



Baire category results for stochastic orders

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Abstract

In the sense of Baire categories, we prove that the elements of a typical pair of univariate distribution functions (defined on a bounded subset of \mathbb{R}) cannot be compared in the sense of the usual stochastic order, the increasing convex order and the mean residual lifetime order. A similar result is also proved in the class of copulas, i.e. multivariate distribution functions with standard uniform marginals, equipped with the orthant order.

Keywords Baire category · Copula · Orthant dependence · Stochastic order · Weak convergence

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1 Introduction

A large literature has appeared in recent decades about stochastic orders, i.e. partial orders introduced in the class of distribution functions (see, for instance, Müller and Stoyan [7] and Shaked and Shanthikumar [14] for an excellent overview of the state-of-art). Given a partial order \leq_* and two distribution functions F and G , one may wonder whether either $F \leq_* G$ or $G \leq_* F$. Such a question has relevance, for instance, when F and G represent risk distributions (see [7]).

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However, roughly speaking, it is not clear *how many* pairs of distribution functions can be compared according to a given partial order \leq_* . Here, by using the topological description of a set given by Baire categories (see [9]), we are going to show that the set of all pairs of univariate distributions that can be compared by means of some popular stochastic orders is topologically small in the set of all possible pairs of distributions.

Moreover, we discuss an extension of this problem in the class of copulas (see, e.g., Durante and Sempi [4]; Joe [6]; Nelsen [8]) equipped with the lower orthant order by showing that two pairs of copulas are typically not comparable.

2 Preliminaries

Let \mathbb{I} denote the interval $[0, 1]$. Let \mathcal{F} be the class of (right-continuous) distribution functions F with support on \mathbb{I} and such that $F(0^+) \geq 0$ and $F(1) = 1$. Given $F \in \mathcal{F}$, we denote by \bar{F} the associated survival function given by $\bar{F}(x) = 1 - F(x)$ for every $x \in \mathbb{I}$. Moreover, the associated mean residual life function is defined as

$$m_F(x) = \frac{1}{F(x)} \int_x^1 \bar{F}(t) dt$$

for every $x \in \mathbb{I}$ such that $\bar{F}(x) > 0$.

On \mathcal{F} , we consider the weak convergence so that, given $F \in \mathcal{F}$ and a sequence $(F_n)_{n \in \mathbb{N}} \subseteq \mathcal{F}$, F_n tends to F weakly (write $F_n \xrightarrow{w} F$) whenever $F_n(x) \rightarrow F(x)$, as $n \rightarrow \infty$, for all the continuity points x of F . Thanks to Helly-Bray Theorem (see, e.g., Dudley [3]), $F_n \xrightarrow{w} F$ whenever

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} \phi(x) dF_n(x) = \int_{\mathbb{R}} \phi(x) dF(x),$$

for all continuous and bounded functions $\phi : \mathbb{R} \rightarrow \mathbb{R}$.

The Lévy metric d_L on \mathcal{F} is a metrization of the weak convergence, that is $F_n \xrightarrow{w} F$, as n tends to ∞ , if and only if, $d_L(F_n, F) \rightarrow 0$. The metric d_L is defined, for every $F, G \in \mathcal{F}$, as

$$d_L(F, G) = \inf \{ \varepsilon > 0 : \forall x \in \mathbb{R} \ F(x - \varepsilon) - \varepsilon \leq G(x) \leq F(x + \varepsilon) + \varepsilon \}, \tag{1}$$

and it makes (\mathcal{F}, d_L) a complete space (see, e.g., Schweizer and Sklar [11]; Sempi [12]). Given $F \in \mathcal{F}$ and $\varepsilon > 0$, we shall denote by $\bar{B}_{d_L}(F, \varepsilon)$ (respectively, $B_{d_L}(F, \varepsilon)$) the closed (respectively, open) ball of radius ε centered on F . Thus

$$\bar{B}_{d_L}(F, \varepsilon) = \{ G \in \mathcal{F} : \forall x \in \mathbb{R} \ F(x - \varepsilon) - \varepsilon \leq G(x) \leq F(x + \varepsilon) + \varepsilon \}.$$

The following definitions can be found, e.g., in Müller and Stoyan [7]; Shaked and Shanthikumar [14].

Definition 1 Given $F, G \in \mathcal{F}$, we say that

- (a) $F \leq_{st} G$ in the usual stochastic order if $F(x) \geq G(x)$ for every $x \in \mathbb{I}$, or, equivalently, if $\bar{F}(x) \leq \bar{G}(x)$ for every $x \in \mathbb{I}$.
- (b) $F \leq_{icx} G$ in the increasing convex order if

$$\int_{\mathbb{R}} \phi(x) dF(x) \leq \int_{\mathbb{R}} \phi(x) dG(x) \tag{2}$$

for every increasing convex function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ such that the integrals in (2) exist.

(c) $F \leq_{mrl} G$ in the mean residual life order if

$$m_F(x) \leq m_G(x) \tag{3}$$

for every $x \in \mathbb{I}$ such that $\overline{F}(x) > 0$ and $\overline{G}(x) > 0$.

The three partial orders on \mathcal{F} introduced above are linked together as follows.

Proposition 1 *Let $F, G \in \mathcal{F}$. Then:*

- (a) $F \leq_{st} G$ implies $F \leq_{icx} G$;
- (b) $F \leq_{mrl} G$ implies $F \leq_{icx} G$.

We recall that part (a) of Proposition 1 follows from the fact that $F \leq_{st} G$ is equivalent to

$$\int_{\mathbb{R}} \phi(x) dF(x) \leq \int_{\mathbb{R}} \phi(x) dG(x) \tag{4}$$

for every increasing function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ such that the expectations in (4) exist (see [7, Theorem 1.2.8]). Part (b) follows from Shaked and Shanthikumar [14, Theorem 4.A.26].

Notice that, in general, \leq_{st} and \leq_{mrl} are not comparable (see [14, Sect. 2.A.2]).

The following multivariate stochastic order will be also considered (see [7, 14]).

Definition 2 Let $d \geq 2$. Let F and G be d -dimensional distribution functions. We say that $F \leq_{lo} G$ in the lower orthant order if $F(\mathbf{x}) \geq G(\mathbf{x})$ for every $\mathbf{x} \in \mathbb{R}^d$.

Notice that \leq_{lo} is closed under weak convergence and, it is implied by the multivariate version of the order \leq_{st} (see [7, Sect. 3.3]).

Finally, we recall some terminology from Baire category (see [9]). Given a topological space (X, d) , a subset $A \subseteq X$ is called nowhere dense if its closure has empty interior, i.e., $\overline{A} = \emptyset$. If A can be expressed as countable union of nowhere dense subsets of X , then A is called *meager*, or of *first category*. Subsets of X that are not of first category are called of *second category*. If $A^c = X \setminus A$ is meager, then A is called *co-meager*. Loosely speaking, we will also refer to the elements of a co-meager set as typical.

An important result that we need to recall is *Baire Category Theorem*, which asserts that a complete metric space (X, d) is a *Baire space* (we refer to Charalambos and Border [2] for a proof of this result). Baire spaces are, by definition, metric spaces in which every nonempty open set is of second category. Obviously, if X is a Baire space, X itself is of second category. A relevant observation is that, in Baire spaces, e.g. complete metric spaces, a co-meager set $A \subseteq X$ is necessarily of second category, otherwise we would express $X = A \cup A^c$ as a countable union of nowhere dense sets, hence X would be of first category, a contradiction.

3 Stochastic orders of univariate distribution functions

Now, we consider the set $\mathcal{F}^2 := \mathcal{F} \times \mathcal{F}$ equipped with the topology induced by the metric $d_L \times d_L$. For a given stochastic order \leq_* on \mathcal{F} we denote by

$$\mathcal{F}_*^2 = \{(F, G) \in \mathcal{F}^2 : (F \leq_* G) \vee (G \leq_* F)\}$$

the set of all pairs of distribution functions that are comparable according to the given order. Clearly, the complement set of \mathcal{F}_*^2 in \mathcal{F}^2 is given by

$$(\mathcal{F}_*^2)^c = \{(F, G) \in \mathcal{F}^2 : (F \not\leq_* G) \wedge (G \not\leq_* F)\}.$$

Loosely speaking, we are going to show that, in a “typical” pair in $\mathcal{F} \times \mathcal{F}$, the two distribution functions are not comparable with respect to any of the stochastic orders introduced in Sect. 2.

Now, we consider some preliminary results.

Lemma 2 *Let $\leq_* \in \{\leq_{st}, \leq_{icx}, \leq_{mrl}\}$ be a stochastic order on \mathcal{F} . Then \mathcal{F}_*^2 is closed.*

Proof First, note that $\mathcal{F}_*^2 = (\mathcal{F}_*^2)_1 \cup (\mathcal{F}_*^2)_2$, where

$$(\mathcal{F}_*^2)_1 := \{(F, G) \in \mathcal{F}^2 : F \leq_* G\}$$

and

$$(\mathcal{F}_*^2)_2 := \{(F, G) \in \mathcal{F}^2 : G \leq_* F\}.$$

We claim that both $(\mathcal{F}_*^2)_1$ and $(\mathcal{F}_*^2)_2$ are closed, hence \mathcal{F}_*^2 is closed. We only prove that $(\mathcal{F}_*^2)_1$ is closed, since the proof for $(\mathcal{F}_*^2)_2$ is analogous.

Consider a sequence $((F_n, G_n))_{n \in \mathbb{N}} \subseteq (\mathcal{F}_*^2)_1$ such that $F_n \xrightarrow{w} F$ and $G_n \xrightarrow{w} G$. We need to show that $F \leq_* G$. If $\leq_* = \leq_{st}$, then $F \leq_{st} G$ because of Müller and Stoyan [7, Theorem 1.2.14]. If $\leq_* = \leq_{mrl}$, then $F \leq_{mrl} G$ follows from Ahmed [1].

Assume that $\leq_* = \leq_{icx}$. In view of the characterization of \leq_{icx} provided in Müller and Stoyan [7, Theorem 1.5.7], it holds that, for every fixed $a \in \mathbb{I}$ and for every $n \in \mathbb{N}$,

$$\int_a^1 \overline{F}_n(x) \, dx \leq \int_a^1 \overline{G}_n(x) \, dx.$$

Thus, by the Dominated Convergence Theorem,

$$\int_a^1 \overline{F}(x) \, dx \leq \int_a^1 \overline{G}(x) \, dx,$$

that is, $F \leq_{icx} G$, so that the pair (F, G) belongs to the set $(\mathcal{F}_{icx}^2)_1$. □

Remark 1 Notice that, in the space of all distribution functions on \mathbb{R} , the stochastic order \leq_{icx} is not closed with respect to weak convergence (see, for instance, Müller and Stoyan [7, Example 1.5.8]).

Lemma 3 *The set \mathcal{F}_{icx}^2 is nowhere dense.*

Proof We need to prove that \mathcal{F}_{icx}^2 cannot contain any interior points. So let us assume, by way of contradiction, that $(F, G) \in \mathcal{F}_{icx}^2$ is an interior point of \mathcal{F}_{icx}^2 , that is, there exists $\varepsilon > 0$ such that, for every pair $(H, K) \in B_{d_L}(F, \varepsilon) \times B_{d_L}(G, \varepsilon)$, either $H \leq_{icx} K$ or $K \leq_{icx} H$. Without loss of generality, we can assume that $F \leq_{icx} G$.

Now, for every, $\delta, h \in \mathbb{I}$, one can define two functions $F_\delta : \mathbb{R} \rightarrow \mathbb{I}$ and $G_h : \mathbb{R} \rightarrow \mathbb{I}$ in the following way:

$$F_\delta(x) := \begin{cases} \min(1 - \delta, F(x)), & \text{if } x < 1; \\ 1, & \text{if } x \geq 1; \end{cases} \tag{5}$$

and

$$G_h(x) := \begin{cases} G(x), & \text{if } x < h; \\ 1, & \text{if } x \geq h. \end{cases} \tag{6}$$

Notice that both F_δ and G_h belong to \mathcal{F} for all $\delta, h \in \mathbb{I}$.

We claim that it is possible to choose δ and h such that $(F_\delta, G_h) \in B_{d_L}(F, \varepsilon) \times B_{d_L}(G, \varepsilon)$, i.e., there exists $\alpha \in (0, \varepsilon)$ such that, for every x ;

$$F_\delta(x - \alpha) - \alpha \leq F(x) \leq F_\delta(x + \alpha) + \alpha, \tag{7}$$

and

$$G_h(x - \alpha) - \alpha \leq G(x) \leq G_h(x + \alpha) + \alpha, \tag{8}$$

for a suitable choice of δ and h . Indeed, fix any $\alpha \in (0, \varepsilon)$. Note that the left-hand inequality in (7) is true, since $F_\delta(x) \leq F(x)$ for every $x \in \mathbb{I}$. To prove the right-hand inequality, it is enough to choose δ such that $1 - \delta + \alpha \geq 1$, i.e., $\delta \leq \alpha$, that is the only restriction one has on the choice of δ . As far as h , since $G_h(x) \geq G(x)$ for every x , the right-hand inequality in (8) is always true; the left-hand inequality holds if and only if, for every $x \in [\alpha + h, 1)$, one has $G_h(x - \alpha) - \alpha = 1 - \alpha \leq G(x)$, that is, if and only if, h fulfills

$$G(\alpha + h) \geq 1 - \alpha,$$

which is always true if we choose $h \geq 1 - \alpha$. To summarize, we have to choose

$$\delta < \varepsilon, \quad h > 1 - \varepsilon.$$

Moreover, notice that, for every possible choice of δ and for every $a \in \mathbb{I}$, since $F_\delta(x) \leq 1 - \delta$ for all $x \in \mathbb{I}$, one has

$$\int_a^1 \overline{F}_\delta(x) \, dx \geq \delta(1 - a) > 0.$$

Now, without loss of generality, we can assume $F \neq G$, so that there exists some $t \in (0, 1)$ such that

$$\int_t^1 \overline{F}(x) \, dx < \int_t^1 \overline{G}(x) \, dx. \tag{9}$$

Furthermore, we can assume that

$$\int_a^1 \overline{F}(x) \, dx > 0, \quad \text{for every } a \in [0, 1),$$

since we have showed that F and G can be approximated arbitrarily well by pairs of functions that satisfy all of these properties and that belong to $B_{d_L}(F, \varepsilon) \times B_{d_L}(G, \varepsilon)$. Now, let us focus on the pair $(F, G_h) \in B_{d_L}(F, \varepsilon) \times B_{d_L}(G, \varepsilon) \subseteq \mathcal{F}_{icx}^2$. Notice that, for every $x \in [h, 1)$, we have $G_h(x) = 1$, or, equivalently, $\overline{G}_h(x) = 0$. So, if we fix any $a \in [h, 1)$, then

$$\int_a^1 \overline{F}(x) \, dx > 0 = \int_a^1 \overline{G}_h(x) \, dx,$$

that is, $F \not\prec_{icx} G_h$. In order to prove that $G_h \not\prec_{icx} F$, we set

$$\varepsilon_0 := \int_t^1 \overline{G}(x) \, dx - \int_t^1 \overline{F}(x) \, dx > 0$$

(where $t \in (0, 1)$ is the same from (9)), and without loss of generality we assume $h > \max(t, 1 - \varepsilon_0)$ (otherwise, just increase h). Hence we have:

$$\begin{aligned} \int_t^1 \overline{G}_h(x) dx &= \int_t^h \overline{G}(x) dx = \varepsilon_0 + \int_t^1 \overline{F}(x) dx - \int_h^1 \overline{G}(x) dx \\ &\geq \varepsilon_0 + \int_t^1 \overline{F}(x) dx - (1 - h) > \int_t^1 \overline{F}(x) dx, \end{aligned}$$

which shows that $G_h \not\leq_{icx} F$. We have just proved that $(F, G_h) \notin \mathcal{F}_{icx}^2$, which is absurd. Hence \mathcal{F}_{icx}^2 is nowhere dense. \square

Theorem 4 Let $\leq_* \in \{\leq_{st}, \leq_{icx}, \leq_{mrl}\}$ be a stochastic order on \mathcal{F} . Then $(\mathcal{F}_*^2)^c$ is co-meager.

Proof In view of Lemma 3, \mathcal{F}_{icx}^2 is nowhere dense.

Suppose, by way of contradiction, that \mathcal{F}_{st}^2 is not nowhere dense. Since \mathcal{F}_{st}^2 is closed (see Lemma 2), we have $\overline{\mathcal{F}_{st}^2} = \mathcal{F}_{st}^2$, hence $\overset{\circ}{\mathcal{F}_{st}^2} = \overset{\circ}{\mathcal{F}_{st}^2} \neq \emptyset$, i.e., there exists $(F, G) \in \mathcal{F}_{st}^2$ such that $B_{d_L}(F, \varepsilon) \times B_{d_L}(G, \varepsilon) \subseteq \mathcal{F}_{st}^2$, for sufficiently small $\varepsilon > 0$. Then every pair $(H, K) \in B_{d_L}(F, \varepsilon) \times B_{d_L}(G, \varepsilon)$ fulfills $H \leq_{st} K$. In view of Proposition 1, if we consider pairs in \mathcal{F}_{st}^2 , then they are ordered in the \leq_{icx} sense. Thus $\mathcal{F}_{st}^2 \subseteq \mathcal{F}_{icx}^2$, hence we have

$$B_{d_L}(F, \varepsilon) \times B_{d_L}(G, \varepsilon) \subseteq \mathcal{F}_{st}^2 \subseteq \mathcal{F}_{icx}^2,$$

which is absurd since we have assumed that \mathcal{F}_{st}^2 does not contain any interior points.

In the same way, we can prove that \mathcal{F}_{mrl}^2 is not nowhere dense.

Summarizing, $\mathcal{F}_{\leq_*}^2$ is co-meager for every $\leq_* \in \{\leq_{st}, \leq_{icx}, \leq_{mrl}\}$. \square

Remark 2 In Theorem 4, we can replace \mathcal{F} by the class of all distribution functions F supported on $[a, b]$, where $a, b \in \mathbb{R}$, with $F(a^+) \geq 0$ and $F(b) = 1$. In general, however, we cannot directly use the proof of Theorem 4 to the case of all distribution functions on \mathbb{R} since, in such a case, \leq_{icx} is not closed under weak convergence.

Remark 3 Theorem 4 can be extended to any stochastic order that implies the increasing convex order and is closed under weak convergence; consider, for instance, the hazard rate order \leq_{hr} (see [7, Sect. 1.3]). Then it can be proved in a similar way that $(\mathcal{F}_{\leq_{hr}}^2)^c$ is co-meager.

4 A bivariate extension for copulas

Now, we consider an extension to the bivariate case by considering the Fréchet class $\mathcal{F}(F_1, F_2)$ of continuous two-dimensional distribution functions with fixed univariate marginals F_1, F_2 . As known (see, e.g., Rüschendorf [10]), each element F of such a class can be uniquely represented as $F = C(F_1, F_2)$, where C is a copula, i.e. a two-distribution function with standard uniform marginals (see, e.g., Durante and Sempi [4]). Moreover, it holds that:

- for every $F, G \in \mathcal{F}(F_1, F_2)$, $F \leq_{lo} G$ if, and only if, $C_F(x, y) \geq C_G(x, y)$ for every $(x, y) \in \mathbb{I}^2$, where C_F and C_G are the copulas associated with F and G , respectively;
- weak convergence in $\mathcal{F}(F_1, F_2)$ is equivalent to uniform convergence of the corresponding copulas (see, e.g., Sempi [13]).

In view of the one-to-one correspondance between elements of the Fréchet class and copulas, we shall only consider the space of all bivariate copulas, which as usual we denote by \mathcal{C}_2 , equipped with the metric d , defined for every $A, B \in \mathcal{C}_2$ as

$$d(A, B) := \max_{(x,y) \in \mathbb{I}^2} | A(x, y) - B(x, y) | .$$

We recall that the metric space (\mathcal{C}_2, d) is complete (and compact); see, e.g., Durante and Sempi [4].

Given $A, B \in \mathcal{C}_2$, A is less than or equal to B in the pointwise order ($A \leq B$, in symbols), whenever $A(x, y) \leq B(x, y)$ for every $(x, y) \in \mathbb{I}^2$; hence $A \geq B$ if and only if $A \leq_{lo} B$. In particular, for every $C \in \mathcal{C}_2$, it holds

$$W(x, y) := \max(x + y - 1, 0) \leq C(x, y) \leq M(x, y) := \min(x, y)$$

for every $(x, y) \in \mathbb{I}^2$. Notice that, the upper orthant order could be considered as well.

For every $C \in \mathcal{C}_2$, we shall denote by δ_C the *diagonal section* of C , that is, the function $\delta_C : \mathbb{I} \rightarrow \mathbb{I}$ defined by $\delta_C(t) := C(t, t)$, for every $t \in \mathbb{I}$. Note that, for every $t \in \mathbb{I}$ and every copula $C \in \mathcal{C}_2$, one has $\delta_C(t) = C(t, t) \leq C(t, 1) = t$. We can now state the following result.

Theorem 5 *Let C be a fixed copula in \mathcal{C}_2 . The following statements hold:*

- (a) *If there exists some $\varepsilon \in (0, 1)$ such that either $\delta_C(t) < t$ holds for all $t \in (1 - \varepsilon, 1)$ or for all $t \in (0, \varepsilon)$, then the set*

$$\mathcal{A}_C := \{G \in \mathcal{C}_2 : C \leq_{lo} G\}$$

is nowhere dense in (\mathcal{C}_2, d) .

- (b) *If there exists some $\varepsilon \in (0, 1/2]$ such that $\delta_C(t) = t$ holds for all $t \in [0, \varepsilon] \cup [1 - \varepsilon, 1]$, then the set \mathcal{A}_C is of second category in (\mathcal{C}_2, d) .*

Proof First of all, it is straightforward to verify that \mathcal{A}_C is closed with respect to the metric d . Thus, in (a), the goal is to show that \mathcal{A}_C does not contain any interior points, whereas in the proof of (b) we need to prove that \mathcal{A}_C contains at least an interior point (with respect to the metric d). In fact, in this latter case, the assertion will follow from the fact that the metric space (\mathcal{C}_2, d) is a Baire space.

As for (a), let us first assume that $\delta_C(t) < t$ holds for all $t \in (1 - \varepsilon, 1)$ for some $\varepsilon \in (0, 1)$ and that, by way of contradiction, there exist $G \in \mathcal{A}_C$ and $\delta > 0$ such that $\overline{B}_d(G, \delta) \subset \mathcal{A}_C$. Without loss of generality, we can assume $\delta < \varepsilon/2$. Note that every copula $D \in \mathcal{A}_C$ fulfills $\delta_D(t) \leq \delta_C(t)$ for every $t \in \mathbb{I}$, and $\delta_D(t) < t$ for all $t \in (1 - \varepsilon, 1)$. Having set $\gamma := \delta/2$, we define D as the ordinal of G and M with respect to the intervals $[0, 1 - \gamma]$ and $[1 - \gamma, 1]$ (for an overview of ordinal sums, see, e.g., Durante et al. [5]), i.e., D is the copula given, for every $(x, y) \in \mathbb{I}^2$, by

$$D(x, y) := \begin{cases} (1 - \gamma)G \left(\frac{x}{1 - \gamma}, \frac{y}{1 - \gamma} \right), & \text{if } (x, y) \in (0, 1 - \gamma)^2; \\ \min(x, y), & \text{elsewhere.} \end{cases} \tag{10}$$

Since $\gamma < \varepsilon$, we have $1 - \gamma > 1 - \varepsilon$, hence

$$\delta_D(1 - \gamma) = D(1 - \gamma, 1 - \gamma) = 1 - \gamma,$$

which implies that $D \notin \mathcal{A}_C$. We shall prove that $D \in \overline{B}_d(G, \delta) \subset \mathcal{A}_C$ and that will be the contradiction we need. Let $(x, y) \in \mathbb{I}^2$. We have to distinguish several cases.

- If $(x, y) \in (0, 1 - \gamma)^2$ and $D(x, y) \geq G(x, y)$, then, because of the Lipschitz property of G , we have

$$\begin{aligned} |D(x, y) - G(x, y)| &= D(x, y) - G(x, y) \\ &= (1 - \gamma)G\left(\frac{x}{1 - \gamma}, \frac{y}{1 - \gamma}\right) - G(x, y) \\ &\leq G\left(\frac{x}{1 - \gamma}, \frac{y}{1 - \gamma}\right) - G(x, y) \leq (x + y)\left(\frac{1}{1 - \gamma} - 1\right) \\ &\leq 2(1 - \gamma)\left(\frac{\gamma}{1 - \gamma}\right) \leq 2\gamma = \delta. \end{aligned}$$

- If $(x, y) \in (0, 1 - \gamma)^2$ and $D(x, y) < G(x, y)$, then

$$\begin{aligned} |D(x, y) - G(x, y)| &= G(x, y) - D(x, y) \\ &= G(x, y) - (1 - \gamma)G\left(\frac{x}{1 - \gamma}, \frac{y}{1 - \gamma}\right) \\ &\leq G(x, y) - (1 - \gamma)G(x, y) = \gamma G(x, y) \leq \gamma = \frac{\delta}{2}. \end{aligned}$$

- If $\max(x, y) \geq 1 - \gamma$ and $\min(x, y) \geq \gamma$, then

$$\begin{aligned} |D(x, y) - G(x, y)| &= \min(x, y) - G(x, y) \\ &\leq M(x, y) - W(x, y) = 1 - \max(x, y) \leq \gamma = \frac{\delta}{2}. \end{aligned}$$

- If $\max(x, y) \geq 1 - \gamma$ and $\min(x, y) < \gamma$, then

$$|D(x, y) - G(x, y)| = \min(x, y) - G(x, y) \leq M(x, y) < \gamma = \frac{\delta}{2}.$$

Putting all together, we have $d(G, D) \leq \delta$, which proves (a) in this case.

Let us now prove (a) in the other case, assuming that $\delta_C(t) < t$ holds for all $t \in (0, \varepsilon)$ for some $\varepsilon \in (0, 1)$ and that, by way of contradiction, there exist $G \in \mathcal{A}_C$ and $\delta > 0$ such that $\overline{B}_d(G, \delta) \subset \mathcal{A}_C$. Again, without loss of generality, we can assume $\delta < \varepsilon/2$ and we set $\gamma := \delta/2$. Note that every copula $D \in \mathcal{A}_C$ fulfills $\delta_D(t) \leq \delta_C(t)$ for every $t \in \mathbb{I}$, and $\delta_D(t) < t$ for all $t \in (0, \varepsilon)$. In this case, we define D as the ordinal sum of M and G with respect to the intervals $[0, \gamma]$ and $[\gamma, 1]$, i.e., the copula given, for every $(x, y) \in \mathbb{I}^2$, by

$$D(x, y) := \begin{cases} \gamma + (1 - \gamma)G\left(\frac{x - \gamma}{1 - \gamma}, \frac{y - \gamma}{1 - \gamma}\right), & \text{if } (x, y) \in (\gamma, 1)^2; \\ \min(x, y), & \text{elsewhere.} \end{cases} \tag{11}$$

Notice that, for all $t \in [0, \gamma] \subset [0, \varepsilon]$, we have $\delta_D(t) = D(t, t) = \min(t, t) = t$, hence $D \notin \mathcal{A}_C$. We shall prove that $D \in \overline{B}_d(G, \delta) \subset \mathcal{A}_C$ and that will be the contradiction we need. Let $(x, y) \in \mathbb{I}^2$. Again, we have to distinguish several cases.

- Assume $(x, y) \in (\gamma, 1)^2$ and $D(x, y) \geq G(x, y)$. First of all, note that

$$\frac{x - \gamma}{1 - \gamma} \leq x \quad \text{and} \quad \frac{y - \gamma}{1 - \gamma} \leq y.$$

Then:

$$|D(x, y) - G(x, y)| = D(x, y) - G(x, y)$$

$$\begin{aligned}
 &= \gamma + (1 - \gamma)G\left(\frac{x - \gamma}{1 - \gamma}, \frac{y - \gamma}{1 - \gamma}\right) - G(x, y) \\
 &\leq \gamma (1 - G(x, y)) \leq \gamma = \frac{\delta}{2}.
 \end{aligned}$$

- Assume $(x, y) \in (\gamma, 1)^2$ and $D(x, y) < G(x, y)$. Then, because of the Lipschitz property of G :

$$\begin{aligned}
 |D(x, y) - G(x, y)| &= G(x, y) - D(x, y) \\
 &= G(x, y) - \gamma - (1 - \gamma)G\left(\frac{x - \gamma}{1 - \gamma}, \frac{y - \gamma}{1 - \gamma}\right) \\
 &= G(x, y) - G\left(\frac{x - \gamma}{1 - \gamma}, \frac{y - \gamma}{1 - \gamma}\right) + \gamma \left(G\left(\frac{x - \gamma}{1 - \gamma}, \frac{y - \gamma}{1 - \gamma}\right) - 1\right) \\
 &\leq G(x, y) - G\left(\frac{x - \gamma}{1 - \gamma}, \frac{y - \gamma}{1 - \gamma}\right) \leq x - \frac{x - \gamma}{1 - \gamma} + y - \frac{y - \gamma}{1 - \gamma} \\
 &= \frac{x - x\gamma - x + \gamma + y - y\gamma - y + \gamma}{1 - \gamma} = \frac{\gamma(1 - x + 1 - y)}{1 - \gamma} \\
 &< 2\gamma \frac{1 - \gamma}{1 - \gamma} = 2\gamma = \delta.
 \end{aligned}$$

- Assume $\min(x, y) \leq \gamma$. In this last case:

$$|D(x, y) - G(x, y)| = \min(x, y) - G(x, y) \leq M(x, y) \leq \gamma = \frac{\delta}{2}.$$

Putting all together, we have $d(G, D) < \delta$, so that completes the proof of (a).

As for the proof of (b), suppose that there exists some $\varepsilon \in (0, 1/2]$ such that $\delta_C(t) = t$ holds for all $t \in [0, \varepsilon] \cup [1 - \varepsilon, 1]$. We shall prove that $B_d(W, \varepsilon/2) \subset \mathcal{A}_C$. Fix any $G \in B_d(W, \varepsilon/2)$, that is, any copula G such that, for every $(x, y) \in \mathbb{I}^2$, it holds

$$G(x, y) < \max(x + y - 1, 0) + \frac{\varepsilon}{2}. \tag{12}$$

In order to prove that $C(x, y) \geq G(x, y)$ holds for every $(x, y) \in \mathbb{I}^2$, we have to distinguish several cases. So, let $(x, y) \in \mathbb{I}^2$.

- If $\min(x, y) \leq \varepsilon$, then, it holds

$$C(x, y) \geq C(M(x, y), M(x, y)) = \delta_C(M(x, y)) = M(x, y) \geq G(x, y).$$

- If $\min(x, y) > \varepsilon$ and $\max(x, y) \in (\varepsilon, 1 - \varepsilon)$, then we distinguish two subcases.

◊ $W(x, y) = 0$. In this subcase, (12) yields $G(x, y) < \varepsilon/2$. Moreover:

$$C(x, y) \geq C(M(x, y), M(x, y)) \geq C(\varepsilon, \varepsilon) = \delta_C(\varepsilon) = \varepsilon > \frac{\varepsilon}{2} > G(x, y).$$

◊ $W(x, y) > 0$. In this subcase, (12) yields

$$G(x, y) < x + y - 1 + \frac{\varepsilon}{2}.$$

Moreover, because of the Lipschitz property of C , we have

$$C(1 - \varepsilon, 1 - \varepsilon) - C(x, y) \leq 1 - \varepsilon - x + 1 - \varepsilon - y = 2 - 2\varepsilon - x - y,$$

which is equivalent, since $\delta_C(1 - \varepsilon) = 1 - \varepsilon$, to

$$C(x, y) \geq x + y - 1 + \varepsilon,$$

hence

$$C(x, y) \geq x + y - 1 + \varepsilon > x + y - 1 + \frac{\varepsilon}{2} > G(x, y).$$

- If $\min(x, y) > \varepsilon$ and $\max(x, y) \geq 1 - \varepsilon$, then, as in the first case, we shall prove that $C(x, y) = M(x, y)$. Indeed, having set $z := \max(x, y)$, because of the Lipschitz property of C one has

$$C(z, z) - C(x, y) \leq z - x + z - y,$$

which is equivalent, since $\delta_C(z) = z$, to

$$C(x, y) \geq x + y - z = x + y - \max(x, y) = \min(x, y).$$

Since $C(x, y) = M(x, y) \geq G(x, y)$, the proof of (b) is completed. □

The following result focuses on $\mathcal{C}_2 \times \mathcal{C}_2$ and basically states that, in a “typical” pair of copulas (C_1, C_2) , C_1 and C_2 are not comparable.

Theorem 6 *The set*

$$\mathcal{P} := \{(C_1, C_2) \in \mathcal{C}_2 \times \mathcal{C}_2 : (C_2 \leq_{lo} C_1) \vee (C_1 \leq_{lo} C_2)\}$$

is nowhere dense in $\mathcal{C}_2 \times \mathcal{C}_2$ with respect to the metric $d \times d$.

Proof First of all, it is easy to prove that \mathcal{P} is closed with respect to the metric $d \times d$: indeed, note that

$$\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2,$$

where

$$\mathcal{P}_1 := \{(C_1, C_2) \in \mathcal{C}_2 \times \mathcal{C}_2 : C_2 \leq_{lo} C_1\}$$

and

$$\mathcal{P}_2 := \{(C_1, C_2) \in \mathcal{C}_2 \times \mathcal{C}_2 : C_1 \leq_{lo} C_2\},$$

both \mathcal{P}_1 and \mathcal{P}_2 being closed with respect to the product distance $d \times d$. The assertion will follow from proving that \mathcal{P} does not have any interior points, i.e., every product of open balls of the kind $B_d(C_1, \gamma) \times B_d(C_2, \gamma)$ contains at least one pair (S_1, S_2) of copulas that are not comparable (that is, $S_1 \not\leq_{lo} S_2$ and $S_1 \not\geq_{lo} S_2$), for every possible choice of $(C_1, C_2) \in \mathcal{P}$ and $\gamma \in (0, 1]$. Fix any pair $(C_1, C_2) \in \mathcal{P}$ and choose any $\rho \in (0, \gamma/3)$. We define S_1 as the ordinal sum of W , C_1 , and M with respect to the intervals $[0, \rho]$, $[\rho, 1 - \rho]$ and $[1 - \rho, 1]$, i.e., for every $(x, y) \in \mathbb{I}^2$,

$$S_1(x, y) := \begin{cases} \rho W\left(\frac{x}{\rho}, \frac{y}{\rho}\right), & \text{if } (x, y) \in (0, \rho)^2; \\ \rho + (1 - 2\rho)C_1\left(\frac{x - \rho}{1 - 2\rho}, \frac{y - \rho}{1 - 2\rho}\right), & \text{if } (x, y) \in (\rho, 1 - \rho)^2; \\ \min(x, y), & \text{elsewhere,} \end{cases}$$

and S_2 as the ordinal sum of M , C_2 and W with respect to the same intervals, , i.e., for every $(x, y) \in \mathbb{I}^2$,

$$S_2(x, y) := \begin{cases} \rho + (1 - 2\rho)C_2\left(\frac{x - \rho}{1 - 2\rho}, \frac{y - \rho}{1 - 2\rho}\right), & \text{if } (x, y) \in (\rho, 1 - \rho)^2; \\ 1 - \rho + \rho W\left(\frac{x + \rho - 1}{\rho}, \frac{y + \rho - 1}{\rho}\right), & \text{if } (x, y) \in (1 - \rho, 1)^2; \\ \min(x, y), & \text{elsewhere.} \end{cases}$$

Note that

$$S_1\left(\frac{\rho}{2}, \frac{\rho}{2}\right) = 0 < \frac{\rho}{2} = S_2\left(\frac{\rho}{2}, \frac{\rho}{2}\right),$$

hence $S_1 \not\leq S_2$, and

$$S_2\left(\frac{2 - \rho}{2}, \frac{2 - \rho}{2}\right) = 1 - \rho < \frac{2 - \rho}{2} = S_1\left(\frac{2 - \rho}{2}, \frac{2 - \rho}{2}\right),$$

hence $S_1 \not\leq S_2$. Thus, the copulas S_1 and S_2 are not comparable. We need to prove that

$$(S_1, S_2) \in B_d(C_1, \gamma) \times B_d(C_2, \gamma).$$

For both the proofs of $S_1 \in B_d(C_1, \gamma)$ and $S_2 \in B_d(C_2, \gamma)$, we have to consider several cases, depending on the “position” of an arbitrarily chosen pair $(x, y) \in \mathbb{I}^2$.

STEP 1: $S_1 \in B_d(C_1, \gamma)$. Let $(x, y) \in \mathbb{I}^2$.

- If $(x, y) \in (0, \rho)^2$, then

$$|S_1(x, y) - C_1(x, y)| \leq M(\max(x, y), \max(x, y)) < \rho < \gamma.$$

- If $(x, y) \in (\rho, 1 - \rho)^2$ and $S_1(x, y) \geq C_1(x, y)$, then we distinguish four subcases.

◊ Assume $(x, y) \in (\rho, 1/2]^2$. Since in this subcase

$$\frac{x - \rho}{1 - 2\rho} \leq x \quad \text{and} \quad \frac{y - \rho}{1 - 2\rho} \leq y,$$

we have:

$$\begin{aligned} |S_1(x, y) - C_1(x, y)| &= S_1(x, y) - C_1(x, y) \\ &= \rho + (1 - 2\rho)C_1\left(\frac{x - \rho}{1 - 2\rho}, \frac{y - \rho}{1 - 2\rho}\right) \\ &\quad - C_1(x, y) \leq \rho + (1 - 2\rho)C_1(x, y) - C_1(x, y) \\ &= \rho - 2\rho C_1(x, y) \leq \rho < \gamma. \end{aligned}$$

◊ Assume $y \in (\rho, 1/2]$ whereas $x > 1/2$. Since in this subcase

$$\frac{x - \rho}{1 - 2\rho} > x \quad \text{and} \quad \frac{y - \rho}{1 - 2\rho} \leq y,$$

we have, because of the Lipschitz property of C_1 :

$$\begin{aligned} |S_1(x, y) - C_1(x, y)| &= S_1(x, y) - C_1(x, y) \\ &= \rho + (1 - 2\rho)C_1\left(\frac{x - \rho}{1 - 2\rho}, \frac{y - \rho}{1 - 2\rho}\right) \end{aligned}$$

$$\begin{aligned}
 & -C_1(x, y) \leq \rho + (1 - 2\rho)C_1\left(\frac{x - \rho}{1 - 2\rho}, y\right) - C_1(x, y) \\
 & \leq \rho + \frac{x - \rho}{1 - 2\rho} - x = \rho + \frac{2\rho x - \rho}{1 - 2\rho} = \rho\left(1 + \frac{2x - 1}{1 - 2\rho}\right) \\
 & < \rho\left(1 + \frac{2 - 2\rho - 1}{1 - 2\rho}\right) = 2\rho < \gamma.
 \end{aligned}$$

◊ Assume $x \in (\rho, 1/2]$ whereas $y > 1/2$. This subcase is analogous to the previous one, so the reader can easily verify that, again, it holds

$$|S_1(x, y) - C_1(x, y)| \leq 2\rho < \gamma.$$

◊ Assume $(x, y) \in (1/2, 1 - \rho)^2$. Since in this subcase

$$\frac{x - \rho}{1 - 2\rho} > x \quad \text{and} \quad \frac{y - \rho}{1 - 2\rho} > y,$$

we have, because of the Lipschitz property of C_1 :

$$\begin{aligned}
 |S_1(x, y) - C_1(x, y)| &= S_1(x, y) - C_1(x, y) \\
 &= \rho + (1 - 2\rho)C_1\left(\frac{x - \rho}{1 - 2\rho}, \frac{y - \rho}{1 - 2\rho}\right) \\
 &\quad - C_1(x, y) \leq \rho(1 - 2C_1(x, y)) \\
 &\quad + C_1\left(\frac{x - \rho}{1 - 2\rho}, y\right) - C_1(x, y) \\
 &\leq \rho(1 - 2C_1(x, y)) + \frac{x - \rho}{1 - 2\rho} - x + \frac{y - \rho}{1 - 2\rho} - y \\
 &= \rho\left(1 - 2C_1(x, y) + \frac{2x - 1}{1 - 2\rho} + \frac{2y - 1}{1 - 2\rho}\right) \\
 &\leq \rho\left(1 - 2C_1(x, y) + 2\frac{2(1 - \rho) - 1}{1 - 2\rho}\right) \leq 3\rho < \gamma.
 \end{aligned}$$

- Assume $(x, y) \in (\rho, 1 - \rho)^2$ and $S_1(x, y) < C_1(x, y)$. In this case:

$$|S_1(x, y) - C_1(x, y)| = C_1(x, y) - \rho - (1 - 2\rho)C_1\left(\frac{x - \rho}{1 - 2\rho}, \frac{y - \rho}{1 - 2\rho}\right),$$

and again we need to distinguish four subcases, in which we apply the monotonicity, with respect to both arguments, as well as the Lipschitz property, of C_1 .

◊ $(x, y) \in (\rho, 1/2]^2$. We obtain:

$$\begin{aligned}
 |S_1(x, y) - C_1(x, y)| &\leq \rho\left(\frac{1 - 2x}{1 - 2\rho} + \frac{1 - 2y}{1 - 2\rho} - 1 + 2C_1(x, y)\right) \\
 &\leq \rho\left(2\frac{1 - 2\rho}{1 - 2\rho} - 1 + 2C_1(x, y)\right) \leq 3\rho < \gamma.
 \end{aligned}$$

◊ $y \in (\rho, 1/2]$ whereas $x > 1/2$. Since in this subcase

$$\frac{x - \rho}{1 - 2\rho} > x \quad \text{and} \quad \frac{y - \rho}{1 - 2\rho} \leq y,$$

we have:

$$\begin{aligned}
 |S_1(x, y) - C_1(x, y)| &\leq C_1\left(\frac{x - \rho}{1 - 2\rho}, y\right) - \rho - (1 - 2\rho)C_1\left(\frac{x - \rho}{1 - 2\rho}, \frac{y - \rho}{1 - 2\rho}\right) \\
 &\leq \rho\left(\frac{1 - 2y}{1 - 2\rho} - 1 + 2C_1\left(\frac{x - \rho}{1 - 2\rho}, \frac{y - \rho}{1 - 2\rho}\right)\right) \leq 2\rho < \gamma.
 \end{aligned}$$

◊ $x \in (\rho, 1/2]$ whereas $y > 1/2$. This subcase is analogous to the previous one, so the reader can easily verify that, again, it holds $|S_1(x, y) - C_1(x, y)| \leq 2\rho < \gamma$.

◊ $(x, y) \in (1/2, 1 - \rho)^2$. In this last subcase, we have:

$$\begin{aligned}
 |S_1(x, y) - C_1(x, y)| &\leq C_1(x, y) - C_1(x, y) + \rho\left(2 - C_1\left(\frac{x - \rho}{1 - 2\rho}, \frac{y - \rho}{1 - 2\rho}\right)\right) \\
 &\leq \rho < \gamma.
 \end{aligned}$$

- $(x, y) \notin (0, \rho)^2 \cup (\rho, 1 - \rho)^2$. In this case one has $S_1(x, y) = M(x, y) \geq C_1(x, y)$, and there are two subcases.

◊ $\min(x, y) \leq \rho$. In this subcase:

$$|S_1(x, y) - C_1(x, y)| = M(x, y) - C_1(x, y) \leq M(x, y) \leq \rho < \gamma.$$

◊ $\min(x, y) > \rho$. Since in this subcase we necessarily have $\max(x, y) \geq 1 - \rho$, and because of the Lipschitz property of C_1 , it holds

$$\begin{aligned}
 |S_1(x, y) - C_1(x, y)| &= M(x, y) - C_1(x, y) \\
 &\leq 1 - \max(x, y) \leq 1 - (1 - \rho) = \rho < \gamma.
 \end{aligned}$$

Since we covered all the possible cases and subcases, we have finally proved that $d(S_1, C_1) < \gamma$, so that $S_1 \in B_d(C_1, \gamma)$.

STEP 2: $S_2 \in B_d(C_2, \gamma)$. Let $(x, y) \in \mathbb{I}^2$.

- If $(x, y) \in (\rho, 1 - \rho)^2$, then we just replace S_1 with S_2 and C_1 with C_2 in the analogous case covered in STEP 1, and it turns out that

$$|S_2(x, y) - C_2(x, y)| \leq 3\rho < \gamma.$$

- If $(x, y) \in (1 - \rho, 1)^2$, then note that it holds

$$\frac{x + \rho - 1}{\rho} \leq x \quad \text{and} \quad \frac{y + \rho - 1}{\rho} \leq y.$$

If $S_2(x, y) \geq C_2(x, y)$, we have, because of the Lipschitz property of C_2 :

$$\begin{aligned}
 |S_2(x, y) - C_2(x, y)| &= 1 - \rho + \rho W\left(\frac{x + \rho - 1}{\rho}, \frac{y + \rho - 1}{\rho}\right) - C_2(x, y) \\
 &\leq 1 - \rho + \rho C_2(x, y) - C_2(x, y) \leq 1 - \rho + \rho - C_2(x, y) \\
 &\leq C_2(1, 1) - C_2(1 - \rho, 1 - \rho) \leq 2(1 - (1 - \rho)) = 2\rho < \gamma.
 \end{aligned}$$

If, instead, $S_2(x, y) < C_2(x, y)$, then it holds:

$$\begin{aligned}
 |S_2(x, y) - C_2(x, y)| &= C_2(x, y) - 1 + \rho - \rho W\left(\frac{x + \rho - 1}{\rho}, \frac{y + \rho - 1}{\rho}\right) \\
 &\leq \rho\left(1 - W\left(\frac{x + \rho - 1}{\rho}, \frac{y + \rho - 1}{\rho}\right)\right) \leq \rho < \gamma.
 \end{aligned}$$

- If $(x, y) \notin (\rho, 1 - \rho)^2 \cup (1 - \rho, 1)^2$, then one has $S_2(x, y) = M(x, y) \geq C_2(x, y)$, and we distinguish two subcases.

◊ $\min(x, y) \leq \rho$. In this subcase:

$$|S_2(x, y) - C_2(x, y)| = M(x, y) - C_2(x, y) \leq M(x, y) \leq \rho < \gamma.$$

◊ $\min(x, y) > \rho$. Since in this subcase we necessarily have $\max(x, y) \geq 1 - \rho$, and because of the Lipschitz property of C_2 , it holds

$$|S_2(x, y) - C_2(x, y)| = M(x, y) - C_2(x, y) \leq 1 - \max(x, y) \leq 1 - (1 - \rho) = \rho < \gamma.$$

Since we covered all the possible cases and subcases, we have finally proved that $d(S_2, C_2) < \gamma$, so that $S_2 \in B_d(C_2, \gamma)$. Hence

$$(S_1, S_2) \in B_d(C_1, \gamma) \times B_d(C_2, \gamma),$$

which means that every possible product of open balls includes pairs of copulas that are not comparable. This proves the assertion, that is, \mathcal{P} is nowhere dense in $\mathcal{C}_2 \times \mathcal{C}_2$ with respect to the metric $d \times d$. \square

Thus the following result holds.

Corollary 7 (\mathcal{P})^c is co-meager. Thus, two elements of the same Fréchet class $\mathcal{F}(F_1, F_2)$ of continuous distribution functions are typically not comparable in the \leq_{lo} sense.

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