Symmetric Spaces, Geometric Manifolds and Geodesic Completeness

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Contents

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Dedication

To my family.

Thank you for always supporting me.

Abstract

This thesis explores topics related to the geodesic completeness of compact locally symmetric Lorentzian manifolds. In particular, it discusses some important results relating to locally and globally symmetric spaces as well as the theory of geometric manifolds. These results are used to present a proof of a key proposition in Klingler (1996), which proves the geodesic completeness of compact Lorentzian manifolds with constant curvature. We also prove a new result, that compact Lorentzian manifolds which are locally isometric to the product of Cahen-Wallach space and flat Riemannian space are geodesically complete by extending methods used in Leistner & Schliebner (2016). These results may be helpful in the study of geodesic completeness of compact locally symmetric Lorentzian manifolds more generally as they reduce the number of open cases.

Chapter 1

Introduction

1.1 Background and motivation

An important aspect of geometry is the study of curves with constant speed, known as geodesics. Locally, geodesics are described by a system of second order ODEs, called the geodesic equations. In Riemannian geometry, geodesics are distance minimising curves. We say that a manifold is *geodesically complete* if every maximal geodesic is defined on $\mathbb{R}$, i.e. the curves can continue for an infinite amount of time. One reason why we are specifically interested in the geodesic completeness of compact locally symmetric manifolds is that they are exactly the compact quotients of globally symmetric spaces. This result follows from the fact that geodesically complete, simply connected locally symmetric semi-Riemannian manifolds are symmetric (8.21 in O'Neill (1983)).

It follows from the Hopf-Rinow theorem that any compact Riemannian manifold is geodesically complete (this can be directly proven without the use of Hopf-Rinow also). This result does not extend to Lorentzian manifolds; a standard example of a non-complete compact Lorentzian manifold is the Clifton-Pohl Torus, which is constructed in Chapter 2. In light of this counterexample, there have been attempts to impose additional conditions on compact Lorentzian manifolds in order to ensure geodesic completeness. These conditions can be broadly classified as either global or local.

We shall first discuss a result which imposes global conditions on the manifold. The geodesic completeness of compact homogeneous manifolds with arbitrary signature was shown in Marsden (1973). This is quite a general result and so many subsequent results impose either local or a combination of local and global conditions.

Now we shall consider some local conditions which ensure the geodesic completeness of compact Lorentzian manifolds. Most of the results with local conditions are restrictions on the Riemann curvature tensor. Firstly Carrière (1989) showed that flat, i.e. vanishing curvature tensor, compact Lorentzian manifolds are geodesically complete. This result was then extended upon in Klingler (1996), which proves that compact Lorentzian manifolds of constant sectional curvature are geodesically complete. It then seems natural to ask *"are compact Lorentzian manifolds with covariantly constant curvature ($\nabla R = 0$) geodesically complete?"*. This is still an open question. There was some progress towards an answer in Lafuente-López (1988), which proves that if a (not necessarily compact) Lorentzian manifold with covariantly constant curvature has the property that if all geodesics of a single causal characteristic are complete, then the manifold is geodesically complete.

Next we shall discuss results which impose both local and global conditions. The geodesic completeness of a compact Lorentzian manifold of constant curvature (a local condition) admitting a timelike Killing vector field (a global condition) was shown in Kamishima et al. (1993). This result was later generalised in Romero & Sánchez (1995) with no curvature conditions imposed and only requiring a timelike vector field which is conformally Killing. A more recent result which has a combination of global and local assumptions, Leistner & Schliebner (2016), showed that compact Lorentzian manifolds admitting a global null parallel vector field V and having a curvature tensor such that $R(U, W) = 0$ for all U, W in the orthogonal complement to V are geodesically complete. Manifolds that satisfy the previous two properties are called *pp-wave*. Being pp-wave is completely separate set of conditions to the previous cases as in general, manifolds admitting such a null vector field will not necessarily be homogeneous, have constant curvature or admit a timelike conformally Killing vector field. Finally Hau & Sánchez (2016) showed any compact affine manifolds with precompact holonomy group are geodesically complete.

In light of these results, we will focus on the geodesic completeness of compact Lorentzian manifolds with covariantly constant curvature. One property that may suggest the geodesic completeness of compact Lorentzian manifolds with covariantly constant curvature (a property also known as being locally symmetric), is that they are locally isometric to an important class of geodesically complete manifolds, symmetric spaces. Symmetric spaces are manifolds with an involution symmetry at each point. Classification results in Wu (1964) and Cahen & Wallach (1970) have shown that each Lorentzian symmetric spaces is isometric to the product of either flat Minkowski space, or an indecomposable symmetric Lorentzian space and a Riemannian symmetric space. There are two classes of indecomposable Lorentzian symmetric spaces. The first are those with constant curvature, which consists of positively curved de Sitter space and negatively curved Anti-de Sitter space. The second have non-constant curvature but covariantly constant curvature and are called Cahen-Wallach space. The previous results verify that a compact locally symmetric Lorentzian manifold which is locally isometric to an indecomposable symmetric space is complete as Klingler (1996) covers the constant curvature cases and being locally isometric to Cahen-Wallach is a special case of being pp-wave and is thus complete by Leistner & Schliebner (2016). Hence the remaining cases are locally symmetric compact manifolds which are locally isometric to the product of an indecomposable or flat Lorentzian symmetric space and a Riemannian symmetric space.

1.2 Summary of thesis

This thesis aims to provide the tools required to obtain a clear understanding of the relevant proofs in Klingler (1996) and Leistner & Schliebner (2016). It also slightly generalises a result in Leistner & Schliebner (2016) in order to show that compact locally symmetric Lorentzian manifolds which are locally isometric to the product of Cahen-Wallach space and Riemannian flat space are complete before finally giving a detailed presentation of a key result in Klingler (1996) then discussing attempts to generalise this result. In order to achieve this goal we must first discuss some prerequisite topics, namely symmetric spaces and (G, X)-structures. We first consider symmetric, and locally symmetric, spaces in order to develop an understanding of the types of manifolds we are considering. Importantly, we give the classification of Lorentzian symmetric spaces up to isometry and locally symmetric spaces up to local isometry. Additionally, we show that locally symmetric spaces must be locally isometric to a symmetric space, this result is required for both sections in Chapter 5 and also allows us to equip locally symmetric manifolds with (G, X)-structures. (G, X)-structures are a particular notion of being locally homogeneous. The theory of (G, X)-structures is related to geodesic completeness by a particular local isometry called the development map from the universal cover of the (G, X)-manifold to X. In the locally symmetric case, it is a fact that such a manifold is geodesically complete if and only if this development map is a covering, which is how geodesic completeness is shown in both Carrière (1989) and Klingler (1996). We then show that any locally symmetric space can be equipped with a (G, X)-structure where X is the corresponding symmetric space and G is the isometry group of X. After these results we are able to present results from Klingler (1996) and Leistner & Schliebner (2016) in enough detail to provide a slight extension to the latter and discuss attempts at extending the former.
An outline of the thesis is as follows:

Chapter 2: *Semi-Riemannian geometry.* This chapter covers some preliminary standard results that will be used in the later chapters. Many of the standard results throughout the thesis are in O'Neill (1983).

Chapter 3 *Symmetric Spaces.* This chapter begins by defining symmetric spaces. We then show that symmetric spaces are geodesically complete by taking any geodesic and applying an appropriate symmetry to extend it. Next, we discuss some symmetric spaces: semi-Euclidean space, Cahen-Wallach space and Hyperquadrics are defined and the symmetries, geodesics, curvature and isometry groups of these spaces are calculated. Hyperquadrics are defined in terms of an arbitrary metric so as to cover both the Riemannian sphere and hyperbolic space as well as the Lorentzian de Sitter and Anti de Sitter spaces. A brief overview of Lie triples and symmetric triples is then given in order to present two important results. The first result: A manifold M is locally symmetric if and only if there exists a simply connected symmetric space S such that M is locally isometric to

S. This is a well known result, but most sources only present the proof for Riemannian spaces. We broadly follow the methods used in Neukirchner (2003). An outline of the proof is below. If M is locally isometric to S, then since isometries preserve curvature, and symmetric spaces are locally symmetric, the pushforward of ∇R shows M is locally symmetric. Now suppose M is locally symmetric, for any point $p \in M$ we can construct a local isometry as such:

1. Show $(T_pM, g_p, R_{|p})$ is a Lie triple.
2. Construct a symmetric triple system from $(T_pM, g_p, R_{|p})$.
3. Construct a symmetric space S from the symmetric triple.
4. Extend isometry at p to local isometry.

The second key result is the classification of indecomposable simply connected Lorentzian symmetric spaces. We say that a simply connected Lorentzian symmetric space is indecomposable if it is not isometric to the product of two other manifolds. Because the metric signature of the product of two manifolds is equal to the sum of their respective signatures we immediately see that if a Lorentzian manifold is a product, it must be the product of a Lorentzian manifold and a Riemannian manifold. The classification follows Cahen & Wallach (1970) and relies on classifying the symmetric triples rather than the spaces directly. Importantly, when these two results are combined, we see that each locally symmetric Lorentzian manifold is locally isometric to the product of an indecomposable Lorentzian symmetric space and a Riemannian symmetric space.

Chapter 4: *Geometric manifolds.* This chapter discusses the theory of (G, X)-manifolds and follows Ratcliffe (2006) and Goldman (2021). Given a Lie group G that acts transitively on a manifold X, M is said to be a (G, X)-manifold if it can be equipped with an atlas of charts $\{\phi_i : U_i \to X\}$ such that the transition maps $\phi_{ij} = \phi_i \circ \phi_i^{-1}$ locally agree with an element of G. From this, we eventually construct a *development map* $D : \tilde{M} \to X$. M is called (G, X)-complete if D is a covering map, in particular, in the locally symmetric case, if G is the isometry group of X then (G, X)-completeness is equivalent to geodesic completeness. This theory was used in Carrière (1989) to show compact, flat Lorentzian manifolds can be given a complete $(O(n-1,1) \ltimes \mathbb{R}^{\ltimes}, \mathbb{R}^{n-1}_1)$- structure, and was extended by Klingler (1996) to show the completeness of compact Lorentzian manifolds with constant curvature. In addition to discussing the general theory of (G, X)-structures the chapter concludes with a key result which allows Carrière (1989) and Klingler (1996) to use (G, X)-theory: if M is a locally symmetric Lorentzian manifold, locally isometric to a symmetric space S, then M can be given a $(Iso(S), S)$-structure.

Chapter 5: *Completeness of compact Lorentzian locally symmetric spaces.* This chapter uses the tools presented throughout the thesis in order to explain and then attempt to generalise two theorems on geodesic completeness of compact locally symmetric Lorentzian manifolds. The first theorem is from Leistner & Schliebner (2016) and the second is from Klingler (1996). The geodesic completeness of compact locally Cahen-Wallach manifolds was shown in Leistner & Schliebner (2016) as a corollary to a more general theorem about pp-waves. A Lorentzian manifold (M, g) is called *pp-wave* if it admits a global parallel null vector field $V \in \Gamma(TM)$, i.e. $V \neq 0, g(V, V) = 0$ and $\nabla V = 0$, and if its curvature tensor R satisfies

$$R(U, W) = 0, \quad \text{for all } U, W \in V^{\perp}$$

Theorem 2 in Leistner & Schliebner (2016) shows that every compact pp-wave (M, g) is geodesically complete. They later show that compact locally symmetric Lorentzian manifolds which are locally isometric to Cahen-Wallach space have a time-orientable cover that is a compact pp-wave and is hence complete. By considering the bundle of the kernel of the curvature endomorphism and intersecting it with its orthogonal complement we are able to extend this result and prove that compact locally symmetric Lorentzian manifolds which are locally isometric to the product Cahen-Wallach space with $\mathbb{R}^n$ are geodesically complete.

Next we discuss Klingler (1996), who proves that compact Lorentzian manifolds with constant sectional curvature are geodesically complete. This result uses both the theory of symmetric spaces and (G, X)-manifolds, in particular, it shows that the developing map is surjective. In our attempt to generalise this result, we focus on the central proposition of Klingler (1996), which describes the convexity of the image of geodesic stars under the development map.

1.3 Outlook

One may aim to show that compact locally symmetric Lorentzian manifolds are geodesically complete by considering all the possible Lorentzian symmetric spaces separately, which are products of indecomposable Lorentzian symmetric spaces and Riemannian symmetric spaces as shown in Chapter 3.

It may seem like a natural choice to attempt to extend the methods in Section 5.1 to compact Lorentzian manifolds which are locally isometric to the product of Cahen-Wallach space and a non-flat Riemannian symmetric space. Unfortunately, it appears that this approach is not possible without significant changes, as the manifold will no longer be a pp-wave. As before, we can define a global parallel null vector field V by intersecting the bundle defined by the kernel of the curvature endomorphism with its complement, however there will now be vector fields in $V^{\perp}$ such that

$$R(U, W) \neq 0$$

because of the non-flat Riemannian factor. Therefore the manifold is not pp-wave.

We would eventually like to show that compact locally symmetric manifolds which are locally isometric to the product of a constant curvature Lorentzian symmetric space and a Riemannian symmetric space are geodesically complete by extending the results of Klingler (1996). A first step towards this would be generalising Proposition 1 in Klingler (1996) to these cases. In Section 5.2.3 we provide a list of properties which are used in the proof and show that the constant curvature cases satisfy these properties. It remains to be seen if any additional manifolds satisfy all of the properties, if any such manifolds do exist then it would be an encouraging step towards showing that they are geodesically complete, as Proposition 1 is considered the central proposition of Klingler (1996), however further work would be needed to extend the later results in the paper. In particular we would first like to extend these methods to the product of a constant curvature Lorentzian manifold with a constant curvature Riemannian manifold.

There are other approaches which might prove fruitful, however these are outside the scope of this thesis so this is purely conjecture. In light of Hau & Sánchez (2016) proving that compact manifolds with precompact holonomy are geodesically complete, it would be interesting to discover if compact manifolds with discompacity 0 holonomy are geodesically complete, as all precompact groups have discompacity 0. The proof of Hau & Sánchez (2016) directly uses the precompactness of the holonomy group, so such a proof would be difficult to discover. However if such a result were true, then by the de Rham-Wu theorem we would have that compact manifolds which are locally isometric to the product of Cahen-Wallach space and any Riemannian symmetric space with are geodesically complete as all Riemannian symmetric spaces have compact holonomy groups.

Chapter 2

Basic notions in semi-Riemannian geometry

We assume that the reader is familiar with the basics of semi-Riemannian geometry and in particular, has some understanding of semi-Riemannian metrics. Throughout this thesis we will always be considering semi-Riemannian manifolds (M, g) equipped with the Levi-Civita connection, written ∇.
This chapter states some standard results presented mostly without proof. The reader is most likely familiar with many, if not all, of these results and so the purpose of this chapter is to provide references used in the later chapters. The majority of these results can be found in O'Neill (1983).

2.1 Isometries and local isometries

Definition 2.1 (Isometry)**.** Let (M, g_M) and (N, g_N) be Lorentzian manifolds, an *isometry* from M to N is a diffeomorphism $\phi : M \to N$ that preserves the metric tensors: $\phi^*(g_N) = g_M|_p$. More explicitly:

$$g_N|_{\phi(p)}(d\phi|_p(v), d\phi|_p(w)) = g_M(v, w)$$

for all $v, w \in T_pM$ for any $p \in M$.

From this definition, we can immediately see that:

1. The identity map is an isometry.

2. The composition of isometries is an isometry.

3. The inverse of an isometry is an isometry.

So isometry endomorphisms form a group and we will call this *the group of isometries of* M, written $Iso(M)$.

Definition 2.2 (Isotropy subgroup of isometry group). Given a semi-Riemannian manifold (M, g) with isometry group $Iso(M)$, let $p \in M$, then the *isotropy subgroup of* $Iso(M)$ *fixing* p is the subgroup

$$Iso_p(M) := \{\phi \in Iso(M) | \phi(p) = p)\} \subset Iso(M).$$

This definition can be extended to subgroups of $G \subset Iso(M)$. We can define the *isotropy group of* G *fixing* p as such

$$Iso_p(G) := \{\phi \in Iso(G) | \phi(p) = p)\} \subset Iso(G).$$

Lemma 2.3 (3.7 in O'Neill (1983)). *If* $\psi : V \to W$ *is a linear isometry of scalar product spaces, then* $\psi : V \to W$ *is an isometry.*

Proposition 2.4 (3.59 in O'Neill (1983)). *If* $\phi : M \to \hat{M}$ *is an isometry, then* $d\phi(\nabla_X Y) = \hat{\nabla}_{d\phi X}(d\phi Y)$.

Definition 2.5. A smooth map $\phi : M \to N$ of semi-Riemannian manifolds is a *local isometry* if for each point $p \in M$, there exists a neighbourhood U of p such that $\phi|_U$ is an isometry onto a neighbourhood of $\phi(p)$.

Lemma 2.6 (3.60 O'Neill (1983)). *A smooth map* $\phi : M \to N$ *of semi-Riemannian manifolds is a local isometry if and only if each differential map* $d\phi : T_pM \to T_{\phi(p)}N$ *is a linear isometry.*

Proposition 2.7 (3.62 O'Neill (1983)). *Let* $\phi, \psi : M \to N$ *be local isometries of a connected semi-Riemannian manifold* M*. If there is a point* $p \in M$ *such that* $d\phi_p = d\psi_p$ *(and hence* $\phi(p) = \psi(p)$*), then* $\phi = \psi$.

2.2 Curvature

Definition 2.8 (Riemannian curvature tensor). Let M be a semi-Riemannian manifold with Levi-Civita connection ∇, we define

$$\begin{aligned} R : \Gamma(TM)^3 &\to \Gamma(TM) \\ R(X,Y)Z &:= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \end{aligned}$$

Proposition 2.9 (Properties of the Riemannian curvature tensor). *If* $x, y, z, w, v \in T_pM$ *then:*

1. *R is a multi-linear map.*

2. $R(x, y) = -R(y, x)$.

3. $g(R(x, y)v, w) = -g(R(x, y)w|v)$.

4. $R(x, y)z + R(y, z)x + R(z, x)y = 0$ *(First Bianchi Identity).*

5. $g(R(x, y)v, w) = g(R(v, w)x, y)$.

6. $(\nabla_z R)(x, y) + (\nabla_x R)(y, z) + (\nabla_y R)(z, x) = 0$ *(Second Bianchi Identity).*

Proposition 2.10 (Describing Riemann curvature tensor locally). *If $x^1, \ldots, x^n$ is a coordinate system, then on a neighborhood,*

$$\begin{aligned} R(\partial_i, \partial_j)\partial_k &= R_{ijk}{}^l \partial_l, \\ \text{where } R_{ijk}{}^l &= \frac{\partial}{\partial x^j}\Gamma^l_{ik} - \frac{\partial}{\partial x^l}\Gamma^i_{jk} - \Gamma^l_{ip}\Gamma^p_{jk} - \Gamma^l_{jp}\Gamma^p_{ik}. \end{aligned}$$

Furthermore, we can lower the index by the metric g in order to obtain the $(4, 0)$-Riemann curvature tensor, which is locally described as such:

$$\begin{aligned} g(R(\partial_i, \partial_i)\partial_k, \partial_l) &= R_{ijkl}, \\ \text{where } R_{ijkl} &= (\frac{\partial}{\partial x^i}\Gamma^p_{jk} - \frac{\partial}{\partial x^j}\Gamma^p_{ik} + \Gamma^q_{jk}\Gamma^p_{iq} - \Gamma^q_{ik}\Gamma^p_{jq})g_{pl}. \end{aligned}$$

A two dimensional subspace of the tangent space T_pM is called a *tangent plane* of M at p. For tangent vectors v, w we define

$$Q(v, w) = g(v, v)g(w, w) - (g(v, w))^2.$$

We say that Π is non-degenerate if $Q(v, w) \neq 0$ for any (and hence every) basis of Π.

Lemma 2.11 (3.39 in O'Neill (1983)). *If Π is a non-degenerate tangent plane of M at p with basis vectors v, w. We define the* sectional curvature $K(\Pi)$ *as such:*

$$K(v, w) = \frac{g(R(v, w)v, w)}{Q(v, w)}.$$

It is independent of choice of basis for Π.

Proposition 2.12 (3.41 in O'Neill (1983)). *If $K = 0$ at $p \in M$, then $R = 0$ at p.*

We say that a manifold has *constant curvature* if K is a constant function independent of point p and choice of tangent plane.

2.3 Geodesics

Definition 2.13 (Geodesic)**.** Given a semi-Riemannian manifold M, a curve $\gamma : I \to M$ is a *geodesic* if its acceleration is zero, i.e.

$$\frac{\nabla \dot{\gamma}(t)}{dt} = 0.$$

Lemma 2.14 (Geodesic Equation)**.** *Let $x^1, \ldots, x^n$ be a coordinate system on a neighbourhood U. Then a curve γ in U is a geodesic if and only if*

$$0 = \ddot{\gamma}^k + \dot{\gamma}^i \dot{\gamma}^j \Gamma^k_{ij} \circ \gamma$$

for all $k = 1, \ldots, n$ where $\gamma^k = x^k \circ \gamma$.

This is a system of non-linear second order ordinary differential equations.

Lemma 2.15. *If $v \in T_pM$, then there exists an interval I containing* 0 *such that there exists a unique geodesic $\gamma : I \to M$ with $\gamma'(0) = v$.*

Lemma 2.16 (3.23 in O'Neill (1983))**.** *If $\gamma, \eta : I \to M$ are geodesics such that there is some $a \in I$ such that $\dot{\gamma}(a) = \dot{\eta}(a)$, then $\gamma = \eta$.*

Lemma 2.17 (3.32 in O'Neill (1983))**.** *A semi-Riemannian manifold M is connected if and only if any two of its points can be connected by a piecewise geodesic curve.*

Definition 2.18. Let (M, g) be a semi-Riemannian manifold, we say that M is *geodesically complete*, or complete, if every maximal geodesic is defined on all of $\mathbb{R}$.

The key theorem about geodesic completeness of Riemannian manifolds is the Hopf-Rinow theorem:

Theorem (Hopf-Rinow)**.** *Let (M, g) be a connected Riemannian manifold. Then the following are equivalent:*

1. *The closed and bounded subsets of M are compact,*
2. *M is a complete metric space,*
3. *M is geodesically complete.*

This theorem does not extend to Lorentzian manifolds, with one example of a non-complete compact Lorentzian manifold being the Clifton-Pohl torus.

Example (Clifton-Pohl Torus)**.** [7.16 in O'Neill (1983)] Consider the Lorentzian manifold $(M = \mathbb{R}^2 \setminus \{0\}, g = \frac{2dudv}{u^2+v^2})$. Then the non-zero Christoffel symbols of M are $\Gamma^u_{uu} = \frac{-2u}{u^2+v^2}$, $\Gamma^v_{vv} = \frac{-2v}{u^2+v^2}$

Then by the geodesic equation: $0 \equiv \ddot{\gamma}^k + \dot{\gamma}^i\dot{\gamma}^j\Gamma^k_{ij} \circ \gamma$, a curve $\gamma(t) = (u(t), v(t))$, γ is a geodesic if and only if it satisfies:

$$\ddot{u} = (\dot{u})^2 \frac{2u}{u^2+v^2} \text{ and } \ddot{v} = (\dot{v})^2 \frac{2v}{u^2+v^2}.$$

Now notice that the curve $\alpha(t) = (\frac{1}{1-t}, 0)$ for $t \in (-\infty, 1)$ satisfies the geodesic equation, so it is a geodesic. Then $\alpha(t)$ is inextendible to $t = 1$ and hence M is geodesically incomplete. This is not surprising as we have the non-compact manifold $\mathbb{R}^2$ with a point removed. However, it is somewhat surprising that we can take the quotient of M by a group of isometries to obtain a compact Lorentzian manifold which is geodesically incomplete. This coset manifold is defined as such. First notice that the metric g is preserved by scalar multiplication:

$$\frac{2(dcu)(dcv)}{(cu)^2 + (cv)^2} = \frac{2c^2du\ dv}{c^2(u^2+v^2)} = \frac{2du\ dv}{u^2+v^2}.$$

So the map $\mu : (u, v) \mapsto (2u, 2v)$ is an isometry of M. The group generated by μ, written $\Gamma := \{\mu^n\}$ has a properly discontinuous action on M and so the quotient $T = M/\Gamma$ is a Lorentz surface. T is topologically a torus as it is identified with the closed annulus of M of radii $1 \leq r \leq 2$ with the boundary points identified and hence it is compact. Since Γ is a group of isometries α projects to a geodesic on T, which is inextendible so T is a compact Lorentzian manifold which it not geodesically complete.

2.4 Submanifolds

Definition 2.19. A manifold M is a *submanifold* of a manifold $\bar{M}$ provided:

1. M is a topological subspace of $\bar{M}$.
2. The inclusion map $i : M \hookrightarrow \bar{M}$ is smooth and at each point its differential di is injective.

Definition 2.20. Let M be a submanifold of a semi-Riemannian manifold $(\bar{M}, g)$, then M is a *semi-Riemannian submanifold* when the pullback metric i^*g is non-degenerate.

If M is a semi-Riemannian submanifold, then each tangent space T_pM must be a non-degenerate subspace of $T_p\bar{M}$. It turns out there is the direct sum decomposition

$$T_p\bar{M} = T_pM \oplus T_pM^{\perp}.$$

Vectors in T_pM are *tangent* to M and vectors in $T_pM^\perp$ are *normal* to M. We can define the following orthogonal projections

$$\begin{aligned} tan &: T_p\bar{M} \to T_pM \\ nor &: T_p\bar{M} \to T_pM^\perp. \end{aligned}$$

We can extend this concepts to vector fields. Given $X \in \Gamma(M, T\bar{M})$, we can apply *tan* and *nor* at each point to define *tan* $X \in \Gamma(M, TM)$ and *nor* $X \in \Gamma(M, TM)^\perp \subset \Gamma(M, T\bar{M})$. Similarly we can extend the above orthogonal projections to

$$\begin{aligned} tan &: \Gamma(M, T\bar{M}) \to \Gamma(M, TM) \\ nor &: \Gamma(M, T\bar{M}) \to \Gamma(M, TM)^\perp. \end{aligned}$$

In order to understand the relationship between geodesics on manifolds and geodesics on submanifolds, we must first discuss the relationship between their respective (Levi-Civita) connections.

Given the Levi-Civita connection on $\bar{M}$, $\nabla^{\bar{M}}$ we can define the *induced connection* from $\bar{M}$ on M, which is a smooth function from $\bar{\nabla}^M : \Gamma(M, TM) \times \Gamma(M, T\bar{M}) \to \Gamma(M, T\bar{M})$. It is defined as such: given $V \in \Gamma(M, TM), X \in \Gamma(M, T\bar{M})$, then there exists local extension of V and X to vector fields on $\bar{M}$, they are written $\bar{V}$ and $\bar{X}$. Finally we define:

$$\bar{\nabla}^M_V X := \nabla^{\bar{M}}_{\bar{V}} \bar{X}.$$

Lemma 2.21 (4.1 in O'Neill (1983))**.** $\bar{\nabla}^M_V X$ *is well defined with respect to choice of local extension.*

Lemma 2.22 (4.3 in O'Neill (1983))**.** *If* $V, W \in \Gamma(M, TM)$*, then*

$$\nabla^M_V W = tan\ \bar{\nabla}^M_V W.$$

Definition 2.23. We can define a function called the *shape tensor* of $M \subset \bar{M}$

$$\begin{aligned} II &: \Gamma(TM) \to \Gamma(TM)^\perp \\ II(V, W) &= nor\ \bar{\nabla}^M_V W. \end{aligned}$$

In particular, this allows us to write $\bar{\nabla}^M = \nabla^M + II(\cdot, \cdot)$. Let Y be a tangent vector field to M, which is defined along some curve $\alpha \subset M$, then we adopt the following notation

$$\dot{Y} = \frac{\nabla Y}{dt}, \qquad Y' = \frac{\nabla Y}{dt}.$$

Proposition 2.24 (4.8 in O'Neill (1983))**.** *Let* Y *be a vector field as defined above. Then*

$$\dot{Y} = Y' + II(\alpha', Y).$$

Corollary 2.25 (4.9 in O'Neill (1983)). *Let α be a curve in $M \subset \bar{M}$. If $\ddot{\alpha}$ is the acceleration of α in $\bar{M}$ and α' is the acceleration of α in M then*

$$\ddot{\alpha} = \alpha'' + II(\alpha', \alpha').$$

Since geodesics are curves such that $\frac{\nabla \gamma}{dt} = 0$, we can then immediately deduce the following corollary.

Corollary 2.26. *A curve α of $M \subset \bar{M}$ is a geodesic of M if and only if $\ddot{\alpha}$ is normal to M at each point.*

Definition 2.27. If a submanifold $M \subset \bar{M}$ has a shape tensor that is identically equal to zero, $II = 0$, then we say M is a *totally geodesic submanifold* of $\bar{M}$.

Proposition 2.28 (4.13 in O'Neill (1983)). *Let M be a submanifold of $\bar{M}$. The following are equivalent.*

1. *M is totally geodesic in $\bar{M}$.*
2. *Every geodesic in M is also a geodesic in $\bar{M}$.*
3. *If $v \in T_p\bar{M}$ is tangent to M, then there is some interval J such that the geodesic of $barM$ starting at p with initial velocity v, $\gamma_v : I \to \bar{M}$ is a geodesic of M, i.e. $\gamma_v|_J$ is a geodesic in M.*
4. *If α is a curve in M and $v \in T_{\alpha(0)}M$, then the parallel transport of v along α is the same in both M and $\bar{M}$.*

Lemma 2.29 (4.14 in O'Neill (1983)). *Let M and N be complete, connected, totally geodesic semi-Riemannian submanifolds of $\bar{M}$. If there is a point $p \in M \cap N$ at which $T_pM = T_pN$, then $N = M$.*

Definition 2.30. A point p of $M \subset \bar{M}$ is *umbillic* if there is a normal vector $z \in T_pM^\perp$ such that

$$II(v, w) = g(v, w)z \quad \text{for all } v, w \in T_pM.$$

Then z is called the *normal curvature vector* of M at p.

A semi-Riemannian submanifold M of $\bar{M}$ is said to be *totally umbillic* if each point of M is umbillic.

2.5 Hypersurfaces

Definition 2.31. Let M be a semi-Riemannian submanifold of $\bar{M}$. If M has codimension 1 in $\bar{M}$ we say that M is a *semi-Riemannian hypersurface* of $\bar{M}$. The co-index of M, the index of all one dimensional normal subspaces to T_pM be either 0 or 1.

We are particularly interested in submanifolds of codimension 1 because the semi-Riemannian manifolds with constant curvature can be described as a hypersurface of semi-Euclidean space.

Definition 2.32. The sign θ of a semi-Riemannian hypersurface M of $\bar{M}$ is:

- $+1$ if the co-index of M is 0. i.e. $\langle z, z\rangle > 0$ for every normal vector $z \neq 0$.
- -1 if the co-index of M is 1. i.e. $\langle z, z\rangle < 0$ for every normal vector $z \neq 0$.

In the Riemannian case every hypersurface will have a sign $+1$ but in the Lorentzian case both signs are possible.

Proposition 2.33 (4.17 in O'Neill (1983))**.** *Let $f : \bar{M} \to \mathbb{R}$ and let c be a value of f. Then $M = f^{-1}(c)$ is a semi-Riemannian hypersurface of $\bar{M}$ if and only if $g(grad\ f, grad\ f)$ is non-zero on M. In this case $g(grad\ f, grad\ f)$ will be either exclusively greater than or less than zero, the sign of $g(grad\ f, grad\ f)$ will be the sign of M. $U = \frac{grad\ f}{|grad\ f|}$ is a unit normal vector field on M.*

When considering hypersurfaces, the shape tensor can be simplified as such.

Definition 2.34. Let U be a unit normal vector field on a semi-Riemannian hypersurface $M \subset \bar{M}$. The $(1,1)$-tensor field S on M such that

$$g(S(V), W) = g(II(V, W), U) \quad \text{for all } V, W \in \Gamma(TM)$$

is called the *shape operator of $M \subset \bar{M}$ derived from U.*

At each point S defines a linear operator $T_pM \to T_pM$.

Lemma 2.35 (4.19 in O'Neill (1983))**.** *If S is the shape operator derived from U, then $S(v) = -\bar{\nabla}_v U$, and at each point the linear operator S on T_pM is self-adjoint.*

Corollary 2.36. *(4.20 in O'Neill (1983)) Let S be the shape operator of a semi-Riemannian hypersurface $M \subset \bar{M}$. If v, w span a non-degenerate tangent plane on M, then*

$$K(v, w) = \bar{K}(v, w) + \theta \frac{g(S, v)g(Sw, w) - g(Sv, w)^2}{g(v, v)g(w, w) - g(v, w)^2},$$

where θ is the sign of $M \subset \bar{M}$.

Lemma 2.37 (4.21 in O'Neill (1983))**.** *A semi-Riemannian hypersurface $M \subset \bar{M}$ is totally umbillic if and only if its shape operator is scalar.*

2.6 Covering spaces and covering maps

In order to classify symmetric spaces and explore (G, X)-manifolds we require some results relating to covering maps and the fundamental group.

Definition 2.38. Let M be a manifold and x, y be two points of M. If $\alpha, \beta : I \to M$ are two paths from x to y then an *endpoint fixing homotopy from α to β* is a continuous map $H : I \times I \to M$ such that for all $s, t \in I$

$$\begin{aligned} H(t,0) &= \alpha(t), \quad & H(t,1) &= \beta(t), \\ H(0,s) &= p, \quad & H(1,s) &= q. \end{aligned}$$

If such a homotopy exists we say that α and β are *homotopic* and write $\alpha \simeq \beta$.

Lemma 2.39 (A.2 in O'Neill (1983))**.** *Endpoint fixing homotopy $\simeq$ is an equivalence relation on paths between two fixed points. The homotopy equivalence class of a curve α is written $[\alpha]$.*

Definition 2.40. Let M be a manifold and x, y, z be points of M. Then if α is a path from x to y and β is a path from y to z then we can define a path from x to z called the *concatenation of α and β* by:

$$(\alpha \# \beta)(t) := \begin{cases} \alpha(2t) & 0 \le t \le \frac{1}{2} \\ \beta(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}.$$

Paths with the same start and end point are called *loops*.

Proposition 2.41 (A.4 in O'Neill (1983))**.** *Write the set of homotopy equivalence classes of loops at x as $\pi_1(M, x)$. This set, equipped with the operation $[\alpha][\beta] = [\alpha \# \beta]$ forms a group called* the fundamental group of M at x*.*

When M is connected the fundamental groups at any two points are isomorphic, so when considering the fundamental group of a connected manifold we can describe the fundamental group of M, written $\pi_1(M)$. If M is connected and has trivial fundamental group then M is said to be *simply connected*.

Definition 2.42. Let M and N be manifolds. A surjective smooth map $k : N \to M$ is called a *covering map* if for each point $x \in M$ there exits some neighbourhood U of x such that each connected component of $k^{-1}(U)$ is diffeomorphic to U. We say that N is a cover of M.

Definition 2.43. A *deck transformation* of a covering map $k : N \to M$ is a diffeomorphism $\phi : N \to N$ such that $k \circ \phi = k$.

Deck transformations must therefore map a fibre $k^{-1}(p)$ to itself.

Lemma 2.44 (A.8 in O'Neill (1983)). *If $k : N \to M$ is a covering map and M is connected then the number of points in $k^{-1}(p)$ is the same for all points $p \in M$. This number is called the* multiplicity *of the covering.*

Definition 2.45. Given manifolds M, N and P and maps $\pi : N \to M$ and $\phi : P \to M$, we say that a function $\tilde{\phi} : N \to M$ is a *lift* of ϕ via π if $\pi \circ \tilde{\phi} = \phi$.

Lemma 2.46 (A.9 in O'Neill (1983)). *Let $k : N \to M$ be a covering map. Let $\alpha : I \to M$ be a smooth. curve, and let $k(q) = \alpha(0)$. Then there is a unique smooth lift $\tilde{\alpha} : J \to \tilde{M}$ of α via k such that $\tilde{\alpha}(0) = q$.*

This means paths can be uniquely lifted to any level of the cover.

Corollary 2.47 (A.10 in O'Neill (1983)). *Let $k : N \to M$ be a covering, and let α and β be fixed point homotopic paths in M. If $\tilde{\alpha}$ and $\tilde{\beta}$ are lifts of α and β by k such that $\tilde{\alpha}(0) = \tilde{\beta}(0)$ then $\tilde{\alpha}$ and $\tilde{\beta}$ are fixed-endpoint homotopic. In particular $\tilde{\alpha}(1) = \tilde{\beta}(1)$.*

Proposition 2.48 (A.11 in O'Neill (1983)). *Let $k : N \to M$ be a covering map and $\phi : P \to M$ be a smooth map. Let $p_0 \in P$ and $q_0 \in N$ such that $\phi(p_0) = k(q_0)$. Then*

1. *if P is connected, there is at most one lift $\tilde{\phi}$ of ϕ by k such that $\tilde{\phi}(p_0) = q_0$,*
2. *if P is simply connected, such a lift exists.*

Proposition 2.49 (1.30 in Hatcher (2002)). *Given a covering space $k : \tilde{M} \to M$, a homotopy $f_t : N \to M$ and a map $\tilde{f}_0 : N \to \tilde{M}$ lifting f_0, then there exists a unique homotopy $\tilde{f}_t : N \to \tilde{M}$ of f_0 that lifts f_t.*

Theorem 2.50 (A.12 in O'Neill (1983)). *Every connected manifold M has a simply connected covering. It is called the* universal cover *and is written $\tilde{M}$.*

A covering $k : \tilde{M} \to M$ is *trivial* if each component of M is evenly covered by k.

Corollary 2.51 (A.14 in O'Neill (1983)). *Every covering of a simply connected manifold is trivial.*

Proposition 2.52 (7.4 in O'Neill (1983)). *If $\pi : \tilde{M} \to M$ is a simply connected covering then its deck transformation group is isomorphic to the fundamental group $\pi_1(M)$ of M.*

Definition 2.53. If N and M are semi-Riemannian manifolds then a *semi-Riemannian covering map* $k : N \to M$ is a covering map which is also a local isometry.

Corollary 2.54 (7.12 in O'Neill (1983)). *If Γ is a properly discontinuous group of isometries of a semi-Riemannian manifold M, then there is a unique way to make M/Γ a semi-Riemannian manifold such that $k : M \to M/\Gamma$ is a semi-Riemannian covering. If M is connected, then the deck transformation group is Γ.*

Proposition 2.55 (7.27 in O'Neill (1983)). *If $k : \tilde{M} \to M$ is a semi-Riemannian covering with $\tilde{M}$ connected and M simply connected, then k is an isometry.*

As we are particularly interested in geodesics, we briefly discuss some results about lifting geodesics to geodesics in the covering space.

Theorem 2.56 (7.28 in O'Neill (1983)). *Let $\phi : M \to N$ be a local isometry with N connected. Let $\gamma : I \to N$ be an arbitrary geodesic and let $p \in N$ be a point such that $\phi(p) = \gamma(0)$. If there exists a lift $\tilde{\gamma} : I \to M$ of γ through ϕ starting at p then ϕ is a semi-Riemannian covering map.*

Corollary 2.57 (7.29 in O'Neill (1983)). *Let $\phi : M \to N$ be a local isometry, with N connected. Then M is complete if and only if N is complete and ϕ is a semi-Riemannian covering map.*

2.7 Product manifolds

Definition 2.58. Let (M, g_M) and (N, g_N) be semi-Riemannian manifolds, we can define the *product manifold* $(M \times N, g_M \oplus g_N)$.

Lemma 2.59 (1.43 in O'Neill (1983)). *The tangent space $T_{(p,q)}(M \times N)$ is the direct sum of it subspaces $T_{(p,q)}M$ and $T_{(p,q)}N$; that is each element of $T_{(p,q)}(M \times N)$ has a unique expression as*

$$x + v, \quad \text{where } x \in T_{(p,q)}M \text{ and } v \in T_{(p,q)}N.$$

Similarly, we can decompose a vector field X of $(M \times N)$ into $X^M + X^N$ where X^M is the lift of a vector field on M and X^N is the lift of a vector field on N.

Proposition 2.60 (3.56 in O'Neill (1983)). *Let X^M, Y^M and V^N, W^N be lifts of vector fields X, Y on M and V, W on N respectively to $(M \times N)$, then*

1. $\nabla_{X^M} Y^M$ *is the lift of* $\nabla^M_X Y \in \Gamma(TM)$.
2. $\nabla_{V^N} W^N$ *is the lift of* $\nabla^N_V W \in \Gamma(TN)$.
3. $\nabla_{V^N} X^M = 0 = \nabla_{X^M} V^N$.

Corollary 2.61 (3.57 in O'Neill (1983)). *A curve $\gamma(t) = (\alpha(t), \beta(t))$ in $M \times N$ is a geodesic if and only if its projections α in M and β in N are both geodesics. Furthermore, $M \times N$ is geodesically complete if and only if both M and N are complete.*

Corollary 2.62 (3.58 in O'Neill (1983)). *On $M \times N$ the curvature tensor $R(X, Y)Z = R^M(X^M, Y^M)Z^M + R^N(X^N, Y^N, Z^N)$ where $R^M(X^M, Y^M)Z^M$ is the lift of the curvature tensor on M and $R^N(X^N, Y^N, Z^N)$ is the lift of the curvature tensor on N.*

2.8 Holonomy

We briefly explain what a holonomy group is. See Chapter 10 in Besse (1987) for a more detailed discussion of the topic. Let (M, g) be a semi-Riemannian manifold, and x be an arbitrary point of M, then for any loop γ, we can parallel transport a tangent vector $v \in T_xM$ around the loop γ, we write this as $P_\gamma : T_xM \to T_xM$. Since P_γ is invertible and linear it is an element of $GL(T_xM)$. We define the *holonomy group* of M at x as such:

$$Hol_x(M) := \{P_\gamma \in GL(T_pM) \mid \gamma \text{ is a loop based at } x\}.$$

In particular, the holonomy group at each point is isomorphic as, given any loop γ at x we can define a loop at y by $\eta \circ \gamma \circ \eta^{-1}$ where η is a path from x to y, so we can write $Hol(M)$ for the holonomy group of an arbitrary point. If we consider only loops which are homotopic to the identity we have the *restricted holonomy group*, written $Hol^0(M)$. When M is simply connected we have that $Hol(M) = Hol^0(M)$. There is a canonical homomorphism $\pi_1(M) \to Hol(M)/Hol^0(M)$ which sends equivalence classes of a loop $[\gamma]$ to the equivalence class of parallel transport around that curve $[P_\gamma]$.

Chapter 3

Symmetric spaces

This chapter discusses symmetric spaces, which are an important class of geodesically complete manifolds. Symmetric spaces are particularly nice manifolds to consider as they can be described in a number of ways. Symmetric spaces are defined to be manifolds with a symmetry at each point, but they can also be described as a homogeneous space G/H equipped with a G-invariant metric tensor, or even described purely algebraically by symmetric triples. We begin this chapter with a general discussion of symmetric spaces, showing they are geodesically complete and homogeneous. Next we discuss some specific semi-Riemannian symmetric spaces, flat semi-Euclidean space, the constant sectional curvature hyperquadrics and finally Lorentzian Cahen-Wallach spaces, which have covariantly constant curvature. For each of these examples we calculate the curvature tensors, geodesics, isometry groups and the symmetry at a point. We then briefly discuss some results which describe symmetric spaces in terms of homogeneous spaces before showing that simply connected symmetric spaces are in a one to one correspondence with symmetric triples, which are algebraic objects. Throughout this section we are able to prove Theorem 3.59, which states that a manifold is locally symmetric if and only if it is locally isometric to a symmetric space. Finally, we discuss the results in Cahen & Wallach (1970) which show that every Lorentzian symmetric space is isometric to the product of a Riemannian symmetric space and one of the four Lorentzian symmetric spaces discussed earlier. These two results allow us to locally describe each locally symmetric Lorentzian manifold, which is required for Chapter 4.

3.1 Symmetric spaces

Definition 3.1 (Symmetry)**.** Let (M, g) be a semi-Riemannian manifold. A *symmetry* s_x at $x \in M$ is an isometry with the properties

$$s_x(x) = x, \quad (ds_x)_x = -I,$$

where I is the identity map of T_xM.

Definition 3.2 (Symmetric space)**.** A *symmetric space* is a semi-Riemannian manifold (M, g) where for all points x, there is a symmetry s_x.

Lemma 3.3. *A semi-Riemannian symmetric space is geodesically complete.*

Proof. Let M be a symmetric manifold, γ be some geodesic defined on $[0, b)$ and choose some $c \in (\frac{b}{2}, b)$. Let $s = s_{\gamma(c)}$ be the symmetry at $\gamma(c)$. So then, since s is an isometry, $s \circ \gamma : [0, b) \to M$ is a geodesic with $s \circ \gamma(c) = \gamma(c)$ and $(s \circ \gamma)'(c) = -\gamma'(c)$. We can then affinely reparameterise the geodesic as such:

$$\begin{aligned} \eta : & \quad (0, b] \to M, \\ \eta(t) := & \quad s \circ \gamma(b - t), \\ \eta'(t) = & \quad -s \circ \gamma'(b - t). \end{aligned}$$

Then $\eta(b-c) = \gamma(c)$ and $\eta'(b-c) = \gamma'(c)$, so by the uniqueness of geodesics (Lemma 2.16) we have that η is an extension of γ. Since we can arbitrarily extend any geodesic using this method M is geodesically complete. $\square$

This argument uses a useful property of symmetries and geodesics, namely that at some point $\gamma(s)$,

$$s_{\gamma(s)}\gamma(s + \epsilon) = \gamma(s - \epsilon),$$

and so in general

$$s_{\gamma(s)}\gamma(t) = s_{\gamma(s)}\gamma(s + (t - s)) = \gamma(s - (t - s)) = \gamma(2s - t). \tag{3.1}$$

Definition 3.4 (Transvection)**.** An isometry $\phi : M \to M$ is a *transvection* along a geodesic γ if:

1. ϕ translates along γ; i.e. $\phi(\gamma(s)) = \gamma(s + c)$ for any $s \in \mathbb{R}$ and some $c \in \mathbb{R}$.
2. $d\phi$ parallel translates along γ; i.e. if $x \in T_{\gamma(s)}M$, then $d\phi(x) \in T_{\gamma(s+c)}M$ is the parallel translation of x along γ.

Lemma 3.5 (8.30 in O'Neill (1983))**.** *Let γ be a geodesic in a symmetric space and let $s_{\gamma(t)}$ be the symmetry at $\gamma(t)$. Then for any c, the isometry $s_{\gamma(\frac{c}{2})}s_{\gamma(0)}$ is a transvection along γ that translates $\gamma(t)$ by c.*

Proof. First, notice Equation (3.1) shows that $s_{\gamma(\frac{c}{2})} \circ s_{\gamma(0)}$ translates along γ:

$$s_{\gamma(\frac{c}{2})} \circ s_{\gamma(0)}\gamma(t) = s_{\gamma(\frac{c}{2})}\gamma(-t) = s_{\gamma(\frac{c}{2})}\gamma(\frac{c}{2} - (t + \frac{c}{2})) = \gamma(t + c). \tag{3.2}$$

Now to show that transvections act by parallel transport. Let X be a parallel vector field on γ. Since $s_{\gamma(t)}$ is an isometry, $ds_{\gamma(t)}X$ is a parallel vector field on $s_{\gamma(t)}\gamma$, which is a reparametrisation of γ. If $x \in T_{\gamma(t)}M$, then let y be the parallel transport along γ of x to $T_{\gamma(0)}M$. Then $(ds_{\gamma(0)})_{\gamma(t)}(x)$ will be parallel transported along γ to $(ds_{\gamma(0)})_{\gamma(0)}(y) = -y$ at $\gamma(0)$, and it will also be parallel transported along γ to some vector z at $\gamma(\frac{c}{2})$. So therefore, $ds_{\gamma(\frac{c}{2})}ds(\gamma(0))(x)$ will be parallel to $ds_{\gamma(\frac{c}{2})}z = -z$. So $ds_{\gamma(\frac{c}{2})}ds(\gamma(0))(x)$ is parallel transported along γ to $-z$ in $T_{\gamma(\frac{c}{2})}M$, which is parallel transported to y in $T_{\gamma(0)}M$, which is the parallel transport of x in $T_{\gamma(t)}M$. □

It follows that all the transvections in a symmetric space can be given by the composition of two symmetries. We define the *transvection group*, written $G(M)$ as the group generated by the elements $s_x \circ s_y$ for all points x, y in M.

Corollary 3.6. *Let M be a connected symmetric space, then the transvection group $G(M)$ acts transitively on M.*

Proof. As M is connected, any two of its points can be connected by a piecewise geodesic curve by Lemma 2.17. Let p, q be two points of M connected by the piecewise geodesic curve $\gamma_1 \# \dots \# \gamma_n$, then for each γ_i, there is a transvection ϕ_i which maps $\gamma_i(0)$ to $\gamma_i(1) = \gamma_{i+1}(0)$, so the composition $\phi_1 \circ \dots \circ \phi_{n-1}$ maps p to q. □

Definition 3.7 (Locally symmetric space)**.** A *locally symmetric space* is a semi-Riemannian manifold M with parallel curvature tensor, i.e., with $\nabla R = 0$.

Lemma 3.8. *Let S be a symmetric space, then S is locally symmetric.*

Proof. Our proof follows Theorem 5 in Eschenburg (2012). Let p be an arbitrary point of S. By applying Proposition 2.4 to the isometry s_p from S to itself we see that $ds_p(\nabla_X Y) = \nabla_{ds_p(X)} ds_p(Y)$ for all $X, Y \in \Gamma(TS)$. As R is a tensor we evaluate it at a single point, p. Now let

$$w = \nabla_{v_1}(R(v_2, v_3)v_4)$$

for $v_i \in T_pM$. Then by applying ds_p,

$$\begin{aligned} (ds_p)_p w &= (ds_p)_p \nabla_{v_1}(R(v_2, v_3)v_4) \\ -w &= -I(\nabla_{v_1}(R(v_2, v_3)v_4)) = \nabla_{-v_1}(-I(R(v_2, v_3)v_4)). \end{aligned}$$

Then expanding and simplifying $-I(R(v_2, v_3)v_4)$,

$$\begin{aligned} &-I(R(v_2, v_3)v_4) \\ &= -I(\nabla_{v_2}\nabla_{v_3}v_4 - \nabla v_3 \nabla_{v_2}v_4 - \nabla_{[v_2,v_3]}v_4) \\ &= \nabla_{-v_2} - I(\nabla_{v_3}v_4) - \nabla_{-v_3} - I(\nabla_{v_2}v_4) - \nabla_{-[v_2,v_3]} - v_4 \\ &= \nabla_{-v_2}(-\nabla_{-v_3} - v_4) - \nabla_{-v_3}(-\nabla_{-v_2}(-v_4)) - \nabla_{-[v_2,v_3]} - v_4 \\ &= \nabla_{v_2}\nabla_{v_3}v_4 - \nabla_{v_3}\nabla_{v_2}v_4 - \nabla_{[v_2,v_3]}v_4 \\ &= R(v_2, v_3)v_4. \end{aligned}$$

So, $w = -w$ and hence $\nabla R = w = 0$. □

Since all symmetric spaces are locally symmetric, it is natural to ask the following question: *Given a locally symmetric space M, can we relate M to some globally symmetric space S?* This question will be answered by Theorem 3.59, which states that M is locally symmetric if and only if it is locally isometric to some (unique) simply connected symmetric space S.

The following results provide tools to extend linear isometries between (locally) symmetric spaces to (local) isometries.

Theorem 3.9 (8.14 in O'Neill (1983)). *Let M and N be locally symmetric semi-Riemannian manifolds, and let $L : T_pM \to T_qN$ be a linear isometry that preserves curvature. Then if $\mathcal{U}$ is a sufficiently small neighborhood of p, there is a unique isometry ϕ of $\mathcal{U}$ onto a normal neighborhood $\mathcal{V}$ of q such that $d\phi_p = L$.*

Theorem 3.10 (8.17 in O'Neill (1983)). *Let M and N be complete, connected, locally symmetric semi-Riemannian manifolds, with M simply connected. If $L : T_pM \to T_qN$ is a linear isometry that preserves curvature, then there is a unique semi-Riemannian covering map $\phi : M \to N$ such that $d\phi_p = L$.*

Corollary 3.11 (8.21 in O'Neill (1983)). *A complete, simply connected, locally symmetric semi-Riemannian manifold is symmetric.*

Proof. Given any point $p \in M$, then the linear isometry $-I$ of T_pM preserves the Riemann curvature tensor by the calculation shown above in Lemma 3.8. So then by Theorem 3.10, there exists a unique semi-Riemannian covering map $s_p : M \to M$ with $(ds_p)_p = -I$. Since M is simply connected, all covering maps must be trivial by Corollary 2.51 and hence s_p is an isometry. $\square$

In the next sections we will give some examples of symmetric spaces, describe some properties relating to geodesics and calculate their isometry groups and isotropy subgroups. We will discuss semi-Euclidean spaces, hyperquadrics and Cahen-Wallach spaces. Minkowski space is the Lorentzian semi-Euclidean space, while Lorentzian hyperquadrics are either de Sitter space or anti de Sitter space. This chapter will then conclude by showing that any simply connected Lorentzian symmetric space is isometric to the product of either Minkowski space, (the universal cover of) de Sitter space, anti de Sitter space or a Cahen-Wallach space and a Riemannian symmetric space.

3.2 Flat semi-Riemannian model spaces

The first example of a symmetric space, is the simplest possible semi-Riemannian manifold, $\mathbb{R}^n$ equipped with an arbitrary index generalisation of the Euclidean metric.

3.2.1 Vector spaces as manifolds

Before discussing semi-Euclidean space, we first make a brief detour to arbitrary vector spaces. Let V be a real n-dimensional vector space. Then there is a unique way to make V a manifold such that every linear isomorphism $\xi : V \to \mathbb{R}^n$ is a coordinate system. We will introduce some conventions that will be used throughout this thesis.

Definition 3.12. If $p, v \in V$, then let $v_p \in T_pV$ be the initial velocity $\alpha'(0)$ of the curve $\alpha(t) = p + tv$.

Lemma 3.13 (1.46 in O'Neill (1983)). *If $x^1, \ldots, x^n$ is a linear coordinate system on V, then*

$$v_p = x^i(v)\partial_i|_p.$$

Note v_p is the tangent vector at p with the same coordinates as $v \in V$. Therefore:

1. for a fixed $p \in V$ the function $v \mapsto v_p$ is a linear isomorphism $V \approx T_pV$.
2. For $p, q \in V$ the function $v_p \mapsto v_q$ is a linear isomorphism $T_pV \approx T_qV$.

These canonical isomorphisms identify v with v_p.

The *position vector field* $P \in \Gamma(TM)$ assigns each $p \in V$ to the tangent vector $p_p \in T_pV$.

Remark 3.14 (Scalar product spaces as manifolds). Let V be an n-dimensional vector space equipped with inner product $\langle \cdot, \cdot \rangle$. Then V can be equipped with a natural semi-Riemannian metric tensor g defined as such

$$g(v_p, w_p) := \langle v, w \rangle.$$

This metric tensor will be used when discussing manifolds described as submanifolds of vector spaces, in particular, semi-Euclidean spaces and hyperquadrics.

3.2.2 Semi-Euclidean Space

Definition 3.15 (Semi-Euclidean space). Semi-Euclidean space, denoted $\mathbb{R}^n_m$ is the space $(\mathbb{R}^n, g)$ where $\mathbb{R}^n$ is the vector space spanned by the orthonormal basis $\{e_1, \ldots, e_n\}$ and g is the standard semi-Riemannian metric of index m;

$$\begin{gathered}\text{for } v = v^i e_i \text{ and } w = w^i e_i \\ g(v, w) = -\Sigma_{i=m}^n v^i w^i + \Sigma_{j=m+1}^n v^j w^j.\end{gathered}$$

In this standard basis $g_{ij} = \pm\delta_{ij}$, and so the Levi-Civita connection is simply the directional derivative, i.e. $\nabla = \partial$.

When $m = 1$, $\mathbb{R}^n_1$ is a Lorentzian manifold called *Minkowski space.*

Lemma 3.16. *Semi-Euclidean space is flat.*

Proof. The Christoffel symbols Γ^k_{ij} are equal to 0, so it immediately follows that $R \equiv 0$. □

Corollary 3.17. *Semi-Euclidean Space is locally symmetric.*

Lemma 3.18. *The geodesics of semi-Euclidean space are linear parameterisations of straight lines.*

Proof. Since the Christoffel symbols are all 0, the geodesic equations become:

$$0 = \ddot{\gamma}^k.$$

Which we integrate twice to see:

$$a^k t + b^k = \gamma^k.$$

So the geodesics in Semi-Euclidean space are straight lines. □

In particular, we can extend these geodesics indefinitely by increasing t, so Semi-Euclidean space is geodesically complete.

Lemma 3.19. *Semi-Euclidean space is symmetric.*

Proof. We will show that semi-Euclidean space is symmetric by finding the symmetry at an arbitrary point x. Let x be a point of $\mathbb{R}^n_m$ then the symmetry at x is defined as such:

$$\begin{aligned} s_x(x+y) :=& \quad x - y, \quad \text{or equivalently,} \\ s_x(p) =& \quad s_x(x + (p - x)) = 2x - p. \end{aligned}$$

First, notice $s_x(x) = x$. Additionally, $(ds_x)_p = -I$ at each point so in particular, $(ds_x)_x = -I$. Since metrics are bilinear, s_x is an isometry and thus s_x is the symmetry at x. □

Now we will describe the full isometry and isotropy groups of semi-Euclidean space. This is of particular importance as the tangent space at a point of any semi-Riemannian manifold is isometric to semi-Euclidean space of the same dimension and index.

3.2.3 Indefinite orthogonal groups

In Euclidean geometry, the group of linear isometries of $\mathbb{R}^n$ is of particular importance. It is called the *orthogonal group* of dimension n and is written $O(n)$. In particular, a matrix A is orthogonal if and only if its inverse is equal to its transpose; $A^\top A = AA^\top = I$. This idea can be generalised to inner product of arbitrary index.

Consider $\mathbb{R}^n_m$ equipped with the standard basis of $\mathbb{R}^n$, then the inner product can be represented by the diagonal matrix ε with diagonal entries $\varepsilon_1 = \ldots = \varepsilon_m = -1$, $\varepsilon_{m+1} = \ldots = \varepsilon_n = 1$.

The group of transformations which preserve the matrix ε under conjugation, i.e.

$$A^\top \varepsilon A = \varepsilon$$

is called the *indefinite orthogonal group* of index m and dimension n, written $O(m, n-m)$.

Lemma 3.20. *[9.8 in O'Neill (1983)] Let A be a $n \times n$ matrix. The following are equivalent:*

1. $\langle Av, Aw\rangle = \langle v, w\rangle$ *for all* $v, w \in \mathbb{R}^n$.
2. $A \in O(m, n-m)$.
3. *A maps orthonormal bases of* $\mathbb{R}^n_m$ *to orthonormal bases.*

Proof. (1) $\iff$ (2).

$$\begin{aligned}
\langle Av, Aw\rangle &= \langle v, w\rangle \\
\langle \varepsilon Av, Aw\rangle &= \langle \varepsilon v, w\rangle \text{ (by the definition of } \varepsilon) \\
\langle A^\top \varepsilon Av, w\rangle &= \langle \varepsilon v, w\rangle \text{ (the transpose of a matrix is its adjoint for the dot product.)} \\
A^\top \varepsilon Av &= \varepsilon v \text{ (the above holds for all } w) \\
A^\top \varepsilon A &= \varepsilon \text{ (the above holds for all } v) \\
\varepsilon^{-1} A^\top \varepsilon &= A^{-1} \\
\varepsilon A^\top \varepsilon &= A^{-1} \ (\varepsilon^2 = I).
\end{aligned}$$

(1) $\Rightarrow$ (3). Let $e_1, \ldots, e_n$ be an orthonormal basis of $\mathbb{R}^n_m$, then $\langle e_i, e_j\rangle = \varepsilon_i \delta_{ij}$. Since A preserves the inner product $\langle Ae_i, Ae_j\rangle = \langle e_i, e_j\rangle = \varepsilon_i\delta_{ij}$ so $Ae_1, \ldots, Ae_n$ is an orthonormal basis of $\mathbb{R}^n_m$.
(3) $\Rightarrow$ (1). Now suppose that $e_1, \ldots, e_n$ and $Ae_1, \ldots, Ae_n$ are two orthonormal bases of $\mathbb{R}^n_m$, then it immediately follows that $\langle e_i, e_i\rangle = \langle Ae_i, Ae_i\rangle$, and since A is a matrix, $\langle Ax, Ay\rangle = \langle x, y\rangle$. □

The indefinite orthogonal group describes all the linear isometries of $\mathbb{R}^n_m$, but the full isometry group can also contain translations.

Proposition 3.21 (9.10 in O'Neill (1983))**.** *The isometry group of $\mathbb{R}^n_m$ is $O(m, n-m) \ltimes \mathbb{R}^n_m$. We can write each isometry as a pair (A, c) such that $A \in O(m, n-m)$ and $c \in \mathbb{R}^n_m$ with action $(A, c)x = Ax + c$. The composition of elements is as such*

$$(B, d) \cdot (A, c) = (BA, Bc + d).$$

Proof. Let ϕ be an isometry of $\mathbb{R}^n_m$. First, we will show that if $\phi(0) = 0$, then $\phi \in O(m, n-m)$. Since ϕ is an isometry, $d\phi_0 : T_0\mathbb{R}^n_m \to T_0\mathbb{R}^n_m$ is a linear isometry by Lemma 2.6. In particular, since $T_0\mathbb{R}^n_m$ is identified with $\mathbb{R}^n_m$, $d\phi_0$ is identified with some $A \in O(m, n-m)$. Then since A is linear $A = dA_0$, so $A(0) = 0$ and by construction $dA_0 = d\phi_0$ so A and ϕ are equal at 0 and have the same differential at 0, thus by Proposition 2.7, $A = \phi$.

Now suppose $\phi(0) = p$. Then the map $\psi(x) = \phi(x) - p$ is an isometry such that $\psi(0) = \phi(0) - p = 0$. Then by the previous argument $\psi = A$ for some $A \in O(m, n-m)$, and therefore $\phi(x) = Ax + p$. Now we can see that this is unique: if $(A, c) = (B, d)$ then

$$c = (A, c)(0) = (B, d)(0) = d.$$

Therefore, $Ae_i = Be_i$ for each basis vector, so $A = B$. □

Lemma 3.22. *The isotropy subgroup of $Iso(\mathbb{R}^n_m)$ is $O(m, n-m)$.*

Proof. First, notice that the isotropy subgroup of $Iso(\mathbb{R}^n_m)$ at 0 is $O(m, n-m)$. This is immediate from the definition of the isometry group action. We can then construct the isotropy group at some point p by conjugating with translations. If $(A, 0)$ fixes the point 0 then,

$$(I, p) \cdot (A, 0) \cdot (I, -p) = (A, p - Ap)$$

is an isometry that fixes the point p. □

Lemma 3.23. *The transvection subgroup $Iso(\mathbb{R}^n_m)$ is the group of translations $\mathbb{R}^n$.*

Proof. By Lemma 3.5 we know that transvections are of the form $s_{\gamma(\frac{c}{2})} \circ s_{\gamma(0)}$, which act on a point p as such

$$s_{\gamma(\frac{c}{2})} \circ s_{\gamma(0)}(p) = s_{\gamma(\frac{c}{2})}(2\gamma(0) - x) = 2\gamma(\frac{c}{2}) - (2\gamma(0) - x).$$

Since geodesics have the form $\gamma = at + b$ this simplifies to

$$2\gamma(\frac{c}{2}) - (2\gamma(0) - x) = (at + 2b) - 2b + x = x + at.$$

So the group of transvections in $\mathbb{R}^n_m$ are the translations. □

3.3 Hyperquadrics

Symmetric spaces with constant curvature can be described as hypersurfaces of semi-Euclidean space. In this section we will discuss semi-Riemannian hyperquadrics of any index, as we will consider both the Lorentzian and Riemannian cases.

Consider $\mathbb{R}^{n+1}_m$ and define the following quadratic form $q(v) = \langle v, v\rangle$:

$$q(v) = -\Sigma_{i=1}^m (v^i)^2 + \Sigma_{j=m+1}^{n+1} (v^j)^2.$$

If P is the position vector field, we induce a metric tensor $\bar{g}(P(u), P(v)) := \langle u, v\rangle$ on $\mathbb{R}^{n+1}_m$. In particular, $q(u) = \bar{g}(P(u), P(u))$. So grad $q = 2P$ since for all V,

$$\bar{g}(\text{grad } q, V) = Vq = V\bar{g}(P, P) = 2\bar{g}(\nabla_V P, P) = 2\bar{g}(V, P).$$

Therefore, $\bar{g}(\text{grad } q, \text{grad } q) = 4q$. Thus, it follows from Proposition 2.33 that the level sets $q^{-1}(\theta r^2)$ are semi-Riemannian hypersurfaces with unit normal $U = \frac{P}{r}$ and sign θ.

3.3.1 Definition and curvature

Definition 3.24. Let $n \geq 2$ and $0 \leq m \leq n$. Then:

The *pseudosphere* of radius $r > 0$ in $\mathbb{R}^{n+1}_m$ is the hyperquadric

$$S^n_m(r) = q^{-1}(r^2) = \{p \in \mathbb{R}^{n+1}_m : q(p) = r^2\}$$

with dimension n and index m. If $m = 0$ this is the Riemannian sphere, when $m = 1$ it is called *de Sitter Space*.

Pseudohyperbolic space of radius $r > 0$ in $\mathbb{R}^{n+1}_{m+1}$ is the hyperquadric

$$H^n_m(r) = q^{-1}(-r^2) = \{p \in \mathbb{R}^{n+1}_{m+1} : q(p) = -r^2\}$$

with dimension n and index m. If $m = 0$ this is hyperbolic space. When $m = 1$ it is called *Anti de Sitter Space*.

In particular, there are the unit hyperquadrics, $S^n_m := S^n_m(1)$ and $H^n_m := H^n_m(1)$. For any r, any hyperquadric $S^n_m(r)$ and $H^n_m(r)$ is homothetic to the unit hyperquadrics via the map $x \mapsto \frac{x}{r}$. Thus, any discussion can be simplified to a discussion of unit hyperquadrics.

Lemma 3.25 (4.24 in O'Neill (1983))**.** *The mapping $\sigma : \mathbb{R}^{n+1}_m \to \mathbb{R}^{n+1}_{n-m+1}$ given by*

$$\sigma(p_1, \ldots, p_{n+1}) = (p_{m+1}, \ldots, p_{n+1}, p_1, \ldots, p_m)$$

is an anti-isometry that carries each S^n_m anti isometrically onto H^n_{n-m} and vice versa.

Proof. Since σ is a linear isomorphism and

$$\langle \sigma(p), \sigma(q) \rangle_{\mathbb{R}^{n+1}_m} = -\Sigma_{m+1}^{n+1}(p_j)^2 + \Sigma_1^m (p_i)^2 = -\langle p, p \rangle_{\mathbb{R}^{n+1}_m} \tag{3.3}$$

it follows by an equivalent argument to the proof of Lemma 2.3 that σ is an anti isometry. Equation (3.3) shows that σ maps S^n_m to H^n_{n-m} and vice versa, thus $\sigma|_{S^n_m}$ is a diffeomorphism and hence and anti-isometry. $\square$

Lemma 3.26 (4.25 in O'Neill (1983)). *S^n_m is diffeomorphic to $\mathbb{R}^m \times S^{n-m}$ and H^n_m is diffeomorphic to $S^m \times \mathbb{R}^{n-m}$.*

Proof. First, consider S^n_m. Let $x \in \mathbb{R}^m$ and $p \in S^{m-n}$, define the map

$$\phi(x, p) = (x, (1 + |x|^2)^{\frac{1}{2}} p) \in \mathbb{R}^m_m \times \mathbb{R}^{n+1-m} = \mathbb{R}^{n+1}_m.$$

Now notice that

$$\langle \phi(x, p), \phi(x, p) \rangle = -|x|^2 + (1 + |x|^2) = 1.$$

So ϕ maps $\mathbb{R}^m \times S^{n-m}$ into S^m_n. ϕ is evidently smooth and has an inverse map $(x, q) \mapsto (x, (1 + |x|^2)^{\frac{-1}{2}} q)$, so it is a diffeomorphism. This map can be composed with σ from Lemma 3.25 to obtain a diffeomorphism from $S^m \times \mathbb{R}^{n-m}$ to H^n_m. $\square$

It follows from this lemma that S^n_1 is simply connected for all $n \geq 3$. More generally, S^n_m is simply connected for all $n - m \geq 2$.

Lemma 3.27 (4.27 in O'Neill (1983)). *The hyperquadric $Q = q^{-1}(\theta) \subset R^{n+1}_m$ of sign θ is totally umbillic, with shape operator $S = -I$ derived from the outward unit normal P.*

Proof. If $V \in \Gamma(TQ)$, then $S(V) = -\bar{\nabla}_V(P) = -V$. $\square$

Proposition 3.28 (4.29 in O'Neill (1983)). *Let $n \geq 2$ and $0 \leq m \leq m$.*

1. *The pseudosphere S^n_m has constant positive curvature $K = 1$.*

2. *The pseudohyperbolic space H^n_m has constant negative curvature $K = -1$.*

Proof. Let X, Y be coordinate vectors so that $[X, Y] = 0$. Using the fact that $\nabla_X Y = \bar{\nabla}_X Y - II(X, Y)$ and that the scalar curvature of semi-Euclidean space is equal to 0, we calculate:

$$\begin{aligned} 0 &= \langle \bar{R}(X, Y)X, Y \rangle = \langle \bar{\nabla}_X \bar{\nabla}_Y X - \bar{\nabla}_Y \bar{\nabla}_X X \rangle \\ &= \langle \bar{\nabla}_X \nabla_Y X - \bar{\nabla}_X (II(X, Y)) - \bar{\nabla}_Y \nabla_X X + \bar{\nabla}_Y (II(X, X)), Y \rangle \\ &= \langle \bar{\nabla}_X \nabla_Y X - \bar{\nabla}_Y \nabla_X X, Y \rangle + \langle -\bar{\nabla}_X (II(X, Y)) + \bar{\nabla}_Y (II(X, X)), Y \rangle. \end{aligned}$$

Since Y is tangent to the hypersurface, $\langle \bar{\nabla}_A B, Y\rangle = \langle \nabla_A B, Y\rangle$. Additionally, we use the compatibility of the Levi-Civita connection with the metric to simplify the rightmost inner product to see:

$$\begin{aligned}\langle \bar{R}(X,Y)X,Y\rangle &= \langle \nabla_X\nabla_Y X - \nabla_Y\nabla_X X, Y\rangle + -\langle II(X,Y),Y\rangle \\ &\quad + \langle II(X,Y), \bar{\nabla}_X Y\rangle + Y\langle II(X,X),Y\rangle - \langle II(X,X), \bar{\nabla}_Y Y\rangle.\end{aligned}$$

Similarly, since $II(A,B)$ is normal to the hypersurface we have:

$$\langle \bar{R}(X,Y)X,Y\rangle = \langle R(X,Y)X,Y\rangle + \langle II(X,Y), II(X,Y)\rangle - \langle II(X,X), II(Y,Y)\rangle,$$

i.e.

$$\langle R(X,Y)X,Y\rangle = \langle II(X,X), II(Y,Y)\rangle - \langle II(X,Y), II(X,Y)\rangle.$$

Now utilising the fact that $II(V,W) = \langle S(V),W\rangle U = -\langle V,W\rangle \frac{P}{r}$, where S is the shape operator, U is the unit normal and P is the position vector field, the above equation becomes

$$\begin{aligned}\langle R(X,Y)X,Y\rangle &= \langle\langle X,X\rangle\frac{P}{r}, \langle Y,Y\rangle\frac{P}{r}\rangle - \langle\langle X,Y\rangle\frac{P}{r}, \langle X,Y\rangle\frac{P}{r}\rangle \\ &= \frac{\langle P,P\rangle}{r^2}(\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2).\end{aligned}$$

Hence the sectional curvature K is equal to $\frac{\theta}{r^2}$, where θ is the sign of the hyperquadric, therefore S^n_m has constant positive curvature $K = 1$ and H^n_m has constant negative curvature $K = -1$. □

3.3.2 Geodesics

Proposition 3.29. *Let γ be a non-constant unit velocity or lightlike geodesic of $S^n_m \subset \mathbb{R}^{n+1}_m$. Then γ has one of the following forms:*

1. $\gamma(t) = \sin(t)\dot{\gamma}(0) + \cos(t)\gamma(0)$,
2. $\gamma(t) = \sinh(t)\dot{\gamma}(0) + \cosh(t)\gamma(0)$,
3. $\gamma(t) = \gamma(0) + \dot{\gamma}(0)t$.

Proof. Choose an arbitrary point $\gamma(0) \in S^n_m$ and velocity $\dot{\gamma}(0) \in T_{\gamma(0)}S^n_m$ as initial conditions for some geodesic γ.

Writing γ as a curve in $\mathbb{R}^{n+1}_m$ it follows from Corollary 2.26 that γ is a geodesic of S^n_m if and only if $pr^{\perp}_{S^n_m}(\ddot{\gamma}) = 0$. Then there are three possible cases.

1. If $\langle \dot\gamma(t), \dot\gamma(t)\rangle > 0$, then consider the curve
$$\alpha(t) = \sin(t)\dot\gamma(0) + \cos(t)\gamma(0).$$
Notice that this curve has $\alpha(0) = \gamma(0)$ and $\dot\alpha(0) = \dot\gamma(0)$ and that
$$\ddot\alpha(t) = -\sin(t)\dot\gamma(0) - \cos(t)\gamma(0) = -\alpha(t).$$
So $\ddot\alpha$ is normal to S^n_1 and is hence a geodesic by Corollary 2.26 and therefore by Lemma 2.16 $\gamma = \alpha$.

2. If $\langle \dot\gamma(t), \dot\gamma(t)\rangle < 0$, then consider the curve
$$\alpha(t) = \sinh(t)\dot\gamma(0) + \cosh(t)\gamma(0).$$
Then $\alpha(0) = \gamma(0)$ and $\dot\alpha(0) = \dot\gamma(0)$ Finally:
$$\ddot\alpha(t) = \sinh(t)\dot\gamma(0) + \cosh(t)\gamma(0) = \alpha(t).$$
So $\ddot\alpha$ is normal to S^n_m, so α is a geodesic of S^n_m by Corollary 2.26 and therefore by Lemma 2.16 $\gamma = \alpha$.

3. If $\langle \dot\gamma(t), \dot\gamma(t)\rangle = 0$, then consider the curve
$$\alpha(t) = \gamma(0) + \dot\gamma(0)t.$$
Then $\alpha(0) = \gamma(0)$ and $\dot\alpha(0) = \dot\gamma(0)$ Finally:
$$\ddot\alpha(t) = 0,$$
so α is a geodesic and by Lemma 2.16 $\alpha = \gamma$.

Notice that in each case these geodesics can be extended indefinitely and hence S^n_m is geodesically complete. □

Remark 3.30. Notice that each geodesic remains in a plane, P, of $\mathbb{R}^n_m$ containing $0, \gamma(0)$ and $\dot\gamma(0)$. The causal characteristic of γ will determine the signature of the inner product restricted to P, and by the parametrisations of γ given above we can describe the intersection of P with S^n_m.

1. When $\langle \dot\gamma(t), \dot\gamma(t)\rangle > 0$, $\langle \cdot, \cdot\rangle|_P$ is positive definite and $P \cap S^n_m$ is diffeomorphic to a circle.

2. When $\langle \dot\gamma(t), \dot\gamma(t)\rangle < 0$, $\langle \cdot, \cdot\rangle|_P$ is non-degenerate, with index 1 and $P \cap S^n_m$ is diffeomorphic to a hyperbola of two branches.

3. When $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle = 0$, $\langle \cdot, \cdot \rangle|_P$ is degenerate with a 1-dimensional nullspace and $P \cap S^n_m$ is diffeomorphic to two parallel straight lines in P.

From the above proposition and the anti-isometry of Lemma 3.25 we see that the geodesics of pseudohyperbolic space will correspond to the geodesics of pseudospheres with opposite index. In H^n_m the timelike geodesics are ellipses and the spacelike geodesics are branches of hyperbolae.

We can now discuss the geodesic connectedness of hyperquadrics.

Proposition 3.31 (5.38 in O'Neill (1983)). *Let p and q be distinct non-antipodal points of S^n_m. Then:*

1. *If $\langle p, q \rangle > 1$ then p and q lie on a unique geodesic, which is timelike and one-to-one.*
2. *If $\langle p, q \rangle = 1$ then p and q lie on a unique geodesic, which is also a null geodesic of $\mathbb{R}^{n+1}_m$.*
3. *If $-1 < \langle p, q \rangle < 1$, then p and q lie on a unique geodesic, which is spacelike and periodic.*
4. *If $\langle p, q \rangle \leq -1$, then there is no geodesic joining p and q.*

Proof. Since p and q are non-antipodal, they lie on a unique plane $P \subset \mathbb{R}^{n+1}_m$ containing p, q and 0. From the proof of Proposition 3.29 it is known that each geodesic remains in a plane of $\mathbb{R}^n_m$, so the only possible geodesic joining p and q must be a parameterisation of the one dimensional manifold $P \cap S^n_m$. Now consider the three possible cases for the plane P.

1. P is positive definite, so it has an orthonormal basis e_1, e_1. Then $P \cap S^n_m$ is a circle of radius 1. So $-1 < \langle p, q \rangle < 1$ and are connected by a periodic spacelike geodesic.
2. P is non-degenerate with index one, so it has an orthonormal basis e_0, e_1 with e_0 being timelike. Then $P \cap S^n_m$ is a hyperbola of two branches. Then p and q are on the same branch if and only if $\langle p, q \rangle > 1$ and are on opposite branches if and only if $\langle p, q \rangle < -1$.
3. P is degenerate. Then $P \cap S^n_m$ consists of two parallel null straight lines of $\mathbb{R}^n_m$. As in Proposition 3.29 we write $p = a_1 e_0 + b_1 v$, $q = a_2 e_0 + b_2 v$ where e_0 is a spacelike normal vector and v is a null vector. Then recall $a_i = \pm 1$, and p and q will lay on the same line if and only if $a_1 = a_2$, i.e. $\langle p, q \rangle = 1$. They will lie on parallel lines, and thus not be connected if $a_1 = -a_2$ i.e. $\langle p, q \rangle = -1$.

□

Corollary 3.32. *If p and q are antipodal points of S^n_m, for $m < n$, then p and q are joined by infinitely many geodesics.*

Proof. Since p and q are antipodal, they are on a single affine line of $\mathbb{R}^{n_1}_m$, therefore there are infinitely many planes P_α containing p and q and thus infinitely many geodesics joining p and q in $P_\alpha \cap S^n_m$. □

As before, the anti-isometry from Lemma 3.25 provides an equivalent result for H^n_m with the sign of $\langle p, q\rangle$ and therefore causal characteristic swapped.

3.3.3 Isometry groups and symmetries

Proposition 3.33 (9.8 in O'Neill (1983))**.** *(Isometry group of pseudospheres) $Iso(S^n_m)$, is equal to $O(m, n-m+1)$ and the isometry group of pseudohyperbolic space, $Iso(H^n_m)$, is $O(m+1, n-m)$.*

Proof. The proof is both cases are almost identical. We will present the pseudosphere case and then describe the changes to make the proof valid in the pseudohyperbolic case. First, we show that $O(m, n-m+1)$ is a subset of $Iso(S^n_m)$. Since $O(m, n-m+1)$ is both a linear isometry and an isometry on $\mathbb{R}^{n+1}_m$, it follows that each element of $O(m, n-m+1)$ will map S^n_m to itself, and as S^n_m is a semi-Riemannian submanifold of $\mathbb{R}^{n+1}_m$ each element of $O(m, n-m+1)$ is also an isometry of S^n_m. So $O(m, n-m+1) \subset Iso(S^n_m)$.

Now we show $Iso(S^n_m) \subset O(m, n-m+1)$. Let ϕ be an isometry of S^n_m such that $\phi(p) = q$. Let $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ be tangent frames on S^n_m at points p and q respectively. The position vector p_p at $p \in S^n_m$ is normal to S^n_m. Then if $\tilde{e}_i$ is the element of $\mathbb{R}^n_m$ which canonically corresponds to the tangent vector e_i, then $\tilde{e}_1, \ldots, \tilde{e}_n, p$ is an orthonormal basis for $\mathbb{R}^n_m$. Then there is a unique linear map A that maps the orthonormal basis $\tilde{e}_i$, to $\tilde{f}_i$ and p_p to p_q, and by Lemma 3.20 $A \in O(m, n-m)$. Then since S^n_m is connected whenever $m < n$, $A|_{S^n_m}(p) = \phi(p)$ and $(dA|_{S^n_m})_p = d\phi_p$ they must be equal by Proposition 2.7. So $Iso(S^n_m) = O(m, n-m+1)$.

The proof is the same in the H^n_m case except we consider $\mathbb{R}^{n+1}_{m+1}$. □

Proposition 3.34. *The isotropy subgroup of $Iso(S^n_m)$ is isomorphic to $O(m, n-m)$ and the isotropy subgroup of $Iso(H^n_m)$ is isomorphic to $O(m, n-m)$.*

Proof. As S^n_m and H^n_m are homogeneous spaces, their isometry groups act transitively on them, so we can calculate their isotropy subgroups at a point.

We will first calculate the isotropy subgroup of $Iso(S^n_m)$. Consider the point $e_{n+1} \in S^n_m$:

$$e_{n+1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Then consider $A \in Iso_{e_{n+1}}(S^n_m)$, since $A \in Iso(S^n_m) = O(m, n-m+1)$ Lemma 3.20 shows A maps orthonormal bases to other orthonormal bases, and since $Ae_{n+1} = e_{n+1}$, we see that A must be of the form

$$A = \begin{bmatrix} B & 0 \\ 0 & 1 \end{bmatrix}. \tag{3.4}$$

Additionally, since A maps the orthonormal basis $\{e_i : i = 1, \dots, n-1\}$ to an orthonormal basis, B must do also, and thus by Lemma 3.20, $B \in O(m, n-m)$. It is immediately evident that any isometry A in the block diagonal form shown above fixes e_{n+1} and therefore $Iso_{e_{n+1}}(S^n_m) = O(m, n-m)$. An equivalent argument holds in the case of H^n_m with the following substitutions: consider $e_1 \in H^n_m \subset R^n{+}1_{m+1}$, then $C \in O(m+1, n-m)$ will have block diagonal form

$$C = \begin{bmatrix} 1 & 0 \\ 0 & D \end{bmatrix}.$$

With $D \in O(m, n-1)$. □

Proposition 3.35. *The hyperquadrics S^n_m and H^n_m are symmetric spaces.*

Proof. As with the isotropies, we will calculate the symmetry at a point. First, consider S^n_1 with point e_{n+1}. As $s_{e_{n+1}} \in Iso_{e_{n+1}}(S^n_m)$ it must be of the form described in Equation (3.4). As this is a linear group, $(ds_{e_{n+1}})_{e_{n+1}} = s_{e_{n+1}}$. As $s_{e_{n+1}}$ is a symmetry $(ds_{e_{n+1}})_{e_{n+1}}|_{T_{e_{n+1}}S^n_m} = -I$. In particular, the position vector field is normal to S^n_m so $T_{e_{n+1}}S^n_m \simeq span\{e_1, \dots, e_n\}$ so $s_{e_{n+1}}$ will have the form

$$s_{e_{n+1}} = \begin{bmatrix} -I & 0 \\ 0 & 1 \end{bmatrix}.$$

An equivalent argument holds for H^n_m by making the same substitutions as in the previous proposition. □

3.4 Cahen-Wallach spaces

3.4.1 Definitions and curvature

Definition 3.36. Let S be a symmetric $n \times n$ matrix. Then we can define the Lorentzian manifold:

$$CW_{n+2}(S) := (M = \mathbb{R}^{n+2}, g = 2dx^+dx^- + x^\top Sx(dx^+)^2 + dx^\top dx)$$

for $x^+, x^- \in \mathbb{R}, x \in \mathbb{R}^n$. In particular, if S has non-zero determinant we call $CW_{n+2}(S)$ a *Cahen-Wallach space.*

We choose to use the coordinates $x^+, x^-, x^1, \dots, x^n$, within which we can represent the metric g by the following matrix:

$$[g_{ij}] = \begin{bmatrix} x^\top Sx & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & [\ I\ ] \end{bmatrix}$$

with inverse matrix

$$[g^{ij}] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -(x^\top Sx) & 0 \\ 0 & 0 & [\ I\ ] \end{bmatrix}.$$

We adopt the notation that Greek indices μ, ν run from $1, \dots, n$ and Latin indices i, j ect run from $+, -, 1, \dots, n$. Notice we can write $x^\top Sx = x^\mu x^\nu S_{\mu\nu}$. Since every other term is constant, we notice: $\frac{\partial}{\partial x^k}(g_{ij}) = \begin{cases} 0 & \text{if } k = +, -, \\ 0 & \text{if } (i,j) \neq (+,+), \\ 2x^\nu S_{k\nu} & \text{if } k = 1, \dots, n, \text{ and } (i,j) = (+,+). \end{cases}$

Let $\mu = 1, \dots, n$. From direct calculation the only non-zero Christoffel symbols are:

$$\begin{aligned} \Gamma^-_{\mu+} = \Gamma^-_{+\mu} &= x^\nu S_{\mu\nu}, \\ \Gamma^\mu_{++} &= -x^\nu S_{\mu\nu}. \end{aligned}$$

Now to calculate the local Riemann curvature tensor:

$$R_{ijkl} = (\frac{\partial}{\partial x^i}\Gamma^p_{jk} - \frac{\partial}{\partial x^j}\Gamma^p_{ik} + \Gamma^q_{jk}\Gamma^p_{iq} - \Gamma^q_{ik}\Gamma^p_{jq})g_{pl}.$$

Consider the cases $i = +, -, \mu$ for $\mu = 1, \dots, n$ it follows from direct calculation that $\Gamma^q_{jk}\Gamma^p_{iq} - \Gamma^q_{ik}\Gamma^p_{jq} \equiv 0$. From there, further calculations show that the non-zero cases are:

$$S_{\mu\nu} = R_{+\mu+\nu} = R_{\mu+\nu+} = -R_{+\mu\nu+} = -R_{\mu++\nu}.$$

Equivalently we can describe the $(3,1)$-Riemann curvature tensor locally as such:

$$R_{ijk}{}^l = \frac{\partial}{\partial x^i}\Gamma^l_{jk} - \frac{\partial}{\partial x^j}\Gamma^l_{ik} + \Sigma_p(\Gamma^q_{jk}\Gamma^p_{iq} - \Gamma^q_{ik}\Gamma^p_{jq}).$$

With an equivalent set of calculations showing

$$S_{\mu\nu} = R_{\mu+\nu}{}^- = R_{+\nu+}{}^\mu = -R_{\mu\nu+}{}^- = -R_{++\nu}{}^\mu.$$

Proposition 3.37. *The only non-zero component of the Ricci tensor of $CW_{n+2}(S)$ is Ric_{++}, which is equal to $-tr(S)$, where $tr(S)$ is the trace of S.*

Proof. First, recall the definition of the Ricci tensor

$$Ric_{ij} = R_{kij}{}^{k}.$$

The only non-zero components of this form are $R_{\mu++}{}^{\nu} = -R_{+\mu+}{}^{\nu}$ with

$$Ric_{++} = -\Sigma_{\mu} R_{+\mu+}{}^{\mu} = -\Sigma_{\mu} S_{\mu\mu} = -tr(S).$$

□

Proposition 3.38. *The scalar curvature of $CW_{n+2}(S)$ is equal to 0.*

Proof. By definition

$$Sc = g^{ij} Ric_{ij} = g^{++} Ric_{++} = 0.$$

□

Proposition 3.39. *Let S be a symmetric matrix, then $CW_{n+2}(S)$ is locally symmetric i.e., the Riemannian curvature tensor, R is covariantly constant, $\nabla R \equiv 0$.*

Proof. Since ∇R is a tensor we will evaluate it in terms of standard basis of T_pM, e^i. For the sake of brevity we will write $x^{\nu} S_{\mu\nu} = c_{\mu}$. Recall that,

$$(\nabla_W R)(X,Y)Z = \nabla_W(R(X,Y)Z) - R(\nabla_W X, Y)Z - R(X, \nabla_W Y)Z - R(X,Y)(\nabla_W Z).$$

This will be used to make the calculations simple.

We will now proceed by considering the following cases, which, after application of the second Bianchi identity and the symmetries of the Riemann curvature tensor, cover all possible combinations of basis vectors.

Firstly, if any of the basis vectors are e^-, it immediately immediately follows from the Christoffel symbols that $\nabla R = 0$. In the case with only e^+ it follows that $\nabla_{e^+} R(e^+, e^+)e^+ = 0$ from the skew-symmetry of the tensor.

Now consider the case with one e^{μ} and three e^+:

$$\begin{aligned}
&(\nabla_{e^+} R)(e^+, e^{\mu})e^+ \\
&\quad = \nabla_{e^+}(R(e^+, e^{\mu})e^+) - R(\nabla_{e^+} e^+, e^{\mu})e^+ - R(e^+, \nabla_{e^+} e^{\mu})e^+ - R(e^+, e^{\mu})(\nabla_{e^+} e^+) \\
&\quad = \nabla_{e^+}(S_{\mu\nu} e^{\nu}) - R(-c_{\nu} e^{\nu}, e^{\mu})e^+ - R(e^+, c_{\mu} e^-)e^+ - R(e^+, e^{\mu})(-c_{\nu} e^{\nu}) \\
&\quad = S_{\mu\nu} c_{\nu} e^- - c_{\nu} S_{\mu\nu} e^- - 0 - 0 \\
&\quad = 0.
\end{aligned}$$

Now consider the cases with two e^{μ} and two e^+:

$$\begin{aligned}
&(\nabla_{e^+} R)(e^+, e^{\nu})e^{\mu} \\
&\quad = \nabla_{e^+}(R(e^+, e^+)e^{\mu}) - R(\nabla_{e^+} e^+, e^+)e^{\mu} - R(e^+, \nabla_{e^+} e^+)e^{\mu} - R(e^+, e^+)(\nabla_{e^+} e^{\mu}) \\
&\quad = \nabla_{e^+}(-S_{\mu\nu} e^{\nu}) - R(-c_{\nu} e^{\nu}, e^+)e^{\mu} - R(e^+, -c_{\mu} e^{\mu})e^+ - R(e^+, e^+)(c_{\mu} e^-) \\
&\quad = -S^{\mu\nu} c_{\nu} e^- + c_{\nu} S_{\mu\nu} e^- - 0 - 0 \\
&\quad = 0.
\end{aligned}$$

Then we have that $(\nabla_{e^+}R)(e^\nu, e^+)e^\mu = 0$ by the skew-symmetry of R and hence

$$(\nabla_{e^+}R)(e^\nu, e^\mu)e^+ = 0,$$

by the first Bianchi identity.

Now notice that $\nabla_{e^\nu}X = c_\mu X^-$, so we immediately have that

$$0 \;=\; -R(\nabla_{e^\nu}X, Y)Z - R(X, \nabla_{e^\nu}Y)Z - R(X,Y)(\nabla_{e^\nu}Z).$$

So for the following cases we will only have to evaluate $\nabla_{e^\nu}(R(X,Y)Z)$.

$$(\nabla_{e^\nu}R)(e^+, e^\mu)e^+ = \nabla_{e^\nu}(R(e^+, e^\mu)e^+) = \nabla_{e^\nu}(S_{\mu\eta}e^\eta) = 0.$$

Now consider the cases with three e^μ and one e^+:

$$\begin{aligned}
&(\nabla_{e^+}R)(e^\mu, e^\nu)e^\eta \\
&= \nabla_{e^+}(R(e^\mu, e^\nu)e^\eta) - R(\nabla_{e^+}e^\mu, e^\nu)e^\eta - R(e^\mu, \nabla_{e^+}e^\nu)e^\eta - R(e^\mu, e^\nu)(\nabla_{e^+}e^\eta) \\
&= 0 - R(c_\mu e^-, e^\nu)e^\eta - R(e^\mu, c_\nu e^-)e^\eta - R(e^\mu, e^\nu)(c_\eta e^-) \\
&= 0.
\end{aligned}$$

Then it immediately follows that $0 = \nabla_{e^\mu}R(e^+, e^\nu)e^\eta = \nabla_{e^\mu}R(e^\nu, e^+)e^\eta$ by the skew-symmetry of R and the second Bianchi identity. So then $(\nabla_{e^\mu}R)(e^\nu, e^\eta)e^+ = 0$ by the first Bianchi identity.

And finally consider the case with only e^μ:

$$\begin{aligned}
&(\nabla_{e^\mu}R)(e^\nu, e^\eta)e^\gamma \\
&= \nabla_{e^\mu}(R(e^\nu, e^\eta)e^\gamma) - R(\nabla_{e^\mu}e^\nu, e^\eta)e^\gamma - R(e^\nu, \nabla_{e^\mu}e^\eta)e^\gamma - R(e^\nu, e^\eta)(\nabla_{e^\mu}e^\gamma) \\
&= 0 - 0 - 0 - 0 \\
&= 0.
\end{aligned}$$

This covers every possible combination of basis vectors, and therefore $\nabla R \equiv 0$. □

3.4.2 Geodesics

We will now specifically consider Cahen-Wallach spaces, i.e. $CW_{n+2}(S)$ for a symmetric matrix S with non-zero determinant.

Proposition 3.40. *A Cahen-Wallach space $CW_{n+2}(S)$ is geodesically complete.*

Proof. A curve $\gamma = (\gamma^+, \gamma^-, \gamma^i)$, where $\gamma^+ = x^+ \circ \gamma$, $\gamma^- = x^- \circ \gamma$ and $\gamma^i = x^i \circ \gamma$, is a geodesic in a Cahen-Wallach space if and only if

1. $\ddot{\gamma}^+ = 0$,

2. $\ddot{\gamma}^\mu = (\dot{\gamma}^+)^2 S_{\mu\nu}\gamma^\nu$,
3. $\ddot{\gamma}^- = -2\dot{\gamma}^\mu\dot{\gamma}^+ S_{\mu\nu}\gamma^\nu$.

From this it follows that $\gamma^+ = at + b$ and therefore $\ddot{\gamma}^\mu = a^2 S_{\mu\nu}\gamma^\nu$. We can then write the system of second order homogeneous linear equations with constant coefficients:

$$\ddot{\tilde{\gamma}} = a^2 S\tilde{\gamma}.$$

Hence, it has solutions defined on all of $\mathbb{R}$. Finally, $\ddot{\gamma}^-$ does not depend on γ^- and so we can integrate twice to get solutions that are defined on all of $\mathbb{R}$. Hence Cahen-Wallach spaces are geodesically complete. □

And so we are able to immediately conclude:

Corollary 3.41. *Cahen-Wallach spaces are symmetric spaces.*

Proof. Cahen-Wallach are complete, simply connected (as they are topologically $\mathbb{R}^{n+2}$), locally symmetric Lorentzian manifolds and so they are symmetric spaces by Corollary 3.11. □

Since Cahen-Wallach spaces are symmetric, they are homogeneous, so it suffices to describe the geodesic connectedness at a single point, for simplicity we choose the point 0.

Proposition 3.42. *Consider the Cahen-Wallach space $CW_{n+2}(S)$. There exists a geodesic from 0 to (p^+, p^-, p) if and only if $p = 0$ or $p^+\sqrt{-\lambda_\mu}$ is not equal to an integer multiple of π for all negative eigenvalues λ_μ of S. If a geodesic exists, then it is unique.*

Proof. Since S is symmetric it is diagonalisable, so choose a basis of $\mathbb{R}^n$ such that S is diagonalised. Let p be a point in $CW(S)$ and suppose γ is a geodesic with $\gamma(0) = 0$ and $\gamma(1) = p$. We can then use the geodesic equations as described in Proposition 3.40 to examine the existence of such a γ.

Firstly we have that $\ddot{\gamma}^+ = 0$, and γ^+ must be of the form $at + b$. By imposing the boundary conditions we get the unique solution $a = p^+$ and $b = 0$, i.e. $\gamma^+ = p^+t$.

Next we consider γ^μ for $\mu = 1, \ldots, n$. As in Proposition 3.40, let $\tilde{\gamma}$ be the vector with entries γ^μ from the geodesic equations we get

$$\ddot{\tilde{\gamma}} = a^2 S\tilde{\gamma}.$$

Since S is diagonal, this is equivalent to

$$\ddot{\gamma}^\mu = a^2 S_{\mu\mu}\gamma^\mu = a^2\lambda_\mu\gamma^\mu.$$

Where λ_μ is the μ-th eigenvalue of S.

Which we can solve:

$$\gamma^\mu = \begin{cases} c_1^\mu e^{a\sqrt{\lambda_\mu}t} + c_2^\mu e^{-a\sqrt{\lambda_\mu}t} & \lambda_\mu > 0, \\ c_1^\mu \sin(a\sqrt{-\lambda_\mu}t) + c_2^\mu \cos(a\sqrt{-\lambda_\mu}t) & \lambda_\mu < 0. \end{cases}$$

Whenever $\lambda_\mu > 0$ this has a unique solution once imposing our boundary conditions.

So now suppose that $\lambda_\mu < 0$, since $\gamma(0) = 0$ the above equation simplifies to

$$\gamma^\mu = c_1 \sin(a\sqrt{-\lambda_\mu}t).$$

Which is equal to 0 whenever $a\sqrt{-\lambda_\mu}$ is an integer multiple of π, so γ^μ will not have a solution whenever $p^\mu \neq 0$. When $a\sqrt{-\lambda_\mu} \neq k\pi$ we have the unique solution $c_1^\mu = \frac{p^\mu}{a\sqrt{-\lambda_\mu}}$

Finally, we consider γ^-, supposing we can find a solution for each γ^μ. From the geodesic equations we have

$$\ddot{\gamma}^- = -2a\dot{\gamma}^\mu S_{\mu\nu}\gamma^\nu.$$

As S is symmetric this simplifies to

$$\ddot{\gamma}^- = -2a\lambda_\mu \dot{\gamma}^\mu \gamma^\mu,$$

which does not depend on γ^-, so we can integrate twice and apply boundary conditions to obtain a unique solution. □

3.4.3 Isometry groups and symmetries

The isometry group of Cahen-Wallach spaces have been known since their construction in Cahen & Wallach (1970), with more details discussed in many sources such as Kath & Olbrich (2019), however most of these descriptions have been heavily algebraic. We will use as formulation as found in 4.2.1 of Teisseire (2021) because it explicitly describes how the isometry group acts.

Proposition 3.43 (Isometry group of Cahen-Wallach space)**.** *The isometries of Cahen-Wallach space are of the form*

$$\phi := \begin{pmatrix} x^+ \\ x^- \\ x \end{pmatrix} \mapsto \begin{pmatrix} ax^+ + c \\ a(x^- + b - \langle \dot{\beta}(x^+), Ax + \frac{1}{2}\beta(x^+)\rangle) \\ Ax + \beta(x^+) \end{pmatrix}.$$

Where $a = \pm 1, b, c \in \mathbb{R}$, $A \in Z_{O(n)}(S)$ is an orthogonal matrix commuting with S and $\beta : \mathbb{R} \to \mathbb{R}^n$, satisfying the second order ODE $\ddot{\beta} = S\beta$, where $\ddot{\beta}$ denotes $\frac{\partial \beta}{\partial x^+}$ and $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product on $\mathbb{R}^n$.

Proof. This proof is an involved calculation in Teisseire (2021). $\square$

As before, we will calculate the isotropy subgroup of $Iso(CW_{n+2}(S))$ at a point, as $CW_{n+2}(S)$ is a homogeneous space. We will consider the point 0.

Lemma 3.44. *The isotropies of Cahen-Wallach space that fix* 0 *are of the form:*

$$\psi := \begin{pmatrix} x^+ \\ x^- \\ x \end{pmatrix} \mapsto \begin{pmatrix} ax^+ \\ a(x^- - \langle \dot{\beta}(x^+), Ax + \frac{1}{2}\beta(x^+)\rangle) \\ Ax + \beta(x^+) \end{pmatrix}.$$

where $a = \pm 1$, $A \in Z_{O(n)}(S)$ is an orthogonal matrix commuting with S and $\beta : \mathbb{R} \to \mathbb{R}^n$ satisfies the second order ODE $\ddot{\beta} = S\beta$ with initial condition $\beta(0) = 0$.

Proof. Let ψ be an isometry that fixes 0, then:

$$\psi(0) = \begin{pmatrix} c \\ a(b - \langle \dot{\beta}(0), \frac{1}{2}\beta(0)\rangle) \\ \beta(0) \end{pmatrix} = 0.$$

It is then immediate that $c = 0$ and $\beta(0) = 0$. Then

$$0 = a(b - \langle \dot{\beta}(0), \frac{1}{2}\beta(0)\rangle) = ab,$$

so $b = 0$. So ψ must have the form

$$\psi := \begin{pmatrix} x^+ \\ x^- \\ x \end{pmatrix} \mapsto \begin{pmatrix} ax^+ \\ a(x^- - \langle \dot{\beta}(x^+), Ax + \frac{1}{2}\beta(x^+)\rangle) \\ Ax + \beta(x^+) \end{pmatrix}.$$

Where $a = \pm 1$, $A \in Z_{O(n)}(S)$ is an orthogonal matrix commuting with S and $\beta : \mathbb{R} \to \mathbb{R}^n$ satisfies the second order ODE $\ddot{\beta} = S\beta$ with initial condition $\beta(0) = 0$. $\square$

Remark 3.45. An isometry ϕ_p that maps 0 to the point (p^+, p^-, p) must have the form

$$\phi_p : \begin{pmatrix} x^+ \\ x^- \\ x \end{pmatrix} \mapsto \begin{pmatrix} ax^+ + p^+ \\ a(x^- + \frac{p^- - \langle \dot{\beta}(0), \frac{1}{2}\beta(0)\rangle}{a} - \langle \dot{\beta}(x^+), Ax + \frac{1}{2}\beta(x^+)\rangle) \\ Ax + \beta(x^+) \end{pmatrix}.$$

Where $\beta(0) = p$.

We can directly calculate the symmetries of Cahen-Wallach space as such:

Lemma 3.46. *The symmetry s_p at the point (p^+, p^-, p) in $CW_{n+2}(S)$ is of the form*

$$s_p : \begin{pmatrix} x^+ \\ x^- \\ x \end{pmatrix} \mapsto \begin{pmatrix} -x^+ + 2p^+ \\ -x^- + 2p^- - \langle \dot\beta(x^+), -x + \frac{1}{2}\beta(x^+)\rangle \\ -x + \beta(x^+) \end{pmatrix}.$$

Where $\beta(p^+) = 2x$ and $\dot\beta(p^+) = 0$.

Proof. First, recall that s_p fixes p, i.e.

$$\begin{pmatrix} p^+ \\ p^- \\ p \end{pmatrix} = s_p(p) = \begin{pmatrix} ap^+ + c \\ a(p^- + b - \langle \dot\beta(p^+), Ap + \frac{1}{2}\beta(p^+)\rangle) \\ Ap + \beta(p^+) \end{pmatrix}.$$

From the first row we immediately deduce that $a = 1$ and $c = 0$ or $a = -1$ and $c = 2p^+$. Additionally, we have that $(ds_p) = -I$, so $a = -1$ and $c = 2p^+$. Similarly we have that $A = -I_n$ where I_n is the $n \times n$ identity matrix and that $\beta(p^+) = 2p$ and $\dot\beta(p^+) = 0$. Finally, we have that

$$\begin{aligned} p^- &= a(p^- + b - \langle \dot\beta(p^+), Ap + \frac{1}{2}\beta(p^+)\rangle) \\ &= -(p^- + b - \langle 0, -p + \frac{1}{2}2x\rangle). \end{aligned}$$

So $b = -2p^-$. $\square$

Remark 3.47. An arbitrary transvection is of the form $s_p \circ s_q$, so we can directly calculate it using the preceding lemma. First, we have that

$$\begin{aligned} s_p \circ s_q : \begin{pmatrix} x^+ \\ x^- \\ x \end{pmatrix} &\overset{s_q}{\mapsto} \begin{pmatrix} -x^+ + 2q^+ \\ -x^- + 2q^- - \langle \dot\beta_q(x^+), -x + \frac{1}{2}\beta_q(x^+)\rangle \\ -x + \beta_q(x^+) \end{pmatrix} \\ &\overset{s_p}{\mapsto} \begin{pmatrix} x^+ + c \\ (x^- + b - \langle \dot\beta(x^+), x + \frac{1}{2}\beta(x^+)\rangle) \\ x + \beta(x^+) \end{pmatrix}. \end{aligned}$$

Where $c = 2p - 2q$, $\beta(x^+) = \beta_p(2q^+ - x^+) - \beta_q(x^+)$ and $b = 2p^- - 2q^- + \frac{1}{2}(\langle \dot\beta_p(2q^+ - x^+), \beta_q(x^+)\rangle) - \langle \dot\beta_q(x^+), \beta_p(2q^+ - x^+)\rangle)$. Notice this is a constant:

$$\begin{aligned} &\frac{\partial}{\partial x^+}(\langle \dot\beta_p(2q^+ - x^+), \beta_q(x^+)\rangle) - \langle \dot\beta_q(x^+), \beta_p(2q^+ - x^+)\rangle) \\ &= \langle \ddot\beta_p(2q^+ - x^+), \beta_q(x^+)\rangle + \langle \dot\beta_p(2q^+ - x^+), \dot\beta_q(x^+)\rangle \\ &\quad - \langle \ddot\beta_q(x^+), \beta_p(2q^+ - x^+)\rangle - \langle \dot\beta_q(x^+), \dot\beta_p(2q^+ - x^+)\rangle \\ &= \langle S\beta_p(2q^+ - x^+), \beta_q(x^+)\rangle - \langle S\beta_q(x^+), \beta_p(2q^+ - x^+)\rangle \\ &= 0 \end{aligned}$$

since S is self-adjoint, because it is symmetric.

3.5 Symmetric spaces as homogeneous spaces

Recall that a semi-Riemannian manifold M is *homogeneous* if its isometry group $Iso(M)$ acts transitively on M.

Lemma 3.48. *Let M be a symmetric space, then M is homogeneous.*

Proof. Since the transvection group $G(M)$ is a subset of $Iso(M)$ it follows from Corollary 3.6 that M is homogeneous. $\square$

A particularly appealing property of a homogeneous manifold M is that it can be described as coset manifolds $Iso(M)/Iso_p(M)$. In the previous sections, we calculated the isometry and isotropy groups of some semi-Riemannian symmetric spaces, which we will use in Chapter 5 when presenting Klingler (1996).

In general, viewing a symmetric space S as a homogeneous space allows for an algebraic description of S which provides powerful tools to calculate its geodesics and curvature.

Theorem 3.49 (11.29 in O'Neill (1983))**.** *Let H be a closed subgroup of a connected Lie group G. Let Σ be an involutive automorphism of G such that $Fix(\Sigma)_0 \subset H \subset Fix(\Sigma)$ where $Fix(\Sigma)$ is the set of points in G fixed by Σ and $Fix(\Sigma)_0$ is the connected component containing the identity. Then any G-invariant metric tensor on M makes $M = G/H$ a semi-Riemannian symmetric space such that $s_0 \circ \pi = \pi \circ \Sigma$ where s_0 is the symmetry at 0 and π is the projection $G \to M$.*

Lemma 3.50 (11.30 in O'Neill (1983))**.** *Let H be a closed subgroup of a connected Lie group G, with respective Lie algebras $\mathfrak{h} \subset \mathfrak{g}$, and Σ be an involutive automorphism of G such that $Fix(\Sigma)_0 \subset H \subset Fix(\Sigma)$. We will write $\sigma := d\Sigma$. Then*

1. $\mathfrak{h} = \{X \in \mathfrak{g} \mid \sigma(X) = X\}$,

2. $\mathfrak{g}$ *is the direct sum of* $\mathfrak{h}$ *and the subspace* $\mathfrak{m} := \{X \in \mathfrak{g} \mid \sigma(X) = -X\}$,

3. $Ad_h(\mathfrak{m}) \subset \mathfrak{m}$ *for all* $h \in H$,

4. $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$, *and* $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$.

Proposition 3.51 (11.31 in O'Neill (1983))**.** *Let $M = G/H$ be a semi-Riemannian symmetric space. Then*

1. *The geodesics starting at $O = eH$ with initial velocity $d\pi|_O X$ for $X \in \mathfrak{m}$ are given by*

$$\gamma(t) = \alpha(t)0 = \pi\alpha(t) \quad \textit{for all } t,$$

where $\alpha(t) = exp(tX)$ is the one-parameter subgroup of $X \in \mathfrak{m}$.

2. *The curvature tensor at* 0 *is given by* $R(x,y)z = d\pi[[X,Y],Z]$*, where* $x,y,z \in T_0M$ *corresponds under* $d\pi$ *to* $X,Y,Z \in \mathfrak{m}$*.*

Notice that in Theorem 3.49 the involutive automorphism Σ is defined to be a lift of the symmetry s_0, so it would appear that the useful algebraic description of symmetric spaces requires a manifold to be globally symmetric. In the following section, we will generalise this algebraic data to *symmetric triple systems* and proceed to construct symmetric triples from *locally* symmetric spaces. From there, we will prove two main results; that a manifold is locally symmetric if and only if it is locally isometric to a symmetric space and that there is a one to one correspondence between simply connected symmetric spaces and symmetric triple systems.

3.6 Symmetric triples and locally symmetric spaces

Definition 3.52 (Symmetric triple system)**.** A symmetric triple $(\mathfrak{g}, \sigma, B)$ consists of a finite dimensional real Lie algebra $\mathfrak{g}$, an involutive automorphism σ of $\mathfrak{g}$ and a non-degenerate symmetric bilinear form B on $\mathfrak{g}$ such that the following properties hold:

1. The decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ induced by the eigenspace decomposition $\mathfrak{h} = Eig_\sigma(+1)$ and $\mathfrak{m} = Eig_\sigma(-1)$ of σ satisfies $\mathfrak{h} = [\mathfrak{m}, \mathfrak{m}]$.

2. B is invariant under σ and $\mathfrak{ad}(\mathfrak{g})$, i.e. $\sigma \in SO(B, \mathfrak{g})$ and $\mathfrak{ad}(\mathfrak{g}) \subset \mathfrak{so}(B, \mathfrak{g})$.

Lemma 3.53. *Given a symmetric triple* $(\mathfrak{g}, \sigma, B)$*,* $\mathfrak{h}$ *is a sub-algebra of* $\mathfrak{g}$*.*

Proof. As $\mathfrak{h}$ is the 1-eigenspace decomposition of σ, it is a subspace of $\mathfrak{g}$, so all that remains is to show it is closed under the Lie brackets. Let $h_1, h_2 \in \mathfrak{h}$, then:

$$\sigma([h_1, h_2]) = [\sigma(h_1), \sigma(h_2)] = [h_1, h_2].$$

So therefore, $[h_1, h_2]$ is in the 1-eigenspace of σ i.e. $[h_1, h_2] \in \mathfrak{h}$. $\square$

Theorem 3.54. *Let* $(\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}, \sigma, B)$ *be a symmetric triple. Then there exists a symmetric space* $S = G/H$*, where* G *is the unique simply connected Lie group with Lie algebra* $\mathfrak{g}$ *and* H *is the unique connected subgroup of* G *that has Lie algebra* $\mathfrak{h}$*.*

Proof. This construction follows Neukirchner (2003). First, recall Lie's third theorem: If $\mathfrak{g}$ is a finite dimensional real Lie algebra, then there exists a simply connected Lie group G with $\mathfrak{g}$ as its Lie algebra. So let G be the unique simply connected Lie group with Lie algebra $\mathfrak{g}$. Recall that if $\phi : LA(G) \to LA(H)$ is a Lie algebra homomorphism and G is simply connected, then there exists a unique Lie group homomorphism $f : G \to H$ such that $df = \phi$ (see 3.27 in Warner (1983) for a proof). So then we then lift $\sigma \in Aut(\mathfrak{g})$ to an involutive Lie group automorphism $\Sigma \in Aut(G)$.

Since $LA(G) = T_0G = \mathfrak{g}$, we can define a G-invariant metric $\tilde{B}$ on G by $\tilde{B}|_0 = B$. Then consider the fixed points of G under Σ, $H = \{g \in G \mid \Sigma(g) = g\}$, evidently $\exp(\mathfrak{h}) \subset H$ and by (Koh 1965, p.293) H is closed and connected. Then we define $S = G/H$ with projection $\pi : G \to S$, the $\mathfrak{ad}(\mathfrak{h})$ invariance of B ensures that $\tilde{B}$ is $ad(H)$ invariant, so $\tilde{B}$ descends to a well defined G-invariant metric $\hat{B}$ on S. Then the G-invariant metric on the homogeneous space S makes it a symmetric space by Theorem 3.49. Finally, notice that S is simply connected by the short exact sequence

$$1 = \pi_1(G) \to \pi_1(G/H) \to \pi_0(H) = 1.$$

□

Definition 3.55 (Lie Triple System)**.** A Lie triple system (W, b, T) consists of a vector space W over $\mathbb{R}$, a bilinear form $b : W \times W \to W$ and a trilinear form $T : W \times W \times W \to W$ with the following properties:

1. b is symmetric and non-degenerate.

2. T is skew-symmetric in first two variables: $T(X, Y)Z = -T(Y, X)Z$.

3. T satisfies Bianchi's identity: $T(X, Y)Z + T(Y, Z)X + T(Z, X)Y = 0$.

4. The endomorphism $T_{U,V} : W \to W$ defined by $T_{U,V}(X) := T(U, V)X$ acts as a derivation of the triple product, i.e,
$$T_{U,V}(T(X, Y)Z) = T(T_{U,V}X, Y)Z + T(X, T_{U,V}Y)Z + T(X, Y)(T_{U,V}Z).$$

5. T is skew-adjoint for b: $b(T_{U,V}X, Y) + b(X, T_{U,V}Y) = 0$.

Theorem 3.56. *Given a locally symmetric manifold (M, g) with curvature tensor R, then (T_pM, g_p, R_p) forms a Lie triple system.*

Proof. Firstly, notice that T_pM is a vector space and g_p is a non-degenerate bilinear form. Now notice that R is a trilinear map $T_pM \times T_pM \times T_pM \to T_pM$ such that

$$R(X, Y)Z = -R(Y, X)Z$$

by definition, and

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$$

by the 1st Bianchi identity. Also, by the skew-symmetric property of R,

$$g(R(U, V)X, Y) + g(X, R(U, V)Y) = 0.$$

So all that remains is to show $R(U,V)$ is a derivation of $R(X,Y)Z$. Where the curvature endomorphism is:

$$\begin{aligned} R(U,V) &\in \mathrm{End}(TM) \\ R(U,V) : W &\mapsto R(U,V)W. \end{aligned}$$

Writing $\nabla^2_{U,V}W = \nabla_U\nabla_V W - \nabla_{\nabla_U V}$, then $R(U,V) = \nabla^2_{U,V} - \nabla^2_{V,U}$. Since M is locally symmetric,

$$\begin{aligned} &(\nabla_W R)(X,Y,Z) \\ &= \nabla_W(R(X,Y)Z) - R(\nabla_W X,Y)Z - R(X,\nabla_W Y)Z - R(X,Y)(\nabla_W Z) \\ &= 0. \end{aligned}$$

So, for any $W,X,Y,Z \in \Gamma(TM)$, we have that

$$\nabla_W(R(X,Y)Z) = R(\nabla_W X,Y)Z + R(X,\nabla_W Y)Z + R(X,Y)(\nabla_W Z).$$

We can then take the derivative with respect to $U \in \Gamma(TM)$ and compute

$$\nabla^2_{UW}R(X,Y)Z = \nabla_U(\nabla_W(R(X,Y)Z)) - \nabla_W(\nabla_U(R(X,Y)Z)),$$

which can then be expanded via the Leibniz rule and then simplified to equal

$$R(U,W)(R(X,Y)Z) = R(R(U,W)X,Y)Z + R(X,R(U,W)Y)Z + R(X,Y)(R(U,W)Z). \tag{3.5}$$

Hence $R(U,W)$ acts as a derivation on $R(X,Y)Z$ and therefore (T_pM, g_p, R) is a Lie triple system. □

Notice that conditions $1,2,3$ and 5 are satisfied for all semi-Riemannian manifolds, however condition 4 does not hold in general and required that $\nabla R = 0$ in this case.

Let (W,b,T) be a Lie triple system, we can construct a corresponding symmetric triple $(\mathfrak{g},\sigma,B)$ as follows:

$$\begin{aligned} &\mathfrak{m} := W, \\ &\mathfrak{h} := \mathrm{Span}\{T(U,V) : U,V \in \mathfrak{m}\} \subset \mathfrak{so}(\mathfrak{m}), \\ &\mathfrak{g} := \mathfrak{m} \oplus \mathfrak{h}. \end{aligned}$$

Where we define the Lie brackets on $\mathfrak{g}$ as such: let $X,Y \in \mathfrak{m}$, and $T(U,V), T(W,Z) \in \mathfrak{h}$, then define

$$\begin{aligned} [X,Y] &:= T(X,Y) \in \mathfrak{h}, \\ [T(U,V),X] &:= T(U,V)X, \\ [T(U,V),T(W,Z)] &:= [T(U,V),T(W,Z)]_{\mathfrak{so}(\mathfrak{m})}. \end{aligned}$$

Furthermore, define an involutive automorphism of $\mathfrak{g}$

$$\sigma: \quad \sigma(T(U,V)) = T(U,V), \quad \sigma(X) = -X$$

and extend linearly.

Finally, define a bilinear form B by considering $B|_{\mathfrak{m}\times\mathfrak{m}}$, $B|_{\mathfrak{m}\times\mathfrak{h}}$ and $B|_{\mathfrak{h}\times\mathfrak{h}}$ separately.

$$\begin{aligned} B|_{\mathfrak{m}\times\mathfrak{m}}(X,Y) &:= b(X,Y), \\ B|_{\mathfrak{m}\times\mathfrak{h}}(X,T(U,V)) &:= 0, \\ B|_{\mathfrak{h}\times\mathfrak{h}}(T(U,V),T(X,Y)) &:= B(T(U,V)X,Y) \\ &= b(T(U,V)X,Y). \end{aligned}$$

Notice that $B|_{\mathfrak{h}\times\mathfrak{h}}$ is well defined: suppose $T(X,Y) = T(\tilde{X},\tilde{Y})$ and $T(U,V) = T(\tilde{U},\tilde{V})$ then

$$\begin{aligned} B(T(\tilde{X},\tilde{Y}),T(\tilde{U},\tilde{V})) &= b(T(\tilde{X},\tilde{Y})\tilde{U},\tilde{V}) = b(T(X,Y)\tilde{U},\tilde{V}) \\ &= b(T(\tilde{U},\tilde{V})X,Y) = b(T(U,V)X,Y) = b(T(X,Y)U,V), \end{aligned}$$

by the symmetries of T. B is symmetric since b is symmetric, T is skew-symmetric and T is skew-adjoint for b.

Theorem 3.57. *Let (W,b,T) be a Lie triple system. Then the triple $(\mathfrak{g},\sigma,B)$ defined as above is a symmetric triple.*

Proof. Immediately we have that $[\cdot,\cdot]$ is skew-symmetric and $[\cdot,\cdot]_{\mathfrak{so}(\mathfrak{m})}$ and T satisfy the Bianchi identity, so all that remains is to check that the Bianchi identity is satisfied in the mixed cases.

First, consider the case of one element in $\mathfrak{h}$ and two elements in $\mathfrak{m}$:

$$\begin{aligned} &[T(U,V),[Y,Z]] + [Y,[Z,T(U,V)]] + [Z,[T(U,V),Y]] \\ &= [T(U,V),T(Y,Z)] + [Y,-T(U,V)Z] + [Z,T(U,V)Y] \\ &= [T(U,V),T(Y,Z)] - T(Y,T(U,V)Z) + T(Z,T(U,V)Y) \\ &= T(U,V)T(Y,Z) - T(Y,Z)T(U,V) - T(Y,T(U,V)Z) + T(Z,T(U,V)Y). \end{aligned}$$

Then expanding $T(U,V)T(Y,Z)$ by Equation (3.5)

$$T(U,V)(T(Y,Z)) = T(T(U,V)Y,Z) + T(Y,T(U,V)Z) + T(Y,Z)(T(U,V))$$

and hence

$$\begin{aligned} &T(U,V)(T(Y,Z)) - T(Y,T(U,V)Z) + T(Z,T(U,V)Y) \\ &= T(T(U,V)Y,Z) + T(Y,T(U,V)Z) + T(Y,Z)(T(U,V)) \\ &\quad + T(Z,T(U,V)Y) - T(Y,T(U,V)Z) \\ &= T(Y,Z)(T(U,V)). \end{aligned}$$

So this case satisfies the Bianchi identity.

Now consider the case with two elements in $\mathfrak{h}$ and one in $\mathfrak{m}$:

$$\begin{aligned}
&[T(U,V),[T(X,Y),Z]]+[T(X,Y),[Z,T(U,V)]]+[Z,[T(U,V),T(X,Y)]]\\
&= \quad [T(U,V),T(X,Y)Z]+[T(X,Y),-T(U,V)Z] \quad +[Z,[T(U,V),T(X,Y)]]\\
&= \quad T(U,V)(T(X,Y)Z)-T(X,Y)(T(U,V)Z) \quad -[T(U,V),T(X,Y)](Z)\\
&= \quad T(U,V)(T(X,Y)Z)-T(X,Y)(T(U,V)Z)\\
&\quad -T(U,V)(T(X,Y)Z)+T(X,Y)(T(U,V)Z)\\
&=0.
\end{aligned}$$

So $[\cdot,\cdot]$ satisfies the Bianchi identity and $(\mathfrak{g},[,])$ defines a Lie algebra.

Now consider σ. First, we will show that σ is an involution:

$$\begin{aligned}
\sigma(\sigma(X)) &= \sigma(-X) &= X\\
\sigma(\sigma(T(U,V))) &= \sigma(T(U,V)) &= T(U,V)
\end{aligned}$$

where $\mathfrak{h}$ is the $+1$ eigenspace of σ and $\mathfrak{m}$ is the -1 eigenspace of σ. By definition $\mathfrak{h} = \mathrm{Span}\{T(U,V)=[U,V] : U,V \in \mathfrak{m}\}$. Now we will show σ is a Lie algebra homomorphism.

$$\begin{aligned}
\sigma([X,Y]) &= \sigma(T(X,Y)) = T(X,Y)\\
&= [X,Y] = [-X,-Y]\\
&= [\sigma(X),\sigma(Y)],\\
\sigma([T(U,V),X]) &= \sigma(T(U,V)X) = -T(U,V)X\\
&= -[T(U,V),X] = [T(U,V),-X]\\
&= [\sigma(T(U,V)),\sigma(X)],\\
\sigma([T(U,V),T(X,Y)]) &= [T(U,V),T(X,Y)]\\
&= [\sigma(T(U,V)),\sigma(T(X,Y))].
\end{aligned}$$

Finally, consider B. We must show that B is non-degenerate. As $g|_p$ is non-degenerate all that remains is to show $B|_{\mathfrak{h}\times\mathfrak{h}}$ is non-degenerate. Suppose there is some $T(U,V)$ such that $B(T(U,V),T(X,Y))=0$ for every $T(X,Y)$, then

$$0 = B(T(U,V),T(X,Y)) = g(T(U,V)X,Y).$$

Since this holds for all $T(X,Y)$, it must hold for all X,Y, and so by the non-degeneracy of b,

$$0 = T(U,V)X$$

for all X. As $T(U,V)X=0$, then $T(U,V)=0$. Finally, we show that B is σ and $\mathfrak{ad}(\mathfrak{g})$ invariant. First, we show σ invariance. Since $\mathfrak{h}$ is the $+1$-eigenspace of σ it is immediately invariant. Since σ preserves $\mathfrak{h}$ and $\mathfrak{m}$, $B|_{\mathfrak{m}\times\mathfrak{h}}$ is σ-invariant also. Now notice for $B|_{\mathfrak{m}\times\mathfrak{m}}$:

$$b(\sigma(X), \sigma(Y)) = b(-X, -Y) = b(X, Y)$$

so B is σ-invariant. Now we will show the $\mathfrak{ad}(\mathfrak{g})$-invariance.

First, consider $B|_{\mathfrak{m}\times\mathfrak{m}}$:

$$\begin{aligned} B([T(U,V)X], Y) + B(X, [T(U,V), Y]) &= B(T(U,V)X, Y) + B(X, T(U,V)Y) \\ &= b(T(U,V)X, Y) + b(X, T(U,V)Y) \\ &= 0. \end{aligned}$$

Now consider $B|_{\mathfrak{m}\times\mathfrak{h}}$:

$$\begin{aligned} &B([X,Y], Z) + B(Y, [X, Z]) \\ &\quad = B(T(X,Y), Z) + B(Y, T(X,Z)) \\ &\quad = 0, \\ &B([X, T(U,V)], T(W,Z)) + B(T(U,V), [X, T(W,Z)]) \\ &\quad = B(-T(U,V)X, T(W,Z)) + B(T(U,V), -T(W,Z)X) \\ &\quad = 0, \\ &B([T(U,V)X], T(W,Z)) + B(X, [T(U,V), T(W,Z)]) \\ &\quad = 0. \end{aligned}$$

Finally, $B|_{\mathfrak{h}\times\mathfrak{h}}$:

$$\begin{aligned} &B([X, T(U,V)], Z) + B(T(U,V), T(X,Z)) \\ &\quad = B(-T(U,V)X, Z) + B(T(U,V), T(X,Y)) \\ &\quad = 0, \\ &B([T(U,V), T(W,Z)], T(A,B)) + B(T(W,Z), [T(U,V), T(A,B)]) \\ &\quad = 0. \end{aligned}$$

So B is σ and $\mathfrak{ad}(\mathfrak{g})$ invariant and therefore $(\mathfrak{g}, \sigma, B)$ is a symmetric triple. $\square$

Remark 3.58. In particular, the combination of Theorem 3.56 and Theorem 3.57 show that for a locally symmetric space (M, g), the triple (T_pM, g_p, R) forms a symmetric triple with

$$\begin{aligned} \mathfrak{m} &:= T_pM, \\ \mathfrak{h} &:= \mathrm{Span}\{R(U,V) : U, V \in \mathfrak{m}\} \subset \mathfrak{so}(\mathfrak{m}), \\ \mathfrak{g} &:= \mathfrak{m} \oplus \mathfrak{h}. \end{aligned}$$

We will call this the *corresponding symmetric triple* for (M, g).

Theorem 3.59. *Let (M, g) be a semi-Riemannian manifold. Then (M, g) is locally symmetric if and only if there exists some simply connected symmetric space S that is locally isometric to M.*

Proof. Let M be a semi-Riemannian manifold with Levi-Civita connection ∇ and Riemann curvature tensor R. Suppose that M is locally isometric to a symmetric space S with Levi-Civita connection $\hat{\nabla}$ and Riemann curvature tensor $\hat{R}$. Since ∇R is a tensor, we can evaluate it at each point. Let p be an arbitrary point of M, then ϕ is a local isometry from some neighbourhood U of p to $\phi(U) \subset S$. Then it follows from Proposition 2.4 that:

$$\begin{aligned}
&d\phi(\nabla_{V_p} R(X_p, Y_p)Z_p) \\
&= \hat{\nabla}_{d\phi V_p} d\phi(R(X_p, Y_p)Z_p) \\
&= \hat{\nabla}_{d\phi V_p} d\phi(\nabla_{X_p}\nabla_{Y_p} Z_p - \nabla_{Y_p}\nabla_{X_p} Z - \nabla_{X_p,Y_p} Z_p) \\
&= \hat{\nabla}_{d\phi V_p}(\hat{\nabla}_{d\phi X_p} d\phi(\nabla_{Y_p} Z) - \hat{\nabla}_{d\phi Y_p} d\phi(\nabla_{X_p} Z_p) - \hat{\nabla}_{d\phi[X_p,Y_p]} d\phi Z_p) \\
&= \hat{\nabla}_{d\phi V_p}(\hat{\nabla}_{d\phi X_p}\hat{\nabla}_{d\phi Y_p} d\phi Z - \hat{\nabla}_{d\phi Y_p}\hat{\nabla}_{d\phi X_p} d\phi Z_p - \hat{\nabla}_{[d\phi X_p, d\phi Y_p]} d\phi Z_p) \\
&= \hat{\nabla}_{d\phi V_p}\hat{R}(d\phi X_p, d\phi Y_p) d\phi Z_p.
\end{aligned}$$

Lemma 3.8 showed that $\hat{\nabla}\hat{R} = 0$ and at each point we see $\hat{\nabla}\hat{R} = \nabla R$ so $\nabla R = 0$ i.e. M is locally symmetric.

Now consider the converse, suppose that M is a semi-Riemannian manifold with a metric g such that $\nabla R = 0$.

It follows from Theorem 3.56 that (T_pM, g_p, R) is a Lie triple system, and then Theorem 3.57 shows this corresponds to a symmetric triple $(\mathfrak{g}, \sigma, B)$.

Now, we construct a simply connected symmetric space S from the symmetric triple in a manner identical to Theorem 3.54.

Now notice that by identifying $p \in M$ with $[0] \in S$, there is a vector space isomorphism from $v \in T_pM$ to $\phi(v) \in T_{[0]}S$ because:

$$\phi : T_pM \simeq \mathfrak{m} \simeq (\mathfrak{m} \oplus \mathfrak{h})/\mathfrak{h} \simeq \mathfrak{g}/\mathfrak{h} \overset{d\pi}{\simeq} T_{[0]}(G/H) \simeq T_{[0]}S.$$

There exists a linear isometry ϕ from T_pM to $T_{[0]}S$. Now notice that ϕ preserves curvature: consider $R^M(X, Y)Z \in T_pM$, which is identified with $R^M(X, Y)Z = [[X, Y]Z] \in \mathfrak{m}$, then by Proposition 3.51, $d\pi[[X, Y]Z] = R^S(d\pi X, d\pi Y)d\pi Z = R^S(\phi(X), \phi(Y))\phi(Z)$ in $T_{[0]}S$, so ϕ preserves curvature. As S is symmetric it is locally symmetric. Since $\phi : T_pM \to T_xS$ is a linear isometry between locally symmetric semi-Riemannian manifolds which preserves curvature so by Theorem 3.9 we can (uniquely) extend this linear isometry to a neighborhood. Hence M is locally isometric to S. $\square$

Remark 3.60. If we additionally require M to be a simply connected symmetric space, the above theorem can be strengthened such that ϕ is a isometry, as we are able to replace Theorem 3.9 with Theorem 3.10, and since M and S are simply connected all covering maps are isometries.

Definition 3.61 (Isomorphic Symmetric Triples)**.** Let $(\mathfrak{g}_1, \sigma_1, B_1)$ and $(\mathfrak{g}_2, \sigma_2, B_2)$ be symmetric triples. We say they are *isomorphic as symmetric triples* if there exists a Lie algebra isomorphism $A : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that:

1. $A \circ \sigma_1 = \sigma_2 \circ A$,
2. $B_1(x, y) = B_2(Ax, Ay)$ for any, $x, y \in \mathfrak{m}_1$.

Theorem 3.62. *Let (M, g) and $(\hat{M}, \hat{g})$ be isometric simply connected symmetric spaces. Then their corresponding symmetric triples $(\mathfrak{g}, \sigma, B)$ and $(\hat{\mathfrak{g}}, \hat{\sigma}, \hat{B})$ are isomorphic as symmetric triples.*

Proof. Let ψ be an isometry from M to $\hat{M}$. Then there exists some base-points $p, \hat{p}$ for the corresponding symmetric triples, i.e. $\mathfrak{m} = T_pM$ and $\hat{\mathfrak{m}} = T_{\hat{p}}\hat{M}$. Then since symmetric spaces are homogeneous, there exists some $\hat{g} \in Iso(\hat{M})$ such that $\hat{g}\psi(p) = \hat{p}$, we will call this isometry $\phi := \hat{g}\psi$. Then we have the linear isomorphism

$$d\phi_p : T_pM \to T_{\hat{p}}\hat{M}.$$

Similarly, since isometries preserve the Riemann curvature tensor, they will preserve the curvature endomorphism and hence we have the linear isomorphism

$$R(U, V) \mapsto \hat{R}(d\phi(U), d\phi(V)).$$

Since $(\mathfrak{g}, \sigma, B)$ and $(\hat{\mathfrak{g}}, \hat{\sigma}, \hat{B})$ are the *corresponding* symmetric triples for their respective symmetric spaces, σ, the Lie brackets and bilinear forms B are defined in equivalent manners, and therefore these maps describe a symmetric triple isomorphism from $(\mathfrak{g}, \sigma, B)$ to $(\hat{\mathfrak{g}}, \hat{\sigma}, \hat{B})$. □

Theorem 3.63. *Let (M, g) and $(\hat{M}, \hat{g})$ be simply connected symmetric spaces. If their corresponding symmetric triples $(\mathfrak{g}, \sigma, B)$ and $(\hat{\mathfrak{g}}, \hat{\sigma}, \hat{B})$ are isomorphic as symmetric triples, then (M, g) and $(\hat{M}, \hat{g})$ are isometric.*

Proof. By Remark 3.60, both M and $\hat{M}$ are isometric to some symmetric space S and therefore are isometric to each other. □

Theorem 3.64. *Let $(\mathfrak{g}, \sigma, B)$ be a symmetric triple. Then the corresponding symmetric space (S, g) has a corresponding symmetric triple $(T_0S \oplus span\{R(U, V) \mid U, V \in T_0S\}, \sigma_1, B_1)$ that is isomorphic as a symmetric triple to $(\mathfrak{g}, \sigma, B)$.*

Proof. As before, we let G be the unique simply connected Lie group with Lie algebra $\mathfrak{g}$. Then since G is simply connected, $\sigma \in \mathfrak{aut}(\mathfrak{g})$ lifts to an involutive automorphism $\Sigma \in Aut(G)$. Since $LA(G) = T_0G = \mathfrak{g}$, we can define a G-invariant metric $\tilde{B}$ on G by $\tilde{B}|_0 = B$. Then let $H = \{g \in G \mid \Sigma(g) = g\}$, evidently $\exp(\mathfrak{h}) \subset H$ and by (Koh

1965, p.293) H is closed and connected. Then we define $S = G/H$ with projection $\pi : G \to S$, the $\mathfrak{ad}(\mathfrak{h})$ invariance of B ensures that $\tilde{B}$ is $ad(H)$ invariant, so $\tilde{B}$ descends to a well defined metric $\hat{B}$ on S. Then we define the corresponding symmetric triple $(T_0S \oplus span\{R(U,V) \mid U,V \in T_0S\}, \sigma_1, B_1)$ as in Remark 3.58. Now define the map

$$\begin{array}{rcl} A : \mathfrak{g} & \to & T_0S \oplus span\{R(U,V) \mid U,V \in T_0S\} \\ \mathfrak{m} \ni X & \mapsto & d\pi|_0(X) \in T_0S, \\ h = \Sigma_i[X_i, Y_i] & \mapsto & \Sigma_i R(AX, AY) \in span\{R(U,V) \mid U,V \in T_0S\}. \end{array}$$

A preserves eigenspace decomposition with respect to σ_1 and σ by construction. Now we will show that A preserves Lie brackets. First, consider

$$A[X,Y] = R(AX, AY) = [AX, AY]_1$$

by the definition of A and the Lie brackets $[\cdot,\cdot]_1$, of the corresponding symmetric triple. Now consider the case:

$$\begin{array}{rcl} A[[X,Y],[Z,W]] & = & R(A[X,Y], A[Z,W]) \\ & = & [A[X,Y], A[Z,W]]_1 \\ & = & [R(AX,AY), R(AZ,AW)]_1 \\ & = & [[AX,AY]_1, [AZ,AW]_1]_1. \end{array}$$

Now consider the final case of $[[X,Y],Z]$, using Proposition 3.51 we have that

$$\begin{array}{rcl} A[[X,Y],Z] & = & d\pi[[X,Y],Z] \\ & = & R(AX,AY)AZ \\ & = & [[AX,AY]_1, AZ]_1. \end{array}$$

So A preserves Lie brackets, so it remains to check that A is an isomorphism and that A preserves B.

First, we will show A is surjective. Let $X \in T_0S$, since π is a projection, there exists some $\tilde{X} \in T_0G = \mathfrak{g}$ such that $d\pi_0(\tilde{X}) = X$. Then $A\tilde{X} = X$. Additionally, it follows from Proposition 3.51 that the geodesics of S starting at 0 are given by

$$\gamma_(d\pi\tilde{X})(t) = \alpha(t)(0) = \pi\alpha(t),$$

where α is the one-parameter subgroup of $\tilde{X} \in \mathfrak{m}$, so we have that $\tilde{X} \in \mathfrak{m}$. Similarly, consider $R(X,Y)$, then there exists $\tilde{X}, \tilde{Y}$ that are mapped to X, Y by $d\pi_0$, then $[\tilde{X}, \tilde{Y}] \in \mathfrak{h}$ and $A[\tilde{X}, \tilde{Y}] = R(X,Y)$.

Now we will show that A is injective. First, we will show A is injective on $\mathfrak{m}$. Suppose that $d\pi|_0(X) = 0$, then $[X] = [0]$ in T_0S, so $X \in \mathfrak{h}$. Now consider $[m_1, m_2]_{\mathfrak{g}} \in \mathfrak{h}$, and let $A[m_1, m_2]_{\mathfrak{g}} = 0$ then

$$0 = A[m_1, m_2]_{\mathfrak{g}} = R(d\pi|_0(m_1), d\pi|_0(m_2)).$$

Then for an arbitrary $d\pi|_0(e) \in T_0S$ we have that

$$0 = R(d\pi|_0(m_1), d\pi|_0(m_2))d\pi|_0(e).$$

Then by applying Proposition 3.51 we have that

$$0 = d\pi[[m_1, m_2], e].$$

So $[[m_1, m_2], e] \in \mathfrak{h}$, but

$$\sigma[[m_1, m_2], e] = [\sigma[m_1, m_2], \sigma(e)] = -[[m_1, m_2], e],$$

so $[[m_1, m_2], e] \in \mathfrak{m}$, and therefore $[[m_1, m_2], e] = 0$. Since e was an arbitrary choice it follows that $[m_1, m_2] = 0$, and because A is linear this suffices to show A is injective.

Finally, we show that $B(m_1, m_2) = \hat{B}(Am_1, Am_2)$ for any $m_1, m_2 \in \mathfrak{m}$. This follows from the definition of each respective metric,

$$\begin{aligned}
\hat{B}(Am_1, Am_2) &= \hat{B}(d\pi|_0(m_1), d\pi|_0(m_2)) \\
&= \tilde{B}(m_1, m_2) \\
&= B(m_1, m_2).
\end{aligned}$$

□

Corollary 3.65. *Let $(\mathfrak{g}, \sigma, B)$ and $(\hat{\mathfrak{g}}, \hat{\sigma}, \hat{B})$ be isomorphic as symmetric triples. Then their corresponding symmetric spaces S and $\hat{S}$ are isometric.*

Proof. Let $(\mathfrak{g}, \sigma, B)$ and $(\hat{\mathfrak{g}}, \hat{\sigma}, \hat{B})$ be isomorphic as symmetric triples. Then it follows from Theorem 3.64 that $(\mathfrak{g}, \sigma, B)$ is isomorphic to the corresponding symmetric triple of the corresponding symmetric space S. The same holds for $(\hat{\mathfrak{g}}, \hat{\sigma}, \hat{B})$ with corresponding symmetric space $\hat{S}$. Then it follows from transitivity of isomorphisms that the corresponding symmetric triples of S and $\hat{S}$ are isomorphic as symmetric triples and hence by Theorem 3.63 S and $\hat{S}$ are isomorphic. □

Remark 3.66. The previous results show a correspondence between simply connected symmetric spaces (up to isometry) and symmetric triples (up to symmetric triple isomorphism). Theorem 3.54 describes how to construct the corresponding symmetric space from a symmetric triple, Theorem 3.63 ensures that if two symmetric triples are isomorphic, then their corresponding symmetric spaces will be isometric. Conversely, given a symmetric space, Remark 3.58 describes the corresponding symmetric triple, and by Theorem 3.62, if two simply connected symmetric spaces are isometric, their corresponding symmetric triples will be isomorphic. Also, by Theorem 3.64, we see that given a symmetric triple $(\mathfrak{g}, \sigma, B)$, the corresponding symmetric triple to the corresponding symmetric space of $(\mathfrak{g}, \sigma, B)$ will be isomorphic to $(\mathfrak{g}, \sigma, B)$. Finally, given a simply connected symmetric space S, the corresponding symmetric space to the corresponding symmetric triple of S will be isometric to S by Remark 3.60.

The above correspondence is vital to the classification of Lorentzian symmetric spaces, which we will outline in the following section.

3.7 Classification of Lorentzian symmetric spaces

Definition 3.67. A semi-Riemannian manifold (M, g) is said to be *decomposable* if for all points $p \in M$, there exists a neighbourhood U of p such that (U, g) is isometric to a product $(M_1, g_1) \times (M_2, g_2)$. We say that (M, g) is *indecomposable* if it is not decomposable.

If (M, g) is a locally symmetric decomposable semi-Riemannian manifold then by Proposition 2.60 and Corollary 2.62 we have that $0 = \nabla R = \nabla^{M_1} R^{M_1} + \nabla^{M_2} R^{M_2}$, so $\nabla^{M_2} R^{M_2} = 0$ and $\nabla^{M_1} R^{M_1} = 0$.

Additionally, if (M, g) is a decomposable Lorentzian manifold we can immediately notice that M must be locally isometric to the product of a Lorentzian manifold and a Riemannian manifold.

Lemma 3.68. *Let (M, g) be a semi-Riemannian symmetric space, then M is decomposable if and only if M is isometric to a product $(M_1, g_1) \times (M_2, g_2)$.*

Proof. Suppose that M is a decomposable semi-Riemannian symmetric space, we will show it is a global product of manifolds. Suppose that (M, g) is locally isometric to the product $(M_1, g_1) \times (M_2, g_2)$. Since each M_i are locally symmetric, it follows from Theorem 3.59 that M_i is locally isometric to a simply connected symmetric space S_i hence M is locally isometric to the simply connected product space $S_1 \times S_2$. By Theorem 3.10 this local isometry extends to a global isometry, so M is a global product of manifolds.

Now suppose that M is isometric to a product $(M_1, g_1) \times (M_2, g_2)$, then M is locally isometric to the product and is hence decomposable. $\square$

Definition 3.69. Let $(\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}, \sigma, B)$ be a symmetric triple. We say it is *decomposable* if $\mathfrak{m}$ is the direct sum of two $\mathfrak{h}$-invariant, B-orthogonal, non-zero subspaces. It is *indecomposable* if it is not decomposable. It is *irreducible* if $\mathfrak{m}$ has no $\mathfrak{h}$ invariant subspace.

Theorem 3.70 (Theorem 2 in Cahen & Wallach (1970))**.** *Let S be a semi-Riemannian symmetric space with universal cover $\tilde{S}$. Then $\tilde{S}$ is indecomposable if and only if its symmetric triple is indecomposable.*

Let $\mathfrak{g}$ be a Lie algebra, we say that $\mathfrak{g}$ is *simple* if it is non-abelian and has no nonzero proper ideals, additionally, $\mathfrak{g}$ is said to be *semi-simple* if $\mathfrak{g}$ is the direct sum of simple Lie algebras. Finally, we say that $\mathfrak{g}$ is *solvable* if its derived series terminates in the zero subalgebra.

Theorem 3.71 (Theorem 3 in Cahen & Wallach (1970))**.** *Let $(\mathfrak{g}, \sigma, B)$ be an indecomposable symmetric triple, then $\mathfrak{g}$ is either semi-simple or solvable.*

We will now consider the different possible cases for $\mathfrak{g}$ separately.

Euclidean Case

First, let consider the case where $\mathfrak{g}$ has trivial -1-eigenspace decomposition with respect to σ. A symmetric triple $(\mathfrak{g}, \sigma, B)$ is said to be *Euclidean* if $\mathfrak{h} = 0$. By the correspondence between symmetric triples and simply connected symmetric space described in Remark 3.66, we see that Minkowski space, $\mathbb{R}^n_1$ has an Euclidean corresponding symmetric triple since the Riemann curvature tensor is equal to zero and thus

$$\mathfrak{h} = span\{R(U,V) \mid U, V \in T_p\mathbb{R}^n_1\} = \{0\}.$$

Semi-Simple Case

Proposition 3.72 (Proposition 1 in Cahen & Wallach (1970))**.** *Let $(\mathfrak{g}, \sigma, B)$ be a semi-simple indecomposable Lorentzian symmetric triple such that the action of $\mathfrak{h}$ on $\mathfrak{m}$ is not irreducible. Then up to isomorphism $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{h}$ is the space of all real diagonal matrices in $\mathfrak{g}$.*

Then we have the following Lie algebra isomorphisms $\mathfrak{sl}(2, \mathbb{R}) \simeq \mathfrak{so}(2,1)$ and $\mathfrak{h} \simeq \mathfrak{so}(1,1)$ which corresponds to $\tilde{H}^2_1 = SO(2,1)/SO(1,1) \simeq SO(1,2)/SO(1,1) = S^2_1$.

Now we can suppose that $(\mathfrak{g}, \sigma, B)$ such that $\mathfrak{g}$ is semi-simple, irreducible and indecomposable, then from Cahen (1998) the two possible cases are:

1. $\mathfrak{g}$ is the direct sum of two isomorphic ideals permuted by σ. The only Lorentzian example of this is

$$\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \quad \mathfrak{h} = \mathfrak{sl}(2, \mathbb{R}).$$

   This corresponds to $\tilde{H}^3_1$ which is the universal cover of $SO(2,2)/SO(1,2) \simeq SL(2,\mathbb{R}) \times SL(2,\mathbb{R})/SL(2,\mathbb{R})$.

2. $\mathfrak{g}$ is a simple real Lie algebra. Then the only Lorentzian signatures in the list of irreducible symmetric spaces from Berger (1957) are

$$\mathfrak{g} = \mathfrak{so}(1,n) \quad \mathfrak{h} = \mathfrak{so}(1, n-1),$$
$$\mathfrak{g} = \mathfrak{so}(2,n) \quad \mathfrak{h} = \mathfrak{so}(1, n).$$

   Which correspond to $S^n_1 = SO(1,n)/SO(1,n-1)$ and $\tilde{H}^n_1 = SO(2,n)/SO(1,n)$ respectively.

Solvable Case

After proving Theorem 3.71, the remainder of Cahen & Wallach (1970) is dedicated to constructing an indecomposable Lorentzian symmetric manifold with a solvable corresponding symmetric triple. Theorem (Cahen & Wallach 1970, 5) shows that such a space is isomorphic to the so-called Cahen-Wallach spaces described earlier in Section 3.4.

The above results allow us to classify locally symmetric Lorentzian manifolds. If M is an locally symmetric Lorentzian manifold then by Theorem 3.59 M is locally isometric to some Lorentzian symmetric space S. Then S is either decomposable or indecomposable. If it is indecomposable it must be one of the four spaces described above. If it is decomposable then by Lemma 3.68, S is the product of a Lorentzian symmetric space and a Riemannian symmetric space, which were completely classified by Cartan.

Chapter 4

Geometric manifolds

This chapter presents a brief introduction to geometric manifolds, which is a particular way of defining and describing locally homogeneous manifolds. In this context a manifold M is said to be locally homogeneous if each point is contained in a coordinate neighbourhood of charts to some fixed homogeneous space $X = G/H$, and given pairs of intersecting neighbourhoods in M, the transition maps between them are locally elements of G. We will primarily follow Chapter 8 of Ratcliffe (2006) and Goldman (2021). It is worth noting that Ratcliffe is working in the context of *geometric spaces*, which are homogeneous, geodesically complete and geodesically connected metric spaces with a continuous map $\varepsilon : E^n \to X$ and some $k > 0$ such that ε maps the ball $B(0, k)$ homeomorphically onto $B(\varepsilon(0), k)$. Additionally, G is a group of similarities of X. As Lorentzian metrics do not induce a metric space structure on the manifold, we do not consider similarities of X. We instead assume that X is a homogeneous semi-Riemannian manifold. Conveniently, many of the results for the case we are consider follow from identical proofs to those in Ratcliffe (2006), with only minor changes required at points. As we are interested in geodesic completeness of manifolds, we are particularly interested in the construction of a particular local isometry called the *development map*. Given a (G, X)-manifold M we will define the development map $D : \tilde{M} \to X$ and show in Proposition 4.19 that if D is a covering map from $\tilde{M}$ to X, where X is geodesically complete, then M must be geodesically complete. Once the general theory of (G, X)-manifolds is outlined, we prove Proposition 4.20, which shows how locally symmetric manifolds can be given (G, X)-structures. This result is key to the results in Carrière (1989) (and Klingler (1996)) who proves the geodesic completeness of flat (and constant sectional curvature) Lorentzian manifolds by showing that the development map must be surjective, and thus a covering map.

4.1 (G, X)-structures and (G, X)-maps

Definition 4.1 ((G, X)-atlas)**.** Let X be an manifold, G be a Lie group acting transitively on X, and M be a smooth manifold. A (G, X)-*atlas* for M is a collection of charts $\phi_i : U_i \to X$ satisfying:

1. The open sets U_i cover M.
2. ϕ_i is a diffeomorphism onto its image.
3. $\phi_{ij} := \phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$ agrees in a neighbourhood of each point of its domain with an element of G, that is, for each $x \in \phi_j(U_i \cap U_j)$, there exists some neighbourhood U of x such that $\phi_{ij}|_U = g|_U$ for some $g \in G$. Functions that satisfy this property are called *locally-*G.

Throughout this chapter the term *charts* will always refer to charts from a (G, X)-atlas and not the charts from M into $\mathbb{R}^n$ which give M a manifold structure.

The following theorem is a variation of 8.3.1 in Ratcliffe (2006).

Theorem 4.2. *Let $\mathcal{A}$ be a (G, X)-atlas for M. Then there is a unique maximal (G, X)-atlas for M containing $\mathcal{A}$.*

Proof. Let $\mathcal{A} = \{\phi_i : U_i \to X\}$ and let $\bar{\mathcal{A}}$ be the set of all functions $\phi : U \to X$ such that

1. The set U is an open connected subset of M.
2. The function ϕ maps U homeomorphically onto an open subset of X.
3. The function
$$\phi \circ \phi_i^{-1} : \phi_i(U_i \cap U) \to \phi(U_i \cap U)$$
is locally-G for each i.

$\bar{\mathcal{A}}$ contains $\mathcal{A}$ by construction. Given $\phi : U \to X$ and $\psi : V \to X$ in $\bar{\mathcal{A}}$, the restriction of $\psi \circ \phi^{-1}$ to U_i for each i is:

$$\psi \circ \phi^{-1} : \phi(U \cap V \cap U_i) \to \psi(U \cap V \cap U_i)$$
$$\psi \circ \phi^{-1} = \psi \circ \phi_i^{-1} \circ \phi_i \circ \phi^{-1} = gh^{-1} \in G.$$

So the restrictions of $\psi \circ \phi^{-1}$ to U_i are locally-G. Since $\{U_i\}$ is an open cover of M, we get that $\psi \circ \phi^{-1}$ is locally-G at each point of its domain. If $\tilde{\mathcal{A}}$ is another (G, X)-atlas containing $\mathcal{A}$ with some chart (W, η), then W must be an open connected subset of M, η must map W homeomorphically into X and $\eta \circ \phi_i$ must be locally-G for each i, so $\tilde{\mathcal{A}} \subset \bar{\mathcal{A}}$. So $\bar{\mathcal{A}}$ is a unique maximal atlas for $\mathcal{A}$. $\square$

Definition 4.3 ((G, X)-manifold)**.** A manifold M equipped with a maximal (G, X)-atlas is called a (G, X)-*manifold*.

Definition 4.4 ((G, X)-map)**.** A function $\xi : M \to N$ between (G, X)-manifolds is a (G, X)-*map* if ξ is smooth and for each chart $\phi : U \to X$ for M and chart $\psi : V \to X$ for N, such that U and $\xi^{-1}(V)$ overlap, the function

$$\psi \circ \xi \circ \phi^{-1} : \phi(U \cap \xi^{-1}(V)) \to \psi(\xi(U) \cap V)$$

agrees in a neighbourhood of each point of its domain with an element of G.

4.2 Semi-Riemannian (G, X)-manifolds

Throughout Ratcliffe (2006), the term metric (X, G)-manifold refers to a (G, X)-manifold such that G is a group of isometries of a metric space X. In particular, since X has a metric space structure that is compatible with a Riemannian metric they are able to induce a compatible metric on M. The ability to induce a metric space structure on M does not generalise to the case of Lorentzian manifolds, we will replace metric (X, G)-manifolds with semi-Riemannian (G, X)-manifolds. Additionally, an important result in Chapter 3 is Theorem 3.59 which states that a locally symmetric manifold M must be locally isometric to a symmetric space. Since we are primarily interested in locally symmetric manifolds we require an additional assumption, that each of the coordinate charts are local isometries.

Definition 4.5. Let (X, g) be a semi-Riemannian homogeneous space with $G \subset Iso(X)$ acting transitively on X. We say M is a *semi-Riemannian* (G, X)-*manifold* if M is a (G, X)-manifold and G is the group of isometries on X.

Definition 4.6. Let M be a semi-Riemannian (G, X)-manifold. Then if each coordinate chart $\phi : U \to X$ is a local isometry, we say that M is a *locally isometric* (G, X)-*manifold*.

The following theorem is analogous to 8.3.3 in Ratcliffe (2006). Ratcliffe uses a Theorem 6.6.10 which states that any geodesically connected and geodesically complete metric space is rigid to prove that any two similarities that agree on a nonempty open subset must be equal. As X is not assumed to be a metric space in this thesis, this result will be replaced by Proposition 2.7, which states that if two isometries are equal at a point and have the same differential at that point, they must be equal. This substitution allows most of the results to hold with almost identical proofs.

Theorem 4.7. *Let $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ be a coordinate change of a semi-Riemannian (G, X)-manifold M. Then $\phi_j \circ \phi_i^{-1}$ is equal to an element of G on each connected component of its domain.*

Proof. Let C be a connected component of $\phi_i(U_i \cap U_j)$. Suppose that w and x are points in C. Then there are open subsets $W_1, \ldots, W_m$ of C such that $w \in W_1$, $x \in W_m$, the sets $W_k \cap W_{k+1}$ contain a nonempty open subset and $\phi_j \circ \phi_j^{-1}$ are locally g_k in each W_k. Then, since g_k and g_{k+1} must be equal on some nonempty open intersection of W_k and W_{k+1}, we have that $g_k = g_{k+1}$ by Proposition 2.7. Therefore, all g_k are equal and hence $\phi_j \circ \phi_i^{-1}$ is equal to g_1 at x and therefore on all of C. □

Lemma 4.8. *Let M and N be locally isometric (G, X)-manifolds. Then any (G, X)-map ξ from N to M is a local isometry.*

Proof. Let x be an arbitrary point of X and let U be a sufficiently small neighbourhood of x such that:

$$(\phi_i \circ \xi \circ \psi_j^{-1})|_U = h|_U$$

for some $h \in G$. Then

$$\begin{aligned}(\phi_i \circ \xi)|_{\psi_j^{-1}(U)} &= (h \circ \psi^j)|_{\psi_j^{-1}(U)} \\ \xi|_{\psi_j^{-1}(U)} &= (\phi_i^{-1} \circ h \circ \psi_j)|_{\psi_j^{-1}(U)}.\end{aligned}$$

So $\xi|_{\psi_j^{-1}(U)}$ is the composition of local isometries and hence is a local isometry. □

4.3 Continuation of curves

As before, let M be a semi-Riemannian (G, X)-manifold. Given a chart $\phi : U \to X$ and a curve $\gamma : [a, b] \to M$ such that $\gamma(a) \in U$, there exists the curve $\phi \circ \gamma_1$, in X where γ_1 is a restriction of γ to some $[a, c] \subset [a, b]$ such that $\gamma([a, c])$ is contained in U. We can define a curve $\hat{\gamma}$ in X which extends $\phi \circ \gamma_1$. It is called the continuation of $\phi \circ \gamma_1$ along γ and is constructed by the following method:

Choose a partition of $[a, b]$ such that

$$a = t_0 < t_1 < \ldots < t_m = b$$

and a set of charts $\{(\phi_i, U_i)\}$ such that $\phi_1 = \phi$ and $\gamma([t_{i-1}, t_i]) \subset U_i$. Let $g_i \in G$ be the element that is equal to $\phi_i \circ \phi_{i+1}^{-1}$ on the connected component of $\phi_{i+1}(U_i \cap U_{i+1})$ that contains $\phi_{i+1}\gamma(t_i)$. Let $\gamma_i = \gamma|_{[t_{i-1}, t_i]}$. Then $\phi_i \circ \gamma_i$ and $g_i \phi_{i+1} \circ \gamma_{i+1}$ are curves in X such that:

$$g_i \phi_{i+1} \circ \gamma(t_i) = \phi_i \circ \phi_{i+1}^{-1} \circ \phi_{i+1} \circ \gamma(t_i) = \phi_i \circ \gamma(t_i).$$

Definition 4.9. Now we define the *continuation (starting at $\phi \circ \gamma_1$) of γ in X*, written $\hat{\gamma}$ by concatenation:

$$\begin{aligned}\hat{\gamma} \ &: [a,b] \to X \\ \hat{\gamma} \ &= (\phi_1 \circ \gamma_1)\#(g_1\phi_2 \circ \gamma_2)\# \cdots \#(g_1 \cdots g_{m-1}\phi_m \circ \gamma_m).\end{aligned}$$

Lemma 4.10. *Let M be a semi-Riemannian (G, X)-manifold. Given a curve γ in M, its continuation in X, $\hat{\gamma}$, is independent of choice of charts and choice of partition.*

Proof. First, we show that $\hat{\gamma}$ does not depend on the choice of charts $\{\phi_i\}$ once a partition of $[a, b]$ has been fixed. Suppose $\{\psi_i : V_i \to X\}$ is another set of charts for M such that $\psi_1 = \phi$ and V_i contains $\gamma([t_{i_1}, t_i])$ for each $i = 1, \dots, m$. Let h_i be the element of G that is equal to $\psi_i \circ \psi_{i+1}^{-1}$ on the connected component of $\psi_{i+1}(V_i \cap V_{i+1})$ containing $\psi_{i+1} \circ \gamma(t_i)$. As $U_i \cap V_i$ contains $\gamma([t_{i-1}, t_i])$, it is enough to show that

$$g_1 \cdots g_{i-1}\phi_1 = h_1 \cdots h_{i-1}\psi_i \tag{4.1}$$

on the connected component of $U_i \cap V_i$ containing $\gamma([t_{i-1}, t_i])$ for each i. This is true in the case $i = 1$ as $\psi_1 = \phi = \phi_1$. So we proceed by induction. Suppose that Equation (4.1) holds in the case $i - 1$, i.e.

$$g_1 \cdots g_{i-2}\phi_1 = h_1 \cdots h_{i-2}\psi_{i-1}.$$

Then $\psi_i \circ \phi_i^{-1}$ is locally-G, so on the connected component of $\psi_i(U_i \cap V_i)$ containing $\psi_i \circ \gamma([t_i - 1, t_i])$ it must be equal to some element $f_i \in G$. Then notice when we restrict to the component of $\phi_i(U_{i-1} \cap V_{i-1} \cap U_i \cap V_i)$ containing $\phi_i \circ \gamma(t_{i-1})$ we see by our inductive hypothesis:

$$f_i = \psi_i \circ \phi_i^{-1} = \psi_i \circ (\psi_{i-1}^{-1} \circ h_{i-2}^{-1} \cdots h_1^{-1})(g_1 \cdots g_{i-2}\psi_{i-1}) \circ \phi_i^{-1}.$$

Additionally, from the definitions of g_i and h_i we see that on the same component:

$$\begin{aligned}&(h_{i-1}^{-1} \cdots h_1^{-1})(g_1 \cdots g_{i-1}) \\ &\quad = \ \psi_i \circ \psi_{i-1}^{-1}(h_{i-2}^{-1} \cdots h_1^{-1})(g_1 \cdots g_{i-2})\phi_{i-1} \circ \phi_i^{-1} \\ &\quad = \ f_i\end{aligned}$$

and hence by Proposition 2.7 $f_i = (h_{i-1}^{-1} \cdots h_1^{-1})(g_1 \cdots g_{i-1})$ on X. So:

$$\begin{aligned}(g_1 \cdots g_{i-1})\phi_i \ &= \ (h_1 \cdots h_{i-1})(h_{i-1}^{-1} \cdots h_1^{-1})(g_1 \cdots g_{i-1})\phi_i \\ &= \ (h_1 \cdots h_{i-1})f_i\phi_i \\ &= \ (h_1 \cdots h_{i-1})\psi_i\end{aligned}$$

on the component of $U_i \cap V_i$ containing $\gamma([t_{i-1}, t_i])$. This completes the induction.

Now we will show that $\hat{\gamma}$ does not depend on the partition of $[a, b]$. Let $\{s_i\}_{i=1}^{l}$ be another partition with charts $\{\psi_i : V_i \to X\}$. Then $\{r_i\} = \{s_i\} \cup \{t_i\}$ is a refinement of the two partitions. Since the charts $\{\phi_i\}$ and $\{\psi_i\}$ can be used separately for the choice of partition of $\{r_i\}$, we deduce that all three partitions determine the same curve $\hat{\gamma}$. □

Theorem 4.11 (8.4.1 in Ratcliffe (2006))**.** *Let $\phi : U \to X$ be a chart for a semi-Riemannian (G, X)-manifold M, let $\alpha, \beta : [a, b] \to M$ be curves with the same initial point in U and the same terminal point in M, and let $\hat{\alpha}, \hat{\beta}$ be the continuations of $\phi \circ \alpha_1, \phi \circ \beta_1$ along α, β respectively. If α and β are homotopic by an endpoint fixing homotopy, then $\hat{\alpha}$ and $\hat{\beta}$ have the same endpoints and are homotopic by an endpoint fixing homotopy.*

Proof. If α and β only differ along a sub-interval (c, d) such that $\alpha([c, d])$ and $\beta([c, d])$ are contained in a simply connected coordinate neighbourhood U, then the result is immediate, as the continuation of the curves will be defined by the same g_i and ϕ_i. Let $H : [a, b]^2 \to M$ be an endpoint fixing homotopy from α to β. As $[a, b]$ is compact, we can partition $[a, b]$ into $a = t_0 < t_1 \ldots t_m = b$ such that $H([t_{i-1}, t_i] \times [t_{j-1}, t_j])$ is contained in a simply connected neighbourhood. Finally, we can take a further refinement to ensure $H([t_{i-1}, t_i] \times [t_{j-1}, t_j])$ is contained in a simply connected coordinate neighbourhood U_{ij}. Let α_{ij} be the curve determined by applying H to the curve shown in fig. 4.1(a), and define β_{ij} be the curve determined by applying H to the curve shown in fig. 4.1(b). Then it immediately follows that $\hat{\alpha}_{ij}$ and $\hat{\beta}_{ij}$ are homotopic by an endpoint fixing homotopy since each $\alpha_i j$ and β_{ij} only differ on a simply connected coordinate neighbourhood. Since being homotopy equivalent is a transitive property, can compose these homotopies by working from left to right, bottom to top in $I \times I$ to see:

$$\hat{\alpha} = \hat{\alpha}_{1,m} \simeq \hat{\beta}_{1,m} = \hat{\alpha}_{1,m-1} \simeq \ldots \simeq \hat{\beta}_{1,2} = \hat{\alpha}_{1,1} \simeq \hat{\beta}_{2,m} = \ldots = \hat{\alpha}_{m-1,1} \simeq \hat{\beta}_{m-1,1} = \hat{\beta}.$$

So there exists an endpoint fixing homotopy from $\hat{\alpha}$ to $\hat{\beta}$. □

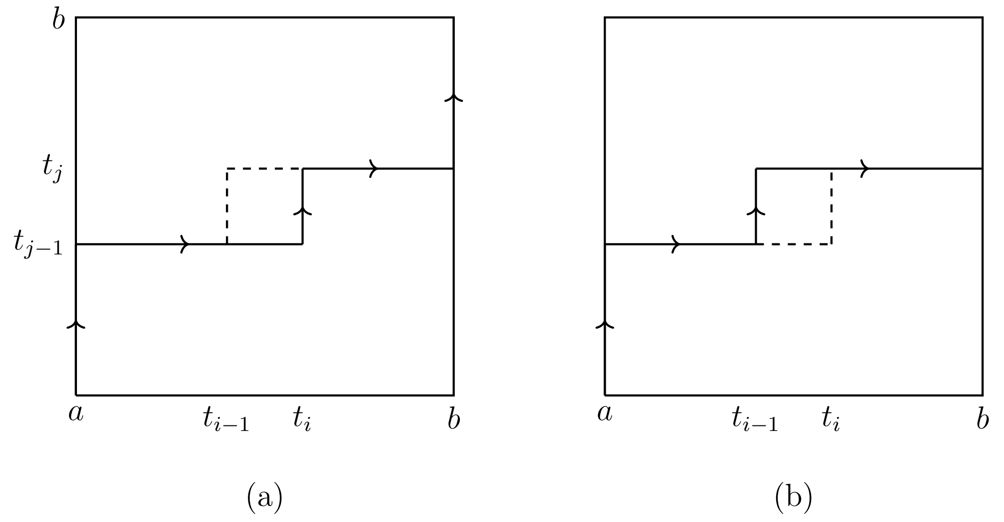
The following theorem is equivalent to Theorem 8.4.2 in Ratcliffe (2006).

Theorem 4.12. *A continuous function $\xi : M \to N$ between semi-Riemannian (G, X)-manifolds is a (G, X)-map if and only if for each point $u \in M$ there is a neighbourhood U of u and corresponding (G, X)-chart $\phi : U \to X$ such that ξ maps U homeomorphically onto an open subset of N and $\phi \circ \xi^{-1} : \xi(U) \to X$ is a chart for N.*

Proof. Suppose that $\xi : M \to N$ is a (G, X)-map and u is an arbitrary point of M. Let $\psi : V \to X$ be a chart of N, such that $\xi(u) \in V$. Since ξ is continuous, there is a chart $\psi : U \to X$ such that $\xi(U) \subset V$. Then

$$\psi \circ \xi \circ \phi^{-1} : \phi(U) \to \psi\xi(U)$$

Figure 4.1: Routes from (a, a) to (b, b) in the square $[a, b]^2$



agrees with an element $g \in G$ since $\phi(U)$ is connected, by Theorem 4.7. Hence ξ maps U homeomorphically onto an open subset of N and $\phi \circ \xi^{-1} : \xi(U) \to X$ agrees with $g^{-1}\psi : V \to X$. Therefore, $\phi \circ \xi^{-1}$ is a chart for N.

Conversely, suppose that for each point $u \in M$ there exists a chart (ϕ, U) of M with $u \in U$ such that ξ maps U homeomorphically onto an open subset of N, and $\phi \circ \xi^{-1} : \xi(U) \to X$ is a chart for N. Then ξ is continuous. Let $\chi : W \to X$ and $\psi : V \to X$ be charts for M and N respectively, such that W and $\xi^{-1}(V)$ have nonempty intersection, and let u be an arbitrary point in $W \cap \xi^{-1}$. Then there is a chart $\phi : U \to X$ such that ξ maps U homeomorphically onto an open subset of N and $\phi \circ \xi^{-1} : \xi(U) \to X$ is a chart for N. Observe that in a neighborhood of $\chi(u)$, the function

$$\psi \circ \xi \circ \chi^{-1} : \chi(W \cap \xi^{-1}(V)) \to \psi(\xi(W) \cap V)$$

agrees with $(\phi \circ \xi \circ \phi^{-1}) \circ (\phi \circ \chi^{-1})$. As $\phi \circ \chi^{-1}$ and $\psi \circ \xi \circ \phi^{-1}$ are coordinate changes for M and N respectively, $\phi \circ \xi \circ \chi^{-1}$ agrees in a neighbourhood of $\chi(u)$ with an element of G. Thus ξ is a (G, X)-map. $\square$

The following theorem is a variation of Theorem 8.4.3 in Ratcliffe (2006).

Theorem 4.13. *Let $\phi : U \to X$ be a chart of a simply connected semi-Riemannian (G, X)-manifold M. Then there is a unique (G, X)-map $\hat{\phi} : M \to X$ extending the chart ϕ, i.e. $\hat{\phi}|_U = \phi$.*

Proof. First, we define $\hat{\phi}$ using continuation of curves. Fix a point $u \in U$, and let v be an arbitrary point of M. M is simply connected, so there is a curve $\alpha : [a, b] \to M$ from u to v. The $\hat{\alpha} : [a, b] \to X$ be the continuation of $\phi \circ \alpha_1$ along α. Let $\hat{\alpha}(b)$ does not depend on the choice of α by Theorem 4.11, since M is simply connected. Hence, we may define a function $\hat{\phi} : M \to X$ by $\hat{\phi}(v) = \hat{\alpha}(b)$. Now we will show that $\hat{\phi}$ is a (G, X)-map using the fact that continuation of curves is independent of choices of path and charts. Let $\psi : V \to X$ be a chart for M with V containing v such that $\psi = \phi$ if $v \in U$. Then there is a partition

$$a = t_0 < t_1 \ldots < t_m = b$$

and a set of charts $\{\phi_i : U_i \to X\}_{i=1}^m$ for M such that $\phi_1 = \phi$ and U_i contains $\alpha([t_{i-1}, t_i])$ for each $i = 1, \ldots, m$ and $\phi_m = \psi$. Let α_i be the restriction of α to $[t_{i-1}, t_i]$ and let g_i be the element of G that is equal to $\phi_i \circ \phi_{i+1}^{-1}$ on the connected component of $\phi_{i+1}(U_i \cap U_{i+1})$ containing $\phi_{i+1} \circ \alpha(t_i)$. Then:

$$\hat{\alpha} = (\phi_1 \circ \alpha)\#(g_1\phi_2 \circ \alpha_2)\# \cdots \#(g_1 \cdots g_{m-1}\phi_m \circ \alpha_m).$$

Let $\beta : [c, d] \to V$ be a curve from v to w and let $g = g_1 \cdots g_{m-1}$. Then $\widehat{\alpha\beta} = \hat{\alpha}\#g\psi \circ \beta$. Hence $\hat{\phi}(w) = \widehat{\alpha\beta}(c) = g\psi(w)$. Therefore, $\phi(\hat{w}) = g\psi(w)$ for all w in V. Hence $\hat{\phi}$ maps V homeomorphically onto the open subset $g\psi(V)$ of X and $\psi \circ \hat{\phi}^{-1} : \hat{\phi}(V) \to X$ is the restriction of g^{-1}. Thus $\hat{\phi}$ is a (G, X)-map by Theorem 4.12 which extends ϕ to all of M. Now we will show that $\hat{\phi}$ is unique, suppose that $\xi : M \to X$ is a (G, X)-map extending ϕ. Without loss of generality, we may assume that the set of charts $\{\phi_i : U_i \to M\}_{i=1}^m$ for M has the property that

$$\phi_i \circ \xi^{-1} : \xi(U_i) \to X$$

is a chart for X by Theorem 4.12. Then $\phi_i \circ \xi^{-1}$ extends to an element h_i^{-1} of G. Hence $\xi(w) = h_i\phi_i(W)$ for all $w \in U_i$. As $\xi(w) = \phi(w)$ for all $w \in U$ we have that $h_1\phi = \phi$ so $h_1 = 1$. We proceed by induction to show that if $\xi|_{U_i} = \phi|_{U_i}$ then $\xi|_{U_{i+1}} = \phi|_{U_{i+1}}$. Suppose that $h_{i-1} = g_1 \cdots g_{i-2}$. Then for each w in U_{i-1} we have:

$$\begin{aligned} \xi(w) &= h_{i-1}\phi_{i-1}(w) \\ &= g_1 \cdots g_{i-2}\phi_{i-1}(w) = \hat{\phi}(w). \end{aligned}$$

Hence

$$h_i\phi_i(w) = \xi(w) = \hat{\phi}(w) = g_1 \cdots g_{i-1}\phi_i(w)$$

for all w in $U_{i-1} \cap U_i$. Therefore, $h_i = g_1 \cdots g_{i-1}$. Hence by induction we have that

$$\xi(v) = h_m\phi_m(v) = g\phi_m = \hat{\phi}(v).$$

Therefore, $\xi = \hat{\phi}$. Thus $\hat{\phi}$ is unique. $\square$

The following theorem is analogous to 8.4.4 in Ratcliffe (2006).

Theorem 4.14. *Let M be a simply connected semi-Riemannian (G, X)-manifold. If $\xi_1, \xi_2 : M \to X$ are (G, X)-maps, then there is a unique element g of G such that $\xi_2 = g\xi_1$.*

Proof. Let $\phi : U \to X$ be a chart for M such that $\phi \circ \xi_i^{-1} : \xi_i(U) \to X$ is a chart for $i = 1, 2$. Then by Theorem 4.13, there is an element g_i of G extending the chart $\phi \circ \xi_i^{-1} : \xi_i(U) \to X$. As $g_i\xi_i$ is a (G, X)-map extending ϕ for $i = 1, 2$, we have that $g_1\xi_1 = g_2\xi_2$ by the uniqueness of $\hat{\phi}$. Let $g = g_2^{-1}g_1$. Then $\xi_2 = g\xi_1$. If h is an element of G such that $\xi_2 = h\xi_1$, then $g\xi_1 = h\xi_2$ so $g = h$ by Proposition 2.7. Thus g is unique. □

4.4 Developing and monodromy

Let M be a connected semi-Riemannian (G, X)-manifold and let $\pi : \tilde{M} \to M$ be the universal covering projection. We will induce a (G, X)-structure on $\tilde{M}$. Let $\{\phi_i : U_i \to X\}$ be an (G, X)-atlas for M such that U_i is simply connected for each i. Then the set U_i is evenly covered by π for each i. Let $\{U_{ij}\}$ be the set of sheets over U_i and let $\pi_{ij} : U_{ij} \to U_i$ be the restriction of π to U_{ij}. Define $\phi_{ij} : U_{ij} \to X$ by $\phi_{ij} = \phi_i \circ \pi_{ij}$. Then ϕ_{ij} maps U_{ij} homeomorphically into $\phi_i(U_i)$ in X. Suppose U_{ij} and U_{kl} overlap. Then U_i and U_k overlap. Consider the function

$$\phi_{ij} \circ \phi_{kl}^{-1} : \phi_{kl}(U_{ij} \cap U_{kl}) \to \phi_{ij}(U_{ij} \cap U_{kl}).$$

For $x \in \phi_{kl}(U_{ij} \cap U_{kl})$:

$$\phi_{ij} \circ \phi_{kl}^{-1} = \phi_i \circ \pi_{ij} \circ \pi_{kl}^{-1} \circ \phi_k^{-1}(x) = \phi_i \circ \phi_k^{-1}(x).$$

Hence $\phi_{ij} \circ \phi_{kl}^{-1}$ is locally G. Therefore, $\{\phi_{ij} : U_{ij} \to X\}$ is an (G, X)-atlas for $\tilde{M}$. Then $\tilde{M}$ is a (G, X)-manifold with the (G, X)-structure described by this atlas.

Observe that π maps the coordinate neighborhood U_{ij} homeomorphically onto U_i and $\phi_{ij} \circ \pi^{-1} : \pi(U_{ij}) \to X$ is the chart $\phi_i : U_i \to X$ for M. Thus π is a (G, X)-map by Theorem 4.12.

Definition 4.15 (Developing map)**.** Let $\phi : U \to X$ be a chart for $\tilde{M}$. Then ϕ extends, via developing, to a unique (G, X)-map $D : \tilde{M} \to X$ by Theorem 4.13. The map

$$D : \tilde{M} \to X$$

is called the *developing map* for M determined by the chart ϕ.

By Theorem 4.13 any two developing maps for $\tilde{M}$ differ only by composition with an element of G. Thus D is unique up to composition with an element of G.

In particular, given a point $\tilde{x} \in \tilde{M}$, then there exists a unique path $\tilde{\gamma}$ (up to homotopy) from $\tilde{x}_0$ to $\tilde{x}$ for any point $\tilde{x}_0$ in U, which we can define the continuation $\phi \circ \tilde{\gamma}$, so

$$D(\tilde{x}) = \phi(\tilde{\gamma}(1)) = g_1 \cdots g_n \phi_n(\tilde{x}).$$

This well defined as the continuation of paths is independent of choice of partitions and the continuation two homotopic paths have an endpoint fixing homotopic continuation. Since the same equation could be used to describe any $\tilde{y}$ in a neighbourhood of $\tilde{x}$, D is a local diffeomorphism by construction.

Lemma 4.16 (p. 358 in Ratcliffe (2006))**.** . *Let M be a (G, X)-manifold. A deck transformation $\tau \in \pi_1(M)$, $\tau : \tilde{M} \to \tilde{M}$ is a (G, X)-map.*

Proof. Let $\tau : \tilde{M} \to \tilde{M}$ be a deck transformation and let $\tilde{u}$ be an arbitrary point of $\tilde{M}$. Then there is an i such that $\pi(\tilde{u}) \in U_i$, hence there is a j such that $\tilde{u} \in U_{ij}$. As τ permutes the sheets over U_i, there is a k such that $\tau(U_{ij}) = U_{ik}$. Observe that $\phi_{ij} \circ \tau^{-1} : \tau(U_{ij}) \to X$ is the chart $\phi_{ik} : U_{ik} \to X$. Therefore, τ is a (G, X)-map. □

Choose a base point u of M and a base point $\tilde{u}$ of $\tilde{M}$ such that $\pi(\tilde{u}) = u$. Let $\alpha : [0, 1] \to M$ be a loop based at u. Then α lifts to a unique curve $\tilde{\alpha}$ in $\tilde{M}$ starting at $\tilde{u}$. Let $\tilde{v}$ be the endpoint of $\tilde{\alpha}$. Then there is a unique deck transformation τ_α such that $\tau_\alpha(\tilde{u}) = \tilde{v}$. τ_α depends only on the homotopy class of α in the fundamental group $\pi_1(M, u)$ by Proposition 2.49. Let $\beta : [0, 1] \to M$ be another loop based at u. Then $\widetilde{\alpha\beta} = \tilde{\alpha}\#(\tau_\alpha \circ \tilde{\beta})$ and so $\tau_{\alpha\beta} = \tau_\alpha \circ \tau_\beta$.

Let $D : \tilde{M} \to X$ be a developing map for M. As $D \circ \tau_\alpha : \tilde{M} \to X$ is a (G, X)-map, there is a unique element g_α of G such that $D\tau_\alpha = g_a D$. Define

$$\begin{aligned} h : \pi_1(M, u) &\to G \\ [\alpha] &\mapsto g_\alpha. \end{aligned}$$

Then h is well defined, since g_α depends only on the homotopy class of α. Observe that

$$D \circ \tau_{\alpha\#\beta} = D \circ \tau_\alpha \circ \tau_\beta = g_\alpha D \circ \tau_\beta = g_\alpha g_\beta D$$

hence,

$$h([\alpha]\#[\beta]) = h([\alpha\#\beta]) = g_\alpha g_\beta = h([\alpha]) \circ h([\beta]).$$

Thus h is a homomorphism.

Definition 4.17 (Monodromy)**.** The homomorphism $h : \pi_1(M) \to G$ is called the *monodromy* (or (G, X)-holonomy) of M determined by the developing map D.

Note, if $D' : \tilde{M} \to X$ is another developing map for M, then there is a $g \in G$ such that $D' = gD$, and therefore

$$D' \circ \tau_\alpha = gD \circ \tau_\alpha = gg_\alpha D = gg_\alpha g^{-1} D'.$$

Hence the monodromy h' of M determined by D' differs from the monodromy of M determined by D by conjugation by g.

4.5 (G, X)-completeness

Definition 4.18 ((G, X)-Completeness)**.** A (G, X)-manifold is said to be (G, X)-complete if the developing map $D : \tilde{M} \to X$ is a covering map.

If X is simply connected this is equivalent to D being a diffeomorphism.

Proposition 4.19. *Let M be a locally isometric (G, X)-manifold. If X is geodesically complete and M is (G, X)-complete then M is geodesically complete.*

Proof. Since D is a (G, X)-map of a locally isometric (G, X)-manifold it is a local isometry by Lemma 4.8. If M is (G, X)-complete then D is a covering map so by Corollary 2.57 $\tilde{M}$ is complete if and only if X is complete. X is complete so $\tilde{M}$ is complete. So by Corollary 2.57 M is complete. □

As X is a homogeneous semi-Riemannian manifold it is not necessarily complete although there are many conditions that ensure completeness. If X is a Riemannian homogeneous manifold then it is complete (See O'Neill (1983) 9.37). If X is a compact homogeneous semi-Riemannian manifold then Marsden (1973) proved it is complete. The case we are most concerned with is when X is a symmetric space, which is complete by Lemma 3.3. The following section discusses how locally symmetric spaces can be equipped with a (G, X)-structure for a symmetric space X.

4.6 Locally symmetric spaces as (G, X)-manifolds

Proposition 4.20. *If (M, g) is a locally symmetric semi-Riemannian manifold, locally isometric to a simply connected symmetric space X. Then it can be given a locally isometric (G, X)-structure.*

Proof. Equip (M, g) with the maximal complete atlas containing $\{(U, \phi)\}$ where ϕ is a local isometry from $M \to X$. Evidently these charts cover M and the ϕ are diffeomorphisms onto their images, so it remains to show that the transition functions are locally-G. Given two charts (U_i, ϕ_i) and (U_j, ϕ_j) with non-empty intersection we can define the map:

$$\phi_j \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j).$$

Which is a local isometry from one neighbourhood of X to another. Fix some $x \in \phi_i(U_i \cap U_j)$ and let $L = d(\phi_j \phi_i^{-1})_x$. As ϕ_j and ϕ_i^{-1} are local isometries, $d(\phi_j)_q$ and $d(\phi_i^{-1})_p$ are linear isometries and hence $dL_p = d(\phi_j)_{\phi_i^{-1}(p)} d(\phi_i^{-1})$ is a linear isometry and thus preserves curvature. Since X is simply connected it follows from Theorem 3.10 that there is a unique covering map $\hat{\phi}$ such that for $d\hat{\phi}_x = L$. Since $\hat{\phi}$ is a global isometry, $\hat{\phi} \in G$. By Proposition 2.7, $\hat{\phi}|_{\phi_i(U_i) \cap U_j} = \phi_j \phi_i^{-1}$. So the transition functions are locally-G. □

Corollary 4.21. *If M is a manifold that is locally isometric to a manifold X with a $(G = Iso(X), X)$ structure, then the developing map $D : \tilde{M} \to X$ is a local isometry.*

Proof. Let $\tilde{p}$ be a point of $\tilde{M}$. Then $\pi(\tilde{p}) = p \in M$ has some neighbourhood V of M and local isometry $\phi : V \to \phi(V)$. By the construction of the (G, X) structure in Proposition 4.20, (V, ϕ) is a (G, X)-chart of M. Since π is a semi-Riemannian covering map, there is a neighbourhood U of $\tilde{p}$ such that $\pi|_U$ is a local isometry. Then $\pi|_U^{-1}(\pi(U) \cap V)$ is a neighbourhood of $\tilde{p}$ such that $\psi := \phi\pi|_U$ is a local isometry. From the construction of the (G, X) structure of a universal cover (in the definition of the developing map) we see that $(\pi|_U^{-1}(\pi(U) \cap V), \psi)$ is a chart of $\tilde{M}$. We can then construct the developing map determined by ψ, D_1, such that for any other developing map $D = gD_1$ for some $g \in G$. So D is a local isometry. $\square$

Chapter 5

Completeness of compact Lorentzian locally symmetric spaces

This chapter contains three sections. The first section uses the more general result of Theorem 2 in Leistner & Schliebner (2016), which shows that compact pp-waves are geodesically complete, to prove that locally symmetric compact manifolds which are locally isometric to the product of Cahen-Wallach space and flat Riemannian space are geodesically complete. This is achieved by slightly extending the methods used in the proof of Corollary 2 in Leistner & Schliebner (2016). The second section is concerned with Klingler (1996). In particular, we aim to present Proposition 1 of Klingler (1996) in a manner which each assumption and step as clear as possible, with an outline of the other results in the paper given in order to provide motivation and context for the proposition. This is in the hope of working towards an extension of both the proposition and the paper as a whole. The chapter concludes with a section discussing and some attempts to extend the methods in Klingler (1996).

5.1 Products of flat space and Cahen-Wallach Space

The geodesic completeness of compact locally Cahen-Wallach manifolds was shown in Leistner & Schliebner (2016) as a corollary to a more general theorem.

Definition 5.1. A Lorentzian manifold (M, g) is called *pp-wave* if it admits a global parallel null vector field $V \in \Gamma(TM)$, i.e. $V \neq 0, g(V, V) = 0$ and $\nabla V = 0$, and if its curvature tensor R satisfies

$$R(U, W) = 0, \quad \text{for all } U, W \in V^{\perp}.$$

Theorem 5.2 (Leistner & Schliebner '16 Theorem 2)**.** *Every compact pp-wave (M,g) is geodesically complete.*

It was then shown that compact locally symmetric Lorentzian manifolds which are locally isometric to Cahen-Wallach space have a time-orientable cover that is a compact pp-wave and are hence complete. By Corollary 2.57 we see that the original manifolds are complete also.
We follow a similar method to show compact locally symmetric Lorentzian manifolds that are locally isometric to the product Cahen-Wallach space with $\mathbb{R}^n$ are covered by compact pp-waves and are hence complete.

Let (M, g) be locally isometric to the product manifold $(CW(S_1), g_1) \times (\mathbb{R}^m, g_2)$, then for each point p of M there exists some neighbourhood U of p and coordinates $x^+, x^-, x^1, \dots, x^{n+m}$ where $x^1, \dots, x^m$ correspond to the standard coordinates on $\mathbb{R}^m$ and $x^+, x^-, x^{m+1}, x^{m+n}$ correspond to the standard coordinates on $CW(S_1)$ as described in Definition 3.36. We will call this choice of coordinates the *standard coordinates* for a manifold which is locally isometric to the product of Cahen-Wallach space and flat Riemannian space.

We can additionally define the $(n+m) \times (n+m)$-dimensional matrix S as such

$$S = \begin{bmatrix} 0 & 0 \\ 0 & S_1 \end{bmatrix}.$$

Then S be a symmetric $n \times n$ symmetric matrix of rank $n - m$. Since $\mathbb{R}^m$ is in the kernel of S it is immediate that the metric

$$g := 2dx^+ dx^- + \vec{x}^\top S \vec{x}(dx^+)^2 + \delta_{\mu\nu} dx^\mu dx^\nu$$

is equal to the product metric (g_1, g_2). for $x^+, x^- \in \mathbb{R}, \vec{x} \in \mathbb{R}^{n+m}, \mu, \nu = 1, \dots, n+m$. This has metric matrix

$$[g_{ij}] = \begin{bmatrix} 0 & \tilde{0} & 1 \\ \tilde{0} & I_n & \tilde{0} \\ 1 & \tilde{0} & \tilde{x}^\perp S \tilde{x} \end{bmatrix}.$$

This formulation is useful as is the same as the definition of a Cahen-Wallach space except without the condition requiring S to have non-zero determinant. So the calculations in Section 3.4 describe the Christoffel symbols and components of the curvature tensor, with $S_{\mu\nu} = 0$ for any $\mu, \nu = 1, \dots, n$.

Before we prove Theorem 5.6, we first require the following lemmas and a definition.

Definition 5.3. Let $\mathcal{E}$ be a sub-bundle of TM. We say that $\mathcal{E}$ is a *parallel* sub-bundle of TM if for all vector fields X and sections $Y \in \Gamma(\mathcal{B})$ the vector field $\nabla_X Y$ is a section of $\mathcal{E}$.

Now we define the bundle $\mathcal{B}$ to be the kernel of the curvature endomorphism. i.e.

$$\mathcal{B} := \{Z \in TM \mid R(X, Y)Z = 0 \quad \forall X, Y \in TM\}.$$

Lemma 5.4. *Let (M, g) be a manifold which is locally isometric to the product of Cahen-Wallach space $CW_{n+2}(S_1)$ and Euclidean space $\mathbb{R}^m$. Then the bundle $\mathcal{B}$ is parallel.*

Proof. Let p be an arbitrary point of M, then we can equip some neighbourhood U of p with standard coordinates x^+, x^-, x^i. First, we will show that

$$\mathcal{B}|_U = span\{\partial_-, \partial_1, \dots, \partial_m\}.$$

First, recall the Christoffel symbols from Definition 3.36:

$$\begin{aligned} \Gamma^-_{\mu+} = \Gamma^-_{+\mu} &= x^\nu S_{\mu\nu}, \\ \Gamma^\mu_{++} &= -x^\nu S_{\mu\nu}. \end{aligned}$$

So:

$$\begin{aligned} \nabla_X \partial_- &= X^i \Gamma^j_{i-} \partial_j \\ &= 0, \text{ where } i, j = +, -, 1, \dots, m+n, \\ \nabla_X \partial_\mu &= X^i \Gamma^j_{i\xi} \partial_j = X^i x^\nu S_{\xi\nu} \partial_- \\ &= 0, \text{ for } \xi = 1, \dots, m. \end{aligned}$$

Since these vector fields are parallel, they are contained in the kernel of the curvature endomorphism, i.e. $\mathcal{B}|_U \supset span\{\partial_-, \partial_\xi \mid \xi = 1, \dots, m\}$. Now consider some arbitrary $X \in \mathcal{B}$, we can write

$$X = a\partial_+ + b\partial_- + c^\xi \partial_\xi + f^\kappa \partial_\kappa$$

for $\xi = 1, \dots, m$ and $\kappa = m+1, \dots, n$. Then

$$\begin{aligned} 0 = R(U, V)X &= R(U, V)(a\partial_+ + b\partial_- + c^\xi \partial_\xi + f^\kappa \partial_\kappa) \\ &= R(U, V)(a\partial_+) + R(U, V)(f^\kappa \partial_\kappa). \end{aligned}$$

Since this holds for arbitrary U, V take $U = \partial_\mu$ and $V = \partial_+$. So we have

$$0 = aR(\partial_\mu, \partial_+)\partial_+ + f^\kappa R(\partial_\mu, \partial_+)\partial_\kappa.$$

Now recalling the non-zero components of the Riemann curvature tensor calculated in Definition 3.36

$$R_{\mu+\nu}{}^- = R_{++\nu}{}^\mu = S_{\mu\nu}.$$

We have that

$$0 = aS_{\mu\nu}\partial_\nu - f^\kappa S_{\mu\kappa}\partial_-.$$

Since ∂_ν and ∂_κ are linearly independent and $S_{\mu\kappa}$ is non degenerate (as $\kappa = m+1\dots,n+m$) it follows that $a=0$ and $f^\kappa = 0$. So $\mathcal{B}|_U = span\{\partial_-, \partial_\xi \mid \xi = 1,\dots,m\}$.

Since there is a basis of sections of $\mathcal{B}|_U$ that is parallel it follows that the bundle $\mathcal{B}|_U$ is parallel too. Since the point p was arbitrarily chosen it follows that the full bundle $\mathcal{B}$ is parallel. □

Lemma 5.5. *The elements of $O(1,n+1)$ that stabilise a null line $\mathbb{R}\cdot e_-$ in a basis e_-, e^i, e_+ such that are of the form*

$$B = \begin{pmatrix} a & X & \frac{-a^{-1}}{2}XX^\top \\ 0 & A & -a^{-1}AX^\top \\ 0 & 0 & a^{-1} \end{pmatrix} \textit{ with } A \in O(n), X \in \mathbb{R}^n, a \in \mathbb{R}^*.$$

Proof. Let $B \in O(1,n)$ fix $\mathbb{R}\partial_-$, then we can write B as such.

$$B = \begin{pmatrix} a & X & b \\ 0 & A & U \\ 0 & V & d \end{pmatrix}.$$

Additionally consider the matrix

$$g = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

It is a fact that $B \in O(1,n+1)$ if and only if $B^\top g B = g$ i.e.

$$\begin{pmatrix} 0 & aV & ad \\ aV^\top & X^\top V + A^\top A + V^\top U & dX^\top + A^\top U + bV^\top \\ ad & bV + U^\top A + dX & 2bd + U^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Hence we are able to immediately conclude that $V=0$, $d = a^{-1}$ and therefore $A^\top A = I$. Then $A^\top U = -a^{-1}X^\top$ so $U = -a^{-1}UX^\top$. Finally, we have that $2bd = -U^2 = d^2(UX^\top)^\top UX^\top = d^2 XX^\top$ so $b = \frac{a^{-1}}{2}XX^\top$. So we have that

$$B = \begin{pmatrix} a & X & \frac{-a^{-1}}{2}XX^T \\ 0 & A & -a^{-1}AX^\top \\ 0 & 0 & a^{-1} \end{pmatrix} \text{ with } A \in O(n), X \in \mathbb{R}^n, a \in R^*.$$

□

We denote by $\mathbb{R}^*$ the subgroup generated by fixing $A = Id_n$ and $X = 0$. Similarly $\mathbb{R}^n$ is the subgroup generated by fixing $A = Id_n$ and $a = 1$.
Now notice $B^\top g B = g$ if and only if $gB^\top g = g^{-1}B^\top g = B^{-1}$. If we require that B stabilises the null line as above, we get:

$$B^{-1} = \begin{pmatrix} a^{-1} & (-a^{-1}AX^\top)^\top & \frac{-a^{-1}}{2}XX^\top \\ 0 & A^T & X^\top \\ 0 & 0 & a \end{pmatrix}.$$

We write this group as

$$\mathrm{Stab}_{O(1,n+1)(\mathbb{R}\cdot_-)} = \mathbb{R}^n \rtimes (\mathbb{R}^* \times O(n)).$$

Theorem 5.6 (Completeness of compact Manifolds which are locally isometric to the product Cahen-Wallach Space and Euclidean space)**.** *Let (M, g) be a compact manifold which is locally isometric to the product of Cahen-Wallach space and Euclidean space. Then the time orientable cover of (M, g) is a pp-wave and hence complete. In particular, it therefore follows that (M, g) itself is complete.*

Proof. We prove this by proving that the double cover of (M, g) admits a parallel null vector field. We achieve this by showing that the holonomy group of (M, g) at an arbitrary point $p \in M$ admits a null vector field on which the holonomy group acts as $\mathbb{Z}_2$. Let $p \in M$, then there are local coordinates with that the symmetric matrix S has $S_{\xi\nu} = S_{\nu\xi} = 0$ for $\xi = 1, \dots, n$ and $\nu = 1, \dots, n + m$. As before, define the bundle $\mathcal{B}$ to the kernel of the curvature endomorphism.

$$\mathcal{B}|_p := \{v \in T_pM : R(x, y)v = 0 \ \ \forall x, y \in T_pM\}.$$

By Lemma 5.4, $\mathcal{B}$ is parallel. Since $\mathcal{B}$ is parallel, it is invariant under parallel transport. Therefore, $\mathcal{B}^\perp$ is also invariant under parallel transport. Next notice:

$$\begin{aligned} g(\partial_-, \partial_-) &= 0, \\ g(\partial_-, \partial_\xi) &= 0 \quad \xi = 1, \dots, m, \\ g(\partial_\xi, \partial_\xi) &= 1 \quad \xi = 1, \dots, m. \end{aligned}$$

So we can define a global null line bundle that is invariant under parallel transport:

$$\mathcal{L} := \mathcal{B} \cap \mathcal{B}^\perp.$$

At each point p there exists standard coordinates so that we have

$$\mathcal{L}|_p = \mathbb{R}\partial_-.$$

As $\mathcal{L}$ is invariant under parallel transport, it is invariant under holonomy, and thus by Lemma 5.5 we can write each element of the holonomy group at the point p, written h as such:

$$h = \begin{pmatrix} a & X & \frac{-a^{-1}}{2}XX^T \\ 0 & A & -a^{-1}AX^T \\ 0 & 0 & a^{-1} \end{pmatrix} \text{ with } A \in O(n+m), X \in \mathbb{R}^{n+m}, a \in R^*$$

Additionally, since M is locally isometric to a globally symmetric space, by Theorem 3.59 it is locally symmetric, i.e. $\nabla R = 0$. Therefore, R is invariant under parallel transport (and thus also holonomy) so $h \cdot R = R$. Therefore, we calculate at $p \in M$:

$$\begin{aligned} S_{\mu\nu}\partial_- &= R(\partial_\mu, \partial_+)\partial_\nu \\ &= (h \cdot R)(\partial_\mu, \partial_+)\partial_\nu \quad \text{by } R = h \cdot R \\ &= h(R(h^{-1}\partial_\mu, h^{-1}\partial_+)h^{-1}\partial_\nu). \end{aligned}$$

Since R is a $(3,1)$ tensor, we notice which component transforms covariantly and which transform contravariantly by considering h as a change of basis. From the previous section we can calculate:

$$\begin{aligned} h^{-1}\partial_\mu &= (X_\eta A^\mu_\eta)\partial_- + (A^\mu_\nu)\partial_\nu, \\ h^{-1}\partial_+ &= (\frac{-a^{-1}}{2}XX^\top)\partial_- + X^{\top\nu}\partial_\nu + a\partial_+. \end{aligned}$$

Hence:

$$\begin{aligned} S_{\mu\nu}\partial_- &= R(\partial_\mu, \partial_+)\partial_\nu \\ &= h(R((X_\eta A^\mu_\eta\partial_- + A^\mu_\nu\partial_\nu), (\frac{-a^{-1}}{2}XX^\top\partial_- + X^{\top\nu}\partial_\nu + a\partial_+))(X_\eta A^\gamma_\eta\partial_- + A^\gamma_\nu\partial_\nu)). \end{aligned}$$

By the multi-linearity of R, we check the non-zero components and the above equation simplifies to:

$$\begin{aligned} S_{\mu\nu}\partial_- &= h(A^\mu_\eta a A^\nu_\gamma S_{\eta\gamma}\partial_-) \\ &= a^2 A^\mu_\eta A^\nu_\gamma S_{\eta\gamma}\partial_-. \end{aligned}$$

at a point p. Next notice that the matrix $A^\top SA$ has entries $[A^\top SA]^\mu_\nu = A^\eta_\mu S_{\gamma\eta} A^\gamma_\nu$, so by the above equation we see $S = a^2 A^\top SA$. So by taking the trace on each side:

$$tr(S) = tr(a^2 A^\top SA) = tr(a^2 SA^\top A) = tr(a^2 S) = a^2 tr(S).$$

Hence $a^2 = 1$ and $a = \pm 1$. So therefore the holonomy group acts on the fibres of $\mathcal{B}$ by ± 1. So the time-orientable cover of (M, g) is a compact manifold admitting a global parallel null vector field, so it is a compact *pp-wave*, which is complete, and hence (M, g) is complete also, by Theorem 2 of Leistner & Schliebner (2016). $\square$

5.2 Geodesic completeness of compact spaces of constant curvature

The geodesic completeness of compact flat Lorentzian manifolds was proved in Carrière (1989) by considering them as geometric manifolds. This approach was extended in Klingler (1996) in order to prove the completeness of compact Lorentzian manifolds with constant curvature. In this section we discuss discompacity and geodesic segments, two concepts which are which are required to understand Klingler (1996). The remainder of this chapter will provide a presentation of the proof in Klingler (1996). We conclude by discussing some attempts at extending the methods to Cahen-Wallach spaces and manifolds which are locally isometric to the product of a constant curvature Lorentzian manifolds and a constant curvature Riemannian manifold.

5.2.1 Discompacity

The key idea in Carrière (1989) was that of *discompacity*, a measure of how non-compact a group is. This concept is also vital to Klingler's extension of the results in Carrière (1989), we will define discompacity and calculate the discompacity of some important isotropy groups.
Throughout the rest of this chapter, we will often discuss Euclidean spaces, written E, which can of course be equipped with the standard Euclidean inner product, but additionally can be equipped with a semi-Euclidean inner product as described in Section 3.2.2. We will be considering both products in the following results, but will specify which one we are considering at a given time.

Definition 5.7. Given E with the standard Euclidean metric, and a sequence of closed subsets $F_i \subset E$. We say that the sequence F_i converges if for all closed Euclidean balls B of E such that $F_i \cap B$ is non-empty, the sequence of compact sets $F_i \cap B$ converge, with the Hausdorff metric, to a compact subset of E.

In particular, a sequence of ellipsoids will converge to a (possibly degenerate) ellipsoid, which we will call C, so the dimension of C is well defined. The set of ellipsoids in E is written $\mathcal{E}$.

Definition 5.8 (Discompacity of a sequence of ellipsoids)**.** If a sequence of ellipsoids $\epsilon_i \in \mathcal{E}$ converges, in the sense defined above, to some compact subset C of B^n then we say that the discompacity of the sequence ϵ_i is the co-dimension of C. We write this as $\mathrm{disc}(\epsilon_i)$.

Notice that $\mathrm{disc}(\epsilon_i)$ is equal to the number of principal axes whose lengths tend towards 0. Suppose that $\mathrm{disc}(\epsilon_i) = r$, i.e. the compact set C has co-dimension r, then we can write $E = \mathbb{R}^{n-r} \times \mathbb{R}^r$ where $C \subset \mathbb{R}^{n-r} \times \{0\}$. Therefore it follows from the fact that

ϵ_i converges in the Hausdorff distance to C that the lengths of the r principal axes of each ϵ_i contained in $\mathbb{R}^r$ must approach 0. Furthermore, the minimum length of the $n-r$ principal axes of each ϵ_i contained in $\mathbb{R}^{n-r}$ must not approach 0, since C is of maximum dimension in $\mathbb{R}^{n-r}$ and ϵ_i converges to C.

Definition 5.9 (Discompacity of a set of ellipsoids)**.** Let $\mathcal{A} \subset \mathcal{E}$. The discompacity of $\mathcal{A}$ is the maximum discompacity of any sequence in $\mathcal{A}$ that converges in B^n.

Notice that any set of ellipsoids without a convergent, non-constant subsequence will always have discompacity 0. This can lead to some counter-intuitive examples. For example if we consider the a set of shrinking ellipsoids which are being translated to infinity, then they will have discompacity 0, in spite of the fact that all of the principal axes tend towards length 0. The fact that this is counter-intuitive provides some motivation as to why discompacity will only be defined for linear groups.

Definition 5.10 (Discompacity of a group)**.** Let G be a subgroup of $GL(E)$. Given $\epsilon \in \mathcal{E}$ and $\mathcal{A}(\epsilon) = G \cdot \epsilon$ (the orbit of ϵ under the action of G). The number $\mathrm{disc}(\mathcal{A}(\epsilon))$ is independent of $\epsilon \in \mathcal{E}$, so we define $\mathrm{disc}(G)$ to be this number.

Lemma 5.11. *Let G be a compact group then $disc(G) = 0$.*

Proof. Suppose otherwise, so there exists a sequence A_i such that $A_i B^n$ converges to an ellipsoid of codimension 1 or greater. Since G is compact there exists a limit $A \in G$ such that $A(B^n)$ has co-dimension 1 or greater. Then A has non-trivial kernel and is therefore not invertible, a contradiction. □

Since discompacity of a compact group is equal to zero, discompacity is some kind of measure of how non-compact a group is. The fact that the $O(1, n-1)$ has discompacity equal to 1 is vital to the proof in Carrière (1989). In the constant curvature case Klingler notices that $Iso(H_1^n) = O(2, n-1)$ has discompacity 2, and extends the original result by showing that the discompacity of the isotropy subgroup is what really matters in the proof, not the full isometry group. It is convenient that in all three cases, the isotropy subgroups are the same with $Iso_p(X) = O(1, n-1)$. The following lemma essentially follows from the fact that the discompacity of $O(1, n-1)$ is 1.

Lemma 5.12 (Lemma 2 in Klingler (1996))**.** *Let g be an element of $O(1, n-1)$ and let B^n be a unit Euclidean ball of E. Then gB^n is an ellipsoid whose principal axes have lengths $e^t, e^{-t}, 1, \ldots, 1$ for some $t \geq 0$.*

Proof. Let B^n be a unit Euclidean ball of E. We write the element

$$a_t = \begin{pmatrix} \cosh t & \sinh t & 0 \\ \sinh t & \cosh t & 0 \\ 0 & 0 & I \end{pmatrix}$$

of $O(1, n-1)$. Now let $A^+ = \{a_t : t \in \mathbb{R}^+\}$. We have the Cartan decomposition $O(1, n-1) = O(n-1)A^+O(n-1)$. Let g be an element of $O(1, n-1)$, $g = k_1 a_t k_2$ is its decomposition. Since $O(n-1)$ preserves the semi-Euclidean metric, gB is an ellipsoid whose principal axes have lengths $e^t, e^{-t}, 1, \ldots, 1$ for some $t \geq 0$, and thus the result is deduced. □

Corollary 5.13. *Let g_i be a sequence in $O(1, n-1)$ such that the sequence of ellipsoids $g_i B^n$ converges towards some limit $\epsilon \subset E$. Then ϵ has co-dimension of either 0 or 1. When ϵ has co-dimension 1, it is contained in a co-null hyperplane of E, i.e. a hyperplane of E that is the orthogonal (with respect to the semi-Euclidean inner product) complement of a null vector of E.*

Proof. Since $O(1, n-1)$ preserves the Lorentzian inner product on E, it follows that two principal axes whose Euclidean lengths may change under action of $O(1, n-1)$ are null vectors, and therefore we can immediately deduce the result. □

Example 5.14 (The discompacity of holonomy of Cahen-Wallach space)**.** From the proof of Theorem 5.6, we have that an element h of the holonomy group of a pp-wave can be written as

$$h = \begin{pmatrix} a & X & \frac{-a^{-1}}{2} X X^T \\ 0 & A & -a^{-1} A X^T \\ 0 & 0 & a^{-1} \end{pmatrix} \text{ with } A \in O(n), X \in \mathbb{R}^n, a \in \mathbb{R}^*.$$

We will call the group of all linear transformations of the form above G. Since the discompacity of a subgroup is bounded above by the discompacity of any group containing it, we can calculate an upper bound of the discompacity of pp-waves by calculating the discompacity of G. So h maps a point

$$p = \begin{pmatrix} u \\ y \\ v \end{pmatrix} \mapsto \begin{pmatrix} au - Xy - \frac{a^{-1}}{2} X X^T v \\ Ay - a^{-1} A X^T v \\ a^{-1} v \end{pmatrix}.$$

In particular, we know that for each element of the holonomy for a Cahen-Wallach space that $a = \pm 1$, so the holonomy group of Cahen-Wallach space will be contained in a group consisting of elements of the form

$$h = \begin{pmatrix} \pm 1 & X & \frac{\pm 1}{2} X X^T \\ 0 & A & \pm A X^T \\ 0 & 0 & \pm 1 \end{pmatrix},$$

Since $A \in O(n)$, action by A will preserve principal axes, as will the action of ± 1. So without loss of generality we can consider the subgroup of the holonomy where $A = I$ and $a = 1$, which maps a points:

$$p = \begin{pmatrix} u \\ y \\ v \end{pmatrix} \mapsto \begin{pmatrix} u - Xy - \frac{1}{2}XX^T v \\ y - Xv \\ v \end{pmatrix}.$$

Then if we take a sequence $X = -ny_1$ we see that the two dimensional vector space spanned by v and $y_1 + nv$ converges towards the one dimensional span of v as n gets larger. Hence $\mathrm{disc}(Hol(CW)) \geq 1$, but since we know that $Hol(CW) \subset O(1, n+1)$ which has discompacity 1, it follows that $\mathrm{disc}(Hol(CW)) = 1$.

5.2.2 Geodesic stars in $\tilde{M}$

Let M be a (G, X)-manifold. Let $\tilde{x}$ be a point of its universal cover $\tilde{M}$. Then we write $x = D(\tilde{x}) \in X$, where D is the developing map of Definition 4.15. We will first discuss some local properties of the development map D. Since D is a local isometry, it maps geodesics to geodesics and so, one may want to consider how D acts on the geodesic star of a point. This is a good choice of neighbourhoods, except for the fact that two points of X may be connected by more than one geodesic, which makes the identification of a geodesic star with its image under D more complicated. To simplify the identification we will define *segments* in X and M.

Let $\gamma : I \to X$ be a geodesic, we will call the image $\gamma(I)$ a *geodesic arc* of X. A geodesic arc is said to be closed if it is the image of a geodesic restricted to some closed interval.

Definition 5.15. If two distinct points p, q of X are joined by a unique geodesic, we will call the geodesic arc with endpoints p and q the segment from p to q, written $[p, q]$. For each point p of X we define the constant arc written $[p, p]$.

Notice that when $K = \pm 1$ we have that two antipodal points are connected by infinitely many geodesics and hence are *not* connected by a segment.

Example 5.16. Suppose that X is one of the Lorentzian constant curvature symmetric spaces. Then we have that:

- When $K = 0$ any two points are joined by a unique geodesic, so a segment of X will be a closed line segment.
- When $K \neq 0$ Corollary 3.32 shows that two antipodal points are connected by more than one geodesic and so a segment of X will be a closed geodesic arc that does not contain any antipodal points.

Two points $x, y \in X$ are then joined by at most one segment, written $[x, y]$ if it exists. If such a segment exists we say that x is *segmentally connected* to y, this relation is evidently symmetric.

Definition 5.17. Given some point $x \in X$, we define the *starlike neighbourhood of* x, denoted X_x, to be the set of all points which are segmentally connected to x, i.e.

$$X_x := \{y \in X \mid [x, y] \text{ exists}\}.$$

Example 5.18. Let X be a Lorentzian symmetric space with constant curvature. Then we have that:

- For $K = 0$, i.e. $X = \mathbb{R}^n_1$, $X_x = X$ as $\mathbb{R}^n_1$ is geodesically connected.
- For $K = \pm 1$, i.e. $X = S^n_1$ and $X = H^n_1$, $X_x = \{y \in X \mid K\langle x, y\rangle > 1\}$ by Proposition 3.31 and Lemma 3.25.

It is evident that the star X_x is open in X.
Now we define analogous notions in $\tilde{M}$.

Definition 5.19. A closed geodesic arc of $\tilde{M}$ is a *segment of* $\tilde{M}$ if its image under D is a segment of X.

If two points $\tilde{x}, \tilde{y} \in \tilde{M}$ are connected by a unique segment, it is written $[\tilde{x}, \tilde{y}]$. This relation is symmetric and it open since D is a local diffeomorphism. As before we say that $\tilde{x}$ is *segmentally connected* to $\tilde{y}$ if there exists a segment between them.

Lemma 5.20. *Two points $\tilde{x}, \tilde{y} \in \tilde{M}$ are connected by at most one segment.*

Proof. Suppose we have two geodesics $\tilde{\gamma}$ and $\tilde{\eta}$ with $\tilde{\gamma}(0) = \tilde{\eta}(0) = \tilde{x}$ and $\tilde{\gamma}(1) = \tilde{\eta}(1) = \tilde{y}$. Then $D(\tilde{\gamma}) = D(\tilde{\eta}) = [x, y]$ by the uniqueness of segments in X. Since D is a local diffeomorphism there is a neighbourhood U of $\tilde{x}$ such that $U \simeq D(U)$. In particular, D is injective so $\tilde{\gamma}([0,1]) \cap U = \tilde{\eta}([0,1]) \cap U$. Since $\tilde{\gamma}$ and $\tilde{\eta}$ are equal on an open subset it follows from Lemma 2.16 that $\tilde{\gamma} = \tilde{\eta}$. □

Definition 5.21. Given a point $\tilde{x} \in \tilde{M}$, define the *starlike neighbourhood of* $\tilde{x}$, written $\tilde{M}_{\tilde{x}}$, to be the set of points $\tilde{y} \in \tilde{M}$ that are segmentally connected to $\tilde{x}$. i.e.

$$\tilde{M}_{\tilde{x}} := \{\tilde{y} \in \tilde{M} \mid [\tilde{x}, \tilde{y}] \text{ exists}\}.$$

Since the relation of being segmentally connected in $\tilde{M}$ is open, the star $\tilde{M}_{\tilde{x}}$ is open in $\tilde{M}$.
Notice that by construction $D(\tilde{M}_{\tilde{x}})$ is contained in X_x.

Lemma 5.22 (Lemma 1 in Klingler (1996))**.** *The function D restricted to $\tilde{M}_{\tilde{x}}$ is injective.*

Proof. Take two points $\tilde{y}$ and $\tilde{z}$ of $\tilde{M}_{\tilde{x}}$ with the same image under D. Therefore, by the uniqueness of segments between two points, the segments $[\tilde{x}, \tilde{y}]$ and $[\tilde{x}, \tilde{z}]$ have the same image in X. As they have the same origin and since D is a local diffeomorphism, they coincide. So $\tilde{z} = \tilde{y}$. □

Since D is a local diffeomorphism and hence an open map, $D(\tilde{M}_{\tilde{x}})$ is an open set of X contained in X_x

To show that $D : \tilde{M} \to X$ is a covering map, one would like to first show that D maps a starlike neighbourhood $\tilde{M}_{\tilde{x}}$ in $\tilde{M}$ to the corresponding starlike neighbourhood X_x in X. We will show a weaker result, Proposition 5.28, the proof of which is considerable and will be split into multiple lemmas.

Definition 5.23. A subset C of X (or M) is *convex* is any two of its points are always connected by a segment contained in C.

Definition 5.24. Let C_1 and C_2 be subsets of X such that $C_1 \subset C_2$. We say that C_1 is *convex relative to* C_2 if any segment in C_2 that joins two points of C_1 is also a segment of C_1.

We will show that for each point $\tilde{x} \in \tilde{M}$, the set $D(\tilde{M}_{\tilde{x}})$ is convex relative to the star X_x. The key to this result is the discompacity of the isotropy subgroup of X.

5.2.3 Properties of X

In addition to discompacity, the proof of Proposition 1 in Klingler (1996) utilises certain properties of the model space X. We will first list these properties and then show that the constant curvature spaces satisfies each one. We are unsure if there are other Lorentzian symmetric spaces which satisfy these properties. Proposition 1 in Klingler (1996) states that the image of a star $\tilde{M}_{\tilde{x}}$ under D is convex relative to the star X_x. It is proved by contradiction, by supposing that there exists points a point $\tilde{x}$ such that there are two points $\tilde{y}, \tilde{z} \in \tilde{M}_{\tilde{x}}$ such that the segment $[y, z]$ exists but the segment $D([\tilde{y}, \tilde{x}])$ does not.

This proof uses that the manifold M with model space X satisfy the following properties:

1. M is a compact locally symmetric manifold which is locally isometric to the symmetric space X. In particular, we equip M with the locally isometric (G, X)-structure described in Proposition 4.20.

2. The isotropy group of X has discompacity less than or equal to 1.

3. X is isometrically embedded into some semi-Euclidean space E such that the $\langle \cdot, \cdot \rangle$ (non-positive definite) inner product on E induces the metric on X. Furthermore, we require that the isometry group $G = Iso(X)$ is a subgroup of the linear isometries of E.

4. Let x be a point in X and let y, z be arbitrary points in the star X_x then, there exists a two dimensional submanifold S of X which contains a convex subset C containing the points x, y and z. Additionally, since S is a 2-dimensional submanifold of X,

which in turn is embedded into E, the tangent space T_pS at a point p is identified with a 2-dimensional affine subspace, written $A_pS = T_pS + p$ which is canonically identified with a 2-dimensional affine subspace of E.

5. There exist a projection ϕ from a neighbourhood of X in E to X that satisfies the following properties:

   - ϕ is a G-equivariant projection onto X.
   - If γ is a geodesic of E such that $\gamma(0) \in X$, then there exists some $\varepsilon > 0$ such that $\phi \circ \gamma(-\varepsilon, \varepsilon)$ is equal to the image of a geodesic in X.
   - Let p be a point of S then ϕ, is a local diffeomorphism between a neighbourhood U of p in $A_pS := T_pS + p$ and a neighbourhood V of p in S. It has local inverses for a neighbourhood of each point written ψ_p.

Now let (M, g) be a compact Lorentzian manifold with constant sectional curvature K. We will show that all the properties listed above are satisfied.

1. Since M has constant curvature, it is evidently locally symmetric and by Theorem 3.59 M is locally isometric to a simply connected symmetric space X.

   - If $K < 0$ then X is the universal cover of anti-de Sitter space.
   - If $K = 0$ then X is Minkowski space.
   - If $K > 0$ then X is de Sitter space.

   Since geodesic completeness is preserved by homotheties, we can consider the cases $K = -1, 0, 1$. Then by Proposition 4.20, M can be given a $(G = Iso(X), X)$-structure. Recalling the isometry groups calculated in Chapter 3 we have that:

   - If $K = -1$ then $(G, X) = (O(2, n-1), H^n_{-1})$.
   - If $K = 0$ then $(G, X) = (O(1, n-1) \ltimes \mathbb{R}^n, \mathbb{R}^n_1)$.
   - If $K = 1$ then $(G, X) = (O(1, n), S^n_1)$.

2. The isotropy group in each case is $O(1, n-1)$ by Lemma 3.22 and Proposition 3.34. $O(1, n-1)$ has discompacity 1.

3. For $X = \mathbb{R}^n_1$ we immediately have that $E = \mathbb{R}^n_1$. For $X = S^n_1$ $E = \mathbb{R}^{n+1}_1$ and for $X = H^n_1$, $E = \mathbb{R}^{n+1}_2$ as they are defined in Section 3.3 as hyperquadrics of these semi-Euclidean spaces.

4. In the constant curvature case, we can construct a surface S as such

   - When $X = \mathbb{R}^n_1$, S is the affine plane of E containing x, y, z.

- When $X = S_1^n$ and $X = H_1^n$ the surface S is defined to be the intersection $X \cap span\{x, y, z\}$.

Then we can construct a convex subset of S containing x, y, z. In particular, we construct a *geodesic triangle* which we will call T_{xyz}.

- When $X = \mathbb{R}_1^n$, T_{xyz} is the closed affine triangle of E with vertices x, y, z.
- When $X = S_1^n$ and $X = H_1^n$ we define the triangle T_{xyz} to be the intersection of the closed half-cone with origin 0 and triangular section xyz, written C_{xyz}.

$$T_{xyz} = C_{xyz} \cap X = \{\lambda x + \mu y + \rho z \mid \lambda, \mu, \rho \geq 0\} \cap X.$$

5. In the constant curvature cases we define the map $\phi : E \to X$ as such:
   - If $K = 0$, ϕ is the identity map from $E = X \to X$.
   - If $K = \pm 1$ then ϕ is the projection onto X given by $\phi : x \to \frac{Kx}{\sqrt{K\langle x,x\rangle}}$ defined on a neighbourhood of X in E.

Lemma 5.25. *S defined as above is a two dimensional totally geodesic submanifold, i.e. if $p \in S, v \in T_pS$ then the geodesic $\gamma_v(t)$ remains in S, and T_{xyz} is convex.*

Proof. When $K = 0$ this is immediately true. Consider $K = \pm 1$. Notice that $S = X \cap span\{x, y, z\}$ is two-dimensional. As X is a hypersurface its intersection with a three dimensional vector space is at least two dimensional. Now notice for each point $ax+by+cz$ in S we have that $a^2 + b^2 + c^2 = 1$, so we have only two free variables and hence S must be two dimensional. Now let γ be a geodesic with initial conditions $\gamma(0) = p, \dot{\gamma}(0) = v$, then by Remark 3.30, γ is contained in the plane spanned by p and v, which is contained in $span\{x, y, z\}$ by definition. So S is totally geodesic in X. Now we will show that T_{xyz} is convex. First, we show that any two points p, q of T_{xyz} are connected by a geodesic. By Proposition 3.31 we know that the geodesics of X are given by the intersections of X with planes through the origin. Consider the plane P spanned by p and q, since C_{xyz} is a half-cone with origin 0 we know that P must intersect C_{xyz} as a degenerate conic; in two lines. In particular, we know that P must intersect two of the following planes $span\{x, y\}, span\{x, z\}, span\{y, z\}$ in these lines. Without loss of generality suppose it intersects $span\{x, y\}$ and $span\{x, z\}$ in lines $\lambda(ax + by)$ and $\mu(cx + dz)$ respectively for some fixed $a, b, c, d > 0$ and variable $\mu, \lambda > 0$. In particular, these lines are spacelike:

$$\langle ax + by, ax + by\rangle = a^2 + b^2 + 2ab\langle x, y\rangle > a^2 + b^2 - 2ab = (a - b)^2 \geq 0$$

since $\langle x, y\rangle > -1$, as they are connected by a geodesic. We will re-scale the pairs (a, b) and (c, d) such that the vectors $ax + by$ and $cx + dz$ have length 1.

The intersection of plane P with X is diffeomorphic to either a circle, a hyperbola of two branches of two parallel straight lines. Two points of $P \cap X$ are geodesically connected if they lay on the same connected component. We will show that the points $ax + by$ and $cx + dz$ are are on the same connected component of $P \cap X$ and therefore p and q, which are on the cone spanned by these two vectors, must be in the same component also.

First, notice that since x and y are geodesically connected $\langle x, y \rangle > -1$ and hence

$$1 = \langle ax + by, ax + by \rangle a^2 + b^2 + 2ab\langle x, y \rangle > a^2 + b^2 - 2ab = (a - b)^2.$$

So $a - b > -1$, and an equivalent argument shows that $c - d > -1$. Suppose that at least one of the planes $span\{x, y\}$, $span\{x, z\}$ or $span\{y, z\}$ is not spacelike. Then at least one of the following is true:

1. If $span\{x, y\}$ is not spacelike, then $1 = \langle x, y \rangle^2 \leq 0$, so $\langle x, y \rangle > 1$ and so $1 = \langle ax + by, ax + by \rangle = a^2 + b^2 + 2ab\langle x, y \rangle > a^2 + b^2 + 2ab = (a + b)^2$ and hence $a + b > -1$. So we can calculate

$$\begin{aligned} \langle ax + by, cx + dz \rangle &= ac + ad\langle x, z \rangle + bc\langle x, y \rangle + bd\langle y, z \rangle \\ &> ac - ad + bc - bd = (a + b)(c - d) > -1. \end{aligned}$$

2. If $span\{x, z\}$ is not spacelike then we have $c + d > -1$ by an equivalent argument to the previous case, then we calculate:

$$\begin{aligned} \langle ax + by, cx + dz \rangle &= ac + ad\langle x, z \rangle + bc\langle x, y \rangle + bd\langle y, z \rangle \\ &> ac + ad - bc - bd = (a - b)(c + d) > -1. \end{aligned}$$

3. If $span\{y, z\}$ is not spacelike then:

$$\begin{aligned} \langle ax + by, cx + dz \rangle &= ac + ad\langle x, z \rangle + bc\langle x, y \rangle + bd\langle y, z \rangle \\ &> ac - ad - bc + bd = (a - b)(c - d) > -1. \end{aligned}$$

If $span\{x, y, z\}$ is spacelike, then any planes through $span\{x, y, z\}$ are spacelike and thus p, q will always be connected.

So now suppose that $span\{x, y, z\}$ is not spacelike but $span\{x, y\}$, $span\{y, z\}$ and $span\{x, z\}$ are spacelike. We can therefore write $y = \alpha x + \beta e_1$ and $z = \gamma x + \eta v$ where u, v are orthogonal to x and $\alpha^2 + \beta^2 = 1 = \gamma^2 + \eta^2$, furthermore we choose u, v such that $\beta, \gamma > 0$. Where $\langle x, y \rangle = \alpha$ and $\langle x, z \rangle = \gamma$.

Since $span\{x, y, z\} = span\{x, u, v\}$ is not spacelike and x is orthogonal to u and v it follows that $span\{u, v\}$ is not spacelike as $det(g) = 1 - \langle u, v \rangle^2 < 0$ and hence $\langle u, v \rangle \geq 1$ or $\langle u, v \rangle \leq -1$. We can assume that $\langle u, v \rangle \geq 1$ since if $\langle u, v \rangle \leq -1$ simply replace v with

$-v$. Now we can write $ax+by = \tilde{a}x+\tilde{b}u$ and $cx+dz = \tilde{c}x+\tilde{d}v$ with $\tilde{a}^2+\tilde{b}^2 = 1 = \tilde{c}^2+\tilde{d}^2$, then:

$$\langle \tilde{a}x+\tilde{b}u, \tilde{c}x+\tilde{d}v\rangle = \tilde{a}\tilde{c}+\tilde{b}\tilde{d}\langle u,v\rangle \geq \tilde{a}\tilde{c}+\tilde{b}\tilde{d} > -1,$$

because $\tilde{a}^2+\tilde{b}^2 = 1 = \tilde{c}^2+\tilde{d}^2$.

Then $ax+by$ and $cx+dz$ are connected by a geodesic because $\langle ax+by, cx+dz\rangle > -1$ and hence p and q are geodesically connected also.

Finally, any two points of T_{xyz} are connected by a unique segment, this follows from the fact that T_{xyz} is geodesically connected and the half-cone $\{\lambda x+\mu y+\rho z \mid \lambda,\mu,\rho \geq 0\}$ containing no antipodal points, by definition.

□

Lemma 5.26. *ϕ has the following properties:*

1. *ϕ is a G-equivariant projection onto X.*
2. *If γ is a geodesic of E such that $\gamma(0) \in X$, then there exists some $\varepsilon > 0$ such that $\phi \circ \gamma(-\varepsilon,\varepsilon)$ is the image of a geodesic in X.*

Proof. In the flat case ϕ is the identity so this is immediately true. Now consider the case $K = \pm 1$.

1. Let $x \in X$ and $g \in G$. Then

$$\phi(gx) = \frac{gx}{\sqrt{K\langle gx, gx\rangle}} = \frac{gx}{\sqrt{K\langle x, x\rangle}} = g\phi(x).$$

2. Let $x \in X$, then a geodesic of E through x is of the form $x+tv$. Then

$$\begin{aligned}\langle \phi(x+tp), \phi(x+tp)\rangle &= \langle \frac{x+tp}{\sqrt{K\langle x+tp, x+tp\rangle}}, \frac{x+tp}{\sqrt{K\langle x+tp, x+tp\rangle}}\rangle \\ &= \frac{1}{K} = K.\end{aligned}$$

So $\phi(x+tp) \in X$. Finally, $\phi(x+tp)$ is contained in the intersection of X and the plane spanned by x and p so by the proof of Proposition 3.31, $\phi(x+tv)$ is a geodesic of X. □

Since X is a hypersurface of E we can describe the tangent space to a point of X as:

$$T_pX = \{w \in E \mid \langle w, p\rangle = 0\}.$$

Therefore, we can describe the tangent space of S as such:

$$T_pS = p^{\perp} \cap span\{x,y,z\}.$$

Since $p \in span\{x,y,z\}$ we can write the affine plane:

$$A_pS = T_pS + p = \{w \mid \langle w-p, p\rangle = 0\} \cap span\{x,y,z\}.$$

Lemma 5.27. *Let p be a point of S then ϕ is a local diffeomorphism between a neighbourhood U of p in A_pS and a neighbourhood V of p in S.*

Proof. For each $p \in S$ define the map $\psi_p : V \to U, q \mapsto \frac{q}{K\langle q,p\rangle}$. Since both $\psi_p : S \to A_pS$ and $\phi : A_pS \to S$ are smooth, all that remains is to show ψ_p is the inverse of ϕ. First, consider some point $q \in S$, then:

$$\phi(\psi_p(q)) = \phi(\frac{q}{K\langle q,p\rangle}) = \frac{\frac{q}{K\langle q,p\rangle}}{\sqrt{K\langle \frac{q}{K\langle q,p\rangle}, \frac{q}{K\langle q,p\rangle}\rangle}} = q$$

since $K\langle x,x\rangle = 1$. Now consider some $\gamma'(0) + p \in U$. Recall that $\langle \gamma'(0), p\rangle = 0$ then:

$$\begin{aligned}
\psi_p(\phi(\gamma'(0)+p)) &= \psi_p(\frac{\gamma'(0)+p}{\sqrt{K\langle\gamma'(0)+p,\gamma'(0)+p\rangle}}) \\
&= \psi_p(\frac{\gamma'(0)+p}{\sqrt{K(\langle\gamma'(0),\gamma'(0)\rangle + 2\langle\gamma'(0),p\rangle) + \langle p,p\rangle)}}) \\
&= \psi_p(\frac{\gamma'(0)+p}{\sqrt{K(\langle\gamma'(0),\gamma'(0)\rangle + K)}}) \\
&= \frac{\frac{\gamma'(0)+p}{\sqrt{K(\langle\gamma'(0),\gamma'(0)\rangle+K)}}}{K\langle\frac{\gamma'(0)+p}{\sqrt{K(\langle\gamma'(0),\gamma'(0)\rangle+K)}},p\rangle} \\
&= \frac{\frac{\gamma'(0)+p}{\sqrt{K(\langle\gamma'(0),\gamma'(0)\rangle+K)}}}{K\langle\frac{p}{\sqrt{K(\langle\gamma'(0),\gamma'(0)\rangle+K)}},p\rangle} \\
&= \gamma'(0)+p.
\end{aligned}$$

So ϕ has an inverse and is hence a diffeomorphism between U and V. □

5.2.4 Relative convexity of $D(\tilde{M}_{\tilde{x}})$

We can now present a detailed proof of the central proposition in Klingler (1996).

Proposition 5.28. *Let M be a compact locally symmetric Lorentzian manifold such that the list of properties given in Section 5.2.3 is satisfied. Then for all points $\tilde{x} \in \tilde{M}$ consider the star $\tilde{M}_{\tilde{x}}$ in $\tilde{M}$. Its image under D, $D(\tilde{M}_{\tilde{x}})$ is convex relative to the star X_x.*

The proof of this proposition is quite long and involves the construction of multiple objects and discussion of some of their properties. Therefore, the proof will be split into a number of lemmas.

Suppose that $D(\tilde{M}_{\tilde{x}})$ is *not* convex relative to the star X_x. Then there exists two points $\tilde{y}, \tilde{z} \in \tilde{M}_{\tilde{x}}$ such that their images y and z are segmentally connected in X_x but $[y,z]$ is not contained in $D(\tilde{M}_{\tilde{x}})$.

5.2.4.1 Working in the surface S

Here we assume Item 1 and Item 4 in Section 5.2.3, i.e. that M is compact locally symmetric manifold equipped with a locally isometric (G, X)-structure and that for any three points x, y and z where y and z are contained in the star X_x, there exists a two dimensional submanifold S of X containing a convex subset C containing the points x, y and z.

Lemma 5.29. *Consider the segment $[y, z] \not\subset D(\tilde{M}_{\tilde{x}})$ as described above. Then there exists points $y_{T_1} \in [x, y]$ and $z_{T_1} \in [x, z]$ such that $[y_{T_1}, z_{T_1}] \not\subset D(\tilde{M}_{\tilde{x}})$ and $[x, v) \subset D(\tilde{M}_{\tilde{x}})$ for all points $v \in [y_{T_1}, z_{T_1}]$, i.e., without loss of generality, we can assume that for all points v on the segment $[y, z]$, the segment $[x, v)$ is contained in the image $D(\tilde{M}_{\tilde{x}})$.*

Proof. Parameterise each point on the segment $[x, y]$ by y_t, $t \in [0, 1]$, such that $y_0 = x$ and $y_1 = y$. We can parameterise the points of $[x, z]$ by z_t defined in the same manner.

Then define the interval

$$I := \{T \in (0, 1] \mid [y_t, z_t] \subset D(\tilde{M}_{\tilde{x}}) \text{ for all } t \leq T\}.$$

Now notice that I is connected by definition and since D is an open map I is non-empty. Now we will show that I is open in $(0, 1]$. Suppose that I is not open in $(0, 1]$. Then by the previous two results I is a non-empty, connected subset of $(0, 1]$, so I must be of the form $I = (0, T_1]$ for some $T_1 < 1$. So for any $T_2 \in (T_1, 1]$ so that $T_1 < T_2$ we have that $[y_{T_2}, z_{T_2}]$ is not contained in $D(\tilde{M}_{\tilde{x}})$. Now parameterise the segment $[y_{T_1}, z_{T_1}]$ by p_s, with $s \in [0, 1]$ and $p_0 = y_{T_1}, p_1 = z_{T_1}$, and since $D(\tilde{M}_{\tilde{x}})$ is open, there exists open balls B_s at each p_s with some radius ε_s such that $B_s \subset D(\tilde{M}_{\tilde{x}})$. Since $[0, 1]$ is closed, the set of radii ε_s must have a minimum value $\varepsilon > 0$, this contradicts the fact that $[y_{T_2}, z_{T_2}]$ is not contained in $D(\tilde{M}_{\tilde{x}})$ for T_2 arbitrarily close to T_1. Hence I is open.

So I must be of the form $(0, T_1)$ or $(0, 1]$. If it were the latter then $[y_1, z_1]$ would be contained in $D(\tilde{M}_{\tilde{x}})$ which contradicts our original assumption. Now relabel $y = y_{T_1}$ and $z = z_{T_1}$.

Now we will show that for any point $v \in [y, z]$ the segment $[x, v]$ is contained in the union of the segments $[y_t, z_t]$ for $t \in [0, 1]$.

In particular, since C is convex we know that the segment $[x, v]$ is contained in S. Furthermore, by the uniqueness of segments we have that $[x, v] \cap [x, y_t] = \emptyset = [x, v] \cap [x, z_t]$ for all $t \in (0, 1)$. If there was some point p in this intersection we would have that $[x, p]$ is contained in $[x, v] \cap [x, y]$ or $[x, v] \cap [x, z]$. By the definition of v we have that $v \in [x, v] \cap [x, y]$ and therefore by uniqueness of geodesics we would have that $[x, y] \subset [x, v]$ but $v \in [y, z]$ so the segments $[y, z]$ and $[x, v]$ have non-empty intersection, so it would follow that $v \in [x, z] = [x, y] \cup [y, z]$ contradicting the fact that $v \notin D(\tilde{M}_{\tilde{x}})$. This argument shows that any two segments in C meet at either a unique point or their union is a segment.

Then, since S is a surface it follows that $[x, v]$ is either contained in $\bigcup_{t\in[0,1]}[y_t, z_t]$ or only intersects at the points x and v. Now we define the interval

$$J = \{R \in (0, 1] \mid [x, v_r] \subset \bigcup_{t\in[0,r]} [y_t, z_t] \text{ for all } v_r \in [y_r, z_r] \text{ for all } r \leq R\}.$$

J is connected by definition. J is non-empty because for sufficiently small T for any point $v_T \in [y_T, z_T]$, we know that the segment $[x, v_T]$ must be contained in $\bigcup_{t\in[0,T]}[y_t, z_t]$ because the exponential map is a local diffeomorphism. So J is an interval, we will now show that $J = (0, 1]$. Suppose that $J \neq (0, 1]$. If J is open there exists $R_0 = \sup J$ and some point v_{R_0} on (y_{R_0}, z_{R_0}) such that the segment (x, v_{R_0}) shares no points with $\bigcup_{t\in[0,R_0]}[y_t, z_t]$. However there also exists a sequence of points $v_r \in [y_r, z_r]$ converging to v_{R_0} such that the sequence of sets $[x, v_r] \subset \bigcup_{t\in[0,R_0]}[y_t, z_t]$, then since geodesics vary continuously with respect to initial conditions and since S is a surface we have that some segment $[x, v_s]$ must intersect either $[x, y]$ or $[x, z]$, a contradiction. If J is closed with $R_1 = \max J$ we can make an equivalent argument by taking a sequence of points $v_r \in [y_r, z_r]$ for $r \in (R_1, 1)$ converging towards some $v_{R_1} \in [y_{R_1}, z_{R_1}]$. So $J = (0, 1]$ and hence each $[x, v] \subset \bigcup_{t\in[0,1]}[y_t, z_t]$.

Now fix a particular point v on the segment $[y, z]$ such that $v \notin D(\tilde{M}_{\tilde{x}})$. Furthermore, since $D(\tilde{M}_{\tilde{x}})$ is open, we can choose a v such that the segment $[y, v) \subset D(\tilde{M}_{\tilde{x}})$ i.e. v is the first point along the segment $[y, z]$ to leave $D(\tilde{M}_{\tilde{x}})$. Then since $[x, v] \subset \bigcup_{t\in[0,1]}[y_t, z_t]$ we have that $[x, v) \subset D(\tilde{M}_{\tilde{x}})$. $\square$

Then by Lemma 5.29 the segment $[x, v)$ is contained in $D(\tilde{M}_{\tilde{x}})$ and we can choose a sequence of points $v_k \in [x, v)$ such that v_k converges to v as k goes to infinity. Furthermore, since $D|_{\tilde{M}_{\tilde{x}}}$ is injective, we can define the sequence $\tilde{v}_k := D^{-1}(v_k) \in \tilde{M}_{\tilde{x}}$.

Lemma 5.30. *Consider the sequence of points $v_l \in [x, v)$ defined above, then there exists a subsequence $\tilde{v}_{k_l}$ of $\tilde{v}_k$ and a sequence of distinct fundamental group elements $\rho_{k_l} \in \pi_1(M)$ with $\rho_{k_l} \neq \rho_{k_j}$ when $l \neq j$, such that the sequence $\tilde{w}_{k_l} := \rho_{k_l}\tilde{v}_{k_l}$ converges to some point $\tilde{w} \in \tilde{M}$.*

Proof. First, consider the projection of $\tilde{v}_k$ to M, $\pi(\tilde{v}_k)$. Since M is compact, the sequence $\pi(\tilde{v}_k)$ must have a convergent subsequence $\pi(\tilde{v}_{k_l})$ that converges to some point $\hat{w}$ in M. Let $\tilde{w}$ be a lift of $\hat{w}$ to $\tilde{M}$. Since π is a local diffeomorphism, for sufficiently large k_l, there are points $\tilde{w}_{k_l}$ in the fibres over $\pi(\tilde{v}_{k_l})$ such that $\tilde{w}_{k_l}$ converges to $\tilde{w}$. As the fundamental group $\pi_1(M)$ acts transitively on the fibres of $\tilde{M}$, there exists some $\rho_{k_l} \in \pi_1(M)$ such that $\rho_{k_l}\tilde{v}_{k_l} = \tilde{w}_{k_l}$.

Now we will ensure that the ρ_{k_l} are distinct. If there are finitely many equal $\rho_i = \rho_j$, we simply take a subsequence that does not contain any repeated terms. Now suppose that there are finitely many distinct ρ_k. Then there exists the constant subsequence $\rho_{k_l} = \rho$. Then the sequence $\tilde{v}_{k_l}$ converges to $\rho^{-1}\tilde{w}$, and since D is continuous we have that $D(\rho^{-1}\tilde{w}) = v$ so $\rho^{-1}\tilde{w} \notin \tilde{M}_{\tilde{x}}$. Additionally, we have that $D^{-1}([x,v)) = [\tilde{x}, \rho^{-1}\tilde{w})$ is contained in $\tilde{M}_{\tilde{x}}$ and that $[\tilde{x}, \rho^{-1}\tilde{w})$ is the image of some geodesic $\tilde{\gamma}([0,b))$ which is mapped under D to the geodesic $\gamma([0,b))$. As C is convex this geodesic extends to the point $\gamma(b) = v$. Since v is contained in the image of D, and D is a local isometry the velocity vector $r := (dD|_v)^{-1}(\gamma'(b)) \in T_{\rho^{-1}\tilde{w}}\tilde{M}$ is well defined. Now let $\tilde{\eta} : (-\epsilon, \epsilon) \to \tilde{M}$ be the unique geodesic with initial conditions $(\tilde{w}, r)$. Now notice that there is some non-empty neighbourhood where the geodesics $D(\tilde{\eta}(t))$ and $D(\tilde{\gamma}(t))$ are equal and therefore by Lemma 2.16 we have that $\tilde{\eta}$ is an extension of $\tilde{\gamma}$, which contradicts the fact that $\rho^{-1}\tilde{w} \notin \tilde{M}_{\tilde{x}}$. □

For simplicity of notation we will relabel the sequences v_{k_l} as v_k. Define $g_k := h(\rho_k^{-1}) \in G$ where h is the monodromy homomorphism in Definition 4.17. As before, we write w_k for the image $w_k := D(\tilde{w}_k)$ and $w := D(\tilde{w})$.

Lemma 5.31. *The monodromy element g_k maps w_k to v_k.*

Proof.

$$g_k w_k \quad = \quad g_k D(\tilde{w}_k) = h(\rho_k^{-1}) D(\rho_k \tilde{v}_k) = D(\rho_k^{-1} \rho_k \tilde{v}_k) = D(\tilde{v}_k) = v_k.$$

□

Lemma 5.32. *There exists a compact subset $C \subset G$ and two sequences c_k, b_k in C and an element $g \in G$ such that*

1. $w_k = c_k^{-1} w$.
2. $v_k = b_k v$.
3. $v = gw$.

Moreover, for each g_k we can write $g_k = a_k o_k c_k$ where a_k, c_k are contained in compact subsets and o_k is contained in the isotropy group of the point w.

Proof. Recall that X is a homogeneous space and can be written as the quotient of $G = Iso(X)$ by $G_p = Iso_p(X)$. In particular, we have that the projection map $\pi_p : G \to G/G_p$ such that $\pi_p : g \mapsto gp$ is an open map. Now let U be an open set in G containing the identity I.

Then $\pi_w(U)$ is an open set in X containing $I(w) = w$. Additionally, since w_k converges towards w we have that for sufficiently large k, $w_k \in \pi_w(U)$ i.e. there exists $c_k^{-1} \in U$ such that $c_k^{-1} w = w_k$. Similarly we have that $\pi_v(U)$ contains v and since v_k converges to v, for sufficiently large k we have $b_k \in U$ such that $b_k v = v_k$.

Now the closure $\bar{U}$ is a compact set in G containing appropriate c_k^{-1} and b_k for all but finitely many k. Now define C to be the union of $\bar{U}$ with the finite set consisting g and the finitely many c_k^{-1} and b_k that were not contained in $\bar{U}$. C is the union of a compact set and a finite (and hence compact) set, so it is compact.

Now define $a_k := b_k g$. Then recall from Lemma 5.31 that $g_k w_k = v_k$ and calculate:

$$\begin{aligned}(a_k^{-1} g_k c_k^{-1})w &= (g^{-1} b_k^{-1} g_k c_k^{-1})w \\ &= g^{-1} b_k^{-1} g_k w_k \\ &= g^{-1} b_k^{-1} v_k \\ &= g^{-1} v \\ &= w.\end{aligned}$$

So, $(a_k^{-1} g_k c_k^{-1})$ is in the isotropy group of the point w and hence we can write $g_k = a_k o_k c_k$ where o_k is contained in the isotropy group of the point w. □

5.2.4.2 Projecting onto S

One of the key techniques used to generalise Carriére's proof of geodesic completeness for compact flat Lorentzian manifolds to the case with compact constant curvature Lorentzian manifolds was the introduction of a $G = Iso(X)$-equivariant projection from the ambient Euclidean space E to X. We are now supposing that M satisfies all of the conditions in Section 5.2.3 with the addition of item 3, item 5.

Now let $\tilde{B}$ be a compact neighbourhood of $\tilde{w}$ in $\tilde{M}$ such that

$$\text{for all } \rho \in \pi_1(M) \setminus \{Id\}, \quad \rho\tilde{B} \cap \tilde{B} = \emptyset. \tag{5.1}$$

The existence of such a neighbourhood is ensured by the properness of the action of $\pi_1(M)$ on $\tilde{M}$.

Now choose some sufficiently small $r > 0$ such that

$$\phi(B(w, 2r)) \subset D(\tilde{B})$$

where $B(w, r)$ is the closed Euclidean ball of E with centre w and radius r. For sufficiently large k the euclidean distance between w_k and w is bounded above by r and therefore the ball $B_k := B(w_k, r)$ is contained in the ball $B(w, 2r)$. In particular, $\phi(B_k) \subset D(\tilde{B})$. We define $\tilde{B}_k$ to be the cover of $\phi(B_k)$ in $\tilde{B}$ and then define $\tilde{C}_k := \rho_k^{-1}\tilde{B}_k$ where ρ_k are elements of $\pi_1(M)$ as defined in Lemma 5.30. $\tilde{C}_k$ is a compact neighbourhood of $\tilde{v}_k$ and by Lemma 5.31, $D(\tilde{C}_k) = \phi(g_k B_k)$. Finally, the compact neighbourhoods $C_k = D(\tilde{C}_k)$ are pairwise disjoint by the property described in Equation (5.1).

Finally, note that the $g_k B_k$ are ellipsoids centred at v_k as they are the image of a unit ball under an orthonormal transformation g_k. Additionally, by Lemma 5.32 we can write $g_k = a_k o_k c_k$ and since a_k and c_k are contained in a compact set, the discompacity of the sequence $(g_k)_k$ is equal to the discompacity of the sequence $(o_k)_k = Iso_w(X) \subseteq O(1, n-1)$. So by Lemma 5.12 we have that $g_k B_k$ is an ellipsoid such that all bar one of its principal axes are bounded below by some constant r' with $0 < r' < r$.

Lemma 5.33. *The intersection $A_{v_k}S \cap g_k B_k$ contains an affine segment of E with centre v_k and length $2r'$. We will call this segment σ_k.*

Proof. Consider the codimension 1 ellipsoid D_k centered at v_k with principal axes equal to the principal axes of $g_k B_k$ whose lengths are bound below by r'. Then $D_k \subset g_k B_k$. Now consider the intersection $A_{v_k}S \cap D_k$. The plane $A_{v_k}S$ cuts through D at its centre, the point v_k, so $A_{v_k}S \cap g_k B_k$ must have dimension greater than or equal to 1. Since the principal axes of D_k are bound below by r' any line through the centre of D_k must have length greater than or equal to $2r'$. □

We write $\delta_k := \phi(\sigma_k)$ for the projection of σ_k onto X.

Lemma 5.34. *The segment δ_k is contained in S and the intersections $\delta_k \cap D(\tilde{M}_{\tilde{x}})$ are disjoint.*

Proof. Since $\sigma_k \in A_{v_k}S$ and ϕ is a local diffeomorphism between neighbourhoods of $A_{v_k}S$ and S it is immediate that $\delta_k = \phi(\sigma_k) \in S$.

Now we will show that each $\delta_k \cap D(\tilde{M}_{\tilde{x}})$ are pairwise disjoint. This follows from the fact that the compact sets $\tilde{C}_k$ are disjoint. First, notice

$$\delta_k = \phi(\sigma_k) \subset \phi(A_{v_k}S \cap g_k B_k) \subset \phi(A_{v_k}S) \cap \phi(g_k B_k) = \phi(A_{v_k}S) \cap D(\tilde{C}_k),$$

and hence

$$\delta_k \cap D(\tilde{M}_{\tilde{x}}) \subset \phi(A_{v_k}S) \cap D(\tilde{C}_k) \cap D(\tilde{M}_{\tilde{x}}) \subset D(\tilde{C}_k \cap \tilde{M}_{\tilde{x}}).$$

Since D restricted to $\tilde{M}_{\tilde{x}}$ is injective, we have that $D(\tilde{C}_k) \cap D(\tilde{M}_{\tilde{x}})$ are pairwise disjoint, and hence $(\delta_k \cap D(\tilde{M}_{\tilde{x}})) \cap (\delta_l \cap D(\tilde{M}_{\tilde{x}})) = \emptyset$ for $k \neq l$. □

Now we will write $\sigma_k = v_k + \sigma_k^0$ where $\sigma_k^0 \in T_{v_k}S$ is an affine segment of E with centre 0 and length $2r'$.

Since each σ_k^0 is contained in the compact set $B(0, 2r)$, there exists a convergent subsequence $\sigma_{k_l}^0$ which converges towards an affine segment of E with centre 0 and length $2r'$. This is possible because each point on each affine segment σ_k is equal to some $\{p_k t \mid t \in [-r', r']\}$ for and p_k having Euclidean length 1. Then since the p_k are contained in the compact set $B(0, 2r)$ the must exist a subsequence which converges to some point p, which determines and affine segment as the image of pt. We will relabel this subsequence σ_k^0 for ease of notation. For any given t, the point $p_k t \in \sigma_k$ converges towards $pt \in \sigma$.

Lemma 5.35. *The sequence of segments σ_k converges towards the segment $\sigma = v + \sigma^0$. Furthermore, σ is contained in $A_v S$.*

Proof. Since v_k converges to v and σ_k^0 converges to σ^0 it follows from continuity that $\sigma_k = v_k + \sigma_k^0$ converges to $v + \sigma^0 = \sigma$. Furthermore, since each σ_k is contained in the vector space $span\{x, y, z\}$ the limit σ must be contained in $span\{x, y, z\}$ also, additionally since each σ_k is tangent to S it follows from the continuity of the bilinear form $\langle \cdot, \cdot \rangle$ that σ is tangent to S therefore $\sigma \subset T_v S$. □

The following lemma uses the fact that S is 2-dimensional.

Lemma 5.36. *The intersection $\delta \cap \tilde{M}_{\tilde{x}}$ is non-empty. In particular, we can choose a point $s \in \delta \cap \tilde{M}_{\tilde{x}}$ and a sequence of distinct points $s_k \in \delta_k \cap \tilde{M}_{\tilde{x}}$ which converge to s.*

Proof. Recall the diffeomorphic neighbourhoods of the point v, $U \subset A_vS$ and $V \subset S$ of v. Now notice that the closure of $D(\tilde{M}_{\tilde{x}})$, written $\overline{D(\tilde{M}_{\tilde{x}})}$, contains the segment $[y, z]$ and so $\overline{D(\tilde{M}_{\tilde{x}})} \cap V$ contains $[y, z] \cap V$. Also, the straight lines $\psi_v([y, z] \cap V)$ and $\psi_v(\delta) = \rho$ are contained in A_vS and pass through the point v. Since S is two dimensional, A_vS is a plane and so $\psi_v(\delta)$ must intersect either the image of the interior $\psi_v((D(\tilde{M}_{\tilde{x}})^\circ)$ or the image of the boundary $\psi_v([y, z])$ in a line. Furthermore, since the segment $[y, v)$ is contained in $D(\tilde{M}_{\tilde{x}})$ it follows that $\psi_v(\delta)$ intersects $\psi_v(D(\tilde{M}_{\tilde{x}}))$ in a line. So now choose some point, besides v on $\psi_v(D(\tilde{M}_{\tilde{x}})) \cap \psi_v(\delta)$, written $v + t_0p$ and call its image under ϕ s. Additionally we can define points $v_k + t_0p_k \in \sigma_k$ and call $s_k := \phi(v_k + t_0p_k)$. As v_k converges to v and p_k converges to p it follows from the continuity of ϕ that s_k converges to s. It remains to show that s_k is contained in $D(\tilde{M}_{\tilde{x}})$. Since $s \in D(\tilde{M}_{\tilde{x}})$ it follows from the openness of $D(\tilde{M}_{\tilde{x}})$ that there are no subsequence of s_k that are contained in its complement $D(\tilde{M}_{\tilde{x}})^C$, as it is closed, therefore there are finitely many $s_k \in D(\tilde{M}_{\tilde{x}})^C$. We just take the subsequence without these points and relabel this sequence s_k.

We write $\tilde{s}$ and $\tilde{s}_k$ in $\tilde{M}_{\tilde{x}}$ for $D^{-1}(s)$ and $D^{-1}(s_k)$ respectively. $\tilde{s}_k$ converges to $\tilde{s}$ because D is a local diffeomorphism on $\tilde{M}_{\tilde{x}}$. $\square$

5.2.4.3 Proof of the proposition

Now we are able to prove Proposition 5.28; that the image of the star $\tilde{M}_{\tilde{x}}$ under D, $D(\tilde{M}_{\tilde{x}})$ is convex relative to the star X_x.

Proof. Suppose this is not the case, then we are able to construct the objects described in the previous lemmas in Section 5.2.4.1 and Section 5.2.4.2. In particular, we have the convergent sequence $\tilde{s}_k$ in $\tilde{M}$. Since s_k is a point of the segment δ_k and δ_k is contained in $\phi(g_kB_k) = D(\tilde{C}_k)$ it follows that $\tilde{s}_k \in \tilde{C}_k$. Now recall the sequence of fundamental group elements ρ_k from Lemma 5.30. Then $\rho_k s_k \in \rho_k\tilde{C}_k = \rho_k\rho_k^{-1}\tilde{B}_k = \tilde{B}_k$ which is contained in the compact neighbourhood $\tilde{B}$. Since $\tilde{B}$ is compact we can find a convergent subsequence of $\rho_k s_k$ which converges to some point $\tilde{m} \in \tilde{B}$.

It follows from the properness of the action of $\pi_1(M)$ on $\tilde{M}$ that since $\tilde{s}_k$ converges to $\tilde{s}$ and $\rho_k\tilde{s}_k$ converges to $\tilde{m}$ then ρ_k must converge to some $\rho \in \pi_1(M)$. Therefore, $\tilde{s} \in \rho^{-1}\tilde{B}$. Then by Equation (5.1)

$$(\rho_k\rho^{-1})\tilde{B} \cap \tilde{B} = \emptyset$$

and therefore

$$\rho^{-1}\tilde{B} \cap \rho_k\tilde{B} = \emptyset$$

and hence $\tilde{s}_k$ is not an element of $\rho^{-1}\tilde{B}$. So $\rho^{-1}\tilde{B}$ is a neighbourhood of $\tilde{s}$ which does not contain any $\tilde{s}_k$, which contradicts the fact that $\tilde{s}_k$ converges to $\tilde{s}$. □

5.2.5 Rest of the proof

For completeness we will now proceed to present an outline of the rest of Klingler (1996) before discussing some attempts to extend Proposition 5.28. Since these attempts are focused on the key proposition above rather than the subsequent results the remaining results will be presented with minimal details, so that the reader can have a more complete understanding of the whole of Klingler (1996). Interested readers can look at Klingler (1996) (in French) and Chapter 4 in Lundberg (2015), which contains a complete proof (in English).

Corollary 5.37. *Let $\tilde{\partial}\tilde{M}_{\tilde{x}}$ be the boundary of the star $\tilde{M}_{\tilde{x}}$ in $\tilde{M}$, its image under D is contained in the boundary $\bar{X}_x \backslash X_x$.*

Proof. Let $\tilde{y}$ be a point of $\tilde{\partial}\tilde{M}_{\tilde{x}}$, and let $\tilde{B}_{\tilde{y}}$ be a convex neighbourhood of $\tilde{y}$. Evidently the point y is in the closure of X_x. If y belongs to the star X_x, since X_x is open, we can chose $\tilde{B}_{\tilde{y}}$ with image included in X_x. Then $\tilde{B}_{\tilde{y}} \cup \tilde{M}_{\tilde{x}} \neq \emptyset$ and by Proposition 5.28, $D(\tilde{M}_{\tilde{x}})$ is convex relative to X_x. So by applying Lemma 3 of Klingler (1996), which is a technical lemma that follows from Proposition 5.28, D restricted to $\tilde{M}_{\tilde{x}} \cup \tilde{B}_{\tilde{y}}$ is injective. The segment $[x, y]$ then lifts to a segment $[\tilde{x}, \tilde{y}]$ in $\tilde{M}$, $\tilde{x}$ is segmentally connected to $\tilde{y}$, and hence $\tilde{y} \in \tilde{M}_{\tilde{x}}$ a contradiction. So $y \in \bar{X} \backslash X$. □

Notice that Corollary 5.37 is an encouraging step towards proving that M is complete: it indicates that the boundary in $\tilde{M}$ of a star $\tilde{M}_{\tilde{x}}$ is sent by D to the boundary of the corresponding star X_x. However $\tilde{\partial}\tilde{M}_{\tilde{x}}$ does not contain enough information on the geometry of $\tilde{M}_{\tilde{x}}$ for us to finish the proof. We still have to understand how the points of $\tilde{M}_{\tilde{x}}$ can go to infinity in $\tilde{M}$ while their images in X converge. To do this, you have to work in a space larger than $\tilde{M}$.

We define this larger space as such. Equip X with the complete Riemannian metric g_0 induced from the canonical Euclidean metric on E. Pullback g_0 to $\tilde{M}$ via D, so D^*g_0 is a Riemannian metric on $\tilde{M}$, and finally take the metric space completion of $\tilde{M}$ with respect to D^*g_0, written $\hat{M}$. Since D is continuous, we can extend it to $\hat{M}$.

We can define segments in $\hat{M}$. Let $\alpha : [0, 1] \to \hat{M}$ be a curve with $\alpha(0) = \tilde{x}$ and $\alpha(1) = \hat{y}$, then if $\alpha([0, 1)) \subset \tilde{M}$ and $D(\alpha([0, 1)))$ is a segment in X we say that α is a segment in $\hat{M}$, written $[\tilde{x}, \hat{y}]$. With this definition we define the stars of $\hat{M}$ as expected:

$$\hat{M}_{\tilde{x}} := \{\hat{y} \in \hat{M} \mid [\tilde{x}, \hat{y}] \text{ exists }\}.$$

It is then shown that each star $\hat{M}_{\tilde{x}}$ is the union of the star $\tilde{M}_{\tilde{x}}$ and the boundary $\hat{\partial}\tilde{M}_{\tilde{x}} \cap D^{-1}(X_x)$ where $\hat{\partial}\tilde{M}_{\tilde{x}} = \partial\tilde{M}_{\tilde{x}} \cap (\hat{M} \backslash \tilde{M})$ is the boundary of $\tilde{M}_{\tilde{x}}$ that is contained exclusively in $\hat{M}$. The boundary of $\tilde{M}_{\tilde{x}}$ is written $\tilde{\partial}\tilde{M}_{\tilde{x}} := \partial\tilde{M}_{\tilde{x}} \cap \tilde{M}$. The proof of this requires Proposition 5.28. Furthermore, it is shown that the set of stars $\hat{M}_{\tilde{x}}$ cover $\hat{M}$ and in particular, each star $\hat{M}_{\tilde{x}}$ is an open set in $\hat{M}$ and D restricted to $\hat{M}_{\tilde{x}}$. So as before we have that $D : \hat{M} \to X$ is locally injective. We now describe the boundary $\hat{\partial}\tilde{M}$.

Definition 5.38. A subset of $\hat{M}$ is called a *totally geodesic co-null set* if locally, its image under D is identified a connected component of $V \cap X$ where V is a co-null hyperplane of X.

Proposition 5.39. *The boundary $\hat{\partial}\tilde{M}$ is either a totally geodesic co-null set, or it is empty.*

This detour into $\hat{M}$ is required for the proof of the final proposition in Klingler (1996):

Proposition 5.40. *Suppose that M is not complete, then the image of $\tilde{M}$ under D is a connected component Ω contained in $X \setminus H$, where H is a co-null hyperplane of X.*

Finally, one can prove that compact Lorentzian manifolds with constant sectional curvature are complete. Suppose M is a compact n-dimensional Lorentzian manifold with constant sectional curvature which is not complete. Then by Proposition 5.40, $D(\tilde{M})$ is contained in an open set Ω with boundary being some co-null hyperplane H. In the flat case this means that the holonomy group Γ leaves an affine hyperplane of $\mathbb{R}^n$ invariant, which contradicts Theorem 2.8 [p. 644] in Goldman & Hirsch (1984). In the case $K = \pm 1$, the argument follows as such:

- Let e be an orthogonal vector to H.
- Define the isometry subgroup that fixes Ω. $G_1 := \{g \in G \mid g\Omega = \Omega\}$ and an isotropy subgroup of G_1 that fixes a point in Ω, written H_1. Then $\Omega = G_1/H_1$.
- Define the G_1-invariant vector field Y_1 on Ω at any point $y \in \Omega$ by

$$Y_{1y} := \frac{e}{\langle e, y\rangle} - y \in T_y\Omega = \{\mathbb{R}y\}^{\perp}.$$

- Let ω_1 be the Lorentzian volume form on Ω. It is also G_1-invariant.
- The Lie derivative $L_{Y_1}\omega_1$ is also G_1-invariant and proportional to ω_1, i.e. $L_{Y_1}\omega_1 = \lambda\omega_1$ for some $\lambda \in \mathbb{R}$.
- By calculating the divergence of Y_1 with respect to ω_1 we see that $L_{Y_1}\omega_1 = -(n-1)\omega_1$. So $\lambda = -(n-1)$.
- Since Y_1 and ω_1 are G_1-invariant they induce a vector field Y and a volume form ω respectively on M.

- Since M has no boundary, we can apply Stokes theorem to the following and notice:

$$\int_M L_y\omega = \lambda \int_M \omega = \lambda \int_{\partial M} d\omega = 0.$$

  Therefore, $\lambda = 0$, a contradiction so M must be complete.

5.2.6 Discussion of Klingler's method

It is worth noting once again that the primary differences between the proofs in Klingler (1996) and Carrière (1989) are the utilization of the projection ϕ, as when $K = \pm 1$, X is a hypersurface of Euclidean space rather than $X = E$, and Lemma 5.32 which shows us that we are concerned with the discompacity of the isotropy group. Carrière (1989) considers the discompacity of the holonomy group. In constant curvature spaces we can not consider the full isometry group because the isometry group of anti-de Sitter space, $Iso(H_1^n) = O(2, n-1)$, has discompacity 2 while in the flat case the isometry group is $O(1, n-1) \times \mathbb{R}_1^n$ which is not linear.

As we would like to extend the methods in Klingler (1996) to arbitrary compact Lorentzian locally symmetric spaces, we should first consider the cases which are most similar to those already proved. Evidently, Klingler (1996) does not consider any decomposable cases, because even the product of two non-flat manifolds with the same constant sectional curvature will no longer have constant sectional curvature as any tangent planes not contained in the tangent space of a single factor will have sectional curvature of 0 by Corollary 2.62. More generally, the product of two non-flat manifolds will have non-constant curvature.

If we would like to extend the results in Klingler (1996) to other cases, we should first extend Proposition 5.28. It should be noted that if Proposition 5.28 can be extended to some manifold M, we do not necessarily know that M is geodesically complete as the later results in Klingler (1996) do not follow from Proposition 5.28 alone. In particular, Proposition 5.40 may cause difficulties as its proof is rather involved with each constant curvature case treated separately. This is not discussed in detail in this thesis because we were unable to verify if any cases extend Proposition 5.28.

Recall the list of properties in Section 5.2.3:

1. M is a compact locally symmetric manifold which is locally isometric to the symmetric space X. In particular, we equip M with the locally isometric (G, X)-structure described in Proposition 4.20.

2. The isotropy group of X has discompacity less than or equal to 1.

3. X is isometrically embedded into some semi-Euclidean space E such that the $\langle \cdot, \cdot \rangle$ (non-positive definite) inner product on E induces the metric on X. Furthermore,

we require that the isometry group $G = Iso(X)$ is a subgroup of the linear isometries of E.

4. Let x be a point in X and let y, z be arbitrary points in the star X_x then, there exists a two dimensional submanifold S of X which contains a convex subset C containing the points x, y and z. Additionally, since S is a 2-dimensional submanifold of X, which in turn is embedded into E, the tangent space at a point T_pS is identified with a 2-dimensional affine subspace, written $A_pS = T_pS + p$ with is canonically identified with a 2-dimensional affine subspace of E.

5. There exist a projection ϕ from a neighbourhood of X in E to X that satisfies the following properties:

   - ϕ is a G-equivariant projection onto X.
   - If γ is a geodesic of E such that $\gamma(0) \in X$, then there exists some $\varepsilon > 0$ such that $\phi \circ \gamma(-\varepsilon, \varepsilon)$ is equal to the image of a geodesic in X.
   - Let p be a point of S then ϕ, is a local diffeomorphism between a neighbourhood U of p in $A_pS := T_pS+p$ and a neighbourhood V of p in S. It has local inverses for a neighbourhood of each point written ψ_p.

In Section 5.2.4 we showed that Proposition 5.28 holds for any (G, X)-manifold which satisfied these properties.

In an attempt to extend find a space which satisfies these properties we considered the simplest decomposable Lorentzian symmetric spaces, the product of a flat space and a positively curved space (as H_1^n is not simply connected it is slightly more complicated). For this discussion we will explicitly consider the product of Minkowski space and the sphere, $\mathbb{R}_1^n \times S^m$, however identical arguments hold for the product of de Sitter space and flat Riemannian space, $S_1^n \times \mathbb{R}^m$.

There are a few things that indicate such an extension may be possible. Firstly we have that $\mathbb{R}_1^n \times S^m$ is a hypersurface of the Euclidean space $E = \mathbb{R}_1^{n+m+1}$ and furthermore, the isometry group $Iso(\mathbb{R}_1^n \times S^m) = Iso(\mathbb{R}_1^n) \times Iso(S^m)$ and in particular, the isotropy subgroup is equal to $Iso_p(\mathbb{R}_1^n) \times Iso_p(S^m) = O(1, n-1) \times O(m)$, which has discompacity equal to 1 as $O(m)$ is a compact group. Furthermore, it seems natural to define a projection $\phi : E \to X$ as such:

$$\begin{aligned} \phi : E &\to X \\ (p_1, p_2) &\mapsto (p_1, \frac{p_2}{|p_2|_2}), \end{aligned}$$

for $p_1 \in \mathbb{R}_1^n$, $p_2 \in S^m$, where $|p_2|_2$ is the Euclidean norm on $E^{m+1} \supset S^m$. Then $\pi_i \circ \phi$ maps affine lines through X to pregeodesics in each X_i. For example, suppose that each

v_i has length 1, to simplify calculations, then:

$$\phi(p_1 + tv_1, p_2 + tv_2) = (p_1 + tv_1, \frac{p_2 + tv_2}{\sqrt{1+t^2}}) := (\alpha_1(t), \alpha_2(t)).$$

Then $\alpha_1(t)$ is a geodesic and $\alpha_2(t)$ is a pregeodesic. If we reparameterise these curves by $f(t) = \sqrt{\frac{1}{cos^2(t)} - 1}$ for $t \geq 0$ we get that $\alpha_2(f(t)) = cos(t)p_2 + sin(t)v_2$, which is a geodesic of S^m but $\alpha_1(f(t)) = p_1 + v_1\sqrt{\frac{1}{cos^2(t)} - 1}$, which is not parameterised as a geodesic in $\mathbb{R}^n_1$. Therefore, we must either attempt to generalise the proof using only this weaker assumption or try and find a different projection, both of these options appear to require significant work.

More difficulties arise when trying to construct an appropriate generalisation to the surface S containing the convex set C for the points $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2)$. One idea would be to take the affine plane $P(x, y, z)$ and project it onto X via ϕ. Such an approach be intuitive in order to ensure compatibility with ϕ. It follows from the property that ϕ maps geodesics of E, which pass through a point of X to the image geodesics of X that each point of X is geodesically connected to the points x, y and z, but in particular, we do not have that any set C containing x, y, z is convex, as two points a, b on different segments $[x, y], [x, z]$ or $[y, z]$ which are not x, y or z may not be geodesically connected.

A second approach may be to consider an appropriate S_i in each factor manifold and then take their product $S := S_1 \times S_2$ in the product space $X_1 \times X_2$. As in Klingler (1996), we could choose S_1 to be the plane in $\mathbb{R}^n_1$ containing x_1, y_1 and z_1, written $P_1(x_1, y_1, z_1)$ and in S^m we could choose $S_2 = S^m \cap span\{x_2, y_2, z_2\}$ analogously to the de Sitter case. As each S_i is a submanifold of E_i containing a convex set $C_i \ni x_i, y_i, z_i$ it immediately follows that S is submanifold of $E = E_1 \times E_2$ containing the convex set $C = C_1 \times C_1$. Unfortunately, this approach immediately fails the to satisfy item 4, as it is 4-dimensional. The fact that S is two dimensional is crucial in some arguments throughout the proof such as Lemma 5.29 and Lemma 5.36. A variation of the proof may still work with S having dimension 4, however this would take some work.

Alternatively, one may want to generalise the results of Klingler (1996) to work for locally Cahen-Wallach spaces in order to have a unified proof of the geodesic completeness of compact indecomposable locally symmetric Lorentzian manifolds. In order to achieve such a proof a few difficulties must be overcome. Firstly, Klingler utilises isometric embeddings of the model spaces into ambient Euclidean space, in particular, these embeddings have codimension 1 (or 0 in the flat case). For Cahen-Wallach space we are unaware of any such global embedding, however (Blau et al. 2002, 9.2) describes a codimension 2 local isometric embedding of $CW_{n+2}(S)$ into $\mathbb{R}^{n+4}_2$. A particularly nice property of this embedding is that it is described by the intersection of two quadrics. Besides the immediate issues of only having a local embedding, the higher codimension of the embedding would require more care to be taken when constructing a surface S. An alternative approach may be to equip Cahen-Wallach space with a Euclidean inner product to define ellipsoids, however this method would cause difficulties as the metric and inner product will no longer be compatible.

Alternatively, instead of considering the isotropy group of Cahen-Wallach space, one could consider the holonomy group, as in Carrière (1989), which by Example 5.14 has discompacity 1. In order to such a method to work the relationship between the holonomy homomorphism $\pi_1(M) \to Hol(M)/Hol^0(M)$ and the monodromy homomorphism $\pi_1(M) \to G$ would have to be explored.

It would appear that any further attempts to generalise Klingler (1996) would require significant effort in order to either construct an appropriate S and C or a modification to the proof such that we no longer require the exact conditions imposed by Section 5.2.3.

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