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"Comments on a paper of Idrisi, Ullah and Sikkandhar (Effect of Perturbations in Coriolis and Centrifugal Forces on Libration Points in the Restricted Six-Body Problem: JAS (2021) 68:4-25)"

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In a recent paper, Idrisi, Ullah and Sikkandhar [1] considered some aspects of the Restricted Six-Body Problem. The problem consists of the motion of an infinitesimal mass under the gravity field of five bodies in a very special configuration: There are four equal points with equal masses m located at the vertices of a regular square that is rotating on its own plane about its center of mass with constant angular velocity ω. Besides, there is a fifth body of mass m_0 placed at the center of the square.

It is not said in Idrisi et al. [1], hereafter IUS Paper, but this a just a particular example of the so-called *Maxwell ring* or *N-gon problem*, which consists of an arbitrary number of equal bodies placed at the vertices of a regular N-gon. Maxwell [2] built this model to simulate Saturn's rings and already studied the stability of the discrete particle ring. Soon after, Tisserand [3, Ch. 12] reformulated Maxwell's analysis and obtained a relation between the mass of each ring particle and the number of them for the system be linearly stable. The problem was tackled again by Scheeres [4] who investigated this configuration including Hill stability, invariant transformations, equilibrium points, periodic orbits, and other features. Later on, independently, Kalvouridis[5] reintroduced the problem, and after then, many papers on this topic have been published. A quite complete list of related publications may be found in [6].

What is important to remark is that the $N+1$ configuration is a relative configuration, that is, the configuration becomes equilibrium of Newton's differential equations in uniformly rotating coordinates. Two principal facts have to be considered in this problem. Firstly, the question of its stability. Moeckel [7] proved that the configuration is linearly stable for $N \geq 7$, and unstable for $N \leq 6$. This is the explanation why most of the considered cases in the literature are for $N = 6$ or $N = 7$. It is worth to mention that when other central forces (different of the pure Newtonian ones) are considered, this condition for the stability changes [8].

Besides, because of the symmetry of the configuration, it is sufficient to restrict the study for the equilibria to the line joining the central primary (or the origin)

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and one of the peripheral bodies (usually the x-axis) and also to one of the bisectors of two peripheral bodies, which reduces the computations.

Secondly, the constant angular velocity ω of the synodic system, is unequivocally determined by the masses. That is to say, once the masses are fixed, ω also remains fixed and cannot be taken as a free parameter as recently some authors assume, e.g. [9–17]. This is the case of IUS Paper. Although authors present correctly the equations of motion in Eq. (1) and also the value of the angular velocity ω, some lines below they decide to change its value. I quote from IUS Paper, page 7:

"We consider the parameters $\alpha = 1 + \nu$ and $\beta = 1 + \rho$, $|\nu| \ll 1$, $|\rho| \ll 1$, as the perturbation in the Coriolis and centrifugal forces respectively. Consequently, Eqns.(1) take the form

$$\ddot{x} - 2\omega\alpha\dot{y} = U_x, \qquad \ddot{y} + 2\omega\alpha\dot{x} = U_y, \tag{2}$$

$$U = \omega^2\beta(x^2 + y^2) + \frac{1 - 4\mu}{r_0} + \sum_{i=1}^{4} \frac{\mu}{r_i}, \qquad ''$$

That is to say, they arbitrarily modify the angular velocity $\omega \mapsto \alpha\omega$, but as it was above said, the angular velocity must be fixed, otherwise the configuration is no longer of relative equilibrium. Besides, as is known from elemental Mechanics courses (see e.g. [18, p. 55]), the Coriolis force (per unit mass) $\mathbf{C}$ with angular velocity vector $\boldsymbol{\omega}$ perpendicular to the plane Oxy is

$$\mathbf{C} = -2\boldsymbol{\omega} \times \mathbf{v} = -2\omega(-\dot{y}, \dot{x}, 0),$$

and the centrifugal force $\mathbf{Z}$ is

$$\mathbf{Z} = -\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = \omega^2(x, y, 0).$$

Hence, if they put in the Coriolis term $\omega' = \alpha\omega$, the corresponding term in the centrifugal force would be $\omega'^2 = \alpha^2\omega^2$, that is, in their paper there should be $\beta = \alpha^2$, which means that β is determined by α, and cannot be a different parameter. All that assuming that it possible to modify the value of the angular velocity, which is not the case.

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Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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