SYMMETRIZATION INEQUALITIES IN THE FRACTIONAL CASE AND BESOV EMBEDDINGS

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Abstract. We prove new extended forms of the Pólya-Szegö symmetrization principle in the fractional case. As a consequence we determine new results for rearrangement invariant hulls of generalized Besov spaces.

1. Introduction

Recently sharp forms of the Sobolev embedding theorem have been obtained using new symmetrization inequalities. In [2] it was shown that the oscillation of the decreasing rearrangement of f, $f_o^*(t) = f^{**}(t) - f^*(t)$ can be estimated by

$$f_o^*(t) \leq c_n t^{1/n} \left| \nabla f \right|^{**}(t),\ f \in C_0^\infty(\mathbb{R}^n) \tag{1.1}$$

where $f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds$, and f^* is the non-increasing rearrangement of f.

The formulation of inequalities in terms of the oscillation $f_o^*(t)$ leads to general forms of the Sobolev embedding theorem that are sharp up to the endpoints and particularly useful in the study of higher order Sobolev inequalities (see [2], [20], [22], [25].)

In [18] we studied the fractional case and we obtained the following estimate: Let $X(\mathbb{R}^n)$ be a rearrangement-invariant Banach function space (r.i. space) and $f \in X(\mathbb{R}^n)$. Then

$$f_o^*(t) \leq c\frac{\omega_X(f, t^{1/n})}{\phi_X(t)}, \tag{1.2}$$

where $\phi_X(t)$ denotes the fundamental function of $X(\mathbb{R}^n)$ and $\omega_X(f,t)$ is the modulus of continuity of $f \in X(\mathbb{R}^n)$ with respect to the $X(\mathbb{R}^n)$–norm (see Section 2 below).

For higher order derivatives the Pólya-Szegö symmetrization principle, which underlies the validity of (1.1) and (1.2), fails. Nevertheless, it was shown in [19] that starting from (1.1) one can develop an iteration argument that leads to a sharp higher order version of (1.2) when one works on $\mathbb{R}^n$, but for domains an estimate like (1.2) is still unknown, even in the case of functions that vanish at the boundary.

The main purpose of this paper is to study Besov-type inequalities involving r.i. spaces. We do this by determining estimates like (1.2) on domains.

The paper is organized as follows. Section 2 contains background material on r.i. spaces, together with the definitions and results required later on.

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In Section 3 we shall obtain rearrangement inequalities involving the moduli of continuity on domains. Our main result of this Section (see Theorem 1 below) states that given Ω an open bounded domain in $\mathbb{R}^n$ with Lipschitz boundary (for the sake of simplicity we fix $|\Omega|=1$, and we write $\Omega\in \mathrm{Lip}_1$), $X(\Omega)$ a r.i. space and $k\in\mathbb{N}$, $k\geq 0$, then for all $f\in W^{k,X}(\Omega)$ and $0<t<1$ we have that

(1.3)
$$f^{**}(t)\leq c\sum_{|\alpha|=k}\int_t^1 \frac{s^{\frac{k}{n}}\left(\omega_{X(\Omega)}(D^\alpha f,s^{\frac{1}{n}})+s^{\frac{1}{n}}\left\|D^\alpha f\right\|_X\right)}{\phi_X(s)}\frac{ds}{s}+\sum_{j=0}^k\sum_{|\alpha|=j}\left\|D^\alpha f\right\|_{X(\Omega)},$$

where the constant $c>0$ is independent of f.

Here $W^{k,X}(\Omega)$, is the Sobolev space defined by

$$W^{k,X}(\Omega)=\left\{f: D^\alpha f\in X(\Omega),\text{ for all }\alpha,\ |\alpha|\leq k\right\},$$

endowed with the norm

$$\|f\|_{W^{k,X}(\Omega)}=\sum_{0\leq|\alpha|\leq k}\left\|D^\alpha f\right\|_{X(\Omega)},$$

($W^{k,X}(\Omega)=X(\Omega)$, if $k=0$).

We will also use the following notation for the the $X(\Omega)-$modulus of continuity of f:

$$\omega_{X(\Omega)}(f,t)=\sup_{0<|h|\leq t}\left\|(f(\cdot+h)-f(\cdot))\chi_{\Omega(h)}\right\|_{X(\Omega)},$$

with $\Omega(h)=\{x\in\Omega: x+\rho h\in\Omega,\ 0\leq\rho\leq 1\}$ and $h\in\mathbb{R}^n$.

In Section 4 the estimate (1.3) will be used in order to obtain sharp embedding results for generalized Besov space $B^\rho_{X(\Omega),q}$ (see Section 2 below), i.e. the function space endowed with the norm

$$\|f\|_{B^\rho_{X,q}}:=\|f\|_X+\left(\int_0^1\left(\frac{\omega_X(f,t)_{k+1}}{\rho(t)}\right)^q\frac{dt}{t}\right)^{1/q},$$

where $\omega_X(f,t)_{k+1}$ is the $(k+1)-$modulus of continuity of $f\in X(\Omega)$, ρ a function in Λ_k (i.e $\rho(t)/t^k$ is equivalent to a quasiconcave function and there are $0<\gamma<\varepsilon<1$ such that $\rho(t)/t^{k+\varepsilon}$ is almost decreasing and $\rho(t)/t^{k+\gamma}$ is almost increasing[1]) and $1\leq q\leq\infty$. For example, if $\rho(t)=t^\sigma$, $t\in(0,1]$, $0<\sigma<\infty$, and $X=L^p$, then $B^\rho_{X,q}$ is the classical Besov spaces $B^\sigma_{p,q}$.

Our main result of this Section is Theorem 2 which states that we can associate with the Besov space $B^\rho_{X(\Omega),q}$ a weight $w=w_{\rho,X,q}$ such that

$$B^\rho_{X(\Omega),q}\subset\Gamma^q(w), \tag{1.4}$$

where $\Gamma^p(w)$ is the Lorentz space defined by those measurable functions f, such that the functional

$$\|f\|_{\Gamma^p(w)}=\left(\int_0^1 f^{**}(s)^p w(s)ds\right)^{1/p}$$

is finite ($1\leq p\leq\infty$, with the usual change if $p=\infty$). Moreover in Theorem 4 we shall see that (1.4) is optimal among the possible target r.i. spaces, in the sense that if $Y=Y(\Omega)$ is any r.i. space, then

$$B^\rho_{X(\Omega),q}\subset Y\Leftrightarrow\Gamma^q(w)\subset Y.$$

[1]A typical example is provided by $\rho(t)=t^\sigma(1+|\log t|)^b$, $k<\sigma<k+1$, $b\in\mathbb{R}$.

We finish this Section considering some "Hardy–type inequalities" (in the sense of [28]) and with two applications of Theorems 2 and 4 in the particular cases of classical Besov spaces and Besov spaces of generalized smoothness.

Finally in Section 5, following the ideas introduced in [22], we consider rearrangement invariant sets expressed in terms of the oscillation of f^*, defined as follows

$$S^q(v) = \left\{ f \in \mathcal{M}(\Omega) : \|f\|_{S^q(v)} = \left(\int_0^1 f_o^*(t)^q v(t) dt \right)^{1/q} + \|f\|_{L^1} < \infty \right\}.$$

We shall see that if $X = X(\Omega)$ is a r.i. space satisfying some mild conditions, then we can associate to the generalized Besov space $B^{\rho}_{X,q}$ a weight $v = v_{\rho,X,q}$ such that $B^{\rho}_{X,q} \subset S^q(v)$. Moreover if $Y = Y(\Omega)$ is a r.i. space such that $B^{\rho}_{X,q} \subset Y$, then $S^p(v) \subset Y$. This result extends and sharpens, in the limiting case, some results given in the previous section.

As usual, the symbol $f \simeq g$ will indicate the existence of a universal constant $c > 0$ (independent of all parameters involved) so that $(1/c)f \leq g \leq c\, f$, while the symbol $f \preceq g$ means that $f \leq c\, g$.

2. Preliminaries

We shall briefly collect some definitions, notations and properties about functions and function spaces involved in our discussion.

Fundamental Indices of functions. (See [3] and [16]). Let ψ be an increasing function on $(0, 1)$ such that $\psi(0^+) = 0$. The **fundamental indices** of ψ are defined by

$$\overline{\beta}_{\psi} = \inf_{t>1} \frac{\ln M_{\psi}(t)}{\ln t} \quad \text{and} \quad \underline{\beta}_{\psi} = \sup_{0<t<1} \frac{\ln M_{\psi}(t)}{\ln t},$$

where

$$M_{\psi}(t) = \sup_{s \in (0, \min(1,1/t))} \frac{\psi(ts)}{\psi(s)}, \ t > 0.$$

It is well known (see [16]) that $0 \leq \underline{\beta}_{\psi} \leq \overline{\beta}_{\psi} \leq \infty$.

Lemma 1. *Let ψ be a quasiconcave function defined on $(0, 1)$, such that $\psi(0^+) = 0$. Then $0 \leq \underline{\beta}_{\psi} \leq \overline{\beta}_{\psi} \leq 1$. Moreover*

(1) If $\overline{\beta}_{\psi} < 1$, then for every $\overline{\beta}_{\psi} < \gamma < 1$ the function $\psi(s)/s^{\gamma}$ is almost decreasing (i.e. $\exists c > 0$ s.t. $\psi(s)/s^{\gamma} \leq c\psi(t)/t^{\gamma}$ whenever $t \leq s$).
(2) If $\underline{\beta}_{\psi} > 0$, then for every $0 < \gamma < \underline{\beta}_{\psi}$ the function $\psi(s)/s^{\gamma}$ is almost increasing (i.e. $\exists c > 0$ s.t. $\psi(s)/s^{\gamma} \leq c\psi(t)/t^{\gamma}$ whenever $t \geq s$).
(3) If $\underline{\beta}_{\psi} > 0$, there exist a concave function $\hat{\psi}$ and constant $c > 0$ such that

$$c^{-1}\psi(t) \leq \hat{\psi}(t) \leq c\psi(t) \quad \text{and} \quad c^{-1}\hat{\psi}(t)/t \leq \frac{\partial}{\partial t}\hat{\psi}(t) \leq c\hat{\psi}(t)/t.$$

Proof. Parts (1) and (2) are a simple exercise. For example to see (1), it follows from the definition of fundamental indices that if $\gamma > \overline{\beta}_{\psi}$, then there is $c > 0$ such that

$$M_{\psi}(t) \leq ct^{\gamma}, \ \text{if } t \geq 1.$$

Thus

$$\frac{\psi(ts)}{\psi(s)} \leq ct^{\gamma}, \text{ for all } 0 < s < 1/t.$$

Considering $u = st$, $v = u/t$, then $v \leq u$ and

$$\frac{\psi(u)}{\psi(v)} \leq c\left(\frac{u}{v}\right)^{\gamma}.$$

For part (3) see [26, Theorem 2.4]. □

We will say that a continuos increasing function $\rho : [0,1] \to [0,\infty)$ belongs to the **class** $\mathbf{\Lambda}_k$ ($k \in \mathbb{N}$) if there is a quasiconcave function Ψ such that

$$\rho(t)/t^k \simeq \Psi(t), \quad \text{with} \quad 0 < \underline{\beta}_{\Psi} \leq \overline{\beta}_{\Psi} < 1,$$

Obviously, if $\rho \in \mathbf{\Lambda}_k$, then $k < \underline{\beta}_{\rho} \leq \overline{\beta}_{\rho} < k+1$.

Rearrangement Invariant Spaces. (See [3] and [16]). A rearrangement invariant space (r.i. space) $X(\mathbb{R}^n)$ is a Banach lattice of Lebesgue measurable functions on $\mathbb{R}^n$ endowed with a norm $\|\cdot\|_X$ that satisfies the Fatou property and is such that, if $f \in X$ and $g^* = f^*$, then $g \in X$ and $\|g\|_X = \|f\|_X$.

Given any measurable subset Ω of $\mathbb{R}^n$, if we let

$$X(\Omega) = \{f\chi_{\Omega} : f \in X(\mathbb{R}^n)\},$$

it is obvious that, by defining

$$\|f\|_{X(\Omega)} := \|f\chi_{\Omega}\|_{X(\mathbb{R}^n)},$$

$X(\Omega)$ is a rearrangement invariant space.

The **fundamental function** of a r.i. space $X(\Omega)$ is defined by

$$\phi_X(s) = \|\chi_A\|_X$$

(where A is any measurable subset of Ω with $|A| = s$.)

The **fundamental indices** $\underline{\beta}_X$ and $\overline{\beta}_X$ of X are defined as the fundamental indices of its fundamental function $\phi_X(s)$.

Finally recall that every r.i. space X has a representation as a function space on $\widehat{X}(0,|\Omega|)$ such that

$$\|f\|_{X(\Omega)} = \|f^*\|_{\widehat{X}(0,|\Omega|)}.$$

When the measure space is clear in the context we will "drop the hat" and use the same letter X to indicate the different versions of the space X that we use.

Hardy's operators and weights. We shall make use of the weighted inequalities collected in the next result:

Proposition 1. *(See* [23]*) Let P and Q be the Hardy operators defined by*

$$Ph(t) = \frac{1}{t}\int_0^t h(s)ds; \qquad Qh(t) = \int_t^1 h(s)\frac{ds}{s}.$$

Let w, v be weights (positive and measurable functions) on $(0,1)$*. Then*

(1)
$$P : L^q(w) \to L^q(v) \text{ is bounded}$$
if and only if

$$\begin{cases} \left(\int_r^1 \frac{w(t)}{t^q} dr\right)^{1/q} \left(\int_0^r v(r)^{-q'+1} dr\right)^{1/q'} \leq c, & \text{if } 1 \leq q < \infty, \\ \frac{w(r)}{r} \int_0^r \frac{dt}{v(t)} \leq c, & \text{if } q = \infty. \end{cases} \tag{2.1}$$

(2)
$$Q : L^q(w) \to L^q(v) \text{ is bounded}$$
if and only if

$$\begin{cases} \left(\int_0^r w(t)dt\right)^{1/q} \left(\int_r^1 \frac{v(t)^{-q'/q}}{t^{q'}} dt\right)^{1/q'} \leq c, & \text{if } 1 \leq q < \infty, \\ w(t) \int_r^1 \frac{1}{v(t)} \frac{dt}{t} \leq c, & \text{if } q = \infty, \end{cases} \tag{2.2}$$

where $\|f\|_{L^q(w)} = \left(\int_0^1 |f(t)|^q w(t)dt\right)^{1/q}$ *and, as usual,* $1/q + 1/q' = 1$.

The next two lemmas will be useful in the following sections.

Lemma 2. *Let $\rho \in \Lambda_k$ and $1 \leq q \leq \infty$. Define $w(t) = t^{(k+1)q-1}/\rho(t)^q$. Then $P : L^q(w) \to L^q(w)$ and $Q : L^q(w) \to L^q(w)$ are bounded.*

Proof. We need to check that conditions (2.1) and (2.2) hold. By Lemma 1 we have that

$$\int_t^1 \frac{w(s)}{s^q} ds = \int_t^1 \left(\frac{s^k}{\rho(s)}\right)^q \frac{ds}{s} \simeq \int_t^1 \left(\frac{\rho(s)}{s^k}\right)^{1-q} \frac{\partial}{\partial s}\left(\frac{\rho(s)}{s^k}\right) ds \leq \left(\frac{t^k}{\rho(t)}\right)^q$$

and similarly

$$\int_0^t w(s)^{-q'+1} ds = \int_0^t \left(\frac{\rho(s)}{s^k}\right)^{q'} \frac{ds}{s} \simeq \int_0^t \left(\frac{\rho(s)}{s^k}\right)^{q'-1} \frac{\partial}{\partial s}\left(\frac{\rho(s)}{s^k}\right) ds \preceq \left(\frac{\rho(t)}{t^k}\right)^{q'}.$$

To see condition (2.2) chose $0 < \gamma < 1$ such that $\rho(t)/t^{k+\gamma}$ is almost decreasing. Then

$$\int_0^t w(x)dx = \int_0^t \left(\frac{x^{k+\gamma}}{\rho(x)}\right)^q \frac{dx}{x^{1-q+\gamma q}} \preceq \left(\frac{t^{k+\gamma}}{\rho(t)}\right)^q t^{q(1-\gamma)} = \left(\frac{t^{k+1}}{\rho(t)}\right)^q$$

and

$$\int_t^1 \frac{w(x)^{-q'+1}}{x^{q'}} dx = \int_t^1 \left(\frac{\rho(x)}{x^{k+\gamma}}\right)^{q'} \frac{dx}{x^{q'-\gamma q'+1}} \preceq \left(\frac{\rho(t)}{t^{k+1}}\right)^{q'}.$$

□

Lemma 3. *Let $\rho \in \Lambda_k$ and X be a r.i. space. Consider the function*

$$v(t) = \begin{cases} \left(\frac{\phi_X(t)}{\rho(t^{1/n})}\right)^q \frac{1}{t}, & \text{if } 1 \leq q < \infty, \\ \frac{\phi_X(t)}{\rho(t^{1/n})}, & \text{if } q = \infty. \end{cases}$$

If $\rho \in \Lambda_k$ *(*$k \geq 1$*) and*

$$\underline{\beta}_X = \overline{\beta}_\rho / n,$$

then

$$Q : L^q(t^{q/n} v(t)) \to L^q(t^{q/n} v(t)) \quad (1 \leq q \leq \infty)$$

is bounded. Moreover for $1 \leq q < \infty$ *we have that*

$$\int_0^1 t^{q/n} v(t) dt < \infty \ .$$

Proof. To establish the boundedness of Q on $L^q(t^{q/n}v(t))$ we need to check that conditions (2.2) are fulfilled. To this end we first prove that if $\psi \in \Lambda_k, 1 \leq q < \infty$ and

$$\underline{\beta}_X > \overline{\beta}_\psi / n, \tag{2.3}$$

then

$$\int_0^r \left(\frac{\phi_X(s)}{\psi(s^{1/n})} \right)^q \frac{ds}{s} \simeq \left(\frac{\phi_X(r)}{\psi(r^{1/n})} \right)^q, \tag{2.4}$$

and

$$1 + \int_r^1 \left(\frac{\psi(s^{1/n})}{\phi_X(s)} \right)^q \frac{ds}{s} \simeq \left(\frac{\psi(r^{1/n})}{\phi_X(r)} \right)^q. \tag{2.5}$$

Effectively, since $0 < \bar{\beta}_\psi / n < \underline{\beta}_X$, Proposition 1 ensures that $\frac{\partial}{\partial s} \phi_X(s) \simeq \phi_X(s)/s$. Thus

$$\begin{aligned}
\int_0^r \left(\frac{\phi_X(s)}{\psi(s^{1/n})} \right)^q \frac{ds}{s} &\geq \left(\frac{1}{\psi(r^{1/n})} \right)^q \int_0^r \phi_X(s)^q \frac{ds}{s} \\
&\simeq \left(\frac{1}{\psi(r^{1/n})} \right)^q \int_0^r \phi_X(s)^{q-1} \frac{\partial}{\partial s} \phi_X(s) ds \\
&\simeq \left(\frac{\phi_X(r)}{\psi(r^{1/n})} \right)^q .
\end{aligned}$$

Conversely, using that $\psi \in \Lambda_k$ and (2.3), Lemma 1 allows us to choose

$$\underline{\beta}_X > \beta > \frac{\gamma + k}{n} > \frac{\bar{\beta}_\psi}{n} > 0$$

such that $\phi_X(s)/s^\beta$ is almost increasing and $\psi(s)/s^{k+\gamma}$ is almost decreasing. Therefore

$$\begin{aligned}
\int_0^r \left(\frac{\phi_X(s)}{\psi(s^{1/n})} \right)^q \frac{ds}{s} &= \int_0^r \left(\frac{\phi_X(s)}{s^\beta} \right)^q \left(\frac{s^{\frac{\gamma+k}{n}}}{\psi(s^{1/n})} \right)^q s^{\left(\beta - \frac{\gamma+k}{n}\right) q} \frac{ds}{s} \\
&\preceq \left(\frac{\phi_X(r)}{r^\beta} \right)^q \left(\frac{r^{\frac{\gamma+k}{n}}}{\psi(r^{1/n})} \right)^q \int_0^r s^{\left(\beta - \frac{\gamma+k}{n}\right) q} \frac{ds}{s} \\
&\preceq \left(\frac{\phi_X(r)}{\psi(r^{1/n})} \right)^q .
\end{aligned}$$

Finally, to see (2.5) set $\phi(r) = \int_0^r \left(\frac{\phi_X(s)}{\psi(s^{1/n})} \right)^q \frac{ds}{s}$, then by (2.4), we have that

$$\int_r^1 \left(\frac{\psi(s^{1/n})}{\phi_X(s)} \right)^q \frac{ds}{s} \simeq \int_r^1 \frac{1}{\phi(r)} \frac{ds}{s} \simeq \int_r^1 \frac{\phi(r)'}{\phi(r)^2} ds = \frac{1}{\phi(r)} - \frac{1}{\phi(1)}.$$

Now, to prove Lemma 3, consider $\psi(t)=\rho(t)/t$. Since $\rho\in\Lambda_k$ $(k\geq 1)$ and

$$\underline{\beta}_X=\frac{\overline{\beta}_\rho}{n}>\frac{\overline{\beta}_\rho-1}{n}=\frac{\overline{\beta}_\psi}{n},$$

(2.4) and (2.5) holds for $\psi(t)=\rho(t)/t$, and now, an easy computation shows that conditions (2.2) are fulfilled. Moreover

$$\int_0^1 t^{q/n}v(t)dt=\int_0^r\left(\frac{t^{1/n}\phi_X(s)}{\rho(s^{1/n})}\right)^q\frac{ds}{s}\simeq\left(\frac{\phi_X(1)}{\rho(1)}\right)^q<\infty.$$

□

Generalized Besov spaces. (See [3], [14], [24] and the references quoted therein). Let $\Omega\subset\mathbb{R}^n$ be an open domain, and $X=X(\Omega)$ be a r.i. space. We set

$$\Omega(h)=\{x\in\Omega: x+\rho h\in\Omega,\ 0\leq\rho\leq 1\},\quad h\in\mathbb{R}^n.$$

Given $r=1,2,\ldots$, the $r-$**modulus of continuity** $\omega_{X(\Omega)}(f,t)_r$ of a function $f\in X(\Omega)$ is defined by

$$\omega_{X(\Omega)}(f,t)_r=\sup_{0<|h|\leq t}\left\|\Delta_h^r f\chi_{\Omega(rh)}\right\|_X,$$

where

$$\Delta_h^1 f(x)=f(x+h)-f(x)\quad\text{and}\quad\Delta_h^{r+1}f(x)=\Delta_h^1(\Delta_h^r)f(x),\quad r=1,2,3,\cdots$$

If $r=1$ we write $\omega_{X(\Omega)}(f,t)$ instead of $\omega_{X(\Omega)}(f,t)_1$, and in what follows we shall write X instead of $X(\Omega)$, whenever it is clear which subset we are working with.

Let $\rho\in\Lambda_k$, let $X=X(\Omega)$ be a r.i. space, and let $1\leq q\leq\infty$. The **generalized Besov space** $B^\rho_{X,q}$ is the function space endowed with the norm

$$\|f\|_{B^\rho_{X,q}}:=\|f\|_X+\left(\int_0^1\left(\frac{\omega_X(f,t)_{k+1}}{\rho(t)}\right)^q\frac{dt}{t}\right)^{1/q}=\|f\|_X+|f|_{B^\rho_{X,q}},\tag{2.6}$$

(with the usual change if $q=\infty$).

Example 1. *If $\rho(t)=t^\sigma$, $t\in(0,1]$, $0<\sigma<\infty$, and $X=L^p$, then $B^\rho_{X,q}$ is the classical Besov space $B^\sigma_{p,q}$.*

If $\rho(t)=t^\sigma\Psi(t)$, $t\in(0,1]$, $0<\sigma<\infty$, Ψ is a slowly varying function (see Section 4 below) and $X=L^p$, then

$$B^\rho_{X,q}=B^{(\sigma,\Psi)}_{p,q}$$

where the space $B^{(\sigma,\Psi)}_{p,q}$ is the Besov space of generalized smoothness (see [15], [21], [8], [17], [10] *and the references quoted therein.)*

Proposition 2. *Let $\rho\in\Lambda_k$, $X=X(\Omega)$ be a r.i. space and $1\leq q\leq\infty$. If r varies over all positive integers with $r\geq k+1$, then*

$$\|f\|_{B^\rho_{X,q}}\simeq\|f\|_X+\left(\int_0^1\left(\frac{\omega_X(f,t)_r}{\rho(t)}\right)^q\frac{dt}{t}\right)^{1/q}.\tag{2.7}$$

Moreover, considering the norm defined by

$$\|f\|^*_{B^\rho_{X,q}}:=\sum_{0\leq|\alpha|\leq k}\|D^\alpha f\|_X+\sum_{|\alpha|=k}\left(\int_0^1\left(\frac{t^k\omega_X(D^\alpha f,t)}{\rho(t)}\right)^q\frac{dt}{t}\right)^{1/q}\tag{2.8}$$

we get that $\|\cdot\|^*_{B^\rho_{X,q}}$ *and* $\|\cdot\|_{B^\rho_{X,q}}$ *are equivalent. In particular*

$$B^\rho_{X,q} \subset W^{k,X}(\Omega).$$

Proof. Assume that $\rho \in \Lambda_k$ for some $k \geq 1$ (in the case $k = 0$ (2.6) and (2.8) coincide). Obviously condition $\|f\|^*_{B^\rho_{X,q}} < \infty$ implies $f \in W^{k,X}(\Omega)$. Thus (see [14, formula (2.4)])

$$\omega_X(f,t)_{k+1} \leq ct^k \sum_{|\alpha|=k} \omega_X(D^\alpha f, t),$$

and therefore

$$\|f\|_{B^\rho_{X,q}} \leq c \|f\|^*_{B^\rho_{X,q}}.$$

For the converse, first of all notice that for all $1 \leq j \leq k$, Lemma 1 implies that

$$\int_0^1 \left(\frac{\rho(t)}{t^j}\right)^{q'} \frac{dt}{t} \leq \int_0^1 \left(\frac{\rho(t)}{t^k}\right)^{q'} \frac{dt}{t} \simeq \int_0^1 \left(\frac{\rho(t)}{t^k}\right)^{q'-1} \frac{\partial}{\partial t}\left(\frac{\rho(t)}{t^k}\right) dt < \infty.$$

Suppose $\|f\|_{B^\rho_{X,q}} < \infty$, then for all $1 \leq j \leq k$, we have that

$$\int_0^1 \frac{\omega_X(f,t)_{k+1}}{t^j}\frac{dt}{t} \leq \left(\int_0^1 \left(\frac{\omega_X(f,t)_{k+1}}{\rho(t)}\right)^q \frac{dt}{t}\right)^{1/q} \left(\int_0^1 \left(\frac{\rho(t)}{t^j}\right)^{q'} \frac{dt}{t}\right)^{1/q'} \leq c\|f\|_{B^\rho_{X,q}} < \infty, \tag{2.9}$$

which implies that (see [14, formula (2.7)])

$$\omega_X(D^\alpha f, t) \preceq \int_0^t \frac{\omega_X(f,s)_{k+1}}{s^k}\frac{ds}{s} = P(\omega_X(f,s)_{k+1}/s^{k+1})(t) \qquad |\alpha| = k \tag{2.10}$$

and (see [14, formula (2.6)])

$$\|D^\alpha f\|_X \preceq \|f\|_X + \int_0^1 \frac{\omega_X(f,s)_{k+1}}{s^j}\frac{ds}{s} \quad 1 \leq |\alpha| \leq k. \tag{2.11}$$

Thus

$$\begin{aligned}
\|f\|^*_{B^\rho_{X,q}} &\preceq \|f\|_X + \int_0^1 \frac{\omega_X(f,s)_{k+1}}{s^j}\frac{ds}{s} + \sum_{|\alpha|=k}\left(\int_0^1 \left(\frac{t^k\omega_X(D^\alpha f,t)}{\rho(t)}\right)^q \frac{dt}{t}\right)^{1/q} && \text{(by (2.11))} \\
&\preceq \|f\|_{B^\rho_{X,q}} + \sum_{|\alpha|=k}\left(\int_0^1 \left(\frac{t^k\omega_X(D^\alpha f,t)}{\rho(t)}\right)^q \frac{dt}{t}\right)^{1/q} && \text{(by (2.9))} \\
&\preceq \|f\|_{B^\rho_{X,q}} + \sum_{|\alpha|=k}\left(\int_0^1 \left(\frac{t^{k+1}P(\omega_X(f,s)_{k+1}/s^{k+1})(t)}{\rho(t)}\right)^q \frac{dt}{t}\right)^{1/q} && \text{(by (2.10))} \\
&\preceq \|f\|_{B^\rho_{X,q}} + \sum_{|\alpha|=k}\left(\int_0^1 \left(\frac{\omega_X(f,t)_{k+1}}{\rho(t)}\right)^q \frac{dt}{t}\right)^{1/q} && \text{(by Lemma 2)} \\
&\preceq \|f\|_{B^\rho_{X,q}}.
\end{aligned}$$

Claim (2.7) follows easily from Marchaud-type inequality (see [14, formula (2.5)])

$$\omega_X(f,t)_j \preceq t^j\|f\|_X + t^j\int_t^1 \frac{\omega_X(f,t)_r}{t^j}\frac{dt}{t} \qquad (1 \leq j \leq r-1)$$

and Lemma 2. $\square$

Inequalities for higher order derivatives. We shall need some estimates from [19] which we state here for the benefit of the reader. Given a vector $u \in \mathbb{R}^N$ we denote by $|u|$ its $\ell^2(\mathbb{R}^N)$-norm.

If g is a locally integrable function having weak derivatives of order $r \in \mathbb{N}$, we denote by $d^r g$ the vector $\left(D^\alpha g\right)_{|\alpha|=r}$ of all derivatives of order $|\alpha| = r$.

Lemma 4. *Let $\Omega \in \mathrm{Lip}_1$. Suppose $f \in W^{k,X}(\Omega)$, $k \geq 1$.*

(1) *If $k = 1$, then*

$$f_o^*(t) \leq ct^{1/n} \left|\nabla f\right|^{**}(t), \quad (0 < t < 1/2).$$

(2) *If $k \geq 2$, then*

$$f_o^*(t) \leq ct^{1/n} \left(\int_t^{1/2} s^{\frac{k-1}{n}} \left(d^k f\right)^{**}(s) \frac{ds}{s} + \sum_{j=1}^{k-1} \left\| \left|d^j f\right| \right\|_{L^1} \right), \quad (0 < t < 1/2),$$

where the constant $c := c(n,k) > 0$ is independent of f.

3. Symmetrization type inequalities in the fractional case

In this section we shall obtain rearrangement inequalities for moduli of continuity. We start with an extension of (1.2) for domains. To this end, let us see that if $\Omega \in \mathrm{Lip}_1$ and $X(\Omega)$ is a r.i. space, then there exists an operator E such that

$$E : X(\Omega) \to X(\mathbb{R}^n) \text{ and } E : W^{1,X}(\Omega) \to W^{1,X}(\mathbb{R}^n)$$

is linear, bounded and

$$Ef(x) = f(x), \qquad x \in \Omega,$$

(in what follows the operator E will be called an **extension operator**.)

To prove this claim, recall that since Ω has Lipschitz boundary (see [27] and [4] for more information about extension methods) there exists an extension operator E such that

$$Ef(x) = f(x), \qquad x \in \Omega.$$

Moreover

$$E : L^1(\Omega) \to L^1(\mathbb{R}^n), \qquad E : W^{1,1}(\Omega) \to W^{1,1}(\mathbb{R}^n)$$

and

$$E : L^\infty(\Omega) \to L^\infty(\mathbb{R}^n), \qquad E : W^{1,\infty}(\Omega) \to W^{1,\infty}(\mathbb{R}^n)$$

is linear and bounded.

So, by interpolation we obtain the following inequalities in terms of $K-$functionals[2]

$$K(t, Ef, L^1(\mathbb{R}^n), L^\infty(\mathbb{R}^n)) \leq cK(t, f, L^1(\Omega), L^\infty(\Omega)) \tag{3.1}$$

and

$$K(t, Ef, W^{1,1}(\mathbb{R}^n), W^{1,\infty}(\mathbb{R}^n)) \leq cK(t, f, W^{1,1}(\Omega), W^{1,\infty}(\Omega)). \tag{3.2}$$

[2]Recall that given a compatible pair of Banach spaces (X_0, X_1) the $K-$functional $K(f, t, X_0, X_1)$ is defined for $f \in X_0 + X_1$ and $t > 0$ by

$$K(f, t, X_0, X_1) = \inf_{f = f_0 + f_1} \left\{ \left\|f_0\right\|_{X_0} + t \left\|f_1\right\|_{X_1} \right\}.$$

We refer the reader to [3] and [16] for further information about interpolation theory.

Inequality (3.1) is equivalent to (see [3, Chapter 5, Theorem 1.6])

$$\int_0^t (Ef)^*(s)ds \leq c\int_0^t f^*(s)ds = \int_0^t ((Ef)\,\chi_\Omega)^*(s)ds \tag{3.3}$$

and inequality (3.2) is equivalent to (see [3, Chapter 5, Theorem 5.11] for the left hand side, and [9, Theorem 2] for the right hand side)

$$\begin{aligned}\int_0^t \left((Ef)^*(s) + (\nabla(Ef))^*(s)\right) ds &\leq c\int_0^t \left(f^*(s) + (\nabla f)^*(s)\right) ds \\ &= c\int_0^t \left(((Ef)\chi_\Omega)^*(s) + (\nabla((Ef)\,\chi_\Omega))^*(s)\right) ds.\end{aligned} \tag{3.4}$$

It follows from (3.3) (see [3, Chapther 2, Corollary 4.7]) that for any rearrangement invariant space $X(\mathbb{R}^n)$

$$\|Ef\|_{X(\mathbb{R}^n)} = \|E(f\chi_\Omega)\|_{X(\mathbb{R}^n)} \preceq \|f\|_{X(\Omega)}.$$

Similarly, from (3.4) we obtain

$$\|Ef\|_{W^{1,X}(\mathbb{R}^n)} = \|E(f\chi_\Omega)\|_{W^{1,X}(\mathbb{R}^n)} \preceq \|f\|_{W^{1,X}(\Omega)}.$$

Let us also recall that since Ω has Lipschitz boundary (see [14, Theorem 1] and [3, Chapter 5, exercise 13, pag. 430]) we get that

$$\begin{aligned}K(t,g;X(\Omega), W^{1,X}(\Omega)) &= \inf_{g\in W^{1,X}(\Omega)}\left(\|f-g\|_{X(\Omega)} + t\,\|g\|_{W^{1,X}(\Omega)}\right) \\ &\simeq \omega_X(g,t) + t\,\|g\|_X\,,\ 0<t<1.\end{aligned} \tag{3.5}$$

In the next set of results we shall obtain pointwise estimates that will play a central role in what follows.

Lemma 5. *Let $\Omega \in \mathrm{Lip}_1$ and $f \in X(\Omega)$. Then*

$$(Ef)^{**}(t) - (Ef)^*(t) \leq c\frac{\omega_X(f,t^{1/n}) + t^{1/n}\,\|f\|_X}{\phi_X(t)}, \qquad 0<t<1, \tag{3.6}$$

where E is any extension operator.

Proof. Since the extension operator E is linear and bounded we have

$$\begin{aligned}K(t,Ef,X(\mathbb{R}^n), W^{1,X}(\mathbb{R}^n)) &= \inf_{g\in W^{1,X}(\mathbb{R}^n)}\left(\|Ef-g\|_{X(\mathbb{R}^n)} + t\,\|g\|_{W^{1,X}(\mathbb{R}^n)}\right) \\ &\leq \inf_{g\in W^{1,X}(\Omega)}\left(\|Ef-Eg\|_{X(\mathbb{R}^n)} + t\,\|Eg\|_{W^{1,X}(\mathbb{R}^n)}\right) \\ &\leq c\inf_{g\in W^{1,X}(\Omega)}\left(\|f-g\|_{X(\Omega)} + t\,\|g\|_{W^{1,X}(\Omega)}\right) \\ &= cK(t,f;X(\Omega), W^{1,X}(\Omega)).\end{aligned}$$

Obviously,

$$\inf_{g\in W^{1,X}(\mathbb{R}^n)}\left(\|Ef-g\|_{X(\mathbb{R}^n)} + t\,\|\nabla g\|_{X(\mathbb{R}^n)}\right) \leq K(t,Ef,X(\mathbb{R}^n), W^{1,X}(\mathbb{R}^n)).$$

But (see [3, page 341])

$$\omega_{X(\mathbb{R}^n)}(Ef,t) \simeq \inf_{g\in W^{1,X}(\mathbb{R}^n)} \left(\|Ef-g\|_{X(\mathbb{R}^n)} + t\,\|\nabla g\|_{X(\mathbb{R}^n)} \right)$$

and (see [18, Theorem 2])

$$(Ef)^{**}(t) - (Ef)^{*}(t) \leq c\frac{\omega_{X(\mathbb{R}^n)}(Ef,t^{1/n})}{\phi_X(t)}.$$

Thus we conclude that

$$\begin{aligned}(Ef)^{**}(t) - (Ef)^{*}(t) &\leq c\frac{\omega_{X(\mathbb{R}^n)}(Ef,t^{1/n})}{\phi_X(t)} \preceq \frac{K(t^{1/n}, Ef, X(\mathbb{R}^n), W^{1,X}(\mathbb{R}^n))}{\phi_X(t)} \\ &\preceq \frac{K(t^{1/n}, f; X(\Omega), W^{1,X}(\Omega))}{\phi_X(t)} \\ &\preceq \frac{\omega_{X(\Omega)}(f,t^{1/n}) + t^{1/n}\,\|f\|_{X(\Omega)}}{\phi_X(t)} \quad \text{(by (3.5))}.\end{aligned}$$

□

Corollary 1. *Let $\Omega \in \mathrm{Lip}_1$. Let $X = X(\Omega)$ be a r.i. space such that Hardy's operators P and Q are bounded on X. Then for all $f \in C_0^\infty(\Omega)$*

$$f^{**}(t) - f^{*}(t) \leq c\frac{\omega_X(f,t^{1/n}) + t^{1/n}\,\|f\|_X}{\phi_X(t)}, \quad 0<t<1.$$

Proof. Let E be the Calderón extension operator (see [1, Theorem 5.28] and [12]). Since $f \in C_0^\infty(\Omega)$

$$Ef(x) = \tilde{f}(x)$$

where $\tilde{f}$ is the zero extension of f. Therefore

$$(Ef)^{*}(t) = (\tilde{f})^{*}(t) = f^{*}(t)$$

and Lemma 5 applies. □

The following result is the counterpart of Lemma 5, for $k \geq 1$.

Lemma 6. *Let $\Omega \in \mathrm{Lip}_1$. Assume that $k \geq 1$ and $f \in W^{k,X}(\Omega)$. Then*

$$f_o^*(t) \leq ct^{1/n}\left(\sum_{|\alpha|=k} \int_t^1 \frac{s^{\frac{k-1}{n}}(\omega_X(D^\alpha f, s^{\frac{1}{n}}) + s^{\frac{1}{n}}\,\|D^\alpha f\|_X)}{\phi_X(s)}\frac{ds}{s} + \sum_{j=0}^{k} \left\| \left| d^j f \right| \right\|_X \right),$$

where the constant $c := c(n,k) > 0$ is independent of f.

Proof. We first assume that $k = 1$. By Lemma 4 we know that

$$f_o^*(t) \leq c_n t^{1/n} \left|\nabla f\right|^{**}(t), \quad 0<t<1/2.$$

For $1/2 \leq t < 1$, we get that

$$f_o^*(t) \leq f^{**}(t) \leq 2f^{**}(1) = 2\,\|f\|_{L^1}. \tag{3.7}$$

Combining both inequalities we obtain

$$f_o^*(t) \preceq t^{1/n} \left|\nabla f\right|^{**}(t) + \|f\|_{L^1}, \quad 0<t<1. \tag{3.8}$$

Let E be an extension operator, and let

$$h = \left|\nabla f\right|.$$

From $Eh(x) = h(x)$ for a.e. $x \in \Omega$, it follows that $h^{**} \leq (Eh)^{**}$. Thus, by (3.6) we get that

$$|\nabla f|^{**}(s) \leq (Eh)^{**}(s) = \int_s^1 ((Eh)^{**}(x) - (Eh)^{*}(x)) \frac{ds}{s} + (Eh)^{**}(1) \tag{3.9}$$
$$\leq \int_s^1 \left(\frac{\omega_X(h, t^{1/n}) + t^{1/n} \|h\|_X}{\phi_X(t)} \right) \frac{dt}{t} + \|Eh\|_{L^1(\mathbb{R}^n)}.$$

Obviously,

$$\|Eh\|_{L^1(\mathbb{R}^n)} \leq c \|h\|_{L^1} = \||\nabla f|\|_{L^1}. \tag{3.10}$$

Combining (3.8), (3.9) and (3.10) we conclude that

$$f_o^*(t) \leq ct^{1/n} \left(\sum_{|\alpha|=1} \int_t^1 \frac{(\omega_X(D^\alpha f, s^{\frac{1}{n}}) + s^{\frac{1}{n}} \|D^\alpha f\|_X)}{\phi_X(s)} \frac{ds}{s} + \sum_{j=0}^{1} \left\| |d^j f| \right\|_X \right).$$

In the case $k > 1$, Lemma 4 shows that

$$f_o^*(t) \leq ct^{1/n} \left(\int_t^{1/2} s^{\frac{k-1}{n}} \left| d^k f \right|^{**}(s) \frac{ds}{s} + \sum_{j=1}^{k-1} \left\| |d^j f| \right\|_{L^1} \right), \quad 0 < t < 1/2,$$

whence, using again (3.7), we obtain

$$f_o^*(t) \preceq t^{1/n} \left(\int_t^{1} s^{\frac{k-1}{n}} \left| d^k f \right|^{**}(s) \frac{ds}{s} + \sum_{j=0}^{k-1} \left\| |d^k f| \right\|_{L^1} \right), \quad 0 < t < 1.$$

Now, as in the previous case, let E be an extension operator, and let $h = \left| d^k f \right|$. Since $Eh(x) = h(x)$ a.e. $x \in \Omega$, we have $h^{**} \leq (Eh)^{**}$ and again by (3.6)

$$\left| d^k f \right|^{**}(s) \leq (Eh)^{**}(s) = \int_s^1 ((Eh)^{**}(x) - (Eh)^{*}(x)) \frac{ds}{s} + (Eh)^{**}(1)$$
$$\leq \int_s^1 \left(\frac{\omega_X(h, t^{1/n}) + t^{1/n} \|h\|_X}{\phi_X(t)} \right) \frac{dt}{t} + \|Eh\|_{L^1(\mathbb{R}^n)},$$

with $\|Eh\|_{L^1(\mathbb{R}^n)} \leq c \|h\|_{L^1} = \left\| \left| d^k f \right| \right\|_{L^1}$. Finally by Fubini's theorem

$$f_o^*(t) \preceq t^{1/n} \int_t^1 \left(s^{\frac{k-1}{n}} \int_s^1 \left(\frac{\omega_X(h, z^{1/n}) + z^{1/n} \|h\|_X}{\phi_X(z)} \right) \frac{dz}{z} \right) \frac{ds}{s} + t^{1/n} \sum_{j=0}^{k} \left\| |d^j f| \right\|_{L^1}$$
$$\preceq t^{1/n} \left(\int_t^1 \frac{s^{\frac{k-1}{n}} \left(\omega_X(h, s^{1/n}) + s^{1/n} \|h\|_X \right)}{\phi_X(s)} \frac{ds}{s} + \sum_{j=0}^{k} \left\| |d^j f| \right\|_{L^1} \right).$$

□

Theorem 1. *Let $\Omega \in \mathrm{Lip}_1$. Let $k \geq 0$, and $f \in W^{k,X}(\Omega)$. Then*

$$f^{**}(t) \preceq \int_t^1 \frac{s^{\frac{k}{n}} \left(\sum_{|\alpha|=k} \left(\omega_X(D^\alpha f, s^{\frac{1}{n}}) + s^{\frac{1}{n}} \|D^\alpha f\|_X \right) \right)}{\phi_X(s)} \frac{ds}{s} + \sum_{j=0}^{k} \left\| |d^j f| \right\|_X.$$

Proof. Case $k = 0$. Given $f \in X(\Omega)$, by Lemma 5 we get

$$(Ef)^{**}(t) - (Ef)^{*}(t) \preceq \frac{\omega_X(f, t^{1/n}) + t^{1/n}\|f\|_X}{\phi_X(t)}.$$

Thus

$$(Ef)^{**}(t) \preceq \int_t^1 \left(\frac{\omega_X(f, s^{1/n}) + s^{1/n}\|f\|_X}{\phi_X(s)} \right) \frac{ds}{s} + (Ef)^{**}(1).$$

Since $Ef(x) = f(x)$ a.e. $x \in \Omega$, we have that $f^{**} \le (Ef)^{**}$, and obviously

$$(Ef)^{**}(1) \le \|Ef\|_{L^1(\mathbb{R}^n)} \preceq \|f\|_{L^1} \preceq \|f\|_X.$$

Thus

$$f^{**}(t) \preceq \int_t^1 \left(\frac{\omega_X(f, s^{1/n}) + s^{1/n}\|f\|_X}{\phi_X(s)} \right) \frac{ds}{s} + \|f\|_X.$$

Case $k \ge 1$. Given $f \in W^{k,X}(\Omega)$ the result follows easily using again that

$$f^{**}(t) = \int_t^1 f_o^*(s) \frac{ds}{s} + f^{**}(1),$$

Lemma 6, and Fubini's theorem. □

4. Embedding Theorems of generalized Besov spaces into r.i. spaces

The principal goal of this section is to prove embedding theorems of generalized Besov spaces into r.i. spaces. Throughout what follows we shall assume that $\Omega \in \mathrm{Lip}_1$, $X = X(\Omega)$ is a r.i. space, $\rho \in \Lambda_k$ $(k \in \mathbb{N})$ and $1 \le q \le \infty$.

Definition 1. *Associated with the generalized Besov space $B^{\rho}_{X,q}$ we consider the function*

$$m_{B^{\rho}_{X,q}}(r) = m(r) := \begin{cases} \displaystyle\int_r^1 \left(\frac{\rho(s^{1/n})}{\phi_X(s)} \right)^{q'} \frac{ds}{s}, & \text{if } \ 1 < q \le \infty, \\ \displaystyle\sup_{s \in [r,1)} \frac{\rho(s^{1/n})}{\phi_X(s)}, & \text{if } \ q = 1. \end{cases}$$

m will be called the ***associated function*** *of $B^{\rho}_{X,q}$.*

Associated with m we considerer the function

$$w_{B^{\rho}_{X,q}}(r) = w(r) = \begin{cases} \dfrac{1}{(1+m(t))\,t}, & \text{if } \ q = 1, \\ \dfrac{|m'(t)|}{(1+m(t))^{q}}, & \text{if } \ 1 < q \le \infty, \\ \dfrac{1}{1+m(t)}, & \text{if } \ q = \infty. \end{cases}$$

w will be called the ***associated weight*** *of $B^{\rho}_{X,q}$.*

Theorem 2. *Let $1 < q \le \infty$. Then,*

$$B^{\rho}_{X,q} \subset \Gamma^{q}(w),$$

where w is the associated weight of $B^{\rho}_{X,q}$.

Proof. Let $\rho \in \Lambda_k$. Given $f \in B^{\rho}_{X,q} \subset W^{k,X}(\Omega)$ consider

$$H(s^{\frac{1}{n}}) = s^{\frac{k}{n}} \sum_{|\alpha|=k} \left(\omega_X(D^{\alpha} f, s^{\frac{1}{n}}) + s^{\frac{1}{n}} \left\| D^{\alpha} f \right\|_X \right).$$

By Theorem 1, we get

$$f^{**}(t) \preceq \int_t^1 \frac{H(s^{\frac{1}{n}})}{\phi_X(s)} \frac{ds}{s} + \sum_{j=0}^{k} \left\| \left| d^j f \right| \right\|_X . \tag{4.1}$$

In the case that $1 < q < \infty$, let

$$v(t) := \left(\frac{\phi_X(t)}{\rho(t^{1/n})} \right)^q \frac{1}{t}.$$

An easy computation shows that

$$\left(\int_0^r w(t)\, dt \right)^{1/q} \left(\int_r^1 \frac{v(t)^{-q'/q}}{t^{q'}}\, dt \right)^{1/q'} \preceq \left(\frac{m(r)}{1+m(r)} \right)^{1/q'} \leq 1,$$

i.e. condition (2.2) holds. Combining (4.1) and Propositions 1 and 2 we conclude that

$$\begin{aligned} \|f\|_{\Gamma^q(w)} &\preceq \left(\int_0^1 \left(\int_t^1 \frac{H(s^{\frac{1}{n}})}{\phi_X(s)} \frac{ds}{s} \right)^q w(t)\, dt \right)^{1/q} + \left(\int_0^1 w(t)\, dt \right)^{1/q} \sum_{j=0}^{k} \left\| \left| d^j f \right| \right\|_X \\ &\preceq \left(\int_0^1 \left(\frac{H(s^{\frac{1}{n}})}{\phi_X(s)} \right)^q v(s) ds \right)^{1/q} + \sum_{j=0}^{k} \left\| \left| d^j f \right| \right\|_X \preceq c \|f\|_{B^{\rho}_{X,q}} . \end{aligned}$$

When $q = \infty$, let

$$v(t) := \frac{\phi_X(t)}{\rho(t^{1/n})}.$$

Since

$$w(r) \int_r^1 \frac{1}{v(t)} \frac{dt}{t} = \frac{m(r)}{1+m(r)} \leq 1,$$

condition (2.2) holds, and again (4.1) and Propositions 1 and 2 yield

$$\begin{aligned} \|f\|_{\Gamma^{\infty}(w)} &\preceq \sup_{0<t<1} \left(\left(\int_t^1 \frac{H(s^{\frac{1}{n}})}{\phi_X(s)} \frac{ds}{s} \right) w(t) \right) + \left(\sup_{0<t<1} w(t) \right) \sum_{j=0}^{k} \left\| \left| d^j f \right| \right\|_X \\ &\preceq \sup_{0<t<1} \left(\frac{H(s^{\frac{1}{n}})}{\phi_X(s)} v(t) \right) + \sum_{j=0}^{k} \left\| \left| d^j f \right| \right\|_X \preceq c \|f\|_{B^{\rho}_{X,q}} . \end{aligned}$$

□

We shall see now that our result is the best possible when we work in the scale of r.i. spaces.

Theorem 3. *Let $Y = Y(\Omega)$ be a r.i. space. Then*

$$B^{\rho}_{X,q} \subset Y \Leftrightarrow \Gamma^q(w) \subset Y, \tag{4.2}$$

where w is the associated weight of $B^{\rho}_{X,q}$.

Proof. It is clear that Theorem 2 holds for a bounded domain of measure 1 with Lipschitz boundary if and only if it holds for all bounded domains of measure 1 with Lipschitz boundary. Thus we can assume that $\Omega = (0,1)^n$. To prove (4.2), let us see first that there is $0 < \varepsilon < 1$ such that the function $\rho(t)/\phi_X(t^n)^{\varepsilon}$ is almost increasing: We know that $\phi_X(t)/t$ is decreasing and by Lemma 1 we can choose $0 < \beta < 1$, such that $\rho(t)/t^{\beta+k}$ is almost increasing. Hence if $v \leq u$, then

$$\frac{\rho(v)}{\phi_X(v^n)^{\varepsilon}} = \frac{\rho(v)}{v^{\beta+k}}\left(\frac{v^n}{\phi_X(v^n)}\right)^{\varepsilon} v^{\beta+k-n\varepsilon} \leq c\frac{\rho(u)}{\phi_X(u^n)^{\varepsilon}}\left(\frac{v}{u}\right)^{\beta+k-n\varepsilon}.$$

Therefore, taking any $0 < \varepsilon < \frac{\beta}{n}$, we get that

$$\frac{\rho(v)}{\phi_X(v^n)^{\varepsilon}} \leq c\frac{\rho(u)}{\phi_X(u^n)^{\varepsilon}}.$$

Hence, by [24, Theorem 1 and Remark 3], there is $c > 0$ such that

$$\left\{g : \exists f \in B^{\rho}_{X,q} \text{ with } \|f\|_{B^{\rho}_{X,q}} \leq 1 \text{ such that } g^*(t) \leq f^*(t)\right\} = \left\{g : \|g\|_{\Gamma^q(w)} \leq c\right\}.$$

We are now ready to prove (4.2). Given $g \in \Gamma^q(w)$ with $\|g\|_{\Gamma^q(w)} \leq c$, there exists $f \in B^{\rho}_{X,q}$, with $\|f\|_{B^{\rho}_{X,q}} \leq 1$ such that $g^*(t) \leq f^*(t)$. Thus

$$\|g\|_Y \leq \|f\|_Y \preceq \|h\|_{B^{\rho}_{X,q}} \leq 1$$

i.e. $\Gamma^q(w) \subset Y$.

Conversely, given $g \in \Gamma^q(w)$ with $\|g\|_{\Gamma^q(w)} \leq c$, there exists $f \in B^{\rho}_{X,q}$, with $\|f\|_{B^{\rho}_{X,q}} \leq 1$ such that $g^*(t) \leq f^*(t)$. Then

$$\|g\|_Y \preceq \|g\|_{\Gamma^q(w)} \leq \|f\|_{\Gamma^q(w)}. \tag{4.3}$$

By Theorem 2 we know that $B^{\rho}_{X,q} \subset \Gamma^q(w)$, and therefore it follows from (4.3) that $B^{\rho}_{X,q} \subset Y$. □

Let us briefly consider the case $q = 1$.

Theorem 4. *Let w be the associated weight of $B^{\rho}_{X,1}$. Assume that*

$$\int_0^r \frac{dt}{w(t)} \preceq \frac{\phi_X(r)}{\rho(r^{1/n})}. \tag{4.4}$$

Then

$$B^{\rho}_{X,1} \subset \Gamma^1(w). \tag{4.5}$$

Moreover, if Y is a r.i. space, then

$$B^{\rho}_{X,1} \subset Y \Leftrightarrow \Gamma^1(w) \subset Y. \tag{4.6}$$

Proof. The proof is similar to the one given in the case $1 < q \leq \infty$. To see (4.5) let

$$v(t) := \frac{\phi_X(t)}{\rho(t^{1/n})t}.$$

From (4.4) we conclude that

$$\frac{1}{r}\int_0^r \frac{1}{(1+m(s))}\frac{ds}{s} \preceq v(r),$$

i.e. condition (2.2) holds. Using again (4.1) and Proposition 1 we get

$$\|f\|_{\Gamma^1(w)} \preceq \left(\int_0^1 \left(\int_t^1 \frac{H(s^{\frac{1}{n}})}{\phi_X(s)}\frac{ds}{s}\right) w(t)\,dt\right) + \left(\int_0^1 w(t)\,dt\right)\sum_{j=0}^{1} \left\||d^j f|\right\|_X$$

$$\preceq \int_0^1 \frac{H(s^{\frac{1}{n}})}{\phi_X(s)}^q v(s)\,ds + \sum_{j=0}^{1} \left\||d^j f|\right\|_X \preceq c\,\|f\|_{B^{\rho}_{X,q}}.$$

With the same argument given in the proof of Theorem 3 the claim (4.6) follows. □

Remark 1. *With the same proof of Theorem 3 it is easily seen that*

$$B^{\rho}_{X,1} \subset Y \Rightarrow \Gamma^1(w) \subset Y$$

holds without assuming condition (4.4).

Remark 2. *A similar result to (4.2) and (4.6) was established in* [24, Proposition 3]*, but under the stronger restriction that in a certain sense Y is "separated" from X.*

Remark 3. *Theorem 2 (resp. Theorem 4) remains true if instead of X we take any rearrangement invariant space F with $\phi_F \simeq \phi_X$. Thus we have the following self improving property: if X, Y are r.i. spaces, then*

$$B^{\rho}_{X,q} \subset Y \Rightarrow B^{\rho}_{M(X),q} \subset Y,$$

where $M(X) = \{f : \|f\|_{M(X)} = \sup_{t>0}\{f^{**}(t)\phi_X(t)\} < \infty\}$ is the Marcinkiewicz space associated with X.

Following [28] let us briefly consider some "Hardy–type inequalities".

Given a generalized Besov space $B^{\rho}_{X,q}$, set $\psi(t) = 1/(1+m(t))^{1/q'}$, where m is its associated function. An easy computation shows that if $1 < q \le \infty$, and w is its associated weight, then

(4.7)

$$\|f\|_{\Gamma^q(w)} = \left(\int_0^1 \left(\frac{f^{**}(t)}{1+m(t)}\right)^q |m'(t)|\,dt\right)^{1/q} \simeq \left(\int_0^1 (\psi(t)f^{**}(t))^q\,\frac{\psi'(t)}{\psi(t)}dt\right)^{1/q}.$$

Moreover, since ψ is increasing, by [28, Proposition 12.2],

(4.8)

$$\sup_{0<t<1}\psi(t)f^{**}(t) \preceq \left(\int_0^1 (\psi(t)f^{**}(t))^{u_1}\,\frac{\psi'(t)}{\psi(t)}dt\right)^{1/u_1} \preceq \left(\int_0^1 (\psi(t)f^{**}(t))^{u_0}\,\frac{\psi'(t)}{\psi(t)}dt\right)^{1/u_0}$$

provided $1 \le u_0 < u_1 < \infty$.

In fact,

$$\left(\int_0^1 (\psi(t)f^{**}(t))^{u_1}\,\frac{\psi'(t)}{\psi(t)}dt\right)^{1/u_1} \preceq \left(\int_0^1 (\psi(t)f^{**}(t))^{u_0}\,\frac{\psi'(t)}{\psi(t)}dt\right)^{1/u_0} \tag{4.9}$$

if and only if $1 \le u_0 \le u_1 \le \infty$.

Theorem 5. *Let $1 < q \le \infty$, let $B^{\rho}_{X,q}$ be a generalized Besov space and let $\psi(t) = 1/(1+m(t))^{1/q'}$. Let $\varkappa(t)$ be a positive decreasing function on $[0,1]$, and let $1 \le u \le \infty$. Then there is a constant $c > 0$ such that*

$$\left(\int_0^1 (\varkappa(t)\psi(t)f^{**}(t))^u \frac{\psi'(t)}{\psi(t)}dt\right)^{1/u} \le c\,\|f\|_{B^{\rho}_{X,q}}, \quad \forall f \in B^{\rho}_{X,q} \tag{4.10}$$

if and only if $\varkappa$ is bounded and $1 < q \le u \le \infty$.

Proof. By Theorem 2, (4.7) and (4.8) we get that

$$\sup_{0<t<1} \psi(t)f^{**}(t) \preceq \|f\|_{\Gamma^q(w)} \preceq \|f\|_{B^{\rho}_{X,q}}. \tag{4.11}$$

Inequality (4.8) implies

$$\sup_{0<t<1} \varkappa(t)\psi(t)f^{**}(t) \preceq \left(\int_0^1 (\varkappa(t)\psi(t)f^{**}(t))^u \frac{\psi'(t)}{\psi(t)}dt\right)^{1/u}.$$

Thus if (4.10) holds, then

$$\sup_{0<t<1} \varkappa(t)\psi(t)f^{**}(t) \le c \text{ for all } \|f\|_{B^{\rho}_{X,q}} \le 1.$$

Whence, by (4.11), we have that $\varkappa(t) \le c'$ uniformly with respect to t. The fact that $1 < q \le u \le \infty$ follows from (4.9). □

Remark 4. *In the case $q = 1$ and $\underline{\beta}_X > \overline{\beta}_{\rho}/n$, the associated function satisfies (see Remark 7 below)*

$$m(r) \simeq 1 + \int_r^1 \frac{\psi(s^{1/n})}{\phi_X(s)}\frac{ds}{s}. \tag{4.12}$$

Considering $\psi(t) = 1/(1+m(t))t$, it follows from (4.12) that

$$\|f\|_{\Gamma^1(w)} = \int_0^1 \frac{f^{**}(t)}{1+m(t)}\frac{dt}{t} \simeq \int_0^1 \psi(t)f^{**}(t)\frac{\psi'(t)}{\psi(t)}dt,$$

and then Theorem 5 holds for $1 \le q \le \infty$.

4.1. The embedding $B^{\rho}_{X,q} \subset L^{\infty}$.

Theorem 6. *Let $B^{\rho}_{X,q}$ be a generalized Besov space. Suppose its associated function m satisfies that $m(0) < \infty$. Then*

$$B^{\rho}_{X,q} \subset L^{\infty}. \tag{4.13}$$

Proof. Given $\rho \in \Lambda_k$, $k \in N$, we know that $B^{\rho}_{X,q} \subset W^{k,X}(\Omega)$ therefore by Theorem 1, given $f \in B^{\rho}_{X,q}$, we have that

$$\begin{aligned} f^{**}(t) &\preceq \int_t^1 \frac{s^{\frac{k}{n}}\left(\sum_{|\alpha|=k}\left(\omega_X(D^{\alpha}f, s^{\frac{1}{n}}) + s^{\frac{1}{n}}\|D^{\alpha}f\|_X\right)\right)}{\phi_X(s)}\frac{ds}{s} + \sum_{j=0}^{k}\left\|\left|d^j f\right|\right\|_X \\ &= I + II. \end{aligned}$$

Let $H(s^{1/n}) = s^{\frac{k}{n}}\sum_{|\alpha|=k}\left(\omega_X(D^{\alpha}f, s^{\frac{1}{n}}) + s^{\frac{1}{n}}\|D^{\alpha}f\|_X\right)$. By Hölder inequality

$$I \le \left(\int_0^1 \left(\frac{H(s^{1/n})}{\rho(s^{1/n})}\right)^q \frac{ds}{s}\right)^{1/q} \left(\int_t^1 \left(\frac{\rho(s^{1/n})}{\phi_X(s)}\right)^{q'} \frac{ds}{s}\right)^{1/q'} = J(H)m(t)^{1/q'}.$$

Moreover

$$J(H) \preceq \sum_{|\alpha|=k}\left(\left(\int_0^1\left(\frac{\omega_X(D^\alpha f,t)}{\rho(t)/t^k}\right)^q\frac{dt}{t}\right)^{1/q}+\|D^\alpha f\|_X\left(\int_0^1\left(\frac{t^{k+1}}{\rho(t)}\right)^q\frac{dt}{t}\right)^{1/q}\right).$$

Since $\rho\in\Lambda_k$, by Lemma 1, there is $0<\varepsilon<1$ such that $\frac{\rho(t)}{t^{k+\varepsilon}}$ is almost decreasing, so

$$\int_0^1\left(\frac{t^{k+1}}{\rho(t)}\right)^q\frac{dt}{t}=\int_0^1\left(\frac{t^{k+\varepsilon}}{\rho(t)}\right)^q\frac{t^{(1-\varepsilon)q}dt}{t}\preceq\left(\frac{1}{\rho(1)}\right)^q\int_0^1\frac{t^{(1-\varepsilon)q}dt}{t}<\infty. \tag{4.14}$$

Thus $I\leq\|f\|^*_{B^\rho_{X,q}}\preceq\|f\|_{B^\rho_{X,q}}$, and

$$\|f\|_{L^\infty}=\sup_{0<t<1}f^{**}(t)\preceq m(0)^{1/q'}\|f\|_{B^\rho_{X,q}}.$$

□

The relation between condition $m(0)<\infty$ and indices is given in the next result.

Proposition 3. *Let m be the associated function of $B^\rho_{X,q}$. Suppose $m(0)<\infty$. Then*

(1) $\underline{\beta}_X<\overline{\beta}_\rho/n$ *if* $1<q\leq\infty$.
(2) $\underline{\beta}_X\leq\overline{\beta}_\rho/n$ *if* $q=1$.

Proof. 1) Recall that for $1<q\leq\infty$, the associated function of $B^\rho_{X,q}$ is

$$m(t)=\int_t^1\left(\frac{\rho(s^{1/n})}{\phi_X(s)}\right)^{q'}\frac{ds}{s}.$$

Thus, for all $0<t<1$

$$\begin{aligned}m(0)^{1/q'}&>\left(\int_0^t\left(\frac{\rho(s^{1/n})}{\phi_X(s)}\right)^{q'}\frac{ds}{s}\right)^{1/q'}=\left(\int_0^1\left(\frac{\rho((zt)^{1/n})}{\phi_X(zt)}\right)^{q'}\frac{dz}{z}\right)^{1/q'}\\&=\left(\int_0^1\left(\frac{\rho((zt)^{1/n})}{\rho(z^{1/n})}\frac{\phi_X(z)}{\phi_X(zt)}\frac{\rho(z^{1/n})}{\phi_X(z)}\right)^{q'}\frac{dz}{z}\right)^{1/q'}\\&\geq\inf_{0\leq z\leq1}\frac{\rho((zt)^{1/n})}{\rho(z^{1/n})}\inf_{0\leq z\leq1}\frac{\phi_X(z)}{\phi_X(zt)}m(0)^{1/q'}.\end{aligned}$$

Whence,

$$\begin{aligned}1&>\inf_{0\leq z\leq1}\frac{\rho((zt)^{1/n})}{\rho(z^{1/n})}\inf_{0\leq z\leq1}\frac{\phi_X(z)}{\phi_X(zt)}=\inf_{0\leq s\leq1}\frac{\rho(st^{1/n})}{\rho(s)}\inf_{0\leq z\leq1}\frac{\phi_X(z)}{\phi_X(zt)}\\&=\frac{1}{M_\rho(1/t^{1/n})}\frac{1}{M_{\phi_X}(t)},\end{aligned}$$

which implies

$$\log M_\rho(1/t^{1/n})+\log M_{\phi_X}(t)>0.$$

Now, since

$$\frac{\log M_{\phi_X}(t)}{\log t}+\frac{\log M_\rho(1/t^{1/n})}{\log t}=\frac{\log M_{\phi_X}(t)}{\log t}-\frac{1}{n}\frac{\log M_\rho(1/t^{1/n})}{\log 1/t^{1/n}}<0,$$

we conclude that

$$\lim_{t\to 0}\frac{\log M_{\phi_X}(t)}{\log t}-\frac{1}{n}\lim_{t\to 0}\frac{\log M_\rho(1/t^{1/n})}{\log 1/t^{1/n}}=\underline{\beta}_X-\frac{\overline{\beta}_\rho}{n}<0.$$

2) In this case $m(t)=\sup_{s\in[t,1)}\dfrac{\rho(s^{1/n})}{\phi_X(s)}$. Thus, since $m(0)<\infty$, there is $c>0$ such that

$$\rho(s^{1/n})\le c\phi_X(s)\quad 0\le s\le 1.$$

Therefore

$$\frac{\rho((ts)^{1/n})}{\rho(s^{1/n})}\frac{\phi_X(s)}{\phi_X(ts)}\le c\frac{\phi_X(s)}{\rho(s^{1/n})}$$

and then

$$\frac{1}{M_\rho(1/t^{1/n})}\frac{1}{M_{\phi_X}(t)}=\left(\inf_{0\le s\le 1}\frac{\rho((ts)^{1/n})}{\rho(s^{1/n})}\right)\left(\inf_{0\le s\le 1}\frac{\phi_X(s)}{\phi_X(ts)}\right)\le c\frac{1}{m(0)},$$

and now we finish the proof as in the previous case. □

4.2. **Examples.**

4.2.1. *Classical Besov spaces.* For $\rho(t)=t^\sigma$, $t\in(0,1]$, $0<\sigma<\infty$, and $X=L^p(\Omega)$, the space $B^\rho_{X,q}$ is the classical Besov space $B^\sigma_{p,q}$. When Theorems 2 and 4 are particularized to this case we obtain:

Theorem 7. *Let $1\le p<\infty$. The following assertions are true:*

(1) *If $\sigma<\frac{n}{p}$ and $r=\frac{1}{\frac{1}{p}-\frac{\sigma}{n}}$, then*

$$\|f\|_{\Gamma^q(t^{q/r-1})}=\|f\|_{L_{r,q}}=\left(\int_0^1\left(f^{**}(t)t^{1/r}\right)^q\frac{dt}{t}\right)^{1/q}\le c\,\|f\|_{B^\sigma_{p,q}}\ (1\le q\le\infty.)$$

(2) *If $\sigma=\frac{n}{p}$, then*

$$\|f\|_{\Gamma^q(w)}=\left(\int_0^1\left(\frac{f^{**}(t)}{1+|\ln t|}\right)^q\frac{dt}{t}\right)^{1/q}\le c\,\|f\|_{B^{\frac{n}{p}}_{p,q}}\ (1<q<\infty)$$

and

$$\|f\|_{\Gamma^\infty(w)}=\sup_{0<t<1}\frac{f^{**}(t)}{1+|\ln t|}\le c\,\|f\|_{B^{\frac{n}{p}}_{p,\infty}}.$$

(3) *Let $1\le q\le\infty$. Then*

$$B^\sigma_{p,q}\subset L^\infty\Leftrightarrow\begin{cases}\sigma>\frac{n}{p}, & \textit{or}\\ \sigma=\frac{n}{p} & \textit{and } q=1.\end{cases}\tag{4.15}$$

Proof. Part (1). It is a simple matter to see that the associated function of $B^\sigma_{p,q}$ satisfies

$$m(t)\simeq\begin{cases}t^{\left(\frac{\sigma}{n}-\frac{1}{p}\right)q'}-1, & \text{if}\quad 1<q\le\infty,\\ t^{\left(\frac{\sigma}{n}-\frac{1}{p}\right)}, & \text{if}\quad q=1.\end{cases}$$

Now (1) is a direct application of Theorems 2 and 4.

Part (2). The associated function of $B^\sigma_{p,q}$ is $m(t)=|\ln t|$ and as in the previous case, Theorems 2 and 4 apply.

Part (3). Assume that conditions on the right hand side of (4.15) hold. Then the associated function of $B^{\sigma}_{p,q}$ satisfies $m(0) < \infty$. Therefore by Theorem 3, $B^{\sigma}_{p,q} \subset L^{\infty}$. Conversely, suppose $B^{\sigma}_{p,q} \subset L^{\infty}$, but $\sigma < \frac{n}{p}$ (or $\sigma = \frac{n}{p}$ and $q \neq 1$). By part (1) $\|f\|_{\Gamma^q(t^{q/r-1})} \subset B^{\sigma}_{p,q}$. Since $B^{\sigma}_{p,q} \subset L^{\infty}$, Theorem 3 ensures that $\Gamma^q(t^{q/r-1}) \subset L^{\infty}$, which is a contradiction. □

4.2.2. *Besov spaces of generalized smoothness.* A function $\Psi : [0,1] \to [0,\infty)$, such that $\Psi(1) = 1$ is said to be slowly varying ($\Psi \in SV(0,1)$) if for each $\varepsilon > 0$, the function $t^{-\varepsilon}\Psi(t)$ is almost decreasing and $t^{\varepsilon}\Psi(t)$ is almost increasing.

Proposition 4. *(See* [11]*) Let $\Psi \in SV(0,1)$. Then*

(1) $\Psi^r \in SV(0,1)$ for each $r \in \mathbb{R}$.
(2) If $\varepsilon > 0$, then there are positive constants $c = c(\varepsilon)$ and $C = C(\varepsilon)$ such that for every $t > 0$

$$c\min(t^{-\varepsilon}, t^{\varepsilon}) \leq \sup_{s \in (0,\min(1,1/t))} \frac{\Psi(ts)}{\Psi(s)} = M_{\Psi}(t) \leq C\max(t^{-\varepsilon}, t^{\varepsilon}). \tag{4.16}$$

(3) If $\alpha > 0$ and $1 \leq q \leq \infty$, then for all $0 < r < 1$

$$\int_0^r (t^{\alpha}\Psi(t))^q \frac{dt}{t} \simeq (r^{\alpha}\Psi(r))^q \quad \text{and} \quad 1 + \int_r^1 \frac{1}{(t^{\alpha}\Psi(t))^q}\frac{dt}{t} \simeq \frac{1}{(r^{\alpha}\Psi(r))^q}.$$

Remark 5. *It follows readily from (4.16) that functions $\Psi \in SV(0,1)$ satisfy $\underline{\beta}_{\Psi} = \overline{\beta}_{\Psi} = 0$.*

Let $\rho(t) = t^{\sigma}\Psi(t)$, $t \in (0,1]$, $0 < \sigma < \infty$, where Ψ is a slowly varying function, and $X = L^p(\Omega)$. Let $B^{\rho}_{X,q} = B^{(\sigma,\Psi)}_{p,q}$ be the Besov space of generalized smoothness. Then we have the following (see [5], [6] and [13] for related results):

Theorem 8. *Let $1 \leq p < \infty$. Then the following assertions are true:*

(1) *Suppose $\sigma < \frac{n}{p}$ and $r = \frac{1}{\frac{1}{p} - \frac{\sigma}{n}}$, then*

$$\|f\|_{L_{(r,\Psi),q}} = \left(\int_0^1 \left(f^{**}(t)\frac{t^{1/r}}{\Psi(t^{1/n})}\right)^q \frac{dt}{t}\right)^{1/q} \leq c\,\|f\|_{B^{(\sigma,\Psi)}_{p,q}}, \quad \textit{if} \;\; 1 \leq q \leq \infty.$$

(2) *Suppose $\sigma = \frac{n}{p}$ and $\int_0^1 \Psi(t)^{q'}\frac{dt}{t} = \infty$. Then*
  (a) *If $1 < q < \infty$, then*

$$\|f\|_{\Gamma^q(w)} = \left(\int_0^1 \left(\frac{f^{**}(t)}{1 + \int_t^1 \Psi(s)^{q'}\frac{ds}{s}}\right)^q \frac{dt}{t}\right)^{1/q} \leq c\,\|f\|_{B^{(\frac{n}{p},\Psi)}_{p,q}}.$$

  (b) *If $q = \infty$, then*

$$\|f\|_{\Gamma^{\infty}(w)} = \sup_{0<t<1} \frac{f^{**}(t)}{1 + \int_t^1 \Psi(s)\frac{ds}{s}} \leq c\,\|f\|_{B^{(\frac{n}{p},\Psi)}_{p,\infty}}.$$

  (c) *If $q = 1$ and*

$$\int_0^r \frac{dt}{\left(1 + \sup\limits_{s\in[t,1)} \Psi(s^{1/n})\right)t} \preceq \frac{1}{\Psi(r^{1/n})},$$

then

$$\|f\|_{\Gamma^1(w)} = \int_0^1 \frac{f^{**}(t)}{1+\sup_{s\in[t,1)} \Psi(s^{1/n})} \frac{dt}{t} \leq c\,\|f\|_{B_{p,1}^{(\frac{n}{p},\Psi)}}.$$

(3) *Let* $1 \leq q \leq \infty$, *then*

$$B_{p,q}^{(\sigma,\Psi)} \subset L^\infty \Leftrightarrow \begin{cases} \sigma > \frac{n}{p}, & or \\ \sigma = \frac{n}{p} & and\ \Psi \in L^{q'}(dt/t). \end{cases}$$

Proof. Claim (1). By Proposition 4 the associated function of $B_{p,q}^{(\sigma,\Psi)}$ satisfies that

$$1 + m(t) = \int_t^1 \left(s^{\frac{\sigma}{n}-\frac{1}{p}}\Psi(s^{1/n})\right)^{q'} \frac{ds}{s} \simeq \left(t^{\frac{\sigma}{n}-\frac{1}{p}}\Psi(t^{1/n})\right)^{q'}, \quad 1 < q \leq \infty$$

and

$$m(t) = \sup_{s\in[t,1)} s^{\frac{\sigma}{n}-\frac{1}{p}}\Psi(s^{1/n}) \simeq t^{\frac{\sigma}{n}-\frac{1}{p}}\Psi(t^{1/n}), \quad q = 1.$$

Whence (1) follows from Theorems 2 and 4.

Claim (2). Here the associated function of $B_{p,q}^{(\sigma,\Psi)}$ is

$$m(t) = \int_t^1 \Psi(s^{1/n})^{q'} \frac{ds}{s}, \quad 1 < q \leq \infty$$

and

$$m(t) = \sup_{s\in[t,1)} \Psi(s^{1/n}), \quad q = 1,$$

and again Theorems 2 and 4 apply.

Claim (3). Conditions on indices ensure that $m(0) < \infty$, thus by Theorem 3 $B_{p,q}^{(\sigma,\Psi)} \subset L^\infty$. Conversely, assume that $B_{p,q}^{(\sigma,\Psi)} \subset L^\infty$ but $\sigma < \frac{n}{p}$ (or $\sigma = \frac{n}{p}$ and $\Psi \notin L^{q'}(dt/t)$). Then Theorem 3 and Remark 1 imply that $\Gamma^q(w) \subset L^\infty$, where $\Gamma^q(w)$ denotes the Lorentz space that appears in parts (1) and (2). Since obviously $L^\infty \subset \Gamma^q(w)$ we get that $L^\infty = \Gamma^q(w)$ which is not possible. □

5. Embedding Theorems on rearrangement invariant sets

Our aim in this section will be to extend Theorem 2 to more general r.i. sets. Since Theorem 3 states that our conditions are optimal in the target of r.i. spaces, following the ideas introduced in [22], we shall modify the definition of the Lorentz spaces $\Gamma^q(v)$ by replacing f^{**} by the quantity

$$f_o^*(t) = f^{**}(t) - f^*(t)$$

which measures the oscillation of f^*.

Definition 2. *Let* $\rho \in \Lambda_k$ *and* X *be a r.i. space. Consider the function*

$$v(t) = \begin{cases} \left(\dfrac{\phi_X(t)}{\rho(t^{1/n})}\right)^q \dfrac{1}{t}, & if\ 1 \leq q < \infty, \\ \dfrac{\phi_X(t)}{\rho(t^{1/n})}, & if\ q = \infty. \end{cases}$$

The rearrangement invariant set $S^q(v)$ *is defined by*

$$S^q(v) = \left\{ f \in \mathcal{M}(\Omega) : \|f\|_{S^q(v)} = \left(\int_0^1 (f^{**}(t) - f^*(t))^q\, v(t)dt\right)^{1/q} + \|f\|_{L^1} < \infty \right\},$$

with the usual changes when $q=\infty$. (Functional properties of $S^p(w)$ were studied in [7]*).*

Theorem 9. *Let $B^{\rho}_{X,q}$ be a generalized Besov space. Then*

$$\|Ef\|_{S^q(v)} \preceq \|f\|_{B^{\rho}_{X,q}}, \quad \text{for all } f \in B^{\rho}_{X,q},$$

(where E is any extension operator (see Section 3)).

Moreover if X is a r.i. space such that the operators P and Q are bounded on X, then

$$\|f\|_{S^q(v)} \preceq \|f\|_{B^{\rho}_{X,q}}, \quad \text{for all } f \in C_0^{\infty}(\Omega).$$

Proof. By Lemma 5 we have that for all $f \in B^{\rho}_{X,q}$

$$(Ef)^{**}(t) - (Ef)^{*}(t) \leq c\frac{\omega_X(f,t^{1/n}) + t^{1/n}\|f\|_X}{\phi_X(t)},$$

which easily implies that

$$\|Ef\|_{S^q(v)} \preceq \|f\|_{B^{\rho}_{X,q}}.$$

If $f \in C_0^{\infty}(\Omega)$, then by Corollary 5 we get that

$$f^{**}(t) - f^{*}(t) \leq c\frac{\omega_X(f,t^{1/n}) + t^{1/n}\|f\|_X}{\phi_X(t)}.$$

□

Theorem 10. *Let $B^{\rho}_{X,q}$ be a generalized Besov space with $\rho \in \Lambda_k$ $(k \geq 1)$. Assume that*

$$\underline{\beta}_X = \overline{\beta}_{\rho}/n. \tag{5.1}$$

Then

$$B^{\rho}_{X,q} \subset S^q(v).$$

Proof. Let $\rho \in \Lambda_k$, $k \geq 1$. Set

$$H(s) = s^{\frac{k-1}{n}} \sum_{|\alpha|=k} \left(\frac{\omega_X(D^{\alpha}f, s^{\frac{1}{n}}) + s^{\frac{1}{n}}\|D^{\alpha}f\|_X}{\phi_X(s)} \right).$$

In the case that $1 \leq q < \infty$, Lemma 6 ensures that for all $f \in B^{\rho}_{X,q}$ the following inequality holds:

$$f_o^*(t) \leq ct^{1/n}\left(\int_t^1 H(s)\frac{ds}{s} + \sum_{j=0}^{k} \left\| \left| d^j f \right| \right\|_X \right), \ \ 0<t<1.$$

Thus

$$\begin{aligned}\|f\|_{S^q(v)} &\preceq \left(\int_0^1 (QH(t))^q\, t^{q/n} v(t)dt \right)^{1/q} + \sum_{j=0}^{k} \left\| \left| d^j f \right| \right\|_X \left(\int_0^1 t^{q/n} v(t)dt \right)^{1/q} \\ &= I + II.\end{aligned}$$

By Lemma 3,

$$I \preceq \sum_{|\alpha|=k}\left[\left(\int_0^1\left(\frac{t^k\omega_X(D^\alpha f,t)}{\rho(t)}\right)^q\frac{dt}{t}\right)^{1/q}+\|D^\alpha f\|_X\left(\int_0^1\left(\frac{t^{k+1}}{\rho(t)}\right)^q\frac{dt}{t}\right)^{1/q}\right]$$
$$\preceq \sum_{|\alpha|=k}\left[\left(\int_0^1\left(\frac{t^k\omega_X(D^\alpha f,t)}{\rho(t)}\right)^q\frac{dt}{t}\right)^{1/q}+\|D^\alpha f\|_X\right] \quad \text{(by (4.14))}$$

and

$$II \preceq \sum_{j=0}^{k}\left\||d^jf|\right\|_X .$$

Collecting both estimates, we obtain

$$\|f\|_{S^q(v)} \preceq \|f\|_{B^\rho_{X,q}} .$$

The case $q=\infty$ is similar to the previous one. □

Let us now see a result that shows that our conditions are best posible.

Theorem 11. *Let $B^\rho_{X,q}$ a generalized Besov spaces. Suppose Y is a r.i. space such that*

$$B^\rho_{X,q}\subset Y.$$

Then

$$S^q(v)\subset Y.$$

Proof. In the case that $1<q\le\infty$, since $B^\rho_{X,q}\subset Y$ by Theorem 3 it follows that $\Gamma^q(w)\subset Y$, (where w is the associated weigh of $B^\rho_{X,q}$). Thus it is enough to see that $S^q(v)\subset\Gamma^q(w)$. Since $\frac{\partial}{\partial t}f^{**}(t)=(f^{**}(t)-f^*(t))/t$, by the fundamental theorem of Calculus

$$f^{**}(t)=\int_t^1(f^{**}(s)-f^*(s))\frac{ds}{s}+\int_0^1 f^*(s)ds.$$

Therefore,

$$\|f\|_{\Gamma^q(w)}=\|f^{**}\|_{L^q(w)}\le\|Q(f^{**}-f^*)\|_{L^q(w)}+\left(\int_0^1 w(t)\,dt\right)^{1/q}\int_0^1 f^*(s)\,ds$$
$$\preceq\|(f^{**}-f^*)\|_{L^q(v)}+\int_0^1 f^*(s)\,ds \quad \text{(by Proposition 1)}$$
$$=\|f\|_{S^q(v)} .$$

For $q=1$, the same proof works, but using Theorem 2 instead of Theorem 3. □

Example 2. *Let $B^{(\sigma,\Psi)}_{p,q}$ be a Besov space of generalized smoothness. Let $1<p<\infty$. Assume that $\sigma=\frac{n}{p}$. Then for all $f\in C_0^\infty(\Omega)$ we have that*

$$\left(\int_0^1\left(\frac{f^{**}(t)-f^*(t)}{\Psi(t)}\right)^q\frac{dt}{t}\right)^{1/q}\le c\,\|f\|_{B^{(\frac{n}{p},\Psi)}_{p,q}} \qquad (1\le q<\infty)$$

and

$$\sup_{0<t<1}\frac{f^{**}(t)-f^*(t)}{\Psi(t)}\le c\,\|f\|_{B^{(\frac{n}{p},\Psi)}_{p,\infty}} .$$

We end this section by showing that $S^q(v)=\Gamma^q(w)$.

Proposition 5. *Let $\rho \in \Lambda_k$ and X be a r.i. space. Suppose*

$$\underline{\beta}_X > \overline{\beta}_\rho / n. \tag{5.2}$$

Then, we have that

$$S^q(v) = \Gamma^q(w).$$

Proof. If $1 < q \leq \infty$, then it follows from the proof of Lemma 3 that

$$1 + \int_r^1 \left(\frac{\rho(s^{1/n})}{\phi_X(s)} \right)^{q'} \frac{ds}{s} \simeq \left(\frac{\rho(r^{1/n})}{\phi_X(r)} \right)^{q'}.$$

For $q = 1$, obviously

$$\sup_{s\in[r,1)} \frac{\rho(s^{1/n})}{\phi_X(s)} \geq \frac{\rho(r^{1/n})}{\phi_X(r)}.$$

Taking

$$\underline{\beta}_X > \beta > \frac{\gamma + k}{n} > \frac{\bar{\beta}_\rho}{n} > 0,$$

we get

$$\begin{aligned} \sup_{s\in[r,1)} \frac{\rho(s^{1/n})}{\phi_X(s)} &= \sup_{s\in[r,1)} \left(\frac{\rho(s^{1/n})}{s^{\frac{\gamma+k}{n}}} \frac{s^\beta}{\phi_X(s)} s^{\frac{\gamma+k}{n} - \beta} \right) \\ &\preceq \left(\frac{\rho(r^{1/n})}{r^{\frac{\gamma+k}{n}}} \frac{r^\beta}{\phi_X(r)} \right) \sup_{s\in[r,1)} s^{\frac{\gamma+k}{n} - \beta} \\ &= \frac{\rho(r^{1/n})}{\phi_X(r)}. \end{aligned}$$

Summarizing, we have proved that

$$\begin{cases} 1 + m(r) \simeq \left(\frac{\rho(r^{1/n})}{\phi_X(r)} \right)^{q'}, & \text{if } 1 < q \leq \infty, \\ \sup\limits_{s\in[r,1)} \dfrac{\rho(s^{1/n})}{\phi_X(s)} \simeq \dfrac{\rho(r^{1/n})}{\phi_X(r)}, & \text{if } q = 1. \end{cases}$$

And from here it is easy to see that $\Gamma^q(w) = \Gamma^q(v)$. In fact, by Theorem 11, and taking into account that $\Gamma^q(w) \subset L^1$ (since $\Gamma^q(w)$ is a r.i. space on $(0,1)$) we have that

$$\begin{aligned} \|f\|_{\Gamma^q(v)} &\simeq \|f\|_{\Gamma^q(w)} \preceq \|(f^{**} - f^*)\|_{L^q(v)} + \|f\|_{L^1} \\ &= \|f\|_{S^q(w)} \leq \|f^{**}\|_{L^q(v)} + \|f\|_{L^1} \\ &= \|f\|_{\Gamma^q(v)} + \|f\|_{L^1} \preceq \|f\|_{\Gamma^q(v)}. \end{aligned}$$

□

Remark 6. *The previous result states that under the condition $\underline{\beta}_X > \overline{\beta}_\rho/n$, $S^q(v)$ is a r.i. space. Then Theorems 2 and 4 imply that $\|f\|_{\Gamma^q(w)} \simeq \|f\|_{S^q(v)}$. Similarly, (see Proposition 3) $\underline{\beta}_X < \overline{\beta}_\rho/n$ is closely related with the embedding $B^\rho_{X,q} \subset L^\infty$. Therefore Theorem 10 is useful in order to measure the type of essential unboundedness of functions of $B^\rho_{X,q}$ when (5.1) holds.*

Remark 7. *Let m be the associated function of $B^{\rho}_{X,1}$. If $\underline{\beta}_X > \overline{\beta}_{\rho}/n$, then we have seen in Proposition 5 that $m(r) \simeq \dfrac{\rho(r^{1/n})}{\phi_X(r)}$. Moreover by (2.5) we also have that $m(r) \simeq 1 + \int_r^1 \frac{\psi(s^{1/n})}{\phi_X(s)} \frac{ds}{s}$.*

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References

[1] R. A. Adams, *Sobolev Spaces*, Pure and Applied Mathematics, Vol. 140. Academic Press, 2003.

[2] J. Bastero, M. Milman and F. Ruiz, *A note in $L(\infty, q)$ spaces and Sobolev embeddings*, Indiana Univ. Math. J. **52** (2003), 1215–1230.

[3] C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, Boston, 1988.

[4] V. I. Burenkov, *Sobolev Spaces on Domains*, B.G. Teubner, Stuttgart, 1998.

[5] A. M. Caetano and S. Moura, *Local growth envelopes of spaces of generalized smoothness: the subcritical case*, Math. Nachr. **273** (2004), 43–57.

[6] A. M. Caetano and S. Moura, *Local growth envelopes of spaces of generalized smoothness: the critical case*, Math. Inequal. Appl. **7** (2004), no. 4, 573–606.

[7] M. J. Carro, A. Gogatishvili, J. Martín and L. Pick, *Functional properties of rearrangement invariant spaces defined in terms of oscillations*, J. Funct. Anal. **229** (2005), 375-404.

[8] F. Cobos and D. L. Fernandez, *Hardy-Sobolev spaces and Besov spaces with a function parameter*, Function Spaces and Applications (Lund, 1986), Lecture Notes in Math., **1302**, 158–170, Springer, Berlin, 1988.

[9] R. DeVeore and K. Scherer, *Interpolation of linear operators on Sobolev spaces*, Ann. Math. **109** (1979), 583-599.

[10] D. Edmunds and H. Triebel, *Spectral theory for isotropic fractal drums*, C. R. Acad. Sci. Paris **326** (1998), no. 11, 1269–1274.

[11] W. D. Evans, B. Opic and L. Pick, *Real interpolation with logarithmic functors*, J. Inequal. Appl. **7** (2002), no. 2, 187–269.

[12] W. Farkas, *A Calderon-Zygmund extension theorem for abstract Sobolev spaces*, Stud. Cerc. Mat. **47** (1995), no. 5-6, 379–395.

[13] D. D. Haroske and S. Moura, *Continuity envelopes of spaces of generalised smoothness, entropy and approximation numbers*, J. Approx. Theory **128** (2004), no. 2, 151–174.

[14] H. Johnen and K. Scherer, *On the equivalence of the $K-$functional and moduli of continuity and some applications*, in Constructive Theory of Functions of Several Variables, pp. 119-140, Lecture Notes in Math. **571**, Springer, Berlin, 1977.

[15] G. A. Kalyabin and P. I. Lizorkin, *Spaces of functions of generalized smoothness*, Math. Nachr. **133** (1987), 7–32.

[16] S. G. Krein, Yu. I. Petunin and E. M. Semenov, *Interpolation of linear operators*, Transl. Math. Monogr. Amer. Math, Soc. **54**, Providence, 1982.

[17] H-G. Leopold, *Embeddings and entropy numbers in Besov spaces of generalized smoothness*, Function spaces (Poznań, 1998), Lecture Notes in Pure and Appl. Math., **213**, 323–336, Dekker, New York, 2000.

[18] J. Martín and M. Milman, *Symmetrization inequalites and Sobolev embeddings*, Proc. Amer. Math. Soc. **134** (2006), 2335-2347.

[19] J. Martín and M. Milman, *Higher order symmetrization inequalities and applications*, J. Math. Anal. Appl. **330** (2007), 91-113.

[20] J. Martín, M. Milman and E. Pustylnik, *Sobolev inequalities: symmetrization and self-improvement via truncation*. J. Funct. Anal. **252** (2007), 677–695.

[21] C. Merucci, *Applications of interpolation with a function parameter to Lorentz, Sobolev and Besov spaces*, Interpolation Spaces and Allied Topics in Analysis (Lund, 1983), Lecture Notes in Math. **1070**, 183–201. Springer, Berlin, 1984.

[22] M. Milman and E. Pustylnik, *On sharp higher order Sobolev embeddings*, Comm. Cont. Math. 6 (2004), 1-17.
[23] B. Mukenhoupt, *Hardy's inequalities with weights*, Studia Math. **44** (1972), 985-992.
[24] Yu. V. Netrusov, *Imbedding theorems of Besov spaces in Banach lattices*, J. Soviet Math. **47** (1989), 2871-2881.
[25] E. Pustylnik, *Sobolev type inequalities in ultrasymmetric spaces with applications to Orlicz-Sobolev embeddings*, J. Funct. Spaces Appl. **3** (2005), 183–208.
[26] R. Sharpley, *Spaces* $\Lambda_\alpha(X)$ *and interpolation*, J. Funct. Anal. **11** (1972), 479-513.
[27] E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, New Jersey, 1970.
[28] H. Triebel, *The Structure of Functions*, Monographs in Mathematics, vol. **97**, Birkhäuser Verlag, Basel, 2001.

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