma 5.4, we check that

$$\left| \left\langle i\frac{\vec{c'}}{|\vec{c'}|} \cdot \nabla Q, Q\psi \right\rangle \right| + \left| \left\langle i\frac{\vec{c'}^{\perp}}{|\vec{c'}|} \cdot \nabla Q, Q\psi \right\rangle \right| \leqslant K \ln\left(\frac{1}{c}\right) \|\varphi\|_{\mathcal{C}}.$$

With (7.10) and (7.11), we deduce that

$$\begin{aligned} |\langle i(\vec{c'} - c\vec{e_2}) \cdot \nabla Q, Q\psi\rangle| &\leqslant \left(Kc\ln^2\left(\frac{1}{c}\right) + o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1) \right) \|\varphi\|^2_{\mathcal{C}} \\ &\quad + o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|^2_{H^{\exp}_Q} \\ &\leqslant \left(o_{c\to 0}(1) + o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1) \right) \|\varphi\|^2_{\mathcal{C}} \\ &\quad + o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|^2_{H^{\exp}_Q}. \end{aligned}$$

Now, we look at $\langle i(\vec{c'} - c\vec{e_2}) \cdot H(\psi), Qi\gamma\rangle$. We check that

$$\langle i\nabla Q, Qi\gamma\rangle = \gamma \int_{\mathbb{R}^2} \mathfrak{Re}(\nabla Q\bar{Q}) = \frac{\gamma}{2} \int_{\mathbb{R}^2} \nabla(|Q|^2 - 1) = 0,$$

thus

$$\langle i(\vec{c'} - c\vec{e_2}) \cdot H(\psi), Qi\gamma\rangle = \left\langle i(\vec{c'} - c\vec{e_2}) \cdot \frac{(1-\eta)\nabla(Q\psi) + \eta e^{\psi} Q\nabla\psi}{(1-\eta) + \eta e^{\psi}}, Qi\gamma \right\rangle.$$

In the area $\{\eta \neq 0\}$, since $|\gamma| = o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)$ by Lemma 7.6, since

$$|\vec{c'} - c\vec{e_2}| \leqslant K\left(c\ln\left(\frac{1}{c}\right) + o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1) \right) \|\varphi\|_{\mathcal{C}} + o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|_{H^{\exp}_Q}$$

by estimates (7.10) and (7.11), we check that

$$\int_{\{\eta\neq 0\}} \mathfrak{Re}\left(i(\vec{c'} - c\vec{e_2}) \cdot \frac{(1-\eta)\nabla(Q\psi) + \eta e^{\psi} Q\nabla\psi}{(1-\eta) + \eta e^{\psi}} \overline{Qi\gamma} \right) \leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|^2_{H^{\exp}_Q},$$

and therefore (with Lemma 7.3, which justifies the integrability)

$$|\langle i(\vec{c'}-c\vec{e_2})\cdot H(\psi), Qi\gamma\rangle| \leqslant \left|\gamma(\vec{c'}-c\vec{e_2})\cdot\int_{\mathbb{R}^2}\eta|Q|^2\mathfrak{Re}(\nabla\psi)\right| + o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|^2_{H^{\exp}_Q}.$$

By integration by parts (since $|\mathfrak{Re}(\psi)| \leqslant \frac{K(\lambda,c,\|Z-Q_c\|_{H^{\exp}_{Q_c}},\varepsilon_0,Z)}{(1+r)^2}$ and $|\mathfrak{Re}(\nabla\psi)| \leqslant \frac{K(\lambda,c,\|Z-Q_c\|_{H^{\exp}_{Q_c}},\varepsilon_0,Z)}{(1+r)^3}$ by Lemma 7.3) and Cauchy–Schwarz

$$\begin{aligned}\left|\int_{\mathbb{R}^2}\eta|Q|^2\mathfrak{Re}(\nabla\psi)\right| &\leqslant \left|\int_{\mathbb{R}^2}\nabla\eta|Q|^2\mathfrak{Re}(\psi)\right| + \left|\int_{\mathbb{R}^2}\eta\nabla(|Q|^2)\mathfrak{Re}(\psi)\right|\\ &\leqslant K(c)\|\varphi\|_{H^{\exp}_Q}.\end{aligned}$$

Since $|\gamma| = o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)$ by Lemma 7.6 and $|\vec{c'}-c\vec{e_2}| \leqslant (K(c) + o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1))\|\varphi\|_{H^{\exp}_Q}$ by (7.10), (7.11), and Lemma 6.1, we conclude that

$$|\langle i(\vec{c'}-c\vec{e_2})\cdot H(\psi), Qi\gamma\rangle| \leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|^2_{H^{\exp}_Q}.$$

Step 3. Proof of $|\langle F(\psi), Q(\psi+i\gamma)\rangle| \leqslant (o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1) + K\lambda_0)\|\varphi\|^2_{\mathcal{C}}$.

We recall

$$\begin{aligned}F(\psi) &= Q\eta(-\nabla\psi\cdot\nabla\psi + |Q|^2S(\psi)),\\ S(\psi) &= e^{2\mathfrak{Re}(\psi)} - 1 - 2\mathfrak{Re}(\psi).\end{aligned}$$

First, we look at $\langle F(\psi), Q\psi\rangle$. We have

$$|\langle F(\psi), Q\psi\rangle| \leqslant |\langle Q(1-\eta)\nabla\psi\cdot\nabla\psi, Q\psi\rangle| + |\langle Q(1-\eta)|Q|^2S(\psi), Q\psi\rangle|.$$

We check that $\|\varphi\|_{L^\infty(\mathbb{R}^2)} \leqslant K\|\psi\|_{L^\infty(\mathbb{R}^2\setminus B(0,\lambda))} + K\|\varphi\|_{L^\infty(B(0,\lambda))} \leqslant K\lambda_0 + o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)$

$$|\langle Q\eta\nabla\psi\cdot\nabla\psi, Q\psi\rangle| \leqslant \|\varphi\|_{L^\infty(\mathbb{R}^2)}\int_{\mathbb{R}^2}\eta|\nabla\psi|^2 \leqslant (K\lambda_0 + o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1))\|\varphi\|^2_{\mathcal{C}}.$$

Finally, since $\|\varphi\|_{L^\infty(\mathbb{R}^2)} \leqslant K$ a uniform constant for c and $\|Z-Q_c\|_{H^{\exp}_{Q_c}}$ small enough,

$$|\langle Q\eta|Q|^2S(\psi), Q\psi\rangle| \leqslant \|\varphi\|_{L^\infty(\mathbb{R}^2)}\int_{\mathbb{R}^2}\eta\mathfrak{Re}^2(\psi) \leqslant (K\lambda_0 + o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1))\|\varphi\|^2_{\mathcal{C}}.$$

Now, we compute

$$|\langle F(\psi), Qi\gamma\rangle| \leqslant |\gamma|\left|\int_{\mathbb{R}^2} -\mathfrak{Re}(\eta i\nabla\psi\cdot\nabla\psi)|Q|^2 + \eta|Q|^4\mathfrak{Re}(S(\psi)i)\right|,$$

and since $S(\psi)$ is real-valued, we check that, since $|\gamma|=o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)$ by Lemma 7.6,

$$|\langle F(\psi), Qi\gamma\rangle| \leqslant |\gamma| \int_{\mathbb{R}^2} \eta|\nabla\psi|^2|Q|^2 \leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|^2_{\mathcal{C}}.$$

7.3.3. Conclusion. Combining the steps 1 to 3 and (7.9) in (7.8), we deduce that, taking c small enough, and then $\|Z-Q_c\|_{H^{\exp}_{Q_c}}$ small enough (depending on c and λ), we have

$$\begin{aligned}0 \geqslant\, & K\|\varphi\|^2_{\mathcal{C}} + K(c)\|\varphi\|^2_{H^{\exp}_Q} \\ & - (o_{c\to 0}(1)+K\mu_0 + o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1))\|\varphi\|^2_{\mathcal{C}} - o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|^2_{H^{\exp}_Q},\end{aligned}$$

hence, if μ_0 is taken small enough (independently of any other parameters), then c small enough and $\|Z-Q_c\|_{H^{\exp}_{Q_c}}$ small enough (depending on λ and c),

$$K(c)\|\varphi\|^2_{H^{\exp}_{Q_c}} + K\|\varphi\|^2_{\mathcal{C}} \leqslant 0.$$

We deduce that $\varphi = 0$, thus $Z = Q$. Furthermore, from (7.10) and (7.11) we deduce that $\vec{c'} = c\vec{e_2}$, and since $Z \to 1$ at infinity, we also have $\gamma = 0$ (or else $\|Z-Q_c\|_{H^{\exp}_{Q_c}} = +\infty$). This concludes the proof of Theorem 1.14.

Appendix A. Estimates on the travelling wave

A.1. Proof of Lemma 2.6. From Propositions 5 and 7 of [**10**] (where $\eta = 1-|Q_c|^2$), we have in our case, for $x = r\sigma \in \mathbb{R}^2$ with $r \in \mathbb{R}^+$, $|\sigma| = 1$, $\sigma = (\sigma_1, \sigma_2) \in \mathbb{R}^2$, that

$$r^2(1-|Q_c|^2)(r\sigma) \to c\alpha(c)\left(\frac{1}{1-\frac{c^2}{2}+\frac{c^2\sigma_2^2}{2}} - \frac{2\sigma_2^2}{\left(1-\frac{c^2}{2}+\frac{c^2\sigma_2^2}{2}\right)^2}\right)$$

uniformly in $\sigma \in S^1$ when $r \to +\infty$, where $\alpha(c) > 0$ depends on c and Q_c. Note that our travelling wave is axisymmetric around axis x_2 (and not x_1, for which the results of [**10**] are given), hence the swap between σ_1 and σ_2 between the two papers. We have

$$\frac{1}{1-\frac{c^2}{2}+\frac{c^2\sigma_2^2}{2}} - \frac{2\sigma_2^2}{(1-\frac{c^2}{2}+\frac{c^2\sigma_2^2}{2})^2} = \frac{1-\frac{c^2}{2}-(2-\frac{c^2}{2})\sigma_2^2}{(1-\frac{c^2}{2}+\frac{c^2\sigma_2^2}{2})^2},$$

which shows in particular that $|Q_c| = 1$ when $r \gg \frac{1}{c}$ is possible only in cones around $\sin(\theta) = \sigma_2 = \pm\sqrt{\frac{1-c^2/2}{2-c^2/2}}$. Therefore, for c small enough, for some $\gamma > 0$ small and $R > 0$ large (that may depend on c), we have

$$\int_{\mathbb{R}^2} |1-|Q_c|^2||\varphi|^2 \geqslant K(c,\beta,R)\int_{\mathbb{R}^2\setminus(B(0,R)\cup D(\gamma))} \frac{|\varphi|^2}{(1+r)^2},$$

where $D(\gamma)=\left\{re^{i\theta}\in\mathbb{R}^2, \left|\sin(\theta)\pm\sqrt{\frac{1-c^2/2}{2-c^2/2}}\right|\leqslant\gamma\right\}$. We want to show that, for $\varphi\in H_{Q_c}$,

$$\int_{D(\gamma)\cup(\mathbb{R}^2\setminus B(0,R))}\frac{|\varphi|^2}{(1+r)^2}\leqslant C(c,\gamma,R)\left(\int_{\mathbb{R}^2}|\nabla\varphi|^2+\int_{\mathbb{R}^2\setminus(B(0,R)\cup D(\gamma))}\frac{|\varphi|^2}{(1+r)^2}\right).$$

For θ_0 any of the four angles such that $\sin(\theta)\pm\sqrt{\frac{1-c^2/2}{2-c^2/2}}=0$, we fix $r>0$ and regard $\varphi(\theta)$ as a function of the angle only. We compute, for $\theta\in[\theta_0-2\beta,\theta_0+2\beta]$ ($\beta>0$ being a small constant depending on γ such that $\{x=re^{i\theta}\in\mathbb{R}^2,\ \theta\in[\theta_0+3\beta,\theta_0+\beta]\}\cap D(\gamma)=\emptyset$, and such that $D(\gamma)$ is included in the union of the $[\theta_0-\beta,\theta_0+\beta]$ for the four possible values of θ_0),

$$\varphi(\theta)=\varphi(2\beta+\theta)-\int_\theta^{2\beta+\theta}\partial_\theta\varphi(\Theta)\,d\Theta,$$

hence,

$$|\varphi(\theta)|\leqslant|\varphi(2\beta+\theta)|+\int_{\theta_0-\beta}^{\theta_0+3\beta}|\partial_\theta\varphi(\Theta)|\,d\Theta.$$

This implies that

$$|\varphi(\theta)|^2\leqslant 2|\varphi(2\beta+\theta)|^2+K\int_0^{2\pi}|\partial_\theta\varphi(\Theta)|^2\,d\Theta$$

by Cauchy–Schwarz and integrating between $\theta_0-\beta$ and $\theta_0+\beta$ yields

$$\int_{\theta_0-\beta}^{\theta_0+\beta}|\varphi(\theta)|^2\,d\theta\leqslant 2\int_{\theta_0+\beta}^{\theta_0+3\beta}|\varphi(\theta)|^2\,d\theta+K\int_0^{2\pi}|\partial_\theta\varphi(\theta)|^2\,d\theta.$$

Now multiplying by $\frac{r}{(1+r)^2}$ and integrating in r on $[R,+\infty[$, we infer

$$\begin{aligned}\int_{\theta-\theta_0\in[-\beta,\beta]}\int_{r\in[R,+\infty[}\frac{|\varphi|^2}{(1+r)^2}r\,dr\,d\theta&\\ &\leqslant 2\int_{\theta-\theta_0\in[\beta,3\beta]}\int_{r\in[R,+\infty[}\frac{|\varphi|^2}{(1+r)^2}r\,dr\,d\theta\\ &\quad+K(c,\beta,R)\int_{\mathbb{R}^2}|\nabla\varphi|^2\,dx\\ &\leqslant 2\int_{\mathbb{R}^2\setminus(B(0,R)\cup D(\gamma))}\frac{|\varphi|^2\,dx}{(1+|x|)^2}+K(c,\beta,R)\int_{\mathbb{R}^2}|\nabla\varphi|^2\,dx,\end{aligned}$$

using

$$\frac{|\partial_\theta\varphi|^2}{(1+r)^2}\leqslant\frac{|\partial_\theta\varphi|^2}{r^2}\leqslant|\nabla\varphi|^2.$$

Therefore,

$$\int_{D(\gamma)\cup(\mathbb{R}^2\setminus B(0,R))}\frac{|\varphi|^2}{(1+r)^2}\leqslant K\int_{\mathbb{R}^2\setminus(B(0,R)\cup D(\gamma))}\frac{|\varphi|^2}{(1+r)^2}\,dx$$
$$+K(c,\beta,\gamma,R)\int_{\mathbb{R}^2}|\nabla\varphi|^2\,dx,$$

and thus

$$\int_{\mathbb{R}^2\setminus B(0,R)}\frac{|\varphi|^2}{(1+r)^2}\leqslant K(c,\beta,\gamma,R)\int_{\mathbb{R}^2}|\nabla\varphi|^2+|1-|Q_c|^2||\varphi|^2.$$

We are left with the proof of

$$\int_{B(0,R)}\frac{|\varphi|^2}{(1+r)^2}\leqslant K(c,\beta,R)\left(\int_{\mathbb{R}^2}|\nabla\varphi|^2+\int_{\mathbb{R}^2\setminus B(0,R)}\frac{|\varphi|^2}{(1+r)^2}\right).$$

We argue by contradiction. We suppose that there exists a sequence $\varphi_n\in H_{Q_c}$ such that $\int_{B(0,R)}\frac{|\varphi_n|^2}{(1+r)^2}=1$ and $\int_{\mathbb{R}^2}|\nabla\varphi_n|^2+\int_{\mathbb{R}^2\setminus B(0,R)}\frac{|\varphi_n|^2}{(1+r)^2}\to 0$. Since φ_n is bounded in $H^1(B(0,R+1))$, by Rellich's theorem, up to a subsequence, we have the convergences $\varphi_n\to\varphi$ strongly in L^2 and weakly in H^1 to some function φ in $B(0,R+1)$. In particular, $\int_{B(0,R+1)}|\nabla\varphi|^2=0$, hence φ is constant on $B(0,R+1)$, and with $\int_{B(0,R+1)\setminus B(0,R)}\frac{|\varphi|^2}{(1+r)^2}=0$ we have $\varphi=0$, which is in contradiction with $1=\int_{B(0,R)}\frac{|\varphi_n|^2}{(1+r)^2}\to\int_{B(0,R)}\frac{|\varphi|^2}{(1+r)^2}$ by $L^2(B(0,R+1))$ strong convergence. This concludes the proof of this lemma. □

A.2. Proof of Lemma 2.14. From equations (2.7) and (2.1), Lemma 2.6 of [**4**], Lemma 2.13, and the mean value theorem, in $B(\widetilde{d_c}\overrightarrow{e_1},\tilde{d}_c^{1/2})$,

$$\begin{aligned}|Q_c-\widetilde{V_1}|&\leqslant|Q_c-V|+|V-\widetilde{V_1}|\\&\leqslant o_{c\to0}(1)+|V_1(\cdot-\widetilde{d_c}\overrightarrow{e_1})-\widetilde{V_1}|\\&\leqslant o_{c\to0}(1)+|d_c-\widetilde{d_c}|\|\partial_{x_1}V\|_{L^\infty(\mathbb{R}^2)}\\&\leqslant o_{c\to0}(1),\end{aligned}\tag{A.1}$$

which is the first statement.

For the second statement, we write $Q_c=V_1(\cdot-d_c\overrightarrow{e_1})V_{-1}(\cdot-d_c\overrightarrow{e_1})+\Gamma_c$, and from equation (2.5) (with some margin), we have

$$|\nabla\Gamma_c|\leqslant\frac{o_{c\to0}(1)}{1+\tilde{r}_1}.$$

Furthermore, since $\tilde{V}_1 = V_1(\cdot - \widetilde{d_c}\overrightarrow{e_1})$,

$$\nabla(V_1(\cdot - d_c\overrightarrow{e_1})V_{-1}(\cdot + d_c\overrightarrow{e_1})) - \nabla\widetilde{V_1}$$
$$= \nabla V_1(\cdot - d_c\overrightarrow{e_1})V_{-1}(\cdot + d_c\overrightarrow{e_1}) - \nabla\widetilde{V_1} + V_1(\cdot - d_c\overrightarrow{e_1})\nabla V_{-1}(\cdot + d_c\overrightarrow{e_1}),$$

and from (2.2), in $B(\widetilde{d_c}\overrightarrow{e_1}, \tilde{d}_c^{1/2})$, we have

$$|\nabla V_{-1}(\cdot + d_c\overrightarrow{e_1})| \leqslant \frac{o_{c\to 0}(1)}{1+\tilde{r}_1}.$$

We compute

$$\nabla V_1(\cdot - d_c\overrightarrow{e_1})V_{-1}(\cdot + d_c\overrightarrow{e_1}) - \nabla\widetilde{V_1}$$
$$= \nabla V_1(\cdot - d_c\overrightarrow{e_1})(V_{-1}(\cdot + d_c\overrightarrow{e_1}) - 1) - \nabla\widetilde{V_1} + \nabla V_1(\cdot - d_c\overrightarrow{e_1})$$

and, from (2.1), in $B(\widetilde{d_c}\overrightarrow{e_1}, \tilde{d}_c^{1/2})$, we have $|V_{-1}(\cdot + d_c\overrightarrow{e_1}) - 1| = o_{c\to 0}(1)$. Finally, from Lemmas 2.1 and 2.13, we estimate (with the mean value theorem)

$$|\nabla V_1(\cdot - d_c\overrightarrow{e_1}) - \nabla\widetilde{V_1}| \leqslant |d_c - \widetilde{d_c}| \sup_{d\in[d_c,\tilde{d}_c]\cup[\tilde{d}_c,d_c]} |\nabla^2 V_1(x-d)|$$
$$\leqslant K\frac{|d_c - \widetilde{d_c}|}{(1+\tilde{r}_1)^2} = \frac{o_{c\to 0}(1)}{(1+\tilde{r}_1)^2},$$

hence

$$|\nabla Q_c - \nabla\widetilde{V_1}| \leqslant \frac{o_{c\to 0}(1)}{1+\tilde{r}_1}. \tag{A.2}$$

Now, writing $w = Q_c - \widetilde{V_1}$, in $B(\widetilde{d_c}\overrightarrow{e_1}, 2\tilde{d}_c^{1/2})$, we estimate (since $\mathrm{TW}_c(Q_c) = 0$ and $\Delta\widetilde{V_1} - (|\widetilde{V_1}|^2 - 1)\widetilde{V_1} = 0$)

$$|\Delta w| = |-ic\partial_{x_2}Q_c - (1-|Q_c|^2)Q_c + (1-|\widetilde{V_1}|^2)\widetilde{V_1}| \leqslant \frac{o_{c\to 0}(1)}{1+\widetilde{r_1}}$$

by equations (2.6) to (2.10) and (2.1). Furthermore, by equations (2.6) to (2.2), we have

$$|\nabla(\Delta w)| \leqslant \frac{o_{c\to 0}(1)}{(1+\tilde{r}_1)}.$$

We check, as the proof of (A.1), that, in $B(\widetilde{d_c}\overrightarrow{e_1}, 2\tilde{d}_c^{1/2})$,

$$|w| = o_{c\to 0}(1),$$

and, similarly, with equations (2.2) and (A.2), that

$$|\nabla w| = o_{c\to 0}(1)$$

in $B(\widetilde{d_c}\overrightarrow{e_1}, 2\tilde{d}_c^{1/2})$. By Theorem 6.2 of [**7**] (taking a domain $\Omega = B\big(x - \widetilde{d_c}\overrightarrow{e_1}, \frac{|x-\tilde{d}_c\overrightarrow{e_1}|}{2}\big)$, and $\alpha = 1/2$, but it also holds for any $0 < \alpha < 1$), we have, for $x \in B(\widetilde{d_c}\overrightarrow{e_1}, 2\tilde{d}_c^{1/2})$,

$$(1+\tilde{r}_1)^2|\nabla^2 w(x - \widetilde{d_c}\overrightarrow{e_1})| \leqslant K(\|w\|_{C^1(\Omega)} + (1+\widetilde{r_1})^2\|\Delta w\|_{C^1(\Omega)}),$$

and from the previous estimates, we have $\|w\|_{C^1(\Omega)} = o_{c\to 0}(1)$ and $\|\Delta w\|_{C^1(\Omega)} \leqslant \frac{o_{c\to 0}(1)}{(1+\tilde{r}_1)}$, therefore

$$|\nabla^2(Q_c - \widetilde{V_1})| = |\nabla^2 w| \leqslant \frac{o_{c\to 0}(1)}{(1+\widetilde{r_1})}. \qquad \square$$

Appendix B. Proofs related to the energy space

B.1. Proof of Lemma 3.4. We recall that

$$\|\varphi\|^2_{H_{Q_c}} = \int_{\mathbb{R}^2} |\nabla\varphi|^2 + |1 - |Q_c|^2||\varphi|^2 + \mathfrak{Re}^2(\overline{Q_c}\varphi),$$

and since, for all $\lambda > 0$,

$$K_1(\lambda)\int_{B(0,\lambda)} |\nabla\varphi|^2 + |\varphi|^2 \leqslant \int_{B(0,\lambda)} |\nabla\varphi|^2 + |1 - |Q_c|^2||\varphi|^2 + \mathfrak{Re}^2(\overline{Q_c}\varphi)$$
$$\leqslant K_2(\lambda)\int_{B(0,\lambda)} |\nabla\varphi|^2 + |\varphi|^2,$$

by a standard density argument, we have that $C^\infty_c(\mathbb{R}^2, \mathbb{C})$ is dense in H_{Q_c} for the norm $\|\cdot\|_{H_{Q_c}}$.

We are therefore left with the proof that $C_c^\infty(\mathbb{R}^2\backslash\{\widetilde{d_c}\overrightarrow{e_1}, -\widetilde{d_c}\overrightarrow{e_1}\}, \mathbb{C})$ is dense in $C_c^\infty(\mathbb{R}^2, \mathbb{C})$ for the norm $\|\cdot\|_{H_{Q_c}}$. For that, it is enough to check that $C_c^\infty(B(0,2)\backslash\{0\}, \mathbb{C})$ is dense in $C_c^\infty(B(0,2), \mathbb{C})$ for the norm $\|\cdot\|_{H^1(B(0,2))}$. This result is a consequence of the fact that the capacity of a point in a ball in dimension 2 is 0. For the sake of completeness, we give here a proof of this result.

We define $\eta_\varepsilon \in C^0(B(0,2), \mathbb{R})$ the radial function with $\eta_\varepsilon(x) = 0$ if $|x| \leqslant \varepsilon$, $\eta_\varepsilon(x) = -\frac{\ln(|x|)}{\ln(\varepsilon)} + 1$ if $|x| \in [\varepsilon, 1]$, and $\eta_\varepsilon(x) = 1$ if $2 \geqslant |x| \geqslant 1$. Then, we define $\eta_{\varepsilon,\lambda} \in C^\infty(B(0,2), \mathbb{R})$ a radial regularization of η_ε with $\eta_{\varepsilon,\lambda}(x) = 0$ if $|x| \leqslant \varepsilon/2$ such that $\eta_{\varepsilon,\lambda} \to \eta_\varepsilon$ in $H^1(B(0,2))$ when $\lambda \to 0$. Finally, we define $\eta_{\varepsilon,\lambda,\delta} = \eta_{\varepsilon,\lambda}\big(\frac{x}{\delta}\big)$ for a small $\delta > 0$.

Now, given $\varphi \in C_c^\infty(B(0,2), \mathbb{C})$, $\eta_{\varepsilon,\lambda,\delta}\varphi \in C_c^\infty(B(0,2)\backslash\{0\}, \mathbb{C})$ for all $\varepsilon > 0$, $\lambda > 0$, $\delta > 0$, by dominated convergence, we check that

$$\int_{B(0,2)} |\eta_{\varepsilon,\lambda,\delta}\varphi|^2 \to \int_{B(0,2)} |\varphi|^2$$

when $\delta \to 0$. Furthermore, we compute by integration by parts

$$\begin{aligned}\int_{B(0,2)} |\nabla(\eta_{\varepsilon,\lambda,\delta}\varphi)|^2 &= \int_{B(0,2)} \eta^2_{\varepsilon,\lambda,\delta}|\nabla\varphi|^2 + 2\int_{B(0,2)} \nabla\eta_{\varepsilon,\lambda,\delta}\eta_{\varepsilon,\lambda,\delta}\mathfrak{Re}(\nabla\varphi\bar{\varphi}) \\ &\quad + \int_{B(0,2)} |\nabla\eta_{\varepsilon,\lambda,\delta}|^2|\varphi|^2 \\ &= \int_{B(0,2)} \eta^2_{\varepsilon,\lambda,\delta}|\nabla\varphi|^2 - \int_{B(0,2)} |\varphi|^2\Delta\eta_{\varepsilon,\lambda,\delta}\eta_{\varepsilon,\lambda,\delta}.\end{aligned}$$

Now, extending φ to $\mathbb{R}^2$ by $\varphi = 0$ outside of $B(0,2)$, we have by change of variables

$$\int_{B(0,2)} |\varphi|^2\Delta\eta_{\varepsilon,\lambda,\delta}\eta_{\varepsilon,\lambda,\delta} = \int_{\mathbb{R}^2} |\varphi|^2\Delta\eta_{\varepsilon,\lambda,\delta}\eta_{\varepsilon,\lambda,\delta} = \int_{\mathbb{R}^2} |\varphi|^2(x\delta)\Delta\eta_{\varepsilon,\lambda}\eta_{\varepsilon,\lambda}.$$

When $\delta \to 0$, we have by dominated convergence that $\int_{B(0,2)}\eta^2_{\varepsilon,\lambda,\delta}|\nabla\varphi|^2 \to \int_{B(0,2)} |\nabla\varphi|^2$ and

$$\int_{\mathbb{R}^2} |\varphi|^2(x\delta)\Delta\eta_{\varepsilon,\lambda}\eta_{\varepsilon,\lambda} \to |\varphi|^2(0)\int_{\mathbb{R}^2} \Delta\eta_{\varepsilon,\lambda}\eta_{\varepsilon,\lambda} = -|\varphi|^2(0)\int_{\mathbb{R}^2} |\nabla\eta_{\varepsilon,\lambda}|^2.$$

Now, taking $\lambda \to 0$, we deduce that

$$\lim_{\lambda\to 0}\lim_{\delta\to 0}\int_{B(0,2)} |\nabla(\eta_{\varepsilon,\lambda,\delta}\varphi)|^2 = \int_{B(0,2)} |\nabla\varphi|^2 - |\varphi|^2(0)\int_{\mathbb{R}^2} |\nabla\eta_\varepsilon|^2.$$

From the definition of η_ε, we compute

$$\begin{aligned}\int_{\mathbb{R}^2} |\nabla\eta_\varepsilon|^2 &= \int_\varepsilon^1 \frac{1}{\ln(\varepsilon)^2 r^2} r\,dr \\ &= \frac{1}{\ln(\varepsilon)^2}\int_\varepsilon^1 \frac{1}{r}\,dr \\ &= \frac{-1}{\ln(\varepsilon)} \to 0\end{aligned}$$

when $\varepsilon \to 0$. We deduce that

$$\lim_{\varepsilon\to 0}\lim_{\lambda\to 0}\lim_{\delta\to 0}\int_{B(0,2)} |\nabla(\eta_{\varepsilon,\lambda,\delta}\varphi)|^2 = \int_{B(0,2)} |\nabla\varphi|^2.$$

This conclude the proof of this lemma. □

B.2. Proof of Lemma 4.1. We recall that $L_{Q_c}(\varphi) = -ic\partial_{x_2}\varphi - \Delta\varphi - (1-|Q_c|^2)\varphi + 2\mathfrak{Re}(\overline{Q_c}\varphi)Q_c$. Writing $\varphi = Q_c\psi \in C_c^\infty(\mathbb{R}^2\backslash\{\widetilde{d_c}\overrightarrow{e_1}, -\widetilde{d_c}\overrightarrow{e_1}\}, \mathbb{C})$, we decompose

$$L_{Q_c}(\varphi) = -ic\partial_{x_2}\psi Q_c - \Delta\psi Q_c - 2\nabla Q_c\cdot\nabla\psi + 2\mathfrak{Re}(\psi)|Q_c|^2 Q_c + \mathrm{TW}_c(Q_c)\psi.$$

Since $\mathrm{TW}_c(Q_c) = 0$,

$$\langle L_{Q_c}(\varphi),\varphi\rangle = \langle (1-\eta)L_{Q_c}(\varphi),\varphi\rangle + \langle \eta L_{Q_c}(\varphi), Q_c\psi\rangle$$

$$= \int_{\mathbb{R}^2} (1-\eta)\mathfrak{Re}((-ic\partial_{x_2}\varphi - \Delta\varphi - (1-|Q_c|^2)\varphi + 2\mathfrak{Re}(\overline{Q_c}\varphi)Q_c)\bar{\varphi})$$

$$+ \int_{\mathbb{R}^2} \eta\mathfrak{Re}((-ic\partial_{x_2}\psi Q_c - \Delta\psi Q_c - 2\nabla Q_c\cdot\nabla\psi + 2\mathfrak{Re}(\psi)|Q_c|^2 Q_c)\overline{Q_c\psi}).$$

By integration by parts,

$$\int_{\mathbb{R}^2} (1-\eta)\mathfrak{Re}((-ic\partial_{x_2}\varphi - \Delta\varphi - (1-|Q_c|^2)\varphi + 2\mathfrak{Re}(\overline{Q_c}\varphi)Q_c)\bar{\varphi})$$

$$= \int_{\mathbb{R}^2} (1-\eta)(|\nabla\varphi|^2 - \mathfrak{Re}(ic\partial_{x_2}\varphi\bar{\varphi}) - (1-|Q_c|^2)|\varphi|^2 + 2\mathfrak{Re}^2(\overline{Q_c}\varphi))$$

$$- \int_{\mathbb{R}^2} \nabla\eta\cdot\mathfrak{Re}(\nabla\varphi\bar{\varphi}).$$

Similarly, we compute

$$\int_{\mathbb{R}^2} \eta\mathfrak{Re}((-ic\partial_{x_2}\psi Q_c - \Delta\psi Q_c - 2\nabla Q_c\cdot\nabla\psi + 2\mathfrak{Re}(\psi)|Q_c|^2 Q_c)\overline{Q_c\psi})$$

$$= \int_{\mathbb{R}^2} \eta(\mathfrak{Re}(-ic\partial_{x_2}\psi\bar{\psi}|Q_c|^2) - \mathfrak{Re}(\Delta\psi\bar{\psi})|Q_c|^2 + 2\mathfrak{Re}^2(\psi)|Q_c|^4$$

$$- 2\mathfrak{Re}(\nabla Q_c\cdot\nabla\psi\overline{Q_c\psi}))$$

$$= \int_{\mathbb{R}^2} \eta(c|Q_c|^2(\mathfrak{Im}(\partial_{x_2}\psi)\mathfrak{Re}(\psi) - \mathfrak{Re}(\partial_{x_2}\psi)\mathfrak{Im}(\psi)) + 2\mathfrak{Re}^2(\psi)|Q_c|^4$$

$$- 2\mathfrak{Re}(\nabla Q_c\cdot\nabla\psi\overline{Q_c\psi}))$$

$$+ \int_{\mathbb{R}^2} \eta|\nabla\psi|^2|Q_c|^2 + 2\int_{\mathbb{R}^2} \eta\mathfrak{Re}(\nabla Q_c\overline{Q_c})\cdot\mathfrak{Re}(\nabla\psi\bar{\psi}) + \int_{\mathbb{R}^2} \nabla\eta\cdot\mathfrak{Re}(\nabla\psi\bar{\psi})|Q_c|^2.$$

Continuing, we have

$$- \int_{\mathbb{R}^2} \eta|Q_c|^2\mathfrak{Re}(\partial_{x_2}\psi)\mathfrak{Im}(\psi) = \int_{\mathbb{R}^2} \eta|Q_c|^2\mathfrak{Re}(\psi)\mathfrak{Im}(\partial_{x_2}\psi)$$

$$+ \int_{\mathbb{R}^2} \partial_{x_2}\eta|Q_c|^2\mathfrak{Re}(\psi)\mathfrak{Im}(\psi) + 2\int_{\mathbb{R}^2} \eta\mathfrak{Re}(\partial_{x_2}Q_c\overline{Q_c})\mathfrak{Re}(\psi)\mathfrak{Im}(\psi),$$

as well as

$$\int_{\mathbb{R}^2} \eta\mathfrak{Re}(\nabla Q_c\cdot\nabla\psi\overline{Q_c\psi}) = \int_{\mathbb{R}^2} \eta\mathfrak{Re}(\nabla Q_c\overline{Q_c})\cdot\mathfrak{Re}(\nabla\psi\bar{\psi})$$

$$+ \int_{\mathbb{R}^2} \eta\mathfrak{Im}(\nabla Q_c\overline{Q_c})\mathfrak{Im}(\nabla\psi\bar{\psi}),$$

therefore

$$\int_{\mathbb{R}^2} \eta \mathfrak{Re}((-ic\partial_{x_2}\psi Q_c - \Delta\psi Q_c - 2\nabla Q_c \cdot \nabla\psi + 2\mathfrak{Re}(\psi)|Q_c|^2 Q_c)\overline{Q_c\psi})$$
$$= \int_{\mathbb{R}^2} \eta(|\nabla\psi|^2|Q_c|^2 + 2\mathfrak{Re}^2(\psi)|Q_c|^4 + 2c\mathfrak{Im}(\partial_{x_2}\psi)\mathfrak{Re}(\psi))$$
$$+ \int_{\mathbb{R}^2} \eta(2c\mathfrak{Re}(\partial_{x_2}Q_c\overline{Q_c})\mathfrak{Re}(\psi)\mathfrak{Im}(\psi) - 2\mathfrak{Im}(\nabla Q_c\overline{Q_c})\mathfrak{Im}(\nabla\psi\bar{\psi}))$$
$$+c\int_{\mathbb{R}^2} \partial_{x_2}\eta\mathfrak{Re}(\psi)\mathfrak{Im}(\psi)|Q_c|^2 + \int_{\mathbb{R}^2} \nabla\eta \cdot \mathfrak{Re}(\nabla\psi\bar{\psi})|Q_c|^2.$$

As $ic\partial_{x_2}Q_c = \Delta Q_c + (1-|Q_c|^2)Q_c$, we have $c\mathfrak{Re}(\partial_{x_2}Q_c\overline{Q_c}) = \mathfrak{Re}(i\Delta Q_c\overline{Q_c})$. By integration by parts,

$$2\int_{\mathbb{R}^2} \eta\mathfrak{Re}(i\Delta Q_c\overline{Q_c})\mathfrak{Re}(\psi)\mathfrak{Im}(\psi) = 2\int_{\mathbb{R}^2} \nabla\eta \cdot \mathfrak{Im}(\nabla Q_c\overline{Q_c})\mathfrak{Re}(\psi)\mathfrak{Im}(\psi)$$
$$-2\int_{\mathbb{R}^2} \eta\mathfrak{Im}(\nabla Q_c\overline{Q_c}) \cdot \mathfrak{Re}(\nabla\psi)\mathfrak{Im}(\psi) - 2\int_{\mathbb{R}^2} \eta\mathfrak{Im}(\nabla Q_c\overline{Q_c}) \cdot \mathfrak{Re}(\psi)\mathfrak{Im}(\nabla\psi),$$

and

$$-2\int_{\mathbb{R}^2} \eta\mathfrak{Im}(\nabla Q_c\overline{Q_c})\mathfrak{Im}(\nabla\psi\bar{\psi}) = -2\int_{\mathbb{R}^2} \eta\mathfrak{Im}(\nabla Q_c\overline{Q_c})(\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi)$$
$$- \mathfrak{Im}(\psi)\mathfrak{Re}(\nabla\psi)).$$

Combining these estimates with

$$\int_{\mathbb{R}^2} \nabla\eta \cdot \mathfrak{Re}(\nabla\varphi\bar{\varphi}) = \int_{\mathbb{R}^2} \nabla\eta \cdot (\mathfrak{Re}(\nabla Q_c\overline{Q_c})|\psi|^2 + \mathfrak{Re}(\nabla\psi\bar{\psi})|Q_c|^2),$$

we conclude the proof of

$$\langle L_{Q_c}(\varphi), \varphi\rangle = B_{Q_c}^{\exp}(\varphi).$$

Now, for the proof for $B_{V_1}(\varphi)$, the computations are identical, simply replacing c by 0, η by $\tilde{\eta}$, and Q_c by V_1. □

B.3. Proof of Lemma 6.1. First, let us show (6.1). We have

$$\|\varphi\|_{H^1(\{\tilde{r}\leqslant 10\})} \leqslant K\|\varphi\|_{H_{Q_c}},$$

and, by equation (2.12) and Lemma 2.6, we check that

$$\int_{\{\tilde{r}\geqslant 5\}} \mathfrak{Re}^2(\psi) \leqslant K\|\varphi\|^2_{H_{Q_c}},$$

and also that

$$\int_{\{\tilde{r}\geqslant 5\}} \frac{|\psi|^2}{\tilde{r}^2\ln(\tilde{r})^2} \leqslant K\int_{\{\tilde{r}\geqslant 5\}} \frac{|\varphi|^2}{(1+\tilde{r})^2} \leqslant K(c)\|\varphi\|^2_{H_{Q_c}}.$$

Furthermore, we compute, by equations (2.12) and (3.1) and Theorem 2.5,

$$\int_{\{\tilde{r}\geqslant 5\}} |\nabla\psi|^2 \leqslant K \int_{\{\tilde{r}\geqslant 5\}} |\nabla\psi|^2 |Q_c|^4$$
$$\leqslant K \left(\int_{\{\tilde{r}\geqslant 5\}} |\nabla\varphi|^2 + \int_{\{\tilde{r}\geqslant 5\}} |\nabla Q_c|^2 |\varphi|^2 \right) \leqslant K(c) \|\varphi\|^2_{H_{Q_c}}.$$

We deduce that (6.1) holds, and therefore $H_{Q_c} \subset H^{\exp}_{Q_c}$. Now, we check that

$$\|iQ_c\|^2_{H^{\exp}_{Q_c}} \leqslant \|iQ_c\|^2_{H^1(\{\tilde{r}\leqslant 10\})} + K \int_{\{\tilde{r}\geqslant 5\}} \frac{|i|^2}{\tilde{r}^2 \ln(\tilde{r})^2} + \int_{\{\tilde{r}\geqslant 5\}} |\nabla i|^2 < +\infty.$$

With regard to the definition of $\|\cdot\|_{\mathcal{C}}$, we check easily that

$$\|\varphi\|_{\mathcal{C}} \leqslant \|\varphi\|_{H^{\exp}_{Q_c}}.$$

Finally, we recall the definition of $B^{\exp}_{Q_c}(\varphi)$ from equation (1.4),

$$\begin{aligned} B^{\exp}_{Q_c}(\varphi) = & \int_{\mathbb{R}^2} (1-\eta)(|\nabla\varphi|^2 - \mathfrak{Re}(ic\partial_{x_2}\varphi\bar{\varphi}) - (1-|Q_c|^2)|\varphi|^2 + 2\mathfrak{Re}^2(\overline{Q_c}\varphi)) \\ & - \int_{\mathbb{R}^2} \nabla\eta \cdot (\mathfrak{Re}(\nabla Q_c \overline{Q_c})|\psi|^2 - 2\mathfrak{Im}(\nabla Q_c \overline{Q_c})\mathfrak{Re}(\psi)\mathfrak{Im}(\psi)) \\ & + \int_{\mathbb{R}^2} c\partial_{x_2}\eta |Q_c|^2 \mathfrak{Re}(\psi)\mathfrak{Im}(\psi) \\ & + \int_{\mathbb{R}^2} \eta(|\nabla\psi|^2|Q_c|^2 + 2\mathfrak{Re}^2(\psi)|Q_c|^4) \\ & + \int_{\mathbb{R}^2} \eta(4\mathfrak{Im}(\nabla Q_c\overline{Q_c})\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi) + 2c|Q_c|^2\mathfrak{Im}(\partial_{x_2}\psi)\mathfrak{Re}(\psi)). \end{aligned}$$

For $\lambda > 0$, we have $\|\varphi\|_{H^1(B(0,\lambda))} \leqslant K(c,\lambda)\|\varphi\|_{H^{\exp}_{Q_c}}$, therefore (since $1-\eta$ is compactly supported) we only have to check that the integrands in the last two lines are in $L^1(\mathbb{R}^2)$, and this is a consequence of Cauchy–Schwarz since

$$\begin{aligned} & \int_{\mathbb{R}^2} \eta(|\nabla\psi|^2|Q_c|^2 + 2\mathfrak{Re}^2(\psi)|Q_c|^4 + 4|\mathfrak{Im}(\nabla Q_c\overline{Q_c})\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi)| \\ & \quad + 2c|Q_c|^2|\mathfrak{Im}(\partial_{x_2}\psi)\mathfrak{Re}(\psi)|) \leqslant K \int_{\mathbb{R}^2} \eta(|\nabla\psi|^2 + \mathfrak{Re}^2(\psi)) \leqslant K\|\varphi\|^2_{H^{\exp}_{Q_c}}. \end{aligned}$$

Furthermore, for two cutoffs η, η' such that they are both 0 near the zeros of Q_c and 1 at infinity, we have

$$\begin{aligned}
&B^{\exp}_{Q_c,\eta}(\varphi) - B^{\exp}_{Q_c,\eta'}(\varphi)\\
&\quad= \int_{\mathbb{R}^2} (\eta'-\eta)(|\nabla\varphi|^2 - \mathfrak{Re}(ic\partial_{x_2}\varphi\bar{\varphi}) - (1-|Q_c|^2)|\varphi|^2 + 2\mathfrak{Re}^2(\overline{Q_c}\varphi))\\
&\qquad+\int_{\mathbb{R}^2} \nabla(\eta-\eta')\cdot(\mathfrak{Re}(\nabla Q_c\overline{Q_c})|\psi|^2 - 2\mathfrak{Im}(\nabla Q_c\overline{Q_c})\mathfrak{Re}(\psi)\mathfrak{Im}(\psi))\\
&\qquad\qquad -c\partial_{x_2}(\eta-\eta')|Q_c|^2\mathfrak{Re}(\psi)\mathfrak{Im}(\psi)\\
&\qquad+\int_{\mathbb{R}^2} (\eta'-\eta)(|\nabla\psi|^2|Q_c|^2 + 2\mathfrak{Re}^2(\psi)|Q_c|^4)\\
&\qquad+\int_{\mathbb{R}^2} (\eta'-\eta)(4\mathfrak{Im}(\nabla Q_c\overline{Q_c})\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi) + 2c|Q_c|^2\mathfrak{Im}(\partial_{x_2}\psi)\mathfrak{Re}(\psi))
\end{aligned}$$

and, developing $\varphi = Q_c\psi$ (see the proof of Lemma 4.1) and by integration by parts using that $\eta-\eta'\neq 0$ only in a compact domain far from the zeros of Q_c, we check that it is 0.

Finally, for $\varphi\in H_{Q_c}$, $B_{Q_c}(\varphi)$, and $B^{\exp}_{Q_c}(\varphi)$ are both well defined. We recall

$$\begin{aligned}
B_{Q_c}(\varphi) = \int_{\mathbb{R}^2} &|\nabla\varphi|^2 - (1-|Q_c|^2)|\varphi|^2 + 2\mathfrak{Re}^2(\overline{Q_c}\varphi)\\
&-c\int_{\mathbb{R}^2}(1-\eta)\mathfrak{Re}(i\partial_{x_2}\varphi\bar{\varphi}) - c\int_{\mathbb{R}^2}\eta\mathfrak{Re}i\partial_{x_2}Q_c\overline{Q_c}|\psi|^2\\
&+2c\int_{\mathbb{R}^2}\eta\mathfrak{Re}\psi\mathfrak{Im}\partial_{x_2}\psi|Q_c|^2 + c\int_{\mathbb{R}^2}\partial_{x_2}\eta\mathfrak{Re}\psi\mathfrak{Im}\psi|Q_c|^2\\
&+c\int_{\mathbb{R}^2}\eta\mathfrak{Re}\psi\mathfrak{Im}\psi\partial_{x_2}(|Q_c|^2).
\end{aligned}$$

With the same computation as in the proof of Lemma 4.1, we check that for $\varphi\in C^\infty_c(\mathbb{R}^2\backslash\{\tilde{d}_c\vec{e}_1, -\tilde{d}_c\vec{e}_1\},\mathbb{C})$, we have

$$B_{Q_c}(\varphi) = B^{\exp}_{Q_c}(\varphi).$$

With the same arguments as in the density proof at the end of the proof of Proposition 1.4, we check that this equality holds for $\varphi\in H_{Q_c}$. □

Appendix C. Proofs related to the local uniqueness

C.1. Proof of Lemma 7.3. From Lemma 7.2, for any $\Lambda > \frac{10}{c}$,

$$\begin{aligned}
&\|Q\psi\|_{C^1(B(0,\Lambda))} \leqslant K(\Lambda)\|Z-Q_c\|_{H^{\exp}_{Q_c}}\\
&\qquad+K\left(|X| + \frac{\delta^{|\cdot|}(c\vec{e_2},\vec{c'})}{c^2} + \frac{\delta^{\perp}(c\vec{e_2},\vec{c'})}{c} + |\gamma|\right),
\end{aligned}\tag{C.1}$$

therefore, we only have to check the integrability at infinity of $Q\psi$ to show that $\varphi = Q\psi \in H_Q^{\exp}$. In $\{\eta = 1\}$, we have

$$e^\psi = \frac{Z}{Q}.$$

We have shown in the proof of Lemma 7.2 that $K > \left|\frac{Z}{Q}\right| > \delta/2$ outside of $B(0,\lambda)$ for some $\delta > 0$, and together with (C.1), we check that

$$\|\psi\|_{C^0(\{\eta=1\})} \leqslant K(\lambda, \|Z - Q_c\|_{H^{\exp}_{Q_c}}, \varepsilon_0). \tag{C.2}$$

This implies that

$$\int_{\{\eta=1\}} \frac{|Q\psi|^2}{\tilde{r}^2 \ln(\tilde{r})^2} < +\infty.$$

Similarly, we check that, in $\{\eta = 1\}$, since $e^\psi = \frac{Z}{Q}$,

$$\nabla\psi = \frac{e^{-\psi}}{Q}\nabla(Z - Q) - \frac{\nabla Q}{Q}(1 - e^{-\psi}),$$

therefore

$$|\nabla\psi| \leqslant K(\lambda, \|Z - Q_c\|_{H^{\exp}_{Q_c}}, \varepsilon_0)(|\nabla(Z - Q)| + |\nabla Q|). \tag{C.3}$$

From Theorem 2.5, we have

$$|\nabla Z| + |\nabla Q| \leqslant \frac{K(c, Z)}{(1 + r)^2},$$

therefore,

$$\int_{\{\eta=1\}} |\nabla Q|^2 |\psi|^2 < +\infty$$

and

$$\int_{\{\eta=1\}} |\nabla(Z - Q)|^2 \leqslant \int_{\{\eta=1\}} \frac{K(c, Z)}{(1 + r)^4} < +\infty.$$

We deduce that $\int_{\{\eta=1\}} |\nabla\psi|^2 < +\infty$, and, furthermore, equation (C.3) shows that

$$|\nabla\psi| \leqslant \frac{K(\lambda, c, \|Z - Q_c\|_{H^{\exp}_{Q_c}}, \varepsilon_0, Z)}{(1 + r)^2}$$

in $\{\eta = 1\}$.

Now, still in $\{\eta = 1\}$, we have

$$Qe^\psi = Z,$$

and we deduce that $Qe^{-i\gamma}(e^{\psi+i\gamma} - 1) = Z - Qe^{-i\gamma}$. Now, we recall that $\|\psi\|_{C^0(\{\eta=1\})} \leqslant K(\lambda, \|Z - Q_c\|_{H^{\exp}_{Q_c}}, \varepsilon_0)$, thus $|\mathfrak{Re}(e^{\psi+i\gamma} - 1 - (\psi + i\gamma))| \leqslant K(\lambda, \|Z - Q_c\|_{H^{\exp}_{Q_c}}, \varepsilon_0)|\mathfrak{Re}(e^{\psi+i\gamma} - 1)|$. With (C.1), we deduce from this that, in $\{\eta = 1\}$, with $\frac{1}{4}\|\psi + i\gamma\|_{L^\infty(\mathbb{R}^2)} \leqslant |\mathfrak{Re}(e^{\psi+i\gamma} - 1)| \leqslant K\|\psi + i\gamma\|_{L^\infty(\mathbb{R}^2)}$,

$$\begin{aligned}
|\mathfrak{Re}(\psi)| &= |\mathfrak{Re}(\psi + i\gamma)| \\
&\leqslant |\mathfrak{Re}(e^{\psi+i\gamma} - 1)| + |\mathfrak{Re}(e^{\psi+i\gamma} - 1 - (\psi + i\gamma))| \\
&\leqslant K(\lambda, \|Z - Q_c\|_{H^{\exp}_{Q_c}}, \varepsilon_0)|\mathfrak{Re}(e^{\psi+i\gamma} - 1)| \\
&\leqslant K(\lambda, \|Z - Q_c\|_{H^{\exp}_{Q_c}}, \varepsilon_0)\left|\mathfrak{Re}\left(\frac{(Z - Qe^{-i\gamma})\overline{Qe^{i\gamma}}}{|Q|^2}\right)\right| \\
&\leqslant K(\lambda, \|Z - Q_c\|_{H^{\exp}_{Q_c}}, \varepsilon_0)(|\mathfrak{Re}(Z - Qe^{-i\gamma})| \\
&\quad + |\mathfrak{Im}(Z - Qe^{-i\gamma})\mathfrak{Im}(Qe^{i\gamma} - 1)|).
\end{aligned}$$

From Theorem 2.5,

$$|\mathfrak{Re}(Z - Qe^{-i\gamma})| \leqslant |\mathfrak{Re}(Z - 1)| + |\mathfrak{Re}(1 - Qe^{-i\gamma})| \leqslant \frac{K(c, Z)}{(1 + r)^2}$$

and

$$|\mathfrak{Im}(Z - Qe^{-i\gamma})\mathfrak{Im}(Qe^{i\gamma} - 1)| \leqslant \frac{K(c, Z)}{(1 + r)^2}.$$

We conclude that, in $\{\eta = 1\}$, we have $|\mathfrak{Re}(\psi)| \leqslant \frac{K(\lambda, c, \|Z - Q_c\|_{H^{\exp}_{Q_c}}, \varepsilon_0, Z)}{(1+r)^2}$ hence

$$\int_{\{\eta=1\}} \mathfrak{Re}^2(\psi) < +\infty.$$

This concludes the proof of $\varphi = Q\psi \in H^{\exp}_{Q_c}$. We are left with the proof of the following estimates: $|\Delta\psi| \leqslant \frac{K(\lambda, c, \|Z - Q_c\|_{H^{\exp}_{Q_c}}, \varepsilon_0, Z)}{(1+r)^2}$, $|\mathfrak{Im}(\psi + i\gamma)| \leqslant \frac{K(\lambda, c, \|Z - Q_c\|_{H^{\exp}_{Q_c}}, \varepsilon_0, Z)}{(1+r)}$, and $|\mathfrak{Re}(\nabla\psi)| \leqslant \frac{K(\lambda, c, \|Z - Q_c\|_{H^{\exp}_{Q_c}}, \varepsilon_0, Z)}{(1+r)^3}$ in $\{\eta = 1\}$.

We recall that, in $\{\eta = 1\}$, $\nabla\psi = \frac{e^{-\psi}}{Q}\nabla(Z - Q) - \frac{\nabla Q}{Q}(1 - e^{-\psi})$, from which we compute, by differentiating a second time,

$$\begin{aligned}
\Delta\psi = &-\frac{\nabla\psi \cdot \nabla(Z - Q)}{Q}e^{-\psi} - \frac{\nabla Q}{Q}e^{-\psi} \cdot \nabla(Z - Q) + \frac{e^{-\psi}}{Q}\Delta(Z - Q) \\
&-\frac{\Delta Q}{Q}(1 - e^{-\psi}) + \frac{\nabla Q \cdot \nabla Q}{Q^2}(1 - e^{-\psi}) - \frac{\nabla Q}{Q} \cdot \nabla\psi e^{-\psi}.
\end{aligned}$$

Using Theorem 2.5, $\Delta Q = -i\vec{c'} \cdot \nabla Q - (1-|Q|^2)Q$, $Z = -ic\partial_{x_2} Z - (1-|Z|^2)Z$ and previous estimates on ψ, we check that, in $\{\eta = 1\}$,

$$|\Delta\psi| \leqslant \frac{K(\lambda, c, \|Z - Q_c\|_{H^{\exp}_{Q_c}}, \varepsilon_0, Z)}{(1+r)^2}.$$

We have $Qe^{-i\gamma}(e^{\psi+i\gamma} - 1) = Z - Qe^{-i\gamma}$ in $\{\eta = 1\}$, therefore

$$e^{\psi+i\gamma} - 1 = \frac{Z}{Qe^{-i\gamma}} - 1.$$

We check, since $\|\psi\|_{C^0(\{\eta=1\})} \leqslant K(\lambda, \|Z - Q_c\|_{H^{\exp}_{Q_c}}, \varepsilon_0)$, that we have by Theorem 2.5

$$\begin{aligned}
|\mathfrak{Im}(\psi + i\gamma)| &\leqslant K(\lambda, \|Z - Q_c\|_{H^{\exp}_{Q_c}}, \varepsilon_0)|\mathfrak{Im}(e^{\psi+i\gamma} - 1)| \\
&\leqslant K(\lambda, \|Z - Q_c\|_{H^{\exp}_{Q_c}}, \varepsilon_0)\left|\frac{Z}{\mathrm{Qe}^{-i\gamma}} - 1\right| \\
&\leqslant \frac{K(\lambda, c, \|Z - Q_c\|_{H^{\exp}_{Q_c}}, \varepsilon_0, Z)}{(1+r)}.
\end{aligned}$$

Finally, since $\nabla\psi = \frac{e^{-\psi}}{Q}\nabla(Z - Q) - \frac{\nabla Q}{Q}(1 - e^{-\psi}) = \frac{\nabla Z}{Q}e^{-\psi} - \frac{\nabla Q}{Q}$, we check with Theorem 2.5 that, in $\{\eta = 1\}$,

$$\begin{aligned}
|\nabla\mathfrak{Re}(\psi)| &\leqslant \left|\mathfrak{Re}\left(\frac{\nabla Z}{Q}e^{-\psi}\right)\right| + \left|\mathfrak{Re}\left(\frac{\nabla Q}{Q}\right)\right| \\
&\leqslant \left|\mathfrak{Re}\left(\nabla Z\bar{Z}\frac{e^{-\psi}}{Q\bar{Z}}\right)\right| + \frac{|\mathfrak{Re}(\nabla Q\bar{Q})|}{|Q|^2} \\
&\leqslant \left|\mathfrak{Im}(\nabla Z\bar{Z})\mathfrak{Im}\left(\frac{e^{-\psi}}{Q\bar{Z}}\right)\right| + |\mathfrak{Re}(\nabla Z\bar{Z})|\left|\mathfrak{Re}\left(\frac{e^{-\psi}}{Q\bar{Z}}\right)\right| + \frac{|\nabla(|Q|^2)|}{2|Q|^2} \\
&\leqslant \frac{K(\lambda, c, \|Z - Q_c\|_{H^{\exp}_{Q_c}}, \varepsilon_0, Z)}{(1+r)^2}\left|\mathfrak{Im}\left(\frac{e^{-\psi}}{Q\bar{Z}}\right)\right| \\
&\quad + \frac{K(\lambda, c, \|Z - Q_c\|_{H^{\exp}_{Q_c}}, \varepsilon_0, Z)}{(1+r)^3}.
\end{aligned}$$

We compute in $\{\eta=1\}$, still using Theorem 2.5,

$$\begin{aligned}\left|\mathfrak{Im}\left(\frac{e^{-\psi}}{Q\bar{Z}}\right)\right| &= \frac{1}{|QZ|^2}|\mathfrak{Im}(e^{-\psi-i\gamma}\bar{Q}Ze^{i\gamma})| \\ &\leqslant K(|\mathfrak{Im}(e^{-\psi-i\gamma}-1)\mathfrak{Re}(\bar{Q}Ze^{i\gamma})|+|\mathfrak{Re}(e^{-\psi-i\gamma})\mathfrak{Im}(\bar{Q}Ze^{i\gamma})|) \\ &\leqslant \frac{K(\lambda,c,\|Z-Q_c\|_{H^{\exp}_{Q_c}},\varepsilon_0,Z)}{1+r} \\ &\quad +K(\lambda,c,\|Z-Q_c\|_{H^{\exp}_{Q_c}},\varepsilon_0,Z)|\mathfrak{Im}(\bar{Q}Ze^{i\gamma})| \\ &\leqslant \frac{K(\lambda,c,\|Z-Q_c\|_{H^{\exp}_{Q_c}},\varepsilon_0,Z)}{(1+r)} \\ &\quad +K(\lambda,c,\|Z-Q_c\|_{H^{\exp}_{Q_c}},\varepsilon_0,Z)(|Qe^{-i\gamma}-1|+|Z-1|) \\ &\leqslant \frac{K(\lambda,c,\|Z-Q_c\|_{H^{\exp}_{Q_c}},\varepsilon_0,Z)}{(1+r)}.\end{aligned}$$

This concludes the proof of this lemma. □

C.2. Proof of Lemma 7.4. First, let us show that $L_{Q_c}(\Phi)=Q_cL'_{Q_c}(\Psi)$ if $\Phi=Q_c\Psi\in C^2(\mathbb{R}^2,\mathbb{C})$. With equation (7.1), this implies that $L_Q(\varphi)=QL'_Q(\psi)$. We recall that

$$L_{Q_c}(\Phi)=-\Delta\Phi-ic\partial_{x_2}\Phi-(1-|Q_c|^2)\Phi+2\mathfrak{Re}(\overline{Q_c}\Phi)Q_c,$$

and we develop with $\Phi=Q_c\Psi$,

$$L_{Q_c}(\Phi)=\mathrm{TW}_c(Q_c)\Psi-Q_c\Delta\Psi-2\nabla Q_c\cdot\nabla\Psi-icQ_c\partial_{x_2}\Psi+2\mathfrak{Re}(\Psi)|Q_c|^2Q_c,$$

thus, since $(\mathrm{TW}_c)(Q_c)=0$, we have $L_{Q_c}(\Phi)=Q_cL'_{Q_c}(\Psi)$.

Now, for $\varphi=Q\psi$, we have

$$\begin{aligned}\langle(1-\eta)L_Q(\varphi)+\eta QL'_Q(\psi),(\varphi+i\gamma Q)\rangle &= \int_{\mathbb{R}^2}\mathfrak{Re}((1-\eta)L_Q(\varphi)\overline{(\varphi+i\gamma Q)}) \\ +\int_{\mathbb{R}^2}\eta|Q|^2\mathfrak{Re}\left(\left(-\Delta\psi-2\frac{\nabla Q}{Q}\cdot\nabla\psi+i\overrightarrow{c}\cdot\nabla\psi\right)(\overline{\psi+i\gamma})\right)&+\eta|Q|^4\mathfrak{Re}^2(\psi).\end{aligned}$$

With Lemma 7.3, we check that all the terms are integrable independently (in particular since $\varphi+i\gamma Q=Q(\psi+i\gamma)$ and $\|(\psi+i\gamma)(1+r)\|_{L^\infty(\{\eta=1\})}<+\infty$ by Lemma 7.3). We recall that $L_Q(\varphi)=-\Delta\varphi+i\vec{c}\cdot\nabla\varphi-(1-|Q|^2)\varphi+2\mathfrak{Re}(\bar{Q}\varphi)Q$, and thus

$$\begin{aligned}&\int_{\mathbb{R}^2}\mathfrak{Re}((1-\eta)L_Q(\varphi)\overline{(\varphi+i\gamma Q)}) \\ &\quad=\int_{\mathbb{R}^2}(1-\eta)(\mathfrak{Re}(i\vec{c}\cdot\nabla\varphi\bar{\varphi})-(1-|Q|^2)|\varphi|^2+2\mathfrak{Re}^2(\bar{Q}\varphi)) \\ &\qquad+\int_{\mathbb{R}^2}(1-\eta)\mathfrak{Re}(-\Delta\varphi\bar{\varphi})+\gamma\int_{\mathbb{R}^2}(1-\eta)\mathfrak{Re}(L_Q(\varphi)\overline{iQ}).\end{aligned}$$

We recall that $1-\eta$ is compactly supported and that $\varphi \in C^2(\mathbb{R}^2, \mathbb{C})$. By integration by parts,

$$\int_{\mathbb{R}^2} (1-\eta)\mathfrak{Re}(-\Delta\varphi\bar{\varphi}) = \int_{\mathbb{R}^2} (1-\eta)|\nabla\varphi|^2 - \int_{\mathbb{R}^2} \nabla\eta \cdot \mathfrak{Re}(\nabla\varphi\bar{\varphi}),$$

and we decompose

$$\int_{\mathbb{R}^2} (1-\eta)\mathfrak{Re}(\eta L_Q(\varphi)\overline{iQ}) = \int_{\mathbb{R}^2} (1-\eta)\mathfrak{Re}(-\Delta\varphi\overline{iQ} + \vec{c}\cdot\nabla\varphi\overline{Q})$$
$$- \int_{\mathbb{R}^2} (1-\eta)\mathfrak{Re}((1-|Q|^2)\varphi\overline{iQ}).$$

By integration by parts, we have

$$\int_{\mathbb{R}^2} (1-\eta)\mathfrak{Re}(\vec{c}\cdot\nabla\varphi\overline{Q}) = -\vec{c}\cdot\int_{\mathbb{R}^2} -\nabla\eta\mathfrak{Re}(\varphi\bar{Q}) + (1-\eta)\mathfrak{Re}(\varphi\nabla\bar{Q})$$

and

$$\int_{\mathbb{R}^2}(1-\eta)\mathfrak{Re}(-\Delta\varphi\overline{iQ}) = \int_{\mathbb{R}^2} -\nabla\eta\cdot(\mathfrak{Re}(i\varphi\nabla\bar{Q}) - \mathfrak{Re}(i\nabla\varphi\bar{Q})) + \int_{\mathbb{R}^2}(1-\eta)\mathfrak{Re}(i\varphi\bar{Q}).$$

Combining these computations, we infer

$$\int_{\mathbb{R}^2} \mathfrak{Re}((1-\eta)L_Q(\varphi)\overline{(\varphi+i\gamma Q)})$$
$$= \int_{\mathbb{R}^2} (1-\eta)(|\nabla\varphi|^2 + \mathfrak{Re}(i\vec{c}\cdot\nabla\varphi\bar{\varphi}) - (1-|Q|^2)|\varphi|^2 + 2\mathfrak{Re}^2(\bar{Q}\varphi))$$
$$- \int_{\mathbb{R}^2} \nabla\eta\cdot\mathfrak{Re}(\nabla\varphi\bar{\varphi})\gamma\vec{c}\cdot\int_{\mathbb{R}^2}\nabla\eta\mathfrak{Re}(\varphi\bar{Q})$$
$$-\gamma\left(\int_{\mathbb{R}^2}\nabla\eta\cdot(\mathfrak{Re}(i\varphi\nabla\bar{Q}) - \mathfrak{Re}(i\nabla\varphi\bar{Q}))\right)$$
$$+\gamma\int_{\mathbb{R}^2}(1-\eta)\mathfrak{Re}(\varphi(-\vec{c}\cdot\nabla\bar{Q} + i(1-|Q|^2)\bar{Q} + i\Delta\bar{Q})).$$

Since $-\Delta Q + i\vec{c}\cdot\nabla Q - (1-|Q|^2)Q = 0$, we have $-\vec{c}\cdot\nabla\bar{Q} + i(1-|Q|^2)\bar{Q} + i\Delta\bar{Q} = 0$, therefore

$$\int_{\mathbb{R}^2} \mathfrak{Re}((1-\eta)L_Q(\varphi)\overline{(\varphi+i\gamma Q)})$$
$$= \int_{\mathbb{R}^2} (1-\eta)(|\nabla\varphi|^2 + \mathfrak{Re}(i\vec{c}\cdot\nabla\varphi\bar{\varphi}) - (1-|Q|^2)|\varphi|^2 + 2\mathfrak{Re}^2(\bar{Q}\varphi))$$
$$- \int_{\mathbb{R}^2}\nabla\eta\cdot\mathfrak{Re}(\nabla\varphi\bar{\varphi})$$
$$-\gamma\left(-\vec{c}\cdot\int_{\mathbb{R}^2}\nabla\eta\mathfrak{Re}(\varphi\bar{Q}) + \int_{\mathbb{R}^2}\nabla\eta\cdot(\mathfrak{Re}(i\varphi\nabla\bar{Q}) - \mathfrak{Re}(i\nabla\varphi\bar{Q}))\right).$$

Until now, all the integrals were on a bounded domain (since $1-\eta$ is compactly supported).

Now, by integration by parts (which can be done thanks to Lemma 7.3 and Theorem 2.5),

$$\begin{aligned}\int_{\mathbb{R}^2} \eta |Q|^2 \mathfrak{Re}(-\Delta\psi(\overline{\psi + i\gamma})) = &\int_{\mathbb{R}^2} \nabla\eta \cdot |Q|^2 \mathfrak{Re}(\nabla\psi(\overline{\psi + i\gamma})) \\ &+ \int_{\mathbb{R}^2} \eta \nabla(|Q|^2) \cdot \mathfrak{Re}(\nabla\psi(\overline{\psi + i\gamma})) \\ &+ \int_{\mathbb{R}^2} \eta |Q|^2 |\nabla\psi|^2.\end{aligned}$$

Now, we decompose (and we check that each term is well defined at each step with Lemma 7.3 and Theorem 2.5)

$$\begin{aligned}\int_{\mathbb{R}^2} \eta |Q|^2 \mathfrak{Re}\left(\left(-2\frac{\nabla Q}{Q}\cdot\nabla\psi\right)(\overline{\psi + i\gamma})\right) = &-2\int_{\mathbb{R}^2} \eta \mathfrak{Re}(\nabla Q \bar{Q} \cdot \nabla\psi\bar{\psi}) \\ &-2\int_{\mathbb{R}^2} \eta \mathfrak{Re}(\nabla Q \bar{Q} \cdot \nabla\psi\overline{(i\gamma)}),\end{aligned}$$

with

$$\begin{aligned}-2\int_{\mathbb{R}^2} \eta \mathfrak{Re}(\nabla Q \bar{Q} \cdot \nabla\psi\bar{\psi}) = &-2\int_{\mathbb{R}^2} \eta \mathfrak{Re}(\nabla Q \bar{Q}) \cdot \mathfrak{Re}(\nabla\psi\bar{\psi}) \\ &+2\int_{\mathbb{R}^2} \eta \mathfrak{Im}(\nabla Q \bar{Q}) \cdot \mathfrak{Im}(\nabla\psi\bar{\psi}),\end{aligned}$$

and since $\nabla(|Q|^2) = 2\mathfrak{Re}(\nabla Q\bar{Q})$, we have

$$\begin{aligned}&\int_{\mathbb{R}^2} \eta |Q|^2 \mathfrak{Re}\left(\left(-\Delta\psi - 2\frac{\nabla Q}{Q}\cdot\nabla\psi\right)(\overline{\psi + i\gamma})\right) \\ &\quad = \int_{\mathbb{R}^2} \eta |Q|^2 |\nabla\psi|^2 + 2\int_{\mathbb{R}^2} (1-\eta)\mathfrak{Im}(\nabla Q\bar{Q}) \cdot \mathfrak{Im}(\nabla\psi\bar{\psi}) \\ &\qquad + \int_{\mathbb{R}^2} \nabla\eta \cdot |Q|^2 \mathfrak{Re}(\nabla\psi(\overline{\psi + i\gamma})) + 2\int_{\mathbb{R}^2} \eta \mathfrak{Im}(\nabla Q\bar{Q}) \cdot \mathfrak{Im}(\nabla\psi\overline{(i\gamma)})).\end{aligned}$$

We continue. We have

$$\begin{aligned}2\int_{\mathbb{R}^2} \eta \mathfrak{Im}(\nabla Q\bar{Q}) \cdot \mathfrak{Im}(\nabla\psi\bar{\psi}) = &\,2\int_{\mathbb{R}^2} \eta \mathfrak{Im}(\nabla Q\bar{Q}) \cdot \mathfrak{Re}(\psi)\mathfrak{Im}(\nabla\psi) \\ &-2\int_{\mathbb{R}^2} \eta \mathfrak{Im}(\nabla Q\bar{Q}) \cdot \mathfrak{Re}(\nabla\psi)\mathfrak{Im}(\psi),\end{aligned}$$

and by integration by parts (still using Lemma 7.3 and Theorem 2.5),

$$-2\int_{\mathbb{R}^2} \eta \mathfrak{Im}(\nabla Q\bar{Q}) \cdot \mathfrak{Re}(\nabla\psi)\mathfrak{Im}(\psi) = 2\int_{\mathbb{R}^2} \eta \mathfrak{Im}(\nabla Q\bar{Q}) \cdot \mathfrak{Re}(\psi)\mathfrak{Im}(\nabla\psi)$$
$$+2\int_{\mathbb{R}^2} \eta \mathfrak{Im}(\Delta Q\bar{Q})\mathfrak{Re}(\psi)\mathfrak{Im}(\psi) + 2\int_{\mathbb{R}^2} \nabla\eta \cdot \mathfrak{Im}(\nabla Q\bar{Q})\mathfrak{Re}(\psi)\mathfrak{Im}(\psi).$$

We have $\mathfrak{Im}(\Delta Q\bar{Q}) = \mathfrak{Im}(i\vec{c}\cdot\nabla Q - (1-|Q|^2Q)\bar{Q}) = \mathfrak{Re}(\vec{c}\cdot\nabla Q\bar{Q})$, therefore

$$\int_{\mathbb{R}^2} \eta|Q|^2\mathfrak{Re}\left(\left(-\Delta\psi - 2\frac{\nabla Q}{Q}\cdot\nabla\psi\right)(\overline{\psi+i\gamma})\right)$$
$$= \int_{\mathbb{R}^2} \eta|Q|^2|\nabla\psi|^2 + 4\int_{\mathbb{R}^2} \eta\mathfrak{Im}(\nabla Q\bar{Q})\cdot\mathfrak{Re}(\psi)\mathfrak{Im}(\nabla\psi)$$
$$+2\int_{\mathbb{R}^2} \eta\mathfrak{Im}(\nabla Q\bar{Q})\cdot\mathfrak{Im}(\nabla\psi\overline{(i\gamma)}))$$
$$+2\int_{\mathbb{R}^2} \eta\mathfrak{Re}(\vec{c}\cdot\nabla Q\bar{Q})\mathfrak{Re}(\psi)\mathfrak{Im}(\psi)$$
$$+\int_{\mathbb{R}^2} \nabla\eta(|Q|^2\mathfrak{Re}(\nabla\psi(\overline{\psi+i\gamma})) + 2\mathfrak{Im}(\nabla Q\bar{Q})\mathfrak{Re}(\psi)\mathfrak{Im}(\psi)).$$

Now, we compute

$$\vec{c}\cdot\int_{\mathbb{R}^2} \eta|Q|^2\mathfrak{Re}(i\nabla\psi(\overline{\psi+i\gamma})) = \vec{c}\cdot\int_{\mathbb{R}^2} \eta|Q|^2\mathfrak{Re}(\nabla\psi)\mathfrak{Im}(\psi+i\gamma)$$
$$-\vec{c}\cdot\int_{\mathbb{R}^2} \eta|Q|^2\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi),$$

and by integration by parts (still using Lemma 7.3 and Theorem 2.5),

$$\vec{c}\cdot\int_{\mathbb{R}^2} \eta|Q|^2\mathfrak{Re}(\nabla\psi)\mathfrak{Im}(\psi+i\gamma) = -\vec{c}\cdot\int_{\mathbb{R}^2} \nabla\eta|Q|^2\mathfrak{Re}(\psi)\mathfrak{Im}(\psi+i\gamma)$$
$$-\vec{c}\cdot\int_{\mathbb{R}^2} \eta\nabla(|Q|^2)\mathfrak{Re}(\psi)\mathfrak{Im}(\psi+i\gamma)$$
$$-\vec{c}\cdot\int_{\mathbb{R}^2} \eta|Q|^2\mathfrak{Re}(\psi)\mathfrak{Im}(\nabla\psi).$$

Since $\nabla(|Q|^2) = 2\mathfrak{Re}(\nabla Q\bar{Q})$, we infer

$$\int_{\mathbb{R}^2} \eta|Q|^2\mathfrak{Re}\left(\left(-\Delta\psi - 2\frac{\nabla Q}{Q}\cdot\nabla\psi - i\vec{c}\cdot\nabla\psi\right)(\overline{\psi + i\gamma})\right)$$

$$= \int_{\mathbb{R}^2} \eta(|Q|^2|\nabla\psi|^2 + 4\mathfrak{Im}(\nabla Q\bar{Q})\cdot\mathfrak{Re}(\psi)\mathfrak{Im}(\nabla\psi) - 2\vec{c}\cdot\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi))$$

$$+2\int_{\mathbb{R}^2} \eta\mathfrak{Im}(\nabla Q\bar{Q})\cdot\mathfrak{Im}(\nabla\psi\overline{(i\gamma)}))$$

$$-2\gamma\int_{\mathbb{R}^2} \eta\mathfrak{Re}(\vec{c}\cdot\nabla Q\bar{Q})\mathfrak{Re}(\psi)$$

$$+\int_{\mathbb{R}^2} \nabla\eta\cdot(|Q|^2\mathfrak{Re}(\nabla\psi(\overline{\psi + i\gamma})) + 2\mathfrak{Im}(\nabla Q\bar{Q})\mathfrak{Re}(\psi)\mathfrak{Im}(\psi))$$

$$+\vec{c}\cdot\int_{\mathbb{R}^2} \nabla\eta|Q|^2\mathfrak{Re}(\psi)\mathfrak{Im}(\psi + i\gamma).$$

Combining these computations yields

$$\int_{\mathbb{R}^2} \mathfrak{Re}(L_Q^{\exp}(\varphi)\overline{(\varphi + i\gamma Q)}) = B_Q^{\exp}(\varphi)$$

$$-\gamma\left(-\vec{c}\cdot\int_{\mathbb{R}^2} \nabla\eta\mathfrak{Re}(\varphi\bar{Q}) + \int_{\mathbb{R}^2} \nabla\eta\cdot(\mathfrak{Re}(i\varphi\nabla\bar{Q}) - \mathfrak{Re}(i\nabla\varphi\bar{Q}))\right)$$

$$+2\int_{\mathbb{R}^2} \eta\mathfrak{Im}(\nabla Q\bar{Q})\cdot\mathfrak{Im}(\nabla\psi\overline{(i\gamma)}))$$

$$-2\gamma\int_{\mathbb{R}^2} \eta\mathfrak{Re}(\vec{c}\cdot\nabla Q\bar{Q})\mathfrak{Re}(\psi)$$

$$+\int_{\mathbb{R}^2} \nabla\eta\cdot|Q|^2\mathfrak{Re}(\nabla\psi\overline{(i\gamma)})$$

$$-\vec{c}\cdot\gamma\int_{\mathbb{R}^2} \nabla\eta|Q|^2\mathfrak{Re}(\psi).$$

We compute, by integration by parts (still using Lemma 7.3 and Theorem 2.5), that

$$2\int_{\mathbb{R}^2} \eta\mathfrak{Im}(\nabla Q\bar{Q})\cdot\mathfrak{Im}(\nabla\psi\overline{(i\gamma)})) = -2\gamma\int_{\mathbb{R}^2} \eta\mathfrak{Im}(\nabla Q\bar{Q})\cdot\mathfrak{Re}(\nabla\psi)$$

$$= 2\gamma\int_{\mathbb{R}^2} \nabla\eta\cdot\mathfrak{Im}(\nabla Q\bar{Q})\mathfrak{Re}(\psi)$$

$$+2\gamma\int_{\mathbb{R}^2} \eta\mathfrak{Im}(\Delta Q\bar{Q})\mathfrak{Re}(\psi),$$

and since $\mathfrak{Im}(\Delta Q\bar{Q}) = \mathfrak{Re}(\vec{c}\cdot\nabla Q\bar{Q})$ and $\mathfrak{Re}(\nabla\psi\overline{(i\gamma)}) = \gamma\mathfrak{Im}(\nabla\psi)$, we have

$$\begin{aligned}\int_{\mathbb{R}^2}\mathfrak{Re}(L_Q^{\exp}(\varphi)\overline{(\varphi+i\gamma Q)}) &= B_Q^{\exp}(\varphi)\\ &-\gamma\left(-\vec{c}\cdot\int_{\mathbb{R}^2}\nabla\eta\mathfrak{Re}(\varphi\bar{Q})+\int_{\mathbb{R}^2}\nabla\eta\cdot(\mathfrak{Re}(i\varphi\nabla\bar{Q})-\mathfrak{Re}(i\nabla\varphi\bar{Q}))\right)\\ &+2\gamma\int_{\mathbb{R}^2}\nabla\eta\cdot\mathfrak{Im}(\nabla Q\bar{Q})\mathfrak{Re}(\psi)\\ &+\gamma\int_{\mathbb{R}^2}\nabla\eta\cdot|Q|^2\mathfrak{Im}(\nabla\psi)\\ &-\vec{c}\cdot\gamma\int_{\mathbb{R}^2}\nabla\eta|Q|^2\mathfrak{Re}(\psi).\end{aligned}$$

We check that $\mathfrak{Re}(\varphi\bar{Q}) = |Q|^2\mathfrak{Re}(\psi)$, $\mathfrak{Re}(i\varphi\nabla\bar{Q}) = -\mathfrak{Re}(\nabla Q\bar{Q})\mathfrak{Im}(\psi) + \mathfrak{Im}(\nabla Q\bar{Q})\mathfrak{Re}(\psi)$ and that

$$\begin{aligned}-\mathfrak{Re}(i\nabla\varphi\bar{Q}) &= -\mathfrak{Re}(i\nabla Q_c\bar{Q}\psi) - \mathfrak{Re}(i\nabla\psi)|Q|^2\\ &= \mathfrak{Im}(\nabla Q\bar{Q})\mathfrak{Re}(\psi) + \mathfrak{Re}(\nabla Q\bar{Q})\mathfrak{Im}(\psi) + \mathfrak{Im}(\nabla\psi)|Q|^2,\end{aligned}$$

thus concluding the proof of

$$\int_{\mathbb{R}^2}\mathfrak{Re}(L_Q^{\exp}(\varphi)\overline{(\varphi+i\gamma Q)}) = B_Q^{\exp}(\varphi). \qquad \square$$

C.3. Proof of Lemma 7.6. For $X = (X_1, X_2), \vec{c'} \in \mathbb{R}^2$, we define, as previously, the function

$$Q = Q_{\vec{c'}}(\cdot - X)e^{i\gamma}.$$

We define, to simplify the notations,

$$\Omega := B(\boldsymbol{d}_{\vec{c'},1}, R) \cup B(\boldsymbol{d}_{\vec{c'},2}, R)$$

and

$$\Omega' := B\left(\frac{(\boldsymbol{d}_{\vec{c'},1} + \boldsymbol{d}_{\vec{c'},2})}{2}, R\right),$$

which is between the two vortices. We define

$$G\begin{pmatrix}X_1\\X_2\\\delta_1\\\delta_2\\\gamma\end{pmatrix} := \begin{pmatrix}\mathfrak{Re}\int_\Omega \partial_{x_1}Q\overline{Q\psi^{\neq 0}}\\ \mathfrak{Re}\int_\Omega \partial_{x_2}Q\overline{Q\psi^{\neq 0}}\\ c^2\mathfrak{Re}\int_\Omega \partial_{\boldsymbol{d}}\boldsymbol{V}\overline{Q\psi^{\neq 0}}\\ c\mathfrak{Re}\int_\Omega \partial_{c^\perp}Q\overline{Q\psi^{\neq 0}}\\ \mathfrak{Re}\int_{\Omega'} i\psi\end{pmatrix},$$

where $\vec{c'}$ (used to defined $Q = Q_{\vec{c'}}(\cdot - X)e^{i\gamma}$) is given by $\delta_1 = \delta^{|\cdot|}(c\vec{e_2}, \vec{c'})$ and $\delta_2 = \delta^\perp(c\vec{e_2}, \vec{c'})$.

Here, we use the notation $\partial_c Q$ for $\partial_c Q_{c|c=c'}$. We note from (7.4) and the definition of η that in Ω we have

$$Q\psi = Z - Q.$$

First, we have

$$\begin{aligned} \|Q\psi\|_{C^1(\Omega)} \leqslant\ & o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1) \\ & + K\left(|X| + \frac{\delta^{|\cdot|}(c\overrightarrow{e_2}, \vec{c'})}{c^2} + \frac{\delta^{\perp}(c\overrightarrow{e_2}, \vec{c'})}{c} + |\gamma|\right), \end{aligned} \tag{C.4}$$

which is a consequence of Lemma 7.1. By Lemma 5.1, we compute that

$$\left| G\begin{pmatrix} 0\\0\\0\\0\\0 \end{pmatrix} \right| \leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1).$$

We are going to apply the implicit function theorem on $H = G - G(0)$, and find a point A such that $H(A) = G(0)$ since $G(0)$ is small, which implies $G(A) = 0$.

Let us compute $\partial_{X_2} G$. We recall that $Q\psi \in C^1(\mathbb{R}^2, \mathbb{C})$. Since Ω depends on X, we have

$$\begin{aligned} \partial_{X_2} \mathfrak{Re} \int_\Omega \partial_{x_2} Q \overline{Q\psi^{\neq 0}} = & \int_{\partial\Omega} \mathfrak{Re}(\partial_{x_2} Q \overline{Q\psi^{\neq 0}}) \\ & - \int_\Omega \mathfrak{Re}(\partial^2_{x_2 x_2} Q \overline{Q\psi^{\neq 0}}) \\ & + \int_\Omega \mathfrak{Re}(\partial_{x_2} Q \overline{\partial_{X_2}(Q\psi^{\neq 0})}). \end{aligned}$$

By estimate (C.4), we have

$$\begin{aligned} & \left| \int_{\partial\Omega} \mathfrak{Re}(\partial_{x_2} Q \overline{Q\psi^{\neq 0}}) \right| + \left| \int_\Omega \mathfrak{Re}(\partial^2_{x_2 x_2} Q \overline{Q\psi^{\neq 0}}) \right| \\ & \quad \leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1) + K\left(|X| + \frac{\delta^{|\cdot|}(c\overrightarrow{e_2}, \vec{c'})}{c^2} + \frac{\delta^{\perp}(c\overrightarrow{e_2}, \vec{c'})}{c}\right), \end{aligned}$$

and since $Q\psi = Z - Q$ and $\psi^{\neq 0} = \psi - \psi^{\mathbf{0}}$ in Ω, we check that,

$$\int_\Omega \mathfrak{Re}(\partial_{x_2} Q \overline{\partial_{X_2}(Q\psi^{\neq 0})}) = -\int_\Omega |\partial_{x_2} Q|^2 + \int_\Omega \mathfrak{Re}(\partial_{x_2} Q \overline{\partial_X (Q\psi^{\mathbf{0}})}).$$

Now, using $Q\psi = Z - Q$, we check that, in $B(\boldsymbol{d}_{\vec{c'},1}, R)$, where $x = \boldsymbol{r}_1 e^{i\boldsymbol{\theta}_1}$,

$$\begin{aligned}
2\pi\partial_{X_2}(Q\psi^0) &= \partial_{X_2}\left(Q\int_0^{2\pi}\frac{Z-Q}{Q}\,d\boldsymbol{\theta}_1\right)\\
&= \partial_{x_2}Q\int_0^{2\pi}\frac{Z-Q}{Q}\,d\boldsymbol{\theta}_1\\
&\quad +Q\int_0^{2\pi}\frac{-\partial_{x_2}Q}{Q}\,d\boldsymbol{\theta}_1 + Q\int_0^{2\pi}\frac{-(Z-Q)\partial_{x_2}Q}{Q^2}\,d\boldsymbol{\theta}_1\\
&\quad +Q\int_0^{2\pi}\partial_{x_2}\left(\frac{Z-Q}{Q}\right)d\boldsymbol{\theta}_1.
\end{aligned}$$

Therefore, we estimate (since R is a universal constant)

$$\begin{aligned}
&\left|\int_{B(\boldsymbol{d}_{\vec{c'},1},R)}\mathfrak{Re}(\partial_{x_2}Q\overline{\partial_X(Q\psi^0)})\right|\\
&\qquad\leqslant\left|\int_{B(\boldsymbol{d}_{\vec{c'},1},R)}\mathfrak{Re}\left(\overline{\partial_{x_2}Q}Q\int_0^{2\pi}\frac{-\partial_{x_2}Q}{Q}\,d\boldsymbol{\theta}_1\right)\right| + K\|Z-Q\|_{C^1(\Omega)}.
\end{aligned}$$

Let us show that, in $B(\boldsymbol{d}_{\vec{c'},1}, R)$,

$$Q\int_0^{2\pi}\frac{-\partial_{x_2}Q}{Q}\,d\boldsymbol{\theta}_1 = o_{c\to 0}(1).$$

We have in this domain that $\frac{Q}{\mathbf{V}_1} = 1 + o_{c\to0}(1)$ and $|\nabla Q_c - \nabla\tilde{V}_1| = o_{c\to0}(1)$ by Lemmas 2.14 and 2.15, where $\mathbf{V}_1$ is the vortex centred at $\boldsymbol{d}_{\vec{c'},1}$. We deduce that, in $B(\boldsymbol{d}_{\vec{c'},1}, R)$,

$$Q\int_0^{2\pi}\frac{-\partial_{x_2}Q}{Q}d\boldsymbol{\theta}_1 = \mathbf{V}_1\int_0^{2\pi}\frac{-\partial_{x_2}\mathbf{V}_1}{\mathbf{V}_1}\,d\boldsymbol{\theta}_1 + o_{c\to0}(1).$$

Finally, by Lemma 2.1, we check that $\frac{\partial_{x_2}\mathbf{V}_1}{\mathbf{V}_1}$ has no 0-harmonic around $\boldsymbol{d}_{\vec{c'},1}$, therefore

$$\mathbf{V}_1\int_0^{2\pi}\frac{-\partial_{x_2}\mathbf{V}_1}{\mathbf{V}_1}\,d\boldsymbol{\theta}_1 = 0. \tag{C.5}$$

By symmetry, the same proof holds in $B(\boldsymbol{d}_{\vec{c'},2}, R)$.

Adding up these estimates, we get

$$\begin{aligned}
&\left|\partial_{X_2}\mathfrak{Re}\int_\Omega\partial_{x_2}Q\overline{Q\psi^{\neq0}} + \int_\Omega|\partial_{x_2}Q|^2\right|\\
&\quad\leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to0}(1) + o_{c\to0}(1) + K\left(|X| + \frac{\delta^{|\cdot|}(c\overrightarrow{e_2},\vec{c'})}{c^2} + \frac{\delta^{\perp}(c\overrightarrow{e_2},\vec{c'})}{c} + |\gamma|\right).
\end{aligned}$$

By a similar computation, we have

$$\left|\partial_{X_2}\mathfrak{Re}\int_\Omega \partial_{\boldsymbol{d}}\boldsymbol{V}\overline{Q\psi^{\neq 0}} - \int_\Omega \mathfrak{Re}(\partial_{\boldsymbol{d}}\boldsymbol{V}\overline{\partial_{x_2}Q})\right|$$
$$\leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)+o_{c\to 0}(1)+K\left(|X|+\frac{\delta^{|\cdot|}(c\overrightarrow{e_2},\vec{c'})}{c^2}+\frac{\delta^{\perp}(c\overrightarrow{e_2},\vec{c'})}{c}+|\gamma|\right).$$

By Lemma 5.1 and Theorem 1.1 (for $p=+\infty$), we have

$$\left|\int_\Omega \mathfrak{Re}(\partial_{\boldsymbol{d}}\boldsymbol{V}\overline{\partial_{x_2}Q})\right| \leqslant \left|\int_\Omega \mathfrak{Re}(c^2\partial_c Q\overline{\partial_{x_2}Q})\right| + \left|\int_\Omega \mathfrak{Re}((\partial_{\boldsymbol{d}}\boldsymbol{V}-c^2\partial_c Q)\overline{\partial_{x_2}Q})\right| = o_{c\to 0}(1).$$

Similarly, we check

$$\left|\partial_{X_2}\int_\Omega \partial_{x_1}Q\overline{Q\psi^{\neq 0}}\right| - \left|\int_\Omega \mathfrak{Re}(\partial_{x_1}Q\overline{\partial_{x_2}Q})\right|$$
$$\leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)+o_{c\to 0}(1)+K\left(|X|+\frac{\delta^{|\cdot|}(c\overrightarrow{e_2},\vec{c'})}{c^2}+\frac{\delta^{\perp}(c\overrightarrow{e_2},\vec{c'})}{c}+|\gamma|\right).$$

Still by Lemma 5.1, we have

$$\left|\int_\Omega \mathfrak{Re}(\partial_{x_1}Q\overline{\partial_{x_2}Q})\right| = o_{c\to 0}(1).$$

With the same arguments, we check that

$$\left|\partial_{X_2}\int_\Omega c\partial_{c^\perp}Q\overline{Q\psi^{\neq 0}}\right|$$
$$\leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)+o_{c\to 0}(1)+K\left(|X|+\frac{\delta^{|\cdot|}(c\overrightarrow{e_2},\vec{c'})}{c^2}+\frac{\delta^{\perp}(c\overrightarrow{e_2},\vec{c'})}{c}+|\gamma|\right).$$

Finally, with equations (2.6) to (2.10) and (C.4), we check easily that

$$\partial_{X_2}\left(\mathfrak{Re}\int_{\Omega'} i\psi\right) \leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1) + o_{c\to 0}(1) + K\left(|X| + \frac{\delta^{|\cdot|}(c\overrightarrow{e_2},\vec{c'})}{c^2} + \frac{\delta^{\perp}(c\overrightarrow{e_2},\vec{c'})}{c} + |\gamma|\right).$$

We deduce that

$$\left|\partial_{X_2}G\begin{pmatrix}X_1\\X_2\\\delta_1\\\delta_2\\\gamma\end{pmatrix}+\begin{pmatrix}0\\\int_\Omega|\partial_{x_2}Q|^2\\0\\0\\0\end{pmatrix}\right|\leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)+o_{c\to 0}(1)$$
$$+K\left(|X|+\frac{\delta^{|\cdot|}(c\overrightarrow{e_2},\vec{c'})}{c^2}+\frac{\delta^{\perp}(c\overrightarrow{e_2},\vec{c'})}{c}+|\gamma|\right).$$

We can also check, with similar computations, that

$$\left|\partial_{X_1}G\begin{pmatrix}X_1\\X_2\\\delta_1\\\delta_2\\\gamma\end{pmatrix}+\begin{pmatrix}\int_\Omega|\partial_{x_1}Q|^2\\0\\0\\0\\0\end{pmatrix}\right|\leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)+o_{c\to 0}(1)$$
$$+K\left(|X|+\frac{\delta^{|\cdot|}(c\overrightarrow{e_2},\vec{c'})}{c^2}+\frac{\delta^{\perp}(c\overrightarrow{e_2},\vec{c'})}{c}+|\gamma|\right).$$

We infer that this also holds with a similar proof for the last two directions, namely

$$\left|c^2\partial_{\delta_1}G\begin{pmatrix}X_1\\X_2\\\delta_1\\\delta_2\\\gamma\end{pmatrix}+\begin{pmatrix}0\\0\\\int_\Omega|c^2\partial_cQ|^2\\0\\0\end{pmatrix}\right|\leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)+o_{c\to 0}(1)$$
$$+K\left(|X|+\frac{\delta^{|\cdot|}(c\overrightarrow{e_2},\vec{c'})}{c^2}+\frac{\delta^{\perp}(c\overrightarrow{e_2},\vec{c'})}{c}+|\gamma|\right)$$

(using the fact that $\partial_{\boldsymbol{d}}\boldsymbol{V}$ is differentiable with respect to δ_1, which is not obvious for $c^2\partial_cQ$ and is the reason why we have to use this orthogonality) and

$$\left|c\partial_{\delta_2}G\begin{pmatrix}X_1\\X_2\\\delta_1\\\delta_2\\\gamma\end{pmatrix}+\begin{pmatrix}0\\0\\0\\\int_\Omega|c\partial_{c^\perp}Q|^2\\0\end{pmatrix}\right|\leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)+o_{c\to 0}(1)$$
$$+K\left(|X|+\frac{\delta^{|\cdot|}(c\overrightarrow{e_2},\vec{c'})}{c^2}+\frac{\delta^{\perp}(c\overrightarrow{e_2},\vec{c'})}{c}+|\gamma|\right).$$

We will only show for these directions that, in $B(\boldsymbol{d}_{\vec{c'},1},R)$,

$$\left|Q\int_0^{2\pi}\frac{c^2\partial_cQ}{Q}\,d\boldsymbol{\theta}_1\right|+\left|Q\int_0^{2\pi}\frac{c\partial_{c^\perp}Q}{Q}\,d\boldsymbol{\theta}_1\right|=o_{c\to 0}(1),$$

and the other computations are similar to the ones done for $\partial_{X_2}F$ (using Lemma 5.1).

We recall from Lemma 2.3 that, in $B(\boldsymbol{d}_{\vec{c'},1}, R)$,

$$\|c^2\partial_c Q - \partial_{\boldsymbol{d}}\boldsymbol{V}\|_{C^1(B(\boldsymbol{d}_{\vec{c'},1},R))} = o_{c\to 0}(1),$$

where $\|\partial_{\boldsymbol{d}}\boldsymbol{V} + \partial_{x_1}V_1\|_{C^1(B(\boldsymbol{d}_{\vec{c'},1},R))} = o_{c\to 0}(1)$, V_1 being centred around a point $d_{\vec{c'}} \in \mathbb{R}^2$ such that

$$|d_{\vec{c'}} - \boldsymbol{d}_{\vec{c'},1}| = o_{c\to 0}(1).$$

Therefore, we check that

$$\begin{aligned}\left|Q\int_0^{2\pi} \frac{c^2\partial_c Q}{Q}\,d\boldsymbol{\theta}_1\right| &\leqslant \left|\mathbf{V}_1\int_0^{2\pi}\frac{\partial_{x_1}\mathbf{V}_1}{\mathbf{V}_1}\,d\boldsymbol{\theta}_1\right| + o_{c\to 0}(1)\\ &= o_{c\to 0}(1)\end{aligned}$$

from (C.5). Finally, we have from Lemma 2.7 that $\partial_{c^\perp}Q = -x^{\perp,\delta^\perp(c\overrightarrow{e_2},\vec{c'})}\cdot\nabla Q$, where $x^{\perp,\delta^\perp(c\overrightarrow{e_2},\vec{c'})}$ is $x^\perp$ rotated by an angle $\delta^\perp(c\overrightarrow{e_2},\vec{c'})$. We note that, in $B(\boldsymbol{d}_{\vec{c'},1}, R)$,

$$\left|Q\int_0^{2\pi}\frac{c\boldsymbol{d}_{\vec{c'},1}\cdot\nabla Q}{Q}\,d\boldsymbol{\theta}_1\right| \leqslant \left|\mathbf{V}_1\int_0^{2\pi}\frac{c\boldsymbol{d}_{\vec{c'},1}\cdot\nabla\mathbf{V}_1}{\mathbf{V}_1}\,d\boldsymbol{\theta}_1\right| + o_{c\to 0}(1)$$

and

$$\left|\mathbf{V}_1\int_0^{2\pi}\frac{c\boldsymbol{d}_{\vec{c'},1}\cdot\nabla\mathbf{V}_1}{\mathbf{V}_1}\,d\boldsymbol{\theta}_1\right| = 0$$

by (C.5) and the same result for ∂_{x_1} instead of ∂_{x_2}. Therefore, since $|x^{\perp,\delta^\perp(c\overrightarrow{e_2},\vec{c'})} - \boldsymbol{d}_{\vec{c'},1}| \leqslant K$ in $B(\boldsymbol{d}_{\vec{c'},1}, R)$,

$$\begin{aligned}\left|Q\int_0^{2\pi}\frac{c\partial_{c^\perp}Q}{Q}\,d\boldsymbol{\theta}_1\right| &\leqslant \left|Q\int_0^{2\pi}\frac{c(x^{\perp,\delta^\perp(c\overrightarrow{e_2},\vec{c'})} - \boldsymbol{d}_{\vec{c'},1})\cdot\nabla Q}{Q}\,d\boldsymbol{\theta}_1\right| + o_{c\to 0}(1)\\ &\leqslant Kc + o_{c\to 0}(1)\\ &= o_{c\to 0}(1).\end{aligned}$$

Finally, we infer that

$$\begin{aligned}\left|\partial_\gamma G\begin{pmatrix}X_1\\X_2\\\delta_1\\\delta_2\\\gamma\end{pmatrix} + \begin{pmatrix}0\\0\\0\\0\\\mathfrak{Re}\int_{\Omega'}Q\end{pmatrix}\right| &\leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1) + o_{c\to 0}(1)\\ &\quad + K\left(|X| + \frac{\delta^{|\cdot|}(c\overrightarrow{e_2},\vec{c'})}{c^2} + \frac{\delta^\perp(c\overrightarrow{e_2},\vec{c'})}{c} + |\gamma|\right).\end{aligned}$$

The proof is similar to those of the previous computations, and we will only show that, in Ω,

$$|\partial_\gamma(Q\psi^{\neq 0})| \leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1).$$

We have

$$\begin{aligned}|\partial_\gamma(Q\psi^{\neq 0})| &= |\partial_\gamma(Q\psi) - \partial_\gamma(Q\psi^{\mathbf{0}})| \\ &\leqslant \left| -iQ - \frac{Q}{2\pi}\int_0^{2\pi} \frac{-iQ}{Q}\, d\theta \right| + o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1) \\ &\leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1).\end{aligned}$$

From Theorem 1.1, $\mathfrak{Re}\int_{\Omega'} Q = \mathfrak{Re}\int_{\Omega'} -1 + o_{c\to 0}(1) \leqslant -K < 0$. We conclude, by Lemma 5.1, that, for c and $\|Z-Q_c\|_{H^{\exp}_{Q_c}}$ small enough, dG is invertible in a neighbourhood of $(0,0,0,0,0)$ of size independent of $\|Z-Q_c\|_{H^{\exp}_{Q_c}}$. Therefore, by the implicit function theorem, taking c small enough and $\varepsilon(c,\lambda)$ small enough, we can find $X, \vec{c'} \in \mathbb{R}^2$, $\gamma \in \mathbb{R}$ such that

$$|X| + \frac{\delta^{|\cdot|}(c\overrightarrow{e_2}, \vec{c'})}{c^2} + \frac{\delta^{\perp}(c\overrightarrow{e_2}, \vec{c'})}{c} + |\gamma| \leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1),$$

and satisfying

$$\mathfrak{Re}\int_{B(\boldsymbol{d}_{\vec{c'},1},R)\cup B(\boldsymbol{d}_{\vec{c'},2},R)} \partial_{x_1} Q\overline{Q\psi^{\neq 0}} = \mathfrak{Re}\int_{B(\boldsymbol{d}_{\vec{c'},1},R)\cup B(\boldsymbol{d}_{\vec{c'},2},R)} \partial_{x_2} Q\overline{Q\psi^{\neq 0}} = 0,$$

$$\mathfrak{Re}\int_{B(\boldsymbol{d}_{\vec{c'},1},R)\cup B(\boldsymbol{d}_{\vec{c'},2},R)} \partial_{\boldsymbol{d}} \boldsymbol{V}\overline{Q\psi^{\neq 0}} = \mathfrak{Re}\int_{B(\boldsymbol{d}_{\vec{c'},1},R)\cup B(\boldsymbol{d}_{\vec{c'},2},R)} \partial_{c^\perp} Q\overline{Q\psi^{\neq 0}} = 0,$$

$$\mathfrak{Re}\int_{B((\boldsymbol{d}_{\vec{c'},1}+\boldsymbol{d}_{\vec{c'},2})/2,R)} i\psi = 0. \qquad \square$$

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Received on November 9, 2020.

Accepted on September 6, 2021.