USING EUCLID IN A PRACTICAL CONTEXT: CLAUDE RICHARD'S COURSE ON SECTORS AT THE JESUIT IMPERIAL COLLEGE IN 17TH CENTURY SPAIN

Elena Ausejo
Universidad de Zaragoza

Abstract

This paper looks at two manuscripts kept at the Spanish Royal Academy of History (Madrid) containing the course on sectors that Professor Claude Richard taught at the Jesuit Imperial College in Madrid. The literary form of the course goes beyond practitioners' commonplace books, for it aims at teaching practical geometry on a solid Euclidean basis while claiming that the entire practical geometry consists of the brief and easy use of pantometers, that is, Coignet-type sectors. Actually, the course focuses on checking the solid geometric foundations of the scales graduation, which would justify the numerical consideration of continuous magnitudes as quantities –accepting a margin of error sensorially imperceptible and irrelevant for the purposes of application.

Resumen

Este artículo estudia dos manuscritos conservados en la Real Academia de la Historia (Madrid) sobre el curso de pantómetras que el catedrático Claude Richard impartió en el Colegio Imperial de la Compañía de Jesús en Madrid. El curso trasciende las limitaciones de los manuales de uso de este instrumento, pues Richard aspira a enseñar sobre una sólida base euclídea la geometría práctica, que según él consiste en un uso breve y fácil de las pantómetras –concretamente el compás de proporción que atribuye a Coignet–. De hecho, el curso se concentra en la verificación de la sólida base geométrica de la graduación de las escalas grabadas en el instrumento, lo que justificaría la consideración numérica de las magnitudes continuas como cantidades –aceptando un margen de error sensiblemente imperceptible e irrelevante a efectos prácticos–.

Recibido el 30 de mayo de 2021 — Aceptado el 20 de noviembre de 2021
https://doi.org/10.47101/llull.2022.45.90.ausejo
LLULL, Vol. 45 (N.º 90) 2022 - ISSN: 0210-8615, pp. 35-67

Palabras clave: Claude Richard; Colegio Imperial de la Compañía de Jesús (Madrid, España); Instrumentos matemáticos; Compás de proporción de Coignet.

Key words: Claude Richard; Jesuit Imperial College (Madrid, Spain); Mathematical instruments; Coignet-type sectors.

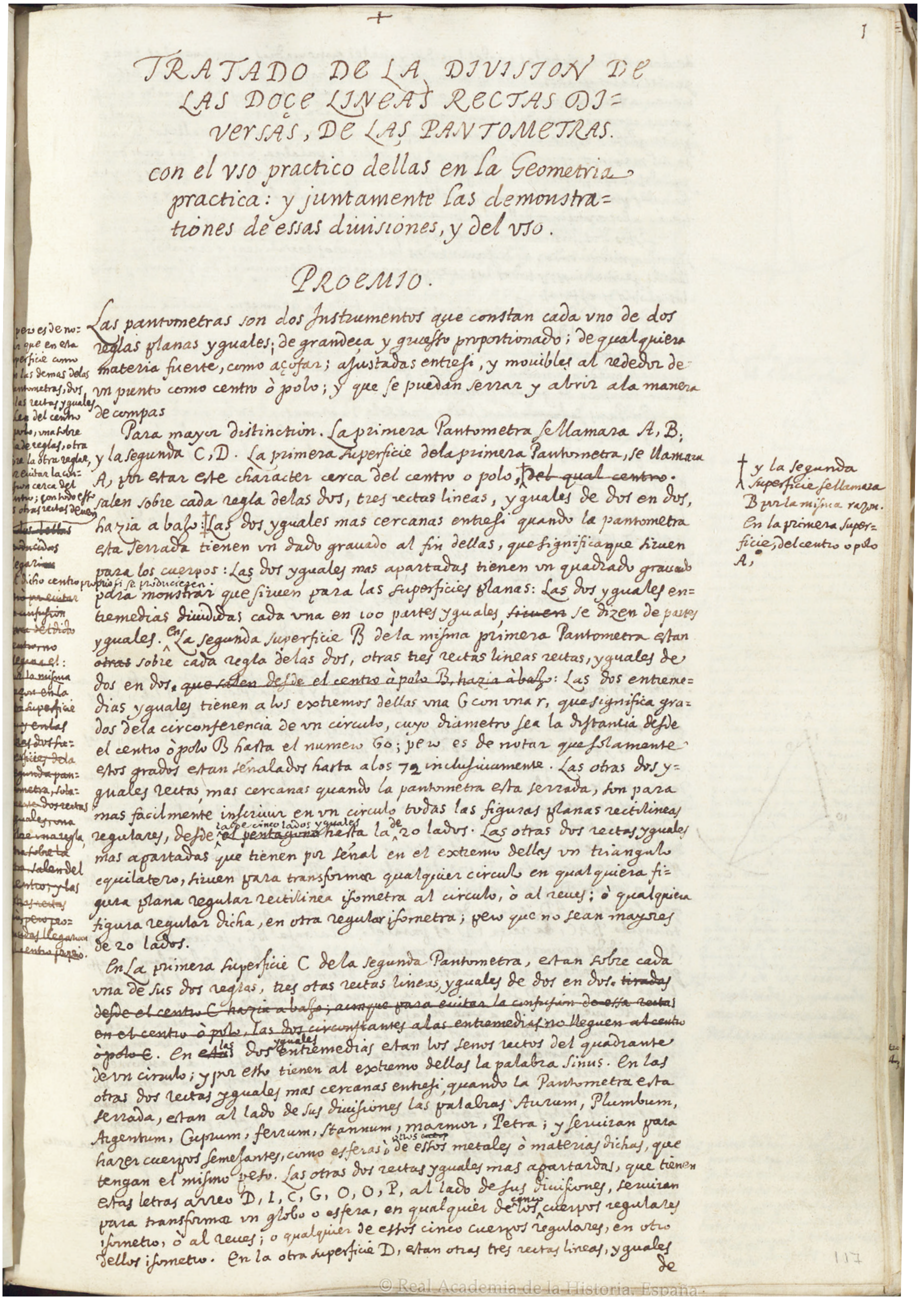
1. INTRODUCTION

Claude Richard (1589–1664), a barely known teacher of mathematics, held his mathematics professorship in the brand-new Royal Studies at the Jesuit Imperial College in Madrid from 1630 until the end of his life. He published *Euclides elementorum geometricorum libros tredecim* [RICHARD, 1645] and *Apollonii Pergaei Conicorum libri IV* [RICHARD, 1655]. Furthermore, the Spanish Royal Academy of History keeps his manuscript legacy, namely teaching handbooks and notes. Among these, a draft copy of Richard's course on sectors has been identified [M-RAH 9/2779a] (figure 1), together with a manuscript copy produced by an anonymous student in 1656 [M-RAH 9/2785] (figure 2) that proves that these contents were actually taught by Richard at the Imperial College[1].

Richard's *Treatise on the division of the twelve diverse straight lines of sectors, with their practical use in practical geometry, and also the proofs of these divisions and the use* [M-RAH 9/2779a] does not only address the brief and easy instrumental practice of geometry, as it was conceived as a textbook going beyond the handbooks for the use of instruments that were frequently sold together. Actually, Richard's course aims at teaching practical geometry on a solid Euclidean basis. Far from collecting a set of practical instructions, Euclid's *Elements* –especially Book VI– are repeatedly referred to in order to check, when possible, that the geometric foundation of the sector scales is well constructed, which would justify the numerical consideration of continuous magnitudes as quantities –accepting a margin of error sensorially imperceptible and irrelevant for the purposes of application.

In the first place, this paper approaches Richard's treatise taking into account the teaching context in the Royal Studies, where professors of mathematics were committed with the Imperial College and the Court, in charge of scholarly mathematics and also of the mathematics of war –particularly fortification [DE LUCCA, 2012]. The incorporation of practical geometry and military arts as part of the teaching program might have been inspired by Italian academies [CAMEROTA, 2004, p. 55-57; 2006, p. 327-330], but it is also worth mentioning that the Jesuit Christoph Clavius (1537-1612) –a competent and well-known mathematician– had already published his *Practical Geometry* [CLAVIUS, 1606] with two chapters devoted to the construction and use of two mathematical instruments. Thus, in addition to the common quadrant, an instrument for easily

1. Both manuscripts are actually anonymous. However, Richard's handwriting can be undoubtedly recognized by comparison with his autograph manuscripts. References to Richard's commented edition of Euclid's *Elements* [RICHARD, 1645] in both texts confirm his authorship.


TRATADO DE LA DIVISION DE LAS DOÇE LINEAS RECTAS DI=VERSAS, DE LAS PANTOMETRAS.

con el vso practico dellas en la Geometria practica: y juntamente las demonstra=tiones de essas diuisiones, y del vso.

PROEMIO.

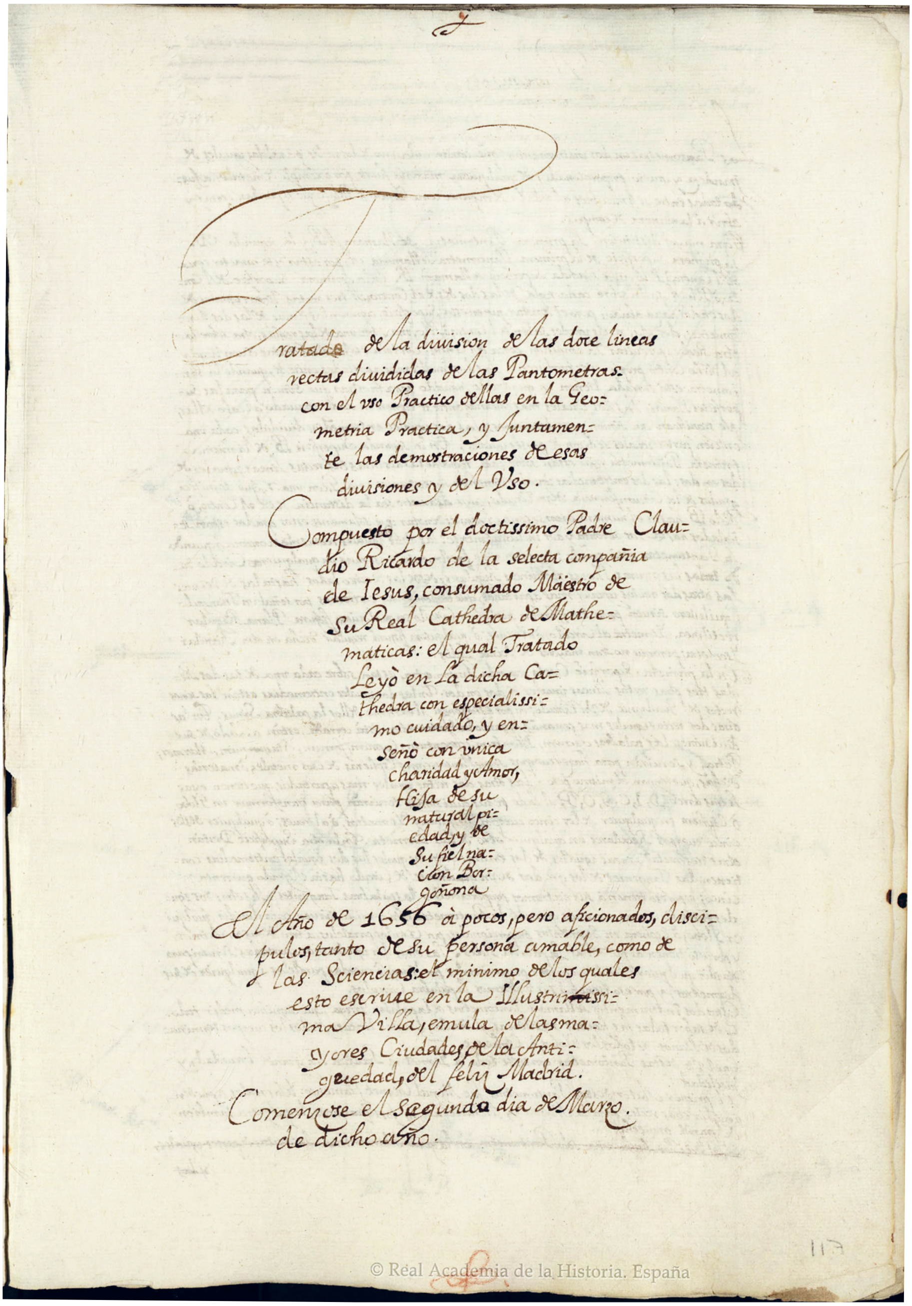
Las pantometras son dos Instrumentos que constan cada vno de dos reglas planas yguales; de grandeça y gruesso proportionado; de qualquiera materia fuerte, como açofar; ajustadas entresi, y mouibles al rededor de vn punto como centro ò polo; y que se puedan serrar y abrir ala manera de compas

Para mayor distinction. La primera Pantometra se llamara A, B, y la segunda C, D. La primera superficie dela primera Pantometra, se llamara A, por estar este character cerca del centro o polo, ~~del qual centro.~~ salen sobre cada regla delas dos, tres rectas lineas, yguales de dos en dos, hazia abaxo: Las dos yguales mas cercanas entresi quando la pantometra esta serrada tienen vn dado grauado al fin dellas, que significa que siruen para los cuerpos: Las dos yguales mas apartadas tienen vn quadrado grauado para monstrar que siruen para las superficies planas: Las dos yguales entremedias diuididas cada vna en 100 partes yguales, ~~siruen~~ se dizen de partes yguales. En La segunda superficie B dela misma primera Pantometra estan ~~otras~~ sobre cada regla delas dos, otras tres rectas lineas rectas, yguales de dos en dos, ~~que salen desde el centro ò polo B, hazia abaxo~~: Las dos entremedias yguales tienen a los extremos dellas vna G con vna r, que significa grados dela circonferencia de vn circulo, cuyo diametro sea la distantia desde el centro ò polo B hasta el numero 60; pero es de notar que solamente estos grados estan señalados hasta alos 72 inclusiuamente. Las otras dos yguales rectas, mas cercanas quando la pantometra esta serrada, son para mas facilmente inscriuir en vn circulo todas las figuras planas rectilineas regulares, desde ~~el pentagono~~ la de cinco lados yguales hasta la de 20 lados. Las otras dos rectas yguales mas apartadas que tienen por señal en el extremo dellas vn triangulo equilatero, siruen para transformar qualquier circulo en qualquiera figura plana regular rectilinea ysometra al circulo, ò al reues; ò qualquiera figura regular dicha, en otra regular ysometra; pero que no sean mayores de 20 lados.

En La primera superficie C dela segunda Pantometra, estan sobre cada vna de sus dos reglas, tres otras rectas lineas, yguales de dos en dos. ~~tiradas desde el centro C hazia abaxo; aunque para euitar la confusion de esta rectas en el centro ò polo, las dos circunstantes alas entremedias no lleguen al centro ò polo C~~. En las dos yguales entremedias estan los senos rectos del quadrante de vn circulo; y por esto tienen al extremo dellas la palabra Sinus. En las otras dos rectas yguales mas cercanas entresi, quando la Pantometra esta serrada, estan al lado de sus diuisiones las palabras Aurum, Plumbum, Argentum, Cuprum, ferrum, Stannum, marmor, Petra; y seruiran para hazer cuerpos semejantes, como esferas, de estos metales ò materias dichas, que tengan el mismo peso. Las otras dos rectas yguales mas apartadas, que tienen estas letras arreo D, I, C, G, O, O, P, al lado de sus diuisiones, seruiran para transformar vn globo o esfera, en qualquier de los cinco cuerpos regulares ysometro, ò al reues; o qualquier de estos cinco cuerpos regulares, en otro dellos ysometro. En la otra superficie D, estan otras tres rectas lineas, yguales de

© Real Academia de la Historia. España

Figure 1. Richard's Course [M-RAH 9/2779a]
[© Reproducción, Real Academia de la Historia]


Tratado de la division de las doze lineas
rectas diuididas de las Pantometras.
con el vso Practico dellas en la Geo-
metria Practica, y juntamen-
te las demostraciones de esas
diuisiones y del Vso.
Compuesto por el doctissimo Padre Clau-
dio Ricardo de la selecta compañia
de Iesus, consumado Maestro de
Su Real Cathedra de Mathe-
maticas: el qual Tratado
Leyò en la dicha Ca-
thedra con especialissi-
mo cuidado, y en-
señò con vnica
charidad y Amor,
Hija de su
natural pi-
edad y de
su fiel na-
cion Bor-
goñona
el Año de 1656 à pocos, pero aficionados, disci-
pulos, tanto de su persona amable, como de
las Sciencias: el minimo de los quales
esto escriue en la Illustrissi-
ma Villa, emula de las ma-
yores Ciudades de la Anti-
guedad, del feliz Madrid.
Comenzose el segundo dia de Marzo.
de dicho año.

117

Figure 2. Student's copy [M-RAH 9/2785]

dividing any line into any number of equal or proportional parts, that he called *Instrumentum Partium* –later *pantometer*– became part of the academic mathematics teaching [CLAVIUS, 1606, p. 4-13]. Consequently, Clavius' and Richard's texts on sectors only showed their use to solve geometric problems, without making explicit their link to applications to military arts.

Our focus next turns to Richard's course on Coignet-type sectors, which were –as many others– the result of the evolution of the proportional compass by Federico Commandino (1568) [CAMEROTA, 2000; 2004, p. 44-61]. However, Richard used the sector to teach practical geometry, and for this purpose he revealed the graduation of the lines –which was omitted in the handbooks for the use of sectors in the 17th century. In this sense, his course is original and rare, for he validated the use of the instrument in practical geometry without disregarding incommensurability, which suggests the influence of Clavius' mathematical thought on Richard –as on Descartes [SASAKI, 2003, p. 45-48].

This paper looks at Richard's course from the point of view of mathematical practice and conceptual change in Early Modern mathematics, as a case study of how mathematical instruments offered materializations of new implicit notions and justifications for new algorithms which only later on would find formal expression.

To this end, the detailed textual analysis of the graduation of the lines provides information on the conceptual and educational aspects of this course, such as the notions of proportionality, number and magnitude, the numerical consideration of geometric magnitudes as quantities, and the acceptance of a numerical approximation to an exact quantity that cannot be found, which suggests the influence of Clavius and Regiomontanus [MALET, 2006, p. 70-71].

2. Mathematics in the Royal Studies at the Imperial College in Madrid

At the beginning of the 17th century the Spanish Monarchy was involved in the *Eighty Years' War* (1568–1648), which resulted in the recognition of the Dutch Republic as an independent country. Actually, the Dutch Republic was recognized by Spain and the major European powers in 1609, at the start of the *Twelve Years' Truce* (1609-1621). In the final years of Philip III's reign, Spain entered the initial part of the *Thirty Years' War* (1618-1648). This would lead his successor, Philip IV, to renew hostilities with the Dutch in 1621. This belligerent context encouraged the interest in instruction in mathematics as applied to the art of war, but restrained the publication of treatises on artillery and fortification.

In 1606 the Court definitely settled in Madrid –the permanent capital of the Monarchy since then. The new Chair on Mathematics and Fortification –set up by the Supreme Council of War for the training of military engineers and artillery men in 1605– enlarged the scientific and technical scope of the Academy of Mathematics and Cosmography –an institution founded in 1582 and ascribed to the Royal and Supreme Council of the Indies since 1591. In 1607, the Major Cosmographer of the Council of Indies assumed the Chair of Mathematics of the Academy of Mathematics and established a three-year course in mathematics: the sphere, planetary theory, and the Alphonsine tables in the first year; the first six books of

Euclid's *Elements* and Ptolemy's *Almagest* in the second; cosmography, navigation, and some instruments in the third and final year [Vicente Maroto, 1991, p. 81-109, 143-152, 173-176, 208-214, 327-333].

Two years after the coronation of Philip IV in 1621, the young king launched a more ambitious project –that was actually part of the reform program of his Favorite, the Count-Duke of Olivares. He informed Muzio Vitelleschi –the Superior General of the Society of Jesus– about his intention to establish "general studies" in Madrid, and offered the Jesuits the direction of these Royal Studies –including the nomination of professors. According to the foundational plan of the Royal Studies at the Jesuit Imperial College in Madrid (1625), the main purpose was the education of the sons of nobles, designed to provide not only a liberal education, but also practical instruction in mathematics, the sciences, and the art of war. The teaching structure consisted of six chairs in Latin grammar for Minor Studies and seventeen chairs for Major Studies, among them two chairs in mathematics, one chair in military art –based on Polybius and Vegetius works–, and one chair in natural history. As for mathematics, one chair was to be assigned to a professor teaching in the morning on the sphere, astronomy, astrology, astrolabe, perspective, and prognosis, and the other to a different professor teaching in the afternoon on geometry, geography, hydrography, and horology [Simón Díaz, 1992, p. 149-157].

That same year, 1625, on the death of the professor of mathematics and major cosmographer of the Council of Indies, the Rector of the Imperial College was to choose Jesuit teachers to teach mathematics at the Academy –the Society of Jesus would receive the full salary of the former professor and cosmographer in exchange [Vicente Maroto, 1991, p. 162, 165-166].

In July 1628, the foundational plan was modified, for it met strong opposition from Castilian Universities –especially Alcalá and Salamanca– throughout 1626 and 1627, which entailed the elimination of the chair of Logic from the Imperial College, the reduction of its financial support, and the impossibility to award any academic degree on the basis of the Royal Studies [Simón Díaz, 1992, p. 157-183]. Soon afterwards, in September, the three-year course on mathematics was transferred from the Academy to the Imperial College. The professor nominated by the Rector had to accept not only his teaching duties –one lesson in the morning and one lesson in the afternoon, plus the translation into Spanish of the necessary books–, but also his tasks as Major Cosmographer. The Rector proposal was to be assessed by the Council of Indies and approved by the King. The appointment as Professor and Major Cosmographer of the Council of Indies included a personal salary of eight hundred ducats a year [Vicente Maroto, 1991, p. 166-167, p. 193-196]. The Jesuit Johann Baptist Cysat (ca. 1586 – 1657) might have been nominated for this post, as he taught at the Imperial College from March 1627 to January 1629, while the Royal Studies were still being organized [O'Neill & Domínguez, 2001, p. 1028; Udías, 2005, p. 373].

Finally, the inaugural ceremony of the Royal Studies took place at the end of February 1629, in the presence of the King, the Queen, and courtiers, and with the premiere of the seven hundred and four verses of *Isagoge a los Reales Estudios de la Compañía de Jesús* by Lope de Vega –the most famous Spanish poet at that time [SIMÓN DÍAZ, 1992, p. 190-208].

Joannes della Faille (1597–1652) was appointed Professor of Mathematics at the Royal Studies –instead of Grégoire de Saint-Vincent, who declined the invitation on grounds of ill-health. He started teaching in March 1629. In 1630 Claude Richard, was appointed to fill the vacant chair of mathematics at the Royal Studies.

Richard was Philip IV's subject, for he was born in Burgundy. In 1606 he joined the Jesuits in Rome. He taught hebrew and mathematics at the Jesuit college in Tournon (1616-21) [DE DAINVILLE, 1954, p. 117], and mathematics at the Jesuit college in Lyon (1622-29) [DE DAINVILLE, 1954, p. 113]. In August 1629 –on his way to China– he arrived in Madrid, where he held his mathematics professorship in the Royal Studies for thirty seven-years –until the end of his life– and became the longest lasting professor of mathematics at the Imperial College in the 17th century [UDÍAS, 2005, p. 427-428].

It should be mentioned that professors of mathematics at the Imperial College were somehow overburdened by the Court. As for Della Faille, he held his mathematics professorship in the Royal Studies for eighteen years (1629-1647) [UDÍAS, 2005, p. 427-428]. He published his work on gravity centers [DELLA FAILLE, 1632], he wrote a treatise on architecture [LAFAILLE, 1636], and he drafted his method of geometry [FAILLE, 1640].[2] As of January 1639, Della Faille was busy with his ordinary lesson in the morning, Father Camassa's lesson on military arts in the afternoon,[3] and private lessons and tutorials in his chamber [VAN DER VYVER, 1977, p. 141]. The mandate to teach military arts and fortifications to the Royal Pages at the Palace followed his appointment as Major Cosmographer on March 23 –daily lessons from 16:00 to 17:00 started in May– [VAN DER VYVER, 1977, p. 145, 148-149]. But from 1641 to 1644 he was sent to the Portuguese Restoration War (1640–1668) –leading to Portugal's regained full sovereignty– as adviser on fortifications to the Duke of Alba [VAN DER VYVER, 1977, p. 160-176; CERRILLO MARTÍN DE CÁCERES, 2017] (figure 3). In 1646, he became Preceptor to Juan José de Austria –Philip IV's illegitimate son–, and remained in his service on his military expeditions to Naples, Sicily, and Catalonia, where he died. The fact that Della Faille spent half of his life in Spain in the service of the Crown provides a vivid image of the job of a competent mathematician in 17th century Spain. As he wrote in January

2. For a complete scientific biography of Della Faille see MESKENS [2005]. He attributes to Della Faille five anonymous manuscripts kept at the Spanish Royal Academy of History –two on astronomy, one on Juan de Roxas astrolabe, one on the telescope, and an abridged translation into Spanish of Giovanni Batista Baliani's *De motu naturali gravium solidorum* (1638) [MESKENS, 2005, p. 69-73], and a chapter devoted to the manuscripts on conic sections [MESKENS, 2005, p. 81-105].
3. Francesco Antonio Camassa (1588–1646) was teaching mathematics in Naples when he was appointed to the chair of *De re military* at the Imperial College (1633-1644), where he also taught mathematics from 1642. However, between 1634 and 1641 he spent most of his time at the service of the Marquis of Leganés, as confessor and expert in military arts and fortification [DAMERI, 2017; UDÍAS, 2005, p. 375, 429].

1645: "I hardly study, because now I am here, now they send me somewhere else. Finally, these are not times for the calmness that our studies require".[4]

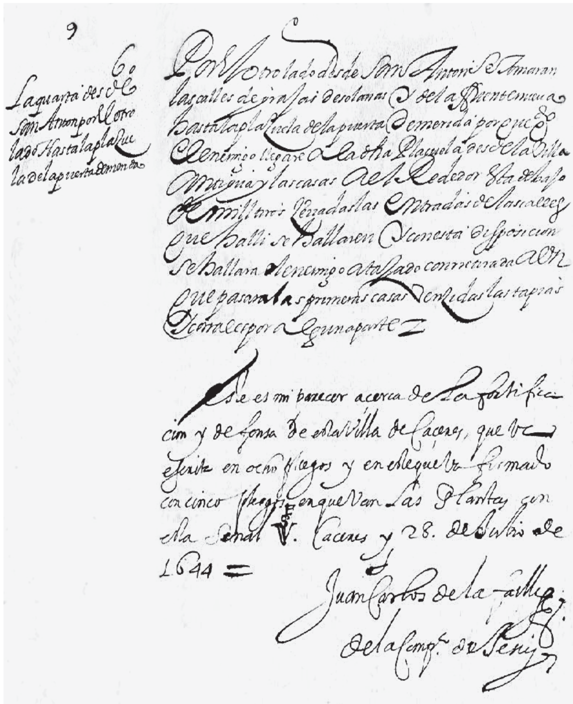


Figure 3. Della Faille's Cáceres Fortification Project (1644)
[Cerrillo Martín de Cáceres, 2017, p. 2661]

As for Claude Richard, he might have been in charge of the lessons of mathematics and cosmography at the Court soon after his arrival in Madrid [Vicente Maroto, 1991, p. 166-168][5] –possibly until Della Faille appointment as Major Cosmographer. He might have also replaced Camassa's at the chair of *De re militari,* for he wrote a treatise on *Ars militaris,* an autograph manuscript[6] dealing with the construction, siege and defense of fortresses, and also with artillery [Udías, 2005, p. 406, 436]. In June 1635, Richard left for Germany as confessor and adviser in military arts and fortification to Alonso Gaspar Fernández de Córdoba y

4. "Yo estudio poco, porque agora estoy aquí, agora me embían a otra parte. Finalmente, estos tiempos no son para el sossiego que nuestros estudios piden" [Van der Vyver, 1977, p. 141].
5. His *Prolegomenon astronomia* is dated 1631. The manuscript is kept at the Spanish Royal Academy of History (shelf mark 9/72779). It consists of 116 folios (32 cm.), including drawings and tables.
6. "Opus P. Claudii Richardi S.I. et propia manu conscriptum" [f. I]. This manuscript is kept at the Spanish Royal Academy of History (shelf mark 9/2687). It consists of 152 folios (27 cm.), 6 plates. On Jesuits in military teaching and consultancy positions see De Lucca [2021].

Alvarado, Marquis of Celada [VAN DER VYVER, 1977, p. 105-107] –who died in Valenza del Po (Italy) on 2 November. On his return to Spain,[7] he drew up reports on the fortification of Malta (1639), San Sebastián (1641) and Pamplona (1643) [DE LUCCA, 2012, p. 149], and also produced a treatise on one of his teaching subjects, namely the sphere [RICHARD, 1640][8] –a work he started early in 1637 and ended in May 1638.[9] The dedication to the king and the sealed approval by the provincial Franciscus Aguado show that this autograph manuscript was ready to be published, but apparently it was never sent to print. Actually, since 1635 paper supply was difficult [VAN DER VYVER, 1977, p. 99-101]. Richard's undated manuscripts on arithmetic and algebra [M-RAH 9/2685; M-RAH 9/2686] [UDÍAS, 2005, p. 435] might belong to this period.

In 1645, trusting the King to get financial support for publications was asking the impossible.[10] This was the case of Father Richard, who spent over eight hundred *reales* of his own money while waiting for the remaining two thousand *reales* he needed in order to publish his commented edition of Euclid's *Elements* in Antwerp [RICHARD, 1645] [VAN DER VYVER, 1977, p. 179].

From 1651 to 1657, Richard was the only professor of mathematics at the Imperial College [UDÍAS, 2005, p. 428] –until Father José Martínez (1603-1668) joined him in 1658. In this period, Richard published his commented edition of Apollonius's *Conics* (I-IV) in Antwerp [RICHARD, 1655], and reported on the comet that appeared on the horizon of Madrid on 20 December 1652 and disappeared ten days later [RICHARD, 1653]. On 2 March 1656, he started teaching a course on the construction and use of sectors at the Imperial College. Two draft copies of this course have been identified, one written by Richard himself [M-RAH 9/2779a], the other by one of his "few but interested in science" students [M-RAH 9/2785].

7. Richard was back in Madrid in February 6, 1636, as it shows his original letter on the siege and liberation of Valenza (Spanish Royal Academy of History, *Copias de cartas y otros documentos sobre Jesuitas en el S. XVII*, shelf mark 9/3699, f. 300-302).
8. It should be mentioned that between 1636 and 1641 an assistant teacher was assigned to the chairs of mathematics at the Imperial College, namely the young Spanish Jesuit Francisco Isasi (1603-1650), who was noted for his works on fortification and architecture [NAVARRO LOIDI, 1999; UDÍAS, 2005, p. 375, 427-428; DE LUCCA, 2012, p. 141-143]. He also taught mathematics from 1636 to 1641.
9. This information is added at the end of the title of a similar manuscript: "Matriti coeptum opus initio anni 1637 et perfectum anno 1638 in fine Maii" (Spanish Royal Academy of History, shelf mark 9/2683).
10. Della Faille to Van Langren: "yo no veo por qué v. merced ha menester a nadie para sacar sus trabajos a luz; si es porque el rey le ayuda [sic] a imprimir, es pedir peras al olmo, que no ay dinero en este mundo agora". Actually, Della Faille was not collecting his salary as Major Cosmographer of the Council of Indies [VAN DER VYVER, 1977, p. 179].

3. Claude Richard's Course on Sectors at the Imperial College

It is not surprising that Richard's course deals with Coignet-type sectors, for the Flemish Michiel Coignet (1549-1623)[11] was at the service of the Habsburg court, as a mathematician and engineer to the Archdukes Albert and Isabella –the governors of the Spanish Low Countries– from 1596 until he died [Meskens, 2013, p. 18-21].

Coignet's works circulated in Spain since the late sixteenth century: His manuscript *L'uso del compasso di Fabricio Mordenti di Salerno mathematico del serenissimo Principe Alessandro Farnese Duca di Parma, composto per Micaelo Coignetto: propositioni geometriche cavate dalli primi sei libri delli elementi d'Euclide* –in Italian– is dated before 1592 [Meskens, 2013, p. 34];[12] a manuscript translation into Spanish of his *Usus trium praecipuorum mathematicorum instrumentorum* was dedicated to Rodrigo Calderón, an ambassador at the Spanish Netherlands in 1612;[13] and a copy of his *Instruction nouvelle des poincts plus excellents & nécessaires, touchant l'art de naviguer* (Antwerp, 1581) was inventoried at the Count of Gondomar's library in 1623.[14] The Southern Low countries and Italy certainly served as sources of knowledge to the Spanish Monarchy in a multilingual context –actually, none of the three Coignet's manuscripts on pantometers in Spanish seems to be kept in Spain [Meskens, 2013, p. 229-231].[15]

Richard's treatise on sectors started with a foreword [M-RAH 9/2779a, p. 1-2] that adopted the term *pantometers* to name Coignet-type sectors, and described the two sides A/B, C/D of two pairs of pivoting plane rulers with twelve engraved scales, namely three equally graduated straight lines on both arms of each side (figure 4).[16] He explained that they were called *Pantometra* after a Greek word meaning "measure all" –namely rectilinear figures and circles, straight lines, polyhedra, globes and spheres–, and claimed that the entire practical geometry consisted of the brief and easy use of pantometers, an instrument first invented by the Flemish Michel Coignet.[17]

---

11. For a complete scientific biography and sectors see Meskens [2013], particularly Meskens [2013, p. 14-21, 113-137].
12. Biblioteca Nacional de España, shelf mark MSS/19709/32.
13. Biblioteca Nacional de España, shelf mark MSS/9213.
14. Real Biblioteca (Madrid, Spain), shelf mark IX/1574.
15. Here we will use Coignet [1618], a very good digital copy. A similar one is Coignet [17th century]. The French posthumous edition is also similar [Coignet, 1626]. For a list of books and manuscripts by Coignet see Meskens [2013, p. 229-231].
16. For this type of sector see Meskens [2013, p. 118-137]. In Coignet's manuscripts titles it was named *Regulae Pantometrae –Reglas Pantometras* in Spanish [Coignet, 1618]. Richard noted that new useful scales for music and architecture could be added to those already described.
17. Richard also mentioned one single pantometer with only six lines made in Paris, which excludes Henrion's four scales sector [Henrion, 1618] but matches engineer Pierre Petit's proportional rule [Petit,1634] –except for the line of regular polyhedral (polygons in Richard's text).

The treatise consists of thirteen chapters –one for each graduated straight line plus one for difficult and curious propositions– on the geometric division and practical use of these lines, both proved with demonstrations. Richard's proofs focus on geometric matters, even if he actually considers the use of trigonometric ratios soundly established.[18] The following textual analysis consequently focuses on the scales graduation –omitted in the handbooks for the use of sectors in the 17th century.

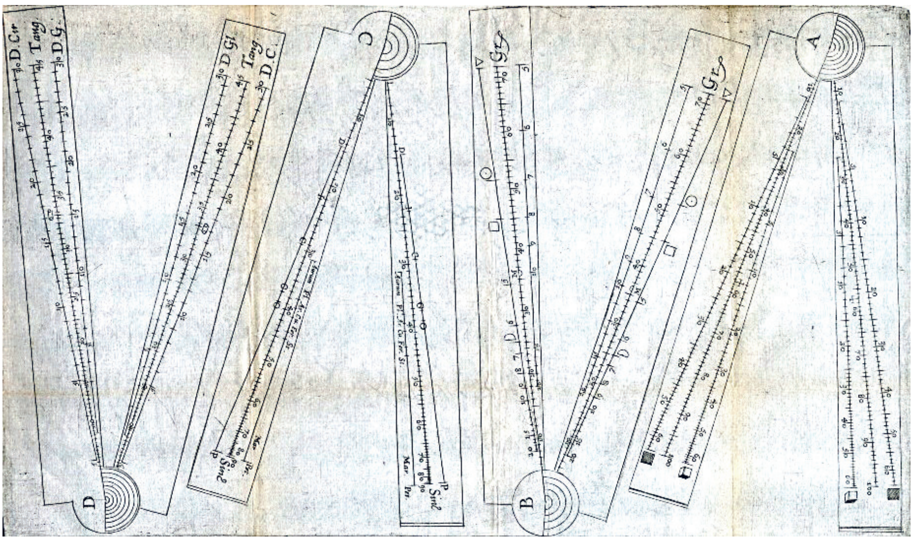


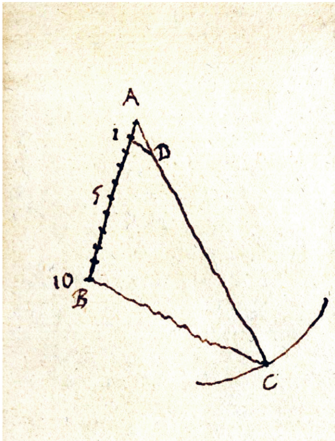
Figure 4. Coignet-type sector [CONNETE, 1626, p. 4-5]

3.1. Side A: Equal parts lines (A_{100})

In the first chapter [M-RAH 9/2779a, p. 2-11e] Richard explained the division into 100 equal parts of the two central lines on the side A of the sector, as follows (figure 5).

Let AB be any straight line. Add 10 equal parts on AB (*Els.*I.3)[19] and mark B at the end of the 10th part. Describe an arc of circle with center A and radius AC, the length of the central line on the first side of the sector. Join BC. Draw a straight line through 1 parallel to BC (*Els.*I.31). Mark D on AC (*Our corollary* 1 to *Els.*I.42)[20] [RICHARD, 1645, p. 46, figure 57]. AD = AC/10 because A1=AB/10 and AB, AC are cut proportionally by 1D parallel to BC (*Els.*VI.2), so AD = A_{100}/10 (*Els.*V.7); similarly the remaining 9 equal parts can be added (*Els.*I.3, V.Def.1).

18. Richard was the author of a treatise on plane trigonometry [M-RAH 9/2779b].
19. This proposition begins the geometric arithmetic of lines in Euclid's *Elements* [JOYCE, 1996]. Richard's corollaries tend to make easier the understanding and use of Euclid's *Elements*. The manuscript always quotes his own edition, but he might have learnt with CLAVIUS [1591], for in 1606 he joined the Jesuits in Rome.
20. *Els.*I.42 begins the application of areas in Euclid's *Elements* [JOYCE, 1996].



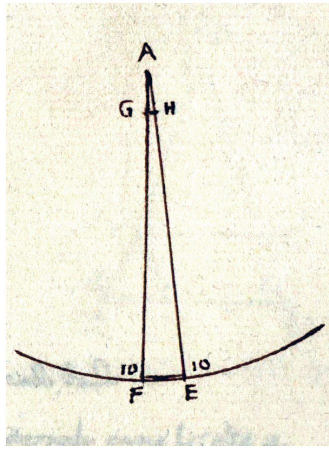


Figure 5. (Left) [M-RAH 9/2779a, p. 2-3]
Figure 6. (Right)
[© Reproducción, Real Academia de la Historia]

This done, the sector had to be opened with a distance of 1 tenth (FE) at the end of the two arms in order to get the length of one-hundredth (GH) between A_1 and A_1 (figure 6). This was explained in Richard's Euclidean style[21] using *Els.*I.3, *Els.*VI.4, VI.2, VI.21, VI. Def.1, *Els.*V.16, and *Els.*V.Def.1. The concepts of equiangular triangles, similarity, and the definition of a magnitude as a *part* –not *parts*– of a magnitude were introduced.

There follow seventeen propositions using the equal parts line in practical geometry. Proposition 9 [M-RAH 9/2779a, p. 10] taught how to find a third proportional to two given straight lines, and Proposition 10 [M-RAH 9/2779a, p. 11a] how to find a fourth proportional to three given straight lines, showing the possibilities of this type of sector as a measuring instrument to be used in the calculation of ratios and proportions. Furthermore, propositions four, five, and twelve are especially interesting, as they show how to deal with incommensurability in practical geometry.

Proposition 4 [M-RAH 9/2779a, p. 5] asks how to determine *in numbers* the *proportion* between two given unequal straight lines AB > CD. This is done by placing the longest line AB between two equal numbers –Richard suggests 100 and 100, and then fitting the shortest line CD crosswise to a pair of numbers other than the former pair –for instance 30 and 30[22]. In this case the *ratio* of AB to CD is as 100 to 30. Richard immediately points out that this proposition and practice is approximately taken for incommensurable straight lines that are

21. By "Euclidean style" I mean that Richard referred to the *Elements*' propositions, definitions or corollaries that supported every step of the division procedure, as in Euclid's *Elements*.
22. Magnitudes were transferred by means of a compass rather than a rule.

studied in detail in *Els*.X. The example also shows that, in modern terminology, Richard's proportions and numerical ratios are positive rational numbers as defined in *Els*.VII.Def.20.[23] This is confirmed in proposition 5 [M-RAH 9/2779a, p. 5], that shows with two examples how to find a straight line CD proportional to a given straight line AB with the ratio of AB to CD given in *unequal numbers*. For AB:CD = 80:15, CD will be the interval between 15 and 15 when AB is placed between 80 and 80; vice versa for AB:CD = 15:80. No comments are needed here, as AB and CD are commensurable magnitudes (*Els*.X.Def.1, X.5, X.6).

Proposition 12 [M-RAH 9/2779a, p. 11b-11c] asks how to mark on both equal parts lines a Greek cross + to be used for determining the size of the radius or diameter of any given circle, and also the third part of the straight line equal to the semi circumference of the given circle, or the sixth part of the straight line equal to the whole circumference of the same circle. Here Richard follows Archimedes' proposition 3 in *Measurement of the Circle* [Heath, 1897, p. 93] by taking the ratio of the circumference to its diameter as 22 to 7 ($\pi \cong 3\frac{1}{7}$), so that the ratio of the diameter to the circumference is as 7 to 22 (*Els*.V.7 Corol.).[24] For a straight line divided into 360 equal parts the diameter of the given circle is $114\frac{12}{22}$ parts of these 360, and the radius of the circle whose semi circumference is a straight line divided into 180 equal parts is $57\frac{6}{22}$ (*Els*.V.11, V.15). Mark a Greek cross + at $57\frac{6}{22}$ on both equal parts lines, where 60 is the third part of the semi circumference or the sixth part of the whole circumference whose radius is $57\frac{6}{22}$. It follows that the first line is proportionally cut in + and 60 as the radius of the circle to the sixth part of his circumference. Propositions 13 to 18 [M-RAH 9/2779a, p. 11c-11e] deal with finding the diameters or radius of circumferences or equal parts of circumferences. This was a most interesting achievement, as measuring arc lengths were needed to tackle circular and spherical segments in practical geometry.

3.2. Side A: Lines of planes ($A_{\square}$)

The second chapter [M-RAH 9/2779a, p. 12-15] is titled *On the geometric division*[25] *into 64 proportional parts of two given equal straight lines, in order to increase and decrease rectilinear planes and circles in the given proportion not greater than 64; and to find out the proportion between them; and to find a mean proportional to two given straight lines*. It consists of seven propositions, but is almost entirely devoted to proposition 1 [M-RAH 9/2779a, p. 12-14], where Richard develops a detailed Euclidean exposition on how to divide into 64 proportional parts the two outer lines on sector side A as follows (figure 7).[26]

Let C be an end of any straight line. Add 65 equal parts on C (*Els*.I.3) and mark D at the end of the first part, and E at the end of the 65th part. DE is 64 times CD. By *Els*.VI.12 –or by the above-mentioned proposition 10– find FG 4th proportional to DE, CD, and $C_{\square}$ –equal to

23. See the Guide's Section *Ratios of various kinds* in Joyce [1996, *Els*.V.Def.3].
24. *Els*.V.4 Corol. is a misprint.
25. Richard uses the expression *geometric division* only in the lines of equal parts and planes.
26. Figures 7 and 8 appear on a single folio inserted between pages 12 and 13, and also in Richard's student manuscript copy [M-RAH 9/2785, p. 126r].

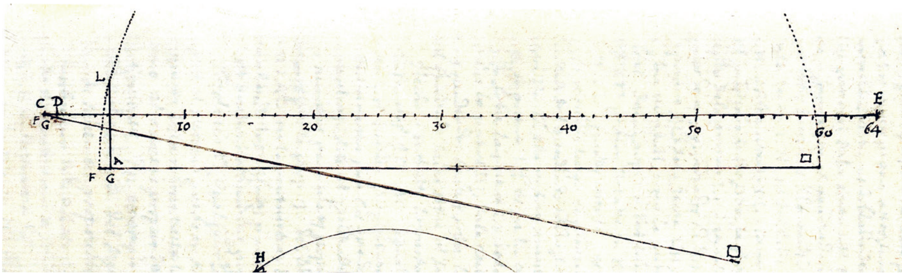


Figure 7. [M-RAH 9/2779a, p. 12-13, © Reproducción, Real Academia de la Historia]

$A_{\square}$ on the sector, so that FG is 64 times smaller than $A_{\square}$. By *Els*.VI.13 find GL mean proportional to $A_{\square}$ and FG, so that $A_{\square}^2 : GL^2 = A_{\square} : FG = 64$ (*Els*.VI.20 Corol.).[27] At this point, Richard first mentions homologous sides, a key concept of the geometric foundations of sectors.

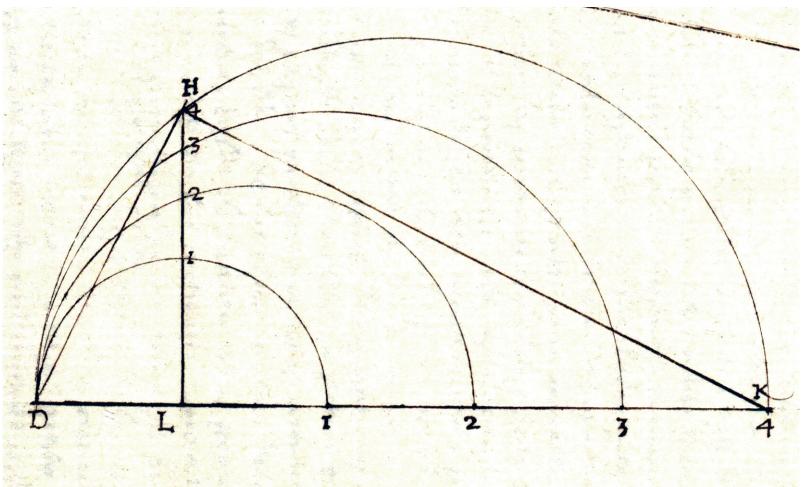


Figure 8. [M-RAH 9/2779a, p. 12-13, © Reproducción, Real Academia de la Historia]

Now (figure 8) make DL equal to GL, draw LH perpendicular to DL (*Els*.I.11), and equal to $A_{\square}$ (*Els*.I.3). Join DH, where H is an acute angle in the right-angled triangle DHL (*Els*.I.17 Corol. 1) [Richard, 1645, p. 29, fig. 38]. Produce DL *ad infinitum* from L and draw a perpendicular HK to DH from H (*Els*.I.11). As LHK is an acute angle –part of the right-angled triangle DHK, and the angle in L is a right angle, HL cuts HK and DL –produced indefinitely from L– making the interior angles on the right side less

27. Richard's text is completely rhetoric, he used no mathematical symbols for ratios or proportions when dealing with magnitudes.

than two right angles, so HK and DL meet in K (*Els.*I.Post.5).[28] Therefore, as a perpendicular HL has been drawn from the right angle in H to the base DK in the right-angled triangle DHK, the HL straight line so drawn is a mean proportional between the segments of the base DL, LK (*Els.*VI.8 Corol.). Then DL : LK = $DL^2 : HL^2$ (*Els.*VI.20 Corol.), where HL equals $A_{\square}$, and DL equals to GL –the square on GL being 64 times smaller than the square on $A_{\square}$, as it was previously proved. Consequently, DL is 64 times smaller than LK, LK can be divided into 64 parts equal to DL (*Els.*I.3), and DL is the sixty-fourth part of LK (*Els.*V.Def.1).[29] The straight line LK can be graduated from L to K with natural numbers 1 to 64 at intervals *n* times DL ($1 \leq n \leq 64$).

Next, bisect DK and describe the circle DK with center in DK/2. The semicircle will circumscribe the right-angled triangle DHK (*Our corollary* to *Els.*III.33) [Richard, 1645, p. 89, fig. 112]. Describe circles on DK with diameter *n* times DL cutting LH in LH_n ($1 \leq n \leq 64$). Now the square on LH_n is to *n* as the square on LK_1 to the *unit*, for $(LH_n)^2 : (LD)^2 = LK_n : LD = (n_* LK_1) : LK_1$ (*Els.*VI.20 Corol.).[30] The square on LH_n is *n* times the square on LD and the lines of planes are graduated in 64 proportional parts.

As a first corollary, Richard proves –by *Els.*V.9 and V.7 Corol.– that the squares on any two segments LH_i, LH_j on LH_n are to each other as *i* to *j*. The second corollary establishes that the first one also applies to circles, for circles are to one another as the squares on their semi diameters (*Our corollary* 7 to *Els*.XII.2) [Richard, 1645, p. 403], so that circles on LH segments are to each other as the ratio between the corresponding numbers (*Els.*V.11).

Propositions 2-3 [M-RAH 9/2779a, p. 14-15] and 5-6 [M-RAH 9/2779a, p. 15_2-15_3][31] use the lines of planes to increase and decrease rectilinear figures and circles in a given ratio –either numerical ratios or magnitudes, and to add and subtract areas–; proposition 4 [M-RAH 9/2779a, p. 15] shows how to find a mean proportional to two given straight lines with the sector. The chapter finishes with proposition 7 [M-RAH 9/2779a, p. 15_3], on how to find the true or approximate square root of any given number with four or five arithmetic figures.[32]

28. Actually Richard referred to *Els.*I.Post.13 [Richard, 1645, p. 14, fig. 18].
29. In Euclidean terms, DL is *a part* of LK –for DL measures LK 64 times, and LK is a *multiple* of DL (*Els.*V.Def.2).
30. LK_n equals n_*DL, and DL equals LK_1 (and LH_1). An additional proof was given with 3 proportional lines LK_n, LH_n, LD, so that $LK_n : LH_n = LH_n : LD$, resulting in $n_*(DL)^2 = (LH_n)^2$ (*Apol.*I Lemma 47, *Els.*VI.17) [Richard, 1655, p. 27].
31. Propositions 5, 6, and 7 appear on a single folio inserted between pages 14 and 15.
32. Proposition 7 is much better explained in Richard's student manuscript copy [M-RAH 9/2785, p. 127v-129r]. For example, to find the root of 4624 take 46,24, open the sector so that the segment A_{80} on one of the equal parts lines fits between the two lines of planes at 64 (since $80^2 = 6400$); without opening or closing the sector take the interval between 46 and 46 on the lines of planes and place it on one of the equal parts lines: the number that this segment marks on the equal parts line (A_{68}) is about the same as the square root of 4624, for $\sqrt{46} : \sqrt{64} = x : 80$ ($x = 10\sqrt{46} = 67{,}82$).

3.3. Side A: Lines of solids (A□)

The third chapter [M-RAH 9/2779a, p. 16-18] consists of six propositions on the two straight lines of solid figures. The first one asks how *to divide the two equal straight lines A into proportional parts in order to increase or decrease solid figures, and other useful things.* For this purpose, Richard proceeds as follows:

Take both lines of solids as long as the equal parts lines,[33] and divide them into continuously proportional parts –referred to equal parts lines A_{100}, so that for any natural number less than 64 the solid (e.g. cube) on A_n be *n* times the similar solid on A_1.

Richard was fully aware of the geometric difficulty of finding two lines continuously proportional to two given lines, as his *Elementorum geometricorum libros tredecim* ended with a section on this subject, namely *Liber de inventione duarum rectarum linearum continue proportionalium, inter duam rectas datas, ex antiquis Geometris & recentioribus* [Richard, 1645, p. 545-563]. He explained here fourteen ways to solve this question by Plato, Archytas of Tarentum, Menaechmus, Eratosthenes, Philo of Byzantium, Hero, Apollonius, Nicomedes, Diocles, Sporus, Johann Werner, Juan Bautista Villalpando –two methods–, and his own solution [Richard, 1645, p. 545-563, fig. 407]. However, he decided to instruct his students with the following simple arithmetic procedure:

Take 25 on the equal parts line A_{100}. For every natural number less than 64 find the cube root of *n* times the cube of 25. These cube roots will be less than 100 –that is the cube root of 64 times the cube of 25. Take these *parts* or *numbers* resulting of the cube roots on the equal parts line A_{100}, and mark them from A on the two inner lines of solids. The lines of solids are graduated in 64 proportional parts with consecutive numbers from 1 to 64 that show the sides of the cubes, which apply to the homologous sides of any solid –polyhedron– or sphere, as they are in the triplicate ratio of their corresponding sides or diameters (*Els.* XI, XII, XIII).

Richard passes over the choice of number 25 for the first proportional part, which results from 100 divided by the cube root of 64, so that the 64^{th} part of the line of solids is four times the first part and equals 100 on the equal parts line A_{100}. Similarly the twenty-seventh part of the line of solids is triple the first part and equals 75 on the equal parts line, and the eighth part of the line of solids is twice the first part and equals 50 on the equal parts line.

This arithmetic procedure suggests the use of tables of squares and cubes[34] –though Richard makes no mention in this chapter. For instance, for $n = 2$, $2 \times 25^3 = 31250$ is between $31^3 = 29791$ and $32^3 = 32768$, so that 31.5 is the approximate cube root for the second proportional part –and the cube root of 2 equal to $(31.5) / 25 = 1.26$.

33. Actually Richard points out that the lines of solids must be referred to any straight line divided into 10^n equal parts, ($n \geq 2$).
34. For instance, those in Clavius' *Geometria practica* [1606, p, 378-388].

Propositions 2-5 [M-RAH 9/2779a, p. 17] use the lines of solids to increase and decrease rectilinear figures and spheres in a given ratio, and to add and subtract volumes. The chapter finishes with proposition 6 [M-RAH 9/2779a, p. 18], on how to find 2 lines continuously proportional to 2 given lines (*Els*.XI.33, XI.37).

3.4. Side B: Lines of degrees (B_{Gr}), Lines of regular polygons (B_5), and Lines of isometric regular polygons (B_Δ)

The fourth chapter [M-RAH 9/2779a, p. 18-21] consists of seven propositions on the two central straight lines on sector side B. These lines are referred to as B_{Gr} –Gr meaning degrees, but they actually are the lines of chords, for proposition 1 asks to mark the chords on these lines of the circle degrees from 1° to 72°.

To begin with, Richard invokes *the doctrine of the chords of the circle and its sines* in order to assert that the chord of any circular arc is twice the *right sine* of half the arc. He also explains that geometers calculate right sines of arcs on the circle quadrant as parts of a semi diameter divided into equal parts of any power of ten. These parts can be reduced to parts of a semi diameter divided into 100 equal parts by taking the total sine equal to these 100 equal parts. For this purpose, geometers give the following rule: remove zeros of the total sine in order to reduce it to 100, and remove from any other sine as many digits on the right as zeros have been removed from the total sine. This rule shows how trigonometry played a role in promoting decimal computation.[35]

An example follows for a radius of 10^7, as it was used by Regiomontanus [PEURBACH & REGIOMONTANUS, 1541, p. 40-57] and Clavius [1586, p. 67-86; 1607]. The chord of 72° –twice the sine of 36°– divided by 10^5 is 118 parts of a radius divided into 100 equal parts.[36] Therefore, it is clear that the tables of sines are going to be used to mark the chords as parts of a radius divided into 100 equal parts on the lines of degrees from 1° to 72°. As it was the case for the diameter of a given circle in the twelfth proposition of the first chapter, chords were *parts* –not *a part*– of a radius (*Els*.VII.Def.4) of proven usefulness since ancient times, and trigonometric ratios were soundly established.

Nonetheless, the length of the lines of degrees is equal to 118 parts of a radius divided into 100 equal parts, and this radius has to be equal to the chord of 60°. To this end, Richard proceeds as follows (figure 9):

35. This role appears again in the following rule: When a digit greater than 5 is removed, a unit must be added to the next digit.
36. 118 is twice 59, the approximate value of the sine of 36° –5877852 as given in the above-mentioned tables of sines– divided by 10^5.

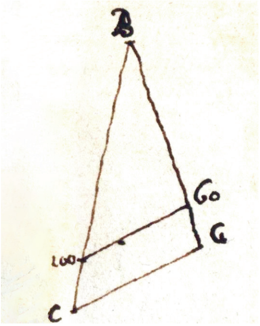


Figure 9. [M-RAH 9/2779a, p. 18, © Reproducción, Real Academia de la Historia]

Let BC be a straight line equal to the addition of the segments A_{100} plus A_{18} on the equal parts line, so that BC be divided into 118 equal parts. Draw BG from B equal to the line of degrees B_{Gr} –BG and BC containing any rectilinear angle, draw CG, draw a parallel to CG from 100 on BC (*Els*.I.31), and mark 60 on BG. BC and BG are proportionally cut at 100 on BC and 60 on BG (*Els*.VI.2), for 100C : B_{100} = 60G : B_{60}; and since the segment 100C is 18 equal parts of B_{100} divided into 100 equal parts, the segment 60G on BG is 18 equal parts of B_{60} divided into 100 equal parts like B_{100}. Therefore, if B60 is conceived as a total sine or radius divided into 100 equal parts of a circle, B_{60} is also the chord of 60° (*Els*.IV.15.Corol.) in a circle where the line BG is the chord of 72°.

By placing the segment B_{60} between A_{100} and A_{100} on the equal parts lines, the interval between the numbers equal to the chord length is the segment to be marked with the corresponding number of degrees on B_{Gr} from B for chords less than 100. For chords greater than 100, the interval between numbers equal to the chord length minus 100 is the segment to be marked with the corresponding number of degrees on B_{Gr} from B_{60}.

Corollary 4 [M-RAH 9/2779a, p. 19] states that the letter M marked on B_{Gr} at 36° is the greater segment to be taken from the radius at 60° to cut it in extreme and mean ratio (*Els*. XIV.4, *Hypsicles* I) [Richard, 1645].[37] No reference is made to *Els*.IV.10, where the side of the decagon –that is, the chord of 36°– cuts the radius of the circle in extreme and mean ratio, possibly because the proof is not easy. However, this is mentioned in proposition 8 –on the use of the sector for this purpose– [M-RAH 9/2779a, p. 19] with reference to *Hypsicles* I.4. [Richard, 1645, p. 472], that offers a simpler argumentation based on Euclid's *Elements*.

Propositions 2-4, 6-7 [M-RAH 9/2779a, p. 20-21] deal with circumferences, circular arcs, and angles. Oddly enough, proposition 5 [M-RAH 9/2779a, p. 20], to inscribe any regular rectilinear figure in a circle, anticipates the next line on the sector.

37. No mention of *Els*. XIV as such appears in Richard [1645].

The fifth chapter [M-RAH 9/2779a, p. 22-23], devoted to the lines of regular polygons (B_5), deals with the two inner lines on sector side B, which are equal in length to the line of chords and benefit from the previously calculated proportional parts. To mark the sides of regular polygons on B_5, the lengths of the chords of 360° divided by the number of sides –from five to twenty– are transferred from the lines of chords. A table of numerical values of degrees is also given.[38] Three propositions show how to use the sector to inscribe regular polygons in a given circle, to construct regular polygons on a given segment, and to circumscribe a circle about a given regular polygon.

In the sixth chapter [M-RAH 9/2779a, p. 24-28], devoted to the lines of isometric regular polygons (B_Δ), the two outer lines on sector side B have to be divided to transform the circle into isometric regular rectilinear figures and vice versa, from the equilateral triangle to the 20-gon. Five lemmas are stated and geometrically proved to show that the sides of isometric polygons can be marked on a line as long as the line of chords B_{Gr}, starting at B and ending at the side of the equilateral triangle isometric to the circle B_Δ. Descending from B, the sides of isometric regular polygons from twenty to four, and the diameter of the circle are marked.[39]

Proposition 1 explains how to divide the lines of isometric regular polygons into proportional parts referring to Richard's treatise on practical geometry:

Transform the equilateral triangle on B_Δ into an isometric circle (chapter 14, problem 39), take the diameter, and mark it on B_Δ as Θ; then transform this circle into an isometric square (chapter 12, problem 2), take the side, and mark it on B_Δ as □; and next transform this square into any isometric regular polygon (chapter 14, problem 37), take the side, and mark it on B_Δ with the number of sides.

No further details are given, but as the side of the equilateral triangle is the chord of 120º (173 for a radius divided into 100 equal parts), its area is $\frac{\sqrt{3}}{4}(173)^2$; therefore, the radius of the isometric circle is $(173/2)\ [(3)^{1/2}/\pi]^{1/2} = 64.24$, and 128.48 the diameter to be marked on B_Δ as Θ.

The proportional mean between the radius of this circle and its semi circumference is the side $[\ \pi\ (64.24)^2]^{\ 1/2} = 113.86$ of the isometric square [BNE Mss 9118, problem 64, p. 30].[40]

Propositions 2, 3 and 4, on the use of the sector to transform isometric figures, show that the intervals between marks *n* on the lines B_Δ –from 5 to 20– give the lengths of the sides of the isometric n-gons to be constructed as shown in chapter 5 proposition 3.

38. Heptagon $51^{3/7}$, 11-gon $32^{8/11}$, 13-gon $17^{9/13}$, 16-gon $22^{1/2}$, 17-gon $21^{3/17}$, 19-gon $19^{18/19}$.
39. The radius of the semicircle is marked between six and seven.
40. The anonymous author of this manuscript introduces himself as Richard's disciple. He says that Richard has been teaching mathematics at the Imperial College for thirty-one years, which dates the manuscript in 1661. The text –that has been written at the request of many gentlemen– consists of 486 problems on practical geometry that he presents as a chapter on geometric practices necessary to trace and measure all kinds of fortifications, to be used as a complement to his manuscript on Military Arts [BNE Mss 9118, 1r-1v]. However, no Richard's treatise on practical geometry has been found yet.

In the absence of further information, a trigonometric approach to the area of regular polygons seems a plausible hypothesis. For any regular n-gon inscribed in a circle, its area is *n* times half the product of its side by the apothem,[41] so that the side of the isometric n-gon is the chord of its central angle α for the radius *r* that makes its area equal to the area of the isometric circle.[42]

3.5. Side C: Lines of the five isometric regular polyhedra (C_P)

The seventh chapter [M-RAH 9/2779a, p. 28-32] consists of six lemmas and five propositions on the two outer straight lines on sector side C, which have to be divided into proportional parts to transform the five regular solid figures –tetrahedron, octahedron, cube, icosahedron and dodecahedron– into an isometric sphere and vice versa.

As in the previous chapter, the sixth lemmas refer to Richard's treatise on practical geometry (chapter 40, problems 40 & 42) to show the order of the marks on the sides of the five regular solid figures on these lines: the side of the tetrahedron and the octahedron are longer than the diameter of the isometric sphere, the side of the octahedron is shorter than the side of the isometric tetrahedron, the side of the cube is shorter than the diameter of the isometric sphere, the side of the icosahedron is shorter than the side of the isometric cube, and the side of the dodecahedron is shorter than the side of the isometric icosahedron [M-RAH 9/2779a, p. 29-30]. On line C_P –starting at C and ending at the side of the tetrahedron (P), the side of the dodecahedron (D), the icosahedron (I), the cube (C), and the octahedron (O) isometric to the sphere (G)[43] can be marked. This is explained in Proposition 1 –again referring to Richard's treatise on practical geometry [M-RAH 9/2779a, p. 30-31]:

Transform the tetrahedron with side CP into an isometric sphere (chapter 40, problem 40), transform the tetrahedron with side CP into an isometric octahedron (chapter 40, problem 48), transform the sphere with diameter CG into an isometric cube (chapter 40, problem 42), transform the cube with side CC into an isometric icosahedron, and transform the icosahedron with side CI into an isometric dodecahedron (chapter 40, problem 48).

No further information is given here. Proposition 2 shows the use of the sector to transform solid isometric figures –including the sphere, and proposition 3 refers again to Richard's treatise on practical geometry to make paper Platonic solids [M-RAH 9/2779a, p. 31; RICHARD, 1645, figs. 295, 298, 300, 303, 306]. Notwithstanding, given the side of any regular solid, proposition 4 geometrically finds the diameter of the circumscribed sphere [M-RAH 9/2779a, p. 31-32; CLAVIUS, 1606, p. 214-215; *Els.*XIII.13, XIII.14, XIII.15, XIII.16, XIII.17].

---

41. Area $A_n = (n/2)\, S_n\, a_n$, where $S_n = 2r_n \sin d_n$, $a_n = r_n \cos d_n$, $\alpha = 2\, d_n = 360/n$, so that $A_n = (n/2)\, 2\, (r_n)^2 \sin d_n \cos d_n = (n/2)\, (r_n)^2 \sin 360/n$.
42. From the former equation de radius of the isometric circle to any n-gon is $r_n = \{ [\pi\, (64.24)^2] / [(n/2) \sin (360/n)] \}^{1/2}$.
43. The diameter of the sphere is marked between the octahedron and the cube with character G meaning *Globe*.

It is worth mentioning that since the square on the diameter of the sphere is one-and-a-half times the square on the side of the inscribed tetrahedron [*Els*.XIII.13], the side of the tetrahedron is 2/3 √6 for a radius of the sphere equal to one unit. Thus the volume of the tetrahedron can be determined, and used to find the radius of the isometric sphere, and the sides of the remaining isometric regular polygons –at least for the cube and the octahedron, for these two figures and the tetrahedron are to one another in rational ratios. Anyway, Richard's expertise on Platonic solids is clear in his commentaries to the thirteenth book of Euclid's *Elements* [RICHARD, 1645, p. 431-462], to Isidore of Miletus' spurious fifteenth book of Euclid's *Elements* [RICHARD, 1645, p. 463-468], to Hypsicles' fourteenth book of Euclid's *Elements* [RICHARD, 1645, p. 469-512], and to François de Foix de Candale's *Euclidis Megarensis Mathematici Clarissimi Elementa* [RICHARD, 1645, p. 512-532].

It should be noted that Richard's student manuscript copy [M-RAH 9/2785, p. 136r-137r] omits the six lemmas and proposition 4, either because he considered them unnecessary –possibly already learnt in the course of Practical Geometry, or because Richard did not explained them in the 1656 course.

3.6. Side C: Lines of sines (C_{Sin})

In the eighth chapter [M-RAH 9/2779a, p. 32-33] the two central straight lines on sector side C have to be divided into proportional parts being the right sines of all the arcs of a circle quadrant with radius equal to 100 parts. For this purpose, tables of sines are used as for the lines of chords in chapter 4. By placing the line C_{Sin} between A_{100} and A_{100} on the equal parts lines,[44] the interval between $A_{Sin(n)}$ and $A_{Sin(n)}$ is equal to the sine of *n* for any natural number from 1 to 90. This length is the segment to be marked with *n* on C_{Sin} from C, so that the segment *Cn* is the sine of *n* degrees for any *n* from 1° to 90°, for a radius equal to the length of C_{Sin} divided into 100 equal parts.

Proposition 2 shows the use of the sector to find the sine, the cosine and the versine of any angle in the first quadrant. A third proposition is added in Richard's student manuscript copy to find the chord [M-RAH 9/2785, p. 138r].

3.7. Side C: Lines of metals (C_{Pet})

In the ninth chapter [M-RAH 9/2779a, p. 33-35] the two inner straight lines on sector side C must be divided into proportional parts for similar solids of a different material and equal weight. Gold, lead, silver, copper, iron, tin, marble and stone are marked with the first characters of their Latin names –*Aurum* (Au), *Plumbum* (Pl), *Argentum* (Ar), *Cuprum* (Cu), *Ferrum* (Fer), *Stannum* (St), *Marmor* (Mar), and *Petra* (Pet).

Eight rectangular right prisms of equal height and weight but different material are needed to graduate the lines of metals. The stone prism immersed in a tub of water up to its

44. This is not necessary if the length of C_{Sin} is equal to the length of A_{100}.

upper surface gives the length of a straight line perpendicular to the base of the tube that has to be divided into one hundred equal parts. Next, the successive immersion of the seven remaining prisms in the tub of water will mark a sequence of numbers on the equal parts line that graduate the lines of metals.[45] For instance, by placing the line C_{Pet} between A_{100} and A_{100} on the equal parts lines, the interval between A_{52} and A_{52} is the segment to be marked with *Au* on C_{Pet} from C.

Proposition 2 [M-RAH 9/2779a, p. 35] shows that intervals between any correspondingly marked pair of points give the diameters of spheres –or the homologous sides of other solid bodies– similar to one another and equal in weight.

Richard adopts a purely experimental method to graduate the lines of metals. No reference to the concept of density is made, but the fact is that similar solids equal in weight, with volumes V_1, V_2 and densities d_1, d_2 have equal mass, so that $V_1 d_1 = V_2 d_2 = m$, and $V_1/V_2 = d_2/d_1$. Consequently, the ratio of the corresponding sides s_1, s_2 of similar solids is $s_1/s_2 = (V_1/V_2)^{1/3} = (d_2/d_1)^{1/3}$. The marks for each material can be calculated from the cube root of its volume, but also from the reciprocal cube root of its density.

3.8. Side D: Lines of tangents (D_{Tang})

In the tenth chapter [M-RAH 9/2779a, p. 35-36], the two central straight lines on sector side D must be divided into proportional parts being the tangents of the arcs of a circle quadrant, with radius equal to the total sine divided into 100 parts, from 1° to 45°. Here again, tables of sines and tangents are used.[46] By placing the line D_{Tang} between A_{100} and A_{100} on the equal parts lines, the interval between $A_{Tan(n)}$ and $A_{Tan(n)}$ is equal to the tangent of *n* for any natural number from 1 to 45. This length is the segment to be marked with *n* on D_{Tang} from D, so that the segment D_n is the tangent of *n* degrees for any *n* from 1° to 45°, for a radius equal to the length of D_{Tang} divided into 100 equal parts. Propositions 2 and 3 show the use of the lines of tangents.

A reference to Richard's treatise on sundials is mentioned at the end of proposition 2 [M-RAH 9/2779a, p. 36][47], a subject where tangents are applied [Coignet, 1618, props. 43-44].

45. The following values are given [M-RAH 9/2779a, p. 34]: gold 52, lead 61, silver 63 1/4, copper 65 1/3, iron 68 2/3, tin 70 1/2, marble 96 1/4, and stone 100. Values differing from these about 20 units are given in [Coignet, 1618, p. 18r].
46. For instance, Clavius' tables of sines, tangents, and secants for a radius equal to 10^7 [Clavius, 1607].
47. For an inventory of anonymous manuscripts on sundials at the Spanish Royal Academy of History see Udías [2005, p. 448]. Among them, *Tratado en el cual se declara por problemas la práctica para describir los reloxes del sol de la hora doze, desde un punto meridiano a otro punto suyo opuesto, en todo género de planos en las espheras oblicuas, por ejemplo, en la oblicua de cuarto grado* is kept in the same file as Richard's student manuscript copy [M-RAH 9/2785 (2)].

3.9. Side D: Lines of circles (D_{Cir}) and Lines of Globes (D_{Gl})

The eleventh chapter deals with the two outer straight lines on sector side D cutting the circle into 60 equal segments –30 for the semicircle, and so does the twelfth chapter with the two inner straight lines cutting the sphere into 60 equal parts [COIGNET, 1618, p. 4v]. In both chapters Richard declares that he will not give the proportional division of these straight lines because he is not aware of its proof, so he will only give their use as explained by Coignet. According to Richard, Coignet supposes the lines of circles divided into 30 proportional parts, the circle whose diameter duplicates the straight lines C_{Cir} (*sic*) divided into 60 equal segments, and the semicircle whose semi diameter is C_{Cir} (*sic*)[48] divided into 30 equal segments [M-RAH 9/2779a, p. 37; COIGNET, 1618, props. 32-34, p. 18v-20r].

Actually, most of Coignet's manuscripts on pantometers in Spanish are handbooks for the use of these instruments, where no precise information on the geometric foundations of the sector scales is given. However, this is implicitly included in the practical use of the sectors scales. Let us see how Coignet proceeded with these lines of circles and globes.

Coignet [1618, p. 4v] introduces Division D_{Cir} dividing a circle into 60 segments equal to each other –30 segments for the semicircle. Then he draws a trigonometric quadrant (figure 10) showing the sines, tangents and secants of fifteen, thirty, and forty five degrees, and the *sinus totus* equal to the radius [COIGNET, 1618, p. 4v-5r].

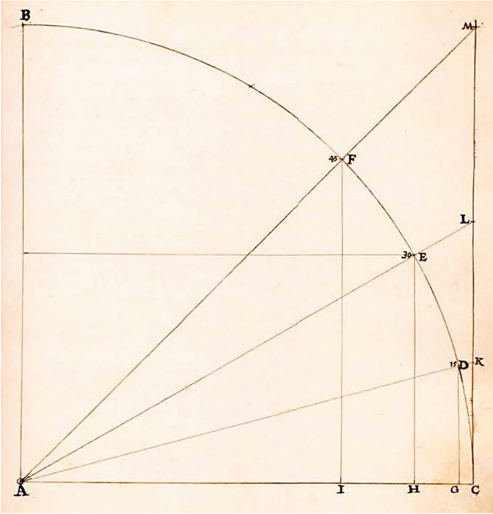


Figure 10. [COIGNET, 1618, p. 13r]

48. These seem to be misprints for D_{Cir} in both cases.

Proposition 4 [Coignet, 1618, p. 6v] explains how to use the equal parts lines A_{100} to find the third part of the semi circumference of radius *r* as the interval A_{60}-A_{60} when segment *r* fits between A_+ and A_+, and also to find half the semi circumference of radius *r* as the interval A_{90}-A_{90} when segment *r* fits between A_+ and A_+. Similarly, proposition 19 uses the line of equal parts to find the length of an arc [Coignet, 1618, p. 12v-13r].

Proposition 20 [Coignet, 1618, p. 12v-13r] shows how to find the areas of circular sectors and segments (figure 11). For any circular sector BAC, he produces AE equal to half the length of arc BOC and perpendicular to the radius AB. With lines AB, AE, he produces the rectangle AEDB[49] that equals the area of the circular sector.

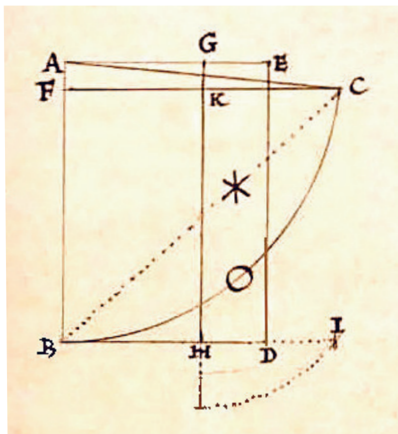


Figure 11. [Coignet, 1618, p. 13r]

He then produces the line B*C that cuts the circular segment BAC into a triangle AB*C and a circular segment B*COB, so that the area of the circular segment will equal the area of the circular sector minus the area of the triangle. He draws CF height of the triangle from F on AB, divides CF in two equal parts at K, divides AE at G so that AG equals FK, and also BD on H. Therefore BH equals FK, and the area of the triangle is then the rectangle AGHB.

Now the area of the circular segment is the rectangle AEDB minus the rectangle AGHB, namely the rectangle GEDH, where GE and DH equal the difference between half the length of arc BOC and half the height of the triangle. Then segment HI is the mean proportional between the radius of the circular sector and HD, namely the side of the square equal to the area of the circular segment.

It should be noticed that Coignet is implicitly working on the fourth quadrant of the circle, so that the height of the triangle is the sine of the arc. However, the construction of circular sectors on central angles makes trigonometric ratios much more clear: the length of

49. ED=AB and parallel to AB, DB=AE and parallel to AE.

the chord is *2r sin* (α/2), the area of the triangle is *(1/2) r^2 sin* (α),[50] and therefore the area of the circular segment is (r^2/2) [(πα/180)- sin α] . Then the sagitta of the circular segment is *2r sin^2* (α/2).[51]

Proposition 32 insists on reminding that D_{Cir} represents the division of a circle into sixty equal segments in order to show how to use this line to cut from any given circle a circular segment whose area is the n^{th} part of 60 –*n* being a natural number– by finding the length of its sagitta –and consequently the length of its chord. Proposition 33 explains the use of D_{Cir} to find the proportion of a circular segment to its circle when its chord and sagitta are given [Coignet, 1618, p. 18v-19r], so that the centre of the circle and its radius can be found.[52] Out of these two propositions we learn that for any natural number (less than 30), the interval D_n-D_n is the sagitta of a circular segment whose area is the n^{th} part of a circle whose radius is fitted crosswise to the interval D_{30}-$D_{30.}$ This results suggest that the lengths of the sines –half the lengths of the chords– should be the segments to be marked with *n* on D_{Cir} for any part *n* of 60 as the square roots of the sagittas.

More information is given in proposition 34 [Coignet, 1618, p. 19v-20r], that shows how to transform a circular segment –defined by its chord and its sagitta– into a circle or any regular polygon. First the center of the circle of which this is a segment is needed in order to find the radius, and then the proportion of the circular segment to its circle in degrees. However, no procedure is given, possibly because the use of the lines of degrees (B_{Gr}) in a similar context was already shown in propositions 17 and 18 –dealing with arcs degrees– [Coignet, 1618, p. 11v-12r]. Out of them it is clear that the radius of the circle is the interval between B_{60} and B_{60} once the chord of the circular segment is fitted crosswise to equal degrees. The proportion of the segment to its circle is as this number of degrees *n* to 60. Next, the lines of planes A□ are used to find a circle whose area equals this number of degrees: the radius of the circle is the interval between $A□_n$ and $A□_n$ once the radius of the former circle is fitted crosswise to numbers 60 –the sagitta being the homologous side. The conclusion is that the area of the latter circle equals the area of the given circular segment of the former circle. This isometric circle can be transformed into any regular polygon using the lines of isometric regular polygons (B_{Δ}).

As for the lines of globes, Coignet [1618, p. 4v] introduces Divisions D_{Gl} as the section of a globe into 60 equal parts –30 for the semicircle. Propositions 35 and 36 are similar to the above-mentioned propositions dealing with the lines of circles,[53] and show that for any

50. *Els.*III.25 can be used to look at segments of a circle with trigonometric eyes, the area of the triangle being *$2r^2$ (1/2) sin* (α/2) *cos* (α/2).
51. The sagitta –the height of the circular segment– equals the *versine* (α) = *r* (1– *cos* (α)) = *r* {1-[cos^2 (α/2) - sin^2 (α/2)]} = *r* [1-cos^2 (α/2) + sin^2 (α/2)] =*r* [sin^2 (α/2) + sin^2 (α/2)].
52. Given a segment of a circle, Coignet [1618, prop. 36, p. 19v] refers to *Els.*III.25 for the construction of the circle of which it is a segment.
53. The data for the spherical cap are the radius of the sphere, the radius of the base of the cap –a circular segment, and the height of the cap.

natural number less than 30, the interval D_n-D_n is the height of an spherical cap equal to the n^{th} part of a globe with radius equal to the interval D_{30}-D_{30} [COIGNET, 1618, p. 19v-20r]. Also similarly, proposition 37 uses the lines of solids instead of the lines of planes to find a sphere equal in volume to the given spherical cap [COIGNET, 1618, p. 20v-21r]. Here again, this isometric sphere can be transformed into any regular solid using the lines of the five isometric regular polyhedra (C_p) [COIGNET, 1618, Prop. 27, p. 16v-17r].

Considering Richard's expertise in Euclid's *Elements* and trigonometry, it seems highly unlikely that he could not follow Coignet's argumentation on circular sectors. The proportion of the circular segment to its circle is based on the intersecting chords theorem [*Els*.III.35] applied to any chord perpendicular to the diameter of the circle.[54] Clavius [1606, p. 201] had also demonstrated that half the chord of a circular segment was a mean proportional to the sagitta and the rest of the diameter. This was applied to a quadrant by Richard [M-RAH 9/2779b, p. 24-25], for the sine of half the central angle is a mean proportional to half the radius and the versine of the central angle.[55] Actually, the two propositions in this chapter prove that Richard trusted Coignet's practical use of this line to deal with proportionality between circular segments and circles [M-RAH 9/2779a, p. 37], but he might have discarded proposing a proportional division of these lines for technical and conceptual reasons. On the one side, the construction of these divisions was hard to check with small pantometers.[56] On the other, the division of the circle was a delicate issue in the area of higher geometry that he possibly considered beyond the interests of students of practical geometry.[57] It is also worth mentioning that no reference to Viète's *Variorum de rebus mathematicis responsorum Liber VIII* (1593) or *Ad Angularium Sectionum Analyticen Theoremata* (1615) has been found up to now among Richard's files at the Spanish Royal Academy of History. [58]

54. When two chords of a circle are cut into two segments at the point of where they intersect, if one chord is cut into two line segments A and B, and the other into the segments C and D, this theorem states that A × B is always equal to C × D no matter where the chords are. Since the sagitta intersects the midpoint of the chord, it is part of a diameter. Using the fact that one part of one chord times the other part is equal to the same product taken along a chord intersecting the first chord, we find that the product of (2*r* – *sagitta*) by the sagitta equals the square of half the bisected chord.

55. $[(1/2)\ crd\ (\alpha)]^2 = [r\ sin\ (\alpha/2)]^2 = r^2\ sin^2\ (\alpha/2) = r.r\ sin^2\ (\alpha/2) = (r/2)\ versine\ (\alpha)$. Richard's proof in proposition 9 of his treatise on plane trigonometry is different.

56. The lines of circles and the lines of globes start on 5 in COIGNET [1618, p. 4r]. By small pantometers we mean between 15 and 18 cm. [Museo Arqueológico Nacional, Madrid, Inventory numbers 56559, 56566].

57. Two manuscripts of Richard's commentaries –and problems– on Archimedes' *De sphaera et cylindro* are kept at the Spanish Royal Academy of History as part of Richard's *Mathesis Varia* [UDÍAS, 2005, p. 405, 434]. Archimedes' *De dimensione circuli, De ysoperimetris, De sphera et cylindro, Quadraturae parabolae, De conoidibus et sphaeroidibus,* and *De lineis spiralibus* can be found in manuscript [M-RAH 9/2787], a manuscript copy of Maurolico's *Theodosii Sphaericorum Elementorum Libri III* (1558).

58. Ten anonymous manuscript pages titled *Angulares sectiones, Vietae* are kept at the Spanish Royal Academy of History (Shelf mark 9/2715).

3.10. Difficult and curious propositions: the ellipse

The aim of this thirteenth and final chapter [M-RAH 9/2779a, p. 38-40] is to show the practical and easy solution of various difficult and curious propositions –eight in all.

Six propositions applying the results of the preceding chapters show the use of the sector to find a straight line equal in square to the sum of rectilinear figures and circles, isometric rectilinear figures or circles on a given line, isometric rectilinear figures or circles to any given circular segment, isometric regular solids or globes to any given spherical cap, isometric rectilinear figures or circles equal in square to the sum of rectilinear figures and circles, and isometric regular solids or globes equal in volume to the sum of regular solids and globes [M-RAH 9/2779a, p. 38-39].

Richard does not solve proposition 7, on sundials, because he considers that this subject has already been practiced in his treatise on sundials [M-RAH 9/2779a, p. 39].

The eighth and last proposition introduces a new geometric figure, the ellipse [M-RAH 9/2779a, p. 40]. It shows the use of the sector to find infinite points of the ellipse with axes AB, FG as follows (figure 12).

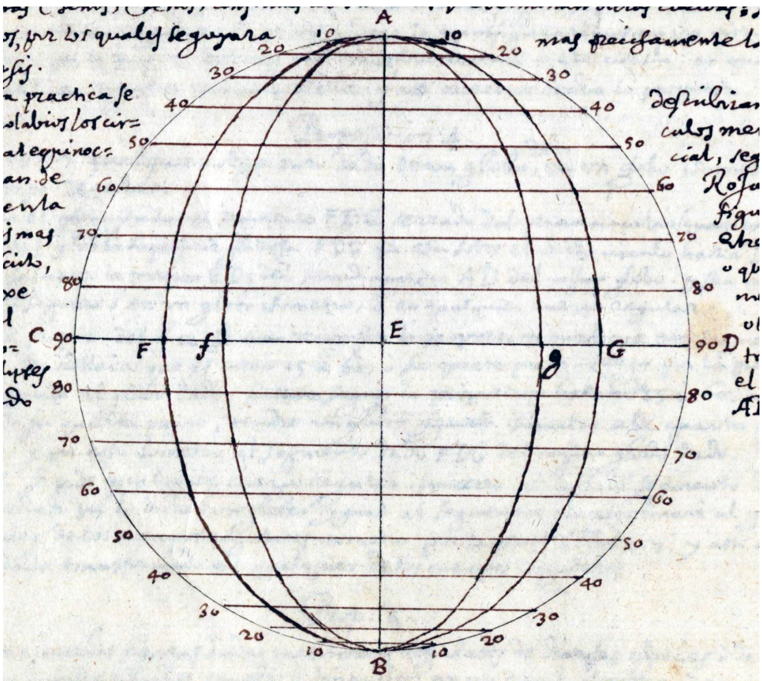


Figure 12. [M-RAH 9/2779a, p. 40, © Reproducción, Real Academia de la Historia]

For any two given perpendicular straight lines AB > FG bisecting at E, draw the circle ACBD with center *E* and radius EA. Fit the radius crosswise to numbers 60 on the lines of degrees (B_{Gr}), and leaving the instrument at this setting take crosswise on the equal parts lines (A_{100}) the distance between points 10-10. With this distance, cut the arcs of the circle AD, DB, BC, and CA into nine equal parts (from ten to ninety), and mark the corresponding chords parallel to CD. Next, fit EG or FE crosswise to numbers 90 on the lines of sines (C_{Sin}), and, leaving the instrument at this setting, take crosswise the distances between points *n*10 - *n*10 ($1 \leq n \leq 8$), place them on the corresponding chord from AE, and mark these points, that are points of the ellipsis with axes AB, FG. Obviously, the four arcs of the circle divided into more equal parts produce a more accurate drawing of the ellipse.

It should be mentioned that Richard was probably familiar with Guidobaldo dal Monte's ellipsograph [Monte, 1579, p. 125-128].[59] However, a closer source might be his colleague della Faille, who also wrote on this same kind of construction of the ellipse, that can be interpreted in terms of parametric equations[60] [Meskens, 2005, p. 66].

4. Conclusions

Richard's interest in practical geometry is not surprising in a Jesuit context. It enabled him to provide the kind of instruction required by the Court, namely mathematics as applied to the art of war. His course on the construction and use of sectors at the Imperial College aimed at facilitating the teaching and learning of the entire practical geometry by means of pantometers. These measuring devices not only simplified and abridged arithmetical operations, but were also particularly suitable for the calculation of ratios and proportions with a wide range of geometric shapes, and furthermore, divisions were independent from the size of objects and units of measurement.

The use of pantometers was not particularly difficult, as it consisted in setting the separation of the arms, taking the distance from the pivot to a point along one of its scales, and taking the crosswise distance between a point and the corresponding point on the other arm [Drake, 1978, p. 11]. However, a proper and reliable use of the instrument required the introduction of a set of basic concepts that Richard explained on a Euclidean basis in the division of the geometric lines (Sections 3.1 and 3.2): equiangular triangles, similarity, the key concept of homologous –or corresponding– sides, and the definition of a magnitude as a *part* –not *parts*– of a magnitude.

Not only: the validation of the instrument in practical geometry could not disregard incommensurability. In modern terminology, Richard's proportions and numerical ratios are positive rational numbers as defined in *Els*.VII.Def.20. He accepts an approximate solution to determine in numbers the proportion for incommensurable straight lines (Section 3.1), he

59. Richard's disciple refers to Dal Monte's ellipsograph in his book on planispheres [BNE Mss 9118, Problem 136, p. 48v].

60. $x(t) = EG \cos(t)$, $y(t) = AE \sin(t)$.

uses the lines of planes to increase and decrease rectilinear figures and circles in a given ratio –either numerical ratios or magnitudes– (Section 3.2), he considers parts or numbers to graduate the lines of solids (Section 3.3), and also chords as parts of a radius (Section 3.4). Richard was clearly ready to reduce incommensurable quantities to nearest commensurable quantities with no significant difference in practice.[61] Actually, the six first propositions in the thirteenth chapter (Section 3.10) show his interest in pantometers as applied to circles, and even to the ellipse, which shows that sectors played a role in the process of assimilation of incommensurability: the numerical consideration of continuous magnitudes as quantities was to give way to the arithmetization of geometry.[62]

The interaction between geometry and trigonometry in this instrument is also remarkable, for it promoted the latter's expansion beyond astronomy. To boot, the trigonometric lines, together with the lines of planes and the lines of solids might have provided with visual and operational inspiration the concept of correspondence –if not function.

As a matter of fact, sectors were appreciated by mathematics practitioners and approved by many scholarly mathematicians, among them Father Richard, whose course on sectors contributed to broaden the horizon of mathematics.

ACKNOWLEDGEMENTS

I am most grateful to Ad Meskens for informing me about the digitized copy of COIGNET [1618]. This work was presented at the History of Science Society Annual Meeting (23 - 27 July 2019, Utrecht, the Netherlands), Session *Practical Mathematics in Early Modern Europe* (Organizer: Thomas Morel). It is financed by the research project HAR2015-70985-P (MINECO/FEDER, UE) –under the direction of Antoni Malet, and by the Research Group on Intellectual and Institutional History of the University of Zaragoza (GIHII, DGA H20_17R, Spain).

BIBLIOGRAPHY

Manuscript sources[63]

• Real Biblioteca (Madrid, Spain)
LAFAILLE, Jean Charles de (S.I.) (1636) *Tratado de la Architectura*. [Shelf mark II/3729].

61. Clavius' construction of the quadratix accepted the intersection of the curve with the axis without a noticeable error, that is, an error that could be perceived by the senses ("sine notabili errore, qui scilicet sum sensum cadat") [CLAVIUS, 1591, vol. VI, p. 350; 1606, p. 321]. See [Bos, 2001, p. 159-166].
62. On notions of proportionality, number and magnitude see [MALET, 1990; 2006].
63. Undated manuscripts are referred to in this paper by means of their shelf marks.

• Real Academia de la Historia (RAH)

Faille, Juan Carlos de la (S.I.) (1640) *Méthodo de la geometría*. [Shelf mark 9/2732].

Richard, Claude (1640) *Sphaerae catholicae liber I de Sphaerae Geometrica: in quo fundamenta geometrica, totius astronomiae, cosmographiae, geographiae et gnomonices iacuntur... ad Philippum IV hispaniarum et indiarum regem catholicum / authore rdo. p. Claudio Richardo Soc. Iesu*. [Shelf mark 9/2684].

Richard, Claude (1653) *Relación del cometa que pareció sobre el orizonte de Madrid a los 20 de diciembre y desaparecio a los 30 del mismo año de 1652 y de su mouimiento, figura y pronóstico conjectural: después con ocasión se trata de la materia de los cielos, estrellas, planetas y cometas y de sus motores y figuras y otros accidentes*. [Shelf mark 9/3638(16)].

[M-RAH 9/2685] *Richardi Arithmética*.

[M-RAH 9/2686] *Arithmeticae practicae compendium / Claudio Richardo*. [3], 191 f. The file also contains: *Quaestiones variae algebricae propositae et solutae* (f. 131-143). [*Propositiones*] (f. 144-151). [*Zetemae*] (f. 156-195v.).

[M-RAH 9/2779a] Richard, Claude (17th century, 1st half) *Tratado de la división de las doce líneas rectas diversas de las pantómetras con el uso práctico dellas en la geometría práctica y juntamente las demonstraciones de essas divisiones y del uso*. [f. 117-140 : il. ; 32 cm. Part of *Mathesis varia* M-RAH 9/2779 (f 117-140) CCPB000918281-0].

[M-RAH 9/2779b] Richard, Claude (17th century, 1st half) *Tratado del conocimiento quantas partes contienen las cuerdas del círculo, de las de su diámetro divididas en un determinado número de iguales* [f. 141-149]; *Tratado del conocimiento quantas partes contienen los senos del círculo de las de su diámetro dividido en un determinado número de iguales* [f. 150-160]; *Tratado del conocimiento de las partes iguales de las rectas tangentes y secantes de cualquier quadrante de círculo, de las de su semidiámetro o seno total determinadas. Juntamente de las analogías entre los senos, tangentes y secantes - Tratado del uso de las tablas de las cuerdas, senos tangentes y secantes - Tratado de los rectángulos rectilíneos en quanto al valor de sus lados, ángulos y proportión entre ellos por medio de los senos, tangentes y secantes* [f. 160-173]. [Part of *Mathesis varia* M-RAH 9/2779 (f. 141-180) CCPB000918281-0].

[M-RAH 9/2785] Richard, Claude (1656) *Tratado de la division de las doce lineas rectas divididas de las pantometras con el uso práctico dellas en la geometría práctica y juntamente las demostraciones de esas divisiones y del uso. Compuesto por el doctísimo Padre Claudio Ricardo de la selecta Compañía de Jesús, consumado maestro de su real cáthedra de mathemáticas, el qual tratado leyó en la dicha cáthedra con especialíssimo cuidado y enseñó con única charidad y amor, hija de su natural piedad y de su fiel nación borgoñona, el año de 1656 a pocos pero aficionados discípulos, tanto de su persona amable como de las sciencias, el mínimo de los quales esto escrive en la illustríssima villa, émula de las mayores ciudades de la Antigüedad, del feliz Madrid, comenzóse el segundo dia de marzo de dicho año*. [f. 117-150 : il. ; 31 cm. Part of a factitious volume M-RAH 9/2785 (4) CCPB000635706-7].

• Erfgoedbibliotheek Hendrik Conscience (Antwerp, Belgium)

Coignet, Michiel (1618) [Coñíetto, Miguel] *El uso de las doze divisiones geometricas, puestas en las dos Reglas Pantometras, por las quales, con ayuda de un Compas commun, se pueden, con gran facilidad, resolver quasi todos los Problemos Mathematicos; Invencion geometrica, sacada por la mayor parte de los ocultos y segretos mysterios que ay en la arte de los numeros, que vulgaramente se llama, el Arte mayor ò Algebra. Platicado y conpuesto por Miguel Coñíetto, natural de la ciudad de Anvers, Mathematico del Serenissimo, muy alto, y poderoso Señor, Alberto, Archiduque de Austria, Duque de*

Borgoña, Brabante, &c. Conde de Habsburg y Flandes. Antwerpen. B 264708, Collectie Stad Antwerpen, Erfgoedbibliotheek Hendrik Conscience

• Bibliothèque Nationale de France

Coignet, Michel (17th century) [Conieto, Miguel] *El uso de las divisiones geometricas puestas en las dos reglas pantometras, por las quales, con aiuda de un conpas comun, se pueden con gran facilidad resolver los problemas mathematicos, invencion geometrica sacada por la maior parte de los ocultos y secretos misterios que ay en la arte de los numeros, que vulgarmente se llama el arte mayor o algebra. Platicado y conpuesto por Miguel Conieto, natural de la ciudad de Anveres y matematico del serenissimo señor archiduque Alberto*.

• Biblioteca Nacional de España (BNE)

[BNE Mss 9118] *Geometría práctica aplicada a la fortificación y arte militar*. S. XVII.

Printed sources

Bos, Henk J.M. (2001) *Redefining Geometrical Exactness: Descartes' Transformation of the Early Modern Concept of Construction*. New York, Springer-Verlag. <doi: 10.1007/978-1-4613-0087-8>.

Camerota, Filippo (2000) *Il compasso di Fabrizio Mordente. Per la storia del compasso di proporzione*. Firenze, Olschki.

Camerota, Filippo (2004) *The geometric and military compass of Galileo Galilei*. Firenze, Istituto e Museo di Storia della Scienza.

Camerota, Filippo (2006) "Teaching Euclid in a Practical Context: Linear Perspective and Practical Geometry". *Science & Education, 15*, 323–334. <doi: 10.1007/s11191-004-3054-1>.

Cerrillo Martín de Cáceres, Enrique (2017) "El proyecto del P. Juan Carlos della Faille S.I. para fortificar Cáceres en 1644". *Revista de Estudios Extremeños, LXXIII (III)*, 2645-2684. <https://www.dip-badajoz.es/cultura/ceex/reex_digital/reex_LXXIII/2017/T.%20LXXIII%20n.%203%202017%20sept.-dic/93068.pdf>. [Consulta: 28-mayo-2021].

Clavius, Christoph (1586) *Theodosii Tripolitae Sphaericorum libri III item eiusdem Christophori Clauii Sinus, lineae tangentes et secantes, triangula rectilinea atque sphaerica*. Romae, ex typographia Dominici Basae.

Clavius, Christoph (1591) *Euclidis Elementorum libri XV: Accessit XVI. de solidorum regularium cuiuslibet intra quodlibet comparatione*. Coloniae, Ioh. Baptistae Ciotti.

Clavius, Christoph (1606) *Christophori Clavii Bambergensis e Societate Iesu Geometria practica*. Moguntia, ex Typographeo Ioannis Albini.

Clavius, Christoph (1607) *Tabulae Sinuum, Tangentium et Secantium ad partes radij 10000000 et ad earum praxim brevis introductio, ex pleniore Tractatu*. Moguntiæ, J. Albini.

Connette, Michel (1626) *La geometrie redvite en une facile et briefve practique, par deux excellens instruments, dont l'un est le pantometre ou compas de proportion de Michel Connette*. Paris, Hulpeau. <http://echo.mpiwg-berlin.mpg.de/MPIWG:3SYMQ7HR>. [Consulta: 28-mayo-2021].

Dameri, Annalisa (2017) "Progettare le difese: il marchese di Leganés e il padre gesuita Francesco Antonio Camassa, esperto di arte militare". In: *Defensive architecture of the Mediterranean: XV to XVIII centuries. Vol. I*. Valencia, Universitat Politècnica de València, 29-36.

De Dainville, François (1954) "L'enseignement des mathématiques dans les Collèges Jésuites de France du XVIe au XVIIIe siècle (suite)". *Revue d'histoire des sciences et de leurs applications*, *7(2)*, 109-123. <doi: 10.3406/rhs.1954.3404>.

De Lucca, Denis (2012) *Jesuits and fortifications: the contribution of the Jesuits to military architecture in the Baroque age*. Boston, Brill.

De Lucca, Denis (2021) *Jesuits, Mathematics and Military Architecture*. European Network for Baroque Cultural Heritage. <doi: 10.14615/enbach21>.

Della Faille, Joannes (1632) *Theoremata de centro gravitatis partium circuli et ellipsis*. Antwerp, Jan van Meurs.

Drake, Stillman (1978) *Galileo Galilei, Operations of the geometric and military compass, 1606. Translated, with an introduction by Stillman Drake*. Washington, D.C., Smithsonian Institution Press.

Heath, Thomas L. (1897) *The Works of Archimedes*. Cambridge, At University Press.

Henrion, Didier (1618) *Usage du compas de proportion*. Paris, chez Michel Daniel, en l'isle du Palais, pres le Cheval blanc.

Joyce, David E. (ed.) (1996) *Euclid's Elements*. Clark University, Department of Mathematics and Computer Science. <http://aleph0.clarku.edu/~djoyce/java/elements/elements.html>. [Consulta: 28-mayo-2021].

Malet, Antoni (1990) "Changing notions of proportionality in pre-modern mathematics". *Asclepio*, *42(1)*, 183–211. <doi: 10.3989/asclepio.1990.v42.1.574>.

Malet, Antoni (2006) "Renaissance notions of number and magnitude". *Historia mathematica*, *33(1)*, 63–81. <doi: 10.1016/j.hm.2004.11.011>.

Meskens, Ad. (2005) *Joannes della Faille S. J. Mathematics, Modesty and Missed Opportunities.* Brussel, Belgisch Historisch Instituut te Rome.

Meskens, Ad. (2013) *Practical mathematics in a commercial metropolis*. Springer, Dordrecht. <doi: 10.1007/978-94-007-5721-9>.

Monte, Guidobaldo dal (1579) *Planisphaeriorum universalium theorica*. Pisavri, Apud Hieronymum Concordiam.

Navarro Loidi, Juan (1999) "Francisco Isasi, jesuita, matemático e ingeniero militar". *Bidasoako Ikaskuntzen Aldizkaria / Boletín de Estudios del Bidasoa*, *19*, 49-76.

O'Neill, Charles & Domínguez, Joaquín Mª. (dirs.) (2001) *Diccionario histórico de la Compañía de Jesús. Vol. II*. Madrid, Universidad Pontificia Comillas.

Petit, Pierre (1634) *L'usage ou le Moyen de Pratiquer par une Regle Toutes les operations du Compas de Proportion*. Paris, chez Melchior Mondiere, dans la Cour du Palais, place du Change, aux deux Viperes.

Peurbach, Georg von & Regiomontanus, Joannes (1541) *Tractatvs Georgii Pevrbachii svper propositiones Ptolemaei de sinubus & chordis. Item compositio tabularum sinuum per Ioannem de Regiomonte. Adiectae sunt & Tabulae sinuum duplices per eundem Regiomontanum. Omnia nunc primum in utilitatem astronomiae studiosis impressa*. Nuremberg, Iohan. Petreium.

Richard, Claude (1645) *Euclides elementorum geometricorum libros tredecim Isidorum et Hypsiclem et Recentiores de Corporibus Regularibus, et Procli propositiones geometricas*. Antverpiæ, Ex Officina Hieronymi Verdvsii.

Richard, Claude (1655) *Apollonii Pergæi Conicorvm libri IV cum commentariis R.P. Claudii Richardi.* Antverpiæ, Ex Officina Hieronymum & Ioannem Bapt. Verdvsem.

Sasaki, Chikara (2003) *Descartes's Mathematical Thought*. Dordecht, Springer Netherlands. <doi: 10.1007/978-94-017-1225-5>.

Simón Díaz, José (1992) *Historia del Colegio Imperial de Madrid*. Madrid, Instituto de Estudios Madrileños.

UDÍAS, Agustín (2005) "Los libros y manuscritos de los profesores de matemáticas del Colegio Imperial". *Archivum Historicum Societatis Iesu, 74,* 369-448.
VAN DER VYVER, Omer (1977) "Lettres de J.Ch. della Faille, S.I., Cosmographe du roi à Madrid, à M.F. Van Langren, cosmographe du roi à Bruxelles, 1634-1645". *Archivium Historicum Societatis Iesu, 46*, 73-183.
VICENTE MAROTO, María Isabel & ESTEBAN PIÑEIRO, Mariano (1991) *Aspectos de la ciencia aplicada en la España del Siglo de Oro*. Salamanca, Junta de Castilla y León, Consejería de Cultura y Bienestar Social.